ON RESTRICTED PROJECTIONS TO PLANES IN $\mathbb{R}^3$

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ABSTRACT. Let $\gamma : [0,1] \to \mathbb{S}^2$ be a non-degenerate curve in $\mathbb{R}^3$, that is to say, $\det\left(\gamma(\theta), \gamma'(\theta), \gamma''(\theta)\right) \neq 0$. For each $\theta \in [0,1]$, let $V_\theta = \gamma(\theta)^\perp$ and let $\pi_\theta : \mathbb{R}^3 \to V_\theta$ be the orthogonal projections. We prove that if $A \subset \mathbb{R}^3$ is a Borel set, then for a.e. $\theta \in [0,1]$ we have $\dim(\pi_\theta(A)) = \min\{2, \dim A\}$.

More generally, we prove an exceptional set estimate. For $A \subset \mathbb{R}^3$ and $0 \leq s \leq 2$, define $E_s(A) := \{\theta \in [0,1] : \dim(\pi_\theta(A)) < s\}$. We have $\dim(E_s(A)) \leq \max\{1 + s - \dim(A), 0\}$. We also prove that if $\dim(A) > 2$, then for a.e. $\theta \in [0,1]$ we have $\mathcal{H}^2(\pi_\theta(A)) > 0$.

CONTENTS

1. INTRODUCTION

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere. Let $\gamma : [0,1] \to \mathbb{S}^2$ be a C^2 curve. We say that γ is non-degenerate if

$$(1) \qquad \det(\gamma(\theta), \gamma'(\theta), \gamma''(\theta)) \neq 0,$$

for every $\theta \in [0,1]$. A model example for the non-degenerate curve is $\gamma_\circ : \theta \mapsto \left(\frac{\cos\theta}{\sqrt{2}}, \frac{\sin\theta}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $(\theta \in [0,1])$.

For a given $\theta \in [0,1]$, let $V_\theta = \gamma(\theta)^\perp$ denote the orthogonal complement of $\gamma(\theta)$ in $\mathbb{R}^3$, and let $\pi_\theta : \mathbb{R}^3 \to \gamma(\theta)^\perp$ denote the orthogonal projection onto $\gamma(\theta)^\perp$. For $\alpha > 0$ and a Borel set $E \subset \mathbb{R}^3$, we will use $\mathcal{H}^\alpha(E)$ to denote the α-dimensional

Hausdorff measure of E. Moreover, we use $\dim X$ to denote the Hausdorff dimension of a set X.

Theorem 1. *Suppose $A \subset \mathbb{R}^3$ is a Borel set of Hausdorff dimension α. For $0 \leq s < 2$, define the exceptional set*
$$E_s = \{\theta \in [0,1] : \dim(\pi_\theta(A)) < s\}.$$
Then we have
$$\dim(E_s) \leq \max\{1 + s - \alpha, 0\}.$$

As an immediate corollary, we have:

Corollary 1. *Suppose $A \subset \mathbb{R}^3$ is a Borel set of Hausdorff dimension α. Then we have*
$$\dim(\pi_\theta(A)) = \min\{2, \alpha\}, \text{ for a.e. } \theta \in [0,1].$$

Theorem 2. *Suppose $A \subset \mathbb{R}^3$ is a Borel set of Hausdorff dimension greater than 2. Then*

$$(2) \qquad\qquad \mathcal{H}^2(\pi_\theta(A)) > 0,$$

for almost every $\theta \in [0,1]$.

1.1. Background of the problems.

The projection theory dates back to Marstrand [18], who showed that if A is a Borel set in $\mathbb{R}^2$, then the projection of A onto almost every line through the origin has Hausdorff dimension $\min\{1, \dim A\}$. This was generalized to higher dimensions by Mattila [19], who showed that if A is a Borel set in $\mathbb{R}^n$, then the projection of A onto almost every k-plane through the origin has Hausdorff dimension $\min\{k, \dim A\}$. It is more general to consider the projection problem when the direction set is restricted to some submanifold of the Grassmannian. To have a better understanding of this restricted projection problem, the first step is to study the problem in $\mathbb{R}^3$. Fässler and Orponen made a conjecture about restricted projections to lines and planes (see Conjecture 1.6 in [5]), and there has been much related research (see for example [5], [2], [16], [15], [17], [21], [22], [23], [13], [14]). For more of an introduction to this problem, we refer to [13]. Recently, Käenmäki-Orponen-Venieri [17] and Pramanik-Yang-Zahl [25] proved one half of the conjecture: restricted projections to lines. In this paper, we resolve another half of the conjecture: restricted projections to planes.

1.2. An overview of the high-low method.
The high-low method is a powerful tool developed recently in Fourier analysis. There are many applications of the high-low method, see for example [11], [3], [12], [10], [6], [7].

In this subsection, we briefly discuss how the high-low method can be used to study projection theory. As a warm-up, we study Marstrand's projection theorem from another point of view, using the high-low method.

Theorem 3 (Marstrand's projection theorem). *For each $\theta \in [0, \pi]$, define $L_\theta := \{x \in \mathbb{R}^2 : \arg(x) = \theta\}$ and let $p_\theta : \mathbb{R}^2 \to L_\theta$ be the projection. Suppose $A \subset \mathbb{R}^2$ is a Borel set, then we have $\dim(p_\theta(A)) = \min\{1, \dim A\}$ for a.e. $\theta \in [0, \pi]$.*

We will frequently use the following definition.

Definition 1. *For a number $\delta > 0$ and any set X, we use $|X|_\delta$ to denote the maximal number of δ-separated points in X.*

Marstrand's projection theorem can be reduced to the following discretized version. We do not show how reduction works in this section, but we will give the full details in the later section when we prove our main theorems.

Proposition 1. *Fix $0 < s < 1$. Fix a small scale $\delta > 0$ and set $\Theta = \delta\mathbb{Z} \cap [0,\pi]$. For each $\theta \in \Theta$, define $L_\theta := \{x \in \mathbb{R}^2 : \arg(x) = \theta\}$ and let $p_\theta : \mathbb{R}^2 \to L_\theta$ be the projection. Suppose $A \subset B^2(0,1)$ is a union of disjoint δ-balls with measure $|A| = \delta^{2-a}$, or equivalently $|A|_\delta \sim \delta^{-a}$. We also assume there is a subset $\Theta' \subset \Theta$ with $\#\Theta' \gtrsim \delta^{-1}$, such that for any $\theta \in \Theta'$, $p_\theta(A)$ (which is a union of line segments of length δ in L_θ) satisfies the s-dimensional condition: For each $r \geq \delta$ and line segment $I_r \subset L_\theta$ of length r, we have*

$$(3) \qquad |p_\theta(A) \cap I_r|_\delta \lesssim (r/\delta)^s.$$

Then,

$$(4) \qquad \delta^{-a} \lesssim_s \delta^{-s}.$$

Proof. For each $\theta \in \Theta'$, let $\mathbb{T}_\theta$ be a set of $\delta \times 1$ tubes that cover $p_\theta^{-1}(p_\theta(A)) \cap B^2(0,1)$ and hence cover A. We can also assume that $\mathbb{T}_\theta$ satisfies a similar s-dimensional condition that is inherited from $p_\theta(A)$. For each θ, let S_θ be a $\delta^{-1} \times 1$-tube centered at the origin whose longest direction forms an angle θ with x-axis. We see that S_θ is dual to the tubes in $\mathbb{T}_\theta$. Now, for each $T_\theta \in \mathbb{T}_\theta$, we choose a bump function ψ_θ satisfying the following properties: $\psi_{T_\theta} \geq 1$ on T_θ, ψ_{T_θ} decays rapidly outside T_θ, and $\mathrm{supp}\widehat{\psi}_{T_\theta} \subset S_\theta$.

Define functions

$$f_\theta := \sum_{T_\theta \in \mathbb{T}_\theta} \psi_{T_\theta} \quad \text{and} \quad f = \sum_{\theta \in \Theta'} f_\theta.$$

By definition, $f(x) \gtrsim \#\Theta'$ for any $x \in A$, so we simply have

$$|A|(\#\Theta')^2 \lesssim \int_A |f|^2.$$

We can do better by performing a high-low decomposition for f.

Let K be a large number that will be determined later. Let $\eta_{low}(\xi)$ be a smooth bump function adapted to $B^2(0, K^{-1}\delta^{-1})$ and let $\eta_{high} = 1 - \eta_{low}$. We have the following high-low decomposition for f:

$$f = f_{low} + f_{high},$$

where $\widehat{f}_{low} = \eta_{low}\widehat{f}$ and $\widehat{f}_{high} = \eta_{high}\widehat{f}$. We will show that the high part dominates on A. Actually, for $x \in A$, we have

$$(5) \qquad \delta^{-1} \lesssim f(x) \leq |f_{high}(x)| + |f_{low}(x)|.$$

By definition, $f_{low}(x) = \eta_{low}^\vee * f(x) = \eta_{low}^\vee * \left(\sum_{\theta \in \Theta'}\sum_{T_\theta \in \mathbb{T}_\theta} \psi_{T_\theta}\right)(x)$. Note that $|\eta_{low}^\vee|$ is morally an L^1-normalized bump function at $B^2(0, K\delta)$. By the s-dimensional condition of $\mathbb{T}_\theta$, we have

$$\eta_{low}^\vee * \left(\sum_{T_\theta \in \mathbb{T}_\theta} \psi_{T_\theta}\right)(x) \lesssim K^{-1}\#\{T_\theta \in \mathbb{T}_\theta : T_\theta \cap B^2(x, CK\delta) \neq \emptyset\} \lesssim K^{s-1}.$$

In the last step, we use the condition (3) in Proposition 1. As a result, we have

$$|f_{low}(x)| \lesssim \#\Theta' K^{s-1} \lesssim \delta^{-1}K^{s-1}.$$

Since $s < 1$, by choosing K large enough (depending on s) and plugging into (5), we see that

$$|f_{high}(x)| \geq C^{-1}\delta^{-1} - |f_{low}(x)| \gtrsim \delta^{-1}.$$

We obtain that

$$(6) \qquad |A|\delta^{-2} \lesssim \int_A |f_{high}|^2.$$

Next, we will use a strong separation property for the high part. Note that $\widehat{f}_{high} = \sum_{\theta \in \Theta'} \eta_{high}\widehat{f}_\theta$, and $\{\mathrm{supp}(\eta_{high}\widehat{f}_\theta)\}_\theta$ is at most $O(K)$-overlapped. We have

$$\int |f_{high}|^2 = \int |\widehat{f}_{high}|^2 \lesssim K^2 \sum_{\theta \in \Theta'} \int |\eta_{high}\widehat{f}_\theta|^2 \lesssim K^2 \sum_{\theta \in \Theta'} \int |f_\theta|^2 \lesssim K^2 \delta^{-s}.$$

In the last inequality, we used the s-dimensional condition of $p_\theta(A)$.

As a result, we have

$$|A|\delta^{-2} \lesssim_s \delta^{-s},$$

which implies $\delta^{-a} \lesssim_s \delta^{-s}$. $\hfill\square$

Notation.

(1) For two positive real numbers R_1 and R_2, we say that $R_1 \lesssim R_2$ if there exists a large constant C, depending on relevant parameters, such that $R_1 \leq CR_2$; we say that $R_1 \ll R_2$ if $R_1 \leq R_2/C$.

(2) We use $\dim(E)$ for the Hausdorff dimension of E.

(3) For a given Borel measure μ supported on $\mathbb{R}^3$ and the projection π_θ, the pushforward measure $\pi_{\theta\#}\mu$, supported on $\gamma(\theta)^\perp$, is defined by

$$(\pi_{\theta\#}\mu)(E) := \mu((\pi_\theta)^{-1}(E)),$$

for every Borel $E \subset \gamma(\theta)^\perp$.

(4) Let μ be a compactly supported Borel measure on $\mathbb{R}^3$. Take $\alpha > 0$. Define

$$c_\alpha(\mu) := \sup_{x \in \mathbb{R}^3, r > 0} \frac{\mu(B(x,r))}{r^\alpha},$$

where $B(x,r)$ is the ball of radius r centered at $x \in \mathbb{R}^3$.

(5) For $r \geq 1$ and a rectangular box $T \subset \mathbb{R}^3$, we use rT to mean the dilation of T by r with respect to the center of T, unless otherwise stated. For $r > 0$ and $E \subset \mathbb{R}^3$, we use $r \cdot E$ to mean $\{rx : x \in E\}$.

(6) We often use $m(E)$ the Lebesgue measure of the set $E \subset \mathbb{R}^3$; that is, $m(E) = \mathcal{H}^3(E)$.

(7) By a measure we always assume that it is non-negative, unless stated otherwise.

2. Proof of Theorem 1

In this section we prove Theorem 1. First, we make some remarks on γ. We can cut γ into several small pieces and work on each of them. From now on, we assume that $\gamma : [0, a] \to \mathbb{S}^2$ is C^2 and non-degenerate, and satisfies

(7) $$\gamma(0) = (0, 0, 1), \quad \gamma'(0) = (1, 0, 0), \; |\gamma'(\theta)| = 1, \forall \theta \in [0, a].$$

Here $a > 0$ is sufficiently small depending on γ. Since the parameter a does not play any role here, we may pretend $a = 1$. We need some notation.

Definition 2 $((\delta, s)$-set$)$. *Let $P \subset \mathbb{R}^n$ be a bounded set. Let $\delta > 0$ be a dyadic number, and let $0 \leq s \leq d$. We say that P is a (δ, s)-set if*

$$|P \cap B_r|_\delta \lesssim (r/\delta)^s,$$

for any B_r being a ball of radius r with $\delta \leq r \leq 1$.

Let $\mathcal{H}^t_\infty$ denote the t-dimensional Hausdorff content which is defined as

$$\mathcal{H}^t_\infty(B) := \inf\{\sum_i r(B_i)^t : B \subset \cup_i B_i\}.$$

Here, each B_i in the covering is a cube and $r(B_i)$ is the length of the cube. We recall the following result (see [5] Lemma 3.13).

Lemma 1. *Let $\delta, s > 0$, and $B \subset \mathbb{R}^n$ with $\mathcal{H}^s_\infty(B) := \kappa > 0$. Then there exists a δ-separated (δ, s)-set $P \subset B$ with cardinality $\#P \gtrsim \kappa\delta^{-s}$.*

Our main effort will be devoted to the proof of the following theorem.

Theorem 4. *Fix $0 < s < 2$. For each $\varepsilon > 0$, there exists $C_{s,\varepsilon}$ so that the following holds. Let $\delta > 0$. Let $H \subset B^3(0, 1)$ be a union of disjoint δ-balls and we use $\#H$ to denote the number of δ-balls in H. Let Θ be a δ-separated subset of $[0, 1]$ such that Θ is a (δ, t)-set and $\#\Theta \gtrsim (\log \delta^{-1})^{-2}\delta^{-t}$ for some $t > 0$. Assume for each $\theta \in \Theta$, we have a collection of $\delta \times \delta \times 1$-tubes $\mathbb{T}_\theta$ pointing in direction $\gamma(\theta)$. Each $\mathbb{T}_\theta$ satisfies the s-dimensional condition:*

(1) $\#\mathbb{T}_\theta \lesssim \delta^{-s}$,
(2) $\#\{T \in \mathbb{T}_\theta : T \cap B_r\} \lesssim (\frac{r}{\delta})^s$, for any B_r being a ball of radius r $(\delta \leq r \leq 1)$.

We also assume that each δ-ball contained in H intersects $\gtrsim (\log \delta^{-1})^{-2}\#\Theta$ many tubes from $\cup_{\theta \in \Theta} \mathbb{T}_\theta$. Then

$$\#\Theta\#H \leq C_{s,\varepsilon}\delta^{-1-s-\varepsilon}.$$

2.1. δ-discretization of the projection problem. In this subsection we show how Theorem 4 implies Theorem 1. Before starting the proof, we state a very useful lemma. We use the following notation. For any $\delta = 2^{-k}$ $(k \in \mathbb{N}^+)$, let $\mathcal{D}_\delta$ denote the lattice of δ-squares in $[0, 1]^2$. For technical reasons, we remove the top edge and the right edge of each δ-square so that they are disjoint.

Lemma 2. *Suppose $X \subset [0, 1]^2$ with $\dim X < s$. Then for any $\varepsilon > 0$, there exist dyadic squares $\mathcal{C}_{2^{-k}} \subset \mathcal{D}_{2^{-k}}$ $(k > 0)$ so that*

(1) $X \subset \bigcup_{k>0} \bigcup_{D \in \mathcal{C}_{2^{-k}}} D$,
(2) $\sum_{k>0} \sum_{D \in \mathcal{C}_{2^{-k}}} r(D)^s \leq \varepsilon$,
(3) $\mathcal{C}_{2^{-k}}$ satisfies the s-dimensional condition: For $l < k$ and any $D \in \mathcal{D}_{2^{-l}}$, we have $\#\{D' \in \mathcal{C}_{2^{-k}} : D' \subset D\} \leq 2^{(k-l)s}$.

Proof of the lemma. Consider all the covering $\mathcal{C}$ of X by dyadic lattice squares that satisfy condition (1), (2) in Lemma 2, i.e., $\mathcal{C} \subset \bigcup_{k>0} \mathcal{D}_{2^{-k}}$, $X \subset \bigcup_{D \in \mathcal{C}} D$ and $\sum_{D \in \mathcal{C}} r(D)^s \leq \varepsilon$. We also assume all the dyadic squares in $\mathcal{C}$ are disjoint. We will define an order "$<$" between any two of such coverings $\mathcal{C}, \mathcal{C}'$. First, we define the k-th covering number of $\mathcal{C}$ by

$$c_k(\mathcal{C}) := \#(\mathcal{C} \cap \mathcal{D}_{2^{-k}}),$$

which is the number of 2^{-k}-squares in the covering $\mathcal{C}$.

We say $\mathcal{C} < \mathcal{C}'$, if they satisfy: (1) There is a maximal $k_0 \geq 0$ such that $\mathcal{C} \cap \mathcal{D}_{2^{-k}} = \mathcal{C}' \cap \mathcal{D}_{2^{-k}}$ ($k < k_0$), and $\mathcal{C} \cap \mathcal{D}_{2^{-k_0}} \subset \mathcal{C}' \cap \mathcal{D}_{2^{-k_0}}$; (2) For any $x \in X$, the square in $\mathcal{C}'$ that covers x contains the square in $\mathcal{C}$ that covers x. It is not hard to check the transitivity: If $\mathcal{C} < \mathcal{C}'$ and $\mathcal{C}' < \mathcal{C}''$, then $\mathcal{C} < \mathcal{C}''$.

Suppose $\mathcal{C}$ is a covering that is maximal with respect to the order $<$. Then we can show that $\mathcal{C}$ satisfies condition (3) in Lemma 2. Suppose by contradiction, there exist $l < k$ and $D \in \mathcal{D}_{2^{-l}}$ so that

$$(8) \qquad \#\{D' \in \mathcal{C} \cap \mathcal{D}_{2^{-k}} : D' \subset D\} > 2^{(k-l)s}.$$

We define another covering $\mathcal{C}'$ by adding D to $\mathcal{C}$ and deleting $\{D' \in \mathcal{C} \setminus \{D\} : D' \subset D\}$ from $\mathcal{C}$. It is easy to check that $\mathcal{C}'$ is still a covering of X. By (8), we can also check $\sum_{D \in \mathcal{C}'} r(D)^s < \sum_{D \in \mathcal{C}} r(D)^s \leq \varepsilon$, so $\mathcal{C}'$ satisfies (2) in Lemma 2. However, $\mathcal{C} < \mathcal{C}'$ which contradicts the maximality of $\mathcal{C}$.

Now, it suffices to find a maximal element among all the coverings that satisfy condition (1), (2) in Lemma 2. First of all, such covering exists by the definition of Hausdorff dimension and $\dim X < s$. By Zorn's lemma, it suffices to find an upper bound for any ascending chain.

Let $\{\mathcal{C}_j\}_{j \in J}$ be an infinite chain of coverings of X. Define

$$\mathcal{C} = \bigcap_{j \in J} \bigcup_{\substack{i \in J \\ \mathcal{C}_i \geq \mathcal{C}_j}} \mathcal{C}_i.$$

We show that $\mathcal{C}$ is an upper bound of the chain. First, we show that $\mathcal{C}$ covers X. For $x \in X$ and j, let $D_x^{(j)}$ be the largest dyadic square in $\bigcup_{\substack{i \in J \\ \mathcal{C}_i \geq \mathcal{C}_j}} \mathcal{C}_i$ containing x. By the definition of the partial order and the fact that chains are totally ordered, $D_x^{(j)} = D_x$ is independent of j, and thus $D_x \in \mathcal{C}$. This shows that $\mathcal{C}$ is a covering of X. It also shows that the squares in $\mathcal{C}$ are disjoint. Let $K \in \mathbb{N}$. Choose $j \in J$ such that $\mathcal{C}_i \cap \mathcal{D}_{2^{-k}} = \mathcal{C}_j \cap \mathcal{D}_{2^{-k}}$ for all $0 \leq k \leq K$ and all $\mathcal{C}_i \geq \mathcal{C}_j$. Then

$$\sum_{k=0}^{K} \sum_{D \in \mathcal{C} \cap \mathcal{D}_{2^{-k}}} r(D)^s \leq \sum_{k=0}^{K} \sum_{D \in \mathcal{C}_j \cap \mathcal{D}_{2^{-k}}} r(D)^s \leq \varepsilon.$$

Letting $K \to \infty$ gives

$$\sum_{D \in \mathcal{C}} r(D)^s \leq \varepsilon.$$

So, $\mathcal{C}$ satisfies condition (2). By definition, it is easy to check $\mathcal{C}_i \leq \mathcal{C}$ for every $\mathcal{C}_i$ in the initial chain. This proves that $\mathcal{C}$ is an upper bound of the chain. $\qquad\square$

Remark 1. Besides $[0,1]^2$, this lemma holds for other compact metric spaces, for example $[0,1]^n$ or $\mathbb{S}^2$. The proof is exactly the same.

Proof that Theorem 4 implies Theorem 1. Suppose $A \subset \mathbb{R}^3$ is a Borel set of Hausdorff dimension α. We may assume $A \subset B^3(0,1)$. Recall the definition of the exceptional set

$$E_s := \{\theta \in [0,1] : \dim\pi_\theta(A) < s\}.$$

Recall the definition of the t-dimensional Hausdorff content is given by

$$\mathcal{H}^t_\infty(B) := \inf\{\sum_i r(B_i)^t : B \subset \cup_i B_i\}.$$

A property for the Hausdorff content is that

$$\dim(B) = \sup\{t : \mathcal{H}^t_\infty(B) > 0\}.$$

We choose $a < \dim(A), t < \dim(E_s)$. Then $\mathcal{H}^t_\infty(E_s) > 0$, and by Frostman's lemma there exists a probability measure ν_A supported on A satisfying $\nu_A(B_r) \lesssim r^a$ for any B_r being a ball of radius r. We may assume $t > 0$, otherwise $\dim(E_s) = 0$. We only need to prove

$$a \leq 1 + s - t,$$

since then we can send $a \to \dim(A)$ and $t \to \dim(E_s)$. As a and t are fixed, we may assume $\mathcal{H}^t_\infty(E_s) \sim 1$ is a constant.

Fix a $\theta \in E_s$. By definition, we have $\dim\pi_\theta(A) < s$. We also fix a small number $\epsilon_\circ$ which we will later send to 0. By Lemma 2, we can find a covering of $\pi_\theta(A)$ by disks $\mathbb{D}_\theta = \{D\}$, each of which has radius 2^{-j} for some integer $j > |\log_2 \epsilon_\circ|$. We define $\mathbb{D}_{\theta,j} := \{D \in \mathbb{D}_\theta : r(D) = 2^{-j}\}$. Lemma 2 yields the following properties:

$$(9) \qquad \sum_{D \in \mathbb{D}_\theta} r(D)^s < 1;$$

For each j and r-ball $B_r \subset V_\theta$ with $2^{-j} \leq r \leq 1$, we have

$$(10) \qquad \#\{D \in \mathbb{D}_{\theta,j} : D \subset B_r\} \lesssim \left(\frac{r}{2^{-j}}\right)^s.$$

For each $\theta \in E_s$, we can find such a $\mathbb{D}_\theta$. We also define the tube sets $\mathbb{T}_{\theta,j} := \{\pi_\theta^{-1}(D) : D \in \mathbb{D}_{\theta,j}\} \cap B^3(0,1)$, $\mathbb{T}_\theta = \bigcup_j \mathbb{T}_{\theta,j}$. Each tube in $\mathbb{T}_{\theta,j}$ has dimensions $2^{-j} \times 2^{-j} \times 1$ and direction $\gamma(\theta)$. One easily sees that $A \subset \bigcup_{T \in \mathbb{T}_\theta} T$. By pigeonholing, there exists $j(\theta)$ such that

$$(11) \qquad \nu_A(A \cap (\cup_{T \in \mathbb{T}_{\theta,j(\theta)}} T)) \geq \frac{1}{10j(\theta)^2}\nu_A(A) = \frac{1}{10j(\theta)^2}.$$

For each $j > |\log_2 \epsilon_\circ|$, define $E_{s,j} := \{\theta \in E_s : j(\theta) = j\}$. Then we obtain a partition of E_s:

$$E_s = \bigsqcup_j E_{s,j}.$$

By pigeonholing again, there exists j such that

$$(12) \qquad \mathcal{H}^t_\infty(E_{s,j}) \geq \frac{1}{10j^2}\mathcal{H}^t_\infty(E_s) \sim \frac{1}{10j^2}.$$

In the rest of the poof, we fix this j. We also set $\delta = 2^{-j} (< \epsilon_\circ)$. By Lemma 1, there exists a δ-separated (δ, t)-set $\Theta \subset E_{s,j}$ with cardinality $\#\Theta \gtrsim (\log \delta^{-1})^{-2}\delta^{-t}$.

Next, we consider the set $S := \{(x,\theta) \in A \times \Theta : x \in \cup_{T \in \mathbb{T}_{\theta,j}} T\}$. We also use μ to denote the counting measure on Θ. Define the section of S:

$$S_x = \{\theta : (x,\theta) \in S\}, \quad S_\theta := \{x : (x,\theta) \in S\}.$$

By (11) and Fubini, we have

$$(13) \qquad\qquad (\nu_A \times \mu)(S) \geq \frac{1}{10j^2}\mu(\Theta).$$

This implies

$$(14) \qquad (\nu_A \times \mu)\left(\left\{(x,\theta) \in S : \mu(S_x) \geq \frac{1}{20j^2}\mu(\Theta)\right\}\right) \geq \frac{1}{20j^2}\mu(\Theta).$$

since

$$(15) \qquad (\nu_A \times \mu)\left(\left\{(x,\theta) \in S : \mu(S_x) \leq \frac{1}{20j^2}\mu(\Theta)\right\}\right) \leq \frac{1}{20j^2}\mathcal{H}^a_\infty(A)\mu(\Theta).$$

By (14), we have

$$(16) \qquad\qquad \nu_A\left(\left\{x \in A : \mu(S_x) \geq \frac{1}{20j^2}\mu(\Theta)\right\}\right) \geq \frac{1}{20j^2}.$$

We are ready to apply Theorem 4. Recall $\delta = 2^{-j}$ and $\#\Theta \gtrsim (\log \delta^{-1})^{-2}\delta^{-t}$. By (16), we can find a δ-separated subset of $\{x \in A : \#S_x \geq \frac{1}{20j^2}\#\Theta\}$ with cardinality $\gtrsim (\log \delta^{-1})^{-2}\delta^{-a}$. We denote the δ-neighborhood of this set by H, which is a union of δ-balls. For each δ-ball B_δ contained in H, we see that there are $\gtrsim (\log \delta^{-1})^{-2}\#\Theta$ many tubes from $\cup_{\theta \in \Theta}\mathbb{T}_{\theta,j}$ that intersect B_δ. We can now apply Theorem 4 to obtain

$$(\log \delta^{-1})^{-4}\delta^{-a-t} \lesssim \#\Theta \#H \leq C_{s,\varepsilon}\delta^{-1-s-\varepsilon}.$$

Letting $\epsilon_\circ \to 0$ (and hence $\delta \to 0$) and then $\varepsilon \to 0$, we obtain $a + t \leq 1 + s$.

$\square$

2.2. **Proof of Theorem 4.** The proof of Theorem 4 is base on the L^6 decoupling inequality for cone which is well-understood. For convenience, we will prove the following version of Theorem 4 after rescaling $x \mapsto \delta^{-1}x$.

Theorem 5. *Fix $0 < s < 2$. For each $\varepsilon > 0$, there exists $C_{s,\varepsilon}$ so that the following holds. Let $\delta > 0$. Let $H \subset B^3(0,\delta^{-1})$ be a union of δ^{-a} many disjoint unit balls so that H has measure $|H| \sim \delta^{-a}$. Let Θ be a δ-separated subset of $[0,1]$ so that Θ is a (δ,t)-set and $\#\Theta \gtrsim (\log \delta^{-1})^{-2}\delta^{-t}$. Assume for each $\theta \in \Theta$, we have a collection of $1 \times 1 \times \delta^{-1}$-tubes $\mathbb{T}_\theta$ pointing in direction $\gamma(\theta)$. $\mathbb{T}_\theta$ satisfies the s-dimensional condition:*

    (1) $\#\mathbb{T}_\theta \lesssim \delta^{-s}$,
    (2) $\#\{T \in \mathbb{T}_\theta : T \cap B_r\} \lesssim r^s$, for any B_r being a ball of radius r $(1 \leq r \leq \delta^{-1})$.
We also assume that each unit ball contained in H intersects $\gtrsim |\log \delta^{-1}|^{-2}\#\Theta$ many tubes from $\cup_\theta \mathbb{T}_\theta$. Then

$$\delta^{-t-a} \leq C_{s,\varepsilon}\delta^{-1-s-\varepsilon}.$$

We first discuss the geometry of γ. Let γ be the non-degenerate curve as discussed in the beginning of this section. We have $|\gamma'(\theta)| = 1$. For convenience, we define

$$(17) \qquad \mathbf{e}_1(\theta) := \gamma(\theta), \quad \mathbf{e}_2(\theta) := \gamma'(\theta), \quad \mathbf{e}_3(\theta) := \gamma(\theta) \times \gamma'(\theta).$$

We see that $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta), \mathbf{e}_3(\theta)\}$ form a Frenet coordinate along γ. Define the corresponding conical surface $\Gamma := \{r\mathbf{e}_3(\theta) : 1/2 \leq r \leq 1, \theta \in [0,1]\}$.

We first show that Γ satisfies the same non-degenerate condition as the standard cone. Note that we have the following formulae for the Frenet coordinate:

$$\mathbf{e}_1'(\theta) = \mathbf{e}_2(\theta), \tag{18}$$

$$\mathbf{e}_2'(\theta) = -\mathbf{e}_1(\theta) + \kappa(\theta)\mathbf{e}_3(\theta), \tag{19}$$

$$\mathbf{e}_3'(\theta) = -\kappa(\theta)\mathbf{e}_2(\theta), \tag{20}$$

where $\kappa(\theta) = \langle \mathbf{e}_2'(\theta), \mathbf{e}_3(\theta)\rangle > 0$.

First, we show that Γ is a C^2 surface. We will do this by finding a reparametrization $s = s(\theta)$ so that $\mathbf{e}_3(\theta(s))$ is a C^2 function of s. Choose

$$s(\theta) = \int_0^\theta \kappa(t)dt,$$

and then $\frac{d\theta}{ds} = \kappa(\theta)^{-1}$. We have

$$\frac{d}{ds}\mathbf{e}_3 = \frac{d\theta}{ds} \cdot \frac{d}{d\theta}\mathbf{e}_3 = -\mathbf{e}_2.$$

Since $\theta = \theta(s)$ is C^1, we have $\frac{d}{ds}\mathbf{e}_3 = -\mathbf{e}_2(\theta(s))$ is C^1 with respect to s, and therefore $\mathbf{e}_3(\theta(s))$ is C^2 with respect to s. Moreover, $\det\left(\mathbf{e}_3(\theta(s)), \frac{d}{ds}\mathbf{e}_3(\theta(s)), \frac{d^2}{ds^2}\mathbf{e}_3(\theta(s))\right) = \theta'(s)\det(\mathbf{e}_3(\theta(s)), -\mathbf{e}_2(\theta(s)), \mathbf{e}_1(\theta(s)))$ by the above, which is nonvanishing since $\theta'(s)$ is nonvanishing.

For any large scale R, there is a standard partition of $N_{R^{-1}}\Gamma$ into planks σ of dimensions $R^{-1} \times R^{-1/2} \times 1$:

$$N_{R^{-1}}\Gamma = \bigcup \sigma.$$

For any Schwartz function f, we define $f_\sigma := (1_\sigma \widehat{f})^\vee$ as usual. We have the following L^6-decoupling inequality for these planks.

Theorem 6 (Bourgain-Demeter [1]). *For any Schwartz f with $\widehat{f} \subset N_{R^{-1}}\Gamma$, we have*

$$\|f\|_6 \lesssim_\varepsilon R^\varepsilon \Big(\sum_{\sigma:R^{-1}\times R^{-1/2}\times 1} \|f_\sigma\|_6^2 \Big)^{1/2}. \tag{21}$$

**Remark 2.** *We will actually apply Theorem 6 to a slightly different cone*

$$\Gamma_{K^{-1}} = \{r\mathbf{e}_3(\theta) : K^{-1} \le r \le 1, \theta \in [0,1]\}, \tag{22}$$

for some $K \sim (\log \delta^{-1})^{O(1)}$. Compared with Γ, we see that $\Gamma_{K^{-1}}$ is at distance K^{-1} from the origin, but we still have a similar decoupling inequality. Instead of (21), we have

$$\|f\|_6 \lesssim_\varepsilon K^{O(1)} R^\varepsilon \Big(\sum_{\sigma:R^{-1}\times R^{-1/2}\times 1} \|f_\sigma\|_6^2 \Big)^{1/2}. \tag{23}$$

The idea is to partition $\Gamma_{K^{-1}}$ into $\sim O(K)$ many parts, each of which is roughly a cone for which we can apply Theorem 6. By triangle inequality, this results in an additional factor $K^{O(1)}$. It turns out that this factor is not harmful, since we will set $K \sim (\log R)^{O(1)}$ which can be absorbed into R^ε.

We are ready to prove Theorem 5.
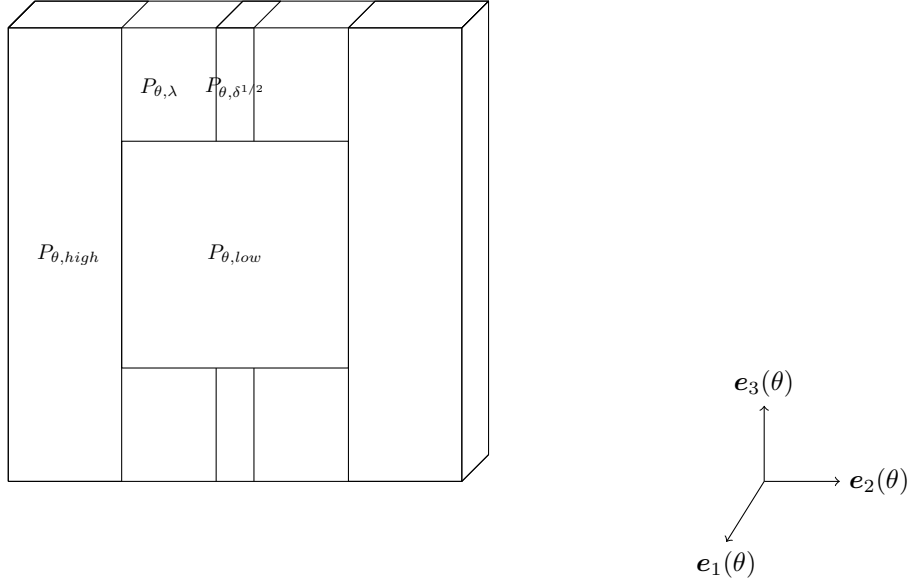
FIGURE 1. High-low decomposition for P_θ

Proof of Theorem 5. Recall that $\mathbb{T}_\theta$ is a collection of $1 \times 1 \times \delta^{-1}$-tubes pointing to direction $\gamma(\theta) = \mathbf{e}_1(\theta)$. We consider the dual of each T_θ in the frequency space. For each θ, we define P_θ to be a slab centered at the origin that has dimensions $1 \times 1 \times \delta$, and its shortest direction is parallel to $\mathbf{e}_1(\theta)$. We see that P_θ is the dual rectangle of each $T_\theta \in \mathbb{T}_\theta$. Now, for each $T_\theta \in \mathbb{T}_\theta$, we choose a bump function ψ_{T_θ} satisfying the following properties: $\psi_{T_\theta} \geq 1$ on T_θ, ψ_{T_θ} decays rapidly outside T_θ, and $\operatorname{supp}\widehat{\psi}_{T_\theta} \subset P_\theta$.

Define functions

$$f_\theta = \sum_{T_\theta \in \mathbb{T}_\theta} \psi_{T_\theta} \qquad \text{and} \qquad f = \sum_{\theta \in \Theta} f_\theta.$$

From our definitions, we see that for any $x \in H$, we have $f(x) \gtrsim \#\{T \in \cup_\theta \mathbb{T}_\theta : x \in T\} \gtrsim (\log \delta^{-1})^{-2} \#\Theta$. Therefore, we obtain

$$(24) \qquad\qquad |H|(\log \delta^{-1})^{-2p}(\#\Theta)^p \leq 2^p \int_H |f|^p,$$

for any p. For our purpose, we just choose $p = 6$, so we have

$$(25) \qquad\qquad |H|(\log \delta^{-1})^{-12}(\#\Theta)^6 \leq 2^6 \int_H |f|^6,$$

Our goal is to find an upper bound for the right hand side of (25). We will decompose P_θ into pieces and estimate the contribution of $\widehat{f}_\theta$ from each piece.

Let us discuss the decomposition for P_θ. Recall that P_θ is a $1 \times 1 \times \delta$-slab centered at the origin with normal direction $\mathbf{e}_1(\theta)$. Recall (17), we can write $P_\theta = \{\sum_{i=1}^3 \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq \delta, |\xi_2| \leq 1, |\xi_3| \leq 1\}$.

Definition 3. *See Figure 1. Let K be a large number which we will choose later. (Actually, we will choose $K \sim (\log \delta^{-1})^{O(1)}$) Define the high part of P_θ as*

$$P_{\theta,high} := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq \delta, K^{-1} \leq |\xi_2| \leq 1, |\xi_3| \leq 1\}.$$

Define the low part of P_θ as

$$P_{\theta,low} := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq \delta, |\xi_2| \leq K^{-1}, |\xi_3| \leq K^{-1}\}.$$

For dyadic numbers $\lambda \in (\delta^{1/2}, K^{-1}]$, define

$$P_{\theta,\lambda} := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq \delta, \frac{1}{2}\lambda \leq |\xi_2| \leq \lambda, K^{-1} \leq |\xi_3| \leq 1\}.$$

In particular, we define

$$P_{\theta,\delta^{1/2}} := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq \delta, |\xi_2| \leq \delta^{1/2}, K^{-1} \leq |\xi_3| \leq 1\}.$$

Remark 3. *We obtain a partition of P_θ as*

$$P_\theta = P_{\theta,high} \bigsqcup P_{\theta,low} \bigsqcup_\lambda P_{\theta,\lambda}.$$

We see that $P_{\theta,\lambda}$ consists of four planks of dimensions $\sim \delta \times \lambda \times 1$ whose longest side is along direction $\mathbf{e}_3(\theta)$. Here λ plays a role of angular parameter in the sense that $P_{\theta,\lambda}$ are roughly those points in $P_\theta \setminus P_{\theta,low}$ so that the lines connecting them with the origin form an angle $\sim \lambda$ with $\mathbf{e}_3(\theta)$.

We choose a smooth partition of unity adapted to this covering which we denote by $\eta_{\theta,high}, \eta_{\theta,low}, \eta_{\theta,\lambda}$, so that

$$(26) \qquad \eta_{\theta,high} + \eta_{\theta,low} + \sum_{\delta^{1/2} \leq \lambda \leq K^{-1}} \eta_{\theta,\lambda} = 1$$

on P_θ. Since $\operatorname{supp}\widehat{f_\theta} \subset P_\theta$, we also obtain a decomposition of f_θ

$$(27) \qquad f_\theta = f_{\theta,high} + f_{\theta,low} + \sum_{\delta^{1/2} \leq \lambda \leq K^{-1}} f_{\theta,\lambda},$$

where $\widehat{f_{\theta,high}} = \eta_{\theta,high}\widehat{f_\theta}, \widehat{f_{\theta,low}} = \eta_{\theta,low}\widehat{f_\theta}, \widehat{f_{\theta,\lambda}} = \eta_{\theta,\lambda}\widehat{f_\theta}$. Similarly, we have a decomposition of f

$$(28) \qquad f = f_{high} + f_{low} + \sum_{\delta^{1/2} \leq \lambda \leq K^{-1}} f_\lambda,$$

where $f_{high} = \sum_\theta f_{\theta,high}, f_{low} = \sum_\theta f_{\theta,low}, f_\lambda = \sum_\theta f_{\theta,\lambda}$.

Recalling (25) and using triangle inequality, we have

$$(29) \quad |H|(\log \delta^{-1})^{-12}(\#\Theta)^6 \lesssim \int_H |f_{low}|^6 + \int_H |f_{high}|^6 + (\log \delta^{-1})^{O(1)} \sum_\lambda \int_H |f_\lambda|^6.$$

We will discuss three cases depending on which term on the right hand side of (29) dominates.

Case 1: Low case

If the first term on the right hand side of (29) dominates, we say we are in the low case. Actually, we will see that we are never in the low case by showing

$$(30) \qquad \int_H |f_{low}|^6 \leq C^{-1}|H|(\log \delta^{-1})^{-12}(\#\Theta)^6,$$

for some large constant C. This means the low term on the right hand side of (29) will not dominate. By properly choosing K, we can show a pointwise bound for f_{low}:

$$(31) \qquad |f_{low}(x)| \leq C^{-1}(\log \delta^{-1})^{-2}\#\Theta.$$

This will immediately imply (30). Let us focus on (31).

Recall that $f_{low} = \sum_\theta f_{\theta,low} = \sum_\theta f_\theta * \eta_{\theta,low}^\vee$. Since $\eta_{\theta,low}$ is a bump function at $P_{\theta,low}$, we see that $\eta_{\theta,low}^\vee$ is a bump function essentially supported in the dual of $P_{\theta,low}$. Denote the dual of $P_{\theta,low}$ by $T_{\theta,K}$ which is a $K \times K \times \delta^{-1}$-tube parallel to $\mathbf{e}_1(\theta)$. One has

$$|\eta_{\theta,low}^\vee| \lesssim \frac{1}{|T_{\theta,K}|}\psi_{T_{\theta,K}}.$$

Here $\psi_{T_{\theta,K}}$ is a bump function $= 1$ on $T_{\theta,K}$ and decays rapidly outside $T_{\theta,K}$.

By definition, $f_\theta = \sum_{T_\theta} \psi_{T_\theta}$ is the sum of bump function of tubes. We have

$$(32) \qquad |f_{low}| \lesssim \sum_\theta \sum_{T_\theta \in \mathbb{T}_\theta} \psi_{T_\theta} * \frac{1}{|T_{\theta,K}|}\psi_{T_{\theta,K}}.$$

If we ignore the rapidly decaying tails, we have

$$(33) \qquad |f_{low}(x)| \lesssim \sum_\theta \frac{1}{K^2}\#\{T_\theta \in \mathbb{T}_\theta : T_\theta \cap B_{100K}(x) \neq \emptyset\}.$$

Recalling the condition (2) in Theorem 5, we have

$$\#\{T_\theta \in \mathbb{T}_\theta : T_\theta \cap B_{100K}(x) \neq \emptyset\} \lesssim (100K)^s.$$

This implies

$$(34) \qquad |f_{low}(x)| \lesssim K^{s-2}\#\Theta.$$

Since $s < 2$, by choosing $K \sim (\log \delta^{-1})^{\frac{2}{2-s}}$, we obtain (31).

In the rest of the proof, we may pretend K is a large constant, since any $(\log \delta^{-1})^{O(1)}$-loss is allowable (see Remark 2).

$\boxed{\text{Case 2: High case}}$

If the second term on the right hand side of (29) dominates, we say we are in the high case.

Since for any $x \in \mathbb{R}^3$ there is at most 1 tube in $\mathbb{T}_\theta$ pass through x, we have $|f_\theta(x)| \lesssim 1$ (here we use $\lesssim 1$ instead of ≤ 1 to take care of the rapidly decaying tail). Recalling the definition $f_{high} = \sum_\theta \eta_{\theta,high}^\vee * f_\theta$ and noting that each $\eta_{\theta,high}^\vee$ is L^1 bounded, we have

$$|f_{high}(x)| \lesssim \sum_\theta |f_\theta(x)| \leq \#\Theta.$$

We see that

$$(35) \qquad \int_H |f_{high}|^6 \lesssim (\#\Theta)^4 \int |f_{high}|^2 = (\#\Theta)^4 \int |\sum_\theta f_{\theta,high}|^2.$$

Next we will show that $\{\operatorname{supp}\widehat{f}_{\theta,high}\}_{\theta\in\Theta}$ are finitely overlapping, i.e., $\{P_{\theta,high}\}_{\theta\in\Theta}$ are finitely overlapping. (Actually they are $O(K)$-overlapping. But since the $K^{O(1)}$-loss are acceptable, we may just pretend $K \lesssim 1$. See also Remark 2.) If this is true, then we have

$$\text{(36)} \qquad \int_H |f_{high}|^6 \lesssim (\#\Theta)^4 \int \sum_\theta |f_{\theta,high}|^2.$$

Since $\int \sum_\theta |f_{\theta,high}|^2 = \int \sum_\theta |\eta_{\theta,high}^\vee * f_\theta|^2 \le \int \sum_\theta |f_\theta|^2 \sim \sum_\theta (\#\mathbb{T}_\theta)\delta^{-1}$. We obtain

$$\text{(37)}$$
$$|H|(\log\delta^{-1})^{-12}(\#\Theta)^6 \lesssim \int |f_{high}|^6 \lesssim (\#\Theta)^4 \sum_\theta (\#\mathbb{T}_\theta)\delta^{-1} \lesssim (\#\Theta)^4 \delta^{-s-t-1},$$

which implies

$$\text{(38)} \qquad \delta^{-a-t} \lesssim \delta^{-s-1-\varepsilon}.$$

Now we prove that $\{P_{\theta,high}\}_{\theta\in\Theta}$ are finitely overlapping. First, recall that

$$P_{\theta,high} = \{\sum_{i=1}^3 \xi_i \mathbf{e}_i(\theta) : |\xi_1| \le \delta, K^{-1} \le |\xi_2| \le 1, |\xi_3| \le 1\}.$$

We see that $P_{\theta,high}$ is contained in the δ-neighborhood of the plane

$$\Pi_{\theta,high} = \{\xi_2\mathbf{e}_2(\theta) + \xi_3\mathbf{e}_3(\theta) : K^{-1} \le |\xi_2| \le 1, |\xi_3| \le 1\}.$$

To show the finitely overlapping property, we just need to show: For any $\theta \in [0,1]$ and $\theta' = \theta + \Delta \in [0,1]$ with $C\delta \le \Delta \le C^{-1}$ (for some bounded C to be determined later), if $\xi \in \Pi_{\theta,high}$, then

$$\operatorname{dist}(\xi, \Pi_{\theta+\Delta,high}) > 10\delta.$$

Write $\xi = a\mathbf{e}_2(\theta) + b\mathbf{e}_3(\theta) = a\gamma'(\theta) + b\gamma(\theta) \times \gamma'(\theta)$, where $|a| \in [K^{-1}, 1]$ and $|b| \le 1$. Since the normal direction of $\Pi_{\theta+\Delta,high}$ is $\gamma(\theta+\Delta)$, it suffices to prove

$$\text{(39)} \qquad |\gamma(\theta+\Delta) \cdot \big(a\gamma'(\theta) + b\gamma(\theta) \times \gamma'(\theta)\big)| \ge 10\delta.$$

By Taylor's expansion, we have $\gamma(\theta+\Delta) = \gamma(\theta) + \Delta\gamma'(\theta) + O(\Delta^2)$. We see the left hand side of (39) is $|a\Delta|\gamma'(\theta)|^2 + O(\Delta^2)| \ge |\big(a - O(\Delta)\big)\Delta| \ge 10\delta$, if C is large enough (depending on K).

$\boxed{\text{Case 3: } \lambda\text{-middle case } (\delta^{1/2} \le \lambda \le K^{-1})}$ If the term $(\log\delta^{-1})^{O(1)} \sum_\lambda \int_H |f_\lambda|^6$ on the right hand side of (29) dominates, we say we are in the λ-middle case. We remark that when λ is close to K^{-1}, $\int_H |f_\lambda|^6$ can be estimated in a similar way as in the High case. We will be interested in the cone

$$\Gamma_{K^{-1}} = \{r\mathbf{e}_3(\theta) : K^{-1} \le r \le 1, \theta \in [0,1]\}.$$

Recall Remark 2 that we still have the decoupling inequality for this cone.

We first discuss the case that $\lambda = \delta^{1/2}$.

$\boxed{\text{Case 3.1: } \lambda = \delta^{1/2}}$

When $\lambda = \delta^{1/2}$, we have $f_{\delta^{1/2}} = \sum_\theta f_{\theta,\delta^{1/2}}$, where each $\widehat{f}_{\theta,\delta^{1/2}}$ is supported in $P_{\theta,\delta^{1/2}}$. Note that $P_{\theta,\delta^{1/2}}$ consists of two pieces: One is

$$P_{\theta,\delta^{1/2}}^+ := \{\sum_{i=1}^3 \xi_i \mathbf{e}_i(\theta) : |\xi_1| \le \delta, |\xi_2| \le \delta^{1/2}, K^{-1} \le \xi_3 \le 1\}.$$

the other is

$$P^-_{\theta,\delta^{1/2}} := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \le \delta, |\xi_2| \le \delta^{1/2}, -1 \le \xi_3 \le -K^{-1}\}.$$

We note that $P^+_{\theta,\delta^{1/2}}$ lies in the δ-neighborhood of $\Gamma_{K^{-1}}$. Symmetrically, $P^-_{\theta,\delta^{1/2}}$ lies in the δ-neighborhood of $\Gamma^-_{K^{-1}}$, which is the reflection of $\Gamma_{K^{-1}}$ with respect to the origin. We can write $f_{\theta,\delta^{1/2}} = f^+_{\theta,\delta^{1/2}} + f^-_{\theta,\delta^{1/2}}$, so that $\mathrm{supp}\widehat{f}^+_{\theta,\delta^{1/2}} \subset P^+_{\theta,\delta^{1/2}}$ and $\mathrm{supp}\widehat{f}^-_{\theta,\delta^{1/2}} \subset P^-_{\theta,\delta^{1/2}}$. We also write $f^+_{\delta^{1/2}} = \sum_\theta f^+_{\theta,\delta^{1/2}}$ and $f^-_{\delta^{1/2}} = \sum_\theta f^-_{\theta,\delta^{1/2}}$, and then $f_{\delta^{1/2}} = f^+_{\delta^{1/2}} + f^-_{\delta^{1/2}}$. We have

$$(40) \qquad\qquad \int |f_{\delta^{1/2}}|^6 \lesssim \int |f^+_{\delta^{1/2}}|^6 + \int |f^-_{\delta^{1/2}}|^6.$$

By symmetry, we only estimate $\int |f^+_{\delta^{1/2}}|^6$.

Note that $P^+_{\theta,\delta^{1/2}}$ and $P^+_{\theta',\delta^{1/2}}$ are essentially the same when $|\theta-\theta'| \lesssim \delta^{1/2}$; $P^+_{\theta,\delta^{1/2}}$ and $P^+_{\theta',\delta^{1/2}}$ are essentially distinct when $|\theta - \theta'| \gtrsim \delta^{1/2}$. We can choose a partition of $N_\delta(\Gamma_{K^{-1}})$ by finitely overlapping planks of dimensions $10\delta \times 10\delta^{1/2} \times 1$, denoted by $\{R\}$. We attach each $P^+_{\theta,\delta^{1/2}}$ to one of the R and denote by $\theta \prec R$, if $P^+_{\theta,\delta^{1/2}} \subset R$. We see that for each R, there are $\lesssim \delta^{-1/2}$ many $P_{\theta,\delta^{1/2}}$ attached to it. We define $f_R := \sum_{\theta \prec R} f^+_{\theta,\delta^{1/2}}$. The Fourier support of f_R is contained in R, by Theorem 6, we have

$$\int |f^+_{\delta^{1/2}}|^6 = \int |\sum_R f_R|^6 \lesssim_\varepsilon \delta^{-\varepsilon}(\#\{R : f_R \ne 0\})^2 \sum_R \|f_R\|_6^6.$$

By pigeonholing, we may pass to a subset of $\{R\}$ so that $\#\{\theta : \theta \prec R\}$ are all comparable. For simplicity, we write $\#\{\theta : \theta \prec R\}$ as $\#\{\theta \prec R\}$, and write the number of $\{R\}$ after pigeonholing as $\#\{R\}$. For each R, we have by triangle inequality:

$$\|f_R\|_6^6 \lesssim \#\{\theta \prec R\}^5 \sum_{\theta \prec R} \|f^+_{\theta,\delta^{1/2}}\|_6^6 \lesssim \#\{\theta \prec R\}^5 \sum_{\theta \prec R} \|f_\theta\|_6^6.$$

We obtain that

$$\int |f^+_{\delta^{1/2}}|^6 \lesssim_\varepsilon \delta^{-\varepsilon} \#\{R\}^2 \#\{\theta \prec R\}^5 \sum_\theta \|f_\theta\|_6^6.$$

Note that $\#\{R\}\#\{\theta \prec R\} \lesssim \#\Theta$, $\#\{\theta \prec R\} \lesssim \delta^{-t/2}$ (by the (δ,t)-spacing of Θ), and $\sum_\theta \|f_\theta\|_6^6 \lesssim \#\Theta\delta^{-1-s}$. We obtain

$$\int |f^+_{\delta^{1/2}}|^6 \lesssim_\varepsilon \delta^{-\varepsilon}(\#\Theta)^3 \delta^{-3t/2-1-s}.$$

Similarly, we have

$$\int |f^-_{\delta^{1/2}}|^6 \lesssim_\varepsilon \delta^{-\varepsilon}(\#\Theta)^3 \delta^{-3t/2-1-s}.$$

As a result, we obtain

$$(41) \qquad\qquad |H|(\#\Theta)^6 \lesssim_\varepsilon \delta^{-2\varepsilon}(\#\Theta)^3 \delta^{-3t/2-1-s}.$$

Combined with $\#\Theta \gtrsim (\log \delta^{-2})^{-2}\delta^{-t}$, we have $\delta^{-a-3t/2} \lesssim \delta^{-\varepsilon-1-s}$ which is even better than what we aimed.

Case 3.2: $\lambda \in (\delta^{1/2}, K^{-1}]$

For λ being a dyadic scale in $(\delta^{1/2}, K^{-1}]$, we see that $\widehat{f}_{\theta,\lambda}$ is supported in $P_{\theta,\lambda}$ which consists of four separated planks ($P_{\theta,\delta^{1/2}}$ only consists of two planks). As in the proof of case $\lambda = \delta^{1/2}$, we will write $f_{\theta,\lambda}$ as the sum of four functions each of which has Fourier support in one of the planks of $P_{\theta,\lambda}$. We will estimate for one of the planks. We define

$$(42) \qquad Q_\theta := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq \delta, \frac{1}{2}\lambda \leq \xi_2 \leq \lambda, K^{-1} \leq \xi_3 \leq 1\}.$$

Roughly speaking, Q_θ is the top-right plank of $P_{\theta,\lambda}$ and the distance between Q_θ and the line $\mathbb{R}\mathbf{e}_3(\theta)$ is $\geq \frac{1}{2}\lambda$. For simplicity, we may assume $f_{\theta,\lambda}$ has Fourier support in Q_θ.

We discuss some geometric properties for the planks $\{Q_\theta\}_{\theta \in \Theta}$. First of all, there is a canonical finitely overlapping covering of $N_{\lambda^2}(\Gamma_{K^{-1}})$ by planks of dimensions $\lambda^2 \times \lambda \times 1$. More precisely, we choose $\Sigma = \lambda\mathbb{Z} \cap [0,1]$ to be a set of λ-lattice points. For each $\sigma \in \Sigma$, define

$$R_\sigma := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\sigma) : |\xi_1| \leq C_1\lambda^2, |\xi_2| \leq C_1\lambda, C_1^{-1}K^{-1} \leq \xi_3 \leq C_1\},$$

where C_1 is a large constant. We see that $\{R_\sigma\}$ form a finitely overlapping covering of $N_{\lambda^2}(\Gamma_{K^{-1}})$. We have the following three properties:

Lemma 3. *For Q_θ, R_σ defined above, we have*

  (1) If $|\theta - \sigma| \lesssim \lambda$, then Q_θ is contained in R_σ.
  (2) If $|\theta - \theta'| \lesssim \lambda^{-1}\delta$, then Q_θ and $Q_{\theta'}$ are essentially the same.
  (3) If $|\theta - \theta'| \gtrsim \lambda^{-1}\delta$, then Q_θ and $Q_{\theta'}$ are disjoint.

Before proving the lemma, we see how it can be used to finish the proof of Theorem 5. Motivated by Property (2) and (3), we define $\mathcal{T} = (\lambda^{-1}\delta)\mathbb{Z} \cap [0,1]$, and for each $\tau \in \mathcal{T}$ define

$$S_\tau := \{\sum_{i=1}^{3} \xi_i \mathbf{e}_i(\tau) : |\xi_1| \leq \delta, |\xi_2| \leq C_2\lambda, C_2^{-1}K^{-1} \leq \xi_3 \leq C_2\},$$

where C_2 is a large constant but much smaller than C_1. Note that S_τ has the same dimensions as Q_θ up to a C_2-dilation.

Now we have three subsets of $[0,1]$:

$$\Theta = \delta\mathbb{Z} \cap [0,1], \quad \mathcal{T} = (\lambda^{-1}\delta)\mathbb{Z} \cap [0,1], \quad \Sigma = \lambda\mathbb{Z} \cap [0,1].$$

We will define a relationship between their elements. For any $\theta \in \Theta$, we attach it to a $\tau \in \mathcal{T}$ such that $|\theta - \tau| \leq \lambda^{-1}\delta$, which we denote by $\theta \prec \tau$. For any $\tau \in \mathcal{T}$, we attach it to a $\sigma \in \Sigma$ such that $|\tau - \sigma| \leq \lambda$, which we denote by $\tau \prec \sigma$. We also write $\theta \prec \sigma$ if there is a τ such that $\theta \prec \tau$ and $\tau \prec \sigma$.

By property (1), if $\theta \prec \sigma$, then $Q_\theta \subset R_\sigma$. By property (2), for a given $\tau \in \mathcal{T}$, all the planks in $\{Q_\theta : \theta \prec \tau\}$ are essentially the same and contained in S_τ. By property (3), if $Q_\theta, Q_{\theta'}$ lie in different S_τ, then $Q_\theta, Q_{\theta'}$ are disjoint.

Before estimating $\int |f_\lambda|^6$, we may apply a pigeonhole argument to pass to subsets of $\Theta, \mathcal{T}, \Sigma$ (still denoted by $\Theta, \mathcal{T}, \Sigma$), so that $\#\{\theta \in \Theta : \theta \prec \tau\}$ are comparable for

$\tau \in \mathcal{T}$, and $\#\{\tau \in \mathcal{T} : \tau \prec \sigma\}$ are comparable for $\sigma \in \Sigma$. For convinience, we write $\#\{\theta \in \Theta : \theta \prec \tau\}$ as $\#\{\theta \prec \tau\}$, and write $\#\{\tau \in \mathcal{T} : \tau \prec \sigma\}$ as $\#\{\tau \prec \sigma\}$.

We define
$$f_\tau := \sum_{\theta \prec \tau} f_{\theta,\lambda},$$
$$f_\sigma := \sum_{\theta \prec \sigma} f_{\theta,\lambda} = \sum_{\tau \prec \sigma} f_\tau.$$

By decoupling for $N_{\lambda^2}(\Gamma_{K^{-1}}) = \bigsqcup_\sigma R_\sigma$, we have

$$(43) \qquad \int |f_\lambda|^6 = \int |\sum_\sigma f_\sigma|^6 \lesssim \delta^{-\varepsilon} \#\{\sigma\}^2 \sum_\sigma \int |f_\sigma|^6.$$

By the trivial decoupling for $R_\sigma = \bigsqcup_{\tau \prec \sigma} S_\tau$, we have

$$(44) \qquad \int |f_\sigma|^6 = \int |\sum_{\tau \prec \sigma} f_\tau|^6 \lesssim \#\{\tau \prec \sigma\}^4 \sum_{\tau \prec \sigma} \int |f_\tau|^6.$$

By Hölder's inequality, we have

$$(45) \qquad \int |f_\tau|^6 = \int |\sum_{\theta \prec \tau} f_{\theta,\lambda}|^6 \lesssim \#\{\theta \prec \tau\}^5 \sum_{\theta \prec \tau} \int |f_{\theta,\lambda}|^6.$$

Combining the three inequalities, we obtain

$$(46) \qquad \int |f_\lambda|^6 \lesssim \delta^{-\varepsilon} \#\{\sigma\}^2 \#\{\tau \prec \sigma\}^4 \#\{\theta \prec \tau\}^5 \sum_\theta \int |f_{\theta,\lambda}|^6.$$

Note that $\#\{\sigma\}\#\{\tau \prec \sigma\}\#\{\theta \prec \tau\} \lesssim \#\Theta$, $\#\{\tau \prec \sigma\}\#\{\theta \prec \tau\} \lesssim (\lambda\delta^{-1})^t$ (by the (δ,t)-spacing of Θ), and $\#\{\theta \prec \tau\} \lesssim \lambda^{-t}$ (by the (δ,t)-spacing of Θ). We also note that $\sum_\theta \int |f_{\theta,\lambda}|^6 \lesssim (\#\mathbb{T})\delta^{-1} \lesssim \#\Theta\delta^{-s-1}$. We obtain

$$(47) \qquad \int |f_\lambda|^6 \lesssim \delta^{-\varepsilon}(\#\Theta)^2(\lambda\delta^{-1})^{2t}\lambda^{-t}(\#\mathbb{T})\delta^{-1} \lesssim \delta^{-\varepsilon}(\#\Theta)^3\lambda^t\delta^{-s-1-2t}.$$

Plugging into (29) and noting $\delta^{-t} \gtrsim \#\Theta \gtrsim (\log\delta^{-1})^{-2}\delta^{-t}$, we obtain
$$\delta^{-a-t} \lesssim \delta^{-2\varepsilon}\lambda^t\delta^{-s-1},$$

which is better than we aimed because of the factor λ^t.

$\square$

It remains to prove Lemma 3. Before proving the lemma, we give some intuition on why the lemma should be true. See Figure (2). We first cover $N_{\lambda^2}\Gamma_{K^{-1}}$ by gray planks R_σ of dimensions $\sim \lambda^2 \times \lambda \times 1$. Fix a R_σ, we draw all the black slabs P_θ of dimensions $\delta \times 1 \times 1$ whose corresponding Q_θ is contained in R_σ. Morally speaking, $P_\theta \cap R_\sigma \approx Q_\theta$ which is a $\delta \times \lambda \times 1$-plank. One tricky thing is that different P_θ may have essentially same Q_θ, which is the reason to introduce S_τ (the thick-black planks in the Figure of dimensions $\delta \times \lambda \times 1$). Suppose we have a partition $R_\sigma = \bigcup_{\tau \prec \sigma} S_\tau$. We see that each $P_\theta \cap R_\sigma$ is contained in one of the S_τ. If so, then we define $\theta \prec \tau$. We can talk about the intuition on the numerology of these planks.

(1) $\#\{R_\sigma\} = \lambda^{-1}$,
(2) $\#\{Q_\theta : Q_\theta \subset R_\sigma\} \sim \#\{\theta \prec \sigma\} = \delta^{-1}\lambda$,
(3) $\#\{S_\tau : S_\tau \subset R_\sigma\} \sim \frac{|R_\sigma|}{|S_\tau|} = \delta^{-1}\lambda^2$,
(4) $\#\{Q_\theta : Q_\theta \subset S_\tau\} \sim \frac{\#\{Q_\theta : Q_\theta \subset R_\sigma\}}{\#\{S_\tau : S_\tau \subset R_\sigma\}} = \lambda^{-1}$.
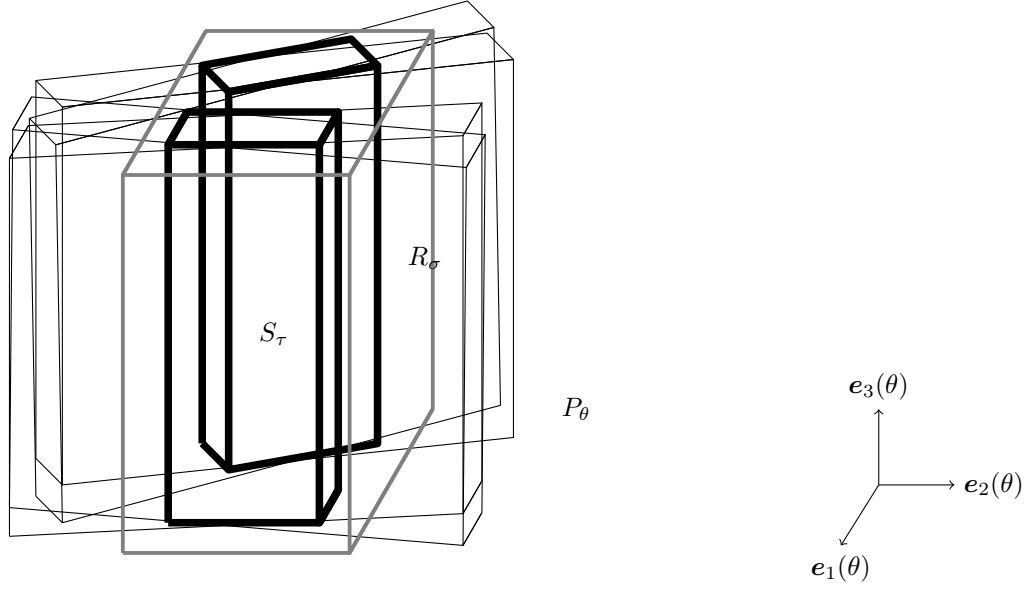
FIGURE 2. Relation between planks

By (2), we see Property (1) in Lemma 3 should be true. By (4), we see Property (2) and (3) in Lemma 3 should be true.

Proof of Lemma 3. Recall $\mathbf{e}_1(\theta) = \gamma(\theta), \mathbf{e}_2(\theta) = \gamma'(\theta), \mathbf{e}_3(\theta) = \gamma(\theta) \times \gamma'(\theta)$. Defining

$$\kappa(\theta) = \langle \mathbf{e}_2'(\theta), \mathbf{e}_3(\theta)\rangle (\gtrsim 1),$$

we have

(48) $$\mathbf{e}_1'(\theta) = \mathbf{e}_2(\theta),$$

(49) $$\mathbf{e}_2'(\theta) = -\mathbf{e}_1(\theta) + \kappa(\theta)\mathbf{e}_3(\theta),$$

(50) $$\mathbf{e}_3'(\theta) = -\kappa(\theta)\mathbf{e}_2(\theta).$$

To prove Property (1), write $\theta = \sigma + \Delta$ with $|\Delta| \lesssim \lambda$. Any $\xi \in Q_\theta$ can be written as $\xi = a\mathbf{e}_1(\theta) + b\mathbf{e}_2(\theta) + c\mathbf{e}_3(\theta)$ with $|a| \leq \delta, |b| \leq \lambda, |c| \leq 1$. By Taylor's expansion, we have

(51) $$\begin{aligned} \xi =& a(\mathbf{e}_1(\sigma) + \Delta\mathbf{e}_2(\sigma)) + b(\mathbf{e}_2(\sigma) - \Delta\mathbf{e}_1(\sigma) + \Delta\kappa(\sigma)\mathbf{e}_3(\sigma)) \\ &+ c(\mathbf{e}_3(\sigma) - \Delta\kappa(\sigma)\mathbf{e}_2(\sigma)) + O(\Delta^2) \\ =&(a - b\Delta)\mathbf{e}_1(\sigma) + (a\Delta + b - c\Delta\kappa(\sigma))\mathbf{e}_2(\sigma) + (b\Delta\kappa(\sigma) + c)\mathbf{e}_3(\sigma) + O(\Delta^2). \end{aligned}$$

One can easily check $\xi \in R_\sigma$. The Property (2) can also be proved by using (51). For the Property (3), we have proved a special case $\lambda \sim 1$ in the High case, but here we need to do more work. We may assume $\lambda << 1$. Consider the plane

$$\Pi_\theta = \{\xi_2\mathbf{e}_2(\theta) + \xi_3\mathbf{e}_3(\theta) : \frac{1}{2}\lambda \leq \xi_2 \leq \lambda, K^{-1} \leq \xi_3 \leq 1\}.$$

We see Q_θ is the δ-neighborhood of Π_θ. We just need to show: For any $\theta \in [0, 1]$ and $\theta' = \theta + \Delta \in [0, 1]$ with $C\lambda^{-1}\delta \leq \Delta \leq C^{-1}$ (for some bounded C to be determined

later), if $\xi \in \Pi_\theta$, then

$$\text{dist}(\xi, \Pi_{\theta+\Delta}) > 10\delta.$$

Write $\xi = a\mathbf{e}_2(\theta) + b\mathbf{e}_3(\theta)$, where $a \in [\frac{1}{2}\lambda, \lambda]$ and $b \in [K^{-1}, 1]$.

We consider two scenarios: (1) $C\lambda^{-1}\delta \leq \Delta \leq C\lambda$, (2) $C\lambda \leq \Delta \leq C^{-1}$. If we are in the first scenario, since the normal direction of $\Pi_{\theta+\Delta}$ is $\mathbf{e}_1(\theta + \Delta)$, it suffices to prove

$$(52) \qquad |\mathbf{e}_1(\theta + \Delta) \cdot (a\mathbf{e}_2(\theta) + b\mathbf{e}_3(\theta))| \geq 10\delta.$$

By Taylor's expansion, we have $\mathbf{e}_1(\theta + \Delta) = \mathbf{e}_1(\theta) + \Delta\mathbf{e}_2(\theta) + \frac{\Delta^2}{2}(-\mathbf{e}_1(\theta) + \kappa(\theta)\mathbf{e}_3(\theta)) + o(\Delta^2)$. We see the left hand side of (52) is $|a\Delta + \frac{\Delta^2}{2}\kappa(\theta)b + o(\Delta^2)| \geq a\Delta - o(\Delta^2) = (a - o(\Delta))\Delta \geq \frac{10}{C}a\Delta \gtrsim 10\delta$, when $\Delta \leq C\lambda$ and $\lambda << 1$. If we are in the second scenario, we show that

$$(53) \qquad \mathbf{e}_2(\theta + \Delta) \cdot (a\mathbf{e}_2(\theta) + b\mathbf{e}_3(\theta)) \geq 10\lambda.$$

By Taylor's expansion, we have $\mathbf{e}_2(\theta+\Delta) = \mathbf{e}_2(\theta) + \Delta(-\mathbf{e}_1(\theta) + \kappa(\theta)\mathbf{e}_3(\theta)) + O(\Delta^2)$. We see the left hand side of (53) is $|a + \Delta\kappa(\theta)b + O(\Delta^2)| \geq \Delta\kappa(\theta)b - a - O(\Delta^2) \geq 10\lambda$ if the constant C is big enough. $\qquad \square$

3. Proof of Theorem 2

For a small positive number δ and $E \subset [0, 1]$, we use $\Lambda_\delta(E)$ to denote a maximal δ-separated subset of E. By definition, $\#\Lambda_\delta(E) \sim |E|_\delta$. If $E = [0, 1]$, then we abbreviate $\Lambda_\delta(E)$ as Λ_δ and just choose it to be the δ-lattice points in $[0, 1]$. A rectangular box of dimensions $\delta \times \delta \times 1$ will be referred to as a δ-tube. For each $\theta \in \Lambda_\delta$, there is a set of finitely overlapping collection of δ-tubes that cover $\mathbb{R}^3$ whose long sides are parallel to $\gamma(\theta)$.

In order to prove Theorem 2, we need the following result about incidence estimate.

3.1. An incidence estimate.

Theorem 7. *Let Λ_δ be a δ-net of $[0, 1]$ for some $\delta > 0$. Given a small constant $\varepsilon > 0$ and $0 < \alpha \leq 2$, let μ be a finite nonzero Borel measure supported in the unit ball in $\mathbb{R}^3$ with $c_\alpha(\mu) := \sup_{x \in \mathbb{R}^3, r > 0} \frac{\mu(B(x,r))}{r^\alpha} < \infty$. Suppose that $\mathbb{W}$ is a set of δ-tubes, with directions in $\{\gamma(\theta) : \theta \in \Lambda_\delta\}$, such that each $\mathbb{W}_\theta$ is disjoint, where we use $\mathbb{W}_\theta$ to denote the subset of tubes in $\mathbb{W}$ that points to direction $\gamma(\theta)$. Suppose also that*

$$(54) \qquad \sum_{T \in \mathbb{W}} \chi_T(x) \gtrsim \delta^{\varepsilon-1}, \qquad \forall\, x \in \text{supp}(\mu).$$

Then

$$(55) \qquad |\mathbb{W}| \geq C_{\varepsilon,\alpha} \cdot \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} \delta^{-(1+\alpha-O(\sqrt{\varepsilon}))},$$

where the constant $C_{\varepsilon,\alpha}$ is allowed to depend on α and ε but not on δ, and $O(\sqrt{\varepsilon})$ can be taken to be $10^{10}\sqrt{\varepsilon}$. We also remark that $C_{\varepsilon',\alpha} \leq C_{\varepsilon'',\alpha}$ for $\varepsilon' \leq \varepsilon''$.

Proof of Theorem 7. The main argument of the proof is similar to that of Theorem 4, except for the "low case".

For a given δ-tube $T_\theta \in \mathbb{W}_\theta$, we denote the dual slab of T_θ by P_θ which is of dimensions $\delta^{-1} \times \delta^{-1} \times 1$ and centered at the origin. Recalling (17), we can write

$$P_\theta = \left\{ \sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq 1, |\xi_2| \leq \delta^{-1}, |\xi_3| \leq \delta^{-1} \right\}.$$

Remark 4. The slab P_θ here is just the δ^{-1}-dilation of that in the proof of Theorem 4. So are the tubes T_θ and $P_{\theta,high}, P_{\theta,low}, P_{\lambda,\theta}$ that we will define right now.

Let $\phi_T : \mathbb{R}^3 \to \mathbb{R}$ be a non-negative function with $\widehat{\phi_T}$ supported on P_θ and $\phi_T(x) \gtrsim 1$ for every $x \in T$. Set

(56) $$f_\theta(x) := \sum_{T \in \mathbb{W}_\theta} \phi_T(x), \quad f(x) := \sum_{T \in \mathbb{W}} \phi_T(x).$$

The assumption (54) implies that

(57) $$f(x) \gtrsim \delta^{\varepsilon-1},$$

for every $x \in \text{supp}(\mu)$.

Next we will do the frequency decomposition for f_θ. Similar to Definition 3, we make the following definitions.

Definition 4. *(See Figure 1.) Let $K = \delta^{-\sqrt{\varepsilon}}$. Define the high part of P_θ as*

$$P_{\theta,high} := \left\{ \sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq 1, K^{-1}\delta^{-1} \leq |\xi_2| \leq \delta^{-1}, |\xi_3| \leq \delta^{-1} \right\}.$$

Define the low part of P_θ as

$$P_{\theta,low} := \left\{ \sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq 1, |\xi_2| \leq K^{-1}\delta^{-1}, |\xi_3| \leq K^{-1}\delta^{-1} \right\}.$$

For dyadic numbers $\lambda \in (\delta^{1/2}, K^{-1}]$, define

$$P_{\theta,\lambda} := \left\{ \sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq 1, \frac{1}{2}\lambda\delta^{-1} \leq |\xi_2| \leq \lambda\delta^{-1}, K^{-1}\delta^{-1} \leq |\xi_3| \leq \delta^{-1} \right\}.$$

In particular, we define

$$P_{\theta,\delta^{1/2}} := \left\{ \sum_{i=1}^{3} \xi_i \mathbf{e}_i(\theta) : |\xi_1| \leq 1, |\xi_2| \leq \delta^{-1/2}, K^{-1}\delta^{-1} \leq |\xi_3| \leq \delta^{-1} \right\}.$$

Similarly to (27) and (28), we have

(58) $$f_\theta = f_{\theta,high} + f_{\theta,low} + \sum_{\delta^{1/2} \leq \lambda \leq K^{-1}} f_{\theta,\lambda},$$

and

(59) $$f = f_{high} + f_{low} + \sum_{\delta^{1/2} \leq \lambda \leq K^{-1}} f_\lambda.$$

Since $f(x) \gtrsim \delta^{\varepsilon-1}$, there are two cases:

$\boxed{\text{Case 1: High case}}$ We can find a Borel set F satisfying

$$\mu(F) \gtrsim \mu(\mathbb{R}^3),$$

and

$$(60) \qquad \delta^{\varepsilon-1} \lesssim \left| f_{high}(x) + \sum_\lambda f_\lambda(x) \right|, \quad \forall\, x \in F.$$

$\boxed{\text{Case 2: Low case}}$ We can find a Borel set F satisfying

$$\mu(F) \gtrsim \mu(\mathbb{R}^3),$$

and

$$(61) \qquad \delta^{\varepsilon-1} \lesssim |f_{low}(x)|, \quad \forall\, x \in F.$$

Assume first that we are in the high case. We raise both sides of (60) to the sixth power, integrate with respect to $d\mu$, and obtain

$$(62) \qquad \mu(\mathbb{R}^3)\delta^{-6(1-\varepsilon)} \lesssim (\log \delta^{-1})^{O(1)} \int |f_{high}|^6 + \sum_\lambda |f_\lambda|^6 d\mu.$$

Since the functions on the right hand side are locally constant on δ-balls, together with the upper density condition on μ, we obtain

$$(63) \quad \mu(\mathbb{R}^3)c_\alpha(\mu)^{-1}\delta^{-6(1-\varepsilon)+3-\alpha} \lesssim (\log \delta^{-1})^{O(1)} \int |f_{high}(x)|^6 + \sum_\lambda |f_\lambda(x)|^6 \, dx.$$

We can just use (37) and (47) with $t = 1$ and $\#\Theta \sim \delta^{-1}$, noting there is a scaling difference, so that the right hand side above is bounded by

$$\delta^{-O(\sqrt{\varepsilon})}\delta^{-2}|\mathbb{W}|.$$

It follows that

$$|\mathbb{W}| \gtrsim \mu(\mathbb{R}^3)c_\alpha(\mu)^{-1}\delta^{-(1+\alpha-O(\sqrt{\varepsilon}))}.$$

This finishes the proof if we are in the high case (60).

Now we assume that we are in the low case (61). For each $T \in \mathbb{W}$, the support of $\phi_T * \widetilde{\eta_{low}}$ is essentially a thickened tube of T with dimensions $1 \times K\delta \times K\delta$ where $K = \delta^{-\sqrt{\varepsilon}}$. We use $\widetilde{T}$ to denote this thickened tube. Let $\widetilde{\mathbb{W}}$ be the collection of these thickened tubes obtained from $\mathbb{W}$, and we only keep those essentially distinct tubes (each tube intersects $\lesssim 1$ other tubes of those whose angle is within $\lesssim K\delta$ of its own, and every $T \in \mathbb{W}$ is contained in some $\widetilde{T}$ from $\widetilde{\mathbb{W}}$).

Write $\widetilde{\mathbb{W}}$ as a disjoint union

$$(64) \qquad \widetilde{\mathbb{W}} = \widetilde{\mathbb{W}}_{\text{heavy}} \bigcup \widetilde{\mathbb{W}}_{\text{light}}$$

where $\widetilde{\mathbb{W}}_{\text{light}}$ is the collection of thickened tubes $\widetilde{T}$ that contain $\leq C^{-1}K^{3-\sqrt{\varepsilon}}$ tubes from $\mathbb{W}$. Here C is a large universal constant which is much larger than the implicit (universal) constant in (61). Note that

$$(65) \qquad \left\| \sum_{\widetilde{T}\in\widetilde{\mathbb{W}}_{\text{light}}} \sum_{T'\in\mathbb{W}:T'\subset\widetilde{T}} \phi_{T'} * \widetilde{\eta_{low}} \right\|_\infty \leq \sum_{\widetilde{T}\in\widetilde{\mathbb{W}}_{\text{light}}} \sum_{T'\in\mathbb{W}:T'\subset\widetilde{T}} \left\| \phi_{T'} * \widetilde{\eta_{low}} \right\|_\infty$$

$$(66) \qquad \qquad \lesssim C^{-1}(\delta^{-1}K^{-1})K^{-2}K^{3-\sqrt{\varepsilon}} = C^{-1}\delta^{-1+\varepsilon}$$

As a consequence, we see that (61) can be upgraded to

$$(67) \qquad \delta^{\varepsilon-1} \lesssim \left| \sum_{\widetilde{T}\in\widetilde{\mathbb{W}}_{\text{heavy}}} \sum_{T'\in\mathbb{W}:T'\subset\widetilde{T}} \phi_{T'} * \widetilde{\eta_{low}}(x) \right|, \quad \forall\, x \in F.$$

Next, by (67) and the fact that if $x \in \widetilde{T}$ then $\#\{T \in \mathbb{W} : x \in T \subset \widetilde{T}\} \lesssim K$, we conclude that

$$
\text{(68)} \qquad \sum_{\widetilde{T} \in \widetilde{\mathbb{W}}_{\text{heavy}}} \chi_{\widetilde{T}}(x) \gtrsim K^{-1} \delta^{\varepsilon-1}, \ \forall x \in F.
$$

Write

$$
\text{(69)} \qquad \delta^{-1+\varepsilon} K^{-1} = (\delta K)^{-1+\widetilde{\varepsilon}}, \ \widetilde{\varepsilon} := \frac{\varepsilon}{1 - \sqrt{\varepsilon}}.
$$

The tubes in $\widetilde{\mathbb{W}}_{\text{heavy}}$ satisfy the induction hypothesis at the scale $K\delta$ with the new parameter $\widetilde{\varepsilon}$. Hence

$$
|\mathbb{W}| \geq |\widetilde{\mathbb{W}}_{\text{heavy}}| K^{3-\sqrt{\varepsilon}} \geq \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} C_{\widetilde{\varepsilon},\alpha} \left(\delta^{-1} K^{-1}\right)^{(\alpha+1-10^{10}\sqrt{\widetilde{\varepsilon}})} K^{3-\sqrt{\varepsilon}}
$$

Elementary computation shows that

$$
\text{(70)} \qquad \sqrt{\widetilde{\varepsilon}} - \sqrt{\varepsilon} = \frac{\widetilde{\varepsilon} - \varepsilon}{\sqrt{\widetilde{\varepsilon}} + \sqrt{\varepsilon}} \leq \frac{\varepsilon}{2(1 - \sqrt{\varepsilon})} \leq \frac{3\varepsilon}{4}.
$$

Therefore,

$$
\text{(71)} \qquad \begin{aligned}
|\mathbb{W}| &\geq C_{\widetilde{\varepsilon},\alpha} \cdot \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} \delta^{-(\alpha+1-10^{10}\sqrt{\varepsilon})} \delta^{10^{10}\frac{3\varepsilon}{4}} K^{3-(1+\alpha-10^{10}\sqrt{\varepsilon})-\sqrt{\varepsilon}} \\
&\geq C_{\widetilde{\varepsilon},\alpha} \cdot \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} \delta^{-(\alpha+1-10^{10}\sqrt{\varepsilon})} K^{2-\alpha} \delta^{10^{10}\frac{3\varepsilon}{4}} \delta^{-10^{10}\varepsilon+\varepsilon}.
\end{aligned}
$$

Since $\alpha \leq 2$, the induction closes. $\qquad\square$

We have the following corollary of Theorem 7.

Corollary 2. *Let* $\alpha \in (0,3)$ *and* $\alpha_1 \in (0, \min\{2, \alpha\})$. *Let* μ *be a finite non-zero Borel measure supported on the unit ball in* $\mathbb{R}^3$ *with* $c_\alpha(\mu) \leq 1$. *Let* $\delta > 0$ *be a small number. Let* Λ_δ *be a* δ-*net of* $[0,1]$. *For each* $\theta \in \Lambda_\delta$, *let* $\mathbb{D}_\theta$ *be a disjoint collection of at most* $\mu(\mathbb{R}^3)\delta^{-\alpha_1}$ *balls of radius* δ *in* $\pi_\theta(\mathbb{R}^3)$. *Then there exists* $\varepsilon > 0$, *depending only on* α *and* α_1, *such that*

$$
\text{(72)} \qquad \delta \sum_{\theta \in \Lambda_\delta} (\pi_{\theta\#}\mu)\Big(\bigcup_{D \in \mathbb{D}_\theta} D \Big) \leq C(\alpha, \alpha_1) \mu(\mathbb{R}^3) \delta^\varepsilon,
$$

where the constant $C(\alpha, \alpha_1)$ *depends on* α, α_1, *but not on* δ.

Proof of Corollary 2. We argue by contradiction. Suppose that for every $\varepsilon > 0$, there exist $\delta > 0$ and $\mathbb{D}_\theta$, a disjoint collection of at most $\mu(\mathbb{R}^3)\delta^{-\alpha_1}$ balls of radius δ in $\pi_\theta(\mathbb{R}^3)$ for each $\theta \in \Lambda_\delta$, such that (72) fails. Note that for each $D_\theta \in \mathbb{D}_\theta$, $\pi_\theta^{-1}(D_\theta) \cap B^3(0,1)$ is a δ-tube. We denote these tubes by

$$
\text{(73)} \qquad \mathbb{W}_\theta := \{\pi_\theta^{-1}(D_\theta) \cap B^3(0,1) : D_\theta \in \mathbb{D}_\theta\},
$$

and write $\mathbb{W} := \cup_\theta \mathbb{W}_\theta$. Define

$$
\text{(74)} \qquad F := \Big\{ x \in \text{supp}(\mu) : \delta \sum_{\theta \in \Lambda_\delta} \sum_{T \in \mathbb{W}_\theta} \chi_T(x) \geq C(\alpha, \alpha_1)\delta^\varepsilon/2 \Big\}
$$

Note that by our contradiction assumption, we have

$$
\text{(75)} \qquad \begin{aligned}
C(\alpha, \alpha_1)\mu(\mathbb{R}^3)\delta^\varepsilon &\leq \delta \sum_{\theta \in \Lambda_\delta} (\pi_{\theta\#}\mu)\Big(\bigcup_{D \in \mathbb{D}_\theta} D \Big) \\
&= \delta \int_{\mathbb{R}^3} \sum_\theta \sum_{T \in \mathbb{W}_\theta} \chi_T(x) d\mu(x).
\end{aligned}
$$

This further implies

$$(76) \qquad \mu(F) \gtrsim C(\alpha, \alpha_1) \mu(\mathbb{R}^3) \delta^\varepsilon.$$

Note that by definition, for every $x \in F$, it holds that

$$(77) \qquad \sum_{T \in \mathbb{W}} \chi_T(x) \gtrsim \delta^{-1+\varepsilon}.$$

We apply Theorem 7 to the measure μ restricted to F and obtain

$$(78) \qquad |\mathbb{W}| \gtrsim \delta^\varepsilon \delta^{-1-\min\{2,\alpha\}+O(\sqrt{\varepsilon})} \mu(\mathbb{R}^3).$$

By pigeonholing, this further implies that there exists θ such that

$$(79) \qquad |\mathbb{W}_\theta| \gtrsim \delta^\varepsilon \delta^{-\min\{2,\alpha\}+O(\sqrt{\varepsilon})} \mu(\mathbb{R}^3).$$

This is a contradiction to the assumption that $|\mathbb{D}_\theta| \lesssim \delta^{-\alpha_1} \mu(\mathbb{R}^3)$ when ε is chosen to be small enough. $\qquad\square$

The corollary below is a special case of Corollary 2. It is recorded below for a later application.

Corollary 3. *Let $\alpha \in [2,3]$ and $\alpha^* \in [0,2)$. Let μ be a finite non-zero Borel measure supported on the unit ball in $\mathbb{R}^3$ with $c_\alpha(\mu) \leq 1$. Let $\delta > 0$ be a small number. Let $\Lambda_{\sqrt{\delta}}$ be a $\sqrt{\delta}$-net of $[0,1]$. For each $\theta \in \Lambda_{\sqrt{\delta}}$, let $\mathbb{D}_\theta$ be a disjoint collection of at most $\delta^{-\frac{\alpha^*+1}{2}} \mu(\mathbb{R}^3)$ rectangles of dimension $\delta \times \sqrt{\delta}$ in $\pi_\theta(\mathbb{R}^3)$ whose long sides point in the $\gamma'(\theta)$ direction. Then there exists $\varepsilon > 0$, depending only on α and α^*, such that*

$$(80) \qquad \sqrt{\delta} \sum_{\theta \in \Lambda_{\sqrt{\delta}}} (\pi_{\theta\#}\mu)\Big(\bigcup_{D \in \mathbb{D}_\theta} D \Big) \leq C(\alpha, \alpha^*) \delta^\varepsilon \mu(\mathbb{R}^3),$$

where the constant $C(\alpha, \alpha^*)$ depends on γ, α, α^*, but not on δ.

Proof of Corollary 3. For each $\theta \in \Lambda_{\sqrt{\delta}}$ and recalling (17), define

$$(81) \qquad U_\theta := \left\{ \sum_{i=1}^3 x_i \mathbf{e}_i(\theta) : |x_1| \leq 1, |x_2| \leq \sqrt{\delta}, |x_3| \leq \delta \right\},$$

which is a $\delta \times \sqrt{\delta} \times 1$-plank. By a simple geometric observation, we have that

$$(82) \qquad \pi_{\theta'}(W_\theta) \subset C\pi_\theta(W_\theta),$$

for every $\theta, \theta' \in [0,1]$ with $|\theta' - \theta| \leq \sqrt{\delta}$, where C is a constant depending only on γ. For each $\theta \in \Lambda_{\sqrt{\delta}}$ we have a set of $\delta \times \sqrt{\delta} \times 1$-planks

$$\mathbb{U}_\theta = \{\pi_\theta^{-1}(D_\theta) \cap B^3(0,1)\}_{D_\theta \in \mathbb{D}_\theta},$$

which are essentially the translates of U_θ. For each $\theta' \in \Lambda_\delta$, we choose a $\theta \in \Lambda_{\sqrt{\delta}}$ with $|\delta - \delta'| \leq \sqrt{\delta}$. We partition each plank in $\mathbb{U}_\theta$ into $\delta \times \delta \times 1$-tubes with direction $\gamma(\theta')$ and denote all these tubes by $\mathbb{W}_{\theta'}$. We also define $\mathbb{D}'_{\theta'} = \pi_\theta(\mathbb{W}_{\theta'})$ which is a collection of at most $\mu(\mathbb{R}^3)\delta^{-\frac{\alpha^*}{2}-1}$ balls of radius δ in $\pi_\theta(\mathbb{R}^3)$.

Now we can apply Corollary 2 to the sets $\{\mathbb{D}'_{\theta'}\}_{\theta' \in \Lambda_\delta}$ with $\alpha_1 = \frac{\alpha^*}{2} + 1$. We obtain

$$\sqrt{\delta} \sum_{\theta \in \Lambda_{\sqrt{\delta}}} (\pi_{\theta\#}\mu)\Big(\bigcup_{D \in \mathbb{D}_\theta} D \Big) \sim \delta \sum_{\theta \in \Lambda_\delta} (\pi_{\theta\#}\mu)\Big(\bigcup_{D \in \mathbb{D}'_\theta} D \Big) \leq C(\alpha, \alpha^*)\mu(\mathbb{R}^3)\delta^\varepsilon.$$

$\qquad\square$

By Frostman's Lemma (see for instance [20, page 112]), Theorem 2 is an immediate consequence of the following theorem.

Theorem 8. *Let $\gamma : [0,1] \to \mathbb{S}^2$ be C^2 and non-degenerate. If μ is a compactly supported Borel measure on $\mathbb{R}^3$ such that $c_\alpha(\mu) < \infty$ for some $\alpha > 2$, then $\pi_{\theta\#}\mu$ is absolutely continuous with respect to $\mathcal{H}^2$, for a.e. $\theta \in [0,1]$.*

To prove Theorem 8, we will cut γ into finitely many pieces with a number depending only on γ, and work on one piece. From now on, we assume that $\gamma : [0,a] \to \mathbb{S}^2$ is C^2 and non-degenerate, and satisfy

$$(83) \qquad \gamma(0) = (0,0,1), \quad \gamma'(0) = (1,0,0), \quad |\gamma'(\theta)| = 1, \forall \theta \in [0,a].$$

Here $a > 0$ is sufficiently small depending on γ.

3.2. **Decomposition of the frequency space.** In this subsection, we discuss the decomposition of the frequency space $\mathbb{R}^3_\xi$. Recall the cone that we are considering:

$$\Gamma = \{r(\gamma \times \gamma')(\theta) : r \in \mathbb{R}, \theta \in [0,a]\}.$$

For any $\xi = r(\gamma \times \gamma')(\theta) \in \Gamma$, there are three directions that we would like to specify: the normal direction $\gamma(\theta)$; the tangent direction $\gamma'(\theta)$; and the flat (or radial) direction $\gamma \times \gamma'(\theta)$. We want to decompose $\mathbb{R}^3_\xi$ into regions according to the distance from the origin, the distance from the cone Γ, and the angular parameter. We give the precise definition below.

For a given integer k, define

$$(84) \qquad \Theta_k := \left(2^{-k/2}\mathbb{N}\right) \cap [0,a].$$

Fix $k \le j$. If $k < j$ and $\square \in \{+,-\}$, then we define the plank in the forward/backward light cone

$$(85) \quad \tau(\theta,j,k,\square) = \Big\{\lambda_1 \left(\gamma \times \gamma'\right)(\theta) + \lambda_2\gamma'(\theta) + \lambda_3\gamma(\theta)$$

$$: |\lambda_1| \sim 2^j, |\lambda_2| \le C_\gamma^{-1}2^{-k/2+j}, |\lambda_3| \sim 2^{-k+j}, \mathrm{sgn}\lambda_1 = \square = -\mathrm{sgn}\lambda_3\Big\}.$$

Here $C_\gamma > 0$ is some large constant that depends only on γ. It is chosen such that the distance from $\tau(\theta,j,k)$ to cone $\{(\gamma \times \gamma')(\theta) : \theta \in [0,a]\}$ is comparable to 2^{j-k}. Let us digest a little bit about the plank $\tau(\theta,j,k)$: $\lambda_1 \sim 2^j$ is the distance from the origin; $|\lambda_3| \sim 2^{-k+j}$ is the distance from the cone Γ; and $|\lambda_2| \lesssim 2^{-k/2+j}$ is the angular parameter of the plank.

If $k = j$, then for $\square \in \{+,-\}$ we define

$$(86) \quad \tau(\theta,j,j,\square) =$$

$$\Big\{|\lambda_1| \left(\gamma \times \gamma'\right)(\theta)+\lambda_2\gamma'(\theta)+\lambda_3\gamma(\theta) : \lambda_1 \sim 2^j, \quad |\lambda_2| \lesssim 2^{j/2}, \quad |\lambda_3| \lesssim 1, \mathrm{sgn}\lambda_1 = \square\Big\}.$$

Let

$$(87) \qquad \Lambda_{j,k} := \{\tau(\theta,j,k,\square) : \theta \in \Theta_k, \square \in \{-,+\}\},$$

and

$$(88) \qquad \Lambda_j := \bigcup_{k \le j} \Lambda_{j,k}, \quad \Lambda := \bigcup_{j \in \mathbb{N}} \Lambda_j.$$

Roughly speaking, for $k < j$, $\Lambda_{j,k}$ forms a canonical covering of the part of $\{\xi \in \mathbb{R}^3 : |\xi| \sim 2^j; \mathrm{dist}(\xi, \Gamma) \sim 2^{-k+j}\}$ outside the cone Γ by $2^j \times 2^{-k/2+j} \times 2^{-k+j}$-planks. Each Λ_j forms a covering of the ~ 1-neighbourhood of $\{\xi \in \Gamma : |\xi| \sim 2^j\}$.

For each $\tau \in \Lambda$, we define $\mathbb{T}_\tau$ to be a set of planks of dual dimensions to τ (but scaled by $2^{k\delta}$ in each direction where $\delta > 0$ and $\tau \in \Lambda_{j,k}$) and forming a finitely overlapping covering of $\mathbb{R}^3$. We will refer to $\mathbb{T}_\tau$ as the wave packets determined by the plank τ. Now, we discuss the wave packet decomposition. For each $\tau \in \Lambda$, we can choose a smooth bump function ψ_τ supported in 2τ and choose a smooth bump function ψ_0 supported in the unit ball, so that we have the partition of the unity

$$\psi_0(\xi) + \sum_{\tau \in \Lambda} \psi_\tau(\xi) = 1,$$

on the union of the τ's. For each $T \in \mathbb{T}_\tau$, we can choose a smooth function η_T which is essentially supported in T (with rapidly decaying tail outside of T), such that $\mathrm{supp}\widehat{\eta}_T \subset \tau$ and

$$\sum_{T \in \mathbb{T}_\tau} \eta_T(x) = 1.$$

For any $T \in \mathbb{T}_\tau$, we define the wave packet

$$M_T \mu := \eta_T\big(\mu * \widetilde{\psi_\tau}\big).$$

Lemma 4. *For $\tau \in \Lambda_{j,k}$ and $T \in \mathbb{T}_\tau$, we have*

$$\|M_T\mu\|_{L^1(\mathbb{R}^3)} \lesssim 2^{3k\delta}\mu(2T) + C_N 2^{-kN}\mu\left(\mathbb{R}^3\right),$$

for every $N \geq 1$.

Proof of Lemma 4. Note that $|\widetilde{\psi_\tau}(x)| \lesssim \phi_{\tau^*}(x)$, where $\phi_{\tau^*}(x)$ is an L^1 normalized function essentially supported in τ^* (the dual plank of τ). So, we have $|\eta_T| * |\widetilde{\psi_\tau}| \lesssim |\eta_T|$. Therefore,

$$\int |M_T\mu| \lesssim \int |\mu|\big(|\eta_T| * |\widetilde{\psi_\tau}|\big) \lesssim \int |\mu||\eta_T| \lesssim 2^{3k\delta}\mu(2T) + C_N 2^{-kN}\mu(\mathbb{R}^3).$$

$\square$

Lemma 5. *For $\tau \in \Lambda_{j,k}$ and $\theta \in [0, a]$ with*

$$(89) \qquad\qquad |\theta - \theta_\tau| \geq 2^{-k(1/2-\delta)},$$

it holds that

$$(90) \qquad\qquad \|\pi_{\theta\#} M_T f\|_{L^1(\mathcal{H}^2)} \lesssim_{\delta, N} 2^{-kN}|\tau|\|f\|_{L^1(\mathbb{R}^3)}.$$

for every $N \geq 1$, $T \in \mathbb{T}_\tau$ and $f \in L^1(\mathbb{R}^3)$.

Proof of Lemma 5. We start by writing

$$(91) \qquad\qquad M_T f(x) = \eta_T(x) \int_{\mathbb{R}^3} \widehat{f}(\xi)\psi_\tau(\xi)e^{i\langle x, \xi\rangle}\,d\xi.$$

By identifying the complex measure $\pi_{\theta\#} M_T f$ with its Radon-Nikodym derivative with respect to $\mathcal{H}^2$, we obtain

$$(92) \qquad \pi_{\theta\#} M_T f(x) = \int_{\mathbb{R}^3} \widehat{f}(\xi)\psi_\tau(\xi)\Big[\int_{\mathbb{R}} \eta_T(x + t\gamma(\theta))e^{it\langle\gamma(\theta),\xi\rangle}\,dt\Big]e^{i\langle x,\xi\rangle}\,d\xi$$

for every $x \in \gamma(\theta)^\perp$. It therefore suffices to show that

$$\left| \int_{\mathbb{R}} \eta_T(x + t\gamma(\theta)) e^{it\langle \xi, \gamma(\theta) \rangle} \, dt \right| \lesssim_N 2^{-kN}, \quad \forall \xi \in \tau, \quad \forall x \in \mathbb{R}^3.$$

Integration by parts will finish the proof. For more details, we refer to Lemma 2.6 in [13]. $\qquad \square$

3.3. Proof of Theorem 8: good part and bad part.
The main idea is to divide the wave packets into two parts, called the good part and the bad part. We will prove an L^1 estimate for the bad part and an L^2 estimate for the good part.

Let $\alpha > 2$ be as in Theorem 8. Let $\epsilon > 0$ be a small number (note ϵ is different from ε) and let $\alpha_0 > 0$ be determined later (we will later let $\alpha_0 \nearrow 2$). For $j \geq k$ and $\tau \in \Lambda_{j,k}$, define

$$\mathbb{T}_{\tau,b} := \left\{ T \in \mathbb{T}_\tau : \mu(4T) \geq C 2^{-k(\alpha_0+1)/2 - \alpha(j-k) + 10^3 j\epsilon} \right\}, \qquad \mathbb{T}_{\tau,g} = \mathbb{T}_\tau \setminus \mathbb{T}_{\tau,b}.$$

Define

(93) $$\mu_b = \sum_{j \in \mathbb{N}} \sum_{k \in [j\epsilon, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} M_T \mu, \quad \mu_g = \mu - \mu_b.$$

We remark that the wave packets of μ_b are those that have heavy μ-mass and not too far away from the cone Γ. We have

$$\mu = \mu_g + \mu_b.$$

We remark that $\mu_g = \mu_{g,\alpha,\epsilon,\alpha_0}$ and $\mu_b = \mu_{b,\alpha,\epsilon,\alpha_0}$ depends on parameters $\alpha, \epsilon, \alpha_0$, but for simplicity we just omit them.

Theorem 8 follows from Lemma 6 and Lemma 7 below.

Lemma 6. *Let $\alpha > 2$ and $\epsilon \ll 1$. Fix $\alpha_0 < 2$. For all Borel measures μ supported on the unit ball in $\mathbb{R}^3$ with $c_\alpha(\mu) \leq 1$, it holds that*

$$\int \int |\pi_{\theta \#} \mu_b| \, d\mathcal{H}^2 d\theta \lesssim 1,$$

where μ_b is defined by (93), and the implicit constant depends on $\alpha, \alpha_0, \epsilon$ and μ.

Lemma 7. *Let $\alpha > 2$. Then for $\alpha_0 < 2$ sufficiently close to 2, and $\epsilon > 0$ small enough depending on α and α_0, and $\delta \ll \epsilon$,*

(94) $$\int \int |\pi_{\theta \#} \mu_g|^2 \, d\mathcal{H}^2 \, d\theta \lesssim 1,$$

where the implicit constant depends on $\alpha, \alpha_0, \epsilon$ and μ.

Proof of Lemma 6. By definition, we first write

(95) $$\int \int |\pi_{\theta \#} \mu_b| \, d\mathcal{H}^2 d\theta \leq \int \sum_{j \in \mathbb{N}} \sum_{k \in [j\epsilon, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} \int |\pi_{\theta \#} M_T \mu| \, d\mathcal{H}^2 d\theta.$$

By the triangle inequality, this is

$$
\text{(96)} \qquad \leq \sum_j \sum_{k \in [j\epsilon, j]} \int \int \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| < 2^{k(-1/2+\delta)}}} \sum_{T \in \mathbb{T}_{\tau,b}} |\pi_{\theta \#} M_T \mu| \, d\mathcal{H}^2 d\theta
$$

$$
\text{(97)} \qquad + \sum_j \sum_{k \in [j\epsilon, j]} \int \int \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \geq 2^{k(-1/2+\delta)}}} \sum_{T \in \mathbb{T}_{\tau,b}} |\pi_{\theta \#} M_T \mu| \, d\mathcal{H}^2 d\theta.
$$

By Lemma 5, the contribution from (97) is

$$
\lesssim_{\delta,\epsilon,N} \sum_j \sum_{k \in [\epsilon j, j]} 2^{3j - \frac{3}{2}k - kN} \mu\left(\mathbb{R}^3\right) \lesssim_{\delta,\epsilon} \sum_j \sum_{k \in [\epsilon j, j]} 2^{-j} \mu\left(\mathbb{R}^3\right) \lesssim \mu(\mathbb{R}^3),
$$

By choosing $N > 100\varepsilon^{-1}$.

To estimate (96), we discretize the integration in θ and bound it by

$$
\text{(98)} \qquad \sum_j \sum_{k \in [j\epsilon, j]} \sum_{\theta \in \Theta_k} 2^{-k/2} \int \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| < 2^{k(-1/2+\delta)}}} \sum_{T \in \mathbb{T}_{\tau,b}} |\pi_{\theta \#} M_T \mu| \, d\mathcal{H}^2
$$

By Lemma 4 the contribution from (96) is

$$
\lesssim \mu\left(\mathbb{R}^3\right) + \sum_j \sum_{k \in [j\epsilon, j]} \sum_\theta 2^{-k/2} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| < 2^{k(-1/2+\delta)}}} \sum_{T \in \mathbb{T}_{\tau,b}} 2^{3j\delta} \mu(2T)
$$

$$
\text{(99)} \qquad \lesssim \mu\left(\mathbb{R}^3\right) + \sum_j \sum_{k \in [j\epsilon, j]} \sum_\theta 2^{-k/2} 2^{100j\delta} \mu(B_{j,k}(\theta)),
$$

where

$$
B_{j,k}(\theta) = \bigcup_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| < 2^{k(-1/2+\delta)}}} \bigcup_{T \in \mathbb{T}_{\tau,b}} 2T.
$$

For fixed j and k, let $\{B_l\}_l$ be a finitely overlapping cover of the unit ball in $\mathbb{R}^3$ by balls of radius $2^{-(j-k)}$. For each θ and l let

$$
B_{j,k,l}(\theta) = \bigcup_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| < 2^{k(-1/2+\delta)}}} \bigcup_{\substack{T \in \mathbb{T}_{\tau,b}: \\ 2T \cap B_l \neq \emptyset}} 2T.
$$

Let $\mu_{j,k}$ be the pushforward of μ under $x \mapsto 2^{j-k-2k\delta} x$. Denote

$$
\text{(100)} \quad B'_{j,k,l}(\theta) := 2^{j-k-2k\delta} \cdot B_{j,k,l}(\theta), \quad \widetilde{B}_l = \left\{ 2^{j-k-2k\delta} b_l + y : |y| \leq 1, y \in \mathbb{R}^3 \right\},
$$

with b_l the centre of B_l, and define

$$
\text{(101)} \qquad \widetilde{\mu}_{j,k,l} = 2^{\alpha(j-k-2k\delta)} \cdot \mu_{j,k} \chi_{\widetilde{B}_l}.
$$

Then
$$
\text{(102)}
$$
$$
\sum_\theta \mu(B_{j,k}(\theta)) \leq \sum_\theta \sum_l \mu(B_{j,k,l}(\theta)) \leq \sum_l 2^{-\alpha(j-k-2k\delta)} \sum_\theta \widetilde{\mu}_{j,k,l}(B'_{j,k,l}(\theta)).
$$

Note that for each θ, the set $B'_{j,k,l}(\theta)$ is contained in a union of planks of dimensions $1 \times 2^{-k/2} \times 2^{-k}$; the number of planks is

$$
\text{(103)} \qquad \lesssim 2^{k\frac{\alpha_0+1}{2} + C\delta k} \widetilde{\mu}_{j,k,l}(\mathbb{R}^3),
$$

for some large constant C, and each plank overlaps $\lesssim 2^{10k\delta}$ of the others. Moreover $c_\alpha(\widetilde{\mu}_{j,k,l}) \leq 1$ and $\widetilde{\mu}_{j,k,l}$ is supported in a ball of radius 1. Therefore, by applying the triangle inequality and Corollary 3, we can find δ', depending only on α and α_0, such that

$$2^{-k/2} \sum_\theta \widetilde{\mu}_{j,k,l}(B'_{j,k,l}(\theta)) \lesssim 2^{-C'\delta'k + C'\delta k} \widetilde{\mu}_{j,k,l}(\mathbb{R}^3)$$

$$\lesssim 2^{-C'\delta'k + C'\delta k} 2^{\alpha(j-k-2k\delta)} \mu_{j,k}(\widetilde{B}_l),$$

for some large constant C', whose precise value is not important. Putting this into (102) yields

(104) $$\qquad\qquad (102) \lesssim 2^{k/2} 2^{-C'\delta'k + C'\delta k} \mu(\mathbb{R}^3).$$

Substituting this into (99) and then (95) gives

(105) $$\qquad (95) \lesssim \sum_j \sum_{k \in [\epsilon j, j]} 2^{100\delta j} 2^{-C'\delta'k + C'\delta k} \mu(\mathbb{R}^3)$$

Recall that δ' depends only on α and α_0. We just need to pick δ to be sufficiently small, and will finish the proof. $\qquad\square$

Proof of Lemma 7. Take $\epsilon \ll \min\{\alpha - 2, 2 - \alpha_0\}$. Given $x \in \pi_\theta(\mathbb{R}^3)$, note that

(106) $$\pi_{\theta\#}\mu_g(x) = \int \mu_g(x + t\gamma(\theta)) dt.$$

Fix the coordinate $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\gamma'(\theta), \gamma(\theta) \times \gamma'(\theta), \gamma(\theta))$. Any $x \in \pi_\theta(\mathbb{R}^3)$ can be written in this coordinate as $x = (x_1, x_2, 0)$. We can also rewrite (106) as

$$\pi_{\theta\#}\mu_g(x_1, x_2) = \int \mu_g(x_1, x_2, t) dt.$$

Doing the Fourier transfom in the $(\mathbf{e}_1, \mathbf{e}_2)$-plane, we have

$$(\pi_{\theta\#}\mu_g)^\wedge(\eta_1, \eta_2) = \int \mu_g(x_1, x_2, t) e^{-i(x_1\eta_1 + x_2\eta_2)} dt$$

$$= \widehat{\mu_g}(\eta_1, \eta_2, 0) = \widehat{\mu_g}(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)).$$

By Plancherel's theorem,

(107) $$\int\int |\pi_{\theta\#}\mu_g|^2 \, d\mathcal{H}^2 \, d\theta = \int\int_{\mathbb{R}^2} |\widehat{\mu_g}(\eta_1\gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta.$$

Roughly speaking,

$$\mu_g = \mu_0 + \mu_{g,1} + \mu_{g,2},$$

where μ_0 is roughly $\mu(\mathbb{R}^3)\mathbf{1}_{B^3(0,1)}$ with rapidly decaying tail outside $B^3(0,1)$, $\mu_{g,1}$ is the sum of good wave packets which have controlled mass, and $\mu_{g,2}$ is the sum of wave packets which are far away from the cone Γ. A formula for $\mu_{g,1}$ is

$$\mu_{g,1} = \sum_{j \in \mathbb{N}} \sum_{k \in [j\epsilon, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} M_T \mu.$$

The above used that $\mathbb{T}_{\tau,b}$ is empty when $\tau \in \Lambda_{j,k}$, $k \leq j\epsilon$ and j is sufficiently large, which follows from the Frostman condition on μ.

Claim 1. *Let $\tau \in \Lambda_{j,k}$ with $k < j$. If there exist $\theta \in [0, a]$ and (η_1, η_2) satisfying*

$$(108) \qquad \eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta) \in \operatorname{supp}(\psi_\tau),$$

then it holds that $|\eta_2| \sim 2^j$ and $|\eta_1| \sim 2^{j-k/2}$.

Proof of Claim 1. Recalling (85), we may assume

$$\tau = \tau(\theta', j, k)$$
$$= \left\{ \lambda_1 \gamma'(\theta') + \lambda_2 (\gamma \times \gamma')(\theta') + \lambda_3 \gamma(\theta') : |\lambda_1| \lesssim 2^{-k/2+j}, \lambda_2 \sim 2^j, |\lambda_3| \sim 2^{-k+j} \right\}.$$

If $\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta) \in \tau(\theta', j, k)$, we discuss some geometric observations. Noting that $|\lambda_3| \sim 2^{-k+j}$ in the definition of $\tau(\theta', j, k)$, we see that $\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta) \notin \tau(\theta', j, k)$ if $\theta' = \theta$; we also note that if $|\theta' - \theta| \gg 2^{-k/2}$ are too far apart, then $\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta) \notin \tau(\theta', j, k)$. Therefore we must have $|\theta' - \theta| \sim 2^{-k/2}$. In this case, in order for $\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta) \in \tau(\theta', j, k)$, we must have $|\eta_2| \sim 2^j$ and $|\eta_1| \sim 2^{j-k/2}$, which finishes the proof. $\square$

By Claim 1, we see that (107) is bounded by

$$(109) \qquad 1 + \int \int_{\{|\eta_1| \geq |\eta_2|^{1-\epsilon}\}} |\widehat{\mu}(\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta$$

$$(110) \qquad + \int \int_{\{|\eta_1| < |\eta_2|^{1-\epsilon}\}} |\widehat{\mu_{g,1}}(\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta.$$

For the first term, the change of variables

$$(111) \qquad \xi = \xi(\eta, \theta) = \eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta)$$

has Jacobian

$$(112) \qquad \left| \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\eta_1, \eta_2, \theta)}(\eta_1, \eta_2, \theta) \right| = |\eta_1| \, |\det((\gamma \times \gamma')(\theta), \gamma'(\theta), \gamma''(\theta))|$$
$$= |\eta_1| \, |\langle \gamma(\theta), \gamma''(\theta) \rangle| = |\eta_1|,$$

where in the last step we used

$$(113) \qquad \langle \gamma(t), \gamma'(t) \rangle \equiv 0 \implies \langle \gamma(t), \gamma''(t) \rangle \equiv -1.$$

Applying this change of variables to (109) gives

$$(109) \lesssim 1 + \int_{|\xi| \geq 1} |\xi|^{\epsilon - 1} |\widehat{\mu}(\xi)|^2 \, d\xi \lesssim 1 + I_{\alpha - \epsilon}(\mu) \lesssim 1.$$

Here $I_{\alpha - \epsilon}(\mu) = \int |\xi|^{\alpha - \epsilon - 3} |\widehat{\mu}(\xi)|^2 \, d\xi$ is the $(\alpha - \epsilon)$-energy of μ and we used the fact that $\alpha > 2$ and ϵ is sufficiently small. The last step is because $c_\alpha(\mu) < \infty$.

It remains to bound the contribution from $\mu_{g,1}$, in (110). By frequency disjointness,

$$(114) \qquad \begin{aligned} &\int \int_{\mathbb{R}^2} |\widehat{\mu_{g,1}}(\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta))|^2 \, d\eta \, d\theta \\ &\lesssim \sum_j \sum_{k \in [\epsilon j, j]} \sum_{\tau \in \Lambda_{j,k}} \int \int_{\mathbb{R}^2} \left| \sum_{T \in \mathbb{T}_{\tau, g}} \widehat{M_T \mu}(\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta)) \right|^2 d\eta \, d\theta. \end{aligned}$$

Consider the case $k < j$ and $k = j$ separately. In the former case, we apply the change of variables as in (111) and obtain

$$(115) \quad \sum_{\tau \in \Lambda_{j,k}} \int \iint_{\mathbb{R}^2} \Big| \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu} \left(\eta_1 \gamma'(\theta) + \eta_2 \left(\gamma \times \gamma' \right)(\theta) \right) \Big|^2 d\eta \, d\theta$$

$$\lesssim \sum_{\tau \in \Lambda_{j,k}} 2^{-j+k/2} \int_{\mathbb{R}^3} \Big| \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(\xi) \Big|^2 d\xi$$

$$\lesssim \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} 2^{-j+k/2} 2^{O(\delta)k} \int_{\mathbb{R}^3} \Big| M_T \mu(x) \Big|^2 dx.$$

When $k = j$, we show that (115) holds as well. To see this, we first observe that for each fixed T, in order for

$$(116) \qquad \widehat{M_T \mu} \left(\eta_1 \gamma'(\theta) + \eta_2 \left(\gamma \times \gamma' \right)(\theta) \right)$$

not to vanish, θ has to take values on an interval of length $2^{-j/2}$; next, we apply the two dimensional Plancherel's theorem in the η_1 and η_2 variables for every fixed θ, and (115) follows from the uncertainty principle.

We continue to estimate (115) and do not distinguish $k < j$ and $k = j$ anymore. We have

$$(117) \qquad (115) \lesssim \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} 2^{-j+k/2} 2^{O(\delta)k} \int_{\mathbb{R}^3} \Big| M_T \mu(x) \Big|^2 dx$$

$$(118) \qquad = 2^{-j+k/2} 2^{O(\delta)k} \int \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} f_T \, d\mu$$

where

$$(119) \qquad f_T := (\eta_T M_T \mu) * \widecheck{\psi_\tau},$$

and from (117) to (118) we applied Fubini and expanded the square. We cut the unit ball into small balls B_ι of radius 2^{-j+k} and let ν_ι be the restriction of μ to $2^{10\delta k} B_\iota$. By Cauchy-Schwarz,

$$(120) \qquad (118) \lesssim 2^{-j+k/2} 2^{O(\delta)k} \sum_\iota \mu(2^{10k\delta} B_\iota)^{1/2} \Big(\int \Big| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} f_T \Big|^2 d\nu_\iota \Big)^{1/2}$$

Let ζ_j be a non-negative bump function such that $\widehat{\zeta_j}(\xi) = 1$ for $|\xi| \leq 2^{j+10}$. By the Fourier support information of f_T, we have

$$(121) \qquad \int \Big| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} f_T \Big|^2 d\nu_\iota = \int \Big| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} f_T \Big|^2 d(\nu_\iota * \zeta_j)$$

By pigeonholing, we can find a subset

$$(122) \qquad \mathbb{W}_\iota \subset \bigcup_{\tau \in \Lambda_{j,k}} \{ T \in \mathbb{T}_{\tau,g} : T \cap B_\iota \neq \emptyset \}$$

such that $\big\| f_T \big\|_2$ is constant up to a factor of 2 as T varies over $\mathbb{W}_\iota$, and

$$(123) \qquad (121) \lesssim 2^{O(\delta)j} \int \Big| \sum_{T \in \mathbb{W}_\iota} f_T \Big|^2 d(\nu_\iota * \zeta_j)$$

By pigeonholing again and by Hölder's inequality, there is a disjoint union Y of balls Q of radius 2^{-j}, such that

$$(124) \qquad \int \Big| \sum_{T \in \mathbb{W}_\iota} f_T \Big|^2 d(\nu_\iota * \zeta_j) \lesssim 2^{O(\delta)j} \Big\| \sum_{T \in \mathbb{W}_\iota} f_T \Big\|_{L^p(Y)}^2 \left(\int_Y (\nu_\iota * \zeta_j)^{\frac{p}{p-2}} \right)^{1-\frac{2}{p}},$$

and such that each $Q \subseteq Y$ intersects $\sim M$ planks $3T$ as T varies over $\mathbb{W}_\iota$, for some dyadic number M. By rescaling and then applying the refined decoupling inequality in Theorem 9 from the Appendix A, the first term in (124) satisfies

$$(125) \qquad \Big\| \sum_{T \in \mathbb{W}_\iota} f_T \Big\|_{L^p(Y)} \lesssim 2^{(3j-\frac{3k}{2})(\frac{1}{2}-\frac{1}{p})+O(\epsilon)j} \left(\frac{M}{|\mathbb{W}_\iota|} \right)^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{T \in \mathbb{W}_\iota} \|f_T\|_2^2 \right)^{1/2}.$$

For the second term in (124), the assumption that $c_\alpha(\mu)$ is finite implies that

$$(126) \qquad \|\nu_\iota * \zeta_j\|_\infty \lesssim 2^{j(3-\alpha)}.$$

Hence by Hölder's inequality and the definition of $\mathbb{T}_{\tau,g}$, we have

$$(127) \qquad \int_Y (\nu_\iota * \zeta_j)^{\frac{p}{p-2}} \lesssim 2^{\frac{2j(3-\alpha)}{p-2}} M^{-1} \sum_{T \in \mathbb{W}_\iota} (\nu_\iota * \zeta_j)(3T)$$

$$(128) \qquad \lesssim 2^{\frac{2j(3-\alpha)}{p-2}} M^{-1} |\mathbb{W}_\iota| 2^{-k(\alpha_0+1)/2 - \alpha(j-k)}.$$

Combining (125) and (128), we obtain

$$(129) \qquad (124) \lesssim 2^{O(\epsilon)j} 2^{j(3-\alpha)+k\left(\frac{1}{2}-\frac{1}{p}\right)(-4+2\alpha-\alpha_0)} \sum_{T \in \mathbb{W}_\iota} \|f_T\|_2^2.$$

Note that

$$(130) \qquad \|f_T\|_2 \lesssim \|M_T \mu\|_2$$

for every T. Substituting into (129) and then into (117) yields

$$(131) \qquad \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} \int_{\mathbb{R}^3} \left| M_T \mu(x) \right|^2 dx$$

$$(132) \qquad \lesssim \sum_\iota \mu(2^{10k\delta} B_\iota)^{1/2} 2^{O(\epsilon)j} 2^{\frac{1}{2}j(3-\alpha)+\frac{1}{2}k\left(\frac{1}{2}-\frac{1}{p}\right)(-4+2\alpha-\alpha_0)} \left(\sum_{T \in \mathbb{W}_\iota} \|f_T\|_2^2 \right)^{1/2}.$$

By Cauchy-Schwarz in the sum over ι, we obtain

$$(133) \qquad \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} \int_{\mathbb{R}^3} \left| M_T \mu(x) \right|^2 dx \lesssim 2^{O(\epsilon)j} 2^{j(3-\alpha)+k\left(\frac{1}{2}-\frac{1}{p}\right)(-4+2\alpha-\alpha_0)}.$$

By substituting back into (117), we obtain

$$(134) \qquad (115) \lesssim \sum_j \sum_{\epsilon j \le k \le j} 2^{-j+k/2} 2^{O(\epsilon)j} 2^{j(3-\alpha)+k\left(\frac{1}{2}-\frac{1}{p}\right)(-4+2\alpha-\alpha_0)}.$$

In the end, we pick $p = 4$ and finish the proof. $\square$

APPENDIX A. REFINED DECOUPLING INEQUALITY

The refined decoupling inequality stated here is a natural analogue of the refined decoupling inequality for the paraboloid from [9]. The shortest length of a plank dual to an $R^{-1/2}$-cap in the cone is ≈ 1, rather than $\approx R^{1/2}$ in the case of the paraboloid, so the setup uses unit cubes instead of $R^{1/2}$-cubes. The argument is similar to the paraboloid case, using induction and with Lorentz rescaling in place of parabolic rescaling, but the use of unit cubes requires the induction to be carried out over a finer sequence of scales (similarly to the induction setup in [4]). The refined decoupling inequalities in [13, 14] used tubes rather than planks, and the use of planks here is a significant reason for the improved result on the projection problem (at least in Theorem 2). Much of the proof is similar to [13, 14], with only the differences outlined above.

For each $R \geq 1$ let $\Xi_R = \left\{jR^{-1/2} : j \in \mathbb{Z}\right\} \cap [0, a]$. For each $\theta \in \Xi_R$, let

(135)
$$\tau_R(\theta) = \left\{x_1\gamma(\theta) + x_2\gamma'(\theta) + x_3(\gamma \times \gamma')(\theta) : 1 \leq x_1 \leq 2, \, |x_2| \leq R^{-1/2}, \, |x_3| \leq R^{-1}\right\}.$$

If it is clear from the context which R is used, then we often abbreviate $\tau_R(\theta)$ to $\tau(\theta)$. Let $\mathcal{P}_{R^{-1}} = \{\tau(\theta) : \theta \in \Xi_R\}$. For $\tau = \tau(\theta) \in \mathcal{P}_{R^{-1}}$, denote $\theta_\tau := \theta$. Let

(136)
$$T^\circ_{\tau,0} = \left\{x_1\left(\gamma \times \gamma'\right)(\theta_\tau) + x_2\gamma'(\theta_\tau) + x_3\gamma(\theta_\tau) : |x_1| \leq R, \, |x_2| \leq R^{1/2}, \, |x_3| \leq 1\right\}.$$

Moreover, we will use $\mathbb{T}^\circ_\tau$ to denote the collection of translates of $T^\circ_{\tau,0}$ that cover $B(0, R)$. For a fixed small constant $\delta > 0$, denote $T_{\tau,0} := R^\delta T^\circ_{\tau,0}$, and $\mathbb{T}_\tau := \{R^\delta T : T \in \mathbb{T}^\circ_\tau\}$. For $T \in \mathbb{T}_\tau$, set $\tau(T) = \tau$.

Definition 5. *Fix $T \in \mathbb{T}_\tau$. We say that a function $f_T : \mathbb{R}^3 \to \mathbb{C}$ is a T-function if $\widehat{f_T}$ is supported on $\tau(T)$ and*

(137)
$$\|f_T\|_{L^\infty(B(0,R)\setminus T)} \lesssim_\delta R^{-10000} \|f_T\|_2 .$$

Theorem 9. *Let $\gamma : [a, b] \to \mathbb{S}^2$ be a C^2 curve with $\det(\gamma, \gamma', \gamma'')$ nonvanishing. Let $B \geq 1$ be such that*

(138)
$$|\det(\gamma, \gamma', \gamma'')| \geq B^{-1},$$

and

(139)
$$\|\gamma\|_{C^2[a,b]} \leq B.$$

Let $R \geq 1$ and suppose that

$$f = \sum_{T \in \mathbb{W}} f_T,$$

where each f_T is a T-function and

$$\mathbb{W} \subseteq \bigcup_{\tau \in \mathcal{P}_{R^{-1}}} \mathbb{T}_\tau.$$

Assume that for all $T, T' \in \mathbb{W}$,

(140)
$$\|f_T\|_2 \sim \|f_{T'}\|_2.$$

Let Y be a disjoint union of unit balls in $B(0, R)$, each of which intersects at most M sets $2T$ with $T \in \mathbb{W}$. Then for $2 \leq p \leq 6$,

$$\|f\|_{L^p(Y)} \lesssim_{\epsilon, \delta} R^\epsilon \left(\frac{M}{|\mathbb{W}|}\right)^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{T \in \mathbb{W}} \|f_T\|_p^2\right)^{1/2}.$$

Proof. Assume that $[a, b] = [-1, 1]$. Fix $\epsilon \in (0, 1/2)$, $\delta_0 = \epsilon^{100}$, $\delta \in (0, \delta_0)$,

$$R \geq \min\left\{B^{10^3/\epsilon}, 2^{10^5/\epsilon}\right\},$$

and assume inductively that a (superficially) stronger version of the theorem holds with K^2-cubes instead of unit cubes, where $K = R^{\delta^2}$, for all scales smaller than $\widetilde{R} := R/K^2$, for all curves γ satisfying (138) and (139), and for all $B \geq 1$.

For each $\tau \in \mathcal{P}_{R^{-1/2}}(\Gamma(\gamma))$, let $\kappa = \kappa(\tau) \in \mathcal{P}_{K^{-1}}(\Gamma(\gamma))$ be the element of $\mathcal{P}_{K^{-1}}(\Gamma(\gamma))$ which minimises $|\theta_\tau - \theta_\kappa|$. For each κ, let

$$\Box_{\kappa, 0} = \left\{x_1 \frac{(\gamma \times \gamma')(\theta_\kappa)}{|(\gamma \times \gamma')(\theta_\kappa)|} + x_2 \frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + x_3 \gamma(\theta_\kappa) : \right.$$

$$\left. |x_1| \leq R^{1+\delta}, \ |x_2| \leq R^{1+\delta}/K, \ |x_3| \leq R^{1+\delta}/K^2\right\},$$

and

$$\mathbb{P}_\kappa = \left\{\Box = a\gamma(\theta_\kappa) + b\frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + \Box_{\kappa, 0} : \right.$$

$$\left. a \in \left((1/10)R^{1+\delta}K^{-2}\right)\mathbb{Z}, \quad b \in \left((1/10)R^{1+\delta}K^{-1}\right)\mathbb{Z}\right\}.$$

Let $\mathbb{P} = \bigcup_{\kappa \in \mathcal{P}_{K^{-1}}(\Gamma(\gamma))} \mathbb{P}_\kappa$. Given any τ and corresponding $\kappa = \kappa(\tau)$,

(141) $$|\langle (\gamma \times \gamma')(\theta_\tau), \gamma'(\theta_\kappa)\rangle| \leq B^{-7}K^{-1},$$

and

(142) $$|\langle (\gamma \times \gamma')(\theta_\tau), \gamma(\theta_\kappa)\rangle| \leq B^{-7}K^{-2}.$$

It follows that for each $T \in \mathbb{T}_\tau$, there are ~ 1 sets $\Box \in \mathbb{P}_{\kappa(\tau)}$ with $T \cap 10\Box \neq \emptyset$, and moreover $T \subseteq 100\Box$ whenever $T \cap 10\Box \neq \emptyset$. For each such T let $\Box = \Box(T) \in \mathbb{P}_\kappa$ be some choice such that $T \cap 10\Box \neq \emptyset$, and let $\mathbb{W}_\Box$ be the set of T's associated to $\Box$.

For each κ and $\Box \in \mathbb{P}_\kappa$, let $\{Q_\Box\}_{Q_\Box}$ be a finitely overlapping cover of $100\Box$ by translates of the ellipsoid

$$\left\{x_1 \gamma(\theta_\kappa) + x_2\frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + x_3\frac{(\gamma \times \gamma')(\theta_\kappa)}{|(\gamma \times \gamma')(\theta_\kappa)|} : \right.$$

$$\left. \left(|x_1|^2 + \left(|x_2|K^{-1}\right)^2 + \left(|x_3|K^{-2}\right)^2\right)^{1/2} \leq \widetilde{K}^2\right\}.$$

Using Poisson summation, let $\{\eta_{Q_\Box}\}_{Q_\Box \in \mathcal{Q}_\Box}$ be a smooth partition of unity such that on $10^3\Box$,

$$\sum_{Q_\Box \in \mathcal{Q}_\Box} \eta_{Q_\Box} = 1,$$

and such that each $\eta_{Q_\Box}$ satisfies

$$\|\eta_{Q_\Box}\|_\infty \lesssim 1, \quad \|\eta_{Q_\Box}\|_{L^\infty(\mathbb{R}^3 \setminus Q_\Box)} \lesssim R^{-10000},$$

and

$$|\eta_{Q_\square}(x)| \lesssim \operatorname{dist}(x, Q_\square)^{-10000} \quad \forall x \in \mathbb{R}^3,$$

with $\widehat{\eta_{Q_\square}}$ supported in

$$\left\{\xi_1 \gamma(\theta_\kappa) + \xi_2 \frac{\gamma'(\theta_\kappa)}{|\gamma'(\theta_\kappa)|} + \xi_3 \frac{(\gamma \times \gamma')(\theta_\kappa)}{|(\gamma \times \gamma')(\theta_\kappa)|}\right.$$
$$\left. : |\xi_1| \le \widetilde{K}, \quad |\xi_2| \le \widetilde{K}K, \quad |\xi_3| \le \widetilde{K}K^2 \right\}.$$

By dyadic pigeonholing,

$$\|f\|_{L^p(Y)} \lesssim \log R \left\|\sum_\square \sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(Y)} + R^{-1000} \left(\sum_{T \in \mathbb{W}} \|f_T\|_p^2\right)^{1/2},$$

where, for each $\square$, $Y_\square$ is a union over a subset of the sets $Q_\square$, and $\eta_{Y_\square}$ is the corresponding sum over $\eta_{Q_\square}$, such that each $Q_\square \subseteq Y_\square$ intersects a number $\# \in [M'(\square), 2M'(\square))$ different sets $(1.5)T$ with $T \in \mathbb{W}_\square$, up to a factor of 2. By pigeonholing again,

$$\left\|\sum_\square \sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(Y)} \lesssim (\log R)^2 \left\|\sum_{\square \in \mathbb{B}} \sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(Y)},$$

where $|\mathbb{W}_\square|$ and $M' = M'(\square)$ are constant over $\square \in \mathbb{B}$, up to a factor of 2. By one final pigeonholing step,

$$\left\|\sum_{\square \in \mathbb{B}} \sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(Y)} \lesssim \log R \left\|\sum_{\square \in \mathbb{B}} \sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(Y')},$$

where Y' is a union over K^2-balls $Q \subseteq Y$ such that each ball $2Q$ intersects a number $\# \in [M'', 2M'')$ of the sets $Y_\square$ in a set of strictly positive Lebesgue measure, as $\square$ varies over $\mathbb{B}$. Fix $Q \subseteq Y'$. By the decoupling theorem for generalised C^2 cones, followed by Hölder's inequality,

$$\left\|\sum_{\square \in \mathbb{B}} \sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(Q)}$$

$$\le C_\epsilon B^{100} K^{\epsilon/100} (M'')^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{\square \in \mathbb{B}} \left\|\sum_{T \in \mathbb{W}_\square} \eta_{Y_\square} f_T\right\|_{L^p(2Q)}^p\right)^{1/p}$$

$$+ R^{-900} \left(\sum_{T \in \mathbb{W}} \|f_T\|_p^2\right)^{1/2}.$$

Summing over Q gives

$$\|f\|_{L^p(Y)} \lesssim C_\epsilon (\log R)^{100} B^{100} K^{\epsilon/100} (M'')^{\frac{1}{2} - \frac{1}{p}}$$

$$\times \left(\sum_{\square \in \mathbb{B}} \left\|\sum_{T \in \mathbb{W}_\square} f_T\right\|_{L^p(Y_\square)}^p\right)^{1/p} + R^{-800} \left(\sum_{T \in \mathbb{W}} \|f_T\|_p^2\right)^{1/2}.$$

This will be bounded using the inductive assumption, following a Lorentz rescaling.

For each $\theta \in [-1, 1]$, define the Lorentz rescaling map $L = L_\theta$ at θ by

$$L\left[x_1\gamma(\theta) + x_2\frac{\gamma'(\theta)}{|\gamma'(\theta)|} + x_3\frac{(\gamma \times \gamma')(\theta)}{|(\gamma \times \gamma')(\theta)|}\right]$$
$$= x_1\gamma(\theta) + Kx_2\frac{\gamma'(\theta)}{|\gamma'(\theta)|} + K^2x_3\frac{(\gamma \times \gamma')(\theta)}{|(\gamma \times \gamma')(\theta)|}.$$

Let

$$\widetilde{\gamma}(\phi) = \frac{L(\gamma(\phi))}{|L(\gamma(\phi))|}, \qquad \phi \in [-1, 1].$$

Then for any $\phi \in [-1, 1]$,

$$\widetilde{\gamma}'(\phi) = \frac{\pi_{\widetilde{\gamma}(\phi)^\perp}(L(\gamma'(\phi)))}{|L(\gamma(\phi))|},$$

and

$$\widetilde{\gamma}''(\phi) = \frac{\pi_{\widetilde{\gamma}(\phi)^\perp}(L(\gamma''(\phi)))}{|L(\gamma(\phi))|} - \frac{\langle L(\gamma(\phi)), L(\gamma'(\phi))\rangle\,\pi_{\widetilde{\gamma}(\phi)^\perp}(L(\gamma'(\phi)))}{|L(\gamma(\phi))|^3}.$$

Hence

$$\det(\widetilde{\gamma}, \widetilde{\gamma}', \widetilde{\gamma}'') = \frac{1}{|L \circ \gamma|^3}\det(L \circ \gamma, L \circ \gamma', L \circ \gamma'')$$
$$= \frac{K^3}{|L \circ \gamma|^3}\det(\gamma, \gamma', \gamma'').$$

Let $\varepsilon = (10^5 B^{10})^{-1}$, and for fixed $\theta \in [-1 + \varepsilon, 1 - \varepsilon]$, let

(143) $$\widehat{\gamma}(\phi) = \widetilde{\gamma}(\theta + K^{-1}\phi), \quad \phi \in [-\varepsilon, \varepsilon].$$

The assumption that $\|\gamma\|_{C^2[-1,1]} \leq B$ yields

$$1 \leq |L(\gamma(\phi))| \leq 1 + 10B\varepsilon, \quad \forall \phi \in [\theta - \varepsilon K^{-1}, \theta + \varepsilon K^{-1}].$$

Similarly,

$$|L(\gamma'(\phi)) - L(\gamma'(\theta))| \leq 10\varepsilon BK, \quad \forall \phi \in [\theta - \varepsilon K^{-1}, \theta + \varepsilon K^{-1}].$$

It follows that

$$|\det(\widehat{\gamma}, \widehat{\gamma}', \widehat{\gamma}'')| \geq (2B)^{-1}$$

on $[-\varepsilon, \varepsilon]$, and that

$$\|\widehat{\gamma}\|_{C^2[-\varepsilon, \varepsilon]} \leq 2B.$$

For each $\square \in \mathbb{B}$, given $T \in \mathbb{W}_\square$, let $g_T = f_T \circ L$, where $L = L_{\theta_{\kappa(\square)}}$. Then

(144) $$\left\|\sum_{T \in \mathbb{W}_\square} f_T\right\|_{L^p(Y_\square)} \leq K^{\frac{3}{p}}\left\|\sum_{T \in \mathbb{W}_\square} g_T\right\|_{L^p(L^{-1}Y_\square)}.$$

The inequalities (141) and (142) imply that for each $T \in \mathbb{W}_\square$, the set $L^{-1}(T)$ is a equivalent (up to a factor 1.01) to a plank of length $\widetilde{R}^{1+\delta}$ in its longest direction parallel to $L^{-1}(\gamma \times \gamma')(\theta_{\tau(T)})$, of length $\widetilde{R}^{1/2+\delta}$ in its medium direction, and of length $\widetilde{R}^\delta$ in its shortest direction. The ellipsoids $Q_\square$ are rescaled to $\widetilde{K}^2$-balls $L^{-1}(Q_\square)$. Moreover, it will be shown that

(145) $\quad L(\tau) \subseteq \left\{ x_1 \widetilde{\gamma}(\theta_\tau) + x_2 \frac{\widetilde{\gamma}'(\theta_\tau)}{|\widetilde{\gamma}'(\theta_\tau)|} + x_3 \frac{(\widetilde{\gamma} \times \widetilde{\gamma}')(\theta_\tau)}{|(\widetilde{\gamma} \times \widetilde{\gamma}')(\theta_\tau)|} \right.$

$$\left. : 1 \le x_1 \le 2.01, \ |x_2| \le (1.01)\widetilde{R}^{-1/2}, \ |x_3| \le \widetilde{R}^{-1} \right\}.$$

To prove this, let

$$x = x_1 \gamma(\theta_\tau) + x_2 \frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} + x_3 \frac{(\gamma \times \gamma')(\theta_\tau)}{|(\gamma \times \gamma')(\theta_\tau)|} \in \tau,$$

where

$$x_1 \in [1, 2], \quad |x_2| \le R^{-1/2}, \quad |x_3| \le R^{-1}.$$

The vector $(\widetilde{\gamma} \times \widetilde{\gamma}')(\theta_\tau)$ is parallel to $L^{-1}((\gamma \times \gamma')(\theta_\tau))$, since $L^{-1}((\gamma \times \gamma')(\theta_\tau))$ is orthogonal to $\widetilde{\gamma}(\theta_\tau)$ and $\widetilde{\gamma}'(\theta_\tau)$. The inequality

$$\left| L^{-1}((\gamma \times \gamma')(\theta_\tau)) \right| \ge K^{-2} \left| (\gamma \times \gamma')(\theta_\tau) \right|$$

gives

(146) $$\left| \left\langle Lx, \frac{L^{-1}((\gamma \times \gamma')(\theta_\tau))}{|L^{-1}((\gamma \times \gamma')(\theta_\tau))|} \right\rangle \right| \le \widetilde{R}^{-1}.$$

Moreover,

$$\left| \left\langle Lx, \frac{\pi_{L(\gamma(\theta_\tau))^\perp} \left(L\left(\gamma'(\theta_\tau) \right) \right)}{\left| \pi_{L(\gamma(\theta_\tau))^\perp} \left(L\left(\gamma'(\theta_\tau) \right) \right) \right|} \right\rangle \right|$$

$$= \left| \left\langle x_2 L\left(\frac{\gamma'(\theta_\tau)}{|\gamma'(\theta_\tau)|} \right) + x_3 L\left(\frac{(\gamma \times \gamma')(\theta_\tau)}{|(\gamma \times \gamma')(\theta_\tau)|} \right), \frac{\pi_{L(\gamma(\theta_\tau))^\perp} \left(L\left(\gamma'(\theta_\tau) \right) \right)}{\left| \pi_{L(\gamma(\theta_\tau))^\perp} \left(L\left(\gamma'(\theta_\tau) \right) \right) \right|} \right\rangle \right|$$

(147)

$$\le (1.01)\widetilde{R}^{-1/2}.$$

For the direction $L(\gamma(\theta_\tau))$,

(148) $$\left\langle Lx, \frac{L(\gamma(\theta_\tau))}{|L(\gamma(\theta_\tau))|} \right\rangle = x_1 |L(\gamma(\theta_\tau))| + O(K^2 R^{-1/2}).$$

Combining (146), (147) and (148) gives (145).

Inductively applying the theorem at scale $\widetilde{R}$ gives

$$(144) \lesssim C_{\epsilon,\delta} B^{10^{10}/\epsilon} R^\epsilon K^{-2\epsilon} \left(\frac{M'}{|\mathbb{W}_\square|} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{T \in \mathbb{W}_\square} \|f_T\|_p^2 \right)^{1/2}$$

for each $\square \in \mathbb{B}$. Hence

$$\|f\|_{L^p(Y)}$$

$$\le C_{\epsilon,\delta} B^{10^{10}/\epsilon} K^{-\epsilon} \left(\frac{M'M''}{|\mathbb{W}_\square|} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{\square \in \mathbb{B}} \left(\sum_{T \in \mathbb{W}_\square} \|f_T\|_p^2 \right)^{p/2} \right)^{1/p}.$$

By the dyadically constant property of $\|f_T\|_p$, this is

$$\lesssim C_{\epsilon,\delta} B^{10^{10}/\epsilon} K^{-\epsilon} \left(\frac{M'M''}{|\mathbb{W}|} \right)^{\frac{1}{2} - \frac{1}{p}} \left(\frac{|\mathbb{B}| \, |\mathbb{W}_\square|}{|\mathbb{W}|} \right)^{\frac{1}{p}} \left(\sum_{T \in \mathbb{W}} \|f_T\|_2^2 \right)^{1/2}.$$

The second bracketed term is $\lesssim 1$, since

$$|\mathbb{W}| = \sum_{T\in\mathbb{W}} 1 \geq \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}:\\ \square=\square(T)}} 1 \sim \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}:\\ \square=\square(T)}} 1 \geq |\mathbb{B}|\,|\mathbb{W}_\square|.$$

It remains to show that $M'M'' \lesssim M$. Let $Q \subseteq Y'$ be any $R^{1/2}$-ball. By definition of M,

$$M \gtrsim \sum_{\substack{T\in\mathbb{W}:\\ 2T\cap Q\neq\emptyset}} \sum_{\substack{\square\in\mathbb{B}:\\ \square=\square(T)}} 1$$

$$= \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}:\\ \square=\square(T)\\ 2T\cap Q\neq\emptyset}} 1$$

$$\geq \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}_\square:\\ 2T\cap Q\neq\emptyset}} 1.$$

By definition of M' and M'',

$$M'M'' \sim \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} M'$$

$$\leq \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{Q_\square\subseteq Y_\square} M'\frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)}$$

$$\sim \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{Q_\square\subseteq Y_\square} \sum_{\substack{T\in\mathbb{W}_\square:\\ Q_\square\cap(1.5)T\neq\emptyset}} \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)}$$

$$= \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{T\in\mathbb{W}_\square} \sum_{\substack{Q_\square\subseteq Y_\square:\\ Q_\square\cap(1.5)T\neq\emptyset}} \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)}$$

(149)
$$\leq \sum_{\substack{\square\in\mathbb{B}:\\ m(Y_\square\cap 2Q)>0}} \sum_{\substack{T\in\mathbb{W}_\square:\\ 2T\cap Q\neq\emptyset}} \sum_{Q_\square\subseteq Y_\square} \frac{m(Q_\square\cap 2Q)}{m(Y_\square\cap 2Q)}$$

$$\lesssim \sum_{\square\in\mathbb{B}} \sum_{\substack{T\in\mathbb{W}_\square:\\ 2T\cap Q\neq\emptyset}} 1$$

$$\lesssim M.$$

The inequality (149) above follows from the observation that if $Q_\square \cap 2Q \neq \emptyset$, and if $T \in \mathbb{W}_\square$ is such that $Q_\square \cap (1.5)T \neq \emptyset$, then $2T \cap Q \neq \emptyset$. $\qquad\square$

APPENDIX B. DECOUPLING FOR C^2 CONES

We have been using the decoupling inequality for C^2 cones in $\mathbb{R}^3$ in a few places above, but it may have not been written down in the literature. In the appendix, we state it and sketch its proof. We start with the decoupling for C^2 curves on $\mathbb{R}^2$.

Theorem 10. *Let $\gamma : [-1,1] \to \mathbb{R}$ with $\gamma(0) = \gamma'(0) = 0$ be C^2 and satisfy $\gamma''(t) \neq 0$ for every $t \in [-1,1]$. Then*

$$(150) \qquad \left\| E_{[-1,1]} f \right\|_{L^6(\mathbb{R}^2)} \lesssim_{\gamma,\epsilon} \delta^{-\epsilon} \Big(\sum_{I \subset [0,1], |I| = \delta} \| E_I f \|^2_{L^6(\mathbb{R}^2)} \Big)^{1/2},$$

for every $\epsilon > 0$ and $\delta \in (0,1)$. Here

$$(151) \qquad E_I f(x,y) := \int_I f(t) e^{i(xt + y\gamma(t))} dt,$$

for an interval $I \subset [-1,1]$.

One can follow the same argument as in [8] to prove Theorem 10. We leave out the proof. Via the bootstrapping argument as in Bourgain and Demeter [1] and Pramanik and Seeger [24], one can prove the following decoupling for C^2 cones.

Theorem 11. *Let $\gamma : [-1,1] \to \mathbb{R}$ with $\gamma(0) = \gamma'(0) = 0$ be C^2 and satisfy $\gamma''(t) \neq 0$ for every $t \in [-1,1]$. Then*

$$(152) \qquad \left\| \mathcal{E}_{[-1,1]} f \right\|_{L^6(\mathbb{R}^3)} \lesssim_{\gamma,\epsilon} \delta^{-\epsilon} \Big(\sum_{I \subset [0,1], |I| = \delta} \| \mathcal{E}_I f \|^2_{L^6(\mathbb{R}^3)} \Big)^{1/2},$$

for every $\epsilon > 0$ and $\delta \in (0,1)$. Here

$$(153) \qquad \mathcal{E}_I f(x,y,z) := \int_{[1,2] \times I} f(s,t) e^{i(sx + sty + s\gamma(t)z)} ds dt,$$

for an interval $I \subset [-1,1]$.

We give a sketch of the proof of Theorem 11. By the triangle inequality, we can assume that f is supported on $[1, 1+\delta^\epsilon] \times [0, \delta^\epsilon]$. The key observation in [1] is that the cone

$$(154) \qquad \{ s(1, t, \gamma(t)) : 1 \leq s \leq 1 + \delta^\epsilon, 0 \leq t \leq \delta^\mu \}$$

is in the $\delta^{2\mu+\epsilon}$-neighborhood of the cylinder

$$(155) \qquad \{ (s, t, \gamma(t)) : 1 \leq s \leq 1 + \delta^\epsilon, 0 \leq t \leq \delta^\mu \},$$

for every $\mu \geq \epsilon$. To see this, let us take one point $s(1, t, \gamma(t))$ from (154), and we will show that its distance to $(s, st, \gamma(st))$, which lies in (155), is $\lesssim \delta^{2\mu+\epsilon}$. This amounts to proving

$$(156) \qquad |\gamma(st) - s\gamma(t)| \lesssim \delta^{2\mu+\epsilon}.$$

Note that

$$(157) \qquad |\gamma(st) - s\gamma(t)| \lesssim |1 - s||\gamma(t)| + |\gamma(st) - \gamma(t)|.$$

The desired bound (156) follows from Taylor's expansion and mean value theorems.

After proving (156), one can then apply Theorem 10 iteratively, in the same way as in [1], and finish the proof of Theorem 11. We leave out the iteration step.

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