Statistical properties of sites visited by independent random walks

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The set of visited sites and the number of visited sites are two basic properties of the random walk trajectory. We consider two independent random walks on hyper-cubic lattices and study ordering probabilities associated with these characteristics. The first is the probability that during the time interval $(0, t)$, the number of sites visited by a walker never exceeds that of another walker. The second is the probability that the sites visited by a walker remain a subset of the sites visited by another walker. Using numerical simulations, we investigate the leading asymptotic behaviors of the ordering probabilities in spatial dimensions $d = 1, 2, 3, 4$. We also study the time evolution of the number of ties between the number of visited sites. We show analytically that the average number of ties increases as $a_1 \ln t$ with $a_1 = 0.970508$ in one dimension and as $(\ln t)^2$ in two dimensions.

I. INTRODUCTION

Random walk is an elementary random process which is ubiquitous in several branches of mathematics, physics, chemistry, biology, finance, etc. [1–5]. Questions involving large deviations, persistence, and geometrical characteristics of random walks continue to emerge [6, 7]. Here, we investigate ordering probabilities associated with the set of sites visited by independent random walks.

The maximum position attained by the walk is a basic characteristic of the set of visited sites. The maxima of two one-dimensional random walks remain ordered up to time t with a probability that decays as $t^{-1/4}$ [8, 9]. In general, it is difficult to compute "persistence" exponents for non-Markovian quantities such as the maximal position of a random walk [10, 11]. Nevertheless, the persistence exponent $1/4$ can be derived analytically [8, 9]. Further, it can also be shown that the average number of lead changes $A(t)$ grows logarithmically with time [12]

$$A(t) \simeq \pi^{-1} \ln t \tag{1}$$

The maximal position is (i) not uniquely defined in higher dimensions, and (ii) does not [13, 14] necessarily increase by equal amounts [15]. In this study, we focus on the total number $\mathcal{N}(t)$ of distinct sites visited by a random walk, *range* in short; in one dimension, $\mathcal{N}(t) = M(t) - m(t) + 1$ where $M(t)$ is the maximum, and $m(t)$ is the minimum. Unlike the maximum, the range is well-defined in arbitrary dimension; moreover, it is a piecewise constant function of time that increases by one.

We investigate the "competition" between the ranges of, as well as the sets of sites visited by, two independent random walks. Specifically, we consider two identical random walks with the same starting position on hyper-cubic lattices $\mathbb{Z}^d$ in dimension d. In each step, each random walk moves to one of its $2d$ neighboring sites, a site that is selected randomly and independently. We study survival probabilities associated with the number of visited sites and the set of visited sites in dimensions $d = 1, 2, 3, 4$. (We expect that the asymptotic behavior for $d = 4$ holds for all $d > 4$.) Our extensive numerical simulations reveal a diverse set of asymptotic behaviors ranging from power laws and stretched exponentials to simple exponentials. We also find asymptotic behaviors varying logarithmically with time.

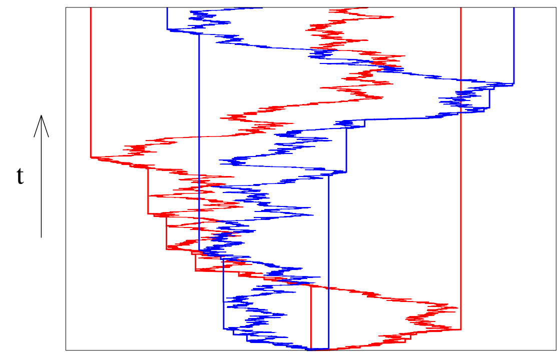

FIG. 1. Spacetime diagrams of two random walkers where the range of the random walker shown in red always exceeds the range of the random walker shown in blue. The maximal and minimal positions of the two walkers are also indicated.

Let $\mathcal{N}_j(t)$ be the number of sites visited by the j^{th} walker: $\mathcal{N}_j(t) = \mathcal{N}_j(t-1) + 1$ if at time t the j^{th} walker hops to a previously unvisited site. Initially $\mathcal{N}_1(0) = \mathcal{N}_2(0) = 1$. The ordering probability associated with the ranges $\mathcal{N}_1$ and $\mathcal{N}_2$ is [see Fig. 1]

$$P(t) = \text{Prob}[\mathcal{N}_1(\tau) \leq \mathcal{N}_2(\tau) \,|\, 0 \leq \tau \leq t] \tag{2}$$

In other words, $P(t)$ is the probability that a random walker never visits more sites than another independent random walker up time t. The random quantities $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent and non-Markovian, and this feature makes determination of the ordering probability $P(t)$ challenging [10, 11].

One can also compare the sets of sites visited by the two walkers, denoted by $S_1(t)$ and $S_2(t)$. For the sets of visited sites, the natural ordering is inclusion [16]. The

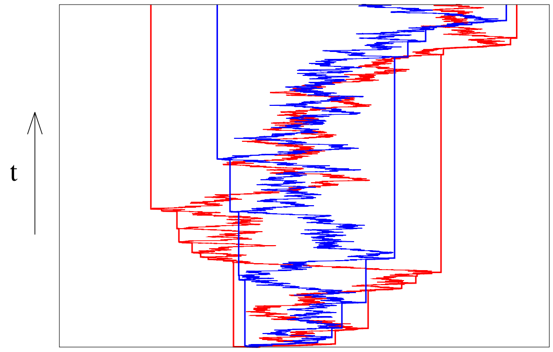

FIG. 2. Spacetime diagrams of two random walkers where each site visited by the random walker shown in blue has been previously visited by the random walker shown in red. The maximal and minimal positions of both walkers are also displayed.

ordering probability associated with the sets $S_1(t)$ and $S_2(t)$ is

$$Q(t) = \text{Prob}[S_1(\tau) \subseteq S_2(\tau) \,|\, 0 \leq \tau \leq t] \tag{3}$$

Hence, $Q(t)$ is the probability that a walker never visits a site that has not been previously visited by another independent walker up to time t. One can visualize this condition as a "matryoshka" arrangement with the set S_1 always remaining a subset of S_2 throughout the time interval $(0, t)$, see Fig. 2.

Since $\mathcal{N}_j = |S_j|$, where $|S|$ denotes the number of elements in set S, the probability $Q(t)$ is bounded from above by $P(t)$:

$$Q(t) \leq P(t) \tag{4}$$

for all $t \geq 0$. Our simulations show that the ordering probabilities $P(t)$ and $Q(t)$ decay algebraically in one dimension

$$P(t) \sim t^{-\beta}, \qquad Q(t) \sim t^{-\gamma} \tag{5}$$

with $\beta = 0.667 \pm 0.002$ and $\gamma = 1.45 \pm 0.03$. The algebraic decays (5) are consistent with the asymptotic behavior of the ordering probability associated with maxima [8, 9].

We also study the number of distinct ties, that is, instances when $\mathcal{N}_1$ equals $\mathcal{N}_2$ and vice versa. Our theoretical results suggest that the average number of ties during the time interval $(0, t)$ grows as

$$A(t) \simeq \begin{cases} a_1 \ln t & d = 1 \\ a_2 (\ln t)^2 & d = 2 \\ a_3\, t^{1/2} (\ln t)^{-1/2} & d = 3 \\ a_d\, t^{1/2} & d \geq 4 \end{cases} \tag{6}$$

In Sec. II, we recall a few basic results about statistics of the range of a random walk. In Sec. III we present the asymptotic behaviors of the ordering probabilities $P(t)$ and $Q(t)$ suggested by numerical simulations, and we also provide heuristic arguments supporting some of these behaviors. In Sec. IV, we study the average number of ties and obtain the growth laws (6) theoretically. Generalizations to multiple independent random walks are outlined in Sec. V. We conclude with a discussion (Sec. VI).

II. RANGE OF A RANDOM WALK

The number of distinct sites visited by a random walk, namely the range, has been the subject of considerable research [17–21]. Statistical properties of the range are well understood in one dimension [22–25], but remain incomplete in higher dimensions [26–37].

We now summarize key statistical properties of the range, which we later use to analyze the growth laws (6). These results apply to a symmetric nearest-neighbor random walk on the hyper-cubic lattice $\mathbb{Z}^d$ in dimension d. The overall hopping rate is set to unity so that the variance in the displacement $\mathbf{r}$ equals time, $\langle \mathbf{r}^2 \rangle - \langle \mathbf{r} \rangle^2 = t$. The leading asymptotic behaviors of the average range $N(t) = \langle \mathcal{N}(t) \rangle$ are

$$N(t) \simeq \begin{cases} \sqrt{\frac{8t}{\pi}} & d = 1 \\ \frac{\pi t}{\ln t} & d = 2 \\ t/W_d & d > 2 \end{cases} \tag{7}$$

where W_d are the so-called Watson integrals [38–41]. For hyper-cubic lattices

$$W_d = \int_0^{2\pi} \cdots \int_0^{2\pi} \left[1 - \frac{1}{d}\sum_{i=1}^{d} \cos q_i\right]^{-1} \prod_{i=1}^{d} \frac{dq_i}{2\pi} \tag{8}$$

when $d \geq 3$. For the cubic lattice, the Watson integral can be expressed [39] via the gamma function,

$$W_3 = \frac{\sqrt{6}}{32\,\pi^3}\, \Gamma\!\left(\frac{1}{24}\right) \Gamma\!\left(\frac{5}{24}\right) \Gamma\!\left(\frac{7}{24}\right) \Gamma\!\left(\frac{11}{24}\right) \tag{9}$$

The asymptotic behavior of the variance, $V = \langle \mathcal{N}^2 \rangle - \langle \mathcal{N} \rangle^2$, is also known

$$V(t) \simeq \begin{cases} (4 \ln 2 - 8/\pi)\, t & d = 1 \\ V_2\, t^2/(\ln t)^4 & d = 2 \\ V_3\, t \ln t & d = 3 \\ V_d\, t & d \geq 4 \end{cases} \tag{10}$$

The amplitudes for square and cubic lattices are

$$V_2 = \pi^2 - \frac{\pi^4}{6} - 2\pi^2 \int_0^1 dx\, \frac{\ln x}{1 - x + x^2} = 16.768\,193\ldots$$
$$V_3 = 4\pi^{-2}(1 - 1/W_3)^4 = 0.005\,450\,284\ldots$$

where W_3 is given by (9), see [27–29, 34] for derivations of the amplitudes V_2 and V_3. However, no compact formulas are available for the amplitude V_d when $d \geq 4$.

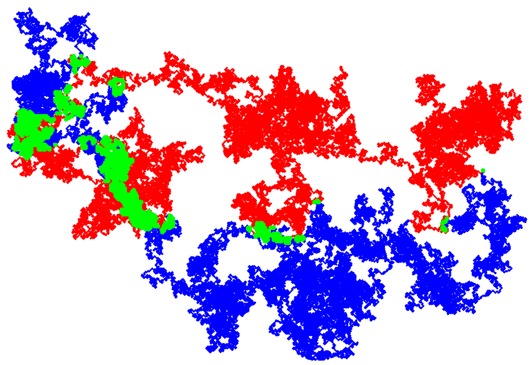

FIG. 3. An illustration of sites visited by two independent random walkers on the square lattice after 10^5 steps. Sites visited only by the first walker are shown in red, sites visited only by the second walker are shown in blue, and sites visited by both walkers are shown in green.

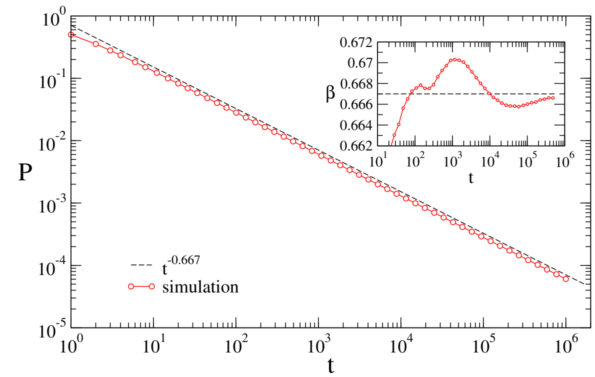

FIG. 4. The ordering probability $P(t)$ versus time t in one dimension. An average over 2^{38} independent runs has been performed. Also shown for reference is a line with slope 0.667. The inset shows the local slope $\beta \equiv -d\ln P/d\ln t$.

Equations (7) and (10) imply that the random quantity $\mathcal{N}(t)$ is non-self-averaging in one dimension and self-averaging when $d \geq 2$. Further, $\mathcal{N}(t)$ is weakly self-averaging in two dimensions since the ratio $\sqrt{V}/N$ vanishes very slowly as $(\ln t)^{-1}$.

The random variable $\mathcal{N}$ is fully characterized by the distribution $P_n(t) = \text{Prob}[\mathcal{N}(t) = n]$. In one dimension, the range distribution converges to [22–25]

$$P_n(t) \simeq \frac{8}{\sqrt{2\pi t}} \sum_{j\geq 1} (-1)^{j-1} j^2 \exp\!\left[-\tfrac{j^2 n^2}{2t}\right] \tag{11}$$

For a random walk on a square lattice, the range distribution is non-Gaussian [31], and a closed explicit expression for the asymptotic range distribution remains elusive. When $d \geq 3$, the range distribution is asymptotically Gaussian [27–29]

$$P_n(t) \simeq \frac{1}{\sqrt{2\pi V(t)}} \exp\!\left\{-\frac{[n-N(t)]^2}{2V(t)}\right\} \tag{12}$$

with $N(t)$ and $V(t)$ given by (7) and (10).

The number of common sites $\mathcal{C} = |S_1 \cap S_2|$ quantifies the overlap between the sites visited by two independent random walkers. The average number of common sites, $C(t) = \langle \mathcal{C}(t) \rangle$, grows according to

$$C(t) \sim \begin{cases} t^{1/2} & d=1 \\ t/(\ln t)^2 & d=2 \\ t^{1/2} & d=3 \\ \ln t & d=4 \\ 1 & d\geq 5 \end{cases} \tag{13}$$

See [42, 43] for the derivation of (13) and generalization to common sites visited by m walkers with arbitrary m. Figure 3 illustrates the number of common sites visited by two walkers on the square lattice.

III. ORDERING PROBABILITIES

Here, we analyze the evolution of the ordering probabilities $P(t)$ and $Q(t)$ using numerical simulations. We implement the random walk process in the standard way:

1. Initially, the random walk is at the origin.

2. At each time step, the walker hops to one of its $2d$ neighboring sites, a site that is chosen at random. Therefore, throughout the evolution, the average displacement remains equal to zero.

3. Time is augmented by one after each step.

With this implementation, the variance of the displacement $\mathbf{r}(t)$ equals time, $\langle \mathbf{r}^2(t) \rangle = t$.

In one dimension, we keep track of three quantities: the current position of the walk, the leftmost position $m(t)$ and the rightmost position $M(t)$; the total number of visited sites is given by $\mathcal{N}(t) = M(t) - m(t) + 1$. Hence, the required computer memory is minimal. In higher dimensions, it is necessary to maintain a physical lattice to indicate which sites were visited by the walker and which remain unvisited. The simulations can be still performed efficiently by keeping track of all sites visited by the walk and resetting *only* the visited sites on the indicator lattice at the end of each run. This approach is especially well suited for measuring survival probabilities.

The Monte Carlo simulation results suggest the following asymptotic behaviors for the ordering probability $P(t)$ associated with the range (2) [see Figs. 4–6]

$$P(t) \sim \begin{cases} t^{-\beta} & d=1 \\ t^{-\beta}\,\ln t & d=2 \\ t^{-1/2}(\ln t)^{-1/2} & d=3 \\ t^{-1/2} & d=4 \end{cases} \tag{14}$$

Simulations suggest a simple rational value $\beta = 2/3$ for

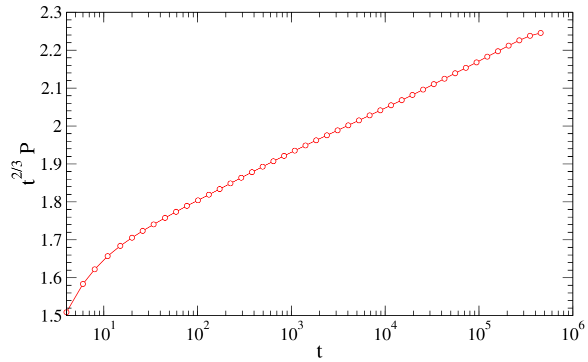

FIG. 5. The quantity $t^{2/3}P(t)$ versus time t on a square lattice. Simulation results represent an average of 2^{34} independent realizations.

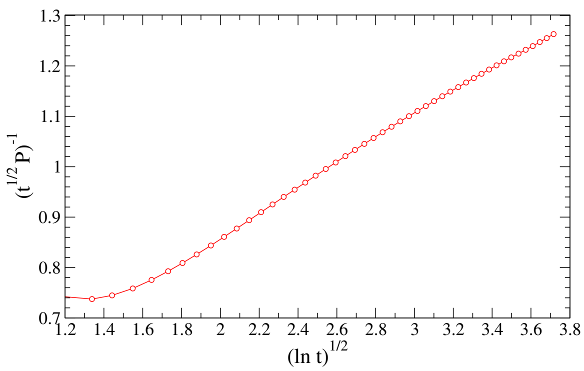

FIG. 6. The quantity $[t^{1/2}P(t)]^{-1}$ versus $\sqrt{\ln t}$ on a cubic lattice. Simulation results represent an average of 2^{30} runs.

the persistence exponent; specifically, we measure

$$\beta = 0.667 \pm 0.002 \tag{15}$$

in one dimension (see Fig. 4). In two dimensions, the effective exponent is only slightly smaller than 2/3, and moreover, the quantity $-d\ln P/d\ln t$ increases slowly with time. These observations indicate a possible logarithmic correction, and indeed, the simulations support the decay $P \sim t^{-2/3}\ln t$, see Fig. 5. A simple $t^{-1/2}$ decay emerges in four dimensions, and we expect this behavior extends to $d > 4$. In three dimensions, the effective exponent is slightly larger than 1/2. Moreover, the quantity $-d\ln P/d\ln t$ decreases slowly with time, again indicating a logarithmic correction. The numerical results support the decay law $P \sim t^{-1/2}(\ln t)^{-1/2}$, see Fig. 6.

When $d \geq 4$, the ranges $\mathcal{N}_1(t)$ and $\mathcal{N}_2(t)$ perform independent directed random walks, so $P \sim t^{-1/2}$. The logarithmic correction to the $t^{-1/2}$ asymptotic in three dimensions is due to the temporal behavior of the variance, see Eq. (10). Comparing Eq. (14) with the variance in the number of visited sites, Eq. (10), we observe that $P \propto V^{-1/2}$ when $d \geq 3$. The first-passage probability for a broad class of one-dimensional Markovian random variables decays as $(\text{variance})^{-1/2}$, see [44–46]. Therefore, the first-passage probability for the random variable $\mathcal{N}_1 - \mathcal{N}_2$, namely $P(t)$, is expected to have this property when $d \geq 3$ as $\mathcal{N}_1$ and $\mathcal{N}_2$ become uncorrelated and Markovian in the asymptotic limit.

In one dimension, the algebraic decay of $P(t)$ is consistent with the behavior found for the maxima [8]. It would be interesting to find a heuristic explanation for the logarithmic enhancement of the ordering probability $P(t)$ in two dimensions. We note that logarithmic terms arise in Eqs. (7), (10), and also characterize the support of the two-dimensional random walk, see [47–56].

The ordering probability associated with the set of visited sites $Q(t)$ is bounded from above by the quantity $P(t)$, as stated in Eq. (4). Indeed, the condition in (3) is significantly more stringent than the condition in (2) . In one dimension

$$Q(t) \sim t^{-\gamma}\,, \quad \gamma = 1.45 \pm 0.03 \tag{16}$$

Both $P(t)$ and $Q(t)$ decay algebraically in one dimension. The inequality $\gamma > \beta$ follows from Eq. (4), yet a more stringent relation, $\gamma > 1 > \beta$, holds. These two inequalities imply that average first-passage time associated with the survival probability $P(t)$ is infinite, while that associated with the quantity $Q(t)$ is finite.

When $d \geq 2$, the ordering probability $Q(t)$ decays faster than any power law. Numerical simulations (Figs. 7–9) support stretched exponential behaviors:

$$\ln[1/Q(t)] \sim \begin{cases} t^{1/2} & d=2 \\ t^{3/4} & d=3 \\ t & d\geq 4 \end{cases} \tag{17}$$

The temporal range probed by the simulations is much larger in dimension $d = 2$ compared with that in dimensions $d = 3$ and $d = 4$. In $d = 2$, the simulation results provide evidence in support of the stretched exponential decay in (17) as the quantity $\nu \equiv d\ln\ln[1/Q(t)]/d\ln t$ saturates at the value $\nu = 0.50 \pm 0.01$; see inset to Fig. 7, the region $t > 80$ was excluded because of poor statistics. The sharp decay of the quantity $Q(t)$ in $d = 3$ and $d = 4$ dimensions makes it difficult to assess the asymptotic behavior and the decays stated in (17) represent our best estimates based on extensive Monte Carlo simulations. For instance, in $d = 3$, the asymptotic behavior quoted in (17) is only slightly better aligned with the simulation results than the decay $\ln[1/Q(t)] \sim -t/\ln t$.

Establishing the asymptotic behaviors stated in (17) theoretically is a formidable challenge. In $d \geq 3$, a random walk hops to an unvisited site with a non-vanishing probability in the long-time limit, suggesting $Q(t)$ decays exponentially [57]. However, we find purely exponential decay only when $d \geq 4$.

To appreciate the plausibility of the exponential decay, consider two random walks with identical trajectories. In this scenario, the sets of sites visited by the two walkers are identical, $S_1 \equiv S_2$. Such a time evolution is realized with probability $(2d)^{-t}$. This argument

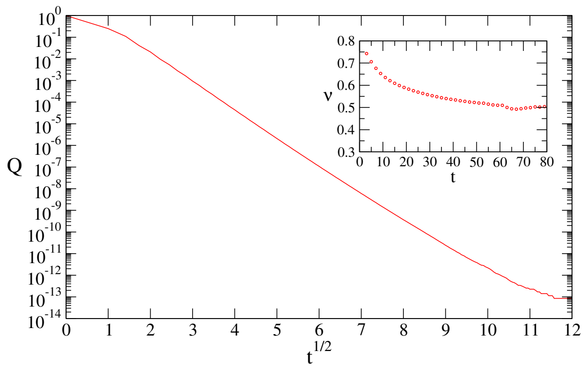

FIG. 7. A semi-log plot of $Q(t)$ versus $t^{1/2}$ in two dimensions. Simulation results represent an average of 2^{45} independent runs. The inset displays the quantity $\nu \equiv d \ln \ln[1/Q]/d \ln t$ versus time t.

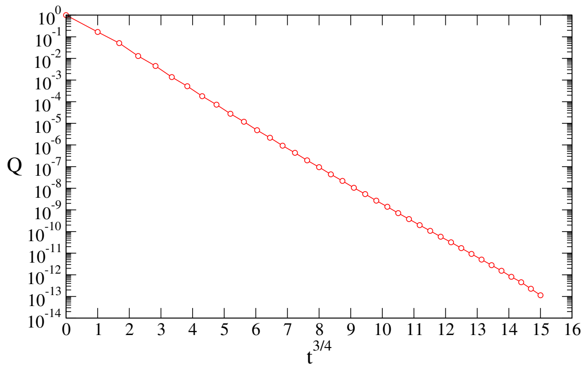

FIG. 8. A semi-log plot of $Q(t)$ versus $t^{3/4}$ in three dimensions. Simulation results represent an average of 2^{46} independent runs.

provides the lower bound $Q(t) \geq (2d)^{-t}$ and the upper bound $b_d \leq \ln(2d)$. In $d = 4$, the numerical simulations give $b_4 = 1.05 \pm 0.05$, while the upper bound is $\ln 8 = 2.079$. We stress that the bound $b_d \leq \ln(2d)$ applies to a discrete-time random walk.

IV. THE NUMBER OF TIES

We now study ties between the ranges $\mathcal{N}_1 = |S_1|$ and $\mathcal{N}_2 = |S_2|$, i.e., instances when the number of sites visited by the two random walkers become equal. The random quantity $\mathcal{N}_1 - \mathcal{N}_2$ is piecewise constant, and changes by unit increments or decrements. Specifically, we are interested in *distinct* ties that occur when $\mathcal{N}_1 - \mathcal{N}_2$ resets to zero. Let $T(t)$ be the number of distinct ties during the time interval $(0, t)$. The initial condition is $T(0) = 1$. We define $\Phi_n(t) = \text{Prob}[T(t) = n]$ to be the probability the number of distinct ties at time t equals n. Our focus is

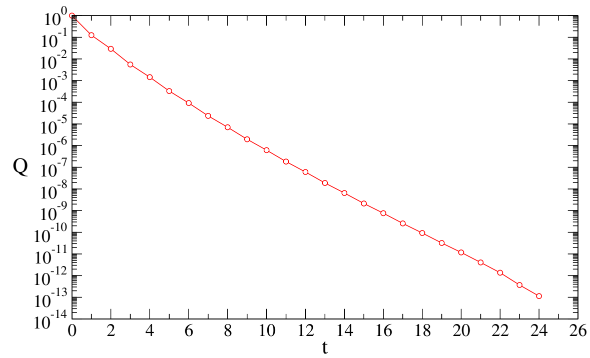

FIG. 9. A semi-log plot of $Q(t)$ versus t in four dimensions. An average over 2^{45} independent runs has been performed.

the average number of ties,

$$A(t) = \sum_{n \geq 1} n \Phi_n(t) \tag{18}$$

To obtain the asymptotic behavior of the average number of ties, we use the general formula [12]

$$\frac{dA}{dt} = 2 \sum_{n \geq 2} P_n \frac{d\mathbb{P}_n}{dt} \tag{19}$$

This equation relates the growth of the average number of ties to the range distribution P_n and its corresponding cumulative distribution

$$\mathbb{P}_n = \sum_{k \geq n} P_k \tag{20}$$

To derive (19) we note that $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent variables. The number of ties can increase only when: (i) these quantities differ by one, say $\mathcal{N}_1 = n-1$ and $\mathcal{N}_2 = n$, and (ii) the smaller quantity $\mathcal{N}_1$ increases, $n-1 \to n$. The factor 2 in (19) accounts for the fact that either random walk may be in the lead. The rate by which the trailing walker makes the jump $n - 1 \to n$, denoted by $W_{n-1,n}$, is the gain term in $\frac{dP_n}{dt}$, viz.

$$\frac{dP_n}{dt} = W_{n-1,n} - W_{n,n+1} \tag{21}$$

Similarly, we have

$$\begin{aligned}
\frac{dP_{n+1}}{dt} &= W_{n,n+1} - W_{n+1,n+2}, \\
\frac{dP_{n+2}}{dt} &= W_{n+1,n+2} - W_{n+2,n+3}, \\
\frac{dP_{n+3}}{dt} &= W_{n+2,n+3} - W_{n+3,n+4},
\end{aligned} \tag{22}$$

etc. By summing (21) and all the successive equations (22) we obtain

$$\frac{d\mathbb{P}_n}{dt} = \sum_{k \geq n} \frac{dP_k}{dt} = W_{n-1,n} \tag{23}$$

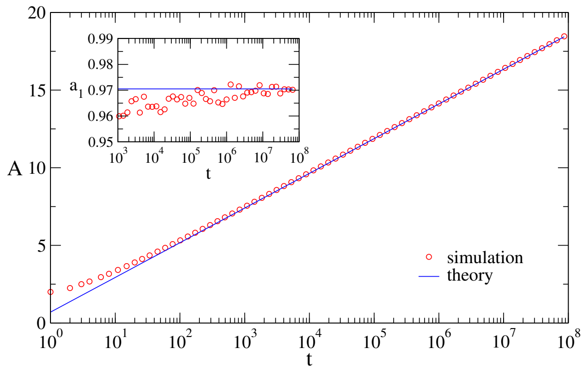

FIG. 10. The average number of ties A versus time t in one dimension. We compare simulation results with theory. Fitting to the form $A \simeq a_1 \ln t$ in the range $10^6 < t < 10^8$ yields the estimate $a_1 = 0.970 \pm 0.001$. The inset compares simulation results for the quantity $a_1(t) \equiv dA/d\ln t$ with theoretical prediction, Eq. (27). Simulation results represent an average of 2^{20} independent runs.

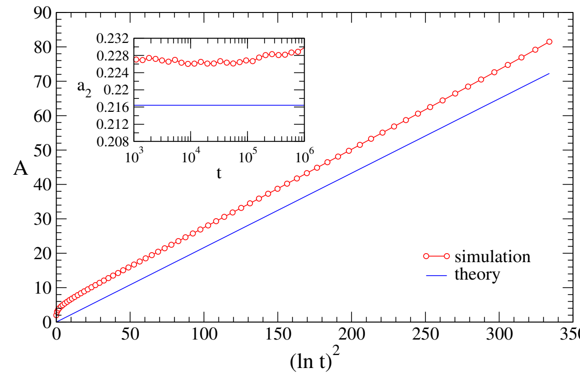

FIG. 11. The average number of ties A versus $(\ln t)^2$ in two dimensions. Simulation results are compared with the theoretical prediction. The inset compares simulation results for the pre-factor $a_2(t) \equiv dA/d(\ln t)^2$ with the approximate value (31). An average over 2^{16} independent runs has been performed.

thereby leading to (19). The rate equation (19), which utilizes continuous time, is suitable for describing the long-time asymptotic behavior.

Using the asymptotic formula (11) and replacing summation with integration, we find

$$\begin{aligned}\mathbb{P}_n &\simeq \frac{8}{\sqrt{2\pi t}}\sum_{j\geq 1}(-1)^{j-1}j^2\int_n^\infty dk\, \exp\Big[-\tfrac{j^2k^2}{2t}\Big]\\ &= 4\sum_{j\geq 1}(-1)^{j-1}j\,\mathrm{Erfc}\left(\frac{j\,n}{\sqrt{2t}}\right)\end{aligned} \tag{24}$$

where $\mathrm{Erfc}(y) = \frac{2}{\sqrt{\pi}}\int_y^\infty dz\, e^{-z^2}$ is the error function. Differentiating (24) yields

$$\frac{d\mathbb{P}_n}{dt} \simeq t^{-1}\,\frac{4}{\sqrt{\pi}}\sum_{j\geq 1}(-1)^{j-1}j^2\nu\, e^{-j^2\nu^2} \tag{25}$$

Substituting (11) and (25) into (19) and replacing summation over n with integration over $\nu = n/\sqrt{2t}$ we arrive at $\frac{dA}{dt} \simeq \frac{a_1}{t}$ [and hence $A(t) \simeq a_1 \ln t$] with

$$a_1 = \frac{64}{\pi}\int_0^\infty d\nu \sum_{i,j\geq 1}(-1)^{i+j}i^2j^2\nu\, e^{-(i^2+j^2)\nu^2} \tag{26}$$

It is possible to simplify the integral over the double sum into a compact sum (Appendix A)

$$a_1 = 16\sum_{j\geq 1}\frac{(-1)^{j-1}\,j^3}{\sinh(\pi j)} = 0.970\,508\,\ldots \tag{27}$$

The simulation results are in excellent agreement with this theoretical prediction: The numerically measured amplitude, $a_1 = 0.970 \pm 0.001$, is within 0.05% of the theoretical value (see also Fig. 10).

When $d \geq 2$, the range is a self-averaging quantity with the asymptotic distribution

$$P_n(t) \simeq \frac{1}{\sqrt{V}}\,\mathcal{P}_d(\sigma), \qquad \sigma = \frac{n-N}{\sqrt{V}} \tag{28}$$

By inserting (28) into (19) we obtain

$$\frac{dA}{dt} \simeq \frac{2}{\sqrt{V}}\,\frac{dN}{dt}\int_{-\infty}^{\infty} d\sigma\,[\mathcal{P}_d(\sigma)]^2 \tag{29}$$

to leading order.

In two dimensions, $\frac{dN}{dt} \simeq \frac{\pi}{\ln t}$ and $V \simeq V_2 t^2/(\ln t)^4$, and therefore, Eq. (29) leads to the asymptotic behavior $A(t) \simeq a_2\,(\ln t)^2$ in (6) with

$$a_2 = \frac{\pi}{\sqrt{V_2}}\int_{-\infty}^{\infty} d\sigma\,[\mathcal{P}_2(\sigma)]^2 \tag{30}$$

The range distribution $\mathcal{P}_2$ is not Gaussian in two dimensions, but it has been probed numerically in Ref. [58] and was found to be close to Gaussian [59]. Substituting $\mathcal{P}_2^{\rm Gauss} = (2\pi)^{-1/2}e^{-\sigma^2/2}$ into (30) yields the uncontrolled approximation (see Fig. 11)

$$a_2^{\rm Gauss} = \sqrt{\frac{\pi}{4V_2}} = 0.216\ldots \tag{31}$$

Numerically we measured $a_2 = 0.227 \pm 0.001$, a value within 5% of (31). Thus, the uncontrolled Gaussian approximation yields a close estimate for the amplitude a_2.

When $d = 3$, the range distribution is Gaussian. Using $\mathcal{P}_3 = (2\pi)^{-1/2}e^{-\sigma^2/2}$, $\frac{dN}{dt} \simeq \frac{1}{W_3}$ and $V = V_3\, t\ln t$ we recast Eq. (29) into

$$\frac{dA}{dt} \simeq \frac{1}{W_3\sqrt{\pi V_3\, t\ln t}} \tag{32}$$

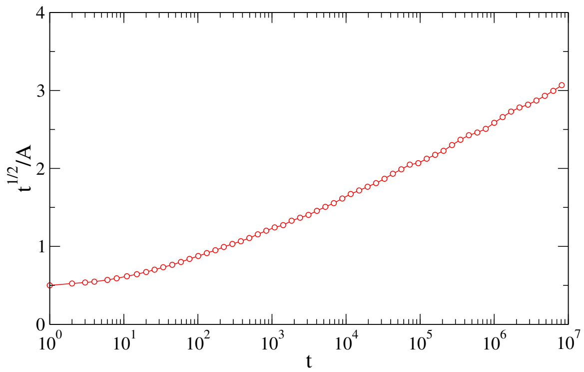

FIG. 12. The quantity $\sqrt{t}/A$ versus time in three dimensions.

Performing the integration yields $A(t) \simeq a_3 \sqrt{t/\ln t}$ with

$$a_3 = \frac{2}{W_3\sqrt{\pi V_3}} = \frac{\sqrt{\pi}\, W_3}{(W_3-1)^2} = 10.079\,423\ldots \tag{33}$$

However, numerically we find that $A \sim \sqrt{t}/\ln t$ provides a significantly better fit to simulation results than the theoretical prediction $A \sim \sqrt{t/\ln t}$, see Fig. 12.

When $d \geq 4$, Eq. (29) becomes

$$\frac{dA}{dt} \simeq \frac{1}{W_d\sqrt{\pi V_d t}} \tag{34}$$

leading to $A(t) \simeq a_d\sqrt{t}$ as in (6) with the amplitude

$$a_d = \frac{2}{W_d\sqrt{\pi V_d}} \tag{35}$$

This completes the derivation of the asymptotic behaviors (6) with the amplitudes (27), (33), and (35). However, the numerical simulation results are at odds with the theoretical prediction for a_4. Numerically, we measured $W_4 = 1.24 \pm 0.01$ in agreement with $W_4 = 1.239467$ [60] that follows from (8). Numerical simulations yield the amplitude $V_4 = 0.26 \pm 0.01$ for the variance. Accordingly, Eq. (35) gives $a_4 = 1.78$ but the numerical simulations yield $a_4 = 1.04 \pm 0.01$.

The diffusive growth $A(t) \sim \sqrt{t}$ for $d \geq 4$, which we verified numerically for $d = 4$, can be deduced using heuristic arguments. In the limit $d \to \infty$, the quantities $\mathcal{N}_1$ and $\mathcal{N}_2$ become Markovian. These two quantities reduce to directed random walks: each directed walk undergoes $+1$ hops with unit rate. Hence, the difference $\mathcal{N}_1 - \mathcal{N}_2$ performs a one-dimensional symmetric random walk as it undergoes ± 1 jumps, both with unit rate. Consequently, the number of ties is equivalent to the number of times a symmetric random walk returns to the origin. As a result, the average number of ties grows diffusively, $A(t) \sim \sqrt{t}$, in the limit $d \to \infty$. While this diffusive growth is formally justified only in infinite dimension, we expect this behavior to hold for all $d \geq 4$.

The logarithmic growth of the number of ties in one dimension resembles the growth law (1) corresponding to ties between maxima. The probability to observe n ties between maxima of two random walks during the time interval $(0,t)$ was found to be Poissonian $\sim t^{-1/4}(\ln t)^n$ [12]. We anticipate a similar functional form holds for ties between the ranges of two random walks,

$$\Phi_n(t) \sim t^{-2/3}(\ln t)^n \tag{36}$$

Numerically, we confirmed (36) for $n = 0, 1, 2, 3$.

V. MULTIPLE RANDOM WALKS

The probability $P_m(t)$ that the ranges of m random walks remain perfectly ordered till time t, defined by

$$P_m(t) = \text{Prob}[\mathcal{N}_1(\tau) \leq \cdots \leq \mathcal{N}_m(\tau)\,|\,0 \leq \tau \leq t] \tag{37}$$

is a straightforward generalization of (2). We compare this quantity with the probability

$$\Pi_m(t) = \text{Prob}[x_1(\tau) \leq \cdots \leq x_m(\tau)\,|\,0 \leq \tau \leq t] \tag{38}$$

that the *positions* of m one-dimensional random walks remain ordered till time t, When $t \to \infty$, these ordering probabilities decay algebraically with time [61–63],

$$\Pi_m(t) \sim t^{-\overline{\beta}_m}\,, \qquad \overline{\beta}_m = \tfrac{1}{4}m(m-1) \tag{39}$$

In high dimensions, the range of a random walk undergoes a one-dimensional directed random walk. Hence, the asymptotic behavior of the ordering probability $P_m(t)$ when $d \geq 4$ is specified in Eq. (39). Based on the asymptotic behaviors of $P_2(t)$ given in Eqs. (14), we conjecture

$$P_m \sim \begin{cases} t^{-\beta_m} & d=1 \\ t^{-\beta_m}(\ln t)^{h_m} & d=2 \\ t^{-m(m-1)/4}(\ln t)^{-g_m} & d=3 \\ t^{-m(m-1)/4} & d\geq 4 \end{cases} \tag{40}$$

The set of algebraic exponents β_m characterizes $P_m(t)$ in one dimension (see Table I), and additionally, the logarithmic exponents h_m and g_m characterize this ordering probability when $d = 2$ and $d = 3$.

We also studied the probability $L_m(t)$ that the range of one walk (the leader) exceeds that of every other walk during the time interval $(0,t)$, that is,

$$L_m(t) = \text{Prob}[\mathcal{N}_1(\tau) \geq \mathcal{N}_j(\tau)\,|\,j = 2,\ldots,m; 0 \leq \tau \leq t]$$

Numerically, we find that the ordering probability exhibits an algebraic decay in one dimension (see Table I)

$$L_m \sim t^{-\alpha_m} \tag{41}$$

This algebraic decay is similar to that of the ordering probability $P_m(t)$, see Eq. (40). In Table I, we also list the set of exponents $\overline{\alpha}_m$ characterizing the decay of the

m	2	3	4	5	6	$m \gg 1$
α_m	0.667	0.947	1.103	1.233	1.315	$(\ln m)/4$
$\overline{\alpha}_m$	1/2	3/4	0.9134	1.03	1.11	$(\ln m)/4$
β_m	0.667	1.91	3.65	6.0	8.3	Bm^2
$\overline{\beta}_m$	1/2	3/2	3	5	15/2	$m(m-1)/4$

TABLE I. The exponents α_m and β_m obtained from numerical simulations of m one-dimensional random walks for $m = 2, \ldots, 6$. The exponents $\overline{\alpha}_m$ and $\overline{\beta}_m$ characterizing similar ordering of the positions of one-dimensional random walks are listed as a reference. The exponents $\overline{\alpha}_2 = \frac{1}{2}$, $\overline{\alpha}_3 = \frac{3}{4}$ are known analytically. The leading large m behaviors is shown in the last column. The asymptotic behavior $\beta_m \simeq Bm^2$ is conjectural, the amplitude B is unknown.

probability that a random walker remains in the lead position [64–70], that is

$$\text{Prob}[x_1(\tau) \geq x_j(\tau)\,|\, j = 2, \ldots, m,\ 0 \leq \tau \leq t] \sim t^{-\overline{\alpha}_m}$$

The exponents presented in Table I indicate $\alpha_m > \overline{\alpha}_m$ and $\beta_m > \overline{\beta}_m$ for all $m \geq 2$. The growth of the exponents α_m and β_m with m resembles that of $\overline{\alpha}_m$ and $\overline{\beta}_m$. The asymptotic growth is $\overline{\alpha}_m \simeq \frac{1}{4}\ln m$ for $m \gg 1$, see [65–67].

Using heuristic arguments, it is possible to show that the leading large-m behaviors of α_m and $\overline{\alpha}_m$ are the same. First, we recall the known derivation for the quantity $\overline{\alpha}_m$. The boundaries of the region visited by random walks other than the leader become more and more deterministic as $m \to \infty$. The region is asymptotically symmetric with respect to the origin, $(-x_*, x_*)$, with x_* estimated from the criterion

$$\int_{x_*}^{\infty} dx\, \frac{m-1}{\sqrt{2\pi t}}\, e^{-x^2/2t} \sim 1 \tag{42}$$

An elementary asymptotic analysis yields

$$x_* \simeq \sqrt{2Ct}, \qquad C = \ln m \tag{43}$$

The leader must stay in the region $x > x_* = \sqrt{C\tau}$ during the time interval $0 < \tau < t$. This problem admits an exact solution [65–67] for arbitrary $C > 0$, namely, the survival probability decays as $t^{-\alpha}$ with $\alpha = \alpha(C)$. We are interested in the $m \gg 1$ behavior, and generally, the deterministic description of the boundaries is asymptotically exact only when $m \gg 1$. Thus $C \gg 1$, and in this situation $\alpha \simeq (C-1)/4$, see [65–67], with our choice of the diffusion coefficient $D = \frac{1}{2}$. From (42) we obtain $C \simeq \ln m$. Thus, we recover $\overline{\alpha}_m \simeq \frac{1}{4}\ln m$ for $m \gg 1$.

The range distribution (11) simplifies to

$$P_n(t) \simeq \frac{8}{\sqrt{2\pi t}}\, \exp\!\left[-\frac{n^2}{2t}\right] \tag{44}$$

in the limit $n \gg \sqrt{t}$. The probability of finding a random walk of range n and other $m-2$ random walks of smaller range is

$$(m-1)P_n(1-\mathbb{P}_n)^{m-2} \simeq (m-1)P_n e^{-(m-2)P_n} \tag{45}$$

with P_n given by (44). The criterion

$$\sum_{n \geq n_*} (m-1)P_n e^{-(m-2)P_n} \sim 1 \tag{46}$$

gives $n_* \simeq \sqrt{2Ct}$ and the same $C \simeq \ln m$ in the leading order. The range is anomalously large, so the leader must stay in the region $x > x_* = n_* = \sqrt{C\tau}$. Thus, we arrive at the same leading behavior $\alpha_m \simeq \frac{1}{4}\ln m$.

We have also investigated the average number of distinct complete ties for three random walks, i.e., instances when $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3$. Using the rate equation approach, we have found that the number of ties saturates at a finite value when $d \leq 2$, while for $d \geq 3$, it grows indefinitely with time. Numerical simulations confirm these qualitative behaviors. For $m \geq 4$ random walks, the number of complete ties remains finite in any dimension.

VI. DISCUSSION

We investigated the competition between sets visited by two identical random walks on hyper-cubic lattices. We also studied the race between the ranges of the walks. Using analytic methods, we studied the asymptotic behaviors (6) for the average number of ties between the ranges of the two walks. We found that the average number of ties grows as $\ln t$ in one dimension and as $(\ln t)^2$ in two dimensions.

We also studied ordering probabilities associated with the number of sites and the set of visited sites. In general, the ordering probabilities decay algebraically in one dimension, and a challenge for future work is an analytic determination of the decay exponents β and γ. The behavior of the ordering probabilities in higher dimensions is much richer. Of special interest is the ordering probability $Q(t)$ associated with the sets of visited sites, viz., the probability that the set of sites visited by one random walker remains a subset of the sites visited by another. Numerically, we find that the ordering probability $Q(t)$ decays as a stretched exponential in $d \geq 2$. Determining the quantity $Q(t)$ analytically is a formidable challenge

The probabilities $P(t)$ and $Q(t)$ may be also studied for Brownian motions [56, 71]. To have well-defined ordering probabilities, we postulate that $S_1(0) \subset S_2(0)$, e.g., $S_2(0) = [-\epsilon, \epsilon]$ and $S_1(0)$ is the origin. The ranges $|S_j(t)|$ are now positive real numbers, and the probabilities $P(t)$ and $Q(t)$ are well defined by (2) and (3). The decay laws (15) and (16) acquire dimensionally consistent form

$$P(t) \sim \left(\frac{\epsilon^2}{Dt}\right)^{\beta}, \qquad Q(t) \sim \left(\frac{\epsilon^2}{Dt}\right)^{\gamma} \tag{47}$$

where D is the diffusion coefficient. For Brownian motion in $d \geq 2$, one can consider a Wiener sausage containing all points within a fixed distance from the Brownian trajectory, i.e., a domain visited by a spherical Brownian particle. For Wiener sausages, the average visited volume and its variance are thoroughly understood [72–79], and qualitatively similar to (7) and (10).

In this study, we addressed the probability that one random walk never visits a site that was not previously visited by another walk. A complementary and related question involves non-intersection probabilities [6, 80, 81] describing realizations when trajectories do not intersect, i.e., any two walks never visit the same site. Conformal field theory, two-dimensional quantum gravity, and Schramm-Löwner evolution have been applied [82–88] to study non-intersection probabilities in two dimensions. Methods used for the analysis of non-intersection probabilities in higher dimensions [6, 80, 81] could perhaps be adapted to the analysis of the ordering probabilities.

In addition to the average number of ties between the ranges of two random walks, one can study further statistical properties of the number of ties. Another natural direction for future work is to investigate ties between the sets of visited sites.

Appendix A: Derivation of Eq. (27)

To simplify the right-hand side of (26) let us reverse the order of summation and integration, i.e., first integrate term by term. The sum in

$$a_1 = \frac{32}{\pi}\sum_{i,j\geq 1}(-1)^{i+j}\,\frac{i^2 j^2}{i^2+j^2} \tag{A1}$$

is formally divergent. Regularization allows us to deduce a finite answer. Rearranging the terms in the sum yields

$$\begin{aligned}\sum_{i,j\geq 1}(-1)^{i+j}\,\frac{i^2 j^2}{i^2+j^2} &= \sum_{i,j\geq 1}(-1)^{i+j} j^2\,\frac{i^2+j^2-j^2}{i^2+j^2}\\ &= \sum_{i,j\geq 1}(-1)^{i+j}\left[j^2-\frac{j^4}{i^2+j^2}\right]\\ &= -\sum_{i,j\geq 1}(-1)^{i+j}\,\frac{j^4}{i^2+j^2}\end{aligned} \tag{A2}$$

In the last step, we used $\sum_{j\geq 1}(-1)^j j^2 = 0$. Indeed, the sum is equal to $7\zeta(-2)=0$, with zeta function $\zeta(s)=\sum_{j\geq 1} j^{-s}$ at $s=-2$ viewed as an analytic continuation of $\zeta(s)$ defined when $\mathrm{Re}(s)>1$. The zeta function vanishes at all even negative integers, $\zeta(-2p)=\sum_{j\geq 1} j^{2p}=0$, as discovered by Euler, see [89, 90].

We now perform the summation in (A2) over $i\geq 1$ using the identity

$$\sum_{i\geq 1}\frac{(-1)^i}{i^2+j^2} = \frac{\frac{\pi j}{\sinh(\pi j)}-1}{2j^2} \tag{A3}$$

Inserting (A2) and (A3) into (A1) we obtain

$$a_1 = \frac{16}{\pi}\sum_{j\geq 1}(-1)^j\left[j^2-\frac{\pi j^3}{\sinh(\pi j)}\right] \tag{A4}$$

Using the identity $\sum_{j\geq 1}(-1)^j j^2=0$ again we simplify (A4) to Eq. (27).

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