

Gröbner Bases for Increasing Sequences

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August 2, 2022

Abstract

Let $q, n \geq 1$ be integers, $[q] = \{1, \dots, q\}$, and \mathbb{F} be a field with $|\mathbb{F}| \geq q$. The set of increasing sequences

$$I(n, q) = \{(f_1, f_2, \dots, f_n) \in [q]^n : f_1 \leq f_2 \leq \dots \leq f_n\}$$

can be mapped via an injective map $i : [q] \rightarrow \mathbb{F}$ into a subset $J(n, q)$ of the affine space \mathbb{F}^n . We describe reduced Gröbner bases, standard monomials and Hilbert function of the ideal of polynomials vanishing on $J(n, q)$.

As applications we give an interpolation basis for $J(n, q)$, and lower bounds for the size of increasing Kakeya sets, increasing Nikodym sets, and for the size of affine hyperplane covers of $J(n, q)$.

⁰The research of LR was supported in part by the Hungarian Ministry of Innovation and Technology NRD Office within the framework of the Artificial Intelligence National Laboratory Program.

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1 Introduction

In the paper $q, n \geq 1$ are integers, and $[q] = \{1, \dots, q\}$. We view $[q]$ as an ordered set with $1 < 2 < \dots < q$ and consider the following set of increasing sequences

$$I(n, q) = \{(f_1, \dots, f_n) \in [q]^n : f_1 \leq f_2 \leq \dots \leq f_n\}.$$

Clearly we have

$$|I(n, q)| = \binom{n+q-1}{q-1},$$

as shown for example by a stars and bars argument. Let \mathbb{F} be a field with $|\mathbb{F}| \geq q$. We can map $[q]$ into a subset of \mathbb{F} with the aid of an injective map $i : [q] \rightarrow \mathbb{F}$. This induces a map of $I(n, q)$ to a subset $J(n, q)$ of the affine space \mathbb{F}^n : a sequence $(v_1, \dots, v_n) \in [q]^n$ is mapped to $(i(v_1), \dots, i(v_n)) \in \mathbb{F}^n$. One may study the algebraic and geometric properties of the set $J(n, q)$. We refer to [16], [18], [19] and references therein for research of this kind on other combinatorially relevant point sets and polynomial ideals.

Most of the properties of $J(n, q)$ considered here are independent of the map i , as long as it is an injection. One exception is grid maps. An injection $i : [q] \rightarrow \mathbb{F}$ is a *grid map* if there exists an $a \in \mathbb{F}$ such that $i(j) = a + j$ for $j \in [q]$. In particular, a grid map of $[q]$ exists iff the characteristic of \mathbb{F} is at least q . Our main technical result is the determination of Gröbner bases, standard monomials and Hilbert functions (for the definitions please see Section 2) of the ideal $\mathbf{I}(J(n, q))$ of polynomials vanishing on $J(n, q)$ (Proposition 3.1, Corollary 3.2 and a useful interpolation basis for $J(n, q)$). We obtained applications of these results in several directions, as it is detailed next.

1.1 Interpolation and covering by hyperplanes

We denote by $\mathbb{F}[x_1, \dots, x_n] = \mathbb{F}[\mathbf{x}]$ the polynomial ring over \mathbb{F} with variables x_1, \dots, x_n . The standard monomials for the ideal $\mathbf{I}(J(n, q))$ form a linear basis of the space of functions from $J(n, q)$ to the ground field \mathbb{F} . Here we exhibit an other natural basis of interpolation which turns out to be useful when we consider coverings by hyperplanes. The polynomials can be given quite explicitly when i is a grid embedding.

Theorem 1.1 *Let $\mathbf{s} \in J(n, q)$. Then there exists a unique polynomial $P_{\mathbf{s}} \in \mathbb{F}[\mathbf{x}]$ such that*

(i) $P_{\mathbf{s}}(\mathbf{s}) = 1$ and $P_{\mathbf{s}}(\mathbf{w}) = 0$ for each $\mathbf{w} \in J(n, q)$, $\mathbf{w} \neq \mathbf{s}$, with $\deg(P_{\mathbf{s}}) = q - 1$, and

(ii) *If i is a grid embedding, then we can write $P_{\mathbf{s}}$ into the form $P_{\mathbf{s}} = \prod_{i=1}^{q-1} L_i$, where the L_i are linear polynomials.*

There are several results on covering discrete subsets of \mathbb{F}^n by hyperplanes (see [2], [4], [7], [6], and [22]), a prominent example being the theorem of Alon and Füredi [2] on the covers of discrete grids with the exception of a point. An analogous, sharp statement for increasing vectors is the following:

Theorem 1.2 *Let $0 < k \leq n$ be an integer and let $\mathbf{s}_1, \dots, \mathbf{s}_k \in J(n, q)$ be increasing vectors. Let $\{H_j : 1 \leq j \leq m\}$ be a set of affine hyperplanes such that*

$$J(n, q) \setminus \{\mathbf{s}_1, \dots, \mathbf{s}_k\} \subseteq \cup_{j=1}^m H_j.$$

Then $m \geq q - 1$.

A similar reasoning gives a sharp (non-constant) bound on the number of hyperplanes covering the whole $J(n, q)$.

Theorem 1.3 *Let $\{H_i : 1 \leq i \leq m\}$ be a set of affine hyperplanes such that*

$$J(n, q) \subseteq \cup_{i=1}^m H_i.$$

Then $m \geq q$.

1.2 Kakeya and Nikodym sets for increasing vectors

Let $\mathbf{a}, \mathbf{v} \in \mathbb{F}_q^n$ be vectors, with $\mathbf{v} \neq \mathbf{0}$. Define the line $\ell(\mathbf{a}, \mathbf{v}) \subseteq \mathbb{F}_q^n$ as

$$\ell(\mathbf{a}, \mathbf{v}) := \{\mathbf{a} + t\mathbf{v} : t \in \mathbb{F}_q\}.$$

A subset $K \subseteq \mathbb{F}_q^n$ is a *Kakeya set*, if for each $\mathbf{0} \neq \mathbf{v} \in \mathbb{F}_q^n$ there exists an $\mathbf{a} \in \mathbb{F}_q^n$ such that $\ell(\mathbf{a}, \mathbf{v}) \subseteq K$.

Wolff proposed the conjecture in [21], that for every $\epsilon > 0$ and for every $n \geq 1$ there exists a constant $c(n, \epsilon)$ such that for any Kakeya set $K \subseteq \mathbb{F}_q^n$ we have

$$|K| \geq c(n, \epsilon)q^{n-\epsilon}.$$

Dvir obtained a stronger bound in his breakthrough work [10]:

Theorem 1.4 *Let $K \subseteq \mathbb{F}_q^n$ denote a Kakeya set. Then*

$$|K| \geq \binom{q+n-1}{n}.$$

In [13] Ganesan obtained lower and upper bounds for the size of local Kakeya sets, where the set of required directions of the lines is local in the sense that it is possibly a proper subset of $\mathbb{F}_q^n \setminus \{0\}$. We consider here some special local Kakeya sets, the so-called increasing Kakeya sets.

A subset $K \subseteq \mathbb{F}_q^n$ is an *increasing Kakeya set*, if for each $\mathbf{0} \neq \mathbf{v} \in J(n, q)$ there exists an $\mathbf{a} \in \mathbb{F}_q^n$ such that the line $\ell(\mathbf{a}, \mathbf{v}) \subseteq K$. Clearly each Kakeya set is an increasing Kakeya set as well. We can prove the same lower bound as in Theorem 1.4 for increasing Kakeya sets. In fact, our results on the standard monomials and the Hilbert function of $J(n, q)$ allow to use the argument of Dvir-Tao-Alon directly for increasing Kakeya sets. This is a strengthening of Dvir's result, as we have a smaller number of conditions (lines) to consider.

Theorem 1.5 *Let $K \subseteq \mathbb{F}_q^n$ be an increasing Kakeya set. Then*

$$|K| \geq \binom{q+n-1}{n}.$$

It was proved on page 3 of [12], that for each prime power q there exists a subset $K \subseteq \mathbb{F}_q^2$ which is a union of q lines with different directions and with

$$|K| = \binom{q+1}{2},$$

providing an optimal increasing Kakeya set in the case $n = 2$. We also have an optimal construction for $n = q = 3$, hence the bound of Theorem 1.5 is sharp if $n = 2$ or $q = 2$ or $n = q = 3$.

In general we have the following simple construction which seems to be good for small values of q :

$$T(n, q) := \cup_{\mathbf{0} \neq \mathbf{v} \in J(n, q)} \ell(\mathbf{0}, \mathbf{v}). \tag{1}$$

Clearly we have

$$|T(n, q)| \leq (q-1) \left(\binom{q+n-1}{n} - (q-1) \right) + 1.$$

It would be interesting to see, if there is a general upper bound on the size of the smallest increasing Kakeya sets, which is better than the best available upper bound for general Kakeya sets.

Nikodym sets are closely related to Kakeya sets. A subset $B \subseteq \mathbb{F}_q^n$ is a *Nikodym set*, if for each $\mathbf{z} \in \mathbb{F}_q^n$ there exists a line $\ell_{\mathbf{z}} \subseteq \mathbb{F}_q^n$ through \mathbf{z} such that $\ell_{\mathbf{z}} \setminus \{\mathbf{z}\} \subseteq B$. A variant of the Dvir-Alon-Tao-argument for Kakeya sets [10], (see also Theorem 2.9 in [15]) gives that the size of a Nikodym set $B \subseteq \mathbb{F}_q^n$ is at least $\binom{q+n-2}{n}$. Here we obtain a version of this result for increasing vectors. We shall consider a local kind of Nikodym sets. $B \subseteq \mathbb{F}_q^n$ is an *increasing Nikodym set* if for each $\mathbf{z} \in J(n, q)$ there exists a line $\ell_{\mathbf{z}}$ through \mathbf{z} such that $\ell_{\mathbf{z}} \setminus \{\mathbf{z}\} \subseteq B$.

Theorem 1.6 *Let $B \subseteq \mathbb{F}_q^n$ be an increasing Nikodym set. Then*

$$|B| \geq \binom{n+q-2}{n}.$$

The inequality of Theorem 1.6 strengthens the above bound for Nikodym sets in that here we have fewer conditions (lines to take care of). We note also that the set $T(n, q)$ from (1) is an increasing Nikodym set.

In Section 2 we collected the preliminaries about Gröbner bases, standard monomials and Hilbert functions. In Section 3 we describe Gröbner bases and standard monomials for the vanishing ideal of increasing sequences $J(n, q)$. We extend these results to some special subsets of $J(n, q)$ and to strictly increasing sequences. Applications are discussed in Section 4.

2 Notation and results from Gröbner theory

A total ordering \prec on the monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ composed from variables x_1, x_2, \dots, x_n is a *term order*, if 1 is the minimal element of \prec , and $\mathbf{u}\mathbf{w} \prec \mathbf{v}\mathbf{w}$ holds for any monomials $\mathbf{u}, \mathbf{v}, \mathbf{w}$ with $\mathbf{u} \prec \mathbf{v}$. Important term orders are the lexicographic order \prec_l and the deglex order \prec_{dl} . We have

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \prec_l x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

iff $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$. For deglex, we have $\mathbf{u} \prec_{dl} \mathbf{v}$ iff either $\deg \mathbf{u} < \deg \mathbf{v}$, or $\deg \mathbf{u} = \deg \mathbf{v}$, and $\mathbf{u} \prec_l \mathbf{v}$.

The *leading monomial* $\text{lm}(f)$ of a nonzero polynomial f from the polynomial ring $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, \dots, x_n]$ is the largest monomial with respect to \prec , which has nonzero coefficient in the standard form of f .

Let I be an ideal of $\mathbb{F}[\mathbf{x}]$. A finite subset $\mathcal{G} \subseteq I$ is a *Gröbner basis* of I if for every $f \in I$ there exists a $g \in \mathcal{G}$ such that $\text{lm}(g)$ divides $\text{lm}(f)$. It is known that such a \mathcal{G} is a basis of I . A fundamental fact is (cf. [8, Chapter 1, Corollary 3.12] or [3, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal I of $\mathbb{F}[\mathbf{x}]$ has a Gröbner basis with respect to any term order \prec .

A monomial $\mathbf{w} \in \mathbb{F}[\mathbf{x}]$ is a *standard monomial* for I if it is not a leading monomial of any $f \in I$. Let $\text{sm}(I, \prec)$ stand for the set of all standard monomials of I with respect to the term-order \prec over \mathbb{F} . It is known (see [8, Chapter 1, Section 4]) that for a nonzero ideal I the set $\text{sm}(I, \prec)$ is a basis of the \mathbb{F} -vector space $\mathbb{F}[\mathbf{x}]/I$. In fact, every $g \in \mathbb{F}[\mathbf{x}]$ can be written uniquely as $g = h + f$ where $f \in I$ and h is a unique \mathbb{F} -linear combination of monomials from $\text{sm}(I, \prec)$. For a subset $X \subseteq \mathbb{F}^n$ we write $\mathbf{I}(X)$ for the ideal of polynomials from $\mathbb{F}[\mathbf{x}]$ which vanish on X . When $X \subseteq \mathbb{F}^n$ is a finite subset, interpolation gives that every $X \rightarrow \mathbb{F}$ function is a polynomial function. The latter two facts imply for $\text{sm}(X, \prec) := \text{sm}(\mathbf{I}(X), \prec)$ that

$$|\text{sm}(X, \prec)| = |X|. \quad (2)$$

A Gröbner basis $\{g_1, \dots, g_m\}$ of I is *reduced* if the coefficient of $\text{lm}(g_i)$ is 1, and no nonzero monomial in g_i is divisible by any $\text{lm}(g_j)$, $j \neq i$. By a theorem of Buchberger ([3, Theorem 1.8.7]) a nonzero ideal has a unique reduced Gröbner basis.

The *initial ideal* $\text{in}(I)$ of I is the ideal in $\mathbb{F}[\mathbf{x}]$ generated by the monomials $\{\text{lm}(f) : f \in I\}$.

The notion of reduction is closely related to Gröbner bases. Let \mathcal{G} be a Gröbner basis of an ideal I of $\mathbb{F}[\mathbf{x}]$ and $f \in \mathbb{F}[\mathbf{x}]$ be a polynomial. We can reduce f by the set \mathcal{G} by subtracting multiples of polynomials $g \in \mathcal{G}$ from f in such a way that the resulting polynomial is composed of \prec -smaller monomials. It is known that this way any $f \in \mathbb{F}[\mathbf{x}]$ can be reduced into a (unique) \mathbb{F} -linear combination of standard monomials. This is related to the fact we have already mentioned: $\text{sm}(I, \prec)$ provides a linear basis of $\mathbb{F}[\mathbf{x}]/I$.

Let I be an ideal of $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \dots, x_n]$. The (affine) *Hilbert function* of the algebra $\mathbb{F}[\mathbf{x}]/I$ is the sequence of natural numbers $h_{\mathbb{F}[\mathbf{x}]/I}(0), h_{\mathbb{F}[\mathbf{x}]/I}(1), \dots$

Here $h_{\mathbb{F}[\mathbf{x}]/I}(m)$ is the dimension over \mathbb{F} of the factor space

$$\mathbb{F}[x_1, \dots, x_n]_{\leq m} / (I \cap \mathbb{F}[x_1, \dots, x_n]_{\leq m})$$

(see [5, Section 9.3]). It is easy to see that $h_{\mathbb{F}[\mathbf{x}]/I}(m)$ is the number of standard monomials of degree at most m , where the ordering \prec is deglex.

For $I = \mathbf{I}(X)$ with some $X \subseteq \mathbb{F}^n$, the number $h_X(m) := h_{\mathbb{F}[\mathbf{x}]/I}(m)$ is the dimension of the linear space of those $X \rightarrow \mathbb{F}$ functions which are polynomials of degree at most m .

3 Gröbner bases for increasing sequences

Via an injective map $i : [q] \rightarrow \mathbb{F}$ we consider $[q]$ as a subset of our ground field \mathbb{F} , in particular we assume that $|\mathbb{F}| \geq q$. We denote by $J(n, q)$ the image of $I(n, q)$ by the map induced by i :

$$J(n, q) = \{(i(v_1), \dots, i(v_n)); (v_1, \dots, v_n) \in I(n, q)\}.$$

The ordering $<$ on $i([q])$ is defined via the map i , as $i(1) < i(2) < \dots < i(q)$. Let \prec be a term order on the monomials of $\mathbb{F}[\mathbf{x}]$

We say that an n -tuple (I_1, \dots, I_n) of subsets of $[q]$ is a good decomposition of $[q]$, if

- (i) $\cup_{j=1}^n I_j = [q]$,
- (ii) if $i < j$, then $x < y$ for each $x \in I_i, y \in I_j$.

In particular $I_k \cap I_l = \emptyset$ if $1 \leq k \neq l \leq n$.

Next we define a polynomial $f_{I_1, \dots, I_n}(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ attached to a good decomposition (I_1, \dots, I_n) of $[q]$. We set

$$f_{I_1, \dots, I_n}(x_1, \dots, x_n) := \prod_{j=1}^n \left(\prod_{t \in i(I_j)} (x_j - t) \right) \in \mathbb{F}[x_1, \dots, x_n].$$

Let

$$\mathcal{G} := \{f_{I_1, \dots, I_n} : (I_1, \dots, I_n) \text{ is a good decomposition of } [q]\}.$$

Proposition 3.1 \mathcal{G} is the reduced Gröbner basis of $J(n, q)$ for any term order \prec . Moreover for the standard monomials we have

$$\text{sm}(J(n, q), \prec) = \{x^u : \deg(x^u) \leq q - 1\}.$$

Proof. Observe first that the leading monomial of f_{I_1, \dots, I_n} is $x_1^{|I_1|} x_2^{|I_2|} \dots x_n^{|I_n|}$ for any term order \prec on the monomials of $\mathbb{F}[\mathbf{x}]$, and it is of degree q . Moreover any monomial of degree q is the leading monomial of a polynomial of the shape f_{I_1, \dots, I_n} for a suitable good decomposition (I_1, \dots, I_n) of $[q]$. It is then sufficient to prove that $f_{I_1, \dots, I_n}(x_1, \dots, x_n)$ vanishes on the vectors of $J(n, q)$. Indeed then it follows that $\text{sm}(J(n, q), \prec) \subseteq \{x^u : \deg(x^u) \leq q - 1\}$. The two sets in the preceding formula have the same size $\binom{n+q-1}{q-1}$, hence they must be equal:

$$\text{sm}(J(n, q), \prec) = \{x^u : \deg(x^u) \leq q - 1\}.$$

This gives the statement about the standard monomials and shows also that the polynomials in \mathcal{G} indeed form a Gröbner basis. These polynomials are monic, and except for the leading monomial they are made of standard monomials. This proves that the Gröbner basis \mathcal{G} is reduced.

It remains to verify that if $\mathbf{v} \in J(n, q)$, then $f_{I_1, \dots, I_n}(\mathbf{v}) = 0$. Suppose that this is not the case, $f_{I_1, \dots, I_n}(\mathbf{v}) \neq 0$. Then straightforward induction on j gives that $v_j \notin i(I_1 \cup \dots \cup I_j)$. This leads in the end to $v_n \notin i([q])$, a contradiction. \square

Please note that the Gröbner basis and the standard monomials are independent of the term order \prec selected. For the the Hilbert function of $J(n, q)$ we have:

Corollary 3.2

$$h_{J(n, q)}(s) = \binom{n + s}{s}$$

for each $0 \leq s \leq q - 1$.

Proof. We apply Proposition 3.1 with \prec being the deglex order. We obtain that the value of $h_{J(n, q)}(s)$ is the number of monomials in $\mathbb{F}[\mathbf{x}]$ of degree at most s . \square

We obtain the following version of the Combinatorial Nullstellensatz [1] for increasing sequences.

Corollary 3.3 *Let $0 \neq f \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial such that $\deg(f) \leq q - 1$. Then there exists a vector $\mathbf{v} \in J(n, q)$ such that $f(\mathbf{v}) \neq 0$.*

Proof. By Proposition 3.1 f is a nontrivial linear combination of standard monomials, hence it can not vanish on the entire set $J(n, q)$. \square .

We now proceed to exhibit Gröbner bases for the ideals $\mathbf{I}(\mathcal{F})$ where $\emptyset \neq \mathcal{F} \subseteq J(n, q)$ is a *downset* in the sense that if we have $\mathbf{u}, \mathbf{v} \in J(n, q)$ with $\mathbf{v} \in \mathcal{F}$ and $\mathbf{u} \leq \mathbf{v}$ (component-wise) then $\mathbf{u} \in \mathcal{F}$. We denote by \mathcal{F}^c the complement of \mathcal{F} in $J(n, q)$. We define first the map

$$\phi : I(n, q) \rightarrow \{\mathbf{v} \in \{0, 1, \dots, q-1\}^n : \sum_{i=1}^n v_i \leq q-1\} :$$

if $\mathbf{f} = (f_1, \dots, f_n) \in I(n, q)$ is an increasing vector, then

$$\phi(\mathbf{f}) := (f_1 - 1, f_2 - f_1, \dots, f_n - f_{n-1}).$$

It is straightforward to see that ϕ is a bijection.

Let $\mathbf{g} = (g_1, \dots, g_n) \in I(n, q)$ be an increasing vector and consider the following system of sets corresponding to \mathbf{g} : $I_1(\mathbf{g}) := \{1, 2, \dots, g_1 - 1\}$ and $I_j(\mathbf{g}) := \{g_{j-1}, \dots, g_j - 1\}$ for each $2 \leq j \leq n$. We define the following polynomial

$$f_{I_1(\mathbf{g}), \dots, I_n(\mathbf{g})}(x_1, \dots, x_n) := \prod_{j=1}^n \left(\prod_{t \in I_j(\mathbf{g})} (x_j - t) \right) \in \mathbb{F}[x_1, \dots, x_n].$$

Here an empty product has value 1. We have $\text{lm}(f_{I_1(\mathbf{g}), \dots, I_n(\mathbf{g})}) = \mathbf{x}^{\phi(\mathbf{g})}$. Set

$$\mathcal{H} := \{(I_1(\mathbf{g}), \dots, I_n(\mathbf{g})) : i(\mathbf{g}) \in \mathcal{F}^c\},$$

and

$$\mathcal{T} := \{(I_1, \dots, I_n) : (I_1, \dots, I_n) \text{ is a good decomposition of } [q]\}.$$

Proposition 3.4 *Let \mathbb{F} be a field, $i([q]) \subseteq \mathbb{F}$ as before, \prec an arbitrary term order on $\mathbb{F}[\mathbf{x}]$. Let $\emptyset \neq \mathcal{F} \subseteq J(n, q)$ be a downset. Then*

$$\mathcal{G} := \{f_{I_1, \dots, I_n} : (I_1, \dots, I_n) \in \mathcal{T} \cup \mathcal{H}\}$$

is a Gröbner basis of the ideal $\mathbf{I}(\mathcal{F})$. Moreover for the standard monomials we have

$$\text{sm}(\mathcal{F}, \prec) = \{\mathbf{x}^{\mathbf{u}} : \mathbf{u} \in \phi(i^{-1}(\mathcal{F}))\}.$$

Please note that the Gröbner basis \mathcal{G} is independent of the term order \prec .

Proof. We observe first that the polynomials in \mathcal{G} all vanish on \mathcal{F} . This was established for f_{I_1, \dots, I_n} when $(I_1, \dots, I_n) \in \mathcal{T}$ in Proposition 3.1. Consider now vectors $\mathbf{v} \in \mathcal{F}$ and $\mathbf{g} \in I(n, q)$ such that $i(\mathbf{g}) \in \mathcal{F}^c$. We have to establish $f_{I_1(\mathbf{g}), \dots, I_n(\mathbf{g})}(\mathbf{v}) = 0$. If $f_{I_1(\mathbf{g}), \dots, I_n(\mathbf{g})}(\mathbf{v}) \neq 0$, then an induction on j gives that $v_j \geq i(g_j)$ holds for $j = 1, \dots, n$. This is immediate for $j = 1$, as v_1 can not be in $i(I_1(\mathbf{g}))$. Also for $j > 1$ the facts $i(g_{j-1}) \leq v_{j-1} \leq v_j$ and $v_j \notin i(I_j(\mathbf{g}))$ give the claim.

On the other hand, by the selection of \mathbf{v} and \mathbf{g} there must be an index j such that $i(g_j) > v_j$, giving a contradiction, which shows that the polynomials from \mathcal{G} indeed vanish on \mathcal{F} ; here we used that \mathcal{F} is a downset.

Next we verify that any monomial $\mathbf{x}^{\mathbf{u}}$ such that $u \notin \phi(i^{-1}(\mathcal{F}))$ is divisible by the leading monomial of a polynomial from \mathcal{G} . Suppose first that $\deg(\mathbf{x}^{\mathbf{u}}) \geq q$. Then there exists a good decomposition $(I_1, \dots, I_n) \in \mathcal{T}$ and a $\mathbf{w} \in \mathbb{N}^n$ such that $\text{lm}(\mathbf{x}^{\mathbf{w}} \cdot f_{(I_1, \dots, I_n)}) = \mathbf{x}^{\mathbf{u}}$. Suppose now that $\deg(\mathbf{x}^{\mathbf{u}}) < q$. As $u \notin \phi(i^{-1}(\mathcal{F}))$, we have $\mathbf{u} \in \phi(i^{-1}(\mathcal{F}^c))$. Let $\mathbf{g} := \phi^{-1}(\mathbf{u})$ and consider $f_{I_1(\mathbf{g}), \dots, I_n(\mathbf{g})}$ whose leading monomial is $\mathbf{x}^{\phi(\mathbf{g})} = \mathbf{x}^{\mathbf{u}}$. We obtained

$$\text{sm}(\mathcal{F}, \prec) \subseteq \{\mathbf{x}^{\mathbf{u}} : \mathbf{u} \in \phi(i^{-1}(\mathcal{F}))\}.$$

We have equality here, as both sides have size $|\mathcal{F}|$. Also, with \mathcal{G} we can reduce any polynomial into a linear combination of standard monomials. This implies that \mathcal{G} is a Gröbner basis. \square

We remark that the statement above implies also that if $\mathcal{F} \subseteq J(n, q)$ is a downset then $\phi(i^{-1}(\mathcal{F}))$ is a downset as well. This fact can be seen more directly, without using Gröbner theory, by the fact that ϕ^{-1} is an order preserving map.

Let $q \geq n \geq 1$ be fixed integers. Next we consider the collection of strictly increasing sequences. We put

$$SI(n, q) = \{f \in [q]^n : f \text{ is strictly increasing}\}.$$

Clearly we have $|SI(n, q)| = \binom{q}{n}$.

Similarly to increasing sequences, we view $[q]$ as an ordered subset of a field \mathbb{F} via an injective map $i : [q] \rightarrow \mathbb{F}$, in particular we assume $|\mathbb{F}| \geq q$. We consider the image $SJ(n, q)$ of $SI(n, q)$ with respect to i :

$$SJ(n, q) := \{(i(g_1), \dots, i(g_n)) : \mathbf{g} \in SI(n, q)\}.$$

The set $SJ(n, q)$ is a subset of the affine space \mathbb{F}^n , and we proceed to determine a Gröbner basis for the ideal of polynomials from $\mathbb{F}[\mathbf{x}]$ which vanish on $SJ(n, q)$. Let \prec denote a term order on the monomials of $\mathbb{F}[\mathbf{x}]$.

Let $1 \leq j_1 < j_2 < \dots < j_{n-1} \leq q$ be integers. The collection (I_1, I_2, \dots, I_n) of subsets of $[q]$ is called a *super decomposition* of $[q]$ if $I_1 = \{1, 2, \dots, j_1 - 1\}$, $I_2 = \{j_1 + 1, \dots, j_2 - 1\}$ and so on, finally $I_n = \{j_{n-1} + 1, \dots, q\}$. If $j_{m+1} = j_m + 1$, then $I_m = \emptyset$. Clearly the sets I_i are mutually disjoint and we have $|\cup_{i=1}^n I_i| = q - n + 1$.

Let

$$\mathcal{M} := \{(I_1, \dots, I_n) : (I_1, \dots, I_n) \text{ is a super decomposition of } [q]\}.$$

For a super decomposition (I_1, \dots, I_n) we consider the polynomial

$$f_{I_1, \dots, I_n}(x_1, \dots, x_n) := \prod_{j=1}^n \left(\prod_{t \in i(I_j)} (x_j - t) \right) \in \mathbb{F}[x_1, \dots, x_n],$$

and set

$$\mathcal{G} := \{f_{I_1, \dots, I_n} : (I_1, \dots, I_n) \in \mathcal{M}\}.$$

We have a statement analogous to Proposition 3.1, with a similar proof.

Proposition 3.5 \mathcal{G} is the reduced Gröbner basis of $SJ(n, q)$. Moreover

$$\text{sm}(SJ(n, q), \prec) = \{\mathbf{x}^{\mathbf{u}} : \text{deg}(\mathbf{x}^{\mathbf{u}}) \leq q - n\}.$$

Proof. We note first that for any monomial $\mathbf{w} \in \mathbb{F}[x_1, \dots, x_n]$ of degree $q - n + 1$ there exists a super decomposition (I_1, \dots, I_n) of $[q]$ such that the leading monomial of f_{I_1, \dots, I_n} is \mathbf{w} . We claim first that it suffices to verify that the polynomials $f \in \mathcal{G}$ all vanish on $SJ(n, q)$. Indeed, then we obtain at once that

$$\text{sm}(SJ(n, q), \prec) \subseteq \{\mathbf{x}^{\mathbf{u}} : \text{deg}(\mathbf{x}^{\mathbf{u}}) \leq q - n\},$$

which implies that the sets on the two sides are equal because they have the same size $\binom{q}{n}$. We conclude from here as in Proposition 3.1, and obtain that \mathcal{G} is a reduced Gröbner basis.

It remains to verify that for every $(I_1, \dots, I_n) \in \mathcal{M}$, and $\mathbf{v} \in SJ(n, q)$ we have $f_{I_1, \dots, I_n}(\mathbf{v}) = 0$. Suppose that \mathbf{v} is a counterexample. A straightforward

induction on ℓ gives that if $f_{I_1, \dots, I_n}(\mathbf{v}) \neq 0$, then $v_\ell \geq i(j_\ell)$ for $\ell = 1, \dots, n-1$, where j_1, \dots, j_{n-1} is the sequence defining the super decomposition I_1, \dots, I_n . This implies that $v_n > v_{n-1} \geq i(j_{n-1})$, and hence $v_n \notin i([q])$ a contradiction. This finishes the proof. \square

Applying the preceding result to a deglex order provides the Hilbert function of $SJ(n, q)$.

Corollary 3.6

$$h_{SJ(n,q)}(s) = \binom{n+s}{s}$$

for each $0 \leq s \leq q - n$. \square

Similarly to Corollary 3.3 we have a non-vanishing statement here as well.

Corollary 3.7 *Let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial such that $\deg(f) \leq q - n$. Then there exists a $\mathbf{v} \in SJ(n, q)$ such that $f(\mathbf{v}) \neq 0$. \square*

4 Applications

4.1 Interpolation and covering

Proof of Theorem 1.1. From Proposition 3.1 we obtain that the function $P_{\mathbf{s}} : J(n, q) \rightarrow \mathbb{F}$ can be obtained as a unique linear combination of standard monomials whose degree is at most $q-1$. The degree of this polynomial $P_{\mathbf{s}}$ can not be smaller than $q-1$, as otherwise the nonzero polynomial $(x_1 - s_1)P_{\mathbf{s}}$ would have degree $\leq q-1$ and vanish on $J(n, q)$, which is impossible by Corollary 3.3. This proves (i).

We explain the proof for (ii) in the case when $\mathbb{F} = \mathbb{Q}$ and i is the identical map of $[q]$. The general case follows similarly, but involves more complicated notation. Let $\mathbf{s} = (s_1, \dots, s_n) \in J(n, q) \subseteq [q]^n$. We shall give the linear factors L_i of $P_{\mathbf{s}}$ in terms of \mathbf{s} . We include the linear polynomials $x_1 - t$ for each $t \in [q]$ such that $t < s_1$ and $x_n - t$ for each $t > s_n$. There are as many as $q - (s_n - s_1 + 1)$ such linear factors. Moreover, for each $i > 1$ such that $s_i - s_{i-1} = k > 0$, we consider the k polynomials

$$(x_i - x_{i-1}), (x_i - x_{i-1} - 1), \dots, (x_i - x_{i-1} - (k - 1)).$$

There are $s_n - s_1$ linear polynomials of this type. The product Q of the preceding $q-1$ linear polynomials does not vanish on \mathbf{s} . Suppose that $\mathbf{t} \in J(n, q)$ is a vector such that $Q(\mathbf{t}) \neq 0$. Then an inductive argument proceeding from $i = 1$ to $i = n$ shows that $\mathbf{t} \geq \mathbf{s}$ (component-wise comparison). A similar argument in the direction from $i = n$ to $i = 1$ gives $\mathbf{t} \leq \mathbf{s}$, hence $\mathbf{t} = \mathbf{s}$. A suitable scalar multiple of Q will be appropriate as $P_{\mathbf{s}}$. Uniqueness follows as in the general case. \square

Example. Let $n = 5$, $q = 5$, and $\mathbf{s} = (1, 2, 2, 4, 4)$. Then we have $Q(\mathbf{x}) = (x_5 - 5)(x_4 - x_3)(x_4 - x_3 - 1)(x_2 - x_1)$.

Remarks. 1. The polynomials Q and $P_{\mathbf{s}}$ in (ii) are explicitly determined by \mathbf{s} in the proof.

2. The uniqueness of $P_{\mathbf{s}}$ in (i) can also be established by a dimension counting argument. The polynomials $P_{\mathbf{s}}$ form a basis the space of $J(n, q) \rightarrow \mathbb{F}$ functions and hence also of the space of polynomials of degree at most $q-1$ in $\mathbb{F}[\mathbf{x}]$. In particular no nonzero polynomial from the latter space can be the identically 0 function on $J(n, q)$. This reasoning gives also an alternative proof of the part of Proposition 3.1 on standard monomials, when \prec is the deglex order, and then of Corollary 3.2.

Lemma 4.1 *Let $0 < k \leq n$ be an integer and let $\mathbf{s}_1, \dots, \mathbf{s}_k \in J(n, q)$ be increasing vectors. Let $P \in \mathbb{F}[\mathbf{x}]$ be a polynomial such that $P(\mathbf{s}_i) \neq 0$ for each i and $P(\mathbf{w}) = 0$ whenever $\mathbf{w} \in J(n, q) \setminus \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$. Then $\deg(P) \geq q - 1$.*

Proof. Suppose for contradiction that there exists a polynomial $P \in \mathbb{F}[\mathbf{x}]$ such that $P(\mathbf{s}_i) \neq 0$ for each i , but $P(\mathbf{w}) = 0$ for each $\mathbf{w} \in J(n, q) \setminus \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$, and $\deg(P) < q - 1$. There exists a hyperplane in \mathbb{F}^n which contains the points \mathbf{s}_i , that is, a linear polynomial $L \in \mathbb{F}[\mathbf{x}]$ such that $L(\mathbf{s}_i) = 0$ for each i . We define the nonzero polynomial $Q := P \cdot L$. Then $Q(\mathbf{w}) = 0$ for each $\mathbf{w} \in J(n, q)$. But $\deg(Q) \leq q - 1$, which contradicts to Corollary 3.3. \square

Proof of Theorem 1.2. Let $L_j \in \mathbb{F}[\mathbf{x}]$ be a linear polynomial whose set of zeros is exactly H_j . We can apply Lemma 4.1 to $P = L_1 \cdots L_m$. \square

Proof of Theorem 1.3. Let L_i be a linear polynomial defining H_i . Then $P = L_1 \cdots L_m$ vanishes on $J(n, q)$ and $\deg P = m$. Here $m < q$ would contradict to Corollary 3.3. \square

4.2 Increasing Kakeya and Nikodym sets

The following statement is a variant of a result of Alon, Dvir and Tao, see for example Theorem 1.5 in [10]. We give the proof for the readers' convenience. Here \prec is an arbitrary term order on the variables x_1, \dots, x_n .

Proposition 4.2 *Let q be a prime power, Let $0 < \ell \leq q - 1$ be an integer. Let $\mathcal{T} \subseteq \mathbb{F}_q^n$ be a subset for which*

$$\{\mathbf{x}^{\mathbf{b}} : \deg(\mathbf{x}^{\mathbf{b}}) \leq \ell\} \subseteq \text{sm}(\mathcal{T}, \prec). \quad (3)$$

Suppose that $K \subseteq \mathbb{F}_q^n$ is a subset such that for each $\mathbf{0} \neq \mathbf{v} \in \mathcal{T}$ there exists a vector $\mathbf{a} \in \mathbb{F}_q^n$ such that $|\ell(\mathbf{a}, \mathbf{v}) \cap K| \geq \ell + 1$. Then

$$|K| \geq \binom{n + \ell}{n}.$$

Proof. Note first that by the assumption $\ell > 0$ we have $|\mathcal{T}| > 1$, hence K is not empty. Suppose for contradiction that

$$|K| < \binom{n + \ell}{n}.$$

Then the monomials from $\{\mathbf{x}^{\mathbf{b}} : \deg(\mathbf{x}^{\mathbf{b}}) \leq \ell\}$ can not be linearly independent as functions on K : there exists a nonzero polynomial $P \in \mathbb{F}_q[\mathbf{x}]$ such that $D := \deg(P) \leq \ell$ and $P(\mathbf{v}) = 0$ for each $\mathbf{v} \in K$. We can write P as a sum of two polynomials:

$$P = P_D + Q,$$

where $P_D \neq 0$ is the homogeneous part of P of degree D and $\deg(Q) < D$. Note that we have $D > 0$, because otherwise P would be a nonzero constant function. This implies that $P_D(\mathbf{0}) = 0$.

Let $\mathbf{0} \neq \mathbf{v} \in \mathcal{T}$. Then there exists a vector $\mathbf{a} \in \mathbb{F}_q^n$ such that we have $|\ell(\mathbf{a}, \mathbf{v}) \cap K| \geq \ell + 1$. Define now a polynomial in the single variable t as follows:

$$P_{\mathbf{v}}(t) := P(\mathbf{a} + t \cdot \mathbf{v}) \in \mathbb{F}_q[t].$$

The coefficient of t^D in $P_{\mathbf{v}}$ is exactly $P_D(\mathbf{v})$ and $\deg(P_{\mathbf{v}}) \leq \ell$.

The condition $|\ell(\mathbf{a}, \mathbf{v}) \cap K| \geq \ell + 1$ implies that there exist different values $t_1, \dots, t_{\ell+1} \in \mathbb{F}_q$ such that $P_{\mathbf{v}}(t_i) = 0$ for each i , implying that $P_{\mathbf{v}}$ is the identically zero polynomial.

From this we obtain that $P_D(\mathbf{v}) = 0$ for each $\mathbf{v} \in \mathcal{T}$. This contradicts to (3), as P_D is a nontrivial linear combination of standard monomials, hence can not vanish on the entire \mathcal{T} . This finishes the proof. \square

Proof of Theorem 1.5. We apply Proposition 4.2. Set $\ell = q - 1 > 0$ and $\mathcal{T} = J(n, q)$. Proposition 3.1 shows that condition (3) is satisfied. Let $K \subseteq \mathbb{F}_q^n$ be an increasing Kakeya set. Proposition 4.2 applies and gives the lower bound

$$|K| \geq \binom{n+q-1}{n}.$$

\square

Example. We describe an increasing Kakeya set $K \subset \mathbb{F}_3^3$ for the set $J(3, 3)$ of nondecreasing vectors. Take the "usual" ordering $0 < 1 < 2$ of \mathbb{F}_3 . The 10 nondecreasing vectors from \mathbb{F}_3^3 appear in two groups:

$$\mathbf{v}^0 = (0, 0, 0), \quad \mathbf{v}^1 = (0, 0, 1), \quad \mathbf{v}^2 = (0, 0, 2), \quad \mathbf{v}^3 = (0, 1, 1),$$

$$\mathbf{v}^4 = (0, 1, 2), \quad \mathbf{v}^5 = (0, 2, 2),$$

these are from the plane $L = \{x_1 = 0\}$, and there are four more points outside L :

$$\mathbf{w}^1 = (1, 1, 1), \quad \mathbf{w}^2 = (1, 1, 2), \quad \mathbf{w}^3 = (1, 2, 2), \quad \mathbf{w}^4 = (2, 2, 2).$$

Among these 10 vectors a maximal projectively inequivalent system is $\mathbf{v}^1, \mathbf{v}^3, \mathbf{v}^4, \mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3$. A Kakeya set for the nondecreasing vectors is then a union of six lines. In our construction we start out with an optimal construction K_0 for the 3 lines in L . Note that this is essentially the case of nondecreasing sequences in \mathbb{F}_3^2 , hence $|K_0| = 6$. We select K_0 as the union of the three lines $\{\lambda \mathbf{v}^1\}, \{\mathbf{v}^1 + \lambda \mathbf{v}^3\}, \{\lambda \mathbf{v}^4\}$. We have then

$$K_0 = \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 2), (0, 2, 0), (0, 2, 1)\}.$$

We add to K_0 three more lines in directions $\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3$ respectively, which intersect the plane L in K_0 . These are

$$\{(0, 0, 1) + \lambda \mathbf{w}^1\},$$

$$\{(0, 0, 0) + \lambda \mathbf{w}^2\},$$

$$\{(0, 2, 0) + \lambda \mathbf{w}^3\}.$$

These three lines all pass through $(1, 1, 2)$; and together they include 3 more points with first coordinate value 2. We conclude that the union K of the six lines has $6 + 1 + 3 = 10$ points, hence K is an optimal Kakeya set for the nondecreasing vectors in \mathbb{F}_3^3 .

Proof of Theorem 1.6. Suppose for contradiction that there exists an increasing Nikodym set B with size

$$|B| < \binom{q+n-2}{n}.$$

Then there exists a nonzero polynomial $P \in \mathbb{F}_q[\mathbf{x}]$ such that $\deg(P) \leq q-2$ and $P(\mathbf{v}) = 0$ for each $\mathbf{v} \in B$.

Let $\mathbf{z} \in J(n, q) \subset \mathbb{F}_q^n$ be an arbitrary element. Then there exists a line $\ell_{\mathbf{z}} = \{\mathbf{z} + t\mathbf{v} : t \in \mathbb{F}_q\}$ with $\mathbf{0} \neq \mathbf{v} \in \mathbb{F}_q^n$ through \mathbf{z} such that $\ell_{\mathbf{z}} \setminus \{\mathbf{z}\} \subseteq B$.

Define the polynomial $Q(t) := P(\mathbf{z} + t\mathbf{v}) \in \mathbb{F}_q[t]$. Then $Q(t) = 0$ for each $t \in (\mathbb{F}_q)^*$, because $\ell_{\mathbf{z}} \setminus \{\mathbf{z}\} \subseteq B$. It follows from $\deg(Q) \leq q-2$ that Q is the identically 0 polynomial, hence $P(\mathbf{z}) = Q(0) = 0$. We obtained that $P(\mathbf{z}) = 0$ for each $\mathbf{z} \in J(n, q)$, and then $\deg(P) \leq q-2$ implies that P is the identically 0 polynomial by Proposition 3.1. This contradiction proves the claim. \square

References

- [1] N. Alon. Combinatorial Nullstellensatz. *Comb., Probability and Computing*, **8(1-2)** (1999), 7-29.
- [2] N. Alon, Z. Füredi. Covering the cube by affine hyperplanes. *European Journal of Combinatorics*, **14(2)** (1993), 79-83.
- [3] W. W. Adams, P. Lounstaunau. *An Introduction to Gröbner Bases*, American Mathematical Society, 1994.
- [4] S. Ball. On intersection sets in Desarguesian affine spaces. *European Journal of Combinatorics*, **21(4)** (2000), 441-446.
- [5] T. Becker, V. Weispfenning. *Gröbner bases - a computational approach to commutative algebra*, Springer-Verlag, Berlin, Heidelberg, 1993.

- [6] A. Blokhuis, A.E. Brouwer, T. Szőnyi. Covering all points except one. *Journal of Algebraic Combinatorics*, **32(1)** (2010), 59-66.
- [7] A.E. Brouwer, A. Schrijver. The blocking number of an affine space. *Journal of Combinatorial Theory, Series A*, **24(2)** (1978), 251-253.
- [8] A. M. Cohen, H. Cuypers, H. Sterk (eds.). *Some Tapas of Computer Algebra*, Springer-Verlag, Berlin, Heidelberg, 1999.
- [9] D. Cox, J. Little and D. O’Shea. *Ideals, Varieties, and Algorithms*, Springer-Verlag, Berlin, Heidelberg, 1992.
- [10] Z. Dvir. On the size of Kakeya sets in finite fields. *Journal of the American Math. Soc.*, **22** (2009), 1093-1097.
- [11] Z. Dvir, S. Kopparty, S. Saraf, M. Sudan. Extensions to the method of multiplicities, with applications to Kakeya sets and mergers. *SIAM J. Comput.*, **42(6)** (2013), 2305-2328.
- [12] X.W.C. Faber. On the finite field Kakeya problem in two dimensions. *Journal of Number Theory*, **124(1)** (2007), 248-257.
- [13] G. Ganesan. Size of local finite field Kakeya sets. *In: Extended Abstracts EuroComb 2021*, (pp. 1-4). Birkhäuser, Cham.
- [14] A. Guo, S. Kopparty, M. Sudan. New affine-invariant codes from lifting. *Proceedings of the 4th conference on Innovations in Theoretical Computer Science* (pp. 529-540) (2013).
- [15] L. Guth. *Polynomial methods in combinatorics*, American Mathematical Soc., 2016.
- [16] J.A. De Loera, C.J. Hillar, P.N. Malkin, M. Omar. Recognizing graph theoretic properties with polynomial ideals. *Electronic Journal of Combinatorics*, **17** R114 (2010).
- [17] B. Lund, S. Saraf, C. Wolf. Finite field Kakeya and Nikodym sets in three dimensions. *SIAM Journal on Disc. Math.*, **32(4)** (2018), 2836-2849.

- [18] T. Mészáros, L. Rónyai. Some combinatorial applications of Gröbner bases. *In: Winkler, F. (eds) Algebraic Informatics. CAI 2011.* Springer LNCS 6742, pp. 65-83. (2011)
- [19] T. Mészáros, L. Rónyai. Standard monomials and extremal vector systems. *Electronic Notes in Discrete Math.*, **61** (2017), 855-861.
- [20] S. Saraf, M. Sudan. An improved lower bound on the size of Kakeya sets over finite fields. *Analysis and PDE*, **1(3)** (2008), 375-379.
- [21] T. Wolff. Recent work connected with the Kakeya problem. *Prospects in Mathematics (Princeton, NJ, 1996)*, **2** (1999), 129-162.
- [22] C. Zanella. Intersection sets in $AG(n, q)$ and a characterization of the hyperbolic quadric in $PG(3, q)$. *Discrete Mathematics*, **255(1-3)** (2002), 381-386.