On Good 2-Query Locally Testable Codes from Sheaves on High Dimensional Expanders

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Abstract

We expose a strong connection between good 2-query locally testable codes (LTCs) and high dimensional expanders. Here, an LTC is called good if it has constant rate and linear distance. Our emphasis in this work is on LTCs testable with only 2 queries. These are are harder to construct than general LTCs, and are of particular interest to theoretical computer science.

The connection we make between 2-query LTCs and high dimensional expanders is done by introducing a new object called a *sheaf* that is put on top of a high dimensional expander. Sheaves are vastly studied in topology. Here, we introduce sheaves on simplicial complexes. Moreover, we define a notion of an *expanding sheaf* that has not been studied before.

We present a framework to get good infinite families of 2-query LTCs from expanding sheaves on high dimensional expanders, utilizing towers of coverings of these high dimensional expanders. Starting with a high dimensional expander and an expanding sheaf, our framework produces an infinite family of codes admitting a 2-query tester. If the initial sheaved high dimensional expander satisfies some conditions, which can be checked in constant time, then these codes form a family of good 2-query LTCs.

We give candidates for sheaved high dimensional expanders which can be fed into our framework, in the form of an iterative process which (conjecturally) produces such candidates given a high dimensional expander and a special auxiliary sheaf. (We could not verify the prerequisites of our framework for these candidates directly because of computational limitations.) We analyze this process experimentally and heuristically, and identify some properties of the fundamental group of the high dimensional expander at hand which are sufficient (but not necessary) to get the desired sheaf, and consequently an infinite family of good 2-query LTCs.

Contents

1 Introduction

Locally Testable Codes. A locally testable code (LTC) is an error correcting code admitting a randomized algorithm — called a *tester* — which, given access to a word, can decide with high probability whether it is close to a codeword or not by querying just a few (i.e. $O(1)$) of its letters. More formally, the tester must accept all codewords, and the probability of rejecting a word outside the code is at least proportional to its Hamming distance from the code. Thus, upon transmitting a codeword along a noisy channel, the receiver can probe just a few letters to assess whether the codeword was significantly corrupted or not, and decide accordingly whether to decode it, or ask for retransmission. The probability of rejecting a word which is very far from the code (i.e. of relative Hamming distance $\geq \eta$) is the called (η-)soundness of the LTC.

LTCs also play a major role in the construction of probabilistically checkable proofs (PCPs), as almost all known PCPs include them as building blocks. The length of such PCPs is related to various properties of the LTC, e.g., its distance, rate and the efficiency of the the testing; see [Gol11] for a survey.

A family of LTCs is called *good* if the codes in that family have constant rate and linear distance.

2-Query LTCs. A subclass of LTCs of particular importance is the 2-query LTCs, i.e., LTCs admitting a tester probing just two letters; the alphabet size may be large. Such LTCs admit an even stronger connection to PCPs, and also to the Unique Games Conjecture (UGC). However, they are also known to be somewhat constrained. Indeed, by [BSGS03], there are no 2-query LTCs with linear distance and constant rate on a binary alphabet, and likewise for *linear* 2-query LTCs on any finite field alphabet. See [KR16, KR12] for further restrictions.

Some Notable Constructions of LTCs. LTCs are generally difficult to construct for the reason that random low-density parity check (LDPC) codes are usually not locally testable. Rather, the parity checks should be designed to admit redundancy. 2-query LTCs are even harder to come by.

Some notable examples of LTCs include Reed–Muller codes, which have linear distance and polylogarithmic message length [FS95], [RS96], and the LTCs of Ben-Sasson–Sudan [BSS08] and Dinur [Din07], which have linear distance and inverse poly-logarithmic rate. It was further studied how the group under which a code is invariant affects its local testability. With respect to that, it was shown that affine invariant codes with the so called "single orbit" property are locally testable [KS08]. Dinur, Evra, Livne, Lubotzky and Mozes [DEL$^+$21] and Panteleev and Kalachev [PK21] have recently and independently of each other constructed infinite families of LTCs with constant (large) query size, linear distance and constant rate.[1] (Panteleev and Kalachev also constructed good LDPC quantum codes in *op. cit.*)

Local-Testability Follows From an Underlying High-Dimensional Expander. The past decade had seen an emerging trend of using *high-dimensional expanders* for constructing locally testable codes and other property testers, e.g., [KL14], [KKL16], [EK17], [DDHR20], [KO21] to name just a few. Loosely speaking, these works share a common theme: one uses a 2-dimensional object, e.g., a 2-dimensional simplicial complex, or a 3-layer partially ordered set, in order to define a code. One then relates expansion properties of the object at hand to the testability of the code, making use of the 3 layers of the object (vertices, edges and triangles in the case of a simplicial complex). See [KO21] for an aximatization of this approach.

Despite the extensive research, so far, the prototypical high dimensional expanders did not give rise to LTCs with linear distance and constant rate. It was particularly expected that the *Ramanujan complexes* of Lubozky, Samuels and Vishne [LSV05a] (see also Li [Li04] and Sarveniazi

[1] An earlier version of this paper was written and circulated independently of these works.

[Sar07]), which are often considered as the prototype of high dimensional expanders, should give rise to LTCs. For comparison, the recent LTCs constructed in [DEL+21] and [PK21] use special square complexes, which do not seem to admit higher-dimensional analogues.

1.1 Main Contributions

This work concerns with the construction of 2-query LTCs basing on high-dimensional expanders, e.g. Ramanujan complexes. Our contributions are the following.

A Framework for Constructing Good 2-Query LTCs from Expanding Sheaves on High Dimensional Expanders. We present a general framework — called the *tower paradigm* — for constructing 2-query LTCs from high dimensional expanders, e.g. Ramanujan complexes, by introducing a new piece of data: a *sheaf* on the expander at hand.

In more detail, our framework takes as input (constant sized) initial data consisting of a "small" high dimensional expander and a sheaf. We also assume that the "small" high dimensional expander admits an infinite family of *coverings*, which is the case for many known high-dimensional expanders. Using the constant-sized initial data and the coverings, we construct an infinite family of codes (with length tending to ∞) admitting a natural 2-query tester. We then show that if the constant sized initial data satisfies a list of conditions, which can be verified by a finite (constant sized) computation, then the entire infinite family of codes is a family of 2-query LTCs with linear distance and constant rate; see §2.5 for more details and Theorem 11.1 for a precise statement.

This result consists of two components of independent interest. The first is a new *local-to-global principle* which allows us to show that a 2-query code arising from an *expanding sheaf* on a high-dimensional expander is locally testable and has linear distance by means of local conditions (Theorem 8.1, Corollary 8.15, Remark 8.16); this is a vast generalization of [KKL16], [EK17], [KM18] which moreover works under milder assumptions. The second is a *rate conservation* method (Theorem 10.3), used to maintain a constant rate among the infinite family of codes we construct.

Examples of 2-Query LTCs With Linear Distance and Conjectural Constant Rate. We construct candidates for the constant-sized initial data — consisting of a (constant sized) expander and an expanding sheaf — required for our framework. To that end, we first construct sheaved high-dimensional expanders fulfilling the conditions guaranteeing testability and linear distance. Then, we present an iterative process which takes such a sheaf and modifies it to create a new sheaf which, conjecturally, also satisfies the conditions guaranteeing a constant rate. This gives rise to explicit infinite families of 2-query LTCs with linear distance and conjectural constant rate. See §2.7 for an explicit example of how such a family of codes might look like, and Theorem 12.11 and Remark 12.12 for precise statements. While verifying that each such family has constant rate could be done in a finite (constant sized) computation involving the initial data, doing so is presently not possible due to computational limitations.

The iterative process works by artificially creating or eliminating *cohomology classes* (in the cohomology of the sheaf); see §12.1. We analysed it using computer simulations, and identified conditions involving the fundamental group of the expander and the sheaf to modified which, once met, guarantee that the process outputs a modified sheaf satisfying all the requirements (Conjecture 12.8). We also justify these conditions with a heuristic theoretical argument. They are not necessary for the success of the process, though.

We remark that our 2-query LTCs do not violate the restrictions proved in [BSGS03] since they are not linear and use a very large alphabet $\Sigma = \mathbb{F}_2^m$. If we treat each letter in the alphabet Σ as an m-letter string in $\mathbb{F}_2$, then they become linear codes over the alphabet $\mathbb{F}_2$. We further note that the soundness of the codes does not depend on the alphabet size. In addition, our LTCs are not *lifted codes*, in contrast to the presently known LTCs.

Some Implications to Quantum Codes. Our framework can also be used to construct infinite families of other low-query LTCs (non-linear, with large alphabet) and (linear) *quantum CSS codes* whose X-side is locally testable and has linear distance. Our rate conservation method applies in these contexts, but securing the required conditions on the initial data is still out of reach.

1.2 Conceptual and Methodological Contributions

Exposing a Connection Between 2-Query LTCs and High Dimensional Expansion. Our results expose a strong relation between 2-query LTCs and high dimensional expanders. Connections between general local testability and high-dimensional expansion was studied previously, but in this work, we show that high dimensional expansion is moreover related to a stronger notion of local testability, namely, to 2-query locally testable codes. This connection was already glimpsed on in the work of the second author and Lubotzky [KL14] under the broader connection between high dimensional expansion (coboundary expansion) and general local testability. Alas, the only 2-query LTC obtained in that work had to have two codewords (a sting of 1s and a string of 0s), and so it was not regarded as a true LTC. In this work we show that by using sheaves, we can (conjecturally) get good 2-query LTCs from high dimensional expanders, namely, the intrinsic barrier that high dimensional expanders can not give 2-LTCs with satisfactory rate is overcome.

Introducing Sheaves on Simplicial Complexes and Expanding Sheaves. Loosely speaking, a sheaf is a layer of linear algebra data that is put on top of a simplicial complex. Sheaves are vastly studied in topology and algebraic geometry. Here, we introduce a discrete variation of the topological definition: sheaves on simplicial complexes. Moreover, we define a notion of an *expanding sheaf* that has not been studied before in topology, nor elsewhere.

Utilizing Coverings of High Dimensional Expanders as a Way to Reduce Obstructions and Getting New Examples. We use *coverings* of high dimensional expanders, and more specifically *towers of coverings*, both in our framework for getting good 2-query LTCs and in finding initial data to feed into the framework.

When establishing the tower paradigm, we use coverings to obtain new examples of expanding sheaves from existing ones, generating infinitely many examples from a single base example. Specifically, given a "big" simplicial complex covering a "small" one and a sheaf on the small complex, we can construct a sheaf on the big complex by *pulling back* the sheaf on the small complex along the covering map. Pullback of sheaves is a well-known construction in topology. At the basis of our framework lies the observation that the pullback of a sheaf inherits many properties, e.g., local expansion conditions, from the original sheaf.

Our second use of coverings is in applying our framework, as they allows us to reduce obstructions. In more detail, the conditions on the initial data for our framework which guarantee constant rate depend on the dimension of the first *cohomology space*. Loosely speaking, the larger it is, the further away we are from satisfying these conditions. We use coverings together with the *pushforward* construction from topology to create sheaves of dimension that is significantly larger than the dimension of the obstructing cohomology space. For certain coverings of high dimensional expanders arising form number theory, this approach results in expanding sheaves of arbitrarily large dimension, but such that the obstruction to rate conservation remains constant, ultimately becoming negligible in dimension to the sheaf. It is those sheaves that we feed into our iterative process, which (conjecturally) eliminates the relatively small obstruction.

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2 Overview of The Main Results

We now survey the main results of the paper. In §2.1, we give some relevant background on high dimensional expanders. We then introduce *sheaves on simplicial complexes* in §2.2, and the important notion of *expanding sheaves* in §2.3. Next, in §2.4, we discuss coverings of high dimensional expanders and highlight their important role in our framework for obtaining good 2-query LTCs from expanding sheaves on high dimensional expanders. The framework itself, called the *tower paradigm*, is then presented in §2.5. In §2.6, we present a method which conjecturally produces the constant-sized initial data (consisting of a sheaved high-dimensional expander) required for our framework; here we use coverings once more. Lastly, in §2.7, we give an explicit family of 2-query LTCs which arises from our framework. This family has linear distance and we conjecture that, for an appropriate choice of parameters, it has constant rate.

The outline of the paper is given in §2.8.

2.1 High Dimensional Expanders

Of the various flavors of high dimensional expansion which have emerged in the past two decades — all of which generalize expansion in graphs — the two most relevant for our purpose are *coboundary expansion* and *cosystolic expansion*.

Some History. Coboundary expansion originated in the works of Linial–Meshulam [LM06] and Meshulam–Wallach [MW09] on the cohomology of random simplicial complexes, and the work of Gromov [Gro10] on the minimum amount of overlapping forced by mapping a simplicial complex to $\mathbb{R}^n$. Cosystolic expansion is a more relaxed version of coboundary expansion developed in [DKW18], [KKL16], [EK17] in order to extend the reach of Gromov's methods.

The first connections between high dimensional expansion and property testing were observed and studied in [KL14].

Cochains, Cocycles and Coboundaries. Let X be a simplicial complex.[2] We write $X(i)$ to denote the set of i-dimensional faces of X, e.g., $X(0)$, $X(1)$ and $X(2)$ stand for the vertices, edges and triangles of X, respectively. The simplicial complex X also has a single empty face, of dimension -1.

Let $i \in \mathbb{N} \cup \{-1, 0\}$. Recall that an *$i$-cochain* on X with coefficients in $\mathbb{F}_2$ is an assignment of an element in $\mathbb{F}_2$ to each i-face of X, i.e., a vector $f \in \mathbb{F}_2^{X(i)}$. We set $C^i = C^i(X, \mathbb{F}_2) = \mathbb{F}_2^{X(i)}$ and write the x-coordinate of $f \in C^i$ as $f(x)$. As usual, the *i-th coboundary map* $d_i : C^i \to C^{i+1}$ is defined by

$$(d_i f)(y) = \sum_{x \text{ is an } i\text{-face of } y} f(x) \tag{2.1}$$

[2] All simplicial complexes are assumed to be finite, unless indicated otherwise.

for all $f \in C^i$, $y \in X(i+1)$. A standard computation shows that $d_{i+1} \circ d_i = 0$. The spaces of *i-coboundaries* and *i-cocycles* are now defined as

$$B^i = B^i(X, \mathbb{F}_2) = \operatorname{im} d_{i-1} \qquad \text{and} \qquad Z^i = Z^i(X, \mathbb{F}_2) = \ker d_i, \tag{2.2}$$

respectively, where $d_{-2} = 0$ by convention. We have $B^i \subseteq Z^i \subseteq C^i$ because $d_i \circ d_{i-1} = 0$, and the quotient space Z^i/B^i is the *i-th cohomology* space $\mathrm{H}^i(X, \mathbb{F}_2)$.[3]

Expansion of Cochains is a Form of Local Testability of The Cocycle Code. Given a simplicial complex X, we may regard the i-cocycles $Z^i = Z^i(X, \mathbb{F}_2)$ as a linear code inside $C^i = C^i(X, \mathbb{F}_2) = \mathbb{F}_2^{X(i)}$. This code is called the *i-cocycle code*, and it admits a natural $(i+2)$-query tester: given $f \in C^i$, choose a face $y \in X(i+1)$ uniformly at random and accept f if $(d_i f)(y) = 0$ (cf. (2.1)). By definition,[4] this tester makes Z^i into an an *ε-testable code* inside C^i if and only if

$$\frac{\|d_i f\|_{\mathrm{Ham}}}{d_{\mathrm{Ham}}(f, Z^i)} \geq \varepsilon \qquad \forall\, f \in C^i - Z^i, \tag{2.3}$$

where $\|\cdot\|_{\mathrm{Ham}}$ and d_{Ham} denote the normalized Hamming norm and distance (in $\mathbb{F}_2^{X(i)}$ or $\mathbb{F}_2^{X(i+1)}$), respectively. As for the distance of Z^i, since B^i typically contains vectors with small support (unless $i = 0$), the best we could for is the existence of $\delta > 0$ such that

$$\|g\|_{\mathrm{Ham}} \geq \delta \qquad \forall\, g \in Z^i - B^i. \tag{2.4}$$

Conditions (2.3) and (2.4) can also be viewed as statements concerning the expansion of i-cochains under d_i. When both of these conditions hold, X is said to be an *(ε, δ)-cosystolic expander* in dimension i.[5] If moreover $Z^i = B^i$ (equivalently $\mathrm{H}^i(X, \mathbb{F}_2) = 0$), then X is said to be an *ε-coboundary expander* in dimension i (the parameter δ plays no role as $Z^i - B^i = \emptyset$). Thus, X is an ε-coboundary expander if and only if the code $B^i \subseteq C^i$ is ε-testable with respect to the natural tester.

Expansion in Dimension 0: The Case of Graphs. Since the 0-cocycle code Z^0 of a simplicial complex X is determined by its underlying graph, we might as well assume that X is a graph. In this case, if $f \in C^0(X, \mathbb{F}_2)$ has support $A \subseteq X(0)$, then the support of $d_0 f$ is precisely the set of edges leaving A. Note further that $B^0 \subseteq \mathbb{F}_2^{X(0)}$ consists of exactly two vectors, namely $(0, \dots, 0)$ and $(1, \dots, 1)$. Consequently, X is an ε-coboundary expander in dimension 0 if and only if X is an ε-expander in usual sense, i.e.,

$$\frac{|E(A, X(0) - A)|}{\min\{|A|, |X(0) - A|\}} \geq \varepsilon \frac{|X(1)|}{|X(0)|} \qquad \forall \emptyset \neq A \subsetneq X(0). \tag{2.5}$$

Similarly, a graph X is an (ε, δ)-cosystolic expander in dimension 0 if and only if each connected component of X is an ε-expander consisting of at least δ-fraction of the vertices in X.

Shifting back our point of view to codes, we also note that X is an (ε, δ)-cosystolic expander if and only if the code $Z^0 \subseteq \mathbb{F}_2^{X(0)}$ is ε-testable with respect to its natural 2-query tester and has relative distance $\geq \delta$. On the other hand, the message length of Z^0 is meager — it is the number of connected components of X, thus bounded from above by $\frac{1}{\delta}$.

[3] What we have defined here is the *reduced* cohomology of X with $\mathbb{F}_2$-coeffients, denoted $\tilde{\mathrm{H}}^i(X, \mathbb{F}_2)$ later in the paper and elsewhere. The ordinary, non-reduced, cohomology is defined in the same manner with the difference that the empty face of X is ignored, i.e., one sets $C^{-1} = 0$ and $d_{-1} = 0$.

[4] Definitions concerning codes are recalled in Section 7.

[5] Warning: The cosystolic expansion considered later in this work and also in other sources involves weights on the faces of X, which we have suppressed here for simplicity. See §5.3 for details.

Expansion in Higher Dimensions. In contrast to the case of 0-cocycle codes, if $i > 0$, then the code $Z^i = Z^i(X, \mathbb{F}_2)$ may have constant rate, but its distance is typically small, becuase B^i usually contains vectors of small support. However, in this case, the code Z^i can be enriched into a *quantum CSS code*. Moreover, if X is an (ε, δ)-cosystolic expander in dimension i, then the X-side of this quantum CSS code is ε-testable and has relative distane $\geq \delta$; see [EK17], or §7.4 for a generalization.

Intrinsic Barrier to Good 'Ordinary' Cocycle Codes. The last two paragraphs demonstrate an intrinsic difficulty in trying to construct good LTCs from high dimensional expanders: either the rate or the distance are small. We will see below that *sheaves* allow us to bypass this natural limitation.

Before we move to present sheaves, we recall an important method for obtaining "global" local testability from "local" local testability in the context of cocycle codes coming from high dimensional expanders. A generalization of this method to sheaves will play a major role in our new framework for constructing good 2-LTCs from high dimensional expanders.

Local Local-Testability Implies Global Local-Testablity. We have seen that the testability of the 0-cocycle code of X, i.e., the cosystolic expansion of X in dimension 0, is determined directly by the expansion of its underlying graph.

For higher dimensions, the most prominent method for proving that a simplicial complex X is a good cosystolic expander in a desired dimension $i \in \{1, \dots, \dim X - 2\}$ is a *local-to-global principle* established in [KKL16] for $i = 1$ and [EK17] in general.

Recall that the *link* of a simplicial complex X at a face $z \in X$ is $X_z = \{x - z \,|\, z \subseteq x \in X\}$. If $z \neq \emptyset$, the link X_z is called a *proper link* of X. The main result of [EK17] states that if each of the proper links X_z is a good coboundary expander in a range of dimensions, and if the underlying graph of X is a sufficiently good expander, then X is an (ε, δ)-coboundary expander in dimension i with ε, δ depending on the implicit expansion constants. In fact, by Oppenheim's Trickling Down Theorem [Opp15, Theorem 1.4], we can replace the expansion condition on the underlying graph of X with requiring that X is connected and all its proper links are sufficiently good coboundary expanders in dimension 0. The main theorem of [EK17] can therefore be summarized as: good coboundary expansion at the links (informally called "local" local-testablity) implies cosystolic expansion (informally called "global" local-testability). Among our main results is a generalization of this principle to *sheaves*.

2.2 Sheaves on Simplicial Complexes

Loosely speaking, a sheaf is a layer of linear-algebra data put on top of a simplicial complex.

We comment about the relation between our sheaves and related notions, e.g., sheaves on topological spaces, Jordan and Livne's local systems [JL97, §2] and Friedman's sheaves on graphs [Fri15] at the end.

Sheaves on Graphs. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-sheaf on a graph X consists of

(1) an $\mathbb{F}$-vector space $\mathcal{F}(x)$ for every $x \in X(0) \cup X(1)$, and

(2) a linear map $\mathrm{res}^{\mathcal{F}}_{e \leftarrow u} : \mathcal{F}(u) \to \mathcal{F}(e)$ for every edge $e \in X(1)$ and vertex $u \in X(0)$ with $u \subseteq e$.

The maps $\mathrm{res}^{\mathcal{F}}_{e \leftarrow u}$ are called *restriction maps*.

Examples. The most basic example of an $\mathbb{F}$-sheaf on a graph X is obtained by taking $\mathcal{F}(x) = \mathbb{F}$ for all $x \in X(0) \cup X(1)$ and setting all the restriction maps to be $\mathrm{id}_{\mathbb{F}}$.

A more interesting example that will be revisited later can be constructed as follows: Let X be a k-regular graph. Given $v \in X(0)$, we write $E(v)$ to denote the set of edges containing v. For every $v \in X(0)$, choose $n(v) \in \{0, 1, \dots, k\}$ and an injective linear transformation $T_v : \mathbb{F}^{n(v)} \to \mathbb{F}^{E(v)} \cong \mathbb{F}^k$, and write $C_v = \operatorname{im} T_v$. Using this data, we define an $\mathbb{F}$-sheaf on X by setting

- $\mathcal{F}(v) = \mathbb{F}^{n(v)}$ for each vertex $v \in X(0)$,
- $\mathcal{F}(e) = \mathbb{F}$ for each edge $e \in X(1)$, and
- $\mathrm{res}^{\mathcal{F}}_{e \leftarrow v} = \mathrm{Proj}_e \circ T_v$ for every edge $e \in X(1)$ and vertex $v \subseteq e$, where $\mathrm{Proj}_e : \mathbb{F}^{E(v)} \to \mathbb{F}$ is the projection onto the e-coordinate.

We shall see below (§2.3) that if all the C_v are good codes, then this example gives rise to a good 0-*cocycle code*. In fact, this is a sheaf-theoretic variation on the famous expander codes of Sipser and Spielman [SS96]; their presentation in [Mes18] demonstrates the similarity.

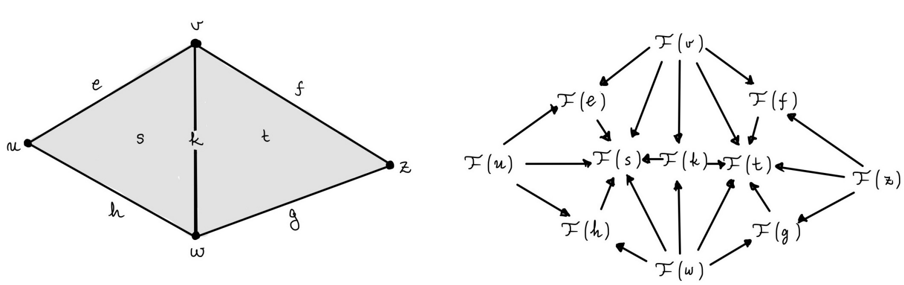
Sheaves on Simplicial Complexes. Sheaves on simplicial complexes are defined in the same manner as sheaves on graphs, with the difference that one needs to impose an extra assumption. Formally, an $\mathbb{F}$-sheaf $\mathcal{F}$ on X consists of

(1) an $\mathbb{F}$-vector space $\mathcal{F}(x)$ for every nonemtpy face $x \in X$, and

(2) a linear map $\mathrm{res}^{\mathcal{F}}_{y \leftarrow x} : \mathcal{F}(x) \to \mathcal{F}(y)$ for every pair of nonempty faces $x, y \in X$ with $x \subsetneq y$,

subject to the requirement $\mathrm{res}^{\mathcal{F}}_{z \leftarrow y} \circ \mathrm{res}^{\mathcal{F}}_{y \leftarrow x} = \mathrm{res}^{\mathcal{F}}_{z \leftarrow x}$ whenever $x \subsetneq y \subsetneq z$. This requirement is vacuous if X is a graph, but is very restrictive if $\dim X > 1$. Figure 1 illustrates of the data of a sheaf on a 2-dimensional simplicial complex, the arrows representing the restriction maps; the extra requirement means that the diagram of vector spaces on the right commutes.

Figure 1: A simplicial complex X (left) and the data of a sheaf $\mathcal{F}$ on X (right).



For the sake of simplicity, we henceforth consider only $\mathbb{F}_2$-sheaves, and call them sheaves for brevity.

As with graphs, one can fix an $\mathbb{F}_2$-vector space V and define a sheaf $\mathcal{F}_V$ on X (denoted V_X later on) by taking $\mathcal{F}_V(x) = V$ for every face $x \in X - \{\emptyset\}$ and setting all the restriction maps to be the identity. Such sheaves are called *constant*. More sophisticated examples will be considered (and needed) below.

Augmented Sheaves. It is sometimes convenient to modify the definition of a sheaf on X by requiring that $\mathcal{F}(x)$ and $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ are also defined when x is the empty face; we call this extended structure an *augmented sheaf*. Two examples of this kind will be important for our discussion.

The first is $\mathcal{F}^{+}_{\mathbb{F}_2}$, obtained by taking $\mathcal{F}(x) = \mathbb{F}_2$ for all $x \in X$ (including the empty face) and setting all the restriction maps to be $\mathrm{id}_{\mathbb{F}_2}$. Replacing $\mathbb{F}_2$ with a general $\mathbb{F}_2$-vector space V gives the *constant augmented sheaf* $\mathcal{F}^{+}_{V}$ (also denoted V_+ later on).

The second example is obtained by restricting a sheaf $\mathcal{F}$ on X to a proper link. Formally, given a nonempty $z \in X$, let $\mathcal{F}_z$ denote the *augmented* sheaf on X_z defined by $\mathcal{F}_z(x) = \mathcal{F}(x \cup z)$ and $\mathrm{res}^{\mathcal{F}_z}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}}_{y\cup z\leftarrow x\cup z}$ for all $x, y \in X_z$ with $x \subsetneq y$.

Cochains, Cocycles and Coboundaries Similarly to §2.1, given a sheaf (resp. augmented sheaf) $\mathcal{F}$ on a simplicial complex X and $i \in \mathbb{N} \cup \{0\}$ (resp. $i \in \mathbb{N} \cup \{0, -1\}$), we can construct vector spaces of i-cochains, i-cocycles and i-coboundaries with coefficients in $\mathcal{F}$. The only difference is that we evoke the restriction maps of $\mathcal{F}$ when defining d_i. Specifically, put $C^i = C^i(X, \mathcal{F}) = \prod_{x\in X(i)} \mathcal{F}(x)$ and define $d_i : C^i \to C^{i+1}$ by

$$(d_i f)(y) = \sum_{x \text{ is an } i\text{-face of } y} \mathrm{res}^{\mathcal{F}}_{y\leftarrow x}(f(x)) \tag{2.6}$$

for every $f \in C^i$ and $y \in X(i + 1)$; cf. (2.1).[6] The vector spaces of i-coboundaries and i-cochains are defined as in (2.2) and denoted $B^i(X, \mathcal{F})$ and $Z^i(X, \mathcal{F})$, respectively, and the i-th cohomology group of $\mathcal{F}$ is $\mathrm{H}^i(X, \mathcal{F}) = Z^i(X, \mathcal{F})/B^i(X, \mathcal{F})$.

If we take $\mathcal{F}$ to be the constant augmented sheaf $\mathcal{F}^{+}_{\mathbb{F}_2}$, then this recovers $C^i(X, \mathbb{F}_2)$, $Z^i(X, \mathbb{F}_2)$ and $B^i(X, \mathbb{F}_2)$ considered in §2.1.

Related Notions to Sheaves. The sheaves we have defined here are inspired by *sheaves on topological spaces*, which are ubiquitous to topology and algebraic geometry, see [MLM94, Chapter II] or [Ive86], for instance. In fact, our sheaves can be reinterpreted as sheaves on certain topological spaces (associated to the simplicial complex at hand) in such a way that the cohomology spaces remain the same; see Appendix A. In contrast, augmented sheaves and their cohomology do not fit nicely into this setting.

The *local systems on graphs* defined by Jordan and Livne [JL97, §2] can be viewed as $\mathbb{R}$-sheaves in which all the restriction maps are isomorphisms. The cohomology theory developed in *op. cit.* then agrees with ours.

Friedman's sheaves on graphs [Fri15] are defined like our sheaves, but with the restriction maps going from the edges to the vertices. From the perspective of our work, they should perhaps be called *co-sheaves*, or sheaves valued in the opposite category of $\mathbb{F}$-vector spaces. They admit a *homology* theory, rather than a cohomology theory.

2.3 Expanding Sheaves

Having introduced sheaves on simplicial complexes, which are discrete versions of the sheaves commonly studied in topology, we turn to present the new concept of an *expanding sheaf*, which has not been previously studied in a topological context. Expanding sheaves will play a pivotal role in our framework for constructing good 2-query LTCs.

Henceforth, X is a simplicial complex and $\mathcal{F}$ is a sheaf or an augmented sheaf on X.

[6] In order to generalize this to $\mathbb{F}$-sheaves with $\mathbb{F}$ a field of characteristic not 2, one should be introduce signs to (2.6); see §4.2.

Expansion of Sheaves. Given $f \in C^i = C^i(X, \mathcal{F})$, let $\operatorname{supp} f = \{x \in X(i) : f(x) \neq 0\}$. The normalized Hamming norm of f is $\|f\|_{\mathrm{Ham}} = \frac{|\operatorname{supp} f|}{|X(i)|}$, and normalized Hamming distance between $f, g \in C^i$ is $d_{\mathrm{Ham}}(f, g) := \|f - g\|_{\mathrm{Ham}}$.

With this notation at hand, the notions of cosystolic expansion and coboundary expansion recalled in §2.1 extend verbatim to sheaves. That is, the sheaf $\mathcal{F}$, or a sheaved complex $(X, \mathcal{F})$, is said be an (ε, δ)-cosystolic expander in dimension i if (2.3) and (2.4) hold for cocycles with coefficients in $\mathcal{F}$, and an ε-cosystolic expander in dimension i if $\mathrm{H}^i(X, \mathcal{F}) = 0$ (equiv. $B^i = Z^i$) and (2.3) holds.

The situation considered in §2.1 now arises as the special case where $\mathcal{F}$ is the constant augmented sheaf $\mathcal{F}^+_{\mathbb{F}_2}$. Note, however, that we have shifted the focus from the expansion of the simplicial complex X to the expansion of the sheaf $\mathcal{F}$. Indeed, $\mathcal{F}$ may have poor expansion even when X is an excellent high-dimensional expander (e.g., take all the restriction maps to be 0).

Considering the expansion of (augmented) sheaves also illuminates an important condition which was transparent in the case of $\mathcal{F}^+_{\mathbb{F}_2}$: coboundary expansion in dimension -1. Suppose that $\mathcal{F}$ is an augmented sheaf such that there is $m \in \mathbb{N}$ with $\mathcal{F}(v) = \mathbb{F}_2^m$ for all $v \in X(0)$, and put $\Sigma = \mathbb{F}_2^m$. Then $d_{-1} : C^{-1} \to C^0$ is an $\mathbb{F}_2$-linear map from $\mathcal{F}(\emptyset)$ to $\Sigma^{X(0)}$. By definition, the augmented sheaf $\mathcal{F}$ is an ε-coboundary expander in dimension -1 if and only if d_{-1} is injective, and its image in $\Sigma^{X(0)}$ is a code with relative distance $\geq \varepsilon$.

0-Cocycle Codes and 2-Query LTCs. Suppose now that $\mathcal{F}$ is a sheaf with $\mathcal{F}(v) = \mathbb{F}_2^m$ for every vertex $v \in X(0)$, where $m \in \mathbb{N}$ fixed, and put $\Sigma = \mathbb{F}_2^m$. Then $C^0 = C^0(X, \mathcal{F}) = \Sigma^{X(0)}$, and we may regard $Z^0 = Z^0(X, \mathcal{F})$ as a code inside $\Sigma^{X(0)}$; we call Z^0 a *0-cocycle code*. There is a natural 2-query tester for Z^0: given $f \in C^0$, choose an edge $e \in X(1)$ uniformly at random and accept f if

$$\operatorname{res}^{\mathcal{F}}_{e \leftarrow u}(f(u)) = \operatorname{res}^{\mathcal{F}}_{e \leftarrow v}(f(v)),$$

where u and v are the vertices of e, cf. (2.6). As in §2.1, $\mathcal{F}$ is an (ε, δ)-coboundary expander in dimension 0 if and only if the code $Z^0 \subseteq \Sigma^{X(0)}$ is ε-testable with respect to this tester, and has relative distance $\geq \delta$ (note that $B^0 = 0$). The rate of Z^0 is $|X(0)|^{-1} \dim_{\mathbb{F}_2} \mathrm{H}^0(X, \mathcal{F})$. Note that we may also view Z^0 as code inside $(\mathbb{F}_2^m)^{X(0)} = \mathbb{F}_2^{X(0) \times \{1, \dots, m\}}$ — the alphabet being $\mathbb{F}_2$ — in which case, we get a $2m$-query ε-testable *linear* code with relative distance $\geq \frac{\delta}{m}$; the rate remains the same.

The 2-query LTCs that we will construct arise as 0-cocycle codes of sheaves with good cosystolic exansion in dimension 0.

Higher-Cocycle Codes and Quantum CSS Codes. We can similarly consider the space of i-coycles $Z^i = Z^i(X, \mathcal{F})$ as a code inside $C^i = C^i(X, \mathcal{F})$ when $i > 0$. If $\mathcal{F}(x) = \Sigma := \mathbb{F}_2^m$ for every $x \in X(i)$, then the alphabet can be taken to be Σ, and this code has an $(i + 2)$-query tester. Such codes have different potential applications depending on whether $B^i(X, \mathcal{F}) = 0$ or $B^i(X, \mathcal{F}) \neq 0$. (The situation $B^i = 0$ with $i > 0$ is impossible if we only consider $\mathbb{F}_2$-valued cocycles as in §2.1, but is possible for general sheaves, e.g., if $\mathcal{F}(x) = 0$ for all $x \in X(i - 1)$.)

If $B^i = 0$, then, as in the case $i = 0$, the code Z^i is ε-testable with relative distance $\geq \delta$ if and only if $\mathcal{F}$ is an (ε, δ)-cosystolic expander in dimension i; its rate is $|X(i)|^{-1} \dim_{\mathbb{F}_2} \mathrm{H}^i(X, \mathcal{F})$. See §9.3 for an example of an infinite family of good 1-cocycle codes. (We do not know if this is a family of LTCs.)

If, on the other hand, $B^i \neq 0$, then by viewing $Z^i(X, \mathcal{F})$ as a *linear* codes inside $C^i(X, \mathcal{F})$ with alphabet $\mathbb{F}_2$, we can enrich it into a *quantum CSS codes* over the alphabet $\mathbb{F}_2$; see §7.4 for details. The rate of this quantum CSS code is again $|X(i)|^{-1} \dim_{\mathbb{F}_2} \mathrm{H}^i(X, \mathcal{F})$, and if $\mathcal{F}$ is an (ε, δ)-coboundary expander in dimension i, then its X-side has relative distance $\geq \delta$ and is ε-testable (up to scaling of the constants).

Local Local-Testablity Implies Global Local-Testibility: a Sheafy Version. Let $(X, \mathcal{F})$ be a sheaved simplicial complex. Our first main result (Theorem 8.1; see also Corollary 8.15, Remark 8.16) is a generalization of the local-to-global principle of Evra–Kaufman [EK17] recalled in §2.1. In more detail, given $i \in \{0, \ldots, \dim X - 2\}$, we show that if for every $z \in X - \{\emptyset\}$, the augmented sheaf $\mathcal{F}_z$ is a good coboundary expander in a range of dimensions, and the underlying graph of X_z is a sufficiently good expander ("local" properties of X and $\mathcal{F}$), then $\mathcal{F}$ is an (ε, δ)-cosystolic expander in dimension i (a "global" property of $\mathcal{F}$) with ε and δ depending on the suppressed expansion constants. This also extends the main result of [KM18] which, in our terminology, addresses the special case of *constant sheaves*.

Our proof is more efficient than [EK17] and [KM18] in the sense that it makes milder assumptions on the expansion of X_z and $\mathcal{F}_z$, and at the same time produces larger expansion constants ε and δ. We also show that the LTCs arising as i-cocycles codes of sheaves to which our theorem applies admit a linear-time decoding algorithm able to correct a linear number of errors (up to a vector in $B^i(X, \mathcal{F})$ if $i > 0$), see Proposition 7.8.

A Side-Application: Sheafy Expander Codes. We return to discuss the sheaf-variation of expander codes defined on page 9, focusing on its 0-cocycle code.

Recall that X is a k-regular graph, and we used codes $C_v \subseteq \mathbb{F}_2^{E(v)}$ to define a sheaf $\mathcal{F}$ on X. Suppose that all the C_v have a common dimension m and relative distance $\geq \varepsilon$. Then, writing $\Sigma = \mathbb{F}_2^m$, we have $C^0(X, \mathcal{F}) = \Sigma^{X(0)}$. Moreover, our assumption on the distance of the C_v says that, for every $v \in X(0)$, the augmented sheaf $\mathcal{F}_v$ (on the link X_v) is an ε-coboundary expander in dimension -1. In other words, "locally", $\mathcal{F}$ has good coboundary expansion in dimension -1.

Since X is merely 1-dimensional (rather than 2-dimensional), this is not enough to apply our Theorem 8.1 to assert that $\mathcal{F}$ is a good cosystolic expander in dimension 0, or equivalently, that $Z^0(X, \mathcal{F}) \subseteq \Sigma^{X(0)}$ is an LTC with linear distance. Indeed, the code $Z^0(X, \mathcal{F})$ is usually *not* testable if $m > \frac{k}{2}$, because removing one of its defining constraints (i.e., removing an edge from X) will typically enlarge $Z^0(X, \mathcal{F})$. Also, even if X were the 1-dimensional skeleton of a triangle complex Y, it is usually not possible to extend $\mathcal{F}$ in a non-redundant way to Y. Indeed, if $\mathcal{F}$ could be extended to a sheaf on Y, then for any triangle $t = \{u, v, w\} \in Y(2)$, we would have $\mathrm{res}^{\mathcal{F}}_{t \leftarrow \{u,v\}} \circ \mathrm{res}^{\mathcal{F}}_{\{u,v\} \leftarrow \{u\}} = \mathrm{res}^{\mathcal{F}}_{t \leftarrow \{u\}} = \mathrm{res}^{\mathcal{F}}_{t \leftarrow \{u,w\}} \circ \mathrm{res}^{\mathcal{F}}_{\{u,w\} \leftarrow \{u\}}$. A simple linear-algebra argument now shows that if C_u contains a word $f \in \mathbb{F}_2^{E(u)}$ with $f_{\{u,v\}} \neq f_{\{u,w\}}$, then $\mathrm{res}^{\mathcal{F}}_{t \leftarrow \{u,v\}}$ and $\mathrm{res}^{\mathcal{F}}_{t \leftarrow \{u,w\}}$ must be 0.

Testability aside, if the second eigenvalue of the adjacency matrix of X is λk ($\lambda \in [-1, 1]$), then we can still infer that $Z^0(X, \mathcal{F}) \subseteq \Sigma^{X(0)}$ has relative distance at least $\varepsilon - \lambda$, see §9.1. If instead we view $Z^0(X, \mathcal{F})$ as a *linear* code inside $\mathbb{F}_2^{X(0) \times \{1, \ldots, m\}}$, then the relative distance is $\geq \frac{\varepsilon - \lambda}{m}$. Since by dimension considerations, the rate of $Z^0(X, \mathcal{F})$ is at least $(1 - \frac{k}{2m})$ (with respect to either alphabet), we conclude that the code $Z^0(X, \mathcal{F})$ is good if $\varepsilon > \lambda$ and $m > \frac{k}{2}$. These bounds are similar to the expander codes of [SS96] (see also [Mes18]). Note, however, that $Z^0(X, \mathcal{F})$ is not a *lifted code*, and thus *not* an expander code in the sense of [SS96].

2.4 Utilizing Coverings

Coverings of high dimensional expanders play an important role in our framework for getting good 2-query LTCs from sheaved high dimensional expanders. Broadly speaking, coverings allow us to produce many expanding sheaves from a single example, and in a different context, provide an “inflation” effect that reduces obstructions. The former will facilitate our framework while the latter would be useful to applying it. We now recall what are coverings, and explain why they are important in our framework.

Henceforth, all simplicial complexes are assumed to be connected.

Coverings of Simplicial Complexes. Let X and Y be (connected) simplicial complexes. Recall that a simplicial map $p: Y \to X$ is called a *covering* map if for every nonempty face $z \in Y$, the restriction of p to the link Y_z defines an bijection between Y_z and the link $X_{p(z)}$. Equivalently, p is a covering map if it induces a covering map of topological spaces between the topological realizations of Y and X. In this case, the connectivity of X implies that the number of faces in Y mapping to a nonempty face $x \in X$ is independent of x; this common number is called the *degree* of p. We say that p is a *double covering*, or that Y is a double covering of X (via p), if the degree of p is 2. In this case, $|Y(i)| = 2|X(i)|$ for all $i \in \mathbb{N} \cup \{0\}$.

It is a standard fact from algebraic topology that there is a one-to-one correspondence between the (connected) coverings of X (considered up to isomorphism over X) and subgroups of the fundamental group $\pi_1(X)$. This restricts to a bijection between the degree-d coverings of X and the index-d subgroups of $\pi_1(X)$.

Pulling Back a Sheaf Along a Covering. If $p: Y \to X$ is a covering map, and $\mathcal{F}$ is a sheaf on X, then we can define a sheaf $p^*\mathcal{F}$ on Y by pulling back $\mathcal{F}$ along p, i.e., by setting

$$p^*\mathcal{F}(y) = \mathcal{F}(p(y)) \qquad \text{and} \qquad \mathrm{res}^{p^*\mathcal{F}}_{y' \leftarrow y} = \mathrm{res}^{\mathcal{F}}_{p(y') \leftarrow p(y)}$$

for all $y, y' \in Y$ with $y \subsetneq y'$. The sheaf $p^*\mathcal{F}$ called the *pullback* of $\mathcal{F}$ along $p: Y \to X$.

Since p is a covering map, it restricts to an isomorphism $Y_z \to X_{p(z)}$ for every nonempty $z \in Y$. Under this isomorphism, the restriction of $p^*\mathcal{F}$ to Y_z, i.e. $(p^*\mathcal{F})_z$, is just $\mathcal{F}_z$. Thus, up to isomorphism, the sheaves $p^*\mathcal{F}$ and $\mathcal{F}$ have the same restrictions to proper links.

Local Local-Testability Lifts Along Coverings. Let $p: Y \to X$ be a covering map and let $\mathcal{F}$ be a sheaf on X. Recall that our Theorem 8.1 says that if the pairs $(X_z, \mathcal{F}_z)_{z \in X - \{\emptyset\}}$ satisfy some expansion conditions (informally called “local” local-testability), then $(X, \mathcal{F})$ will be an (ε, δ)-cosystolic expander in dimension i (“global” local-testability). Since $p^*\mathcal{F}$ and $\mathcal{F}$ have the same restrictions to proper links up to isomorphism, once the assumptions of Theorem 8.1 are satisfied for $\mathcal{F}$, they are also satisfied for $p^*\mathcal{F}$, meaning that $p^*\mathcal{F}$ is also an (ε, δ)-cosystolic expander in dimension i.

We apply this observation in the following context: Let $m \in \mathbb{N}$, $\Sigma = \mathbb{F}_2^m$, and suppose that $\mathcal{F}(v) = \mathbb{F}_2^m = \Sigma$ for all $v \in X(0)$. If $(X, \mathcal{F})$ satisfies the conditions of Theorem 8.1 with $i = 0$, then for every covering $p: Y \to \mathcal{F}$, we have that $p^*\mathcal{F}(u) = \Sigma$ for all $u \in Y(0)$, and $p^*\mathcal{F}$ is an (ε, δ)-cosystolic expander in dimension 0 with $\varepsilon, \delta > 0$ independent of Y, p. Consequently, for every covering $p: Y \to X$, the code $Z^0(Y, p^*\mathcal{F}) \subseteq \Sigma^{Y(0)}$ is ε-testable and has relative distance $\geq \delta$. Otherwise said, the family of codes $\{Z^0(Y, p^*\mathcal{F}) \subseteq \Sigma^{Y(0)}\}_{Y,p}$ with $p: Y \to X$ ranging over the coverings of X is a family of 2-query LTCs with linear distance.

Rate Conservation in Coverings. We continue to assume that $p: Y \to X$ is a covering map and $\mathcal{F}$ is a sheaf on X with $\mathcal{F}(v) = \mathbb{F}_2^m = \Sigma$ for all $v \in X(0)$. Similarly to the situation with testability and distance, we would like to be able to guarantee that “pullback code” $Z^0(Y, p^*\mathcal{F})$ has roughly the same rate as $Z^0(X, \mathcal{F})$.

Suppose that $p: Y \to X$ is of degree ℓ and factors as a composition of double coverings $Y = X_r \to X_{r-1} \to \cdots \to X_0 = X$ (thus $\ell = 2^r$). In Theorem 10.3, we show that in this special case, we have $\dim Z^0(Y, p^*\mathcal{F}) = \Theta(\ell)$, i.e., the rate of $Z^0(Y, p^*\mathcal{F})$ is constant, provided that

$$\dim \mathrm{H}^1(X, \mathcal{F}) < \dim \mathrm{H}^0(X, \mathcal{F}).$$

We call this result *rate conservation*. In particular, if $\{X_r\}_{r\geq 0}$ is an *infinite tower* of connected double coverings of $X_0 = X$, i.e., each X_r is a double covering of X_{r-1}, and if $\mathcal{F}_r$ is the pullback of $\mathcal{F}$ to X_r, then the family of codes $\{Z^0(X_r, \mathcal{F}_r) \subseteq \Sigma^{X_r(0)}\}_{r\geq 0}$ has constant rate.

The integer $\dim \mathrm{H}^0(X, \mathcal{F}) - \dim \mathrm{H}^1(X, \mathcal{F})$ can be considered as measuring the obstruction to rate conservation. Indeed, we can apply rate conservation precisely when it is positive, and the larger it is, the larger the rate of $Z^0(Y, p^*\mathcal{F})$ will be for $p : Y \to X$ as above. We will see in §2.6 that if $\mathcal{F}$ is sheaf on X such that $\dim \mathrm{H}^1(X, \mathcal{F})$ is significantly smaller than m (recall that $\Sigma = \mathbb{F}_2^m = \mathcal{F}(v)$ for $v \in X(0)$), then there is a way to modify $\mathcal{F}$ in order to (conjecturally) decrease $\dim \mathrm{H}^1(X, \mathcal{F})$ even further, thus achieving the threshold for rate conservation.

We also remark that, as stated here, rate conservation is specific to double coverings, $\mathbb{F}_2$-sheaves and 0-cocycle codes. If one wishes to replace $\mathbb{F}_2$ with another field of characteristic $p > 0$, then the requirement that each X_r is a double covering of X_{r-1} should be replaced by $X_r \to X_{r-1}$ being *Galois covering* of degree p, i.e., that $\pi_1(X_r)$ is a normal subgroup of index p in $\pi_1(X_{r-1})$. In order to apply rate conservation to i-cocycle codes with $i > 0$, one needs to add the extra hypothesis $\mathrm{H}^{i-1}(X, \mathcal{F}) = 0$.

2.5 The Tower Paradigm: A Framework for Constructing Good 2-Query LTCs from Expanding Sheaves

We now put together the observations of §2.4 to give a method — the *tower paradigm* — for constructing an infinite family of LTCs with linear distance and constant rate from auxiliary *finite* initial data. This method overcomes the intrinsic barrier in constructing cocycle codes with linear distance and constant rate noted in §2.1.

The Initial Data. The initial data consists of an integer $m \in \mathbb{N}$, a 2-dimensional simplicial complex X, and a sheaf $\mathcal{F}$ on X such that $\mathcal{F}(v) = \mathbb{F}_2^m$ for all $v \in X(0)$. We write $\Sigma = \mathbb{F}_2^m$; this will be the alphabet of the 2-query LTCs that will be constructed from these data.

Requirements on The Initial Data. The initial data $(X, \mathcal{F})$ is required to satisfy the following three requirements:

(t1) There is a sequence of (connected) simplicial complexes $\{X_r\}_{r\geq 0}$ such that $X_0 = X$ and X_r is a double covering of X_{r-1} for all $r \in \mathbb{N}$. We call $\{X_r\}_{r\geq 0}$ a *tower of double coverings* of X.

(t2) For every nonempty $z \in X$, the sheaf $\mathcal{F}_z$ (on the link X_z) is a good coboundary expander and the underlying graph of X_z is a sufficiently good expander; see condition (t2) of Theorem 11.1 for a precise statement. Informally, this means that X is a high-dimensional expander, and $\mathcal{F}$ satisfies "local" local-testability.

(t3) $\dim \mathrm{H}^0(X, \mathcal{F}) > \dim \mathrm{H}^1(X, \mathcal{F})$.

Note that conditions (t2) and (t3) can be verified for a given $(X, \mathcal{F})$ by a finite computation. Condition (t1) does not involve the sheaf $\mathcal{F}$, and can be readily arranged by using existing constructions of high-dimensional expanders, e.g., [LSV05a] or [KO18].

The Induced Family of Good 2-Query LTCs Using the initial data $(X, \mathcal{F})$ and the tower of double coverings $\{X_r\}_{r\geq 0}$, we define an infinite family of codes on the alphabet $\Sigma = \mathbb{F}_2^m$ as follows: Denote by $\mathcal{F}_r$ the pullback of $\mathcal{F}$ along the covering map $X_r \to X_0 = X$. Then $Z_r := Z^0(X_r, \mathcal{F}_r)$ is a code inside $C^0(X_r, \mathcal{F}_r) = \Sigma^{X_r(0)}$. Writing $n_r = |X_r(0)| = 2^r|X(0)|$, this defines a family of codes

$$\{Z_r \subseteq \Sigma^{n_r}\}_{r\in\mathbb{N}}$$

with length tending to infinity.

Theorem 2.1 (Informal; see Theorem 11.1)**.** *If conditions (t1)–(t3) hold, then the codes $\{Z_r \subseteq \Sigma^{n_r}\}_{r\in\mathbb{N}}$ together with their natural 2-query testers form an infinite family of 2-query LTCs with constant rate and linear distance. Moreover, they admit a linear-time decoding algorithm.*

As explained in §2.4, condition (t2) and our local-to-global principle (Theorem 8.1) imply that this is a family of 2-query LTCs with linear distance, and condition (t3) allows us to apply *rate conservation* (Theorem 10.3) to conclude that the rate of the family is constant.

We remark that the soundness of the LTCs $\{Z_r \subseteq \Sigma^{n_r}\}_{r\in\mathbb{N}}$ depends only on the expansion of X and the coboundary expansion of the restriction of $\mathcal{F}$ to the proper links of X. It does not depend on the alphabet size $|\Sigma| = 2^m$.

2.6 Finding Initial Data for The Tower Paradigm

It remains to find examples of initial data for the *tower paradigm* which satisfy all three requirements (t1)–(t3). While we demonstrate that every two of these conditions can be met (see §14.2), finding sheaved high-dimensional expanders satisfying all three is surprisingly difficult, and unfortunately remains open. Instead, we construct candidates satisfying conditions (t1) and (t2), and conjecturally also (t3).

More precisely, we introduce an iterative process which takes a sheaved high dimensional expander satisfying (t1) and (t2) and modifies its sheaf. We show that if the process ends quickly enough, then the resulting sheaf will satisfy (t3) as well. We conjecture that the process will terminate quickly when performed on examples coming from number theory (Conjecture 12.9), hence our aforementioned candidates. Moreover, we identify conditions, phrased by means of representations of the fundamental group of the high-dimensional expander at hand, which imply that the process terminates after just 1 step. What this means in practice is that if one could find an *arithmetic group* with a finite-dimensional $\mathbb{F}_2$*-representation* satisfying certain conditions (see assumption (1) in Theorem 12.11 and the following comment), then they would give rise to initial data for the tower paradigm, and thus to an infinite family of good 2-query LTCs. There exist arbitrarily large *finite* groups with representations meeting these conditions.

We now explain in broad strokes how our candidates for initial data for the tower paradigm are constructed. An example of how the resulting family of codes may look like is given in §2.7.

The Tower. In order to construct the tower $\{X_r\}_{r\geq 0}$, we fix an *affine building* Y of dimension $d \geq 2$, e.g., the affine building of $\mathrm{SL}_{d+1}(\mathbb{Q}_p)$ (see [AB08, §6.9]). Informally, Y is a highly-symmetric d-dimensional *infinite* simplicial complex. Each of the X_r is obtained as a finite quotient $\Gamma_r\backslash Y$, where Γ_r is a group acting freely on Y. By choosing the groups $\{\Gamma_r\}_{r\geq 0}$ to be a decreasing sequence $\Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ such that $[\Gamma_{r-1} : \Gamma_r] = 2$ for all $r \in \mathbb{N}$, the X_r arrange naturally into an infinite tower of double coverings of $X = X_0$. (The covering map $X_r = \Gamma_r\backslash Y \to \Gamma_{r-1}\backslash Y = X_{r-1}$ sends $\Gamma_r y$ to $\Gamma_{r-1} y$.) See §13.4 for particular examples of Y, $\{\Gamma_r\}_{r\geq 0}$. Algorithms for constructing certain quotients $\Gamma_r\backslash Y$ explicitly can be found in [LSV05a], for instance.

Choosing $X := X_0$ to be a quotient of an affine building Y by a group Γ_0 also means that its proper links are *spherical buildings*, which are known to be excellent expanders. This guarantees the that the expansion assumptions on the proper links X_z mentioned in (t2) will hold automatically as soon as Y is *thick* enough. (For example, the thickness of the affine building of $\mathrm{SL}_{d+1}(\mathbb{Q}_p)$ is $p + 1$.) It also has the advantage that $\pi_1(X) = \Gamma_0$ is an *arithmetic group*; a fact that will be put to use later on.

Assuming that $X = X_0$ and the tower $\{X_r\}_{r\geq 0}$ have been chosen, we set to look for a sheaf $\mathcal{F}$ for which conditions (t2) and (t3) hold. We do this in two stages. First, a certain *locally constant sheaf*

$\mathcal{F}$ is chosen. Then, the sheaf $\mathcal{F}$ is modified to produce a sheaf $\overline{\mathcal{F}}$ satisfying (t2) and conjecturally (t3).

Locally Constant Sheaves. A sheaf $\mathcal{G}$ on X is called *locally constant* if for every $v \in X(0)$, the augmented sheaf $\mathcal{G}_v$ is (isomorphic to) a constant augmented sheaf on X_v. This is equivalent to all the restriction maps of $\mathcal{G}$ being isomorphisms. Since X is connected, this means that there is $m \in \mathbb{N} \cup \{0\}$, denoted $\dim \mathcal{G}$ and called the dimension of $\mathcal{G}$, such that $m = \dim \mathcal{G}(x)$ for all $x \in X - \{\emptyset\}$. Locally constant sheaves are abundant: every n-dimensional $\mathbb{F}_2$-representation of $\pi_1(X)$ gives rise to an n-dimensional locally constant sheaf on X, with the trivial representation corresponding to the constant sheaf $\mathcal{F}_{\mathbb{F}_2}$.

Locally Constant Sheaves as Initial Data for The Tower Paradigm. We are interested in locally constant sheaves because condition (t2) is satisfied for any locally constant sheaf $\mathcal{F}$ on the X we chose. Indeed, if $z \in X - \{\emptyset\}$, then $\mathcal{F}_z$ is a constant sheaf on the spherical building X_z. It was shown in [LMM16] and [KM18] (see also [FK21]) that such sheaves are excellent coboundary exapnders in all dimensions (no matter how thick Y is), which means that (t2) holds.

The reason why we do not apply the tower paradigm to locally constant sheaves is because it turns out that conditions (t1) (an infinite tower of double coverings) and (t3) (rate conservation) cannot hold simultaneously for such sheaves (Proposition 11.4). What we suggest to do instead is taking a special locally constant sheaf on X and modifying it slightly so that is satisfies (t3) as well.

We explain the modification process and the choice of the special sheaf separately.

Modifying Locally Constant Sheaves. Let $\mathcal{F}$ be a locally constant sheaf on X of a large dimension m. We think of $\mathcal{F}$ as varying with m as it goes to ∞, but ultimately, both $\mathcal{F}$ and m will be fixed and regarded as "small".

We just observed that $\mathcal{F}$ satisfies (t2) but not (t3). In §12.1, we present an iterative process that takes $\mathcal{F}$ as input and outputs a modified sheaf $\overline{\mathcal{F}}$ which satisfies (t3). If the iterative process terminates quickly, and if $h := \dim \mathrm{H}^1(X, \mathcal{F}) - \dim \mathrm{H}^0(X, \mathcal{F}) + 1$ is very small compared to $\dim \mathcal{F}$, then we can show that the output sheaf $\overline{\mathcal{F}}$ is "very close" to the original $\mathcal{F}$, to the extent that it also satisfies the local expansion condition (t2). If that is indeed the case, then $(X, \overline{\mathcal{F}})$ can serve as initial data for the tower paradigm.

Note that h quantifies how "far" we are from being able to apply rate conservation. Informally, the iterative process eliminates this obstruction when its size is negligible to $\dim \mathcal{F}$.

In more detail, $\overline{\mathcal{F}}$ is constructed as the *quotient* of $\mathcal{F}$ by a "tiny" non-locally constant subsheaf $\mathcal{C}$, chosen to artificially increase $\dim \mathrm{H}^0(X, \mathcal{F}/\mathcal{C})$ and decrease $\dim \mathrm{H}^1(X, \mathcal{F}/\mathcal{C})$. To construct $\mathcal{C}$, we choose a "tiny" subspace $E \subseteq Z^1(X, \mathcal{F})$ and let $\mathcal{C}$ be the smallest subsheaf of $\mathcal{F}$ such that $E \subseteq C^1(X, \mathcal{C})$; see Construction 9.4 or §12.1. The subsheaf $\mathcal{C}$ has the feature that $\mathcal{C}(v)$ is 0 for every vertex $v \in X(0)$ while (typically) $\mathcal{C}(x) \neq 0$ for faces x of dimension > 0. In particular, $\overline{\mathcal{F}}(v) = \mathcal{F}(v)/0 \cong \mathbb{F}_2^m =: \Sigma$ for all $v \in X(0)$. We show in §12.1 that elements in $E \cap B^1(X, \mathcal{F})$ give rise to "new" classes in $\mathrm{H}^0(X, \overline{\mathcal{F}})$ while elements in E which map to a nonzero class in $\mathrm{H}^1(X, \mathcal{F})$ eliminate that class in $\mathrm{H}^1(X, \overline{\mathcal{F}})$. In total, we expect to get

$$\dim \mathrm{H}^0(X, \overline{\mathcal{F}}) - \dim \mathrm{H}^1(X, \overline{\mathcal{F}}) = \dim \mathrm{H}^0(X, \mathcal{F}) - \dim \mathrm{H}^1(X, \mathcal{F}) + \dim E.$$

That is, by passing from $\mathcal{F}$ to $\overline{\mathcal{F}}$, we increase $\dim \mathrm{H}^0(X, -) - \dim \mathrm{H}^1(X, -)$ by $\dim E$. If this prediction works, then we could choose E such that $\dim E = \dim \mathrm{H}^1(X, \mathcal{F}) - \dim \mathrm{H}^0(X, \mathcal{F}) + 1$ and get $\dim \mathrm{H}^0(X, \overline{\mathcal{F}}) > \dim \mathrm{H}^1(X, \overline{\mathcal{F}})$, i.e., (t3) would hold for $\overline{\mathcal{F}}$. However, passing from $\mathcal{F}$ to $\overline{\mathcal{F}}$ may result in "new" cohomology classes in $\mathrm{H}^1(X, \overline{\mathcal{F}})$. We can eliminate these classes by enlarging E further and repeat this process until there are no more excess cohomology classes in $\mathrm{H}^1(X, \overline{\mathcal{F}})$;

this is formalized in Construction 12.2. The resulting sheaf $\overline{\mathcal{F}}$ always satisfies (t3), although not necessarily (t2).

However, we show that if $\dim E \ll \dim \mathcal{F}$ when the process ends — which is what we mean by saying that the process ends quickly —, or if we simply terminate the process when $\dim E \ll \dim \mathcal{F}$, then, with high probability, $\overline{\mathcal{F}}$ still satisfies the necessary local expansion condition (t2); see Corollary 12.5 (which builds on Theorem 9.5).

Condition for The Modification Process to End Quickly. We simulated the iterative modification process for sheaves on 3-dimensional tori, small 2-dimensional 3-thick Ramanujan complexes and other examples.[7] Through the simulations, we came out with formulas that predict the growth of the subspace E when more and more cohomology classes are eliminated. This is formalized in Conjecture 12.6, which, loosely speaking, says that the growth of E is governed by the *cup product* bilinear map $\cup : \mathrm{H}^1(X, \mathbb{F}_2) \times \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{F})$ (see §4.6). In particular, our analysis suggests that:

Conjecture 2.2. *(Simplified; see Conjecture 12.8) If $\mathcal{F}$ is a sheaf on X such that the linear map $\alpha \otimes f \mapsto \alpha \cup f : \mathrm{H}^1(X, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{F})$ is injective and $\mathrm{H}^0(X, \mathcal{F}) \neq 0$, then, once applied to $\mathcal{F}$, the modification process stops after one step (i.e., there are no "new" cohomology classes in $\mathrm{H}^1(X, \overline{\mathcal{F}})$ in the above sense) with high probability. More precisely, when the modification process ends, we have $\dim E = \dim \mathrm{H}^1(X, \mathcal{F}) - \dim \mathrm{H}^0(X, \mathcal{F}) + 1$*

This conjecture (more precisely, the finer Conjecture 12.8) is supported by all of our simulations. Since the universal covering of X is contractible (it is an affine building), $\mathcal{F}$ corresponds to a representation $\rho : \Gamma_0 = \pi_1(X) \to \mathrm{GL}_m(\mathbb{F}_2)$, and the assumption on $\mathcal{F}$ is equivalent to saying that $\mathrm{H}^1(\Gamma_0, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathrm{H}^1(\Gamma_0, \rho) \to \mathrm{H}^2(\Gamma_0, \rho)$ is injective and ρ has nontrivial invariant vectors. We do not know if there is an arithmetic group with an $\mathbb{F}_2$-representation satisfying this condition, but there are arbitrarily large 2-groups for which this holds (see the `MathOverflow` answer [24]).

We also make a bolder conjecture which predicts that the modification process ends quickly if the affine building Y which covers X is sufficiently thick.

Conjecture 2.3. *(Simplified; see Conjecture 12.9) There are $d, q \in \mathbb{N}$ ($d \geq 2$) and a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$ such that if X is covered by a q-thick affine building of dimension d and $\mathcal{F}$ is a sheaf on X, then applying the modification process to $\mathcal{F}$ results in a subspace E such that $\dim E \leq f(\dim \mathrm{H}^1(X, \mathcal{F}))$ with high probability.*

Finding a Locally Constant Sheaf to Modify Using Coverings. If we take Conjectures 2.2 and 2.3 for granted, all that remains in order to find initial data for the tower paradigm is to find a sheaf $\mathcal{F}$ on X such that $\dim \mathrm{H}^1(X, \mathcal{F}) \ll \dim \mathcal{F}$. (In order to use Conjecture 2.2, we also need to require that the additional assumption of that conjecture holds for $\mathcal{F}$.) This sheaf would be modified into a quotient sheaf $\overline{\mathcal{F}}$ satisfying (t2) and (t3).

We show in Theorem 12.10 that there exist finite simplicial complexes X covered by arbitrarily thick affine buildings such that X admits locally constant sheaves $\mathcal{F}$ of arbitrarily large dimension which satisfy $\dim \mathrm{H}^1(X, \mathcal{F}) = 0$.[8] In particular, the requirement $\dim \mathrm{H}^1(X, \mathcal{F}) \ll \dim \mathcal{F}$ can be met. Alternatively, we could start with any locally constant sheaf $\mathcal{G}$ on X, and replace it by $\mathcal{G}_s := \mathcal{G} \times \mathcal{F}^s$ for some large $s \in \mathbb{N}$ in order to increase $\dim \mathcal{G}_s$ without affecting $\dim \mathrm{H}^1(X, \mathcal{G}_s) = \dim \mathrm{H}^1(X, \mathcal{G})$.

The idea behind the construction is to once more utilize coverings. Let $p : X' \to X$ be a covering of degree m, and let $\mathcal{F}$ be the *pushforward* of the the constant sheaf $\mathcal{F}_{\mathbb{F}_2}$ on X' along p

[7] The `Python` code of the simulations was written by the first named author and is attached to the `arXiv` version of this paper.

[8] This forces $\dim \mathrm{H}^1(X, \mathcal{F}) = 0$ if X has an infinite tower of double coverings (Proposition 11.4).

(see §4.3). The sheaf $\mathcal{F}$ is locally constant of dimension m and has the additional property that $\mathrm{H}^i(X,\mathcal{F}) \cong \mathrm{H}^i(X,\mathcal{F}_{\mathbb{F}_2}) = \mathrm{H}^i(X',\mathbb{F}_2)$ (Lemma 4.11). We now put into use the fact that $\pi_1(X)$ is an *arithmetic group*. Using deep facts about such groups, we show in Theorem 13.1, that if the covering building Y is carefully chosen, then X' can be chosen to satisfy $\dim \mathrm{H}^1(X',\mathbb{F}_2) = O(1)$ as m grows. This is already enough if we want sheaves $\mathcal{F}$ satisfying $\dim \mathrm{H}^1(X,\mathcal{F}) = O(1)$ as a function of $\dim \mathcal{F}$, and a more sophisticated construction of this flavor achieves $\dim \mathrm{H}^1(X,\mathcal{F}) = 0$. (Note that this holds despite the fact that X has an infinite tower of double coverings, which means in particular that $\mathrm{H}^1(X,\mathbb{F}_2) \neq 0$.) More generally, it is expected that if Serre's Conjecture on the *congruence subgroup property* (Conjecture 13.11) holds, then for every affine building Y of dimension ≥ 2, one could choose X' with $\dim \mathrm{H}^1(X',\mathbb{F}_2) = O(\log m)$, and thus get $\dim \mathrm{H}^1(X,\mathcal{F}) = O(\log \dim \mathcal{F})$.

Conclusion. We construct candidates for initial data for the tower paradigm as follows: We choose a simplicial complex X covered by a sufficiently thick affine building of dimension ≥ 2, and such that X admits an infinite tower of double coverings (condition (t1)). Using other coverings of X, we find a locally constant sheaf $\mathcal{F}$ such that $\dim \mathrm{H}^1(X,\mathcal{F}) \ll \dim \mathcal{F}$; the pair $(X,\mathcal{F})$ satisfies (t2). We then apply an iterative process to modify $\mathcal{F}$ into a quotient sheaf $\overline{\mathcal{F}} = \mathcal{F}/\mathcal{C}$. However, we terminate the process if $\mathcal{C}$ becomes "close" to $\mathcal{F}$ in dimension, in order to keep the validity of (t2) for $(X,\overline{\mathcal{F}})$. If the process terminated on its own, then $(X,\overline{\mathcal{F}})$ also satisfies (t3). In this case, $(X,\overline{\mathcal{F}})$ are initial data for the tower paradigm, and therefore give rise to an infinite family of good 2-query LTCs; their common alphabet is $\Sigma := \mathbb{F}_2^m$ for $m = \dim \mathcal{F}$. (The soundness of the testing is independent of m, however.)

Our Conjecture 2.3 predicts that the modification process will indeed terminate on its own. Alternatively, our Conjecture 2.2, that is supported by computer simulations, says that this will also be the case if $\mathcal{F}$ satisfies an additional property concerning the cup product. See Theorem 12.11 and Remark 12.12 for precise statements.

2.7 Explicit 2-Query LTCs with Linear Distance and Conjectural Constant Rate

We finish with giving an example of an infinite family of 2-query LTCs with linear distance and conjectural constant rate that arises from our framework. Sheaves are not explicitly mentioned, but are needed for the proofs.

For the example, we use the Ramanujan complexes constructed by Lubotzky, Samuels and Vishne in [LSV05a, §9] as a black box, making reference only to the auxiliary finite field $\mathbb{F}_q$ used in *op. cit.*, which we assume to be of characteristic 2. Alternatively, it is also possible to use the simplicial complexes from Theorem 13.1 below; this has the advantage of not replying on Serre's Conjecture on the congruence subgroup property, and the disadvantage of not having an efficient algorithm to explicitly construct the complexes.

Fix $d \geq 3$. The construction in *op. it.* gives an explicit infinite sequence of d-dimensional simplicial complexes, each mapping into the former:

$$\cdots \to X_r \to \cdots \to X_2 \to X_1 \to X_0,$$

and which can be refined into a tower of *double coverings* of X_0.

Constructing The Codes From Initial Data. Suppose that the following finite set of *initial data* is provided (note that this data is fixed and does not grow with the parameter r):

(1) $m \in \mathbb{N}$,

(2) a linear transformation $T_{e,u} : \mathbb{F}_2^m \to \mathbb{F}_2^m$ for every edge $e = \{u,v\} \in X_0(1)$,

(3) a subspace $C_e \subseteq \mathbb{F}_2^m$ for every edge $e \in X_0(1)$.

The integer m is the same constant m from §2.6. While it is fixed, it will be convenient to think of it as growing.

Once the data (1)–(3) is provided, we construct an infinite family of codes $\{C_r \subseteq \Sigma^{n_r}\}_{r\in\mathbb{N}}$ on the alphabet $\Sigma := \mathbb{F}_2^m$ as follows. Write $n_r = |X_r(0)|$ and identify Σ^{n_r} with $\Sigma^{X_r(0)}$. We write the v-coordinate of $f \in \Sigma^{X_r(0)}$ as $f(v)$. Then $f \in C_r$ if for every edge $e = \{u, v\} \in X_r(1)$ with image $e_0 = \{u_0, v_0\}$ in X_0, we have

$$T_{e_0,u_0}(f(u)) + T_{e_0,v_0}(f(v)) \in C_{e_0}. \tag{2.7}$$

A 2-query tester for C_r is given by choosing $e \in X_r(0)$ uniformly at random and accepting the given $f \in \Sigma^{n_r}$ if (2.7) holds.

Construction of The Initial Data. Provided Conjecture 2.3 holds, a possible choice for the initial data of m, $\{T_{e,v}\}_{e,v}$ and $\{C_e\}_e$ can be as follows. One chooses another simplicial complex X' that is a degree-m covering of X_0, e.g., one of the X_r. The size of m needs to be sufficiently large relative to X_0, but is otherwise fixed (i.e. independent of r). Given a face $x \in X_0 - \{\emptyset\}$, we number the faces in X' mapping onto x as $\hat{x}_1, \dots, \hat{x}_m$. Thus, for every edge $e = \{u, v\} \in X_0(1)$, there is a unique permutation $\sigma_{e,u} : \{1, \dots, m\} \to \{1, \dots, m\}$ such that $\hat{u}_i \in \hat{e}_{\sigma(i)}$ for all $i \in \{1, \dots, m\}$. We take $T_{e,u} : \mathbb{F}_2^m \to \mathbb{F}_2^m$ to be the linear transformation given by

$$T_{e,u}(\alpha_1, \dots, \alpha_m) = (\alpha_{\sigma_{e,u}^{-1}(1)}, \dots, \alpha_{\sigma_{e,u}^{-1}(m)}).$$

To construct the spaces $\{C_e\}_{e\in X_0(1)}$, we view $\mathbb{F}_2^{X'(1)}$ as the space $C^1(X', \mathbb{F}_2)$ of 1-cochains on X' with $\mathbb{F}_2$-coefficients (see §2.1). We then take

$$C_e = \{(h(\hat{e}_1), \dots, h(\hat{e}_m)) \,|\, h \in E\} \subseteq \mathbb{F}_2^m,$$

where E is a subspace of $C^1(X', \mathbb{F}_2)$ corresponding to the space with same name constructed in the iterative process of §2.6 (or formally in Construction 12.2). We terminate the process when $\dim E$ becomes close to m to guarantee that $\dim E \ll m$.

For the sake of simplicity, we only explain what E would be if the process were to end after 1 step (which is not the case here, see §12.2). Let $z_1, \dots, z_t \in Z^1(X', \mathbb{F}_2)$ be 1-cocycles representing a basis of $\mathrm{H}^1(X', \mathbb{F}_2)$. Then, after one step of the process, E will be the $\mathbb{F}_2$-span of $z_1, \dots, z_t$. The next steps of the process add more vectors to E. We would like to choose m be enough in advance so that $\dim E \ll m$. It is expected that if Serre's Conjecture on the congruence subgroup property (Conjecture 13.11) holds, then $t = \dim \mathrm{H}^1(X', \mathbb{F}_2)$ grows logarithmically in m. (However, choosing X_0 and X' to be X and one of the X'_r from Theorem 13.1 guarantees $t = O(1)$ as m grows.)

Validity of The Construction. Provided that our assertion on the logarithmic growth of $\dim \mathrm{H}^1(X', \mathbb{F}_2)$ holds, we have the following:

Theorem 2.4 (informal; see Theorem 12.11, Remark 12.12)**.** *Under the previous assumptions, there is $q_0 \in \mathbb{N}$ such that if $q \geq q_0$, then, with probability $1 - o(1)$ as $m \to \infty$, we have the following:*

(i) The family $\{C_r \subseteq \Sigma^{n_r}\}_{r\in\mathbb{N}}$ is a family of 2-query LTCs with linear distance. The soundness of the testing does not depend on m. If Conjecture 2.3 holds, then the codes in the family have constant rate.

(ii) There is $\eta > 0$ such that each C_r admits a linear-time decoding algorithm able to correct up to ηn_r errors.

In particular, there is $m_0 = m_0(q)$ such that for every $m \geq m_0$, there is a choice of E for which both (i) and (ii) hold.

The dependence on the logarithmic growth of $\dim \mathrm{H}^1(X', \mathbb{F}_2)$ can be avoided by replacing the Ramanujan complexes of [LSV05a] with the complexes from Theorem 13.1 below, but we still rely on Conjecture 2.3 for having constant rate.

2.8 Organization of The Paper

The paper is divided into three chapters and includes two appendices.

Chapter I sets the foundations for the connection between expanding sheaves and codes: The preliminary Section 3 recalls necessary facts about simplicial complexes. The subject matter of Section 4 is sheaves on simplicial complexes, their cohomology, and associated tools such as the pushforward and pullback constructions. In Section 5, we introduce and discuss coboundary and cosystolic expansion of sheaves. Section 6 concerns with the notion of a *locally minimal cochain* and the expansion of such cochains as a mean to get cosystolic expansion. In Section 7, we explain how sheaves give rise to codes with a tester and quantum CSS codes, and formalize the connection between the expansion of the sheaf and various properties of the code.

Chapter II presents the tower paradigm: Section 8 presents a *local-to-global principle* which allows us to establish cosystolic expansion of sheaves from information about their restrictions to proper links. This principle is applied to some examples of cocycle codes in Section 9. In Section 10, we prove our *rate conservation* result. Section 11 puts the previous results together to give the *tower paradigm*, a framework for constructing an infinite family of good 2-query LTCs from a sheaved high-dimensional expander.

Finally, Chapter III concerns with constructing sheaved complexes which can serve as candidates for the initial data of the tower paradigm. In Section 12, we introduce an iterative process which takes special locally constant sheaves as input and produces the desired candidates. Section 13 concerns with constructing simplicial complexes covered by affine buildings with some special properties, e.g., an infinite tower of double coverings. These examples are then used in Section 14 to construct the locally constant sheaves required for the iterative process, as well as other examples of interest.

Appendix A explains the connection between sheaves on simplicial complexes as defined here and the familiar sheaves on topological spaces. Appendix B shows that the sheaf cohomology we define in this work by elementary means is actually a *right derived functor* and thus (from a mathematical point of view) deserves the name “cohomology”.

Chapter I

Foundations

3 Preliminaries

If not indicated otherwise, throughout this work, simplicial complexes are finite, and vector spaces are finite dimensional. We always let X denote a simplicial complex and $\mathbb{F}$ a field.

3.1 Simplicial Complexes

As usual, a simiplicial complex X with vertex set $V = V(X)$ is a nonempty set consisting of finite subsets of V such that $\{v\} \in X$ for all $v \in V$ and every subset of a set in X is also in X. Elements of X are the faces of X and elements of $V(X)$ are the vertices of X. A face with $k+1$ vertices is said to be of dimension k, or a k-face. The set of k-faces of X is denoted $X(k)$. Faces of dimension 1 are called edges, faces of dimension 2 are called triangles, and so on. Note, however, that a 0-face and a vertex are not the thing — the 0-face corresponding to a vertex $v \in V(X)$ is the singleton $\{v\}$. The dimension of X, denoted $\dim X$, is the maximal $k \in \{-1, 0, \infty\} \cup \mathbb{N}$ for which $X(k) \neq \emptyset$.

A graph is a 1-dimensional simplicial complex. The underlying graph of a simplicial complex X is $X(\leq 1) := X(-1) \cup X(0) \cup X(1)$.

If $(Y, \leq)$ is a partially ordered set, we will say that Y is a simplicial complex if there is an isomorphism of partially ordered set $(Y, \leq) \cong (X, \subseteq)$ with X a simplicial complex. We then ascribe all the notation involving X to Y via this isomorphism; the choice of the isomorphism will always be be inconsequential.

The topological realization of a simplicial complex X is denoted $|X|$. We ascribe topological properties of $|X|$ to X, e.g., X is said to be connected if $|X|$ is connected. This condition is equivalent to saying that the underlying graph of X is connected.

Given $A \subseteq X$ and $z \in X$, we write

$$A_{\supseteq z} = \{x \in A \,:\, x \supseteq z\} \qquad \text{and} \qquad A_{\subseteq z} = \{x \in A \,:\, x \subseteq z\}.$$

In particular, $X(k)_{\supseteq z}$ (resp. $X(k)_{\subseteq z}$) is the set of k-faces containing (resp. contained in) z. We further let

$$A_z = \{x - z \,|\, x \in A_{\supseteq z}\}.$$

The set X_z is a simplicial complex known as the *link* of X at z. Note that $X_\emptyset = X$; when $z \neq \emptyset$, we call X_z a *proper link* of X. We say that X is *strongly connected* if all the links of X are connected.

Given $0 \leq i < j$, we define the (i,j)-degree of X to be

$$D_{i,j}(X) = \max\{\# X(j)_{\supseteq z} \,|\, z \in X(i)\},$$

i.e., the largest possible number of j-faces containing a fixed i-face. The degree of X is $D(X) = D_{0,\dim X}(X)$. If every face of X is contained in a d-face, then the degree of X is related to the (i,j)-degree by

$$D_{i,j}(X) \leq \binom{d+1}{j+1} D_{i,d}(X) \leq \binom{d+1}{j+1} D(X). \tag{3.1}$$

An ordered face in X is a face $x \in X$ together with a total ordering of its vertices. Ordered faces will be written as tuples of vertices, e.g. $x = (v_0, \dots, v_i)$, which indicates that $v_0 < \cdots < v_i$. We let X_{ord} denote the set of ordered faces in X, and $X_{\mathrm{ord}}(k)$ the subset of ordered k-faces. If $x \in X_{\mathrm{ord}}(k)$, then we write x_i for the ordered face obtained from $x = (v_0, v_1, \dots, v_k)$ by removing the vertex v_i. We shall freely regard ordered faces as non-ordered faces by forgetting the ordering. If $x = (u_0, \dots, u_i), y = (v_0, \dots, v_j) \in X_{\mathrm{ord}}$ are ordered faces such that $x \cap y = \emptyset$ and $x \cup y \in X$ (here we regarded x, y as non-ordered faces), then the concatenation xy denotes the ordered face $(u_0, \dots, u_i, v_0, \dots, v_j)$.

3.2 Weights

Recall that a simplicial complex X is called *pure of dimension* d ($d \geq 0$), or a *d-complex* for short, if every face of X is contained in a d-face. In this case, following [LMM16], [EK17], [KM18] and other sources, we define the *canonical weight* of a k-face $x \in X(k)$ to be

$$w(x) = w_X(x) = \binom{d+1}{k+1}^{-1} |X(d)|^{-1} |X(d)_{\supseteq x}|.$$

Given $A \subseteq X(k)$, we also write $w(A) = \sum_{x \in A} w(x)$. The weight $w(x)$ is the probability of obtaining x by choosing a d-face $y \in X(d)$ uniformly at random and then choosing a k-face of y uniformly at random. This readily implies that $w(X(k)) = 1$ for all $-1 \leq k \leq d$, and

$$w(X(\ell)_{\supseteq x}) = \binom{\ell+1}{k+1} w(x) \tag{3.2}$$

for all $-1 \leq k \leq \ell \leq d$ and $x \in X(k)$.

Example 3.1. If X is a k-regular graph with n vertices, then X has $\frac{1}{2}nk$ edges and so the canonical weight function of X is given by

$$w(x) = \begin{cases} 1 & x = \emptyset \\ \frac{1}{n} & x \in X(0) \\ \frac{2}{kn} & x \in X(1). \end{cases}$$

Suppose that X is a d-complex and let $z \in X(i)$. Then the link X_z is a $(d-1-i)$-complex. It is straightforward to check that the canonical weight functions of X and X_z are related by the formula

$$w_X(x) = \binom{k+1}{i+1} w_X(z) w_{X_z}(x - z), \tag{3.3}$$

which holds for all $x \in X(k)_{\supseteq z}$ and $k \geq i$.

3.3 Coverings

Let X and Y be simplicial complexes. A morphism of simplicial complexes from Y to X is a function $f : V(Y) \to V(X)$ such that $f(y) := \{f(v) \,|\, v \in y\} \in X$ for all $y \in Y$. The morphism $f : Y \to X$ is *dimension-preserving* if $\dim f(y) = \dim y$ for all $y \in Y$, and a *covering map* if f is onto and it induces a bijection from $Y_{\supseteq y}$ to $X_{\supseteq f(y)}$ for all $y \in Y(0)$. The latter is equivalent to saying that the continuous map $|f| : |Y| \to |X|$ is a covering map of topological spaces. Covering maps are dimension-preserving.

If there exists a covering map $f : Y \to X$, we say that Y covers X. In this case, if Y is a d-complex if and only if X. In addition, if Y is connected, then Y is strongly connected if and only if X is.

A covering $f : Y \to X$ is said to be of *degree* e if $|f^{-1}(v)| = e$ for every $e \in V(X)$; we then write $\deg f = e$ or $[Y : X] = e$ (suppressing f). In this case, for every non-empty face of X, there are exactly e faces in Y which map to it under f. If X is connected and $f : Y \to X$ is a covering, then the size of $|f^{-1}(v)|$ is independent of v, so every covering of a connected simplicial complex has a well-defined degree. A covering map of degree 2 is called a *double covering*.

Let G be a group. A *G-Galois covering* of simplicial complexes consists of a covering map $p : Y \to X$ and G-action $G \times Y \to Y$ such that

(1) for every $g \in G$, the map $v \mapsto gv : V(Y) \to V(Y)$ is an automorphism of Y,

(2) $p(gy) = p(y)$ for all $y \in Y$, and

(3) for every $x \in X$, the action of G on Y restricts to an action on $p^{-1}(x)$, and G acts simply and transitively on $p^{-1}(x)$.

We will often simply say that $p : Y \to X$ is a G-Galois covering, suppressing the G-action.

Condition (3) implies that a G-Galois covering must be of degree $|G|$. The converse is false, however — X may admit coverings of degree $|G|$ which cannot be realized a G-Galois coverings. In more detail, if both X and Y are connected and $p : Y \to X$ is a covering map, then, by fixing a base point $y \in |Y|$, we may realize $\pi_1(X) := \pi_1(|X|, p(y))$ as a subgroup of $\pi_1(Y) = \pi_1(|Y|, y)$. The covering $p : Y \to X$ can be realized as a G-Galois covering if and only if $\pi_1(Y)$ is a normal subgroup of $\pi_1(X)$. In this case, $G \cong \pi_1(X)/\pi_1(Y)$, and the evident map $G \to \mathrm{Aut}(Y/X) := \{f : Y \to Y \,:\, p \circ f = p\}$ is an isomorphism.

Example 3.2. (i) Let C_2 denote the cyclic group with two elements. Every double covering $p : Y \to X$ is C_2-Galois in an unique way. Simply let the nontrivial element of C_2 act on Y by sending $v \in V(Y)$ to the over vertex of Y mapping to $p(v)$.

(ii) If $p : Y \to X$ is a covering map and Y is contractible (and hence connected), then $|Y|$ must coincide with the universal covering of $|X|$. This means that $p : Y \to X$ is Galois with Galois group $\mathrm{Aut}(Y/X) \cong \pi_1(X)$.

(iii) Let G be any group, let Y denote the disjoint union of $|G|$ copies of X and give it the G-action permuting these copies. Let $p : Y \to X$ be the map which restricts to the identity on each copy of X. Then $p : Y \to X$ is a G-Galois covering called the *trivial G-Galois covering* of G. Note that Y is not connected if $|G| > 1$. Moreover, the evident map $G \to \mathrm{Aut}(Y/X)$ is not an isomorphism if $|G| > 2$.

3.4 Skeleton and Spectral Expansion

Let X be a d-complex. Given a set of 0-faces $S \subseteq X(0)$, we write $E(S)$ for the set of edges in X having both of their 0-faces in S. Recall from [KM18, Definition 2.5] that X is said to be an *α-skeleton expander* ($\alpha \in [0, \infty)$) if for every $S \subseteq X(0)$, we have

$$w(E(S)) \leq w(S)^2 + \alpha w(S),$$

where w is the canonical weight function of X defined in §3.2. The complex X is considered more skeleton expanding the smaller α is.

Let $C^0(X,\mathbb{R})$ denote the $\mathbb{R}$-vector space of functions $f : X(0) \to \mathbb{R}$. Following [Opp15], [FK21, §2A] and similar sources, we define the weighed adjacency operator of X to be the linear operator $\mathcal{A} : C^0(X,\mathbb{R}) \to C^0(X,\mathbb{R})$ defined by $(\mathcal{A}f)(x) = \sum_{e\in X(1)_{\supseteq x}} \frac{w(e)}{2w(x)} f(e-x)$ ($f \in C^0(X,\mathbb{R})$, $x \in X(0)$). For example, if X is a k-regular graph, then $\mathcal{A}$ is the usual adjacency operator of X scaled by a factor of $\frac{1}{k}$. Let $C^0_\circ(X,\mathbb{R})$ denote the subspace of $C^0(X,\mathbb{R})$ consisting of functions f with $\sum_{v\in X(0)} f(v) = 0$. Given an interval $I \subseteq [-1, 1]$, we say that the underlying weighted graph of X is a *spectral I-expander* if the spectrum of $\mathcal{A} : C^0_\circ(X,\mathbb{R}) \to C^0_\circ(X,\mathbb{R})$ is contained in I.[1]

It follows readily from the Weighted Expander Mixing Lemma [FK21, Theorem 3.3(ii)] that if the underlying weighted graph of X is a $[-1, \lambda]$-spectral expander for some $\lambda \in [0, 1]$, then X is a λ-skeleton expander.

Given $k \in \{-1, 0, \dots, d-1\}$ and a d-complex X, we say that X is a *k-local $[-1, \lambda]$-spectral expander* (resp. *k-local α-skeleton expander*) if, for every $z \in X(k)$, the underlying weighted graph of X_z is a $[-1, \lambda]$-spectral expander (resp. X_z is an α-skeleton expander).

3.5 Buildings

Buildings are possibly-infinite connected simplicial complexes which have certain remarkable structural properties. They will play a role in some of the examples we consider later on. Contrary to our standing assumption that simplicial complexes are finite, *buildings can be infinite if not otherwise stated*.

We omit the techincal definition of a buidling, which can be found in [AB08], for instance, and satisfy with recalling here some facts about buildings needed for this work. We shall only consider buildings Y admitting a *strongly transitive action* in the sense of [AB08, §6.1.1], and the word "building" will always mean a "building admitting a strongly transitive action". This means that there is a group G acting on Y and satisfying the transitivity properities listed in *op. cit.*.

To every building Y one can attach a Coxeter diagram $T = T(Y)$, called the type of Y, which is a finite undirected graph whose edges are given labels from the set $\{3, 4, 5, \dots\} \cup \{\infty\}$. The complex Y is pure of dimension $|V(T)| - 1$. In fact, there is a labeling $t : V(Y) \to V(T)$ such that every face in Y consists of vertices with different labels. Coxeter diagrams appearing on the list in [AB08, p. 50] are called *spherical*, whereas the ones described in [AB08, Remark 10.33(b)] are called *affine*. We call Y spherical or affine if T is spherical or affine, respectively. If Y is spherical, then $|Y|$ is homotopy equivalent to a bouquet of spheres of dimension $\dim Y$. If Y is affine, then Y is contractible. Finite buildings are spherical.

It will be convenient to treat any nonemtpy 0-dimensional simplicial complex as a spherical building of dimension 0 with Coxeter diagram consisting of a single point.[2]

Let Y be a d-dimensional building. If $z \in Y$ is a face of dimension $\le d-1$, then the link Y_z is also a building. If Y is spherical or affine and $z \ne \emptyset$, then Y_z is spherical (in both cases). The building Y is called q-thick ($3 \le q \in \mathbb{N}$) if every $x \in Y(d-1)$ is contained in at least q d-faces.

Example 3.3. Let $\mathbb{F}$ be a field and let $n \in \mathbb{N}$. Write $A_n(\mathbb{F})$ for the incidence complex of nontrivial subspaces of $\mathbb{F}^{n+1}$. That is, the vertices of $A_n(\mathbb{F})$ are the nonzero proper subspaces of $\mathbb{F}^{n+1}$ and its faces are the sets of vertices which are totally ordered by inclusion. Then $A_n(\mathbb{F})$ is an $(n-1)$-dimenional spherical building. Its type is A_n — the Coxeter diagram consisting of a single path with n vertices and having all edges labeled 3. If $|\mathbb{F}| \ge q$, then $A_n(\mathbb{F})$ is $(q+1)$-thick.

[1] Caution: The spectral expansion of the underlying weighted graph of X takes into account the canonical weight function of X and therefore depends on the higher-dimensional faces of X.

[2] The correct analogue of a 0-dimensional building with a strongly transitive action a *Moufang set*, but this will not be needed in this work.

When $n = 2$, the graph $A_2(\mathbb{F})$ is nothing but the incidence graph of points and lines in the 2-dimensional projective space over $\mathbb{F}$.

We refer the reader to [AB08, §6.9] and [AN02] for the description of some affine buildings. More generally, Bruhat and Tits [BT72] (see also [Tit79]) showed that one can attach to every almost-simple simply-connected algebraic group $\mathbf{G}$ over a local field F an affine building Y equipped with a strongly transitive action by the group $G = \mathbf{G}(F)$. For example, given a prime number p, the group G can be taken to be $\mathrm{SL}_n(\mathbb{Q}_p)$ (with $\mathbf{G} = \mathbf{SL}_n$, $F = \mathbb{Q}_p$), in which case the corresponding affine building is the one described in [AB08, §6.9]. It has type $\tilde{A}_{n-1}$ (a cycle graph on n vertices with all edges labeled 3), dimension $n-1$ and it is $(p+1)$-thick. Moreover, it is locally finite, i.e., every nonempty face is contained in finitely many faces.

We will be particularly interested in finite spherical buildings, and finite simplicial complexes X admitting a covering map $f : Y \to X$ with Y being an affine building. (In the latter case, $|Y|$ is the universal covering of $|X|$, because $|Y|$ is contractible.) Such complexes X are good spectral expanders, and by §3.4, also good skeleton expanders. Formally:

Theorem 3.4 ([FK21, Theorem 7.2])**.** *Let $q \in \{3, 4, 5, \dots\}$ and let X be a (finite) simplicial complex such that one of the following holds:*

(1) X is a finite q-thick spherical building of dimension $d \geq 1$.

(2) There is a covering map $f : Y \to X$ such that Y is a q-thick affine building of dimension $d \geq 2$.

Let L denote the set of edge labels appearing in the Coxeter diagram of the building mentioned in (1) or (2), and let $m = \max(L \cup \{2\})$.[3] *Write $d = \dim X$ and suppose that $q \geq d^2(m-2)$. Then the underlying graph of X is a $[-1, \alpha]$-spectral expander (and thus X is an α-skeleton expander) for*

$$\alpha = \frac{\sqrt{m-2}}{\sqrt{q} - (d-1)\sqrt{m-2}}.$$

The Ramanujan complexes of [LSV05b] and [Li04] are simplicial complexes covered by affine buildings of type $\tilde{A}_n$. However, the spectral (resp. skeleton) expansion of their underlying weighted graph is much better than the bound provided by Theorem 3.4. We demonstrate this in the 2-dimensional case.

Proposition 3.5. *Let Y be the affine building of $\mathrm{SL}_3(F)$, where F is a local non-archimedean field, and let q denote the number of elements in the residue field of F. (The thickness of Y is $q+1$.) If X is a simplicial complex covered by Y and moreover a Ramanujan complex in the sense of [LSV05b] (see also [CSŻ03]), then the underlying weighted graph of X is a $[-1, \frac{3q}{q^2+q+1}]$-spectral expander.*

Proof. The links of X are spherical buildings of the form $A_2(\mathbb{F}_q)$ (notation as in Example 3.3). This means that every vertex is contained in $2(q^2+q+1)$ edges and $(q+1)(q^2+q+1)$ triangles. Now, in the notation of [LSV05b], the weighted adjacency operator of X is $\frac{1}{2(q^2+q+1)}(A_1 + A_2)$. When X is Ramanujan, the joint spectrum of (A_1, A_2) was computed in [LSV05b, Theorem 2.11]. It follows from that computation that the underlying graph of X is a $[-1, \frac{3q}{q^2+q+1}]$-spectral expander. $\square$

4 Sheaves

In this section we introduce sheaves on simplicial complexes and various related notions. Until the end of the section, *simplicial complexes are allowed to be infinite*, and $\mathbb{F}$ denotes a field.

[3] We have $m \leq 8$ if (1) holds and $m \leq 6$ if (2) holds, see [AB08, Chapter 9].

4.1 Sheaves on Simplical Complexes

Let X be a simplicial complex. A *sheaf* $\mathcal{F}$ on X consists of

(1) an abelian group $\mathcal{F}(x)$ for every $x \in X - \{\emptyset\}$, and

(2) a group homomorphism $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} : \mathcal{F}(x) \to \mathcal{F}(y)$ for all $\emptyset \neq x \subsetneq y \in X$,

subject to the condition

$$\mathrm{res}^{\mathcal{F}}_{z\leftarrow y} \circ \mathrm{res}^{\mathcal{F}}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}}_{z\leftarrow x} \tag{4.1}$$

for all $\emptyset \neq x \subsetneq y \subsetneq z \in X$. We also say that $(X, \mathcal{F})$ is a sheaved simplicial complex. Elements of $\mathcal{F}(x)$ are called *x-sections*, and the homomorphisms $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ are called *restriction maps*. If there is no risk of confusion, we will often abbreviate $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} f$ ($f \in \mathcal{F}(x)$) to $\mathrm{res}_{y\leftarrow x} f$, $f|_{x\to y}$ or $f|_y$. Note that condition (4.1) is vacuous if $\dim X \leq 1$.

An *augmented sheaf* $\mathcal{F}$ on X is defined similarly, except we also include the empty face $\emptyset$. That is, $\mathcal{F}(\emptyset)$ and $\mathrm{res}^{\mathcal{F}}_{y\leftarrow\emptyset}$ are defined, and (4.1) is required to told with $x = \emptyset$ as well. We may regard any sheaf $\mathcal{F}$ as an augmented sheaf by setting $\mathcal{F}(\emptyset) = 0$ and $\mathrm{res}^{\mathcal{F}}_{y\leftarrow\emptyset} = 0$ for all $y \in X - \{\emptyset\}$, so all the statements we prove for augmented sheaves also apply to sheaves.

A *sheaf of $\mathbb{F}$-vector spaces*, or an *$\mathbb{F}$-sheaf* for short, on X is a sheaf $\mathcal{F}$ on X such that $\mathcal{F}(x)$ is an $\mathbb{F}$-vector space for all $x \in X - \{\emptyset\}$ and the restriction maps of $\mathcal{F}$ are $\mathbb{F}$-linear. One can define in the same manner (augmented) sheaves of groups, rings, modules, sets, and so on.

Example 4.1. Let X be a simplicial complex.

(i) Given an abelian group A, we define a sheaf $\mathcal{F}_A$ on X by setting $\mathcal{F}_A(x) = A$ for all $x \in X - \{\emptyset\}$ and $\mathrm{res}^{\mathcal{F}_A}_{y\leftarrow x} = \mathrm{id}_A$ for all $\emptyset \neq x \subsetneq y \in X$. The sheaf $\mathcal{F}_A$ is called the *constant sheaf* associated to A. Abusing the notation, we will usually denote $\mathcal{F}_A$ simply as A, or A_X.

(ii) Continuing (i), one can also define an augmented sheaf $\mathcal{F}'_A$ on X by setting $\mathcal{F}'_A(x) = A$ and $\mathrm{res}^{\mathcal{F}'_A}_{y\leftarrow x} = \mathrm{id}_A$ for all $x, y \in X$ with $x \subsetneq y$. We call $\mathcal{F}'_A$ the *constant augmented sheaf* on X and denote it by A_+ when X is clear from the context.

(iii) Fix arbitrary abelian groups $(A_x)_{x\in X}$ and set $\mathcal{F}(x) = A_x$ and $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} = 0 \in \mathrm{Hom}_{\mathbb{Z}}(A_x, A_y)$ for all x, y. Then $\mathcal{F}$ is an augmented sheaf on X (albeit, not a very interesting one). If $A_\emptyset = 0$, then we may regard $\mathcal{F}$ as a sheaf.

(iv) If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, then one can form the product sheaf $\mathcal{F} \times \mathcal{G}$ defined by $(\mathcal{F} \times \mathcal{G})(x) = \mathcal{F}(x) \times \mathcal{G}(x)$ and $\mathrm{res}^{\mathcal{F}\times\mathcal{G}}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}}_{y\leftarrow x} \times \mathrm{res}^{\mathcal{G}}_{y\leftarrow x}$.

(v) Let $\mathcal{F}$ be a sheaf on X. Suppose that we are given subgroups $\mathcal{G}(x) \subseteq \mathcal{F}(x)$ for all $x \in X - \{\emptyset\}$ such that $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}(\mathcal{G}(x)) \subseteq \mathcal{G}(y)$ for all $\emptyset \neq x \subsetneq y \in X$. Then the collection $\{\mathcal{G}(x)\}_{x\in X-\{\emptyset\}}$ can be made into a sheaf $\mathcal{G}$ on X by setting $\mathrm{res}^{\mathcal{G}}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}}_{y\leftarrow x}|_{\mathcal{G}(x)}$. We call such $\mathcal{G}$ a *subsheaf* of $\mathcal{F}$.

(vi) If $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, then we define the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ by setting $(\mathcal{F}/\mathcal{G})(x) = \mathcal{F}(x)/\mathcal{G}(x)$ and $\mathrm{res}^{\mathcal{F}/\mathcal{G}}_{y\leftarrow x}(f + \mathcal{G}(x)) = (\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} f) + \mathcal{G}(y)$ for all $x \in X - \{\emptyset\}$ and $f \in \mathcal{F}(x)$.

(vii) Let $\mathcal{F}$ be an $\mathbb{F}$-sheaf on X and let $\mathbb{K}$ be a field extension of $\mathbb{F}$. The *base change* of $\mathcal{F}$ from $\mathbb{F}$ to $\mathbb{K}$ is the $\mathbb{K}$-sheaf $\mathcal{F}_{\mathbb{K}}$ on X determined by $\mathcal{F}_{\mathbb{K}}(x) = \mathcal{F}(x) \otimes_{\mathbb{F}} \mathbb{K}$ and $\mathrm{res}^{\mathcal{F}_{\mathbb{K}}}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}}_{y\leftarrow x} \otimes \mathrm{id}_{\mathbb{K}}$.

Examples (iii)-(vii) generalize verbatim to augmented sheaves.

Example 4.2. Let X be a connected simplicial complex and let $\mathbb{F}$ be a field. Then every representation $\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbb{F})$ gives rise to an $\mathbb{F}$-sheaf $\mathcal{F} = \mathcal{F}_\rho$. To define it, we must first introduce some general notation.

Write $\Gamma = \pi_1(X)$ and let $\pi : Y \to X$ be the universal covering of X. Then we can (non-canonically) identify Γ with the group of deck transformations of $\pi : Y \to X$ (i.e., the group of automorphisms $g : Y \to Y$ satisfying $\pi \circ g = \pi$). Then Γ acts freely on Y via simplicial automorphisms, and for every non-empty $x \in X$, the preimage $\pi^{-1}(x)$ is an orbit under Γ.

For every nonempty $x \in X$, choose some representative

$$\hat{x} \in \pi^{-1}(x);$$

equivalently, $\{\hat{x} \mid x \in X - \{\emptyset\}\}$ is a set of representatives for $\Gamma \backslash (Y - \{\emptyset\})$. Suppose that $\emptyset \neq x \subsetneq x' \in X$. Then it may not be the case that $\hat{x} \subseteq \hat{x}'$. However, since $\pi : Y \to X$ is a covering, there is a unique $y \in Y$ such that $\hat{x} \subseteq y$ and $\pi(y) = x' = \pi(\hat{x}')$. This means that there is a unique element $\gamma \in \Gamma$ such that $\gamma y = \hat{x}'$. We denote this γ by

$$\gamma(x', x).$$

It is routine to check that if $\emptyset \neq x \subsetneq x' \subsetneq x'' \in X$, then we have

$$\gamma(x'', x')\gamma(x', x) = \gamma(x'', x). \tag{4.2}$$

Now, given a representation $\rho : \Gamma = \pi_1(X) \to \mathrm{End}_{\mathbb{F}}(V)$, where V is an $\mathbb{F}$-vector space, we may define an $\mathbb{F}$-sheaf $\mathcal{F} = \mathcal{F}_\rho$ on X by setting

- $\mathcal{F}(x) = V$ for all $\emptyset \neq x \in X$, and
- $\mathrm{res}^{\mathcal{F}}_{x' \leftarrow x} = \rho(\gamma(x', x)) : V \to V$ for all $\emptyset \neq x \subsetneq x' \in X$.

It follows readily from (4.2) that $\mathcal{F}$ is a sheaf. While $\mathcal{F}(x) = V$ for every $x \in X - \{\emptyset\}$, in general, $\mathcal{F}$ is not the constant sheaf V_X of Example 4.1(i). (In fact, $\mathcal{F}$ is *isomorphic* to V_X if and only if ρ is the trivial representation of Γ on V.)

If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X, then a *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a collection of abelian group homomorphisms $\{\varphi_x : \mathcal{F}(x) \to \mathcal{G}(x)\}_{x \in X - \{\emptyset\}}$ which are compatible with the restriction maps, namely,

$$\varphi_y \circ \mathrm{res}^{\mathcal{F}}_{y \leftarrow x} = \mathrm{res}^{\mathcal{G}}_{y \leftarrow x} \circ \varphi_x$$

for all $\emptyset \neq x \subseteq y \in X$. The collection of all morphisms from $\mathcal{F}$ to $\mathcal{G}$ forms an abelian group with addition given by $\varphi + \varphi' = (\varphi_x + \varphi'_x)_{x \in X - \{\emptyset\}}$ The composition of φ with another morphism $\psi : \mathcal{G} \to \mathcal{H}$ is $\psi \circ \varphi := (\psi_x \circ \varphi_x)_{x \in X - \{\emptyset\}}$. We call φ an *isomorphism* if each φ_x is an isomorphism. If there is an isomorphism $\varphi : \mathcal{F} \to \mathcal{G}$, we say that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and write $\mathcal{F} \cong \mathcal{G}$.

Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves on X, its kernel, $\ker \varphi$, is the subsheaf of $\mathcal{F}$ determined by $(\ker \varphi)(x) = \ker(\varphi_x : \mathcal{F}(x) \to \mathcal{G}(x))$, its image, $\mathrm{im}\, \varphi$, is the subsheaf of $\mathcal{G}$ determined by $(\mathrm{im}\, \varphi)(x) = \mathrm{im}(\varphi_x : \mathcal{F}(x) \to \mathcal{G}(x))$, and its cokernel, $\mathrm{coker}\, \varphi$, is the sheaf $\mathcal{G}/\mathrm{im}\, \varphi$. We call φ injective (resp. surjective) if $\ker \varphi$ is the zero subsheaf of $\mathcal{F}$ (resp. $\mathrm{im}\, \varphi = \mathcal{G}$).

Morphisms of augmented sheaves, their kernels, images and cokernels are defined in the same manner, by including the empty face.

Morphisms of sheaves of $\mathbb{F}$-vector spaces (resp. rings, groups, etc.) are defined similarly with the extra requirement that each φ_x is $\mathbb{F}$-linear (resp. a ring homomorphism, a group homomorphism, etc.). The kernel, image and cokernel of a morphism of $\mathbb{F}$-sheaves are $\mathbb{F}$-sheaves as well.

Remark 4.3. The class of sheaves (resp. $\mathbb{F}$-sheaves) on X together with the morphisms just defined is an *abelian category*, denoted $\mathrm{Sh}(X)$. Similarly, augmented sheaves on X also form an abelian category.

For the relation between the sheaves defined here and the well-known notion of a sheaf on a topological space, see Appendix A.

4.2 Sheaf Cohomology

Sheaf cohomology generalizes ordinary cohomology of simplicial complexes with coefficients in an abelian group. It is defined as follows.

Let $\mathcal{F}$ be an augmented sheaf on a simplicial complex X. Recall (§3.1) that X_{ord} denotes the set of ordered faces in X. For every $k \in \mathbb{N} \cup \{-1, 0\}$, define

$$\tilde{C}^k(X, \mathcal{F}) = \prod_{x \in X_{\mathrm{ord}}(k)} \mathcal{F}(x)$$

(we forget the ordering of x in the expression "$\mathcal{F}(x)$"). Given $f \in \tilde{C}^i(X, \mathcal{F})$ and $x \in X_{\mathrm{ord}}(k)$, we write the x-coordinate of f as $f(x) \in \mathcal{F}(x)$. The group of *$\mathcal{F}$-valued k-cochains* is

$$C^k(X, \mathcal{F}) = \{f \in \tilde{C}^k(X, \mathcal{F}) \,:\, f(\pi x) = \mathrm{sgn}(\pi) f(x) \text{ for all } \pi \in \Sigma_{\{0,\dots,k\}}, x \in X_{\mathrm{ord}}(k)\},$$

where the the permutation group $\Sigma_{\{0,\dots,k\}}$ acts on $X_{\mathrm{ord}}(k)$ by permuting the vertex ordering of every ordered k-face $x = (v_0, \dots, v_k)$.

The *coboundary map* $d_k = d_k^{\mathcal{F}} : C^k(X, \mathcal{F}) \to C^{k+1}(X, \mathcal{F})$ is defined by

$$(d_k f)(y) = \sum_{i=0}^{k+1} (-1)^i \operatorname{res}_{y \leftarrow y_i} f(y_i),$$

where the ordered face y_i is obtained from $y = (v_0, \dots, v_{k+1})$ by removing v_i. It is routine to check that $d_k f$ is in $C^{k+1}(X, \mathcal{F})$ and $d_{k+1} \circ d_k = 0$. The latter is equivalent to saying that

$$0 \to C^{-1}(X, \mathcal{F}) \xrightarrow{d_{-1}} C^0(X, \mathcal{F}) \xrightarrow{d_0} C^1(X, \mathcal{F}) \xrightarrow{d_1} \cdots$$

is a cochain complex. Note that $C^{-1}(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is a sheaf. The *$\mathcal{F}$-valued k-cocycles* and *$\mathcal{F}$-valued k-coboundaries* are

$$Z^k(X, \mathcal{F}) = \ker d_k \qquad \text{and} \qquad B^k(X, \mathcal{F}) = \operatorname{im} d_{k-1},$$

respectively, with the convention that $d_{-2} = 0$. The *k-th cohomology group* of $\mathcal{F}$ is

$$\mathrm{H}^k(X, \mathcal{F}) := Z^k(X, \mathcal{F}) / B^k(X, \mathcal{F}).$$

The cohomology class represented by $f \in Z^k(X, \mathcal{F})$ is denoted $[f]$ or $[f]_{\mathcal{F}}$.

If $\mathcal{F}$ is a sheaf (i.e. $\mathcal{F}(\emptyset) = 0$), then $B^0(X, \mathcal{F})$ is 0 by definition, and thus $\mathrm{H}^0(X, \mathcal{F}) = Z^0(X, \mathcal{F})$. The elements of $Z^0(X, \mathcal{F})$ consist of families $(f(x))_{x \in X(0)} \in \prod_{x \in X(0)} \mathcal{F}(x)$ such that $f(\{u\})|_{\{u,v\}} = f(\{v\})|_{\{u,v\}}$ for every edge $\{u, v\} \in X(1)$. They are called the *global sections* of $\mathcal{F}$.

Example 4.4. (i) Let $\mathcal{F}$ be the augmented sheaf constructed in Example 4.1(iii). Then $\mathrm{H}^k(X, \mathcal{F}) = C^k(X, \mathcal{F}) \cong \prod_{x \in X(k)} \mathcal{F}(x)$, because $d_k = 0$ for all k.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, then we have a canonical isomorphism $C^k(X, \mathcal{F} \times \mathcal{G}) \xrightarrow{\sim} C^k(X, \mathcal{F}) \times C^k(X, \mathcal{G})$, which restricts to isomorphisms $Z^k(X, \mathcal{F} \times \mathcal{G}) \xrightarrow{\sim} Z^k(X, \mathcal{F}) \times Z^k(X, \mathcal{G})$ and $B^k(X, \mathcal{F} \times \mathcal{G}) \xrightarrow{\sim} B^k(X, \mathcal{F}) \times B^k(X, \mathcal{G})$; the details are left to the reader. Consequently, there is a canonical isomorphism $\mathrm{H}^k(X, \mathcal{F} \times \mathcal{G}) \cong \mathrm{H}^k(X, \mathcal{F}) \times \mathrm{H}^k(X, \mathcal{G})$.

Remark 4.5. Fix a linear ordering L on the vertices of X. Then L induces an ordering on the vertices of every face $x \in X$; we write x_L to denote x endowed with this ordering. We can now identify $C^k(X, \mathcal{F})$ with $C^k_L(X, \mathcal{F}) := \prod_{x \in X} \mathcal{F}(x)$ by mapping $f \in C^k(X, \mathcal{F})$ to $(f(x_L))_{x \in X(k)} \in C^k_L(X, \mathcal{F})$.

It is straightforward to check that under this identification, the coboundary map d_k corresponds to $d_{k,L} : C^k_L(X,\mathcal{F}) \to C^{k+1}_L(X,\mathcal{F})$ determined by

$$(d_{k,L}f)(y) = \sum_{x\in X(k)_{\subseteq y}} [y : x]_L \operatorname{res}_{y\leftarrow x} f(x) \tag{4.3}$$

where $y \in X(k+1)$ and $[y : x]_L := (-1)^i$ for the unique $i \in \{0,\dots,k+1\}$ such that x_L is obtained from $y_L = (v_0,\dots,v_{k+1})$ by removing v_i. Consequently, the cohomology of $\mathcal{F}$ can be computed using the the cochain complex

$$0 \to C^{-1}_L(X,\mathcal{F}) \xrightarrow{d_{-1,L}} C^0_L(X,\mathcal{F}) \xrightarrow{d_{0,L}} C^1_L(X,\mathcal{F}) \xrightarrow{d_{1,L}} C^2_L(X,\mathcal{F}) \xrightarrow{d_{2,L}} \dots$$

Since for $x \in X(k)$, the factor $\mathcal{F}(x)$ occurs once in $C^k_L(X,\mathcal{F})$ and $(k+1)!$ times in $\tilde{C}^k(X,\mathcal{F})$, it is sometimes convenient to use $C^k_L(X,\mathcal{F})$ instead of $C^k(X,\mathcal{F})$. The disadvantage of defining $\mathrm{H}^k(X,\mathcal{F})$ using the chain complex $C^\bullet_L(X,\mathcal{F})$ is the ostensible dependency on L.

If $\mathcal{F}$ is augmented $\mathbb{F}_2$-sheaf, then the factor $[y : x]$ in (4.3) has no effect, and can be removed. As a result, $d_{k,L}$ is independent of L, so the isomorphism $C^k(X,\mathcal{F}) \cong C^k_L(X,\mathcal{F}) = \prod_{x\in X(k)} \mathcal{F}(x)$ is also independent of L.

By comparing the description of $\mathrm{H}^k(X,\mathcal{F})$ in Remark 4.5 and the definition of the singular cohomology of $|X|$ with coefficients in an abelian group A, we see that the cohomology of the constant (augmented) sheaf associated to A (Example 4.1) is isomorphic to the (reduced) singular cohomology of $|X|$ with coefficents in A. We record this observation in the following corollary.

Corollary 4.6. *Let A be an abelian group regarded as a constant sheaf on X (see Example 4.1), and let $i \geq 0$. Then $\mathrm{H}^i(X,A) \cong \mathrm{H}^i(|X|,A)$, where the right hand is the singular cohomology of $|X|$ with coefficients in A. Likewise, $\mathrm{H}^i(X,A_+) \cong \tilde{\mathrm{H}}^i(|X|,A)$, where the right hand side denotes the reduced singular cohomology of $|X|$ with coefficients in A.*

As usual, a short exact sequence of sheaves on X is a diagram

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$$

of sheaves on X such that φ is injective, ψ is surjective, and $\operatorname{im}\varphi = \ker\psi$. In this case, there is a long cohomology exact sequence of abelian groups

$$\begin{aligned} 0 \to &\mathrm{H}^0(X,\mathcal{F}) \xrightarrow{\varphi_*} \mathrm{H}^0(X,\mathcal{G}) \xrightarrow{\psi_*} \mathrm{H}^0(X,\mathcal{H}) \xrightarrow{\delta_0} \\ &\mathrm{H}^1(X,\mathcal{F}) \xrightarrow{\varphi_*} \mathrm{H}^1(X,\mathcal{G}) \xrightarrow{\psi_*} \mathrm{H}^1(X,\mathcal{H}) \xrightarrow{\delta_1} \cdots . \end{aligned} \tag{4.4}$$

The map $\varphi_* : \mathrm{H}^i(X,\mathcal{F}) \to \mathrm{H}^i(X,\mathcal{G})$ is defined by sending the cohomology class represented by $f \in Z^i(X,\mathcal{F})$ to the one represented by $(\varphi_x(f(x))_{x\in X_{\mathrm{ord}}(i)} \in Z^i(X,\mathcal{G})$, and ψ_* is defined similarly. The map $\delta_i : \mathrm{H}^i(X,\mathcal{H}) \to \mathrm{H}^{i+1}(X,\mathcal{F})$ is defined as follows: Given $\gamma \in \mathrm{H}^i(X,\mathcal{H})$ represented by some $h \in Z^i(X,\mathcal{H})$, the surjectivity of ψ implies that there is $g \in C^i(X,\mathcal{G})$ such that $h(x) = \psi_x(g(x))$ for all $x \in X_{\mathrm{ord}}(i)$. Using the exactness, one can show that there exists a unique $f \in Z^{i+1}(X,\mathcal{F})$ such that $\varphi_y f(y) = (d_i g)(y)$ for all $y \in X_{\mathrm{ord}}(i+1)$, and we define $\delta_i\gamma := [f]_{\mathcal{F}}$. The proof that δ_i is well-defined and (4.4) is exact is standard and left to the reader.

Remark 4.7. As expected, the functors $\{\mathrm{H}^i(X,-)\}_{i\geq 0}$ are the right derived functors of the left-exact functor $\mathrm{H}^0(X,-)$ from the category of sheaves on X to abelian groups, see Appendix B.

Remark 4.8. Given a short exact sequence of augmented sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ on X, one can define a long exact sequence similar to (4.4), but starting at $\mathrm{H}^{-1}(X, \mathcal{F})$ instead of $\mathrm{H}^0(X, \mathcal{F})$. We omit the details.

If $\mathcal{F}$ is an $\mathbb{F}$-sheaf on X, then the cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are $\mathbb{F}$-vector spaces. When $\mathbb{F}$ is clear from the context, we shall often write

$$h^i(\mathcal{F}) = h^i(X, \mathcal{F}) := \dim_{\mathbb{F}} \mathrm{H}^i(X, \mathcal{F}).$$

Lemma 4.9. *Let $\mathcal{F}$ be an $\mathbb{F}$-sheaf on X and let $\mathbb{K}$ be a field extension of $\mathbb{F}$. Then* $\dim_{\mathbb{F}} \mathrm{H}^i(X, \mathcal{F}) = \dim_{\mathbb{K}} \mathrm{H}^i(X, \mathcal{F}_{\mathbb{K}})$ *for all $i \in \mathbb{N} \cup \{0\}$ (notation as in Example 4.1(vii)).*

Proof. This follows by observing that the cochain complex $C^\bullet(X, \mathcal{F}_{\mathbb{K}})$ is isomorphic to the cochain complex obtained by tensoring $C^\bullet(X, \mathcal{F})$ with $\mathbb{K}$. $\square$

4.3 Pushforward and Pullback

Throughout, let $u : Y \to X$ denote a morphism of simplicial complexes (see §3.3). Given a sheaf $\mathcal{G}$ on Y, there is a natural way of "pushing it" along u to a sheaf on X, and conversely, given a sheaf $\mathcal{F}$ on X, there is a natural way of "pulling it back" along u to a sheaf on Y. We now explain these constructions. They will be extremely useful later on for producing new examples of sheaved complexes from old ones.

Let $\mathcal{F}$ be a sheaf on X. The *pullback* or *inverse image* of $\mathcal{F}$ along $u : Y \to X$ is the sheaf $u^*\mathcal{F}$ on Y defined by

$$u^*\mathcal{F}(y) = \mathcal{F}(u(y)) \qquad \text{and} \qquad \mathrm{res}^{u^*\mathcal{F}}_{y' \leftarrow y} = \mathrm{res}^{\mathcal{F}}_{u(y') \leftarrow u(y)}$$

for all $\emptyset \neq y \subsetneq y' \in Y$, with the convention that $\mathrm{res}^{\mathcal{F}}_{y \leftarrow y} = \mathrm{id}_{\mathcal{F}(y)}$.

Example 4.10. If A_X is the constant sheaf on X associated to the abelian group A (Example 4.1(i)), then u^*A_X is the constant sheaf on Y associated to A, that is, $u^*A_X = A_Y$.

Now let $\mathcal{G}$ be a sheaf on Y and suppose that $u : Y \to X$ is dimension preserving, i.e., $\dim y = \dim u(y)$ for all $y \in Y$. Given $x \in X$, we write $u^{-1}(x)$ for the set $\{y \in Y : u(y) = x\}$. Our assumption on u implies that if $x' \in X$, $y' \in u^{-1}(x')$ and x is a face of x', then there exists a unique face y of y' such that $u(y) = x$; we denote this face y by $y'(x)$. With this notation at hand, we define *pushforward* or *direct image* of a sheaf $\mathcal{G}$ along u to be the sheaf $u_*\mathcal{G}$ on X determined by

$$(u_*\mathcal{G})(x) = \prod_{y \in u^{-1}(x)} \mathcal{G}(y) \qquad \text{and} \qquad \mathrm{res}^{u_*\mathcal{G}}_{x' \leftarrow x}((f_y)_{y \in u^{-1}(x)}) = (\mathrm{res}^{\mathcal{G}}_{y' \leftarrow y'(x)}(f_{y'(x)}))_{y' \in u^{-1}(x')}$$

where $\emptyset \neq x \subsetneq x' \in X$ and $(f_y)_{y \in u^{-1}(x)} \in u_*\mathcal{G}(x) = \prod_{y \in u^{-1}(x)} \mathcal{G}(y)$. It routine to check that the sheaf condition (4.1) is satisfied for $u_*\mathcal{G}$.

One can also define the pushforward $u_*\mathcal{G}$ without assuming that u is dimension preserving. This construction is more involved and explained in Appendix A.3; we will not make use of it in this work.

The following lemma relates the cohomology of $\mathcal{G}$ and $u_*\mathcal{G}$. It can be regarded as a version of Shapiro's Lemma for sheaf cohomology.

Lemma 4.11. *Let $u : Y \to X$ be a dimension-preserving morphism of simplicial complexes and let $\mathcal{G}$ be a sheaf on Y. Then, for all $i \geq 0$, there is an isomorphism $\mathrm{H}^i(Y, \mathcal{G}) \cong \mathrm{H}^i(X, u_*\mathcal{G})$ which is natural in $\mathcal{G}$.*

Proof. It is enough to prove that the cochain complexes $C^\bullet(Y,\mathcal{G})$ and $C^\bullet(X,u_*\mathcal{G})$ are naturally isomorphic, i.e., that there is are isomorphisms $t_{i,\mathcal{G}} : C^i(Y,\mathcal{G}) \to C^i(X,u_*\mathcal{G})$ such that $t_{i+1,\mathcal{G}} \circ d_i^{\mathcal{G}} = d_i^{u_*\mathcal{G}} \circ t_{i,\mathcal{G}}$ and $t_{i,\mathcal{G}'} \circ \varphi_* = \varphi_* \circ t_{i,\mathcal{G}}$ for every morphism of sheaves on Y, $\varphi : \mathcal{G} \to \mathcal{G}'$. The desired isomorphism $t_{i,\mathcal{G}}$ is the restriction of the identification $\tilde{C}^i(Y,\mathcal{G}) = \prod_{y\in X_{\mathrm{ord}}(i)} \mathcal{G}(y) \cong \prod_{x\in X_{\mathrm{ord}}(i)} \prod_{y\in u^{-1}(x)} \mathcal{G}(y) = \prod_{x\in X_{\mathrm{ord}}(i)} u_*\mathcal{G}(x) = \tilde{C}^i(X,u_*\mathcal{G})$ to $C^i(Y,\mathcal{G})$. It is routine to check that it satisfies all the requirements. □

4.4 Restricting Sheaves to The Links

Let X be a simplicial complex and let $z \in X(i)$. Recall (§3.1) that X_z denotes the link of X at z. Every augmented sheaf $\mathcal{F}$ on X restricts to an augmented sheaf $\mathcal{F}_z$ on X_z by setting $\mathcal{F}_z(x) = \mathcal{F}(x\cup z)$ and $\mathrm{res}^{\mathcal{F}_z}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}}_{y\cup z\leftarrow x\cup z}$. (This is how augmented sheaves arise naturally from sheaves!)

Example 4.12. Let A be an abelian group and let A_+ denote the associated augmented sheaf on X (Example 4.1(ii)). Then $(A_+)_z$ is the augmented sheaf on X_z associated to A.

Suppose now that $z \in X_{\mathrm{ord}}(i)$, namely, we are also given an ordering on the vertices of z. With this extra data, it possible to take a cochain $f \in C^k(X,\mathcal{F})$ ($i \le k$) and restrict it to a cochain $f_z \in C^{k-i-1}(X_z,\mathcal{F}_z)$ by setting

$$f_z(x) = f(xz) \qquad \forall x \in X_{z,\mathrm{ord}}(k-i-1).$$

Conversely, given $g \in C^{k-i-1}(X,\mathcal{F}_z)$, there exists a unique cochain $g^z \in C^k(X,\mathcal{F})$ such that

$$g^z(xz) = g(x) \qquad \forall x \in X_{\mathrm{ord}}(k-i-1),$$

and $g^z(y) = 0$ for all $y \in X_{\mathrm{ord}}(k)$ with $z \not\subseteq y$. Clearly, $(g^z)_z = g$.

Lemma 4.13. *In the previous setting, we have $(d_{k-i-1}g)^z = d_k(g^z)$. In particular, if g is a cocycle (resp. coboundary), then so is g^z.*

Proof. Let $x \in X_{\mathrm{ord}}(k+1)$. We need to show that $(d_{k-i-1}g)^z(x) = d_k(g^z)(x)$. If $z \not\subseteq x$ as sets, then $(d_{k-i-1}g)^z(x) = 0 = d_k(g^z)(x)$, so assume that $z \subseteq x$ as sets. By reordering the vertices of x, we may assume that $x = yz$ for some $y \in X_{\mathrm{ord}}(k-i)$. Then $(d_{k-i-1}g)^z(x) = (d_{k-i-1}g)(y) = \sum_{j=0}^{k-i}(-1)^j \mathrm{res}_{x\leftarrow z\cup y_j}\, g(y_j)$. On other hand, since $g^z(x_j) = 0$ if $z \not\subseteq x_j$, we have $d_k(g^z)(x) = \sum_{j=0}^{k+1}(-1)^j \mathrm{res}_{x\leftarrow x_j}\, g^z(x_j) = \sum_{j=0}^{k-i}(-1)^j \mathrm{res}_{x\leftarrow z\cup y_j}\, g(y_j)$, so $(d_{k-i-1}g)^z(x) = d_k(g^z)(x)$. □

Let $\mathcal{P}$ be a property of sheaved simplicial complexes (written "$(X,\mathcal{F})$ is $\mathcal{P}$" when it holds), and let $(X,\mathcal{F})$ be a sheaved simplicial complex. We will say that $(X,\mathcal{F})$ is a *k-local* $\mathcal{P}$ if $(X_z,\mathcal{F}_z)$ is $\mathcal{P}$ for all $z \in X(k)$. If $\mathcal{P}$ also makes reference to a particular dimension i (as in "X is $\mathcal{P}$ in dimension i"), we will say that $(X,\mathcal{F})$ is a *k-local $\mathcal{P}$ in dimension* i if $(X_z,\mathcal{F}_z)$ is $\mathcal{P}$ in dimension $i-k-1$ for all $z \in X(k)$.

4.5 Locally Constant Sheaves

Let X be a simplicial complex. A sheaf $\mathcal{F}$ on X is called *constant* if there is an abelian group A such that $\mathcal{F}$ is isomorphic to the constant sheaf A on X (Example 4.1(i)). Similarly, an augmented sheaf $\mathcal{F}'$ on X is called constant if $\mathcal{F}' \cong A_+$ for some abelian group A. If $\mathcal{F}$ (resp. $\mathcal{F}'$) has the additional structure of an $\mathbb{F}$-sheaf, we further require A to be an $\mathbb{F}$-vector space and the isomorphism $\mathcal{F} \to A$ (resp. $\mathcal{F}' \to A_+$) to be $\mathbb{F}$-linear.

A sheaf $\mathcal{F}$ on X is called *locally constant* if $\mathcal{F}_z$ is a constant augmented sheaf on X_z for every $z \in X - \{\emptyset\}$. Every constant sheaf is locally constant, but the converse is false in general.

Lemma 4.14. *A sheaf $\mathcal{F}$ on a simplicial complex X is locally constant if and only if all the restriction maps* $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ *($\emptyset \neq x \subsetneq y \in X$) are isomorphisms.*

Proof. If $\emptyset \neq x \subsetneq y \in X$ and $\mathcal{F}$ is locally constant, then $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ is equal to $\mathrm{res}^{\mathcal{F}_x}_{y-x\leftarrow\emptyset}$, which is an isomorphism because $\mathcal{F}_x$ is constant. Conversely, if all the restriction maps of $\mathcal{F}$ are isomorphisms and $z \in X - \{\emptyset\}$, take $A = \mathcal{F}(z)$ and note that $(\mathrm{res}_{x\cup z\leftarrow z})_{x\in X_z} : A_+ \to \mathcal{F}_z$ is an isomorphism of augmented sheaves. □

Example 4.15. (i) Let X be a cycle graph on n vertices. Fix an edge $e \in X(1)$ and a 0-face $z \subseteq e$. Define an $\mathbb{R}$-sheaf $\mathcal{F}$ on X by setting $\mathcal{F}(x) = \mathbb{R}$ for every $x \in X - \{\emptyset\}$, $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} = \mathrm{id}_{\mathbb{R}}$ for $(y, x) \neq (e, z)$ and $\mathrm{res}^{\mathcal{F}}_{e\leftarrow z} = -\mathrm{id}_{\mathbb{R}}$. By Lemma 4.14, $\mathcal{F}$ is a locally constant sheaf. However, $\mathcal{F}$ is not constant. Indeed, one readily checks that $Z^0(X, \mathcal{F}) = 0$. However, if $\mathcal{F}$ were constant, then it would be isomorphic to the constant sheaf $\mathbb{R}_X$, and $Z^0(X, \mathbb{R}) \cong \mathbb{R}$.

(ii) Generalizing (i), we can construct locally constant sheaves on any graph X. Simply take an abelian group A, set $\mathcal{F}(x) = A$ for all $x \in X - \{\emptyset\}$ and choose each restriction map $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ ($\emptyset \neq x \subsetneq y \in X(1)$) to be some automorphism of A. As in (i), sheaves obtained in this manner are often not constant.

(iii) If $u : Y \to X$ is a covering map and $\mathcal{F}$ is a locally constant sheaf on Y, then the pushforward $u_*\mathcal{F}$ is a locally constant sheaf on X. The sheaf $u_*\mathcal{F}$ may be non-constant even when $\mathcal{F}$ is.

(iv) Suppose that X is connected and let $\rho : \pi_1(X) \to \mathrm{GL}_{\mathbb{F}}(V)$ be representation of $\pi_1(X)$ on an $\mathbb{F}$-vector space V. Then $\mathbb{F}$-sheaf $\mathcal{F}_\rho$ constructed in Example 4.2 is locally constant. Moreover, it can be shown that $\mathcal{F}_\rho$ is constant if and only if ρ is a trivial representation (i.e., $\rho(\gamma) = \mathrm{id}_V$ for all $\gamma \in \pi_1(X)$).

Remark 4.16. Locally constant $\mathbb{R}$-sheaves on graphs are equivalent as a category to the local sysetms on graphs introduced by Jordan and Livne [JL97].

Let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf on X. If X is connected, then Lemma 4.14 implies that all the vector spaces $\{\mathcal{F}(x)\}_{x\in X-\{\emptyset\}}$ have the same dimension. When the latter holds, we denote this common dimension by

$$\dim \mathcal{F}$$

and call it the dimension of $\mathcal{F}$.

Lemma 4.17. *Let X be a connected simplicial complex and let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf on X. Then* $\dim \mathrm{H}^0(X, \mathcal{F}) \leq \dim \mathcal{F}$.

Proof. Fix some 0-face $x_0 \in X$. It is enough to show that any 0-cocycle $f \in Z^0(X, \mathcal{F})$ is uniquely determined by $f(x_0)$. Indeed, if $y \in X(0)$ is another 0-face, then there exists a sequence of 0-faces $x_0, x_1, \dots, x_n = y$ in X such that $x_{i-1} \cup x_i \in X(1)$ for all i. Since $f \in Z^0(X, \mathcal{F})$, we have $\mathrm{res}_{x_i\cup x_{i-1}\leftarrow x_{i-1}} f(x_{i-1}) = \mathrm{res}_{x_i\cup x_{i-1}\leftarrow x_i} f(x_i)$ for all $i \in \{1, \dots, n\}$. The restriction maps are isomorphisms (Lemma 4.14), so $f(y)$ is uniquely determined by $f(x_0)$. □

Remark 4.18. It is not difficult to see that a every n-dimensional locally constant sheaf $\mathcal{F}$ on a connected simplicial complex X gives rise to a group homomorphism $\pi_1(X) \to \mathrm{GL}_n(\mathbb{F})$. The converse is also true: every representation $\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbb{F})$ gives rise to an n-dimensional locally constant $\mathbb{F}$-sheaf on X, and if the universal covering of X is contractible, then all locally constant sheaves are obtained in this manner, up to isomorphism. Moreover, in this case, $\mathrm{H}^i(X, \mathcal{F})$ and the group cohomology $\mathrm{H}^i(\pi_1(X), \rho)$ are isomorphic. We omit the details as they will not be needed here.

4.6 The Cup Product

The cup product is a well-known operation on cohomology groups in topology. We now present the analogous notion for sheaves on simplicial complexes, which will be needed only for Section 12 below. For the sake of simplicity, we shall restrict the discussion to the cup-product action of $C^i(X,\mathbb{F})$ on $C^j(X,\mathcal{G})$ where $\mathcal{G}$ is an $\mathbb{F}$-sheaf.

As before, X is a simplicial complex. We fix, once and for all, a linear ordering L on $V(X)$ and use it to identify $C^j(X,\mathcal{G})$ with $\prod_{x\in X(j)}\mathcal{G}(x)$ for any sheaf $\mathcal{G}$ as in Remark 4.5. If $v_0,\dots,v_i$ are the vertices of $x\in X(i)$ and $v_0<\cdots<v_i$, then we shall denote the ordered face x_L simply as $v_0v_1\cdots v_i$.

Let $\mathcal{G}$ be an $\mathbb{F}$-sheaf on X. For every $\alpha\in C^i(X,\mathbb{F})$ and $g\in C^j(X,\mathcal{G})$, the *cup product* of α and g is the element $\alpha\cup g\in C^{i+j}(X,\mathcal{G})$ defined by:

$$(\alpha\cup g)(v_0v_1\cdots v_{i+j})=\alpha(v_0\cdots v_i)g(v_i\cdots v_{i+j}).$$

The properties of the cup product that we shall need are summarized in the following proposition.

Proposition 4.19. *Let $\mathcal{G}$ be an $\mathbb{F}$-sheaf on X, and let $f,g\in C^j(X,\mathcal{G})$, $\alpha\in C^i(X,\mathbb{F})$, $\beta\in C^k(X,\mathbb{F})$. Then:*

(i) $\cup : C^i(X,\mathbb{F})\times C^j(X,\mathcal{G})\to C^{i+j}(X,\mathcal{G})$ *is an $\mathbb{F}$-bilinear pairing.*

(ii) $d_{i+j}(\alpha\cup f)=d_i\alpha\cup f+(-1)^i\alpha\cup d_jf$.

(iii) $(\alpha\cup\beta)\cup f=\alpha\cup(\beta\cup f)$.

Moreover, if $\mathcal{G}'$ is another $\mathbb{F}$-sheaf on X and $\varphi:\mathcal{G}\to\mathcal{G}'$ is a morphism, then:

(v) $\varphi_*(\alpha\cup g)=\alpha\cup\varphi_*g$, *where $\varphi_*:C^r(X,\mathcal{G})\to C^r(X,\mathcal{G}')$ is the map induced by φ (see §4.2).*

Proof. Everything is straightforward except for (ii). To prove (ii), suppose that $x=v_0\cdots v_{i+j+1}$ is an ordered $(i+j+1)$-face. Then

$$\begin{aligned}
d_{i+j}(\alpha\cup f)(x) &= \sum_{t=0}^{i+j+1}(-1)^t(\alpha\cup f)(v_0\cdots\hat{v}_t\cdots v_{i+j+1})\\
&=\sum_{t=0}^{i}(-1)^t\alpha(v_0\cdots\hat{v}_t\cdots v_{i+1})f(v_{i+1}\cdots v_{i+j+1})+(-1)^{i+1}\alpha(v_0\cdots v_i\hat{v}_{i+1})f(v_{i+1}\cdots v_{i+j+1})\\
&\quad+(-1)^i\alpha(v_0\cdots v_i)f(\hat{v}_iv_{i+1}\cdots v_{i+j+1})+\sum_{t=i+1}^{i+j+1}(-1)^t\alpha(v_0\cdots v_i)f(v_i\cdots\hat{v}_t\cdots v_{i+j+1})\\
&=d_i\alpha(v_0\cdots v_{i+1})f(v_{i+1}\cdots v_{i+j+1})+(-1)^i\alpha(v_0\cdots v_i)d_jf(v_i\cdots v_{i+j+1})\\
&=((d_i\alpha)\cup f+(-1)^i\alpha\cup(d_jf))(x).
\end{aligned}$$

(Here, $\hat{v}_t$ means that we omit v_t.) As this holds for all x, (ii) follows. □

Proposition 4.19(ii) implies readily that the bilinear pairing

$$[\alpha]\cup[g]\mapsto[\alpha\cup g]:\mathrm{H}^i(X,\mathbb{F})\times\mathrm{H}^j(X,\mathcal{G})\to\mathrm{H}^{i+j}(X,\mathcal{G}),$$

also called the cup product, is well defined. It can further be shown that this pairing is independent of the ordering on $V(X)$.

5 Coboundary and Cosystolic Expansion

In this section we introduce expanding sheaves. In fact, we shall consider two types of expansion — *coboundary expansion* and *cosystolic expansion* — and both make use of an auxiliary *norm* on the sheaf.

5.1 Norms on Abelian Groups

Recall that a *seminorm* on an abelian group A is a function $\|\cdot\| : A \to \mathbb{R}$ such that $\|a\| \geq 0$, $\|a\| = \|-a\|$ and $\|a+b\| \leq \|a\| + \|b\|$ for all $a, b \in A$. If $\|a\| = 0$ implies $a = 0$, we say that $\|\cdot\|$ is *norm* on A. In this case, $(x, y) \mapsto \|x - y\| : A \times A \to \mathbb{R}$ is a translation-invariant metric on A.

A seminorm $\|\cdot\| : A \to \mathbb{R}$ is bounded if $\sup\{\|a\| \mid a \in A\} < \infty$. *All norms and seminorms in this work are assumed to be bounded.*

If $\|\cdot\|_A$ is a seminorm (resp. norm) on A and B is a subgroup of A, then the restriction of $\|\cdot\|_A$ to B is a seminorm (resp. norm) on B. The map $\|\cdot\|_{A/B} : A/B \to \mathbb{R}$ defined by $\|a + B\|_{A/B} = \inf_{b \in B} \|a + b\|_A$ is a seminorm on A/B, called the quotient seminorm. If $\|\cdot\|_A$ is a norm and B is finite, then $\|\cdot\|_{A/B}$ is a norm.

Example 5.1. (i) The *discrete norm* on an abelian group A maps all nonzero elements of A to 1 and the zero element to 0.

(ii) The *Hamming norm* on $\mathbb{F}^n$ sends $v \in \mathbb{F}^n$ to the number of its non-zero coordinates. More generally, if V is a finite dimensional $\mathbb{F}$-vector space with a finite basis B, then the Hamming norm on V relative to B sends the vector $v = \sum_{b \in B} \alpha_b b$ to the number of nonzero α_b-s.

5.2 Normed Sheaves

Let $\mathcal{F}$ be an augmented sheaf on a simplicial complex X. A norm on $\mathcal{F}$ is a collection $\|\cdot\| = \{\|\cdot\|_x\}_{x \in X}$ of norms $\|\cdot\|_x : \mathcal{F}(x) \to \mathbb{R}$. We also say that $(\mathcal{F}, \|\cdot\|)$ is a normed augmented sheaf. In this case, the *mass* of $x \in X$ (relative to $\mathcal{F}$ and $\|\cdot\|$) is defined as $m(x) = \sup\{\|f\|_x \mid f \in \mathcal{F}(x)\}$; it is finite by our standing assumption that all norms are bounded. For $A \subseteq X$, we write $m(A) = \sum_{x \in A} m(x)$ and let $m(i) = m(X(i))$.

In this work, we will be concerned with the following examples of normed augmented sheaves.

Example 5.2 (Weighted support norm)**.** Suppose that X is a d-complex (i.e., pure of dimension d) and let w denote the canonical weight function on X (see §3.2). Let $\mathcal{F}$ be an augmented sheaf on X. The *weighted support norm* of $\mathcal{F}$ is the norm $\|\cdot\|_{\mathrm{ws}} = \{\|\cdot\|_{\mathrm{ws},x}\}_{x \in X}$, where $\|\cdot\|_{\mathrm{ws},x} : \mathcal{F}(x) \to \mathbb{R}$ is defined by $\|f\|_{\mathrm{ws},x} = w(x)$ if $f \neq 0$ and $\|f\|_{\mathrm{ws},x} = 0$ otherwise. Provided that $\mathcal{F}(x) \neq 0$, the mass of x is just $w(x)$. Consequently, if $\mathcal{F}(x) \neq 0$ for all $x \in X(i)$, then $m(i) = w(X(i)) = 1$.

The weighted support norm will be the default norm on every sheaf we consider, and will be denoted simply as $\|\cdot\|$ when there is no risk of confusion. The following norms, however, are more useful for coding theory applications of sheaves.

Example 5.3 (Normalized Hamming norm)**.** The *normalized Hamming norm* on an augmented sheaf $\mathcal{F}$ is the norm $\|\cdot\|_{\mathrm{Ham}} = \{\|\cdot\|_{\mathrm{Ham},x}\}_{x \in X}$, where $\|\cdot\|_{\mathrm{Ham},x} : \mathcal{F}(x) \to \mathbb{R}$ is defined by $\|f\|_{\mathrm{Ham},x} = \frac{1}{|X(\dim x)|}$ if $f \neq 0$ and $\|f\|_{\mathrm{Ham},x} = 0$ otherwise. This is the norm used in Section 2. If $\mathcal{F}(x) \neq 0$ for all $x \in X(i)$, then $m(i) = 1$.

Example 5.4 (Hamming norm relative to a basis)**.** Suppose that $\mathcal{F}$ is an augmented $\mathbb{F}$-sheaf, i.e., an augmented sheaf of $\mathbb{F}$-vector spaces. A *basis* for $\mathcal{F}$ is a collection $B = \{B(x)\}_{x \in X}$ such that $B(x)$

is an $\mathbb{F}$-basis of $\mathcal{F}(x)$ for all $x \in X$. Let $\|\cdot\|_{B,x}$ denote the Hamming norm on $\mathcal{F}(x)$ relative to the basis $B(x)$ (Example 5.1(ii)). Then $\|\cdot\|_B := \{\|\cdot\|_{B,x}\}_{x\in X}$ is a norm on $\mathcal{F}$ called the *Hamming norm* relative to the basis $B = \{B(x)\}_{x\in X}$. Writing $m = m_B$ for the corresponding mass function, we have $m(x) = \dim \mathcal{F}(x)$ and $m(i) = \sum_{x\in X(i)} \dim \mathcal{F}(x) = \dim C^i(X, \mathcal{F})$.

Every norm $\|\cdot\|$ on $\mathcal{F}$ induces a norm on $C^k(X, \mathcal{F})$, also denoted $\|\cdot\|$, given by

$$\|f\| = \|f\|_{C^k} = \sum_{x\in X(k)} \|f(x)\|_x,$$

where in the expression $f(x)$, we regard x as an ordered cell by arbitrarily ordering its vertices. The mass of all i-faces, $m(i)$, is nothing but $\sup\{\|f\| \,|\, f \in C^i(X, \mathcal{F})\}$. For example, if $\|\cdot\|$ is the weighted support norm, then we have $\|f\| \le 1$ for all $f \in C^i(X, \mathcal{F})$.

Example 5.5. (i) Let X be d-complex with weight function $w = w_X$, let $\mathcal{F}$ an augmented sheaf on X and let $f \in C^i(X, \mathcal{F})$. Then, relative to the weighted support norm $\|\cdot\|_{\mathrm{ws}}$ (Example 5.2), we have

$$\|f\|_{\mathrm{ws}} = w(\operatorname{supp} f),$$

where $\operatorname{supp} f := \{x \in X(k) : f(x) \neq 0\}$. (This explains the name "weighted support".) By contrast, with respect to the normalized Hamming norm $\|\cdot\|_{\mathrm{Ham}}$ (Example 5.3), we have

$$\|f\|_{\mathrm{Ham}} = \frac{|\operatorname{supp} f|}{|X(k)|}.$$

Suppose further that there is an abelian group Σ such that $\mathcal{F}(x) \cong \Sigma$ for all $x \in X(i)$. Let us fix a linear ordering on $V(X)$ and use it identify $C^i(X, \mathcal{F})$ with $\prod_{x\in X(i)} \mathcal{F}(x) \cong \Sigma^{X(i)}$ as in Remark 4.5. Then the norm $\|\cdot\|_{\mathrm{Ham}} : C^i(X, \mathcal{F}) \to \mathbb{R}$ coincides with the normalized Hamming norm on $\Sigma^{X(i)}$.

(ii) Let $\mathcal{F}$ be an augmented $\mathbb{F}$-sheaf on X with a basis B, and let $\|\cdot\|_B$ denote the associated Hamming norm (Example 5.4). Again, fix a linear ordering on $V(X)$ and use it to identify $C^k(X, \mathcal{F})$ with $\prod_{x\in X(k)} \mathcal{F}(x)$ as in Remark 4.5. Then, under this identification, $\|\cdot\|_B$ is the Hamming norm of $\prod_{x\in X(k)} \mathcal{F}(x)$ relative to the basis $\bigsqcup_{x\in X} B(x)$.

The norms $\|\cdot\| = \|\cdot\|_{\mathrm{ws}}$, $\|\cdot\|_{\mathrm{Ham}}$, $\|\cdot\|_B$ of Examples 5.2–5.4 are proportional under mild assumptions on X and $\mathcal{F}$.

Proposition 5.6. *Let X be a d-complex, let $k \in \{-1, 0, \dots, d\}$, and let $\mathcal{F}$ be an augmented sheaf on X. Put $Q = D_{k,d}(X)$ (see §3.1). Then,*

(i) $\binom{d+1}{k+1}\frac{|X(d)|}{|X(k)|}Q^{-1}\|f\| \le \|f\|_{\mathrm{Ham}} \le \binom{d+1}{k+1}\frac{|X(d)|}{|X(k)|}\|f\|$ *for all $f \in C^k(X, \mathcal{F})$. Furthermore,* $\binom{d+1}{k+1}^{-1} \le \frac{|X(d)|}{|X(k)|} \le Q$.

If $\mathcal{F}$ is an augmented $\mathbb{F}$-sheaf, B is a basis of $\mathcal{F}$, and $N = \max\{\dim \mathcal{F}(x) \,|\, x \in X(k)\}$, then we moreover have

(ii) $\binom{d+1}{k+1}|X(d)|Q^{-1}\|f\| \le \|f\|_B \le \binom{d+1}{k+1}|X(d)|N\|f\|$ *for all $f \in C^k(X, \mathcal{F})$.*

Proof. (i) The inequality $\binom{d+1}{k+1}^{-1} \le \frac{|X(d)|}{|X(k)|} \le Q$ follows readily from the fact that every d-face contains exactly $\binom{d+1}{k+1}$ k-faces and every k-face is contained in at most Q d-faces.

Let $x \in X(k)$. It is enough to show that for all $g \in \mathcal{F}(x)$, we have $\binom{d+1}{k+1}\frac{|X(d)|}{Q|X(k)|}\|g\|_{\mathrm{ws},x} \le \|g\|_{\mathrm{Ham},x} \le \binom{d+1}{k+1}\frac{|X(d)|}{|X(k)|}\|g\|_{\mathrm{ws},x}$. This is clear if $g = 0$, so assume $g \neq 0$. Then $\|g\|_{\mathrm{ws},x} = w(x)$

whereas $\|g\|_{\mathrm{Ham},x} = \frac{1}{|X(k)|}$. The definition of $w(x)$ in §3.2 implies that $\binom{d+1}{k+1}^{-1}|X(d)|^{-1} \leq w(x) \leq \binom{d+1}{k+1}^{-1}|X(d)|^{-1}Q$. Thus, $\binom{d+1}{k+1}\frac{|X(d)|}{Q|X(k)|}w(x) \leq \frac{1}{|X(k)|} = \binom{d+1}{k+1}\frac{|X(d)|}{|X(k)|} \cdot \binom{d+1}{k+1}^{-1}|X(d)|^{-1} \leq \binom{d+1}{k+1}\frac{|X(d)|}{|X(k)|}w(x)$, which is what we want.

(ii) As in (i), it is enough to show that for all $x \in X(k)$ and $g \in \mathcal{F}(x) - \{0\}$, we have $\binom{d+1}{k+1}|X(d)|Q^{-1}w(x) \leq \|g\|_{B(x)} \leq \binom{d+1}{k+1}|X(d)|Nw(x)$. We observed that $\binom{d+1}{k+1}^{-1}|X(d)|^{-1} \leq w(x) \leq \binom{d+1}{k+1}^{-1}|X(d)|^{-1}Q$. Since $1 \leq \dim \mathcal{F}(x) \leq N$, it follows that $\binom{d+1}{k+1}|X(d)|Q^{-1}w(x) \leq 1 \leq \|g\|_{B(x)} \leq N \leq \binom{d+1}{k+1}|X(d)|Nw(x)$, as required. $\square$

5.3 Coboudary and Cosystolic Expansion

Let $(\mathcal{F}, \|\cdot\|)$ be a normed augmented sheaf on a simplicial complex X and let m be its mass function. Let $k \in \mathbb{N} \cup \{0, -1\}$. The norm $\|\cdot\|_{C^k}$ induces seminorms on $C^k(X,\mathcal{F})/B^k(X,\mathcal{F})$ and $C^k(X,\mathcal{F})/Z^k(X,\mathcal{F})$, which we denote by $\|\cdot\|_{C^k/B^k}$ and $\|\cdot\|_{C^k/Z^k}$, respectively. The subscripts will be dropped when there is no risk of confusion.

Definition 5.7. *Let $\varepsilon, \delta \in [0,\infty)$. We say that $(X, \mathcal{F}, \|\cdot\|)$ is an ε-*coboundary expander *in dimension k if*

(B) $\|d_k f\|_{C^{k+1}} m(k) \geq \varepsilon \|f + B^k(X,\mathcal{F}))\|_{C^k/B^k} m(k+1)$ for all $f \in C^k(X,\mathcal{F})$.

*We say that $(X, \mathcal{F}, \|\cdot\|)$ is an (ε, δ)-*cosystolic expander *in dimension k if*

(C1) $\|d_k f\|_{C^{k+1}} m(k) \geq \varepsilon \|f + Z^k(X,\mathcal{F})\|_{C^k/Z^k} m(k+1)$ for all $f \in C^k(X,\mathcal{F})$, and

(C2) $\|f\|_{C^k} \geq \delta m(k)$ for all $f \in Z^k(X,\mathcal{F}) - B^k(X,\mathcal{F})$.

When X is a d-complex, we say that the pair $(X, \mathcal{F})$ is an ε-coboundary expander, resp. (ε,δ)-cosystolic expander, in dimension i if this holds for $(X, \mathcal{F}, \|\cdot\|)$ with $\|\cdot\|$ being the weighted support norm of $(X, \mathcal{F})$ (Example 5.2). Thus, $(X, (\mathbb{F}_2)_+)$ is an ε-coboundary expander in dimension i if and only if X is an ε-coboundary expander in dimension i in the sense of Lubotzky, Meshulam and Mozes [LMM16, Definition 1.1].

Remark 5.8. The following properties of coboundary and cosystolic expansion are important to note:

(i) The triple $(X, \mathcal{F}, \|\cdot\|)$ is an ε-coboundary expander in dimension k if and only if it is an (ε, δ)-cosystolic expander in dimension k and $\mathrm{H}^k(X, \mathcal{F}) = 0$.

(ii) Scaling the norms $\{\|\cdot\|_x\}_{x \in X(k)}$ by the same constant $c \in \mathbb{R}_+$ does not affect the coboundary and cosystolic expansion in dimension k, and likewise for the for the norms $\{\|\cdot\|_x\}_{x \in X(k+1)}$. More generally, let $\|\cdot\|'$ be another norm on $\mathcal{F}$, and suppose that there are constants $u_k, v_k, u_{k+1}, v_{k+1} \in \mathbb{R}_+$ such that $u_i\|f\|_x \leq \|f\|'_x \leq v_i\|f\|_x$ for all $i \in \{k, k+1\}$, $x \in X(i)$ and $f \in \mathcal{F}(x)$. If $(X, \mathcal{F}, \|\cdot\|)$ is an (ε,δ)-cosystolic expander (resp. ε-coboundary expander) in dimension k, then $(X, \mathcal{F}, \|\cdot\|')$ is a $(\frac{u_k u_{k+1}}{v_k v_{k+1}}\varepsilon, \frac{u_k}{v_k}\delta)$-cosystolic expander (resp. $\frac{u_k u_{k+1}}{v_k v_{k+1}}\varepsilon$-coboundary expander) in dimension k.

(iii) If $\mathcal{F}$ vanishes on $X(k)$ or on $X(k+1)$, equivalently if $m(k) = 0$ or $m(k+1) = 0$, then conditions (B) and (C1) hold with any $\varepsilon \in \mathbb{R}_+$.

Remark 5.9. The normalization by $m(k)$ and $m(k+1)$ in (B), (C1) and (C2) is made in order to keep ε and δ around the interval $[0,1]$. However, it is possible for ε to exceed 1. Indeed, writing Δ_n for the n-dimensional simplex, it is easy to check that the coboundary expansion of

$(\Delta_n, (\mathbb{F}_2)_+, \|\cdot\|_{\mathrm{ws}})$ in dimension 0 is $\frac{n+2-(n \bmod 2)}{n}$.[4] In contrast, if $(X, \mathcal{F}, \|\cdot\|)$ is an (ε, δ)-cosystolic expander in dimesion k and $Z^k(X, \mathcal{F}) \neq B^k(X, \mathcal{F})$, then δ cannot exceed 1.

If we use the weighted support norm, then the coboundary expansion in dimension k cannot exceed $k+2$ by the following lemma.

Lemma 5.10. *Let $(X, \mathcal{F})$ be a sheaved d-complex, let $k \in \{-1, \dots, d-1\}$ and suppose that $\mathcal{F}(x) \neq 0$ for all $x \in X(k+1)$. Then the coboundary expansion of $(X, \mathcal{F})$ in dimension k is at most $k+2$.*

Proof. It is enough to show that $\|d_k f\|m(k) \leq (k+2)\|f\|m(k+1)$ for all $f \in C^k(X, \mathcal{F})$. Our assumptions imply that $m(k+1) = 1$ and $m(k) \leq 1$. Using this and (3.2), we get

$$\begin{aligned} \|d_k f\|m(k) &\leq w(\mathrm{supp}(d_k f)) \leq w\Big(\bigcup_{x \in \mathrm{supp}\, f} X(k+1)_{\supseteq x}\Big) \\ &\leq \sum_{x \in \mathrm{supp}\, f} w(X(k+1)_{\supseteq x}) = (k+2) \sum_{x \in \mathrm{supp}\, f} w(x) = (k+2)\|f\|m(k+1). \end{aligned}$$

□

The meaning of begin an ε-coboundary expander in -1 has been worked out in the Overview section, page 11. We recommend to recall it at this point.

5.4 Some Examples of Coboundary Exapnders

Only a few concrete examples of good coboundary expanders in dimension > 0 are known; see [FK21] for survey. We now recall some of these examples which will be needed in this work.

In contrast, examples of infinite families of cosystolic expanders of the form $(X, A_+, \|\cdot\|_{\mathrm{ws}})$ (A is an abelian group) with $D(X)$ uniformly bounded appear in [KKL16] ($\dim X = 2$, $A = \mathbb{F}_2$), [EK17] ($A = \mathbb{F}_2$) and [KM18]. We shall give more examples in Sections 8 and 9.

We begin with noting that if the underlying weighted graph of a d-complex X is a good spectral expander in the sense of §3.4, then (X, A_+) is a good coboundary expander in dimension 0 (w.r.t. to $\|\cdot\|_{\mathrm{ws}}$) for any abelian group A.

Theorem 5.11 ([FK21, Corollary 5.3])**.** *Suppose that X is d-complex ($d \geq 1$) whose underlying weighted graph is a $[-1, \lambda]$-expander (in the sense of §3.4) for some $\lambda \in [-1, 1]$. Then (X, A_+) is a $(1-\lambda)$-coboundary expander in dimension 0 for every abelian group A.*

Next, we recall that finite buildings (see §3.5) have large (i.e. bounded away from 0) coboundary expansion once endowed with certain sheaves. The following theorem summarizes results from [LMM16], [KM18] and [FK21].

Theorem 5.12. *Let $d \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$. Let X be finite d-dimensional q-thick finite building with Coxeter diagram T and let A be an abelian group. Let L denote the set of edge labels occurring in T and put $m = \max(\{2\} \cup T)$.*[5]

(i) There exists $\varepsilon > 0$, depending only on d, such that (X, A_+) is an ε-coboundary expander in dimensions $-1, 0, \dots, d-1$.

(ii) (X, A_+) is a $\left(1 - \frac{\sqrt{m-2}}{\sqrt{q}-(d-1)\sqrt{m-2}}\right)$-coboundary expander in dimension 0 if $q > (d-1)^2(m-2)$, and a 1-coboundary expander in dimension -1 in general.

[4] As for higher dimensions $k \in \{1, \dots, n-1\}$, Gromov [Gro10] and Meshulam–Wallach [MW09] showed that $(\Delta_n, (\mathbb{F}_2)_+, \|\cdot\|_{\mathrm{ws}})$ is an $\frac{n+1}{n-k}$-coboundary expander in dimension k.

[5] If $q \geq 3$, then $m \leq 8$; see [AB08, Chapter 9].

Proof. The assertions about coboundary expansion in dimension -1 and the case where $\dim X = 0$ are straightforward. As for the rest, (i) is proved in [LMM16] for $A = \mathbb{F}_2$ and in [KM18], for general A, and (ii) is [FK21, Corollary 7.4]. $\square$

Theorem 5.13 ([FK21, Corollary 7.6]). *Let d, q, X, m, A be as in Theorem 5.12 and assume that $q > (d-1)^2(m-2)$. Let $\{A_x\}_{x\in X(0)}$ be subgroups of A such that for every subset $S \subseteq X(0)$ with $|S| \le \lceil \frac{2}{3}|X(0)|\rceil$, the summation map $\bigoplus_{x\in S} A_x \to A$ is injective. Define a subsheaf $\mathcal{C}$ of A_+ by $\mathcal{C}(y) = \sum_{v\in y} A_{\{v\}}$. Then $(X, A_+/\mathcal{C}, \|\cdot\|_{\mathrm{supp}})$ is a ε-coboudary expander in dimension 0 for*

$$\varepsilon = \frac{2d}{5d+2} - \frac{(4d^3+4d)\sqrt{m-2}}{(5d+2)(\sqrt{q}-(d-1)\sqrt{m-2})} - \frac{14d^2+4d}{(5d+2)(q+d-1)} = \frac{2d}{5d+2} - O_{d,m}(q^{-1/2}).$$

The theorems we just recalled concern with coboundary expansion with respect to the weighted support norm (Example 5.2), but they can be adapted to the Hamming norms of Examples 5.3 and 5.4 by means of Proposition 5.6 and Remark 5.8(ii).

6 Locally Minimal Cochains

Locally minimal cochains were introduced in [KKL16] and [EK17] for the augmented sheaf $(\mathbb{F}_2)_+$, and in [KM18] for general constant augmented sheaves as a mean to establish cosystolic expansion. In this section, we extend this notion to all sheaves, and explain how to derive lower bounds on the cosystolic expansion of a sheaved d-complex $(X, \mathcal{F})$ from lower bounds on the expansion of locally minimal cochains.

We work exclusively with the weighted support norm (Example 5.2), which we denote by $\|\cdot\|$.

6.1 Minimal and Locally Minimal Cochains

Let $(X, \mathcal{F})$ be a sheaved d-complex and let $k \in \{0, \dots, d\}$. A cochain $f \in C^k(X, \mathcal{F})$ is called *minimal* if $\|f\|_{C^k} = \|f + B^i(X, \mathcal{F})\|_{C^k/B^k}$ (see §5.3). Given $z \in X$ of dimension $i \in \{0, \dots, k-1\}$, we say that f is *locally minimal at z* if for every $g \in C^{k-i-2}(X_z, \mathcal{F}_z)$, we have $\|f + d_{k-1}(g^z)\| \ge \|f\|$, where g^z is defined as in §4.4 and the vertices of z are given some ordering (the ordering has no effect as g can vary). We say that f is *locally minimal* if it is locally minimal at every $z \in X(0) \cup \dots \cup X(k-1)$.

Clearly, every minimal cochain is locally minimal. Also, vacuously, all 0-cochains are locally minimal.

Proposition 6.1. *Let $(X, \mathcal{F})$ be a sheaved d-complex, let $k \in \{0, \dots, d\}$ and let $f \in C^k(X, \mathcal{F})$. Then:*

- *(i) f is locally minimal at $z \in X_{\mathrm{ord}}(i)$ ($0 \le i < k$) if and only if $f_z \in C^{k-i-1}(X_z, \mathcal{F}_z)$ is minimal.*
- *(ii) If f is locally minimal at some $z \in X(0) \cup \dots \cup X(k-1)$, then f is locally minimal at every $w \in X(0) \cup \dots \cup X(k-1)$ containing z. In particular, f is locally minimal if and only if it is locally minimal at every 0-face of X.*

Proof. (i) Let $g \in C^{k-i-2}(X_z, \mathcal{F}_z)$. The equivalence follows readily once noting that

$$\|f - d_{k-1}(g^z)\| = \|f - (f_z)^z)\| + \|(f_z)^z - d_{k-1}(g^z)\| = \|f - (f_z)^z)\| + \tbinom{k+1}{i+1} w_X(z) \|f_z - d_{k-i-2} g\|_{X_z},$$

where here, $\|\cdot\|_{X_z}$ is the weighted support norm of $\mathcal{F}_z$ and the second equality follows from (3.3)

(ii) Choose orderings on z and w such that $w = uz$ for some $u \in X_{\mathrm{ord}}(j-i-1)$. Then, regarding X_w as the link of u in X_z, we have $(f_z)_u = f_w$. By (i), f_z is minimal, hence locally minimal at u. Applying (i) again, we see that $f_w = (f_z)_u$ is minimal, so f is locally minimal at w. $\square$

Given a minimal cochain, we can produce more minimal cochains by annihilating some of its entries.

Lemma 6.2. *Let $(X,\mathcal{F})$ be a sheaved d-complex, let $k \in \{0,\dots,d\}$ and let $f, g \in C^k(X,\mathcal{F})$. Assume f is minimal. If $\operatorname{supp} g \subset \operatorname{supp} f$ and $g(x) = f(x)$ for all $x \in X_{\mathrm{ord}}(k)$ with $x \in \operatorname{supp} g$, then g is minimal.*

Proof. Let $b \in B^k(X,\mathcal{F})$. We need to prove that $\|g\| \le \|g - b\|$. Our assumptions on g imply that $\|g\| = \|f\| - \|f - g\|$. As f is minimal, $\|f\| - \|f-g\| \le \|f - b\| - \|f - g\| \le \|g - b\|$, hence the lemma. □

The following lemma shows that, under mild assumptions on X, every $(k+1)$-cochain f is equivalent modulo $B^k(X,\mathcal{F})$ to a locally minimal cochain $f' := f - d_k g$, and moreover, that the k-cochain g used for "correcting" f can chosen so that its norm is proportional to that of f.

Lemma 6.3. *Let $(X,\mathcal{F})$ be a sheaved d-complex of degree $Q = D(X)$ (see §3.1). Let $k \in \{-1,\dots,d-1\}$ and $f \in C^{k+1}(X,\mathcal{F})$. Then there exists $g \in C^k(X,\mathcal{F})$ such that:*

(i) $f - d_k g$ *is locally minimal,*

(ii) $\|g\| \le \frac{(k+1)Q}{d+1}\binom{d+1}{k+2}\|f\|$,

(iii) $\|f - d_k g\| \le \|f\|$.

Proof. We define sequences $f_0,\dots,f_r \in C^{k+1}(X,\mathcal{F})$ and $g_0,\dots,g_r \in C^k(X,\mathcal{F})$ by induction as follows: Take $f_0 = f$ and $g_0 = 0$. Assume that f_n and g_n have been defined. If f_n is locally minimal, we stop and let $r = n$. Otherwise, $k \ge 0$ and by Proposition 6.1(ii), there exist $x \in X(0)$ and $g \in C^k(X_x,\mathcal{F}_x)$ such that $\|f_n - d_k(g^x)\| < \|f_n\|$. Take $f_{n+1} = f_n - d_{k-1}(g^x)$ and $g_{n+1} = g^x$.

We claim that $g := g_0 + \dots + g_r$ satisfies the requirements. Indeed, by construction, $f - d_k g = f_r$ is locally minimal and satisfies $\|f - d_k g\| = \|f_r\| < \dots < \|f_0\| = \|f\|$. Furthermore, since the norm of any $(k+1)$-cochain in $C^{k+1}(X,\mathcal{F})$ is an integral multiple of $\binom{d+1}{k+2}^{-1}|X(d)|^{-1}$ (see Example 5.2), we have $r \le \binom{d+1}{k+2}|X(d)|\cdot\|f\|$. On the other hand, each g_n is supported on the k-faces containing a particular vertex $v \in X(0)$ and therefore satisfies $\|g_n\| \le w(X(k)_{\supseteq v}) = \binom{k+1}{1}w(v) \le \binom{k+1}{1}|X(d)|^{-1}\binom{d+1}{1}^{-1}Q = |X(d)|^{-1}\frac{(k+1)Q}{d+1}$ (the first equality is (3.2)). Consequently, $\|g\| \le r|X(d)|^{-1}\frac{(k+1)Q}{d+1} \le \frac{(k+1)Q}{d+1}\binom{d+1}{k+2}\|f\|$. □

6.2 Expansion of Small Locally Minimal Cochains

We continue to assume that $(X,\mathcal{F})$ is a sheaved d-complex. Let $k \in \{0,\dots,d-1\}$ and $\alpha,\beta \in \mathbb{R}_+$. We say that $(X,\mathcal{F})$ *β-expands α-small locally minimal k-cochains* if for every locally minimal $f \in C^k(X,\mathcal{F})$ such that $\|f\| < \alpha$, we have $\|d_k f\| \ge \beta\|f\|$. We say that $(X,\mathcal{F})$ β-expands α-small locally minimal k-cocycles if this condition holds for all locally minimal $f \in Z^k(X,\mathcal{F})$.

Proposition 6.4. *Let $(X,\mathcal{F})$ be a sheaved d-complex of degree $Q = D(X)$ (see §3.1), let $k \in \{0,\dots,d-2\}$, and let $\alpha,\alpha',\beta,\beta' \in \mathbb{R}_+$. Suppose that*

(1) $\mathcal{F}(x) \ne 0$ *for all* $x \in X(k) \cup X(k+1) \cup X(k+2)$,

(2) $(X,\mathcal{F})$ *β-expands α-small locally minimal k-cocycles, and*

(3) $(X,\mathcal{F})$ *β'-expands α'-small locally minimal $(k+1)$-cocycles.*

Then $(X,\mathcal{F})$ *is a* $(\min\{\alpha', \frac{d+1}{(k+1)Q}\binom{d+1}{k+2}^{-1}\}, \alpha)$*-cosystolic expander in dimension* k*. If only (1) and (2) are assumed, then* $\|f\| \geq \alpha$ *for every* $f \in Z^k(X,\mathcal{F}) - B^k(X,\mathcal{F})$*.*

Proof. We need to verify conditions (C1) and (C2) of §5.3 for $\delta = \alpha$ and $\varepsilon = \min\{\alpha', \frac{d+1}{(k+1)Q}\binom{d+1}{k+2}^{-1}\}$. Note that $m(k) = m(k+1) = m(k+2) = 1$ by condition (1).

We begin with (C2). Suppose that $f \in Z^k(X,\mathcal{F}) - B^k(X,\mathcal{F})$. We need to show that $\|f + B^k(X,\mathcal{F})\|_{C^k/B^k} \geq \alpha$. Choose a minimal $f' \in f + B^k(X,\mathcal{F})$. Then f' is locally minimal and nonzero. If $\|f'\| < \alpha$, then by (2), we would have $0 = \|d_k f'\| \geq \beta\|f'\| > 0$, a contradiction. Thus, $\|f + B^k(X,\mathcal{F})\|_{C^k/B^k} = \|f'\| \geq \alpha$.

We turn to (C1). Let $f \in C^k(X,\mathcal{F})$. If $\|d_0 f\| \geq \alpha'$, then $\|d_0 f\| \geq \alpha'\|f\| \geq \varepsilon\|f\|$, as required. Otherwise, $\|d_0 f\| < \alpha'$. We apply Lemma 6.3 to $d_0 f$ to get $g \in C^k(X,\mathcal{F})$ such that $d_k f - d_k g$ is a locally minimal $(k+1)$-cochain, $\|g\| \leq \frac{(k+1)Q}{d+1}\binom{d+1}{k+2}\|d_k f\|$ and $\|d_k f - d_k g\| \leq \|d_k f\| < \alpha'$. The latter and (3) imply that $0 = \|d_{k+1}(d_k f - d_k g)\| \geq \beta'\|d_k f - d_k g\|$, so $d_k(f-g) = 0$, or rather, $g \in f + Z^k(X,\mathcal{F})$. This means that $\|f + Z^k(X,\mathcal{F})\|_{C^k/Z^k} \leq \|g\| \leq \frac{(k+1)Q}{d+1}\binom{d+1}{k+2}\|d_k f\|$, and by rearranging, we get

$$\|d_k f\| \geq \frac{d+1}{(k+1)Q}\binom{d+1}{k+2}^{-1}\|f + Z^k(X,\mathcal{F})\|_{C^k/Z^k} \geq \varepsilon\|f + Z^k(X,\mathcal{F})\|_{C^k/Z^k},$$

which is what we want. □

We will give a criterion for sufficiently small locally minimal cochains to expand in Section 8.

7 Locally Testable Codes and Quantum CSS Codes Arising from Sheaves

We now explain how sheaved complexes which are good cosystolic expanders give rise to locally testable codes and quantum CSS codes. We further show that if the sheaved complex in question expands small locally minimal cochains, then there is an efficient decoding algorithm.

7.1 Conventions

As usual, a code of length n on a finite alphabet Σ consists of a pair $C = (C, \Sigma^n)$ such that C is a subset of Σ^n; we often simply say that C is a code inside Σ^n. Given $f, g \in \Sigma^n$, the Hamming distance and the noramlized Hamming distance of f from g are

$$D_{\mathrm{Ham}}(f,g) = \#\{i \in \{1,\dots,n\} \,:\, f_i \neq g_i\} \qquad \text{and} \qquad d_{\mathrm{Ham}}(f,g) = \frac{1}{n} D_{\mathrm{Ham}}(f,g),$$

respectively. The *distance* of C is $\Delta(C) := \max\{D_{\mathrm{Ham}}(f,g) \,|\, f,g \in C,\, f \neq g\}$ and the *relative distance* of C is $\delta(C) := \frac{1}{n}\Delta(C) = \max\{d_{\mathrm{Ham}}(f,g) \,|\, f,g \in C,\, f \neq g\}$. The *message length* of C is $\log_{|\Sigma|}|C|$ and its *rate* is the message length divided by n, i.e., $\log_{|\Sigma^n|}|C|$.

If Σ is a finite field $\mathbb{F}$ (resp. an abelian group), then the code C is said to be linear (resp. abelian) if C is an $\mathbb{F}$-subspace (resp. subgroup) of Σ^n. In this case, $\delta(C) = \min\{\|f\|_{\mathrm{Ham}} \,|\, f \in C - \{0\}\}$, where $\|f\|_{\mathrm{Ham}} = d_{\mathrm{Ham}}(f, 0)$ is normalized Hamming norm of $f \in \Sigma^n$. In the linear case, the message length of C is $\dim_{\mathbb{F}} C$.

A family of codes $\{(C_i, \Sigma^{n_i})\}_{i\in I}$ is said to be *good* there are $\rho, \delta \in (0,1]$ such that each C_i has rate $\geq \rho$ and relative distance $\geq \delta$. When the latter holds, we also say that the codes $\{(C_i, \Sigma^{n_i})\}_{i\in I}$ have linear distance (indeed, $\Delta(C_i) \geq \delta n_i$ for all $i \in I$).

Let (C, Σ^n) be a code and let $\eta \in [0,1]$. A *decoding algorithm* for (C, Σ^n) able to to correct up to ηn errors, or an η-fraction of errors, is an algorithm which takes a word $f \in \Sigma^n$ with $d_{\mathrm{Ham}}(f, C) < \eta$ and outputs some $g \in C$ with $d_{\mathrm{Ham}}(f, g) < \eta$. If $\eta \leq \frac{1}{2}\delta(C)$, then g is uniquely determined by f.

7.2 Cocycle Codes

We use the following general notation throughout the rest of this section:

- X is a simplicial complex of dimension d,
- $\mathbb{F}$ is a finite field with q elements,
- $\mathcal{F}$ is an $\mathbb{F}$-sheaf on X,
- B is an $\mathbb{F}$-basis of $\mathcal{F}$, i.e., a collection $B = \{B(x)\}_{x \in X - \{\emptyset\}}$ such that $B(x)$ is an $\mathbb{F}$-basis of $\mathcal{F}(x)$.
- $\|\cdot\|$, $\|\cdot\|_{\mathrm{Ham}}$ and $\|\cdot\|_B$ denote the weighted support norm, the normalized Hamming norm, and the (non-normalized) Hamming norm of $\mathcal{F}$ w.r.t. B, respectively (see Examples 5.2, 5.3, 5.4).

We associate the following parameters to with the above data:

- $Q = D(X)$ and $P = D_{k,d}(X)$ (see §3.1),
- $M_k = M_k(\mathcal{F}) = \max\{\dim \mathcal{F}(x) \mid x \in X(k)\}$, and $M = M(\mathcal{F}) = \max\{M_0, \ldots, M_d\}$.

We fix a linear ordering on $V(X)$ and use it to identify $C^k := C^k(X, \mathcal{F})$ with $\prod_{x \in X(k)} \mathcal{F}(x)$; see Remark 4.5. We abbreviate $Z^k(X, \mathcal{F})$ to Z^k and $B^k(X, \mathcal{F})$ to B^k.

Let $k \in \{0, \ldots, d\}$. We use the data of $X, \mathcal{F}, B, k$ to construct a linear code as follows: The ambient space of the code will be $C^k = \prod_{x \in X(k)} \mathcal{F}(x)$, which identify with $\mathbb{F}^{\dim C^k}$ using the basis $\bigsqcup_{x \in X(k)} B(x)$, and the set of code words will be Z^k. Thus, $(Z^k, C^k \cong \mathbb{F}^{\dim C^k})$ is a linear code with alphabet $\mathbb{F}$.

Definition 7.1. *The linear code* $(Z^k(X, \mathcal{F}), C^k(X, \mathcal{F}) \cong \mathbb{F}^{\dim C^k})$ *is the* linear k-cocycle code *of* $(X, \mathcal{F}, B)$.

If there exists $m \in \mathbb{N}$ such that $\dim \mathcal{F}(x) = m$ for all $x \in X(k)$, then we may identify $\mathcal{F}(x)$ with $\Sigma := \mathbb{F}^m$ for every $x \in X(k)$, so that $C^k = \Sigma^{X(k)}$. This allows us to view Z^k as an abelian code inside $\Sigma^{X(k)}$ (rather than $\mathbb{F}^{\dim C^k}$), the alphabet being Σ.

Definition 7.2. *The abelian code* $(Z^k(X, \mathcal{F}), C^k(X, \mathcal{F}) \cong \Sigma^{X(k)})$ *is the* k-cocycle code *of* $(X, \mathcal{F})$.

Henceforth, whenever we refer to the k-cocycle code of $(X, \mathcal{F})$, we tacitly assume that there exists $m \in \mathbb{N}$ such that $\dim \mathcal{F}(x) = m$ for all $x \in X(k)$.

The rate of the k-cocycle code (Z^k, C^k) is $\dim Z^k / \dim C^k$; this is independent of whether we view (Z^k, C^k) as a linear code with alphabet $\mathbb{F}$, or an abelian code with alphabet Σ. Since B^k typically contains k-cochains with small support, the distance of (Z^k, C^k) is poor unless $B^k = 0$, e.g., if $k = 0$, or $\mathcal{F}(x) = 0$ for all $x \in X(k-1)$.

7.3 Locally Testable Codes

Let $C = (C, \Sigma^n)$ be a code on a finite alphabet Σ. Recall that a randomized algorithm Φ which takes a word $f \in \Sigma^n$ and decides whether to accept or reject f is called a *t-query μ-tester* ($t \in \mathbb{N}$, $\mu \in \mathbb{R}_+$) if Φ queries up to t letters from from f, accepts all words $f \in C$, and the probability of rejecting a word $g \in \Sigma^n - C$ is at least $\mu d_{\mathrm{Ham}}(g, C)$. In this case, we call (C, Σ^n, Φ) a *t-query μ-testable* code. We also say that (C, Σ^n, Φ) is a *code with a tester* if we wish to make no reference to t and μ.

A family of codes with testers $(C_i, \Sigma^{n_i}, \Phi_i)_{i \in I}$ is a family of *locally testable codes* (LTCs) if there are $t \in \mathbb{N}$ and $\mu \in \mathbb{R}_+$ such that each $(C_i, \Sigma^{n_i}, \Phi_i)$ is a t-query μ-testable code.

Remark 7.3. The quality of a tester Φ for a code (C, Σ^n) can also be measured by means of *soundness* and *tolerance*. Recall that Φ, or the triple (C, Σ^n, Φ), is said to have *c-soundness* at least ε ($c \in [0, \frac{1}{2}]$, $\varepsilon \in [0, 1]$) if Φ rejects an $f \in \Sigma^n$ satisfying $d_{\mathrm{Ham}}(f, C) > c \operatorname{dist}(C)$ with probability at least ε. It has *c-tolerance* at least ε if Φ accepts an $f \in \Sigma^n$ satisfying $d_{\mathrm{Ham}}(f, C) \le c \operatorname{dist}(C)$ with probability at least ε. Thus, a t-query μ-testable code of relative distance δ has c-soundness $\ge c\delta\mu$ and 0-tolerance 1 (i.e., the tester is perfect).

Keeping the notation of §7.2, we fix $k \in \{0, \dots, d-1\}$, and assume throughout that there exists $m \in \mathbb{N}$ such that $\dim \mathcal{F}(x) = m$ for all $x \in X(k)$. We further let $\Sigma = \mathbb{F}^m$.

The k-cocycle code of $(X, \mathcal{F})$, namely $(Z^k, C^k \cong \Sigma^{X(k)})$, admits a natural $(k+2)$-query tester $\Phi = \Phi(X, \mathcal{F}, k)$: given $f \in C^k$, choose $y \in X(k+1)$ uniformly at random[6] and accept f if $\sum_{x \in X(k)_{\subseteq y}} \operatorname{res}_{y \leftarrow x} f(x) = 0$.

Definition 7.4. *The triple (Z^k, C^k, Φ) is called the k-*cocycle code-with-tester *of $(X, \mathcal{F})$.*

When there is no risk of confusion, we will refer to (Z^k, C^k, Φ) simply as the k-cocycle code $(X, \mathcal{F})$.

Remark 7.5. The tester Φ can also be viewed as a tester for the linear k-cocycle code of $(X, \mathcal{F}, B)$. From this point of view, Φ queries $(k+2)M_k$ letters. One can use this observation to adapt the following discussion to linear k-cocycle codes.

The distance and the testability of (Z^k, C^k, Φ) are tightly related to the coboundary expansion of $(X, \mathcal{F}, \|\cdot\|_{\mathrm{Ham}})$, where $\|\cdot\|_{\mathrm{Ham}}$ is the normalized Hamming norm of $\mathcal{F}$ (Example 5.3). This is expressed in the following proposition, which is immediate from the definitions, see §5.2–5.3.

Proposition 7.6. *With notation as in §7.2, suppose that $\dim \mathcal{F}(x) = m$ for all $x \in X(k)$. Let (Z^k, C^k, Φ) be the k-cocycle code on the alphabet $\Sigma = \mathbb{F}^m$ associated to $(X, \mathcal{F})$, and let $\varepsilon, \delta \in \mathbb{R}_+$. Suppose that $B^k = 0$ (e.g., if $k = 0$) and $\mathcal{F}(x) \neq 0$ for all $x \in X(k) \cup X(k+1)$. Then the following conditions are equivalent:*

(a) $(X, \mathcal{F}, \|\cdot\|_{\mathrm{Ham}})$ is an (ε, δ)-coboundary expander in dimension k;

(b) (Z^k, C^k, Φ) is a $(k+2)$-query ε-testable code with relative distance $\ge \delta$.

We can also relate the distance and testability of (Z^k, C^k, Φ) to the coboundary expansion of $(X, \mathcal{F})$ relative to the weighted support norm $\|\cdot\|$. We loose a factor of $P = D_{k,d}(X)$ in the process.

[6] For the applications considered in this paper, it is better to choose $y \in X(k+1)$ according to the distribution induced by the canonical weight function of X (§3.2). Using this distribution improves the testability by a factor of $Q = D(X)$ in Proposition 7.7, but makes statements more cumbersome elsewhere, so we stick to the uniform distribution.

Proposition 7.7. *Keep the assumptions of Proposition 7.6 and suppose further that X is a d-complex. If $(X,\mathcal{F})$ is an (ε,δ)-coboundary expander in dimension k, then (Z^k,C^k,Φ) a $(k+1)$-query $\frac{\varepsilon}{P^2}$-testable code with relative distance $\geq \frac{\delta}{P}$. Conversely, if (Z^k,C^k,Φ) a $(k+1)$-query ε-testable code with relative distance δ, then $(X,\mathcal{F})$ is an $(\frac{\varepsilon}{P^2},\frac{\delta}{P})$-coboundary expander.*

Proof. This follows from Remark 5.8(ii), Proposition 5.6(i) and Proposition 7.6. □

We have seen in Proposition 6.4 that if $(X,\mathcal{F})$ expands small locally minimal k-cocycles and $(k+1)$-cocycles, then $(X,\mathcal{F})$ is a good coboundary expander in dimension k, and so the associated k-cocycle code (Z^k,C^k,Φ) is an LTC. We now show that these stronger assumptions also guarantee that (Z^k,C^k,Φ) has a linear-time decoding algorithm. The weaker assumption that small locally minimal k-cocycles expand is enough to bound the distance of (Z^k,C^k,Φ) from below.

Proposition 7.8. *With notation as in §7.2, suppose that X is a d-complex, $\dim\mathcal{F}(x)=m$ for all $x\in X(k)$ and $\mathcal{F}(x)\neq 0$ for all $x\in X(k)\cup X(k+1)\cup X(k+2)$. Let (Z^k,C^k,Φ) be the k-cocycle code of $(X,\mathcal{F})$, put $n=|X(k)|$ (the length of the code) and let $\beta,\beta',\gamma,\gamma'\in\mathbb{R}_+$.*

(i) If $B^k=0$ and $(X,\mathcal{F})$ β-expands γ-small locally minimal k-cocycles, then $\delta(Z^k)\geq\frac{\gamma}{P}$.

(ii) If, in addition, $(X,\mathcal{F})$ β'-expands γ'-small locally minimal $(k+1)$-cocycles, then (Z^k,C^k,Φ) is $\frac{1}{P^2}\min\{\frac{d+1}{(k+1)Q}\binom{d+1}{k+2}^{-1},\gamma'\}$-testable and has a decoding algorithm able to correct up to $\frac{1}{(k+2)P}\min\{(\frac{(k+1)Q}{d+1}\binom{d+1}{k+2}+1)^{-1}\gamma,\gamma'\}$-fraction of errors in $O(2^{\binom{d+1}{k+2}Q}Q^4M_{k+1}^2 m\cdot n)=O_{M,Q,d,k}(n)$ operations.

Here and elsewhere in this work, we assume that operations in $\mathbb{F}$ are performed in time $O(1)$. Elements of $\mathcal{F}(x)$ are represented as vectors in $\mathbb{F}^{B(x)}$ via the basis $B(x)$, and the restriction maps $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}:\mathcal{F}(x)\to\mathcal{F}(y)$ are represented by the corresponding $B(y)\times B(x)$-indexed matrix. The complexity of evaluating $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ therefore depends on the number of nonzero entries in this matrix, which is $O(M_{\dim x}M_{\dim y})$. Recall that our assumptions imply $M_k=m$.

Proof. (i) Let $f\in Z^k-\{0\}=Z^k-B^k$. By Proposition 6.4, $\|f\|\geq\gamma$, and by Proposition 5.6(i), this means that $\|f\|_{\mathrm{Ham}}\geq\binom{d+1}{k+1}\frac{|X(d)|}{|X(k)|}P^{-1}\gamma\geq\frac{\gamma}{P}$.

(ii) By Proposition 6.4(i), $(X,\mathcal{F})$ is a $(\min\{\frac{d+1}{(k+1)Q}\binom{d+1}{k+2}^{-1},\gamma'\},\gamma)$-cosystolic expander. The assertion about the testability is therefore a consequence of Proposition 7.7. It remains to show that (Z^k,C^k) has a decoding algorithm as claimed.

Write $\eta:=\frac{1}{(k+2)P}\min\{(\frac{(k+1)Q}{d+1}\binom{d+1}{k+2}+1)^{-1}\gamma,\gamma'\}$. Let $f\in C^k$ and assume that there is $f_0\in Z^k$ with $\|f-f_0\|_{\mathrm{Ham}}<\eta$. We claim that the output of the algorithm in Figure I.1 is f_0.

To see this, observe that by Proposition 5.6(i), $\|f-f_0\|<\binom{d+1}{k+1}^{-1}P\frac{|X(k)|}{|X(d)|}\eta\leq P\eta$. Since $f_0\in Z^k$, this means that

$$\begin{aligned}\|d_kf\|=\|d_k(f-f_0)\|&\leq\sum_{x\in\mathrm{supp}(f-f_0)}w(X(k+1)_{\supseteq x})=\sum_{x\in\mathrm{supp}(f-f_0)}(k+2)w(x)\\&=(k+2)\|f-f_0\|<(k+2)P\eta\leq\gamma',\end{aligned}$$

where in the second equality we used (3.2). By applying Lemma 6.3 to d_kf, we get $g\in C^k$ such that d_kf-d_kg is locally minimal, $\|g\|\leq\frac{(k+1)Q}{d+1}\binom{d+1}{k+2}\|d_kf\|<\frac{(k+1)Q}{d+1}\binom{d+1}{k+2}(k+2)P\eta$ and $\|d_kf-d_kg\|\leq\|d_kf\|<\gamma'$. Moreover, by comparing the proof of Lemma 6.3 with the algorithm in Figure I.1, we see that the output f' is in fact $f-g$ (for a suitable choice of vertices in the proof).

Figure I.1: Decoding Algorithm

1. $f' \leftarrow f$
2. $L \leftarrow$ `empty queue`
3. $B \leftarrow$ `boolean array indexed by` $X(0)$
4. `For each` $z \in X(0)$`:`
   (4a) L`.push(`z`)`
   (4b) $B[z] \leftarrow$ `True //` z `is in` L
5. `While` L `is not empty:`
   (5a) $z \leftarrow L$`.pop(), order the vertices of` z `arbitrarily`
   (5b) $B[z] \leftarrow$ `False //` z `is not in` L
   (5c) `Search for` $h \in C^{k-1}(X_z, \mathcal{F}_z)$ `with` $\|(d_k f')_z - d_{k-1}h\| < \|(d_k f')_z\|$`; set` $h = 0$ `if there is no such` h`.`
   (5d) `If` $h \neq 0$`:`
      i. $f' \leftarrow f' - h^z$`.`
      ii. `For every` $z' \in X(0)$ `adjacent to` z `with` $B[z'] =$ `False:`
         A. L`.push(`z'`)`
         B. $B[z'] \leftarrow$ `True //` z' `is in` L
6. `Return` f'`.`

Since $d_k f - d_k g$ is γ'-small and locally minimal, we have $0 = \|d_k(d_k f - d_k g)\| \geq \beta' \|d_k f - d_k g\|$, so $d_k f - d_k g = 0$, and it follows that $f' = f - g \in Z^k$. Now,

$$\|f' - f_0\| \leq \|f - f'\| + \|f - f_0\| = \|g\| + \|f - f_0\| < (k+2)P\eta \cdot (\frac{(k+1)Q}{d+1}\tbinom{d+1}{k+2} + 1) \leq \gamma.$$

Since $f' - f_0 \in Z^k$ (because $f', f_0 \in Z_k$) and $(X, \mathcal{F})$ β-expands γ-small cochains, Proposition 6.4 tells us that $f' - f_0 \in B^k = 0$, so $f' = f_0$, as required.

We proceed with analyzing the time complexity of the algorithm in Figure I.1. The proof of Lemma 6.3 tells us that the loop `(5)` cannot be executed more than $\binom{d+1}{k+2}|X(d)| = O(|X(k)|P) = O(Qn)$ times (see (3.1)). In order to perform the instruction `(5c)`, we have to enumerate on $O(q^{Qm})$ possible h-s. For each h, the computation of $\|(d_k f')_z - d_{k-1}h\|$ takes $O(QmM_{k+1})$ operations, so naively, `(5c)` requires $O(q^{Qm}QmM_{k+1})$ operations. However, it is better to enumerate on subsets $E \subseteq X_z(k)$ and look for some h such that $(d_k f')_z - d_{k-1}h$ vanishes on E by solving the linear system of equations $\{(d_{k-1}h)(x) = (d_k f')(x \cup z)\}_{x \in E}$. This allows us to perform `(5c)` by solving at most $2^{|X_z(k)|} \leq 2^{D_{1,k+1}(X)} \leq 2^{\binom{d+1}{k+2}Q}$ systems of at most $M_{k+1}|X_z(k)| = O(M_{k+1}Q)$ linear equations in at most $m|X_z(k-1)| = O(mQ)$ variables, which amounts to $O(2^{\binom{d+1}{k+2}Q}Q^3M_{k+1}^2 m)$ operations. The remaining actions inside the loop `(5)` are negligible by comparison, so the instructions `(5a)`–`(5d)` require $O(2^{\binom{d+1}{k+2}Q}Q^3M_{k+1}^2 m)$ operations. The total time complexity of the algorithm is therefore $O(2^{\binom{d+1}{k+2}Q}Q^4M_{k+1}^2 m \cdot n)$. □

7.4 Quantum CSS codes

We proceed with explaining how sheaved complexes give rise to quantum CSS codes. We refer the reader to [LLZ21, §1, §2.2, §2.3, Lemma 13] for a survey of these codes and their significance to quantum computing.

For every $m \in \mathbb{N}$, we endow $\mathbb{F}^m$ with the standard symmetric bilinear form $\langle f, g\rangle = \sum_{i=1}^m f_i g_i$, where f_i is the i-th coordinate of $f \in \mathbb{F}^m$. Given $A \subseteq \mathbb{F}^m$, we write $A^\perp = \{f \in \mathbb{F}^m : \langle f, A\rangle = 0\}$.

For the purposes of this work it is convenient to define a *quantum CSS code* as a quintet $C = (C_X, C_Z, \mathbb{F}^n, \Phi_X, \Phi_Z)$ such that C_X and C_Z are subspaces of $\mathbb{F}^n$, Φ_X is a set of vectors generating $C_X^\perp$, Φ_Z is a set of vectors generating $C_Z^\perp$, and $C_X^\perp \subseteq C_Z$ (equivalently, $C_Z^\perp \subseteq C_X$). In particular, C_X and C_Z are linear codes inside $\mathbb{F}^n$, the alphabet being $\mathbb{F}$. The rate of C is $\frac{1}{n}(\dim C_X - \dim C_Z^\perp) = \frac{1}{n}(\dim C_Z - \dim C_X^\perp)$ and its relative distance is $\min\{d_X, d_Z\}$, where $d_X = \min\{\|w\|_{\mathrm{Ham}} \mid w \in C_X - C_Z^\perp\}$ and $d_Z = \min\{\|w\|_{\mathrm{Ham}} \mid w \in C_Z - C_X^\perp\}$; we call d_X and d_Z the relative X- and Z-distance, respectively. (The distance and message length of C are obtained from the relative distance and rate by multiplying by n, respectively.)

Given $\eta \in [0,1]$, a decoding algorithm for the X-side of $C = (C_X, C_Z, \mathbb{F}^n, \Phi_X, \Phi_Z)$ able to correct up to an η-fraction of errors is a decoding for $(C_X, \mathbb{F}^n)$ able to correct to up to η-fraction of errors. Note that if $f \in \mathbb{F}^n$ satisfies $d_{\mathrm{Ham}}(f, C_X) < \eta$, then there could be numerous $x \in C_X$ with $d_{\mathrm{Ham}}(f, x) < \eta$, because $C_Z^\perp$ may (and often does) contain short vectors. However, if 2η is smaller than the relative X-distance, then the coset $x + C_Z^\perp$ is uniquely determined.

We use the generating set Φ_X to define a natural tester for the linear code $C_X \subseteq \mathbb{F}^n$: given $f \in C_X$, choose $\phi \in \Phi_X$ uniformly at random, and accept f if $\langle f, \phi\rangle = 0$. Abusing the notation, we denote this tester by Φ_X. Likewise, we use the set Φ_Z to define a tester for $C_Z \subseteq \mathbb{F}^n$. Let $q \in \mathbb{N}$ and $\mu \in \mathbb{R}_+$. We say that $C = (C_X, C_Z, \mathbb{F}^n, \Phi_X, \Phi_Z)$ is a *q-query μ-testable* quantum CSS code, if this holds for both linear codes $(C_X, \mathbb{F}^n, \Phi_X)$ and $(C_Z, \mathbb{F}^n, \Phi_Z)$. If this holds only for $(C_X, \mathbb{F}^n, \Phi_X)$, we say that C is *one-sided* q-query μ-testable quantum CSS code.

Let $X, \mathcal{F}, B, d$ be as in §7.2, and let $k \in \{1, \dots, d-1\}$. Write $n = \dim C^k$ and identify $C^k = C^k(X, \mathcal{F})$ with $\mathbb{F}^n$ via the basis $\bigsqcup_{x \in X(k)} B(x)$. Then $Z^k = Z^k(X, \mathcal{F})$ is a linear code inside $\mathbb{F}^n$. We enrich Z^k into a quantum CSS code as follows.

Put $C_X = Z^k$. For every $x \in X - \{\emptyset\}$, we identify $\mathcal{F}(x)$ with $\mathbb{F}^{B(x)}$ via the basis $B(x)$. Under this identification, the standard bilinear form on $\mathbb{F}^{B(x)}$ corresponds to a nondegnerate bilinear form on $\mathcal{F}(x) \times \mathcal{F}(x) \to \mathbb{F}$, denoted $\langle \cdot, \cdot \rangle_x$. The standard bilinear form on $C^k = \mathbb{F}^n$ can now be written as $\langle f, g\rangle = \sum_{x \in X(k)} \langle f(x), g(x)\rangle_x$ for $f, g \in C^k = \prod_{x \in X(k)} \mathcal{F}(x)$. For every $\emptyset \neq x \subsetneq y \in X$, define $\mathrm{res}'_{x \leftarrow y} : \mathcal{F}(y) \to \mathcal{F}(x)$ to be the dual of $\mathrm{res}_{y \leftarrow x} : \mathcal{F}(x) \to \mathcal{F}(y)$ relative to the bilinear pairings on $\mathcal{F}(x)$ and $\mathcal{F}(y)$, that is, $\mathrm{res}'_{y \leftarrow x}$ is determined by the condition $\left\langle \mathrm{res}'_{y \leftarrow x} f, g \right\rangle_y = \langle f, \mathrm{res}_{x \leftarrow y} g\rangle_x$ for all $f \in \mathcal{F}(x)$, $g \in \mathcal{F}(y)$. For $i \in \{1, \dots, d\}$, the i-th *boundary* map $\partial_i : C^i \to C^{i-1}$ is defined by

$$(\partial_i f)(y) = \sum_{x \in X(i)_{\supseteq y}} [x : y]_L \, \mathrm{res}'_{y \leftarrow x} f,$$

for all $f \in C^k$ and $y \in X(i-1)$; the coefficient $[x : y]_L$ is as in Remark 4.5.[7] One readily checks that ∂_i is the dual of $d_{i-1} : C^{i-1} \to C^i$ relative to the bilinear pairings on these vector spaces, i.e.,

$$\langle \partial_i f, g\rangle = \langle f, d_{i-1} g\rangle \tag{7.1}$$

[7] If one does not wish to choose a linear ordering on $V(X)$ and identify $C^i(X, \mathcal{F})$ with $\prod_{x \in X(i)} \mathcal{F}(x)$ as in Remark 4.5, then the formula is given by $(\partial_i f)(y) = \sum_{z \in X(i)_y} \mathrm{res}'_{y \leftarrow x} f(yz)$, where $y \in X_{\mathrm{ord}}(i-1)$.

for all $f \in C^i$, $g \in C^{i-1}$. Since $d_i d_{i-1} = 0$, we have $\partial_{i-1}\partial_i = 0$, with the convention that $\partial_0 = 0$. As expected, the *k-cycles* and *k-boundaries* with coefficients in $\mathcal{F}$ are defined to be the subspaces of C^i given by

$$Z_i = Z_i(X, \mathcal{F}; B) = \ker d_i \qquad \text{and} \qquad B_i = B_i(X, \mathcal{F}; B) = \operatorname{im} d_{i-1},$$

respectively. Set $C_Z = Z_k$.

We now define subsets $\Phi_X \subseteq (Z^k)^\perp$ and $\Phi_Z \subseteq (Z_k)^\perp$ as follows. For every $y \in X(k+1)$ and $b \in B(y)$, define $\phi_{y,b} \in C^k$ to be the unique vector for which $\langle f, \phi_{y,b}\rangle = \langle d_k f(y), b\rangle$ for all $f \in C^k$, and let Φ_X be the set of all $\phi_{y,b}$. Similarly, for every $z \in X(k-1)$ and $b \in B(z)$, define $\phi'_{z,b} \in C^k$ to be the unique vector for which $\left\langle f, \phi'_{z,b}\right\rangle = \langle \partial_k f(z), b\rangle$ for all $f \in C^k$, and let Φ_Z be the set of all $\phi'_{z,b}$. It is clear that Φ_X generates $(Z^k)^\perp$ and Φ_Z generates $(Z_k)^\perp$. The following lemma says that $C_X^\perp \subseteq C_Z$, and thus $C := (C_X, C_Z, C^k, \Phi_X, \Phi_Z)$ is a quantum CSS code.

Lemma 7.9. *In the previous notation, $(Z^k)^\perp = B_k$ and $(Z_k)^\perp = B^k$.*

Proof. It is enough to prove that $B_k^\perp = Z^k$ and $(B^k)^\perp = Z_k$. We have $f \in B_k^\perp$ if and only if $\langle f, \partial_{k+1} g\rangle = 0$ for all $g \in C^{k+1}$, or equivalently $\langle d_k f, g\rangle = 0$ for all $g \in C^{k+1}$. Since the bilinear form on C^{k+1} is nondegenerate, the latter is equivalent to $d_k f = 0$. This proves that $B_k^\perp = Z^k$. The equality $(B^k)^\perp = Z_k$ is shown similarly. $\square$

Definition 7.10. *We call $(C_X = Z^k, C_Z = Z_k, C^k, \Phi_X, \Phi_Z)$ defined above the k-*cocycle quantum CSS code *associated to $(X, \mathcal{F}, B)$.*

At this point, it is convenient to introduce an analogue of cosystolic expansion which uses boundary maps instead of coboundary maps. Let $\varepsilon, \delta \in \mathbb{R}_+$ and let $\|\cdot\|$ be a norm on $\mathcal{F}$ with mass function m (see §5.2). We say that $(X, \mathcal{F}, \|\cdot\|)$ is an *(ε, δ)-systolic expander*, if

(S1) $\|\partial_k f\| m(k) \geq \varepsilon \|f + Z_k(X, \mathcal{F})\|_{C^k/Z_k} m(k-1)$ for all $f \in C^k(X, \mathcal{F})$, and

(S2) $\|f\| \geq \delta m(k)$ for all $f \in Z_k(X, \mathcal{F}) - B_k(X, \mathcal{F})$.

The following proposition relates the cosystolic and systolic expansion of $(X, \mathcal{F}, \|\cdot\|_B)$ to the code-theoretic properties of the k-cocycle quantum CSS code associated to $(X, \mathcal{F}, B)$. Adapting the result to use the weight-support norm instead of $\|\cdot\|_B$ can be done using Proposition 5.6(ii) and Remark 5.8(ii), and is left to the reader.

Proposition 7.11. *Let $X, \mathcal{F}, B, d, Q, P$ be as in §7.2, let $k \in \{1, \dots, d-1\}$ and let $C := (C_X, C_Z, C^k, \Phi_X, \Phi_Z)$ be the k-cocycle quantum CSS code associated to $(X, \mathcal{F}, B)$. Then:*

(i) C is a quantum CSS code, its rate is $\frac{1}{n}\dim \mathrm{H}^k(X, \mathcal{F})$, and the testers Φ_X and Φ_Z query $(k+2)M_k$ and $D_{k-1,k}(X)(d+1-k)M_k$ letters, respectively.

(ii) $(X, \mathcal{F}, \|\cdot\|_B)$ is an (ε, δ)-cosystolic expander and an (ε, δ)-systolic expander in dimension k if and only if C has relative distance $\geq \delta$ and is ε-testable.

(iii) $(X, \mathcal{F}, \|\cdot\|_B)$ is an (ε, δ)-cosystolic expander in dimension k if and only if C has relative X-distance $\geq \delta$ and is one-sided ε-locally testable.

(iv) Suppose that $\mathcal{F}(x) \neq 0$ for all $x \in X(k) \cup X(k+1) \cup X(k+2)$ and there are $\beta, \beta', \gamma, \gamma' \in \mathbb{R}_+$ such that $(X, \mathcal{F})$ β-expands γ-small k-cocycles and β'-expands γ'-small $(k+1)$-cocycles. Then the X-side of C admits an error correcting algorithm able to correct up to $\frac{1}{(k+2)PM_k}\min\{(\frac{(k+1)Q}{d+1}\binom{d+1}{k+2}+1)^{-1}\gamma, \gamma'\}$-fraction of errors in $O_{M,Q,d,k}(n)$ operations.

Proof. (i) It is straightforward to see that every $\phi_{y,b} \in \Phi_X$ is supported on at most $(k+2)M_k$ coordinates. Let $z \in X(k-1)$. Then z is contained in at most $D_{k-1,k}(X)$ d-faces. This means that every $\phi'_{z,b} \in \Phi_Z$ is supported at most $D_{k-1,k}(X)M_k$ coordinates. The assertion about the rate follows from Lemma 7.9.

(ii) and (iii) are immediate from the definitions.

(iv) is shown as in Proposition 7.8 with two differences. First, we use part (ii) of Proposition 5.6 instead of part (i). In particular, writing $\eta := \frac{1}{(k+2)PM_k}\min\{(\frac{(k+1)Q}{d+1}\binom{d+1}{k+2}+1)^{-1}\gamma, \gamma'\}$, the assumption $\|f-f_0\|_B \leq \eta n \leq \eta M_k|X(k)|$ gives $\|f-f_0\| \leq \binom{d+1}{k+1}^{-1}\frac{|X(k)|}{|X(d)|}PM_k\eta \leq PM_k\eta$, and extra M_k is carried throughout the computations. Second, instead of asserting that $f' = f_0$, we only conclude that $f_0 - f' \in B^k \subseteq (Z_k)^\perp = C_Z^\perp$. $\square$

The subject matter of Section 8 is the construction of cosystolic expanders, which in turn give rise to LTCs. Unfortunately, we do not know of analogous results for *systolic expansion*, so our results only give rise to quantum CSS codes whose X-side is an LTC with linear distance, and *a priori* no information on the Z-side.

Chapter II

The Tower Paradigm

In this chapter we provide a method for constructing good infinite families of LTCs from a single sheaved complex $(X, \mathcal{F})$, called the *tower paradigm*. Broadly, the idea is to assume that X admits an infinite tower of double coverings $\cdots \to X_2 \to X_1 \to X_0 = X$, and take the cocycle codes associated to the sheaved complexes $\{(X_r, u_r^*\mathcal{F})\}_{r=0}^{\infty}$, where $u_r : X_r \to X$ is the composition $X_r \to X_{r-1} \to \cdots \to X_0 = X$. We show that if $(X, \mathcal{F})$ satisfies a list of conditions, then this family of codes is a good family of LTCs. Specifically, the main result of Section 8 — a local-to-global principle for cosystolic expansion of sheaves — provides the conditions on $(X, \mathcal{F})$ that would secure linear distance and testability for the codes, and the main result of Section 10 — rate conservation — gives conditions on $(X, \mathcal{F})$ that are sufficient for the rate of the codes to be constant. We pack these results together in Section 11 to give the tower paradigm. The intermediate Section 9 gives examples of cocycle codes to which the local-to-global principle of Section 8 can be applied.

8 A Local-to-Global Principle for Cosystolic Expansion

Let $(X, \mathcal{F})$ be a sheaved connected d-complex. The purpose of this section is to prove the following theorem, which provides *local* condition on $(X, \mathcal{F})$ guaranteeing that $(X, \mathcal{F})$ is a good cosystolic expander in dimension k (see §5.3), and moreover, expands small locally minimal k-cochains and $(k+1)$-cochains (see §6.2). By saying that the conditions are local we mean that they involve only the links $(X_z, \mathcal{F}_z)$ with $z \in X - \{\emptyset\}$.

Following the convention set in §4.4, we say that $(X, \mathcal{F})$ is an *i-local ε-coboundary expander* in dimension k ($-1 \leq i \leq k$) if $(X_z, \mathcal{F}_z)$ is an ε-coboundary expander in dimension $k - i - 1$ for all $z \in X(i)$. Recall also (§3.4) that X is said to be an i-local $[-1, \lambda]$-spectral expander ($-1 \leq i \leq d-2$) if the underlying weighted graph of X_z is a $[-1, \lambda]$-spectral expander for all $z \in X(i)$. Likewise for α-skeleton expansion.

Theorem 8.1. *Let $k \in \mathbb{N} \cup \{0\}$, $Q \in \mathbb{N}$, $\varepsilon_0, \ldots, \varepsilon_k, \varepsilon'_0, \ldots, \varepsilon'_{k+1}$ and $\lambda \in \mathbb{R}_+$. Put*

$$\varepsilon = \min\left\{\frac{(k+2)\varepsilon_i}{k+1-i}\middle| i \in \{0, \ldots, k\}\right\} \quad \text{and} \quad \varepsilon' = \min\left\{\frac{(k+3)\varepsilon'_i}{k+2-i}\middle| i \in \{0, \ldots, k+1\}\right\},$$

and suppose that

$$\lambda \leq \tfrac{1}{d}\min\{(\tfrac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^k}, (\tfrac{\varepsilon'}{(k+2)^2 2^{2k+8}})^{2^{k+1}}, 1\}$$

Let $(X, \mathcal{F})$ be a sheaved strongly connected *d-complex with $d \geq k+2$ such that:*

(1) $(X, \mathcal{F})$ is an i-local ε_i-coboundary expander in dimension k for all $i \in \{0, \ldots, k\}$.

(2) $(X,\mathcal{F})$ *is an* i*-local* ε'_i*-coboundary expander in dimension* $k+1$ *for all* $i \in \{0,\dots,k+1\}$*.*

(3) X *is a* $(d-2)$*-local* $[-1,\lambda]$*-spectral expander.*

(4) $D(X) \le Q$*, i.e., every vertex of* X *is belongs to at most* Q d*-faces,*

(5) $\mathcal{F}(x) \neq 0$ *for all* $x \in X(k) \cup X(k+1) \cup X(k+2)$*.*

Then:

(i) $(X,\mathcal{F})$ $\frac{\varepsilon}{2}$*-expands* $(\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^{k+1}-1}$*-small locally minimal* k*-cochains.*

(ii) $(X,\mathcal{F})$ $\frac{\varepsilon'}{2}$*-expands* $(\frac{\varepsilon'}{(k+2)^2 2^{2k+8}})^{2^{k+2}-1}$*-small locally minimal* $(k+1)$*-cochains.*

(iii) $(X,\mathcal{F})$ *is a* $(\min\{(\frac{\varepsilon'}{(k+2)^2 2^{2k+8}})^{2^{k+2}-1}, \frac{d+1}{(k+1)Q}\binom{d+1}{k+2}^{-1}\}, (\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^{k+1}-1})$*-cosystolic expander in dimension* k*.*

This theorem is an immediate consequence of the following theorem and Proposition 6.4.

Theorem 8.2. *Let* $k \in \mathbb{N} \cup \{0\}$*, let* $\varepsilon_0,\dots,\varepsilon_k \in \mathbb{R}_+$*, and set*

$$\varepsilon := \min\left\{ \frac{(k+2)\varepsilon_i}{k+1-i} \middle| i \in \{0,\dots,k\} \right\} \qquad \text{and} \qquad \lambda = \frac{1}{d}\min\left\{ \left(\frac{\varepsilon}{(k+1)^2 2^{2k+6}}\right)^{2^k}, 1 \right\}.$$

Let $(X,\mathcal{F})$ *be a sheaved* strongly connected d*-complex with* $d \ge k+1$ *such that:*

(1) $(X,\mathcal{F})$ *is an* i*-local* ε_i*-coboundary expander in dimension* k *for all* $i \in \{0,\dots,k\}$*.*

(2) X *is a* $(d-2)$*-local* $[-1,\lambda]$*-spectral expander.*

(3) $\mathcal{F}(x) \neq 0$ *for all* $x \in X(k+1)$*.*

Then $(X,\mathcal{F})$ $\frac{\varepsilon}{2}$*-expands* $(\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^{k+1}-1}$*-small* k*-cochains.*

Before setting to prove Theorem 8.2, a few remarks are in order.

Remark 8.3. (i) Theorems 8.2 and 8.1 were proved for the constant sheaf $\mathbb{F}_2$ (Example 4.1(ii)) in [EK17], with different constants, and this was extended to all constant sheaves in [KM18]. As in these sources, the proof of Theorem 8.2 uses machinery of *heavy faces*[1]. However, we use a different summation argument that makes milder assumptions and gives explicit and asymptotically better expansion constants.

(ii) By Lemma 5.10, the value of ε (resp. ε') in Theorems 8.1 and 8.2 cannot exceed $k+2$ (resp. $k+3$). We do not know if there exist examples approaching this upper bound.

(iii) In Theorem 8.2, condition (2) and the assumption that X is connected can be replaced with the milder assumption that X is an i-local α_i-skeleton expander for for all $i \in \{-1,\dots,k-1\}$ and $\alpha_i \le \Theta_k(\varepsilon^{2^{k-1-i}})$. Condition (3) of Theorem 8.1 can be similarly relaxed. See Theorem 8.11 and the following corollaries for the precise requirements on the α_i, and Corollary 8.15 for a version of Theorem 8.1 where the upper bounds on the α_i are optimized for $k=0$. Our methods cannot increase the order magnitude of the required i-local skeleton expansion of X (as a function of $\varepsilon_0,\dots,\varepsilon_k$) beyond $\Theta_k(\varepsilon^{2^{k-1-i}})$; see Remark 8.17.

[1] Called *fat faces* in [EK17] and [KM18].

(iv) Let $(X, \mathcal{F})$ be a sheaved strongly connected d-complex and let $p : Y \to X$ be a covering map such that Y is connected (and hence strongly connected). Recall (§4.3) that $p^*\mathcal{F}$ denotes the pullback of $\mathcal{F}$ to Y. Then, for every $y \in Y - \{\emptyset\}$, the map p restricts to an isomorphism $Y_y \cong X_{f(y)}$, and under this isomorphism we have $(p^*\mathcal{F})_y = \mathcal{F}_{f(y)}$. This means that the assumptions of Theorem 8.1 (resp. Theorem 8.2) hold for $(X, \mathcal{F})$ if and only if they hold for $(Y, p^*\mathcal{F})$.

The proof of Theorem 8.2 will be given in §8.2, after some after some preliminary results have been established in §8.1. Examples of sheaved complexes satisfying the assumptions of Theorem 8.1 are given in Section 9, and further examples will be given in Section 14.

8.1 Heavy Faces

Fix a sheaved d-complex $(X, \mathcal{F})$. Unless indicated otherwise, $k \in \{0, \dots, d\}$, $f \in C^k(X, \mathcal{F})$ and $\vec{h} = (h_{-1}, h_0, \dots, h_{k-1}) \in (0, 1]^{\{-1,\dots,k-1\}}$. Recall from §3.2 that $w = w_X : X \to \mathbb{R}_+$ denotes the canonical weight function of X.

Generalizing [KM18, §3.3] and [EK17, §3.2], we define for every $i \in \{-1, \dots, k\}$ a set $A_k(f, \vec{h}) \subseteq X(k)$ as follows: Set $A_k(f, \vec{h}) = \operatorname{supp} f$. Assuming $A_i(f, \vec{h})$ was defined, let $A_{i-1}(f, \vec{h})$ consist of the faces $x \in X(i-1)$ such that

$$w(A_i(f, \vec{h})_{\supseteq x}) \geq h_{i-1} w(X(i)_{\supseteq x}).$$

In other words, $x \in A_{i-1}(f, \vec{h})$ if at least h_{i-1}-fraction of the i-faces containing x (counted by weight) are in $A_i(f, \vec{h})$. Elements of $A_i(f, \vec{h})$ are called $(f, \vec{h})$-*heavy* i-faces, or just *heavy* i-faces for short.

Lemma 8.4. *Let X be a d-complex, let $-1 \leq i \leq k \leq d$ and let $A \subseteq X(k)$. Then $\sum_{z \in X(i)} w(A_{\supseteq z}) = \binom{k+1}{i+1} w(A)$.*

Proof. We have

$$\sum_{z \in X(i)} w(A_{\supseteq z}) = \sum_{z \in X(i)} \sum_{x \in A : z \subseteq x} w(x) = \sum_{x \in A} \sum_{z \in X(i) : z \subseteq x} w(x) = \sum_{x \in A} \tbinom{k+1}{i+1} w(x) = \tbinom{k+1}{i+1} w(A). \qquad \square$$

Lemma 8.5. *Let $f \in C^k(X, \mathcal{F})$, $\vec{h} \in (0, 1]^{\{-1,0,\dots,k-1\}}$ and $i \in \{-1, 0, \dots, k\}$. Then $w(A_i(f, \vec{h})) \leq (\prod_{i \leq j < k} h_j^{-1}) \|f\|$.*

Proof. This is clear if $i = k$, so assume $i < k$. In this case, using (3.2), the definition of heaviness, and Lemma 8.4, we see that

$$\begin{aligned} w(A_i(f, \vec{h})) &= \sum_{x \in A_i(f, \vec{h})} w(x) = \sum_{x \in A_i(f, \vec{h})} (i+2)^{-1} w(X(i+1)_{\supseteq x}) \\ &\leq h_i^{-1} (i+2)^{-1} \sum_{x \in A_i(f, \vec{h})} w(A_{i+1}(f, \vec{h})_{\supseteq x}) \leq h_i^{-1} w(A_{i+1}(f, \vec{h})). \end{aligned}$$

Iterating, we find that

$$w(A_i(f, \vec{h})) \leq h_i^{-1} h_{i+1}^{-1} \cdots h_{k-1}^{-1} w(A_k(f, h)) = (\prod_{i \leq j < k} h_j^{-1}) \|f\|. \qquad \square$$

It can happen that the intersection of two $(f, \vec{h})$-heavy i-faces ($0 \leq i \leq k$) is an $(i-1)$-face which is not $(f, \vec{h})$-heavy. We call such pairs $(f, \vec{h})$-*bad*, or just *bad*. Provided $k < \dim X$, we also say that a $(k+1)$-face is $(f, \vec{h})$-bad if it contains a bad pair of faces. The set of $(f, \vec{h})$-bad $(k+1)$-faces is denoted $\Upsilon(f, h)$.

Lemma 8.6. *Let $f \in C^k(X, \mathcal{F})$ with $k \in \{0, \dots, d-1\}$, and let $\vec{h}, \vec{\alpha} \in (0,1]^{\{-1,\dots,k-1\}}$. Suppose that X is an i-local α_i-skeleton expander for all $i \in \{-1, \dots, k-1\}$ (in particular, $X = X_\emptyset$ is an α_{-1}-skeleton expander). Then*

$$w(\Upsilon(f, \vec{h})) \leq \sum_{i=0}^{k} \tbinom{k+2}{i+2}(i+1)(\alpha_{i-1} + h_{i-1})h_i^{-1} \cdots h_{k-1}^{-1}\|f\|.$$

Proof. Fix $i \in \{0, \dots, k\}$ and $z \in X(i-1)$. We call an $(i+1)$-face e z-bad if $e \supseteq z$ and the two i-faces lying between z and e form a bad pair; denote by $B(z)$ the set of z-bad faces. Let $e \in B(z)$ and let x, y be the i-faces between z and e. Then $e - z$ is an edge connecting the 0-faces $x - z$ and $y - z$ in the link X_z. Since both $x - z, y - z \in A_i(f, h)_z$ (because x and y are heavy), our assumption that X_z is an α_{i-1}-skeleton expander implies that

$$w_z(B(z)_z) \leq (w_z(A_i(f, h)_z) + \alpha_{i-1})w_z(A_i(f, h)_z),$$

where $w_z = w_{X_z}$. Since z is not heavy (x, y is a bad pair), $w(A_i(f, h)_{\supseteq z}) \leq h_{i-1}w(X(i)_{\supseteq z})$, which means that $w_z(A_i(f, h)_z) \leq h_{i-1}w_z(X(i)_z) = h_{i-1}$, by (3.3). Thus,

$$w_z(B(z)_z) \leq (\alpha_{i-1} + h_{i-1})w_z(A_i(f, h)_z).$$

Scaling both sides using (3.3), we get

$$w(B(z)) \leq (\alpha_{i-1} + h_{i-1})w(A_i(f, h)_{\supseteq z}).$$

Now, since every face in $\Upsilon(f, \vec{h})$ contains a face in $B(z)$ for some z, we have

$$w(\Upsilon(f, \vec{h})) \leq \sum_{i=0}^{k} \sum_{z \in X(i-1)} \sum_{e \in B(z)} w(X(k+1)_{\supseteq e}).$$

Using (3.2), Lemma 8.4 and Lemma 8.5, the right hand side evaluates to

$$\begin{aligned}
\sum_{i=j}^{k} \sum_{z \in X(i-1)} \sum_{e \in B(z)} \tbinom{k+2}{i+2} w(e) &= \sum_{i=0}^{k} \sum_{z \in X(i-1)} \tbinom{k+2}{i+2} w(B(z)) \\
&\leq \sum_{i=0}^{k} \sum_{z \in X(i-1)} \tbinom{k+2}{i+2}(\alpha_{i-1} + h_{i-1})w(A_i(f, h)_{\supseteq z}) \\
&= \sum_{i=0}^{k} \tbinom{k+2}{i+2}(i+1)(\alpha_{i-1} + h_{i-1})w(A_i(f, h)) \\
&\leq \sum_{i=0}^{k} \tbinom{k+2}{i+2}(i+1)(\alpha_{i-1} + h_{i-1})h_i^{-1} \cdots h_{k-1}^{-1}\|f\|. \qquad \square
\end{aligned}$$

We continue to assume that $f \in C^k(X, \mathcal{F})$ and $\vec{h} \in (0,1]^{\{-1,\dots,k-1\}}$. Given two heavy faces x, y with $x \subseteq y$, we say that y $(f, \vec{h})$-*descends* to x, or x $(f, \vec{h})$-*descends* from y, if there exists a sequence $x = x_i \subseteq x_{i+1} \subseteq \cdots \subseteq x_\ell = y$ with $x_j \in A_j(f, \vec{h})$ for all $j \in \{i, i+1, \dots, \ell\}$. We will simply say that y descends to x if there is no risk of confusion. We say that a $(k+1)$-face y descends to a heavy face x if y contains a heavy k-face descending to x. A face will be called $(f, \vec{h})$-*terminal*, or just terminal, if it is heavy and does not descend to any of its proper faces. It is clear that every heavy face descends to some terminal face. Beware that a terminal face may contain another terminal face.

Lemma 8.7. *Let $f \in C^k(X, \mathcal{F})$ and $\vec{h} \in (0,1]^{\{-1,\dots,k-1\}}$. Let $y \in X(k+1) - \Upsilon(f, \vec{h})$ and let $D(y)$ denote the set of heavy faces which $(f, \vec{h})$-descend from y. If $D(y) \neq \emptyset$, then there exists exactly one terminal face z descending from y. Moreover, every face in $D(y)$ descends to z.*

Proof. Since $D(y) \neq \emptyset$, the face y must descend to some terminal face z. In order to prove the lemma, it is enough to show that every $z' \in D(y)$ descends to z.

By definition, there are sequences $z = x_r \subseteq \cdots \subseteq x_k \subseteq y$ and $z' = x'_s \subseteq \cdots \subseteq x'_k \subseteq y$ such that x_j and x'_j are in $A_j(f, \vec{h})$ for all j. Set $x_{k+1} = x'_{k+1} = y$. We claim that $x_i \cap x'_j$ is heavy for all $i \in \{r, \dots, k, k+1\}$ and $j \in \{s, \dots, k, k+1\}$. We show this by decreasing induction on i and j. The claim is clear if $i = k+1$ or $j = k+1$, so assume $i, j \leq k$. By the induction hypothesis, $x_{i+1} \cap x'_j$ and $x_i \cap x'_{j+1}$ are both heavy. If $x_{i+1} \cap x'_j = x_i \cap x'_j$, or $x_i \cap x'_{j+1} = x_i \cap x'_j$, then $x_i \cap x'_j$ is also heavy. Otherwise, the dimension of both $x_{i+1} \cap x'_j$ and $x_i \cap x'_{j+1}$ is $\dim(x_i \cap x'_j) + 1$, and $(x_{i+1} \cap x'_j) \cap (x_i \cap x'_{j+1}) = x_i \cap x'_j$. Since $y \notin \Upsilon(f, \vec{h})$, the face $x_i \cap x'_j$ must be heavy as well, hence our claim.

To finish, consider the sequence of heavy faces $z = z \cap x'_{k+1} \supseteq z \cap x'_k \supseteq \cdots \supseteq z \cap x'_s = z \cap z'$. The difference between the dimensions of every two consecutive faces in this sequence is either 0 or 1, so z descends to $z \cap z'$. Since z is terminal, we must have $z = z \cap z'$, or rather, $z \subseteq z'$. By a similar argument, z' descends to $z' \cap z = z$, which is what we want. $\square$

Lemma 8.8. *Let $f \in C^k(X, \mathcal{F})$ with $k \in \{0, \dots, d-1\}$, and let $\vec{h} \in (0,1]^{\{-1,\dots,k-1\}}$. Let z be an $(f, \vec{h})$-terminal face, and let*

$$L(z) = \{x \in X(k) \,:\, x \ (f, \vec{h})\text{-descends to } z\},$$
$$L'(z) = \{y \in X(k+1) \,:\, y \ (f, \vec{h})\text{-descends to } z\}.$$

Suppose that f is locally minimal at z (see §6.1), $(X_z, \mathcal{F}_z)$ is an ε-coboundary expander in dimension $k - \dim z - 1$ and $\mathcal{F}(y) \neq 0$ for all $y \in X(k+1)_{\supseteq z}$. Then

$$\frac{(k+2)\varepsilon}{k+1-\dim z} w(L(z)) \leq w([\operatorname{supp}(d_0 f) \cup \Upsilon(f, \vec{h})] \cap L'(z))$$

Proof. Write $i = \dim z$ and fix some ordering on the vertices of z. Define $g \in C^k(X, \mathcal{F})$ by

$$g(x) = \begin{cases} f(x) & x \in L(z) \\ 0 & \text{otherwise}, \end{cases}$$

where $x \in X(k)_{\mathrm{ord}}$. Then $(g_z)^z = g$ (notation as in §4.4). By Proposition 6.1(i), $f_z \in C^{k-i-1}(X_z, \mathcal{F}_z)$ is minimal, and by Lemma 6.2, so is g_z. Write $\|\cdot\|_z$ for the weighted support norm on $\mathcal{F}_z$ and m_z for its associated mass function. Since $\mathcal{F}(y) \neq 0$ for all $y \in X(k+1)_{\supseteq z}$, we have $m_z(k-i) = 1$. Our assumption that $(X_z, \mathcal{F}_z)$ is an ε-coboundary expander in dimension $k-i-1$, therefore implies that

$$\|d_{k-i-1} g_z\|_z \geq \|d_{k-i-1} g_z\|_z m_z(k-i-1) \geq \varepsilon \|g_z\|_z m_z(k-i) = \varepsilon \|g_z\|_z.$$

By Lemma 4.13, $(d_{k-1-1}(g_z))^z = d_k((g_z)^z) = d_k g$. Using this and (3.3), we find that

$$\|d_k g\| \geq \tbinom{k+2}{i+1}\tbinom{k+1}{i+1}^{-1} \varepsilon \|g\| = \frac{(k+2)\varepsilon}{k+1-i} w(L(z)). \tag{8.1}$$

Let $y \in \operatorname{supp}(d_k g)$. By the definition of g, the face y descends to z, that is, $y \in L'(z)$. If $y \notin \Upsilon(f, \vec{h})$, then by Lemma 8.7, every face descended from y also descends to z. In particular, every $x \in (\operatorname{supp} f) \cap X(k)_{\subseteq y}$ descends to z, and thus belongs to $L(z) = \operatorname{supp} g$. It follows that $(d_k f)(y) = (d_k g)(y) \neq 0$, so $y \in \operatorname{supp}(d_k f)$. This shows that

$$\operatorname{supp}(d_k g) \subseteq [\operatorname{supp}(d_k f) \cup \Upsilon(f, \vec{h})] \cap L'(z).$$

Combining this with (8.1) gives the lemma. □

Notation 8.9. We call a collection of subsets $E \subseteq P(\{1,\dots,n\})$ an *n-vine* if:

(1) $\{1,\dots,n\} \in E$,

(2) Every $s \in E$ admits a sequence $s = s_i \subseteq s_{i+1} \subseteq \cdots \subseteq s_n = \{1,\dots,n\}$ such that $s_j \in E$ and $|s_j| = j$ for all j.

We say that $s \in E$ is terminal if no maximal subset of s is in E. (It is possible for non-maximal subsets of s to be in E.) Denote by $T(E)$ the terminal subsets in E. Finally, set

$$U(n) = \max\{\#T(E) \,|\, E \text{ is an } n\text{-vine}\}.$$

Direct computation shows that $U(1) = 1$, $U(2) = 2$ and $U(3) = 3$. In general, we have $\binom{n}{\lfloor n/2 \rfloor} \leq U(n) \leq 2^n - 1$.[2]

Lemma 8.10. *Let $f \in C^k(X, \mathcal{F})$ be locally minimal (see §6.1), let $\vec{h} \in (0,1]^{\{-1,\dots,k-1\}}$ and let $\varepsilon_0, \dots, \varepsilon_k \in \mathbb{R}_+$. Suppose that $\mathcal{F}(x) \neq 0$ for all $x \in X(k+1)$, and that $(X, \mathcal{F})$ is an i-local ε_i-coboundary expander in dimension k for every $i \in \{0,\dots,k\}$. If the empty face of X is not $(f, \vec{h})$-heavy, then*

$$\min\left\{ \frac{(k+2)\varepsilon_i}{k+1-i} \,\middle|\, i \in \{0,\dots,k\} \right\} \|f\| \leq \|d_0 f\| + U(k+2) w(\Upsilon(f, \vec{h})).$$

Proof. Denote by T the set of $(f, \vec{h})$-terminal faces. Given $z \in T$, define $L(z)$ and $L'(z)$ as in Lemma 8.8. We abbreviate $\Upsilon(f, \vec{h})$ to Υ.

Let $z \in T$. By assumption, $z \neq \emptyset$, so f is locally minimal at z and $(X_z, \mathcal{F}_z)$ is an $\varepsilon_{\dim z}$-coboundary expander in dimension $k - \dim z - 1$. Lemma 8.8 now tells us that

$$\begin{aligned} \frac{(k+2)\varepsilon}{k+1-\dim z} w(L(z)) &\leq w([\operatorname{supp}(d_0 f) \cup \Upsilon] \cap L'(z)) \\ &= w([\operatorname{supp}(d_0 f) - \Upsilon] \cap L'(z)) + w(\Upsilon \cap L'(z)). \end{aligned}$$

Summing over all $z \in T$, we get

$$\sum_{z \in T} \frac{(k+2)\varepsilon}{k+1-\dim z} w(L(z)) \leq \sum_{z \in T} w([\operatorname{supp}(d_0 f) - \Upsilon] \cap L'(z)) + \sum_{z \in T} w(\Upsilon \cap L'(z)). \tag{8.2}$$

Since every face in $\operatorname{supp} f$ descends to some terminal face, the left hand side of (8.2) is at least

$$\min\left\{ \frac{(k+2)\varepsilon_i}{k+1-i} \,\middle|\, i \in \{0,\dots,k\} \right\} \|f\|.$$

[2]The number $U(n)$ is larger than $\binom{n}{\lfloor n/2 \rfloor}$ for large n. Indeed, assuming $n = 4k$, consider the n-vine $E = \{s \subseteq \{1,\dots,n\} : |s| \geq 2k\} \cup \{s \subseteq \{1,\dots,2k\} : |s| \geq k\}$. It routine to check that $T(E) = \binom{n}{2k} - (2k)^2 - 1 + \binom{2k}{k}$, which is larger than $\binom{n}{\lfloor n/2 \rfloor}$ as soon as $n \geq 16$.

As for the right hand side of (8.2), by Lemma 8.7, every face in $y \in \operatorname{supp}(d_0 f) - \Upsilon$ descends to a unique terminal face. Thus,

$$\sum_{z \in T} w([\operatorname{supp}(d_0 f) - \Upsilon] \cap L'(z)) = w(\operatorname{supp} d_0 f - \Upsilon) \leq \|d_0 f\|.$$

If $y \in \Upsilon$, then upon identifying y with $\{1, \dots, k+2\}$, the set of faces to which y descends is a $(k+2)$-vine in the sense Notation 8.9. Thus, the number of terminal faces to which y descends is at most $U(k+2)$, meaning that

$$\sum_{z \in T} w(\Upsilon \cap L'(z)) \leq U(k+2) w(\Upsilon).$$

Plugging these observations into (8.2) gives the lemma. □

8.2 Proof of Theorem 8.2

We will deduce Theorem 8.2 from the following more general theorem.

Theorem 8.11. *Let $k \in \mathbb{N} \cup \{0\}$, $\alpha_0, \dots, \alpha_{k-1}, \varepsilon_0, \dots, \varepsilon_k \in \mathbb{R}_+$, and put*

$$\varepsilon := \min \left\{ \frac{(k+2)\varepsilon_i}{k+1-i} \,\middle|\, i \in \{0, \dots, k\} \right\}.$$

Suppose that there are $h_{-1}, \dots, h_{k-1} \in (0,1]$ such that:

$$U(k+2) \sum_{i=0}^{k} \tbinom{k+2}{i+2} (i+1) \frac{\alpha_{i-1} + h_{i-1}}{h_i \cdots h_{k-1}} < \varepsilon, \tag{8.3}$$

where $U(k+2)$ is as in Notation 8.9. Then there exist $\beta, \gamma \in \mathbb{R}_+$ such that the following hold: Let $(X, \mathcal{F})$ be a sheaved d-complex, where $d \geq k+1$. Assume that

(1) $(X, \mathcal{F})$ is an i-local ε_i-coboundary expander in dimension k for all $i \in \{0, \dots, k\}$.

(2) X is an i-local α_i-skeleton expander for all $i \in \{-1, \dots, k-1\}$.

(3) $\mathcal{F}(x) \neq 0$ for all $x \in X(k+1)$.

Then $(X, \mathcal{F})$ β-expands γ-small locally minimal k-cochains. In fact, one can take $\gamma = h_{-1} \cdots h_{k-1}$ and β to be the difference between the right hand side and the left hand side of (8.3).

Proof. Put $\vec{h} = (h_{-1}, \dots, h_{k-1})$ and define β and γ as in the theorem. Let $f \in C^k(X, \mathcal{F})$ be a locally minimal k-cochain such that $\|f\| < \gamma$. We need to prove that $\|d_0 f\| \geq \beta \|f\|$.

We first claim that the empty face is not $(f, \vec{h})$-heavy. Indeed, by Lemma 8.5, $w(A_{-1}(f, \vec{h})) \leq (h_{-1} h_0 \cdots h_{k-1})^{-1} \|f\| < (h_{-1} h_0 \cdots h_{k-1})^{-1} \gamma = 1$. Since the empty face has weight 1, this means that $A_{-1}(f, \vec{h}) = \emptyset$, so the empty face is not heavy.

We may now apply Lemma 8.10, which tells us that

$$\varepsilon \|f\| \leq \|d_0 f\| + U(k+2) w(\Upsilon(f, \vec{h})).$$

By Lemma 8.6, this means that

$$\varepsilon \|f\| \leq \|d_0 f\| + U(k+2) \sum_{i=0}^{k} \tbinom{k+2}{i+2} (i+1) \frac{\alpha_{i-1} + h_{i-1}}{h_i \cdots h_{k-1}} \|f\|,$$

and by rearranging, we get $\beta \|f\| \leq \|d_0 f\|$. □

Proof of Theorem 8.2. Assumption (2) and Oppenheim's Trickling Down Theorem [Opp15, Theorem 1.4] imply that for every $z \in X$ of dimension $i \in \{-1, \dots, d-2\}$, the weighted underlying graph of X_z is a $[-1, \frac{\lambda}{1-(d-2-i)\lambda}]$-spectral expander, and thus X_z is a $\frac{\lambda}{1-(d-2-i)\lambda}$-skeleton expander (see §3.4). By the assumptions on λ, we have

$$\frac{\lambda}{1-(d-2-i)\lambda} \leq \frac{\lambda}{1-(d-2+1)\frac{1}{d}} = d\lambda \leq \left(\frac{\varepsilon}{(k+1)^2 2^{2k+6}}\right)^{2^k}.$$

Setting $\alpha_i := (\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^{k-1-i}}$ for $i \in \{-1, \dots, k-1\}$, we conclude that X is an i-local α_i-skeleton expander for all $i \in \{-1, \dots, k-1\}$.

We now apply Theorem 8.2 with $h_i = \alpha_i$. To see that the inequality (8.3) holds, note that for all $i \in \{0, \dots, k\}$, we have

$$\frac{\alpha_{i-1} + h_{i-1}}{h_i \cdots h_{k-1}} = 2\left(\frac{\varepsilon}{(k+1)^2 2^{2k+6}}\right)^{2^{k-i} - 2^{k-1-i} - \dots - 2^0} = \frac{\varepsilon}{(k+1)^2 2^{2k+5}}.$$

Thus,

$$U(k+2)\sum_{i=0}^{k} \tbinom{k+2}{i+2}(i+1)\frac{\alpha_{i-1} + h_{i-1}}{h_i \cdots h_{k-1}} \leq 2^{k+2}\sum_{i=0}^{k} 2^{k+2}(k+1)\frac{\varepsilon}{(k+1)^2 2^{2k+5}} = \frac{\varepsilon}{2} < \varepsilon.$$

It also follows that the constants β and γ of Theorem 8.2 satisfy $\beta \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$ and $\gamma = h_{-1}h_0 \cdots h_{k-1} = (\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^k + 2^{k-1} + \dots + 2^0} = (\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^{k+1}-1}$. We conclude that $(X, \mathcal{F})$ $\frac{\varepsilon}{2}$-expands $(\frac{\varepsilon}{(k+1)^2 2^{2k+6}})^{2^{k+1}-1}$-small k-cochains. $\square$

In the remainder of this subsection, we analyze the solubility of the inequality (8.3) in the cases $k = 0$ and $k = 1$, deriving specialized versions of Theorem 8.11. We also address the asymptotic behavior of the general case. To begin, we make the following remark.

Remark 8.12. When solving (8.3), we may assume that $h_{-1}, \dots, h_{k-1}$ live in $\mathbb{R}_+$, rather than $(0, 1]$, because (8.3) and assumption (1) of Theorem 8.11 force $h_{-1}, \dots, h_{k-1} \leq 1$. This can be seen by decreasing induction on i. For $i = k-1$, the inequality (8.3) and Remark 8.3(ii) imply that $2(k+1)h_{k-1} \leq U(k+2)\binom{k+2}{k+2}(k+1)h_{k-1} < \varepsilon \leq k+2$, so $h_{k-1} \leq 1$. Assuming $h_{i+1}, \dots, h_{k-1} \leq 1$ for $-1 \leq i < k-1$, the the same reasoning shows that $(k+2)h_i \leq \binom{k+2}{i+3}h_i < \varepsilon \leq k+2$, so $h_i \leq 1$. (In fact, the assumption that $h_i \leq 1$ for all i was never used in the proof of Theorem 8.11.)

Corollary 8.13. *Let $(X, \mathcal{F})$ be a sheaved d-complex ($d \geq 1$), let $\alpha, \varepsilon \in \mathbb{R}_+$ be numbers such that $\alpha < \varepsilon$, and let $h \in [0, \varepsilon - \alpha]$. Suppose that (1) $(X_v, \mathcal{F}_v)$ is an ε-coboundary expander in dimension -1 for every $v \in X(0)$, (2) X is an α-skeleton expander, and (3) $\mathcal{F}(e) \neq 0$ for all $e \in X(1)$. Then $(X, \mathcal{F})$ $2(\varepsilon - \alpha - h)$-expands h-small 0-cochains.*

Proof. When $k = 0$, the inequality (8.3) becomes $2(\alpha_{-1} + h_{-1}) < 2\varepsilon_0$. Setting $h_{-1} = h$, $\varepsilon_0 = \varepsilon$ and $\alpha_{-1} = \alpha$, the statement follows from Theorem 8.11. (Note that every 0-cochain is locally minimal.) $\square$

Corollary 8.14. *Let $\alpha_{-1}, \alpha_0, \varepsilon_0, \varepsilon_1 \in \mathbb{R}_+$ be numbers such that*

$$\alpha_0 < \min\{\frac{\varepsilon_0}{4}, \frac{\varepsilon_1}{2}\} \qquad \textit{and} \qquad \alpha_{-1} < \frac{1}{6}(\min\{\frac{\varepsilon_0}{4}, \frac{\varepsilon_1}{2}\} - \alpha_0)^2.$$

Then there exist $\beta, \gamma \in \mathbb{R}_+$ such that the following hold: Let $(X, \mathcal{F})$ be a sheaved d-complex ($d \geq 2$) such that (1) $(X, \mathcal{F})$ is an i-local ε_i-coboundary expander in dimension 1 for $i \in \{0, 1\}$, (2) X is an i-local α_i-skeleton expander for $i \in \{-1, 0\}$, and (3) $\mathcal{F}(x) \neq 0$ for all $x \in X(2)$. Then $(X, \mathcal{F})$ γ-expands β-small locally minimal 1-cochains.

Proof. When $k = 1$, the inequality (8.3) becomes

$$9 \cdot \frac{\alpha_{-1} + h_{-1}}{h_0} + 6(\alpha_0 + h_0) < \min\{\frac{3}{2}\varepsilon_0, 3\varepsilon_1\}.$$

By treating this as a quadratic inequality in h_0, one finds that it is solvable for $h_0, h_{-1} \in \mathbb{R}_+$ if and only if the inequalities in the corollary are satisfied. The corollary is therefore a special case of Theorem 8.11. $\square$

Corollary 8.15. *Let $\varepsilon_0, \varepsilon_0', \varepsilon_1' \in \mathbb{R}_+$ and $\alpha_{-1}, \alpha_0 \in [0, 1]$ be numbers such that*

$$\alpha_0 < \min\{\frac{\varepsilon_0'}{4}, \frac{\varepsilon_1'}{2}\} \qquad \text{and} \qquad \alpha_{-1} < \min\{\varepsilon_0, \frac{1}{6}(\min\{\frac{\varepsilon_0'}{4}, \frac{\varepsilon_1'}{2}\} - \alpha_0)^2\},$$

and let $Q, d \in \mathbb{N}$ be integers with $d \geq 2$. Then there exist $\beta, \beta', \gamma, \gamma' \in \mathbb{R}_+$, depending on $\varepsilon_0, \varepsilon_0', \varepsilon_1', \alpha_{-1}, \alpha_0$, and $\delta, \varepsilon \in \mathbb{R}_+$, depending on $\varepsilon_0, \varepsilon_0', \varepsilon_1', \alpha_{-1}, \alpha_0, d, Q$, such that the following hold: If $(X, \mathcal{F})$ is a sheaved d-complex such that

(1) $(X, \mathcal{F})$ is a 0-local ε_0-coboundary expander in dimension 0.

(2) $(X, \mathcal{F})$ is a i-local ε_i'-coboundary expander in dimension 1 for $i \in \{0, 1\}$.

(3) X_v is an α_0-skeleton expander for all $v \in X(0)$ and X is an α_{-1}-skeleton expander,

(4) $D(X) \leq Q$, i.e., every vertex of X belongs to at most Q d-faces, and

(5) $\mathcal{F}(x) \neq 0$ for all $x \in X(0) \cup X(1) \cup X(2)$,

then $(X, \mathcal{F})$ is an (ε, δ)-cosystolic expander in dimension 0, β-expands γ-small 0-cochains and β'-expands γ'-small locally minimal 1-cochains.

Proof. By Corollaries 8.13 and 8.14, there are $\beta', \beta, \gamma', \gamma' \in \mathbb{R}_+$, depending on $\varepsilon_0, \varepsilon_0', \varepsilon_1', \alpha_{-1}, \alpha_0$, such that $(X, \mathcal{F})$ β'-expands γ'-small 0-cochains and β-expands γ-small locally minimal 1-cochains. The existence of γ and δ is now a consequence of Proposition 6.4. $\square$

Remark 8.16. As in the proof of Theorem 8.1, we can use Oppenheim's Trickling Down Theorem [Opp15, Theorem 1.4] to replace condition (3) of Corollary 8.15 with

(3′) X is connected and, for all $v \in X(0)$, the underlying weighted graph of X_v is a $[-1, \lambda]$-spectral expander,

where $\lambda \in \mathbb{R}_+$ is required to satisfy the inequalities

$$\lambda < \min\{\frac{\varepsilon_0'}{4}, \frac{\varepsilon_1'}{2}\} \qquad \text{and} \qquad \frac{\lambda}{1-\lambda} < \min\{\varepsilon_0, \frac{1}{6}(\min\{\frac{\varepsilon_0'}{4}, \frac{\varepsilon_1'}{2}\} - \lambda)^2\}.$$

Remark 8.17. We use the notation of Theorem 8.11. It was demonstrated in the proof of Theorem 8.2 that the inequality (8.3) is solvable when

$$\alpha_i \leq \left(\frac{\varepsilon}{(k+1)^2 2^{2k+6}}\right)^{2^{k-1-i}}$$

for all $i \in \{-1, \dots, k-1\}$. The order of magnitude of this upper bound on the i-local skeleton expansion of X (as a function $\varepsilon_0, \dots, \varepsilon_k$) cannot be increased with our present methods. More precisely, if (8.3) is satisfied, then

$$\alpha_i < \left(\frac{\varepsilon}{U(k+2)}\right)^{2^{k-1-i}}$$

for all $i \in \{-1, \dots, k-1\}$, so we must have $\alpha_i = O(\varepsilon^{2^{k-1-i}})$ in order to apply Theorem 8.11. To see this, note that if (8.3) holds for some $h_{-1}, \dots, h_{k-1} \in (0, 1]$, then for all $i \in \{0, \dots, k\}$, we have

$$\varepsilon > U(k+2) \sum_{i=0}^{k} \tbinom{k+2}{i+2} (i+1) \frac{\alpha_{i-1} + h_{i-1}}{h_i \cdots h_{k-1}} \geq U(k+2) \cdot \frac{\alpha_{i-1} + h_{i-1}}{h_i \cdots h_{k-1}}.$$

As a result,

$$\begin{aligned} \max\{\alpha_{k-1}, h_{k-1}\} &< U(k+2)^{-1}\varepsilon, \\ \max\{\alpha_{k-2}, h_{k-2}\} &< U(k+2)^{-1} h_{k-1}\varepsilon, \\ &\vdots \\ \max\{\alpha_{-1}, h_{-1}\} &< U(k+2)^{-1} h_{-1} h_0 \cdots h_{k-1}\varepsilon \end{aligned}$$

These inequalities imply readily that $h_i < (\frac{\varepsilon}{U(k+2)})^{2^{k-1-i}}$ for all i. Plugging this in the right hand side of the inequalities gives $\alpha_i < (\frac{\varepsilon}{U(k+2)})^{2^{k-1-i}}$. It also follows that $\gamma = h_{-1} \cdots h_{k-1}$ (the smallness of locally minimal k-cochains which are guaranteed to β-expand) is smaller than $(\frac{\varepsilon}{U(k+2)})^{2^{k+1}-1}$.

9 Examples of Cocycle Codes

In this section, we give examples of sheaved d-complexes to which Theorems 8.1 and 8.2 can be applied, and analyze the properties of the associated cocycle codes.

Some of the examples make use of simplicial complexes covered affine buildings, recalled in §3.5.

9.1 0-Cocycle Codes of Sheaves on Graphs

We begin by revisiting an example from the Overview section. Fix some $m, k \in \mathbb{N}$ with $\frac{k}{2} < m \leq k$, let X be a k-regular graph, and let $\mathbb{F}$ be a finite field. Given $v \in X(0)$, write $E(v)$ for $X(1)_{\supseteq v}$ and choose an injective $\mathbb{F}$-linear map $T_v : \mathbb{F}^m \to \mathbb{F}^{E(v)} \cong \mathbb{F}^k$. We think of $C_v := \operatorname{im} T_v$ as a code inside $\mathbb{F}^{E(v)}$ with alphabet $\mathbb{F}$ and denote its relative distance by $\delta(C_v)$. In §2.2, we defined a sheaf $\mathcal{F}$ on X by setting $\mathcal{F}(v) = \mathbb{F}^m$ and $\mathcal{F}(e) = \mathbb{F}$ for all $v \in X(0)$, $e \in X(1)$, and $\operatorname{res}^{\mathcal{F}}_{e \leftarrow v} = \operatorname{Proj}_e \circ T_v$ — where $\operatorname{Proj}_e : \mathbb{F}^{E(v)} \to \mathbb{F}$ is projection onto the e-component — whenever $v \subseteq e$. Putting $\Sigma := \mathbb{F}^m$, we form the 0-cocycle code $Z^0(X, \mathcal{F})$ inside $C^0(X, \mathcal{F}) = \Sigma^{X(0)}$ as in §7.3.

Proposition 9.1. *With notation as above, suppose that X is an α-skeleton expander ($\alpha \in \mathbb{R}_+$) and $\varepsilon := \min\{\delta(C_v) \mid v \in X(0)\} > \alpha$. Then the 0-cocycle code $Z^0(X, \mathcal{F}) \subseteq \Sigma^{X(0)}$ has rate $\geq 1 - \frac{k}{2m}$ and relative distance $\geq \varepsilon - \alpha$.*

Proof. We observed in §2.3 that $(X_v, \mathcal{F}_v)$ is a $\delta(C_v)$-coboundary expander in dimension -1. The claim about the relative distance is therefore a consequence of Corollary 8.13 and Proposition 7.8(i). Dimension count implies that $\dim_{\mathbb{F}} Z^0(X,\mathcal{F}) \geq m|X(0)| - |X(1)| = |X(0)|(m - \frac{k}{2})$, hence the lower bound on the rate. □

9.2 Cocycle Codes of Sheaves on Complexes Covered by Affine Buildings

In the following examples we put into use the fact that constant sheaves on finite spherical buildings are good coboundary expanders.

Theorem 9.2. *For every $d \in \mathbb{N} - \{1\}$, there exists $q \in \mathbb{N}$ for which the following hold: Let Y be a q-thick affine building, let $k \in \{0, \dots, d-2\}$, let X be a (finite) simplicial complex covered by Y, and let $\mathcal{F}$ be a nonzero locally constant sheaf on X. Then:*

(i) There are $\varepsilon_0, \dots, \varepsilon_k, \varepsilon'_0, \dots, \varepsilon'_{k+1}, \lambda \in \mathbb{R}_+$, depending only on k and d, and $Q \in \mathbb{N}$, depending only on Y, such that the assumptions of Theorem 8.1 hold for $(X, \mathcal{F})$.

(ii) There are $\beta, \beta', \gamma, \gamma' \in \mathbb{R}_+$, depending only on k and d, and $\delta, \varepsilon \in \mathbb{R}_+$, depending only on Y, such that $(X,\mathcal{F})$ β-expands γ-small locally minimal k-cochains, β'-expands γ'-small locally minimal $(k+1)$-cochains and is an (ε, δ)-coboundary expander in dimension k.

Proof. Part (ii) follows from (i) and Theorem 8.1. We turn to prove (i).

We claim that for every $i \in \{0, \dots, k\}$, there is $\varepsilon_i > 0$, depending only on d, such that $(X, \mathcal{F})$ is an i-local ε_i-coboundary expander in dimension k. Indeed, let $z \in X(i)$. Since $\mathcal{F}$ is locally constant, $\mathcal{F}_z$ is a constant sheaf on X_z. The link X_z is isomorphic to a proper link of Y, so it is a spherical building of dimension $d - i - 1$. Thus, by Theorem 5.12(i), there exists $\varepsilon_i > 0$ (depending only on $\dim X_z$) such that $(X_z, \mathcal{F}_z)$ is an ε_i-coboundary expander in dimensions $k - i - 1$.

A similar argument shows that there are $\varepsilon'_0, \dots, \varepsilon'_{k+1} \in \mathbb{R}_+$, depending only on d, such that $(X, \mathcal{F})$ is an i-local ε'_i-coboundary expander in dimension $k+1$ for all $i \in \{0, \dots, k+1\}$.

Take λ to be the maximal number for which the inequality in Theorem 8.1 holds. Let $Q = D(Y)$; it is finite because Y admits a strongly transitive action (see §3.5). Finally, set $q = \lceil \frac{16}{\lambda^2} \rceil$.

We claim that assumptions (1)–(5) of Theorem 8.1 hold for $(X, \mathcal{F})$ with the parameters we have chosen, provided that Y is q-thick. Indeed, assumptions (1) and (2) are immediate. Assumption (4) holds because $D(X) = D(Y)$ (since Y covers X), and (5) holds because $\mathcal{F}$ is locally constant and nonzero. To see that (3) holds, let $z \in X(d-2)$. Then X_z is isomorphic to a 1-dimensional link of Y and is therefore a spherical building of dimension 1. By Theorem 3.4(i), X_z is a $[-1, \frac{4}{\sqrt{q}}]$-spectral expander, and $\frac{4}{\sqrt{q}} \leq \lambda$ by our choice of q. □

Corollary 9.3. *For every $d \in \mathbb{N} - \{1\}$, there exists $q \in \mathbb{N}$ for which the following hold: Let Y be a q-thick affine building, let $k \in \{0, \dots, d-2\}$, let X be a (finite) simplicial complex covered by Y, let $\mathbb{F}$ be a finite field, let $\mathcal{F}$ be a nonzero locally constant $\mathbb{F}$-sheaf of dimension m on X and let B be an $\mathbb{F}$-basis of $\mathcal{F}$ (see Example 5.4).*

(i) If $k = 0$, then there are $\delta, \varepsilon, \eta \in \mathbb{R}_+$, depending only on Y, such that the 0-cocycle code (Z^0, C^0, Φ) associated to $(X, \mathcal{F})$ (see §7.3; the alphabet is $\mathbb{F}^m$ and the length is $|X(0)|$), is 2-query ε-locally testable of relative distance $\geq \delta$. Furthermore, it admits a decoding algorithm able to correct an η-fraction of errors in $O_{|\mathbb{F}|, \dim \mathcal{F}}(|X(0)|)$ operations.

(ii) If $k > 0$, then there are $\delta, \varepsilon, \eta \in \mathbb{R}_+$ and $r \in \mathbb{N}$, depending only on Y, k and $\dim \mathcal{F}$, such that the X-side of the k-cocycle quantum CSS code $C := (Z^k, Z_k, C^k, \Phi_X, \Phi_Z)$ associated

to $(X, \mathcal{F}, B)$ (see §7.4; the alphabet is $\mathbb{F}$) has relative distance $\geq \delta$, is r-query ε-testable, and admits a decoding algorithm able to correct an η-fraction of errors in $O_{|\mathbb{F}|,\dim \mathcal{F}}(\dim C^k)$ operations.

Proof. This follows from Theorem 9.2(ii) together with Propositions 7.8 and 7.11. Use Proposition 5.6(ii) and Remark 5.8(ii) in order to replace the weighted support norm $\|\cdot\|_{\mathrm{ws}}$ with $\|\cdot\|_B$. $\square$

For every $q, d \in \mathbb{N}$, there are q-thick d-dimensional affine buildings Y which cover arbitrarily large finite d-complexes X (see §13.2, for instance). Each of these quotients X admits an m-dimensional locally constant $\mathbb{F}$-sheaf $\mathcal{F}$, e.g., the constant sheaf $\mathbb{F}^m$ on X. Choosing q large enough in advance and fixing Y and m, Corollary 9.3(i) says that the 0-cocycle codes of the form $(Z^0(X, \mathcal{F}), C^0(X, \mathcal{F}) \cong (\mathbb{F}^m)^{X(0)}, \Phi)$ are an infinite family of 2-query LTCs with linear distance on the alphabet $\Sigma = \mathbb{F}^m$. Unfortunately, the rate of these codes is very poor — at most $\frac{1}{|X(0)|}$ —, because $\dim_{\mathbb{F}} \mathrm{H}^0(X, \mathcal{F}) \leq \dim \mathcal{F} = m$ by Lemma 4.17.

If, instead of considering 0-cocycle codes, we fix $k \in \{1, \ldots, d-2\}$ and $m := \dim \mathcal{F}$, and look at the k-cocycle quantum CSS codes associated to $(X, \mathcal{F}, B)$, with B being some $\mathbb{F}$-basis of $\mathcal{F}$, then, by Corollary 9.3(ii), we get an infinite family of quantum CSS codes whose X-side is locally testable and has linear distance. The rate of these quantum CSS codes is $\frac{1}{|X(k)|} \dim_{\mathbb{F}} \mathrm{H}^k(X, \mathcal{F})$. Very little is known about $\dim_{\mathbb{F}} \mathrm{H}^k(X, \mathcal{F})$, but experts expect that it is polylogarithmic in $|X(k)|$ and linear in the fixed parameter $m = \dim \mathcal{F}$.

Returning to the case of 0-cocycle codes, as demonstrated in §9.1, it is possible to obtain larger rates by considering sheaves that are not locally constant. We now give such an example.

Construction 9.4. Let X be a d-complex ($d \geq 1$), let $\mathcal{F}$ be a locally constant sheaf on X, and let $E \subseteq C^1(X, \mathcal{F})$ be an abelian subgroup. For every edge $e \in X_{\mathrm{ord}}(1)$, let $E(e)$ be the image of E under the projection from $\prod_{x \in X_{\mathrm{ord}}(1)} \mathcal{F}(x)$ to $\mathcal{F}(e)$. The abelian group $E(e) \subseteq \mathcal{F}(e)$ is independent of the ordering on e, so it makes sense to discuss $E(e)$ for unordered edges $e \in X(1)$. We define a subsheaf $\mathcal{C}_E$ of $\mathcal{F}$ by letting

$$\mathcal{C}_E(x) = \sum_{e \in X(1)_{\subseteq x}} \mathrm{res}_{x \leftarrow e} E(e).$$

for all $x \in X$.

Note that $\mathcal{C}_E(v) = 0$ for all $v \in X(0)$, because v contains no edges. The subsheaf $\mathcal{C}_E$ can be characterized as the smallest subsheaf of $\mathcal{F}$ for which $E \subseteq C^1(X, \mathcal{C})$.

We will be interested in the quotient sheaf $\overline{\mathcal{F}} := \mathcal{F}/\mathcal{C}_E$ when $\mathcal{F}$ is a locally constant $\mathbb{F}$-sheaf of dimension m and E is an $\mathbb{F}$-subspace of $C^1(X, \mathcal{F})$. In this case, $\mathcal{C}_E$ and $\overline{\mathcal{F}}$ are $\mathbb{F}$-sheaves. For every $v \in X(0)$, we have $\overline{\mathcal{F}}(v) = \mathcal{F}(v)/0 \cong \mathbb{F}^m$, so we may consider $Z^0(X, \mathcal{F})$ as a code inside $C^0(X, \mathcal{F}) = (\mathbb{F}^m)^{X(0)}$, the alphabet being $\mathbb{F}^m$. As we now show, when X is covered by a sufficiently thick affine building, and E is small and in general position, the 0-cocycle code of $(X, \overline{\mathcal{F}})$ is locally testable and has linear distance. The rate of this code depends on the choice of E and will be studied in Chapter III.

Theorem 9.5. *Let $d \in \mathbb{N} - \{1\}$. There exists $q \in \mathbb{N}$ such that, for every Y and X as in Theorem 9.2 (resp. Corollary 9.3), the conclusions of Theorem 9.2 (resp. Corollary 9.3(i)) continue to hold with $k = 0$ (but with possibly different expansion constants) if the sheaf $\mathcal{F}$ is a replaced by any sheaf of the form $\overline{\mathcal{F}} = \mathcal{F}/\mathcal{C}_E$, where $\mathcal{C}_E$ is as in Construction 9.4, and the subgroup (resp. $\mathbb{F}$-subspace) $E \subseteq C^1(X, \mathcal{F})$ satisfies the following conditions:*

(a1) For every $v \in X(0)$, the map $\sum_e \mathrm{res}_{e \leftarrow v}^{-1} : \bigoplus_e E(e) \to \mathcal{F}(v)$, with e ranging over $X(1)_{\supseteq v}$, is injective.

(a2) For every triangle $t \in X(2)$ with edges e, e', e'', we have $E(e)|_t \subseteq E(e')|_t + E(e'')|_t$.

Example 9.6. Condition (a2) of Theorem 9.2 holds if $E \subseteq Z^1(X, \mathcal{F})$. Indeed, let $t \in X_{\mathrm{ord}}(2)$ and let e, e', e'' denote the ordered edges, obtained by removing the 0-th, 1-st and 2-nd vertex of t, respectively. Then for every $f \in E(e)$, there is $\hat{f} \in E$ such that $\hat{f}(e) = f$. Since $E \subseteq Z^1(X, \mathcal{F})$, we have $d_1 \hat{f}(t) = 0$, which means that $f|_t = \hat{f}(e)|_t = \hat{f}(e')|_t - \hat{f}(e'')|_t \in E(e')|_t + E(e'')|_t$. This shows that $E(e)|_t \subseteq E(e')|_t + E(e'')|_t$.

Condition (a1) typically holds if $\dim E \cdot D_{0,1}(X) \leq \dim \mathcal{F}$ is E is chosen uniformly at random. We make this precise in Proposition 12.4(ii) below.

We first prove the following lemma:

Lemma 9.7. *Let X, $\mathcal{F}$, E and $\mathcal{C} := \mathcal{C}_E$ be as in Construction 9.4 and assume that conditions (a1) and (a2) of Theorem 9.5 hold. Let $v \in X(0)$ and write $A = \mathcal{F}(v)$. For every $u \in X(1)_v$, put $A_u = \mathrm{res}^{-1}_{u \cup v \leftarrow v}(E(u \cup v)) \subseteq A$, and for every $x \in X_v$, define $\mathcal{C}'(x) = \sum_{u \in X_v(0)_{\subseteq x}} A_u$. Then:*

(i) $\mathcal{C}'$ is a subsheaf of the augmented sheaf A_+ on X_v, and the summation map $\bigoplus_{u \in X_v(0)} A_u \to A$ is injective.

(ii) $(\mathcal{F}/\mathcal{C})_v \cong A_+/\mathcal{C}'$ as sheaves on X_v.

Proof. (i) That $\mathcal{C}'$ is a subsheaf of A_+ is straightforward, and the injectivity of $\bigoplus_{u \in X_v(0)} A_u \to A$ is a direct consequence of (a1).

(ii) Write $\overline{\mathcal{F}} = \mathcal{F}/\mathcal{C}$. Then $\overline{\mathcal{F}}_v = \mathcal{F}_v/\mathcal{C}_v$. For every $x \in X_v$, we have

$$\begin{aligned}
\mathrm{res}_{x \cup v \leftarrow v}(\mathcal{C}'(x)) &= \sum_{y \in X_v(0)_{\subseteq x}} \mathrm{res}_{x \cup v \leftarrow v}(A_y) = \sum_{y \in X_v(0)_{\subseteq x}} \mathrm{res}_{x \cup v \leftarrow v} \mathrm{res}^{-1}_{y \cup v \leftarrow v}(E(y \cup v)) \\
&= \sum_{y \in X_v(0)_{\subseteq x}} \mathrm{res}_{x \cup v \leftarrow y \cup v}(E(y \cup v)) = \sum_{e \in X(1) : v \subseteq e \subseteq x \cup v} \mathrm{res}_{x \cup v \leftarrow e}(E(e)) \\
&\subseteq \mathcal{C}(x \cup v) = \mathcal{C}_v(x).
\end{aligned}$$

This allows us to define $\varphi_x : A_+(x)/\mathcal{C}'(x) \to \mathcal{F}'_v(x) = \mathcal{F}_v(x)/\mathcal{C}_v(x)$ by $\varphi_x(f + \mathcal{C}'(x)) = \mathrm{res}_{x \cup v \leftarrow v}(f) + \mathcal{C}_v(x)$ for all $f \in A$. It is routine to check that $\varphi := (\varphi_x)_{x \in X_v} : A_+/\mathcal{C}' \to \mathcal{F}'_v$ is a morphism of sheaves. It remains to prove that each φ_x is bijective, or equivalently, that $\mathrm{res}_{x \cup v \leftarrow v}(\mathcal{C}'(x)) = \mathcal{C}_v(x)$. We already observed that the left hand side is contained in the right hand side. Proving the reverse inclusion amounts to showing that for every $x \in X_{\supseteq v}$ and $e \in X(1)_{\subseteq x}$, we have $\mathrm{res}_{x \leftarrow e} E(e) \subseteq \sum_{y \in X(1) : v \subseteq y \subseteq x} \mathrm{res}_{x \leftarrow y} E(y)$.

Fix $x \in X_{\supseteq v}$, $e \in X(1)_{\subseteq x}$ and $f \in \mathrm{res}_{x \leftarrow e} E(e)$. Then there is $g \in E(e)$ such that $f = \mathrm{res}_{x \leftarrow e}(g)$. If $v \subseteq e$, then $f \in \sum_{y \in X(1) : v \subseteq y \subseteq x} \mathrm{res}_{x \leftarrow y} E(y)$. Otherwise, $t := e \cup v \in X(2)$. Let e' and e'' be the edges of t different from e. Then $v \subseteq e'$ and $v \subseteq e''$. By (a2), there are $f' \in E(e')$ and $f'' \in E(e'')$ such that $g|_t = f'|_t + f''|_t$. This means that $f = g|_x = f'|_x + f''|_x \in \sum_{y \in X(1) : v \subseteq y \subseteq x} \mathrm{res}_{x \leftarrow y} E(y)$, which is what we want. $\square$

Proof of Theorem 9.5. Write $\mathcal{C} = \mathcal{C}_E$. The argument is similar to the proof of Theorem 9.2.

We first show that if q is sufficiently large, then there exists $\varepsilon'_0 > 0$, not depending on q, such that $(X, \overline{\mathcal{F}})$ is a 0-local ε'_0-coboundary expander in dimension 1. Let $v \in X(0)$. Then X_v is a q-thick spherical building of dimension $d - 1$, and Lemma 9.7 and conditions (a1), (a2) say that $\overline{\mathcal{F}}_v$ is isomorphic to a sheaf as in Theorem 5.13. Thus, $(X_v, \overline{\mathcal{F}}_v)$ is a ε'-coboundary expander in dimension 0 for $\varepsilon' = \frac{2(d-1)}{5(d-1)+2} - O_d(\frac{1}{\sqrt{q}})$. Taking q large enough in advance, we get that $(X_v, \overline{\mathcal{F}}_v)$ is a $\frac{1}{4}$-coboundary expander in dimension 0, so $\varepsilon'_0 = \frac{1}{4}$ suffices.

Next, we claim that $(X, \overline{\mathcal{F}})$ is a 1-local ε'_1-coboundary expander in dimension 1 for $\varepsilon'_1 = \frac{1}{2}$. Let $e \in X_{\mathrm{ord}}(1)$ and let $f \in \overline{\mathcal{F}}_e(\emptyset) = \overline{\mathcal{F}}(e) = \mathcal{F}(e)/E(e)$; we shall freely regard f as a member of $C^{-1}(X_e, \overline{\mathcal{F}}_e)$. Fix some 0-face v of e. Then, for every $t \in X(2)_{\supseteq e}$, we have $f|_t = f + \mathcal{C}(t)$. Thanks to (a2), we have $\mathcal{C}(t) = E(e)|_t + E(v \cup (t - e))|_t$. Thus, by condition (a1) and the assumption that $\mathcal{F}$ is locally constant, $f|_t = 0$ if and only if $f \in (E(e) + E(v \cup (t - e)))/E(e)$. Condition (a1) also means that $(E(e) + E(v \cup (t - e))) \cap (E(e) + E(v \cup (t' - e))) = E(e)$ for every $t' \in X(2)_{\supseteq e}$ different from t, so we can have $f|_t = 0$ for at most one $t \in X(2)_{\supseteq e}$. This means that $\mathrm{supp}(d_{-1} f) \supseteq X_e - \{t - e\}$ for some $t \in X(2)_{\supseteq e}$, so $(X_e, \mathcal{F}_e)$ is a $(1 - \xi)$-coboundary expander for $\xi = \max\{w_{X_e}(u) \,|\, u \in X_e(0)\}$. Writing $z = e - v$, equation (3.3) gives $w_{X_v}(u \cup z) = 2 w_{X_v}(z) w_{X_e}(u)$ for every $u \in X_e$. Thus, $\xi = \max\{\frac{1}{2} w_{X_v}(u \cup z) w_{X_v}(z)^{-1} \,|\, u \in X_e(0)\}$. Since X_v is a q-thick spherical building of dimension $d - 1$, [FK21, Lemma 7.5] says that $\xi \leq \frac{1}{2} \cdot \frac{2}{q+d-1} = \frac{1}{q+d-1}$. This means that $(X, \overline{\mathcal{F}})$ is a 1-local $(1 - \frac{1}{q+d-1})$-coboundary expander in dimension 1, and the claim follows since $d \geq 2$ and $q \geq 1$.

As similar argument shows that if q is large enough, then there is $\varepsilon_0 > 0$, not depending on q, such that $(X, \overline{\mathcal{F}})$ is a 0-local ε_0-coboundary expander in dimension 0. Briefly, one similarly finds that this holds for $\varepsilon_0 = 1 - \zeta$ with $\zeta = \max\{w_{X_v}(z) \,|\, v \in X(0), z \in X_v(0)\}$, and by [FK21, Lemma 7.5], $\zeta \leq \frac{2}{q+d-1}$, because X is q-thick.

Now that $\varepsilon'_0, \varepsilon'_1, \varepsilon_0$ have been determined, define λ to be the largest real number for which the inequality of Theorem 8.1 holds, and proceed as in the proof of Theorem 9.2. □

Remark 9.8. In Theorems 9.2 and 9.5, the lower bound on the thickness of the building Y can be lowered by using part (ii) of Theorem 5.12 instead of part (i) whenever possible, and by using Corollary 8.15 instead of Theorem 8.1 when $k = 0$. If X is moreover assumed to be a 2-dimensional Ramanujan complex (in the sense of [CSŻ03], [LSV05a]), then the lower bound on the required thickness can be further lowered by using Proposition 3.5 instead of Theorem 3.4.

9.3 Good 1-Cocycle Codes

We finish this section by giving examples of 1-cocycle codes with linear distance and constant rate. We do not know if these codes are locally testable relative to their natural 3-tester.

Construction 9.9. Let X be a connected d-complex ($d \geq 2$) and assume that there are $Q \in \mathbb{N}$, $\lambda \in [-1, 1]$ and $\kappa \in [1, 2]$ such that

(1) every 0-face of X is contained in exactly Q edges,

(2) the weighted graph underlying X_v is a $[-1, \lambda]$-spectral expander for every $v \in X(0)$,

(3) $\kappa^{-1} w(e) \leq w(e') \leq \kappa w(e)$ for every $e, e' \in X(1)$ sharing a vertex.

Fix integers $0 < r < m$ such that $\frac{\kappa}{Q} < \frac{r}{m} < \frac{2}{Q}$. Let $\mathbb{F}$ be a finite field and let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf of dimension m. Suppose that, for every edge $e \in X(1)$, we are given an $(m - r)$-dimensional subspace $U(e) \subseteq \mathcal{F}(e)$, and that these subspaces are in general position in the following sense:

(4) For every $v \in X(0)$ and any $S \subseteq X_{\supseteq v}$ with $|S| r \geq m$, we have $\bigcap_{e \in S} \mathrm{res}^{-1}_{e \leftarrow v} U(e) = 0$.

Define a subsheaf $\mathcal{G}$ of $\mathcal{F}$ by setting

- $\mathcal{G}(v) = 0$ for every $v \in X(0)$,
- $\mathcal{G}(e) = U(e)$ for every $e \in X(1)$,
- $\mathcal{G}(x) = \mathcal{F}(x)$ for every $x \in X$ of dimension > 1.

Since $\dim_{\mathbb{F}} \mathcal{G}(e) = m - r$ for all $e \in X(1)$, we may form the 1-cocycle code of $(X, \mathcal{G})$ as in §7.3; its alphabet is $\Sigma := \mathbb{F}^{m-r}$.

Remark 9.10. (i) If the subspaces $U(e)$ are chosen uniformly at random, then the probability that condition (4) will hold for particular v and S is at least $\prod_{i=1}^{m}(1 - |\mathbb{F}|^{-i}) > 1 - 2|\mathbb{F}|^{-1}$. In particular, the probability that this holds for all e and S is at last $1 - |X(0)|2^{Q+1}|\mathbb{F}|^{-1}$. This means that we can find subspaces $\{U(e)\}_{e\in X(1)}$ as in (3) if $|\mathbb{F}| > |X(0)|2^{Q}$. (This bound can be improved to $|\mathbb{F}| > \mathrm{poly}(Q)$ with a little more work.)

(ii) Conditions (1)–(4) of Construction 9.9 are local in the sense that they involve only the proper links of X. As a result, if $p : Y \to X$ is a covering and $\mathcal{F}$ and $\{U(e)\}_{e\in X(1)}$ are as in Construction 9.9, then (1)–(4) hold for Y, the sheaf $p^*\mathcal{F}$ (see §4.3) and the subspaces $\{U(e')\}_{e'\in Y(1)}$ defined by $U(e') = U(p(e))$.

Theorem 9.11. *With notation as in Construction 9.9, suppose that*

$$\lambda \leq \min\{\frac{1}{6}(\frac{1}{4}(1-\lambda)(1-\frac{\kappa/Q}{r/m}) - \lambda)^2(1-\lambda), \frac{1}{4}(1-\lambda)(1-\frac{\kappa/Q}{r/m})\}.$$

Then there exists $\delta > 0$, depending on Q, λ, r, m, κ and $D(X)$, such that the 1-cocycle code associated to $(X, \mathcal{G})$ has relative distance $\geq \delta$. The rate of this code is $\geq \frac{2}{Q} - \frac{r}{m} - |X(1)|^{-1}$.

Proof. Note that $B^1(X, \mathcal{G}) = 0$. Thus, by Proposition 7.8(i), in order to prove the lower bound on the relative distance, it is enough to show that $(X, \mathcal{G})$ β-expands γ-small locally minimal 1-cochains for some $\beta, \gamma \in \mathbb{R}_+$ depending only on Q, λ, r, m, κ. To that end, we apply Corollary 8.14 to $(X, \mathcal{G})$.

The fact that $\mathcal{G}$ is a subsheaf of the locally constant sheaf $\mathcal{F}$ implies that all of the restriction maps $\mathrm{res}^{\mathcal{G}}_{y\leftarrow x}$ are injective, so $(X, \mathcal{G})$ is a 1-local 1-coboundary expander in dimension 1.

We claim that $(X, \mathcal{G})$ is a 0-local ε_0-coboundary expander in dimension 1 for $\varepsilon_0 = (1-\lambda)(1-\frac{\kappa/Q}{r/m})$. To see this, fix $v \in X(0)$. Then $\mathcal{G}_v$ is a subsheaf of $\mathcal{F}_v$. Since $\mathcal{F}$ is locally constant of dimension m, we have $\mathcal{F}_v \cong (\mathbb{F}^m)_+$, so we may assume that $\mathcal{F}_v = (\mathbb{F}^m)_+$. In particular, we view $U(v \cup u)$ as a subspace of $\mathbb{F}^m$ for every $u \in X_v$. Let $f \in C^0(X_v, \mathcal{G}_v)$. We need to show that $\|d_0 f\| \geq \varepsilon_0\|f + B^0(X_v, \mathcal{G}_v)\| = \varepsilon_0\|f\|$. If $\|d_0 f\| \geq \varepsilon_0$, then this is clear, so assume $\|d_0 f\| < \varepsilon_0$. Assumption (2) and Theorem 5.11 imply that $(X_v, \mathcal{F}_v)$ is a $(1-\lambda)$-coboundary expander in dimension 0, so there is $g \in B^0(X_v, \mathcal{F}_v) = B^0(X_v, (\mathbb{F}^m)_+)$ such that $\|f - g\| \leq (1-\lambda)^{-1}\|d_0 f\| < \varepsilon_0(1-\lambda)^{-1}$. There is $g_0 \in \mathbb{F}^m$ such that $g(u) = g_0$ for all $u \in X_v(0)$. Put $S = \{u \in X_v(0) : f(u) = g_0\}$. Then $w_{X_v}(S) > 1 - (1-\lambda)^{-1}\varepsilon_0 = \frac{\kappa/Q}{r/m}$. Now, assumption (3) implies that $|S| > \frac{m}{r}$, so by (4), we have $\bigcap_{u\in S} U(v \cup u) = 0$. Since $g_0 \in \bigcap_{u\in S} U(v \cup u)$, we must have $g = 0$, and $\|f\| = \|f - g\| \leq (1-\lambda)^{-1}\|d_0 f\|$. This means that $\varepsilon_0\|f\| \leq (1-\lambda)\|f\| \leq \|d_0 f\|$, as required.

Next, assumption (2) implies that X is a 1-local λ-skeleton expander (see §3.4). Combining (2) with Oppenheim's Trickling Down Theorem [Opp15, Theorem 1.4], we see that the underlying weighted graph of X is a $[-1, \frac{\lambda}{1-\lambda}]$-spectral expander, so X is a $\frac{\lambda}{1-\lambda}$-skeleton expander.

Plugging everything into Corollary 8.14 now gives the existence of γ and β; the inequalities in the corollary hold by our assumptions on λ.

To finish, we show that the rate of $Z^0(X, \mathcal{G})$ is at least $\frac{2}{Q} - \frac{r}{m} - |X(1)|^{-1}$. This is equivalent to showing that $\dim_{\mathbb{F}} Z^0(X, \mathcal{G}) \geq |X(1)|m \cdot (\frac{2}{Q} - \frac{r}{m} - \frac{1}{|X(1)|})$. Observe that

$$\dim B^1(X, \mathcal{F}) = \dim C^0(X, \mathcal{F}) - \dim Z^0(X, \mathcal{F}) \geq |X(0)|m - m = |X(1)|m \cdot \frac{2}{Q} - m,$$

where the inequality follows from Lemma 4.17 and the last equality follows from assumption (1). On the other hand, $\dim C^1(X, \mathcal{G}) = |X(1)|(m - r) = |X(1)|m \cdot (1 - \frac{r}{m})$. Since $Z^1(X, \mathcal{G})$ contains

$C^1(X,\mathcal{G})\cap B^1(X,\mathcal{F})$ and the intersection takes place in the ambient space $C^1(X,\mathcal{F})$, of dimension $|X(1)|m$, it follows that

$$\dim Z^1(X,\mathcal{G}) \geq |X(1)|m\cdot((\frac{2}{Q}-\frac{1}{|X(1)|})+(1-\frac{r}{m})-1) = |X(1)|m\cdot(\frac{2}{Q}-\frac{r}{m}-\frac{1}{|X(1)|}). \quad \square$$

Remark 9.12. The pair $(X,\mathcal{G})$ does not satisfy the assumptions of Theorem 8.1 with $k=1$, so we cannot assert that $(X,\mathcal{G})$ is an (ε,δ)-cosystolic expander in dimension 1 for some $\varepsilon,\delta>0$. Indeed, $(X,\mathcal{G})$ is *not* a 1-local ε-coboundary expander in dimension 2 for every $\varepsilon>0$, because $\mathrm{H}^0(X_e,\mathcal{G}_e)\cong\mathcal{F}(e)/U(e)\neq 0$ for all $e\in X(1)$. Consequently, we cannot assert that the 1-cocycle code of $(X,\mathcal{G})$ (with its natural 3-tester) is μ-testable for $\mu>0$ independent of $(X,\mathcal{G})$. We do not know if such codes are good LTCs in general.

We finish with explaining how to get an infinite family of good 1-cocycle codes using Theorem 9.11.

Example 9.13. Let Y be the affine building of $\mathrm{SL}_3(F)$, where F is a local field with residue field of q elements; it is a 2-dimensional building of type $\tilde{A}_2$. It is well-known that one can find a sequence of simplicial complexes $\{X_s\}_{s\in\mathbb{N}\cup\{0\}}$ covered by Y and such $|X_s|$ tends to ∞ with s and such that each X_s covers X_0; see [LSV05a] for explicit constructions, or §13.2 below. For every $v\in Y$, the link Y_v is isomorphic to the spherical building $A_2(\mathbb{F}_q)$ of Example 3.3. This implies readily that, for every $s\in\mathbb{N}\cup\{0\}$, every 0-face of X_s is contained in exactly $2(q^2+q+1)$ edges, every edge of X_s is contained in exactly $q+1$ triangles, and $(X_s)_v$ is the incidence graph of the projective plane over $\mathbb{F}_q$ for every $v\in X_s(0)$. Thus, conditions (1)–(3) of Construction 9.9 hold for $X=X_s$ with $Q=2(q^2+q+1)$, $\lambda=\frac{\sqrt{q}}{q+1}$ and $\kappa=1$. Let $m=\lfloor\frac{3}{4}Q\rfloor$ and $r=1$, and let $\mathcal{F}_s$ denote the constant sheaf $\mathbb{F}^m$ on X_s. By Remark 9.10(i), for a sufficiently large finite field $\mathbb{F}$, there is a choice of subspaces $\{U(e)\}_{e\in X_0(1)}$ for which condition (4) holds with $(X,\mathcal{F})=(X_0,\mathcal{F}_0)$. Part (ii) of that remark then implies that a choice of $U(e)$-s satisfying (4) exists for every $(X_s,\mathcal{F}_s)$; let $\mathcal{G}_s$ denote the sheaf from Construction 9.9 constructed using this data. Note that λ tends to 0 as q tends to ∞. Thus, if q is sufficiently large, then Theorem 9.11 says that the family of 1-cocycle codes $\{Z^1(X,\mathcal{G}_r)\subseteq C^1(X,\mathcal{G}_r)\}_{r\in\mathbb{N}}$ on the alphabet $\Sigma:=\mathbb{F}^{m-1}$ has linear distance and constant rate.

10 Rate Conservation

Throughout this section, $\mathbb{F}$ denotes a finite field of characteristic $p>0$. Let X be a d-complex, let $k\in\{0,\dots,d-1\}$, let $\mathcal{F}$ be an $\mathbb{F}$-sheaf on X, and let B be a $\mathbb{F}$-basis of $\mathcal{F}$ (see Example 5.4). Recall from Section 7 that if $B^k(X,\mathcal{F})=0$ and $\dim\mathcal{F}(x)=m$ for all $x\in X(k)$, then $(X,\mathcal{F})$ gives rise to a k-cocycle code $Z^k(X,\mathcal{F})\subseteq C^k(X,\mathcal{F})\cong\Sigma^{X(k)}$ with alphabet $\Sigma=\mathbb{F}^m$. If $k>0$ and $B^k(X,\mathcal{F})\neq 0$, then $(X,\mathcal{F},B)$ gives rise to k-cocycle quantum CSS code with alphabet $\mathbb{F}$. In either case, the rate of the code is the ratio $\dim\mathrm{H}^k(X,\mathcal{F})/\dim C^k(X,\mathcal{F})$.

Let $u:Y\to X$ be a covering. Then the pullback sheaf $u^*\mathcal{F}$ similarly gives rise to a k-cocycle code-with-tester or a k-cocycle quantum CSS code. (Note that $B^k(X,\mathcal{F})=0$ implies $B^k(Y,u^*\mathcal{F})=0$.) In general, there is no relation between the rates of the k-cocycle codes associated to $(X,\mathcal{F})$ and $(Y,u^*\mathcal{F})$. However, in this section, we will show that under some assumptions on u and $(X,\mathcal{F})$, we can guarantee that the rate $\dim\mathrm{H}^k(Y,u^*\mathcal{F})/\dim C^k(Y,u^*\mathcal{F})$ is bounded from below by a constant depending only on X and $\mathcal{F}$. This principle, formalized as Theorem 10.3, will be called *rate conservation* in the sequel.

We begin with two lemmas. The cyclic group of order n is denoted C_n. We will make extensive use of C_n-Galois coverings in the sense §3.3. For example, every double covering is a C_2-Galois covering and vice versa (Example 3.2(i)). Recall (§4.2) that if $\mathcal{F}$ is an $\mathbb{F}$-sheaf on X, then $h^k(\mathcal{F})$ or $h^k(X,\mathcal{F})$, denotes $\dim_{\mathbb{F}} \mathrm{H}^k(X,\mathcal{F})$.

Lemma 10.1. *Let p be a prime number, let $u : Y \to X$ be a C_p-Galois covering of simplicial complexes, let $\mathbb{F}$ be a field of characteristic p, and let $\mathcal{F}$ be an $\mathbb{F}$-sheaf. Then there exists a sequence of subsheaves*

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_p = u_*u^*\mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{F}$ for all $i \in \{1,\dots,p\}$.

The proof of the lemma is shorter and more elementary when $p = 2$. To help the reader, we decided to address this special case before proving the lemma in general.

Proof when $p = 2$. Put $\mathcal{F}_2 = u_*u^*\mathcal{F}$ and let $\mathcal{F}_0$ be the zero subsheaf of $\mathcal{F}_2$. For every $x \in X - \{\emptyset\}$, we have $\mathcal{F}_2(x) = u_*u^*\mathcal{F}(x) = \prod_{y\in u^{-1}(x)} u^*\mathcal{F}(y) = \prod_{y\in u^{-1}(x)} \mathcal{F}(x) \cong \mathcal{F}(x)\times\mathcal{F}(x)$. Fix an isomorphism $\mathcal{F}_2(x) \cong \mathcal{F}(x)\times\mathcal{F}(x)$ for every x. If $y \in X_{\supseteq x} - \{x\}$, then the restriction map $\mathrm{res}^{\mathcal{F}_2}_{y\leftarrow x} : \mathcal{F}(x)\times\mathcal{F}(x) \to \mathcal{F}(y)\times\mathcal{F}(y)$ is either $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}\times\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$, or $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}\times\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ followed by swapping the two copies of $\mathcal{F}(y)$. This observation allows us to define a subsheaf $\mathcal{F}_1$ of $\mathcal{F}_2$ by setting $\mathcal{F}_1(x) = \{(f,f)\,|\,f \in \mathcal{F}(x)\}$ for all $x \in X - \{\emptyset\}$.

For every $x \in X-\{\emptyset\}$, define $\varphi_x : \mathcal{F}(x) \to \mathcal{F}_1(x)$ and $\psi_x : \mathcal{F}_2(x)/\mathcal{F}_1(x) \to \mathcal{F}(x)$ by $\varphi_x(f) = (f,f)$ and $\psi_x((f,g)+\mathcal{F}_1(x)) = f-g$. Using the assumption $\mathrm{char}\,\mathbb{F} = 2$, it is straightforward to check that $\varphi = (\varphi_x)_{x\in X-\{\emptyset\}}$ and $\psi = (\psi_x)_{x\in X-\{\emptyset\}}$ determine isomorphisms of sheaves $\varphi : \mathcal{F} \to \mathcal{F}_1 = \mathcal{F}_1/\mathcal{F}_0$ and $\psi : \mathcal{F}_2/\mathcal{F}_1 \to \mathcal{F}$, hence the lemma. □

Proof for general p. Let g denote a generator of the group C_p and let $\mathbb{F}C_p = \mathbb{F}[g\,|\,g^p = 1]$ denote the group algebra of C_p. Let I be the augmentation ideal of $\mathbb{F}C_p$, i.e., $I = (g-1)\mathbb{F}C_p$. Since $\mathrm{char}\,\mathbb{F} = p$, we have $I^p = 0$ (because $(g-1)^p = g^p - 1^p = 0$). In fact, one readily checks that $\dim_{\mathbb{F}} I^n = p-n$ for all $n \in \{0,\dots,p\}$. The quotient $\mathbb{F}C_p$-module I^n/I^{n+1} is spanned as an $\mathbb{F}$-vector space by $(g-1)^n$, and g acts trivially on I^n/I^{n+1} because, for every $a \in I^n$, we have $ga - a = (g-1)a \in (g-1)I^n = I^{n+1}$. In what follows, all tensor products are over $\mathbb{F}$.

For every $x \in X - \{\emptyset\}$, choose some $\hat{x} \in Y$ with $u(\hat{x}) = x$. Since $u : Y \to X$ is a C_p-Galois covering, $u^{-1}(x) = \{\tau\hat{x}\,|\,\tau \in C_p\}$. As a result, for every $y \in X_{\supseteq x}$, there is a unique element $c_{y,x} \in C_p$ such that $c_{y,x}\hat{x} \subseteq \hat{y}$. This also allows us to identify $u_*u^*\mathcal{F}(x) = \prod_{x'\in u^{-1}(x)} \mathcal{F}(x)$ with $\mathbb{F}C_p \otimes_{\mathbb{F}} \mathcal{F}(x)$ via sending $(f_{x'})_{x'\in u^{-1}(x)}$ to $\sum_{\tau\in C_p} \tau\otimes f_{\tau\hat{x}}$. The restriction map $\mathrm{res}^{u_*u^*\mathcal{F}}_{y\leftarrow x} : \mathbb{F}C_p\otimes\mathcal{F}(x) \to \mathbb{F}C_p\otimes\mathcal{F}(y)$ is then given by $a\otimes f \mapsto ac_{y,x}^{-1}\otimes(\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} f)$.

For $n \in \{0,\dots,p\}$, let $\mathcal{F}_n$ denote the subsheaf of $u_*u^*\mathcal{F}$ determined by $\mathcal{F}_n(x) = I^{p-n}\otimes\mathcal{F}(x)$. Fix $n \in \{1,\dots,p\}$. For every $x \in X - \{\emptyset\}$, define $\varphi_x : \mathcal{F}(x) \to \mathcal{F}_n(x)/\mathcal{F}_{n-1}(x) \cong (I^{p-n}/I^{p-n+1})\otimes\mathcal{F}(x)$ by $\varphi_x(f) = ((g-1)^{p-n} + I^{p-n+1})\otimes f$. This is an $\mathbb{F}$-vector space isomorphism. Moreover, since g, and thus all elements of C_p, act trivially on I^{p-n}/I^{p-n+1}, we have $\varphi_y\circ\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} = \mathrm{res}^{\mathcal{F}_n/\mathcal{F}_{n-1}}_{y\leftarrow x}\circ\varphi_x$ whenever $\emptyset \neq x \subsetneq y \in X$. This means that $\varphi = (\varphi_x)_{x\in X-\{\emptyset\}} : \mathcal{F} \to \mathcal{F}_n/\mathcal{F}_{n-1}$ is a an $\mathbb{F}$-sheaf isomorphism, and the lemma follows. □

Lemma 10.2. *Let p be a prime number, let $u : Y \to X$ be a C_p-Galois covering of simplicial complexes, let $\mathbb{F}$ be a field of characteristic p, let $\mathcal{F}$ be an $\mathbb{F}$-sheaf, and let $k \in \mathbb{N}\cup\{0\}$. Suppose that* $\mathrm{H}^{k-1}(X,\mathcal{F}) = 0$ *(this always holds for $k = 0$). Then*

(i) $\mathrm{H}^{k-1}(Y,u^*\mathcal{F}) = 0$, *and*

(ii) $h^k(Y,u^*\mathcal{F}) - h^{k+1}(Y,u^*\mathcal{F}) \geq p(h^k(X,\mathcal{F}) - h^{k+1}(X,\mathcal{F}))$.

Proof. Let $\{\mathcal{F}_i\}_{i=0}^p$ be the sequence of sheaves from Lemma 10.1, and put $N_i = \dim \mathrm{H}^k(X, \mathcal{F}_i) - \dim \mathrm{H}^{k+1}(X, \mathcal{F}_i)$. We will show by increasing induction on $i \in \{1, \dots, p\}$ that $\mathrm{H}^{k-1}(X, \mathcal{F}_i) = 0$ and $N_i \geq iN_1$. Provided this holds, taking $i = p$ and applying Lemma 4.11 to $\mathcal{F}_p = u_* u^* \mathcal{F}$ gives (i) and (ii).

The case $i = 1$ follows from the assumptions of the lemma, because $\mathcal{F}_1 \cong \mathcal{F}$.

Suppose that $i > 1$ and we have shown that $\mathrm{H}^{k-1}(X, \mathcal{F}_{i-1}) = 0$ and $N_{i-1} \geq (i-1)N_1$. The inclusion $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$ gives rise to a short exact sequence $0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$, and thus to a long exact sequence of cohomology groups (see §4.2):

$$\begin{aligned} \cdots \to &\mathrm{H}^{k-1}(X, \mathcal{F}_{i-1}) \to \mathrm{H}^{k-1}(X, \mathcal{F}_i) \to \mathrm{H}^{k-1}(X, \mathcal{F}) \\ \to &\mathrm{H}^{k}(X, \mathcal{F}_{i-1}) \to \mathrm{H}^{k}(X, \mathcal{F}_i) \to \mathrm{H}^{k}(X, \mathcal{F}) \\ \to &\mathrm{H}^{k+1}(X, \mathcal{F}_{i-1}) \to \mathrm{H}^{k+1}(X, \mathcal{F}_i) \to \mathrm{H}^{k+1}(X, \mathcal{F}) \to \cdots \end{aligned}$$

Here, we substituted $\mathcal{F}_i/\mathcal{F}_{i-1}$ with the isomorphic sheaf $\mathcal{F}$. Since both $\mathrm{H}^{k-1}(X, \mathcal{F}_{i-1})$ and $\mathrm{H}^{k-1}(X, \mathcal{F})$ are 0, so is $\mathrm{H}^{k-1}(X, \mathcal{F}_i)$. Write $V = \operatorname{coker}(\mathrm{H}^{k+1}(X, \mathcal{F}_i) \to \mathrm{H}^{k+1}(X, \mathcal{F}))$. Then we have a 7-term exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^{k}(X, \mathcal{F}_{i-1}) \to \mathrm{H}^{k}(X, \mathcal{F}_i) \to \mathrm{H}^{k}(X, \mathcal{F}) \\ &\to \mathrm{H}^{k+1}(X, \mathcal{F}_{i-1}) \to \mathrm{H}^{k+1}(X, \mathcal{F}_i) \to \mathrm{H}^{k+1}(X, \mathcal{F}) \to V \to 0 \end{aligned}$$

This means that

$$h^k(\mathcal{F}_{i-1}) - h^k(\mathcal{F}_i) + h^k(\mathcal{F}) - h^{k+1}(\mathcal{F}_{i-1}) + h^{k+1}(\mathcal{F}_i) - h^{k+1}(\mathcal{F}) + \dim V = 0,$$

and by rearranging, we get

$$N_i = N_{i-1} + N_1 + \dim V \geq (i-1)N_1 + N_1 = iN_i. \qquad \square$$

Theorem 10.3 (Rate Conservation)**.** *Let p be a prime number and let $\mathbb{F}$ be a field of characteristic p. Let X be a simplicial complex of dimension d, let $k \in \{0, \dots, d\}$ and let $\mathcal{F}$ be an $\mathbb{F}$-sheaf on X such that*

$$h^{k-1}(X, \mathcal{F}) = 0 \qquad \text{and} \qquad h^k(X, \mathcal{F}) > h^{k+1}(X, \mathcal{F}).$$

Put

$$\rho = (h^k(X, \mathcal{F}) - h^{k+1}(X, \mathcal{F}))/\dim C^k(X, \mathcal{F}).$$

Let $u : Y \to X$ be a covering map of degree p^r such that u factors as $Y = X_r \to X_{r-1} \to \cdots \to X_0 = X$ and each map $X_i \to X_{i-1}$ is a C_p-Galois covering. Then:

(i) $h^k(Y, u^*\mathcal{F}) - h^{k+1}(Y, u^*\mathcal{F}) \geq \rho \dim C^k(Y, u^*\mathcal{F})$.

(ii) The rate of the k-cocycle code of $(Y, u^*\mathcal{F})$ (resp. the k-cocycle quantum CSS code associated to $(Y, u^*\mathcal{F})$ and some basis of $u^*\mathcal{F}$) is at least ρ.

Proof. Let u_i denote the composition $X_i \to \cdots \to X_0 = X$. Applying Lemma 10.2 to the covering $X_i \to X_{i-1}$ and the sheaf $u_{i-1}^*\mathcal{F}$ with i ranging from 1 to r shows that $h^k(Y, u^*\mathcal{F}) - h^{k+1}(Y, u^*\mathcal{F}) \geq p^r(h^k(X, \mathcal{F}) - h^{k+1}(X, \mathcal{F})) \geq p^r \rho \dim C^k(X, \mathcal{F})$. Since $\dim C^k(Y, u^*\mathcal{F}) = p^r \dim C^k(X, \mathcal{F})$, this means that $h^k(Y, u^*\mathcal{F}) - h^{k+1}(Y, u^*\mathcal{F}) \geq \rho \dim C^k(Y, u^*\mathcal{F})$. This proves (i), and (ii) follows because $\dim Z^k(Y, u^*\mathcal{F}) \geq h^k(u^*\mathcal{F}) \geq h^k(u^*\mathcal{F}) - h^{k+1}(u^*\mathcal{F})$. $\square$

11 What Is Required to Construct an Infinite Family of LTCs?

We now put together the local-to-global principle for cosystolic expansion (Theorem 8.1) and the Rate Conservation Theorem (Theorem 10.3) to give a recipe for constructing infinite families of low-query LTCs with linear distance and constant rate. This is the *tower paradigm* outlined in Section 2.

Theorem 11.1 (Tower Paradigm)**.** *Let p be a prime number and let $\mathbb{F}$ be a finite field of characteristic p. Let X be a strongly connected d-complex, let $k \in \{0, \dots, d-2\}$ and let $\mathcal{F}$ be an $\mathbb{F}$-sheaf on X. Suppose that there is $m \in \mathbb{N}$ such that $\mathcal{F}(x) = \mathbb{F}^m$ for all $x \in X(k)$, that $B^{k-1}(X, \mathcal{F}) = 0$ (e.g., if $k = 0$), that $\mathcal{F}(x) \neq 0$ for all $x \in X(k+1) \cup X(k+2)$, and the following conditions are met:*

- *(t1) X admits an infinite tower of connected C_p-Galois coverings $\cdots \to X_2 \to X_1 \to X_0 = X$ (i.e., $X_r \to X_{r-1}$ is a C_p-Galois covering for all $r \in \mathbb{N}$).*
- *(t2) There exist numbers $\varepsilon_0, \dots, \varepsilon_k, \varepsilon'_0, \dots, \varepsilon'_{k+1}, \lambda \in \mathbb{R}_+$ satisfying the inequality of Theorem 8.1 and such that*
  - *(t2-a) $(X, \mathcal{F})$ is an i-local ε_i-coboundary expander in dimension k for all $i \in \{0, \dots, k\}$,*
  - *(t2-b) $(X, \mathcal{F})$ is an i-local ε'_i-coboundary expander in dimension $k+1$ for all $i \in \{0, \dots, k+1\}$, and*
  - *(t2-c) X is a $(d-2)$-local $[-1, \lambda]$-spectral expander.*
- *(t3) $\dim \mathrm{H}^0(X, \mathcal{F}) > \dim \mathrm{H}^1(X, \mathcal{F})$.*

Write u_r for the composition $X_r \to X_{r-1} \to \cdots \to X_0 = X$, put $\mathcal{F}_r = u_r^* \mathcal{F}$ and $\Sigma = \mathbb{F}^m$, and let

$$(Z^k(X_r, \mathcal{F}_r), C^k(X_r, \mathcal{F}_r) = \Sigma^{X_r(k)}, \Phi_r)$$

denote the k-cocycle code of $(X_r, \mathcal{F}_r)$ with its natural tester (§7.3). Then $\{(Z^k(X_r, \mathcal{F}_r), \Sigma^{X_r(k)}, \Phi_r)\}_{r \geq 0}$ is a family of $(k+2)$-query LTCs with linear distance and constant rate. Moreover, there is $\eta > 0$ such that every code in the family admits a linear-time decoding algorithm able to correct up to an η-fraction of errors.

If only conditions (t1) and (t2) are met, then $\{(Z^k(X_r, \mathcal{F}_r), \Sigma^{X_r(k)}, \Phi_r)\}_{r \geq 0}$ is a family of LTCs with linear distance and the codes admit a decoding algorithm as above. If only conditions (t1) and (t3) are met, then the codes $\{(Z^k(X_r, \mathcal{F}_r), \Sigma^{X_r(k)})\}_{r \geq 0}$ have constant rate.

By default, we will assume that $k = 0$ when talking about the tower paradigm. In this special case, Theorem 11.1 gives a recipe for getting an infinite family of 2-query LTCs with linear distance and constant rate.

Proof. Condition (t3) and Theorem 10.3 imply that the rate of the codes $\{(Z^k(X_r, \mathcal{F}_r), \Sigma^{X_r(k)})\}_{r \geq 0}$ is bounded from below by some $\rho > 0$. The remaining assertions follow from Proposition 7.8 and Theorem 8.1, provided that conditions (1)–(5) of Theorem 8.1 hold for every $(X_r, \mathcal{F}_r)$ with $\varepsilon_0, \dots, \varepsilon_k, \varepsilon'_0, \dots, \varepsilon'_{k+1}, \lambda$ as in (t2) and $Q = D(X)$. These conditions hold for $(X, \mathcal{F})$ by our assumptions, so they also hold for $(X_r, \mathcal{F}_r)$ by Remark 8.3(iv). □

Remark 11.2. In Theorem 11.1, it is possible to replace (t2) and the connectivity assumption in (t1) with milder assumptions by using Theorem 8.11 and its corollaries instead of Theorem 8.1 in the proof. Specifically, in the case $k = 0$, if we use Corollary 8.15 instead of Theorem 8.1, then we can replace (t1) and (t2) with the following: There are $\alpha_{-1}, \alpha_0, \varepsilon_0, \varepsilon'_0, \varepsilon'_1 \in \mathbb{R}_+$ satisfying the inequalities Corollary 8.15 such that (t1′) X admits an infinite tower of C_p-Galois coverings $\cdots \to X_2 \to X_1 \to X_0 = X$, and each X_r is an α_{-1}-skeleton expander.

(t2′) $(X, \mathcal{F})$ is a 0-local ε_0-coboundary expander in dimension 0, an i-local ε'_i-coboundary expander in dimension 1 for $i \in \{0, 1\}$ and a 0-local α_0-skeleton expander.

Thanks to Theorem 11.1, the problem of constructing an infinite family of good 2-query LTCs reduces to the following question:

Question 11.3. *Is there a sheaved d-complex $(X, \mathcal{F})$ satisfying conditions (t1)–(t3) of Theorem 11.1 with $k = 0$?*

Note that we only need a *single* pair $(X, \mathcal{F})$ satisfying (t1)–(t3) with $k = 0$; we will refer to such pairs as *initial data* for the tower paradigm. Note also that once a candidate $(X, \mathcal{F})$ is presented, conditions (t2) and (t3) can be checked by computation, and only condition (t1) needs a theoretical proof.

Finding initial data for the tower paradigm is the subject matter of Chapter III, where we reduce the problem to an experiment-supported conjecture and the existence of certain arithmetic groups. We also show in §14.2 that any two of the conditions (t1)–(t3) are fulfilled for some pair $(X, \mathcal{F})$. Alas, Question 11.3 remains open.

We finish this section by explaining why simplicial complexes with a locally constant sheaf (see §4.5) cannot serve as initial data for the tower paradigm. Note first that if $\mathcal{F}$ is a nonzero locally constant sheaf on X, then the assumption $B^k(X, \mathcal{F}) = 0$ of Theorem 11.1 is satisfied only if $k = 0$. The following proposition says that in this case, conditions (t1) and (t3) cannot hold simultaneously.

Proposition 11.4. *Let X be a d-complex, let $\mathbb{F}$ be a field of characteristic $p > 0$, and let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf on X. Suppose that X admits an infinite tower of connected C_p-Galois coverings $\cdots \to X_2 \to X_1 \to X_0 = X$. Then* $\dim \mathrm{H}^0(X, \mathcal{F}) \leq \dim \mathrm{H}^1(X, \mathcal{F})$.

Proof. Write u_r for the map $X_r \to \cdots \to X_1 \to X$ and put $\mathcal{F}_r = u_r^* \mathcal{F}$. Then $\mathcal{F}_r$ is locally constant of dimension $\dim \mathcal{F}$. Since X_n is connected, Lemma 4.17 tells us that $h^0(X_r, \mathcal{F}_r) \leq \dim \mathcal{F}_r = \dim \mathcal{F}$. Now, if it were the case that $h^0(X, \mathcal{F}) > h^1(X, \mathcal{F})$, then Theorem 10.3 would imply that $h^0(X_r, \mathcal{F}_r)$ tends to ∞ as $r \to \infty$, which contradicts our previous conclusion that $h^0(X_r, \mathcal{F}_r) \leq \dim \mathcal{F}$ for all r. Thus, we must have $h^0(X, \mathcal{F}) \leq h^1(X, \mathcal{F})$. ☐

Remark 11.5. There is an analogue of Theorem 11.1 for quantum CSS codes. That is, we can impose conditions similar to (t1)–(t3) on a sheaved d-complex $(X, \mathcal{F})$ that would give rise to an infinite family of one-sided locally testable quantum CSS codes that have constant rate, linear X-distance, and whose X-side has a linear-time decoding algorithm able to correct a constant-fraction of errors. Simply assume $k > 0$, drop the assumption $B^k(X, \mathcal{F}) = 0$, and replace the use of Proposition 7.8 with Proposition 7.11.

Unlike the case of LTCs, it is seemingly possible for d-complexes with locally constant sheaves to satisfy the required conditions.

Chapter III

Toward Initial Data for The Tower Paradigm

Having the tower paradigm (Theorem 11.1 with $k = 0$) at our disposal to produce good 2-query LTCs, we now set to look for sheaved complexes $(X, \mathcal{F})$ satisfying its three prerequisites (t1)–(t3) (with $k = 0$). We only need one such pair. The purpose of this chapter is to construct sheaved complexes which satisfy (t1) (existence of an infinite tower), (t2) (local expansion conditions) and conjecturally also (t3) (rate conservation). While (t3) could be checked by computation, such a computation is beyond the reach of present computers because of the sheer size of X and $\mathcal{F}$.

In more detail, our approach to constructing initial data for the tower paradigm starts with a d-complex X and a locally constant $\mathbb{F}$-sheaf $\mathcal{F}$ on X of a large (but ultimately fixed) dimension.[1] As in §4.2, we abbreviate

$$h^i(\mathcal{F}) = \dim \mathrm{H}^i(X, \mathcal{F}).$$

In Section 12, we present an iterative process taking $\mathcal{F}$ and producing a subsheaf $\mathcal{C}$ of $\mathcal{F}$ such that $\overline{\mathcal{F}} := \mathcal{F}/\mathcal{C}$ is a (non-locally constant) sheaf with $h^0(\overline{\mathcal{F}}) > h^1(\overline{\mathcal{F}})$, that is, it satisfies the requirement (t3). We show that if the resulting subsheaf $\mathcal{C}$ of $\mathcal{F}$ — which grows with each iteration of the process — is "small" with respect to $\mathcal{F}$, then $(X, \overline{\mathcal{F}})$ will satisfy the local expansion conditions in (t2) when $(X, \mathcal{F})$ satisfies them. (We could also terminate the process while $\mathcal{C}$ is still "small" to secure (t2), and hope that it suffices to get (t3).) Choosing X in advance so that it has an infinite tower of connected double coverings would secure (t1).

Broadly speaking, we expect the process to converge quickly enough when $h^1(\mathcal{F})$ is small with respect to $\dim \mathcal{F}$. We conjecture that this is indeed the case when X is covered by a sufficiently thick affine building (Conjecture 12.9). We show in Theorems 12.10 and **??** that there are simplicial complexes covered by affine buildings which admit $\mathbb{F}_2$-sheaves $\mathcal{F}$ of arbitrarily large dimension such that $h^1(\mathcal{F}) = o(\dim \mathcal{F})$ (even $h^1(\mathcal{F}) = O(1)$ or $h^1(\mathcal{F}) = 0$, in some cases). Thus, if our conjecture holds for just one such X, then there is an $\mathbb{F}_2$-sheaf $\mathcal{F}$ on X such that if we feed it into our process to produce $\overline{\mathcal{F}} = \mathcal{F}/\mathcal{C}$, then $(X, \overline{\mathcal{F}})$ satisfies the requirements (t1)–(t3) of the tower paradigm. In particular, $(X, \overline{\mathcal{F}})$ gives rise to an infinite family of 2-query LTCs with constant rate and linear distance.

We also give strong evidence that if the kernel of the cup product $\cup : \mathrm{H}^1(X, \mathbb{F}) \otimes_{\mathbb{F}} \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{F})$ (see §4.6) is of dimension smaller than $h^0(\mathcal{F})$, then the iterative process stops quickly enough (after one step, in fact). Assuming this, we show that if a sheaf satisfying the said condition exists over a simplicial complex X covered by a sufficiently thick affine building, then there is a locally constant sheaf $\mathcal{F}'$ on $\mathcal{F}$ such that $(X, \overline{\mathcal{F}'})$ satisfies the requirements of the tower paradigm.

[1] Recall that the tower paradigm *cannot* work for locally constant sheaves; see Proposition 11.4.

The iterative process is presented and discussed in Section 12. In Section 14, we show that there exist simplicial complexes X covered by arbitrarily thick affine buildings that admit (1) an infinite tower of double covering, and (2) $\mathbb{F}_2$-sheaves $\mathcal{F}$ of arbitrarily large dimension such that $h^1(\mathcal{F}) = o(\dim \mathcal{F})$. We also demonstrate that any pair of the conditions (t1)–(t3) of the tower paradigm are satisfied by some sheaved complex. The intermediate Section 13 establishes the existence of another tower of coverings (not the tower required for the tower power paradigm) with some special properties that is needed for the construction of the desired sheaves in Section 14. This makes use of deep results about arithmetic groups.

12 Modifying Sheaves to Get Rate Conservation

Throughout this section, $\mathbb{F}$ is a field (of any characteristic) and X is a d-complex with $d \geq 2$. We fix a linear ordering on the vertices of X and use it to identify $C^i(X,\mathcal{F})$ with $\prod_{x\in X(i)} \mathcal{F}(x)$ for every sheaf $\mathcal{F}$ on X and $i \in \{0,\dots,d\}$, see Remark 4.5. We let $\mathcal{F}$ denote a locally constant $\mathbb{F}$-sheaf of dimension m.

12.1 An Iterative Modification Process

Recall from Construction 9.4 that if E is an $\mathbb{F}$-subspace of $C^1(X,\mathcal{F})$, then we can form a subsheaf $\mathcal{C}_E$ of $\mathcal{F}$ by setting:

$$\mathcal{C}_E(x) = \sum_{e\in X(1)_{\subseteq x}} \mathrm{res}_{x\leftarrow e} \mathrm{Proj}_e(E)$$

for all $x \in X$, where $\mathrm{Proj}_e : C^1(X,\mathcal{F}) \to \mathcal{F}(e)$ is the projection $f \mapsto f(e) : C^1(X,\mathcal{F}) \to \mathcal{F}(e)$. Otherwise stated, $\mathcal{C}_E$ is the smallest subsheaf of $\mathcal{F}$ such that $E \subseteq C^1(X,\mathcal{C}_E)$. We will be interested in quotients sheaves of the form

$$\mathcal{F}_E := \mathcal{F}/\mathcal{C}_E.$$

Since $\dim \mathcal{C}_E(v) = 0$ for all $v \in X(0)$, we have $\mathcal{F}_E(v) = \mathcal{F}(v)$ for all $v \in X(0)$. However, if $E \neq 0$, then $\mathcal{C}_E \neq 0$, and as a result, $\mathcal{F}_E$ is not locally constant. When $\dim E \ll \dim \mathcal{F}$, the sheaf $\mathcal{F}_E$ may be regarded as being "close" to $\mathcal{F}$ because $\dim \mathcal{C}(x) \leq \binom{i}{2} \dim E \ll \dim \mathcal{F}$ for every i-face $x \in X$.

With the tower paradigm in mind, our purpose will be to find a (typically small) subspace E of $C^1(X,\mathcal{F})$ such that $(X,\mathcal{F}_E)$ satisfies conditions (t2) and (t3) of Theorem 11.1 (always with $k=0$). We first focus on (t3), which says that $h^1(\mathcal{F}_E) < h^0(\mathcal{F}_E)$.

The effect of E on $\dim \mathrm{H}^i(X,\mathcal{F}_E)$ for $i = 0,1$ is (crudely) described in the following proposition.

Proposition 12.1. *In the previous notation, let $B_E = E \cap B^1(X,\mathcal{F})$, $Z_E = E \cap Z^1(X,\mathcal{F})$ and let $H_E \cong Z_E/B_E$ be the image of Z_E under the quotient map $Z^1(X,\mathcal{F}) \to \mathrm{H}^1(X,\mathcal{F})$. Denote the natural map $\mathrm{H}^2(X,\mathcal{C}_E) \to \mathrm{H}^2(X,\mathcal{F})$ by ω_E. Then:*

(i) $h^0(\mathcal{F}_E) \geq h^0(\mathcal{F}) + \dim B_E$.

(ii) $h^1(\mathcal{F}_E) \leq h^1(\mathcal{F}) - \dim H_E + \dim \mathrm{im}(\omega_E)$.

Proof. Recall from §4.2 that the short exact sequence $0 \to \mathcal{C}_E \to \mathcal{F} \to \mathcal{F}_E \to 0$ gives rise to a long cohomology exact sequence:

$$\begin{aligned}0 =&\mathrm{H}^0(X,\mathcal{C}_E) \to \mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(X,\mathcal{F}_E) \to \\ &\mathrm{H}^1(X,\mathcal{C}_E) \xrightarrow{\alpha} \mathrm{H}^1(X,\mathcal{F}) \to \mathrm{H}^1(X,\mathcal{F}_E) \to \\ &\mathrm{H}^2(X,\mathcal{C}_E) \xrightarrow{\omega} \mathrm{H}^2(X,\mathcal{F})\end{aligned}$$

Since $C^0(X, \mathcal{C}_E) = 0$, we have $\mathrm{H}^0(X, \mathcal{C}_E) = 0$ and $\mathrm{H}^1(X, \mathcal{C}_E) = Z^1(X, \mathcal{C}_E)$. Noting that $Z_E \subseteq Z^1(X, \mathcal{C}_E)$ and $B_E = Z_E \cap B^1(X, \mathcal{F}) \subseteq Z^1(X, \mathcal{C}_E) \cap B^1(X, \mathcal{F})$, we see that $B_E \subseteq \ker \alpha$ and $H_E \subseteq \operatorname{im} \alpha$. The proposition now follows readily from the exactness. $\square$

If we (incorrectly) ignore the factor $\dim \operatorname{im}(\omega_E)$ in Proposition 12.1 (say, if $\omega_E = 0$), then the affect of replacing E with $E + \mathbb{F}f$ for some $f \in C^1(X, \mathcal{F}) - E$ can be summarized as follows:

- if $f \in B^1(X, \mathcal{F})$, then adding f to E increases $h^0(\mathcal{F}_E)$ by 1,
- if $f \in Z^1(X, \mathcal{F}) - B^1(X, \mathcal{F})$, then adding f to E decreases $h^1(\mathcal{F}_E)$ by 1 and leaves $h_0(\mathcal{F}_E)$ unchanged, and
- if $f \in C^1(X, \mathcal{F}) - Z^1(X, \mathcal{F})$, then adding f to E *seemingly* has no affect on $h^0(\mathcal{F}_E)$ and $h^1(\mathcal{F}_E)$.

Recall that our goal is to choose E so that $h^0(\mathcal{F}_E) > h^1(\mathcal{F}_E)$. Taking these thumb rules as facts, we could attempt to achieve this by simply taking E to be a subspace of $Z^1(X, \mathcal{F})$ of dimension $> h^1(\mathcal{F}) - h^0(\mathcal{F})$. Indeed, we can decompose E as $E_1 \oplus E_2$ with $E_1 \subseteq B^1(X, \mathcal{F})$ and $E_2 \cap B^1(X, \mathcal{F}) = 0$. We expect to have $h^0(\mathcal{F}_E) \geq h^0(\mathcal{F}) + \dim E_1$ and $h^1(\mathcal{F}) \leq h^1(\mathcal{F}_E) - \dim E_2$, which together gives $h^0(\mathcal{F}_E) - h^1(\mathcal{F}_E) > 0$.

Unfortunately, $\ker(\omega_E : \mathrm{H}^2(X, \mathcal{C}_E) \to \mathrm{H}^2(X, \mathcal{F}))$ may be nonzero, and its effect on $h^1(\mathcal{F}_E)$ should be taken into account. However, if $\ker \omega_E \neq 0$, then we can attempt to eliminate $\ker \omega_E$ by enlarging E with more 1-cochains. Specifically, choose a subspace $K \subseteq C^2(X, \mathcal{C}_E)$ mapping bijectively onto $\ker \omega_E \subseteq \mathrm{H}^2(X, \mathcal{C}_E)$ and subspace $E' \subseteq C^1(X, \mathcal{F})$ such that d_1 restricts to a bijection $E' \to K$ (it exists because $K \subseteq B^2(X, \mathcal{F})$), and replace $E_0 := E$ with $E_1 := E + E'$. We show in Proposition 12.4(i) below that replacing E by $E + E'$ does not affect $Z_E = E \cap Z^1(X, \mathcal{F})$ and $B_E = E \cap B^1(X, \mathcal{F})$. At the same time, replacing E by $E + E'$ trivializes the cohomology classes in $\ker(\omega_{E_0} : \mathrm{H}^2(X, \mathcal{C}_{E_0}) \to \mathrm{H}^2(X, \mathcal{F}))$, because the natural map $\mathrm{H}^2(X, \mathcal{C}_{E_0}) \to \mathrm{H}^2(X, \mathcal{C}_{E_1})$ vanishes on on $\ker \omega_{E_0}$. The replacement of E by $E + E'$ may result in new cohomology classes in $\ker \omega_E$, so we can repeat this process until $\ker \omega_E = 0$. We therefore arrive at the following iterative process:

Construction 12.2. Let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf on a d-complex X such that $h^1(\mathcal{F}) \geq h^0(\mathcal{F})$.

(1) Set E_0 to be the zero subspace of $C^0(X, \mathcal{F})$.

(2) Let E_0' be a subspace of $Z^1(X, \mathcal{F})$ of dimension $h^1(\mathcal{F}) - h^0(\mathcal{F}) + 1$.[2]

(3) Set $r = 1$ and $E_1 = E_0'$

(4) While $\dim E_{r-1}' > 0$:

   (a) Choose a subspace $E_r' \subseteq C^1(X, \mathcal{F})$ such that $d_1(E_r') \subseteq Z^2(X, \mathcal{C}_{E_r})$ and the composition $E_r' \xrightarrow{d_1} Z^2(X, \mathcal{C}_{E_r}) \to \mathrm{H}^2(X, \mathcal{C}_{E_r})$ maps E_r' bijectively onto $\ker(\omega_{E_r} : \mathrm{H}^2(X, \mathcal{C}_{E_r}) \to \mathrm{H}^2(X, \mathcal{F}))$.

   (b) Set $E_{r+1} = E_r + E_r'$ and increase r by 1.

(5) Set $E = E_r$.

We say that the iterative process stops or converges after n steps if $E = E_n$. If $\mathbb{F}$ is finite, and if not indicated otherwise, we assume that the spaces $E_0', E_1', \ldots$ defined in (2) and (a) are chosen uniformly at random among all eligible subspaces of $C^1(X, \mathcal{F})$.

[2]It is also possible to take a subspace of larger dimension.

In what follows, we abbreviate ω_{E_r} to ω_r and $\mathcal{C}_{E_r}$ to $\mathcal{C}_r$, so that

$$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq C^1(X, \mathcal{F}) \qquad \text{and} \qquad \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{F}.$$

Observe that $\dim E'_r$ is determined by E_r. We shall see in the the following proposition that $E'_r \cap E_r = 0$, and hence $\dim E_{r+1} = \dim E_r + \dim E'_r$. Thus, $\dim E_{r+1}$ determined by E_r. However, the choice of E'_r may affect $\dim E_s$ for $s > r+1$.

Part (ii) of the following proposition says that $\mathcal{F}_E$ satisfies condition (t3) when E is the subspace constructed in the iterative process of Construction 12.2.

Proposition 12.3. *With notation as above:*

(i) $E_r \cap E'_r = 0$, $E_r \cap Z^1(X, \mathcal{F}) = E_1 \cap Z^1(X, \mathcal{F})$ and $E_r \cap B^1(X, \mathcal{F}) = E_1 \cap B^1(X, \mathcal{F})$ for every $r \in \mathbb{N}$.

(ii) The process in Construction 12.2 stops, and the resulting subspace E satisfies $h^0(\mathcal{F}_E) > h^1(\mathcal{F}_E)$.

Proof. (i) By construction, $E'_r \cap d_1^{-1}(B^2(X, \mathcal{C}_r)) = 0$. Since d_1 maps $E_r \subseteq C^1(X, \mathcal{C}_r)$ into $B^2(X, \mathcal{C}_r)$, this means that $E'_r \cap E_r = 0$.

Next, if $r > 1$ and $f \in E_r \cap Z^1(X, \mathcal{F}) = (E_{r-1} + E'_{r-1}) \cap Z^1(X, \mathcal{F})$, then we can write $f = g + g'$ with $g \in E_{r-1}$ and $g' \in E'_{r-1}$. Since $d_1 f = 0$, we have $d_1 g' = -d_1 g \in B^2(X, \mathcal{C}_{r-1})$, so $g' \in d_1^{-1}(B_2(X, \mathcal{C}_{r-1})) \cap E'_{r-1} = 0$. This means that $g' = 0$, hence $f = g \in E_{r-1} \cap Z^1(X, \mathcal{F})$. By induction on r, it follows that $E_r \cap Z^1(X, \mathcal{F}) = E_1 \cap Z^1(X, \mathcal{F})$. Since $B^1(X, \mathcal{F}) \subseteq Z^1(X, \mathcal{F})$, this means that $E_r \cap B^1(X, \mathcal{F}) = E_1 \cap B^1(X, \mathcal{F})$.

(ii) Since $\dim E_r$ is bounded from above by $\dim C^1(X, \mathcal{F})$, in order to prove that the process stops, it is enough to show that $\dim E_{r+1} > \dim E_r$ for all $r \geq 0$ such that $\ker \omega_r \neq 0$. By (i), $\dim E_{r+1} = \dim E'_r + \dim E_r = \dim E_r + \dim \ker \omega_r$, hence our claim.

Suppose now that the process stopped after r steps. Then $E = E_r$ and $\ker \omega_r = \ker \omega_E = 0$. Write $V_1 = E'_0 \cap B^1(X, \mathcal{F})$ and choose a subspace V_2 such that $E'_0 = V_1 \oplus V_2$. By (i), $\dim(E \cap B^1(X, \mathcal{F})) = \dim(E_1 \cap B^1(X, \mathcal{F})) = \dim V_1$ and $\dim(E \cap Z^1(X, \mathcal{F})) = \dim(E_1 \cap Z^1(X, \mathcal{F})) = \dim V_2$. Now, applying Proposition 12.1 to $E = E_r$, we get $h^0(\mathcal{F}_E) - h^1(\mathcal{F}_E) \geq (h^0(\mathcal{F}) + \dim V_1) - (h^1(\mathcal{F}) - \dim V_2 + \dim \ker \omega_r) = \dim E'_0 - (h^1(\mathcal{F}) - h^0(\mathcal{F})) > 0$. $\square$

Now that we know that iterative process of Construction 12.2 outputs a subspace $E \subseteq C^1(X, \mathcal{F})$ such that $\mathcal{F}_E$ satisfies condition (t3) of Theorem 11.1, we turn to check whether $\mathcal{F}_E$ also satisfies the local expansion conditions in (t2). This is *a priori* not true in general. Indeed, the process might stop only when $\mathcal{C}_E(x) = \mathcal{F}(x)$ for all $x \in X - X(0) - X(-1)$, in which case $\mathcal{F}_E$ will be isomorphic to the sheaf obtained from $\mathcal{F}$ by setting $\mathcal{F}_E(x) = \mathcal{F}(x) \cong \mathbb{F}^m$ if $x \in X(0)$ and $\mathcal{F}_E(x) = 0$ otherwise. The pair $(X_v, (\mathcal{F}_E)_v)$ is a poor coboundary expander for every $v \in X(0)$, so condition (t2) of the tower paradigm will not hold for $(X, \mathcal{F}_E)$. Nevertheless, we will now show that if $\dim E \ll \dim \mathcal{F}$ and X is covered by a sufficiently thick affine building, then, with high probability, $\mathcal{F}_E$ satisfies (t2).

To that end, we would like to apply Theorem 9.5. Recall that in order to use this theorem, E must satisfy the following to conditions:

(a1) For every $v \in X(0)$, the map $\sum_e \mathrm{res}_{e \leftarrow v}^{-1} : \bigoplus_e \mathcal{C}_E(e) \to \mathcal{F}(v)$, with e ranging over $X(1)_{\supseteq v}$, is injective.

(a2) For every $t \in X(2)$ with edges e, e', e'', we have $\mathcal{C}_E(e)|_t \subseteq \mathcal{C}_E(e')|_t + \mathcal{C}_E(e'')|_t$.

We would therefore like to choose the spaces $E_0, E_1, \ldots$ of Construction 12.2 such that they all satisfy (a1) and (a2).

Proposition 12.4. *With notation as in Construction 12.2, suppose that $\mathbb{F}$ is a finite field and let $Q = D_{0,1}(X) = \max\{\#X(1)_{\supseteq v} \,|\, v \in X(0)\}$. Then:*

(i) For all $r \in \mathbb{N} \cup \{0\}$, condition (a2) holds for E_r.

(ii) For every $r \in \mathbb{N} \cup \{0\}$, if condition (a1) holds for E_r and

$$\dim E_{r+1} \le \frac{\dim \mathcal{F} - \log_{|\mathbb{F}|} |X(0)|}{Q},$$

then E_{r+1} can be chosen to satisfies (a1). More precisely, if E'_r is chosen uniformly at random, then (a1) is satisfied with probability $> 1 - |X(0)||\mathbb{F}|^{Q \dim E_{r+1} - \dim \mathcal{F}}$.

The non-probabilistic assertions of (ii) also hold if $\mathbb{F}$ is infinite upon replacing $\log_{|\mathbb{F}|} |X(0)|$ with 0.

Proof. (i) We use induction on r. The case $r = 0$ is clear, and the case $r = 1$ follows from Example 9.6.

Suppose that $r > 1$ and we proved that (a2) holds for E_{r-1}. Let $t \in X(2)$ be a triangle with edges e, e', e'', and let $f_e \in \mathcal{C}_r(e)$. We need to show that $f_e|_t \in \mathrm{res}_{t\leftarrow e'} \mathcal{C}_r(e') + \mathrm{res}_{t\leftarrow e''} \mathcal{C}_r(e'')$. There is $f \in E_r$ such that $f_e = f(e)$ (recall that we have fixed a linear ordering on $X(0)$ and used it to identify $C^i(X, \mathcal{F})$ with $\prod_{x \in X(i)} \mathcal{F}(x)$). By construction, $E_r = E_{r-1} + E'_{r-1}$ with $d_1(E'_{r-1}) \subseteq C^1(X, \mathcal{C}_{r-1})$, so $d_1 f \in C^1(X, \mathcal{C}_{r-1})$. Thus, for some choice of signs, $f(e)|_t \in \pm f(e')|_t \pm f(e'')|_t + \mathcal{C}_{r-1}(t)$. By the definition $\mathcal{C}_{r-1}$ and the induction hypothesis, $\mathcal{C}_{r-1}(t) = \mathcal{C}_{r-1}(e)|_t + \mathcal{C}_{r-1}(e')|_t + \mathcal{C}_{r-1}(e'')|_t \subseteq \mathcal{C}_{r-1}(e')|_t + \mathcal{C}_{r-1}(e'')|_t$. It follows that $f(e)|_t \in \pm f(e')|_t \pm f(e'')|_t + \mathcal{C}_{r-1}(e')|_t + \mathcal{C}_{r-1}(e'')|_t \subseteq \mathcal{C}_r(e')|_t + \mathcal{C}_r(e'')|_t$, which is what we want.

(ii) Suppose first that $r > 0$. Fix a subspace $E' \subseteq C^1(X, \mathcal{F})$ such that $d_1 E' \subseteq Z^2(X, \mathcal{C}_r)$ and the composition $E' \xrightarrow{d_1} Z^2(X, \mathcal{C}_r) \to \mathrm{H}^2(X, \mathcal{C}_r)$ is injective with image $\ker \omega_r$ and let $h'_1, \dots, h'_t$ be a basis of E' (so $t = \dim \ker \omega_r$). In order to choose E'_r uniformly at random in Construction 12.2, we can choose $h_i \in h'_i + Z^1(X, \mathcal{F})$ uniformly at random for each $i \in \{1, \dots, t\}$ and take $E'_r = \mathbb{F}h_1 + \dots + \mathbb{F}h_t$.

Fix some $v \in X(0)$ and $i \in \{1, \dots, t\}$. Abbreviate $N(v) = X(1)_{\supseteq v}$. We claim that the collection $\{(\mathrm{res}_{e\leftarrow v})^{-1}(h_i(e))\}_{e \in N(v)}$ distributes uniformly in $\mathcal{F}(v)^{N(v)}$. To see this, write $z_i = h_i - h'_i$. It is enough to show that $\{(\mathrm{res}_{e\leftarrow v})^{-1}(z_i(e))\}_{e \in N(v)}$ distributes uniformly in $\mathcal{F}(v)^{N(v)}$, which, in turn, will follow if we show that the linear transformation $T : Z^1(X, \mathcal{F}) \to \mathcal{F}(v)^{N(v)}$ given by $T(f) = (\mathrm{res}_{e\leftarrow v}^{-1} f(e))_{e \in N(v)}$ is onto. Given $(f_e)_{e \in N(v)} \in \mathcal{F}(v)^{N(v)}$, define $g \in C^0(X, \mathcal{F})$ by

$$g(x) = \begin{cases} -[x \cup v : x] \,\mathrm{res}_{x\cup v\leftarrow x}^{-1} \mathrm{res}_{x\cup v\leftarrow v} f_{x\cup v} & x \in X(1)_v \\ 0 & x \notin X(1)_v, \end{cases}$$

where $[x \cup v : x]$ is 1 if $x < v$ relative to the ordering on $V(X)$, and -1 otherwise. It is straightforward to check that $T(d_0 g) = (f_e)_{e \in N(v)}$, so T is onto, and our claim follows.

Since $h_1, \dots, h_t$ are chosen independently, the previous paragraph implies that, for every $v \in X(0)$, the collection $\{(\mathrm{res}_{y\leftarrow v})^{-1}(h_i(e))\}_{e \in N(v), i \in \{1,\dots,t\}}$ distributes uniformly in $\mathcal{F}(v)^{E(v) \times \{1,\dots,t\}}$. Let $R(v) = \sum_{e \in N(v)} \mathrm{res}_{e\leftarrow v}^{-1} \mathcal{C}_r(e)$; it is a subspace of $\mathcal{F}(v)$. Condition (a1) for E_r implies that $R(v) = \bigoplus_{e \in N(v)} \mathrm{res}_{e\leftarrow v}^{-1} \mathcal{C}_r(e)$. This means that

$$\dim R(v) = \sum_{e \in N(v)} \dim \mathcal{C}_r(e) \le \sum_{e \in N(v)} \dim E_r \le Q \dim E_r.$$

Now, setting $m = \dim \mathcal{F}$, the probability that $\{(\mathrm{res}_{e\leftarrow v})^{-1}(h_i(e))\}_{e \in N(v), i \in \{1,\dots,t\}}$ are linearly independent and span a subspace of $\mathcal{F}(v)$ meeting $R(v)$ only at 0 is $\prod_{j=0}^{tQ-1} (1 - |\mathbb{F}|^{j + \dim R(v) - m}) >$

$1-|\mathbb{F}|^{Qt+Q\dim E_r - m} = 1-|\mathbb{F}|^{Q\dim_{E_{r+1}} - m}$. Consequently, the probability that for every $v \in X(0)$, the collection $\{(\mathrm{res}_{e\leftarrow v})^{-1}(h_i(e))\}_{e\in N(v), i\in\{1,\dots,t\}}$ is linearly independent in $\mathcal{F}(v)$ and its span meets $R(v)$ only at 0 is greater than $1-|X(0)||\mathbb{F}|^{Q\dim E_{r+1}-m}$. This number is non-negative by our assumption on $\dim E_{r+1}$, so we may choose $h_1,\dots,h_t$ to satisfy the last condition, which is easily seen to imply that (a1) holds for E_{r+1}.

The same argument we used for $r>0$ also works for $r=0$. The only difference is that one starts with some subspace E' of $Z^1(X,\mathcal{F})$ of the same dimension as that of E'_0.

Finally, when $\mathbb{F}$ is infinite, an adaptation of the argument shows that if we write $h_i = z_i + h'_i$ with $z_i \in Z^1(X,\mathcal{F})$, then condition (a1) is met if $z_1,\dots,z_t$ are chosen outside of the zero locus of some nonzero multivariate polynomial on $Z^1(X,\mathcal{F})^t \cong \mathbb{F}^N$. Such a choice is possible because $\mathbb{F}$ is infinite. □

Corollary 12.5. *Fix $d \in \mathbb{N}-\{1\}$ and let $q=q(d)$ be as in Theorem 9.5. Let X be a d-complex covered by a q-thick affine building, let $\mathbb{F}$ be a finite field, and let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf on X with $h^1(\mathcal{F}) \geq h^0(\mathcal{F})$. Apply the iterative process of Construction 12.2 to $(X,\mathcal{F})$ and suppose that it stopped after n steps. Let $Q = D_{0,1}(X)$ and let r denote the maximal member of $\{0,1,\dots,n\}$ such that*

$$\dim E_r \leq Q^{-1}(\dim\mathcal{F} - \log_{|\mathbb{F}|}\dim\mathcal{F} - \log_{|\mathbb{F}|}|X(0)|).$$

Then, with probability greater than $1-\frac{r}{\dim\mathcal{F}} \geq 1-\frac{\dim E_r}{\dim\mathcal{F}} \geq 1-\frac{1}{Q}$, the sheaf $\mathcal{F}_{E_r}$ satisfies condition (t2) of the tower paradigm (i.e., Theorem 11.1 with $k=0$).

Proof. Proposition 12.4(ii) and our choice of r imply that for all $0 \leq s \leq r-1$, the probability that E_{s+1} satisfies (a1) provided that E_s satisfies it is greater than $1-|X(0)||\mathbb{F}|^{Q\dim E_{s+1}-\dim\mathcal{F}} \geq 1-|X(0)||\mathbb{F}|^{Q\dim E_r-\dim\mathcal{F}}$. Our assumption on $\dim E_r$ says that the latter quantity is at least $1-|X(0)||\mathbb{F}|^{-\log_{|\mathbb{F}|}\dim\mathcal{F}-\log_{|\mathbb{F}|}|X(0)|} = 1-\frac{1}{\dim\mathcal{F}}$. Since E_0 satisfies (a1), this means that the probability that E_r satisfies (a1) is greater than $1-\frac{r}{\dim\mathcal{F}}$. (We have $1-\frac{r}{\dim\mathcal{F}} \geq 1-\frac{\dim E_r}{\dim\mathcal{F}}$ because $\dim E_r > \dim E_{r-1} > \dots > \dim E_0 = 0$, by Proposition 12.3(i).) Proposition 12.4(i) also tells us that E_r satisfies (a2). Applying Theorem 9.5 to the sheaf $\mathcal{F}$ and the space E_r completes the proof. □

We conclude from Proposition 12.3(ii) and Corollary 12.5 that if X is covered by a sufficiently thick affine building, and if $\mathcal{F}$ is a locally constant $\mathbb{F}$-sheaf for which the iterative process of Construction 12.2 outputs a subspace $E \subseteq Z^1(X,\mathcal{F})$ with $\dim E \ll \dim\mathcal{F}$ with high probability, then $(X,\mathcal{F}_E)$ satisfies conditions (t2) and (t3) with high probability. Since $\dim E_1 = h^1(\mathcal{F}) - h^0(\mathcal{F}) + 1$, in order to have $\dim E \ll \dim\mathcal{F}$, we should start with a sheaf $\mathcal{F}$ such that $h^1(\mathcal{F}) \ll \dim\mathcal{F}$; we will show that such exist in §14. Taking this for granted, we turn to discuss the growth of $\dim E_r$ in the iterative process.

12.2 The Effect of The Cup Product on The Modification Process

As in §12.1, let $\mathcal{F}$ be a locally constant $\mathbb{F}$-sheaf on X such that $h^1(\mathcal{F}) \geq h^0(\mathcal{F})$. We assume that $\mathbb{F}$ is finite, and apply the notation introduced in Construction 12.2.

We conducted a number of computer simulations[3] where we applied Construction 12.2 to a variety of complexes[4] and sheaves. These simulations suggest that the typical behavior of the iterative process can be predicted by means of the cup product (see §4.6), as we now explain.

[3]The `Python` code of the simulations was written by the first named author and is attached to the `arXiv` version of the paper.

[4]So far, we checked different triangulations of a 3-dimensional torus and a 3-thick 2-dimensional Ramanujan complex with 273 vertices. The latter is a quotient of the explicit example in [LSV05a, §10] by a Borel subgroup of $\mathrm{GL}_3(\mathbb{F}_{16})$.

Let $r \in \mathbb{N}$ and suppose that $E_r = E_{r-1} \oplus E'_{r-1}$ of Construction 12.2 has just been defined. We can attempt to construct elements in the kernel of $\omega_r : \mathrm{H}^2(X, \mathcal{C}_r) \to \mathrm{H}^2(X, \mathcal{F})$ as follows. Let $f_1, \ldots, f_t$ be an $\mathbb{F}$-basis for E_{r-1} and let $f'_1, \ldots, f'_s$ be an $\mathbb{F}$-basis for E'_{r-1}. Suppose that there are $\alpha_1, \ldots, \alpha_t, \alpha'_1, \ldots, \alpha'_s \in C^1(X, \mathbb{F})$ such that

$$g := \sum_{i=1}^{t} \alpha_i \cup f_i + \sum_{j=1}^{s} \alpha'_j \cup f'_j \in B^2(X, \mathcal{F}). \tag{12.1}$$

Then $g \in B^2(X, \mathcal{F}) \cap C^2(X, \mathcal{C}_r) \subseteq Z^2(X, \mathcal{C}_r)$, which means that the cohomology class $[g]_{\mathcal{C}_r}$ is in $\ker \omega_r$. Denote by V_r the subspace of $C^1(X, \mathbb{F})^{t+s}$ consisting of tuples $(\alpha_1, \ldots, \alpha_t, \alpha'_t, \ldots, \alpha'_s)$ for which (12.1) holds.

In general, not all elements of V_r give rise to a nonzero class in $\ker \omega_r$. This happens in particular when $\alpha'_1, \ldots, \alpha'_s \in B^1(X, \mathbb{F})$. Indeed, in this case, there are $\beta'_1, \ldots, \beta'_s \in C^0(X, \mathbb{F})$ such that $\alpha'_j = d_0 \beta'_j$ for all $j \in \{1, \ldots, s\}$. By Proposition 4.19, this means that

$$g = \sum_{i=1}^{t} \alpha_i \cup f_i + \sum_{j=1}^{s} d_0 \beta'_j \cup f'_j = \sum_{i=1}^{t} \alpha_i \cup f_i - \sum_{j=1}^{s} \beta'_j \cup d_1 f'_j + \sum_{j=1}^{s} d_1(\beta'_j \cup f'_j).$$

If $r = 1$, then $t = 0$ and $d_1 f'_j = 0$ for all j, so $g = \sum_{j=1}^{s} d_1(\beta'_j \cup f'_j) \in B^2(X, \mathcal{C}_1)$ and $[g]_{\mathcal{C}_1} = 0$. If $r > 1$, then by the construction of E'_r, we have $d_1 f'_j \in C^2(X, \mathcal{C}_{r-1})$, so $\tilde{g} := g - \sum_{j=1}^{s} d_1(\beta'_j \cup f'_j) \in C^2(X, \mathcal{C}_{r-1})$. Since $[\tilde{g}]_{\mathcal{F}} = [g]_{\mathcal{F}} = 0$, we have $[\tilde{g}]_{\mathcal{F}} \in \ker \omega_{r-1}$, so, by the construction of E'_{r-1}, there is $h \in E'_{r-1}$ such that $d_1 h = \tilde{g}$. Consequently, $g = d_1 h + \sum_{j=1}^{s} d_1(\beta'_j \cup f'_j) \in B^2(X, \mathcal{C}_r)$, and $[g]_{\mathcal{C}_r} = 0$.

Similarly, if $\alpha'_1, \ldots, \alpha'_s$ are in the left radical of the pairing $\cup : C^1(X, \mathbb{F}) \times C^1(X, \mathcal{F}) \to C^2(X, \mathcal{F})$, denoted $L(\mathcal{F})$, then $[g]_{\mathcal{C}_r} = 0$.

Let U_r denote the sum of $L(\mathcal{F})$ and the subspace of V_r consisting of tuples $(\alpha_1, \ldots, \alpha_t, \alpha'_1, \ldots, \alpha'_s) \in V_r$ with $\alpha'_1, \ldots, \alpha'_s \in B^1(X, \mathbb{F})$. Our simulations suggest that if $\dim \mathcal{F}$ is big enough with respect to $\dim E_r$, then with high probability,

(i) all elements in $\ker \omega_r$ are obtained from elements of V_r as in (12.1), and

(ii) the elements of V_r which give rise to the zero element in $\ker \omega_r$ are precisely the subspace U_r.

Consequently, $\dim E'_r = \dim V_r - \dim U_r$. Informally, this means that if $\dim E_r \ll \dim \mathcal{F}$, then almost surely, relations coming from the cup product are the only explanation to elements in $\ker \omega_r$.

We now analyse heuristically how big should $\dim \mathcal{F}$ be with respect to $\dim E_r$ in order to make the above estimations valid. Classes in $\ker \omega_r$ which are not explained by the cup product may occur if $\dim \mathrm{H}^2(X, \mathcal{C}_r) > \dim \mathrm{H}^2(X, \mathcal{F})$, so we need to require that $\dim \mathrm{H}^2(X, \mathcal{C}_r) \leq \dim \mathrm{H}^2(X, \mathcal{F})$. The iterative process does not use information from faces of dimension ≥ 3, so we may assume that $\dim X = 2$. Now, since (a2) holds for E_r (Proposition 12.4(i)), we typically have $\dim \mathcal{C}_{E_r}(x) = 2 \dim E_r$ for $x \in X(2)$, so we expect that $\dim C^2(X, \mathcal{C}_r) = 2|X(2)| \dim E_r$. The kernel of $d_1 : C^1(X, \mathcal{C}_r) \to C^2(X, \mathcal{C}_r)$ contains $E_r \cap Z^1(X, \mathcal{F}) = E_1$ (Proposition 12.3(i)), so if $\dim X = 2$, then we have $\dim \mathrm{H}^2(X, \mathcal{C}_r) \geq \dim C^2(X, \mathcal{C}_r) - \dim C^1(X, \mathcal{C}_r) + \dim E_1 = (2|X(2)| - |X(1)|) \dim E_r + (h^1(\mathcal{F}) - h^0(\mathcal{F}) + 1)$. Our simulations suggest that equality holds with high probability. A similar computation shows that when $\dim X = 2$, we have $\dim \mathrm{H}^2(X, \mathcal{F}) \geq (|X(2)| - |X(1)| + |X(0)| - 1) \dim \mathcal{F}$. The requirement $\dim \mathrm{H}^2(X, \mathcal{C}_r) \leq \dim \mathrm{H}^2(X, \mathcal{F})$ is therefore likely to hold if

$$\dim E_r \leq \frac{(|X(2)| - |X(1)| + |X(0)| - 1) \dim \mathcal{F} - (h^1(\mathcal{F}) - h^0(\mathcal{F}) + 1)}{2|X(2)| - |X(1)|.} \tag{12.2}$$

The right hand side is roughly $\frac{1}{2} \dim \mathcal{F}$ if $|X(2)|$ is large w.r.t. $|X(1)|$ and $|X(0)|$.

We summarize our observations in the following conjecture, in which we let $|\mathbb{F}|$ or $\dim \mathcal{F}$ grow. It is supported by all of our simulations.

Conjecture 12.6. *With notation as in Construction 12.2, suppose that E_r has just been constructed, and thus $\dim E'_r = \dim \ker \omega_r$ is determined. Define V_r and U_r as above. Then:*

(i) If (12.2) holds, then $\dim E'_r = \dim V_r - \dim U_r$ with probability $1 - o(1)$ as a function of $|\mathbb{F}|$.

(ii) If (12.2) holds and M is the difference between the right hand side and the left hand side of (12.2), then $\dim E'_r = \dim V_r - \dim U_r$ with probability $1 - o(1)$ as a function of M.

In particular, if $V_r = U_r$, then the iterative process will stop at the r-th step with probability $1 - o(1)$ (in the sense of (i) or (ii)).

Informally, the conjecture means that the "most likely" value of $\dim E_r$ can be predicted purely by means of the cup product action of $C^i(X, \mathbb{F})$ on $C^j(X, \mathcal{F})$ for $i, j \in \{0, 1\}$. (We moreover expect that it is determined by the homotopy type of the differential graded module $C^*(X, \mathcal{F})$ over the differential graded algebra $C^*(X, \mathbb{F})$.)

Example 12.7. Let us use Conjecture 12.6 to predict what $\dim E_2$ will typically be. Recall that $E_1 = E_0 \oplus E'_0$ with $E_0 = 0$ and E'_0 a subspace of $Z^1(X, \mathcal{F})$ of dimension $s := h^1(\mathcal{F}) - h^0(\mathcal{F}) - 1$. (In fact, the analysis that we carry applies to any small subspace of $Z^1(X, \mathcal{F})$.) We assume that (12.2) holds for E_1, or equivalently, that $h^1(\mathcal{F}) - h^0(\mathcal{F}) + 1 \leq \frac{|X(2)| - |X(1)| + |X(0)| - 1}{2|X(2)| - |X(1)| + 1} \dim \mathcal{F}$.

Let $f'_1, \dots, f'_s$ be an $\mathbb{F}$-basis for E'_0. Then V_1 is the space of $(\alpha'_1, \dots, \alpha'_s) \subseteq C^1(X, \mathbb{F})^s$ such that

$$\sum_{j=1}^{s} \alpha'_j \cup f'_j \in B^2(X, \mathcal{F}), \tag{12.3}$$

and $U_1 = V_1 \cap (B^1(X, \mathbb{F}) + L(\mathcal{F}))^s$. Since the $f'_1, \dots, f'_s$ live in $Z^1(X, \mathcal{F})$, and since the cup product of cocycles is a cocycle, it is reasonable to expect that, modulo $L(\mathcal{F})$, (12.3) will hold only if $\alpha'_1, \dots, \alpha'_s \in Z^1(X, \mathbb{F})$; this heuristic is confirmed by our simulations. Replacing V_1 with $V_1 \cap Z^1(X, \mathcal{F})^s$ and U_1 with $B^1(X, \mathcal{F})^s$, so that $\alpha'_1, \dots, \alpha'_s \in Z^1(X, \mathbb{F})$, condition (12.3) is equivalent to having

$$\sum_{j=1}^{s} [\alpha'_j] \cup [f'_j]_{\mathcal{F}} = 0$$

in $\mathrm{H}^2(X, \mathcal{F})$. Since $\dim V_1 - \dim U_1$ equals the image of V_1 in $\mathrm{H}^1(X, \mathbb{F})^s$, it follows that

$$\dim V_1 - \dim U_1 = \ker([\alpha] \otimes f \mapsto [\alpha \cup f] : \mathrm{H}^1(X, \mathbb{F}) \otimes_{\mathbb{F}} E'_0 \to \mathrm{H}^2(X, \mathcal{F})).$$

This leads to Conjecture 12.8 below, which is again supported by our simulations.

The conjectural formula for $\dim E'_1 = \dim V_1 - \dim U_1$ also demonstrates how the choice of $E_1 = E'_0$ might affect $\dim E_2$. For example, if E'_0 is taken to be a subspace of $B^1(X, \mathcal{F})$, then $[f]_{\mathcal{F}} = 0$ for every $f \in E'_0$, and we find that, heuristically,

$$\dim E'_1 = \dim V_1 - \dim U_1 = s \cdot h^1(X, \mathbb{F}).$$

On the other hand, if $\dim E'_0$ is chosen such that $E'_0 \oplus B^1(X, \mathcal{F}) = Z^1(X, \mathcal{F})$, i.e., we are trying to eliminate *all* the cohomology classes in $\mathrm{H}^1(X, \mathcal{F})$ by passing to $\mathcal{F}_{E_1}$, then the map $E'_0 \to \mathrm{H}^1(X, \mathcal{F})$ is a bijection, and we get

$$\dim E'_1 = \ker([\alpha] \otimes [f] \mapsto [\alpha \cup f] : \mathrm{H}^1(X, \mathbb{F}) \otimes_{\mathbb{F}} \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^2(X, \mathcal{F})).$$

Conjecture 12.8. *With notation as in Construction 12.2, suppose that* $M := \frac{|X(2)|-|X(1)|+|X(0)|-1}{2|X(2)|-|X(1)|+1}\dim\mathcal{F}-(h^1(\mathcal{F})-h^0(\mathcal{F})+1)\geq 0$. *Then*

$$\dim E_1' = \ker([\alpha]\otimes f\mapsto[\alpha\cup f]:\mathrm{H}^1(X,\mathbb{F})\otimes_{\mathbb{F}}E_0'\to\mathrm{H}^2(X,\mathcal{F}))$$

with probability $1-o(1)$ *as a function of* M *(resp.* $|\mathbb{F}|$*).*

It possible to continue the analysis of Example 12.7 in order to predict the dimension of $\dim E_r$ for larger values of r. This is manageable for $r=3$, or if one assumes that $h^1(X,\mathcal{F})=0$, but the general case becomes intractable very quickly. We omit the details.

If X has significantly more 2-cells than 1-cells, then dimension considerations suggest that the equations defining V_r will be less and less likely to have nontrivial solutions as r grows. Thus, it may be the case that the iterative process of Construction 12.2 stops after a fixed number of steps if X is covered by a sufficiently thick affine building. We pose it as a conjecture, although we have no computational evidence.

Conjecture 12.9. *There* $q,d\in\mathbb{N}$ *(*$d\geq 2$*) and a function* $f:\mathbb{N}\cup\{0\}\to\mathbb{N}$ *such that if* X *is covered by a* q*-thick* d*-dimensional affine building, then, when the iterative process of Construction 12.2 stops, we have* $\dim E\leq f(h^1(\mathcal{F}))$ *with probability* $1-o(1)$ *as* $|\mathbb{F}|\to\infty$ *or* $\dim\mathcal{F}\to\infty$.

We will see in the sequel how to find infinite families of sheaves on d-complexes covered by q-thick buildings such that $h^1(\mathcal{F})=O(1)$ as a function of $\dim\mathcal{F}$, so any function $f:\mathbb{N}\cup\{0\}\to\mathbb{N}$ will do.

12.3 Candidates for Infinite Families of Good 2-Query LTCs

We conclude this section with showing how a positive answer to the conjectures raised in §12.2 would lead to the existence of an infinite family of good 2-query LTCs. To that end, we use the following theorem, that will be proved in Section 14.

Theorem 12.10. *For every* $q,d\in\mathbb{N}$ *with* $d\geq 2$*, there exists a* d*-complex* X *that is covered by a* q*-thick affine building and a nonzero locally constant* $\mathbb{F}_2$*-sheaf* $\mathcal{G}$ *on* X *such that* $h^0(\mathcal{G})=h^1(\mathcal{G})=0$*. Moreover* X *admits an infinite tower of double coverings* $\cdots\to X_2\to X_1\to X_0\to X$.

Based on this, we show:

Theorem 12.11. *Let* $d\in\mathbb{N}-\{1\}$*, let* $q=q(d)$ *be as in Theorem 9.5, and let* X *be a* d*-complex covered by a* q*-thick affine building as in Theorem 12.10. Let* $\mathbb{F}$ *be a finite field of characteristic* 2 *and suppose that one of the following holds:*

(1) Conjecture 12.8 holds for X*, and there is a nonzero locally constant* $\mathbb{F}$*-sheaf* $\mathcal{F}_0$ *on* X *and a subspace* $E\subseteq\mathrm{H}^1(X,\mathcal{F}_0)$ *of dimension* $h^1(\mathcal{F}_0)-h^0(\mathcal{F}_0)+1$ *such that* $\cup:\mathrm{H}^1(X,\mathbb{F})\otimes_{\mathbb{F}}E\to\mathrm{H}^2(X,\mathcal{F}_0)$ *is injective.*

(2) Conjecture 12.9 is true for X*.*

Then there exists an $\mathbb{F}$*-sheaf* $\mathcal{F}$ *on* X *such that, if we apply the iterative process of Construction 12.2 to* $\mathcal{F}$*, then the pair* $(X,\mathcal{F}_E)$ *satisfies conditions (t1), (t2) and (t3) of the tower paradigm (Theorem 11.1 with* $k=0$*) with probability* >0*. Consequently, there exists initial data for the tower paradigm, and as a result, an infinite family of* 2*-query LTCs with linear distance and constant rate.*

Note that $h^1(\mathcal{F}_0) - h^0(\mathcal{F}_0) + 1 \geq 0$ by Proposition 11.4.

Writing $\Gamma = \pi_1(X)$, the existence of $\mathcal{F}_0$ in (1) is equivalent to the existence of a nonzero representation $\rho : \Gamma \to \mathrm{GL}_m(\mathbb{F}_2)$ and a subspace $E \subseteq \mathrm{H}^1(\Gamma, \rho)$ of dimension $\mathrm{H}^1(\Gamma, \rho) - \mathrm{H}^0(\Gamma, \rho) + 1$ such that $\cup : \mathrm{H}^1(\Gamma, \mathbb{F}_2) \otimes E \to \mathrm{H}^2(\Gamma, \rho)$ is injective. There are representations ρ of arbitrarily large *finite* groups having this property, see the `MathOverflow` answer [24].

Proof. Let $\mathcal{G}$ be the sheaf promised by Theorem 12.10. We replace $\mathcal{G}$ with its base-change from $\mathbb{F}_2$ to $\mathbb{F}$ to assume that $\mathcal{G}$ is an $\mathbb{F}$-sheaf, see Lemma 4.9.

Suppose that (1) holds. For every $s \in \mathbb{N}$, put $\mathcal{F}_s = \mathcal{F}_0 \times \mathcal{G}^s$. By our assumptions on $\mathcal{G}$, the natural map $\mathcal{F}_0 \cong \mathcal{F}_0 \times 0 \to \mathcal{F}_0 \times \mathcal{G}^s = \mathcal{F}_s$ induces maps $\mathrm{H}^i(X, \mathcal{F}_0) \to \mathrm{H}^i(X, \mathcal{F}_s)$ which are bijective for $i \in \{0, 1\}$ and injections for $i \geq 2$. The map $\mathrm{H}^i(X, \mathcal{F}_0) \to \mathrm{H}^i(X, \mathcal{F}_s)$ is compatible with the cup product, so, writing V_s for the image of E in $\mathrm{H}^1(X, \mathcal{F}_s)$, the map $\cup : \mathrm{H}^1(X, \mathbb{F}) \otimes_{\mathbb{F}} V_s \to \mathrm{H}^2(X, \mathcal{F}_s)$ is injective.

Let E_1 be a subspace of $Z^1(X, \mathcal{F}_s)$ of dimension $h^1(\mathcal{F}_s) - h^0(\mathcal{F}_s) + 1 = \dim V_s$, chosen uniformly at random. As s grows, the probability that the image of E in $\mathrm{H}^1(X, \mathcal{F}_s)$ is V_s approaches some $p > 0$. Consequently, the probability that $\cup : \mathrm{H}^1(X, \mathbb{F}) \otimes_{\mathbb{F}} E_1 \to \mathrm{H}^2(X, \mathcal{F}_s)$ is injective is bounded from below by some $p' > 0$. It now follows from Conjecture 12.8 that for all s large enough, the iterative process of Construction 12.2 stops for $\mathcal{F}_s$ after 1 step with probability $p' > 0$. When this happens, the output of the process is the subspace E_1, so $\dim E = h^1(\mathcal{F}_0) - h^0(\mathcal{F}_0) + 1$ is independent of s. Thus, by Corollary 12.5, for all s large enough, $(X, (\mathcal{F}_s)_E)$ satisfies (t2) with probability $p'' > 0$. By Proposition 12.3(ii), $(X, (\mathcal{F}_s)_E)$ satisfies (t3), and (t1) holds by the choice of X in Theorem 12.10. To conclude, we can take $\mathcal{F} = \mathcal{F}_s$ for any s large enough.

The case where (2) holds is handled similarly but with the following differences: One can start with any locally constant sheaf $\mathcal{F}_0$ on X, e.g., the zero sheaf, and one uses Conjecture 12.9 to bound $\dim E$ from above by $f(h^1(\mathcal{F}_0))$. $\square$

Remark 12.12. Suppose that in Theorem 12.11 we take E be E_r of Corollary 12.5 instead of the output of Construction 12.2 (i.e., we terminate the iterative process of the construction when $\dim E_r$ is small w.r.t. to $\dim \mathcal{F}$). The same argument as in the proof of the theorem then shows that for all s large enough there exists a subspace $E \subseteq Z^1(X, \mathcal{F}_s)$ such that $(X, (\mathcal{F}_s)_E)$ satisfies (t1) and (t2) unconditionally, and also (t3) provided that Conjecture 12.9 holds.

13 Arithmetic Groups and Simplicial Complexes Covered by Affine Buildings

The purpose of this section is to prove the following theorem, which will be used in the next section to prove Theorem 12.10 and for a few other purposes.

Theorem 13.1. *Let $q, d \in \mathbb{N}$ and assume that $d \geq 3$. There exist a (finite) simplicial complex X covered by a q-think d-dimensional affine building, a tower of (proper) connected coverings $\cdots \to X'_2 \to X'_1 \to X'_0 = X$ and a constant $C \in \mathbb{R}_+$ such that the following hold:*

(i) Every connected covering of X admits an infinite tower of connected double coverings.

(ii) $[X'_r : X]$ is odd and $\dim \mathrm{H}^1(X'_r, \mathbb{F}_2) \leq C$ for all $r \in \mathbb{N} \cup \{0\}$.

Here, $[X'_r : X]$ denotes the degree of $X'_r \to X$.

The constructions we provide are particular simplicial complexes, which are described in §13.4. The theorem is also true for $d = 2$, but we omit most of the details; see Remark 13.18.

The proof of Theorem 13.1 will make extensive use of arithmetic groups — particularly congruence subgroups — and their actions on affine buildings. While we briefly recall the definitions and facts that we need in §13.1, some knowledge of algebraic number theory is nevertheless assumed. Familiarity with linear algebraic groups and affine group schemes is also recommended. We refer the reader to [PR94] for further details and an extensive discussion of these subjects. A gentle introduction to group schemes is [Wat79, Chapter 1].

Readers who wish to skip the proof of Theorem 13.1 should proceed to Section 14.

13.1 Preliminaries

We begin with recalling necessary facts about algebraic groups, setting notation along the way.

Let R be any commutative ring, let R-Alg denote the category of commutative R-algebras and let Grp denote the category of groups. By a *group scheme*[5] over R we mean a functor G from R-Alg to Grp for which there is a set of multivariate polynomials $f_1, \ldots, f_t \in R[x_1, x_2, \ldots, x_n]$ such that, for every $S \in R$-Alg, the set $G(S)$ is in a *natural* bijection with the solutions of the equations $f_1 = \cdots = f_t = 0$ in S^n. The actual polynomials $f_1, \ldots, f_t$ will rarely matter, and we would only care that they exist. When R is a field, group schemes over R are also called (linear) *algebraic groups* over R.

We will only need the following examples of group schemes.

Example 13.2. (i) The functor $S \mapsto \mathrm{SL}_m(S) : R\text{-}\mathsf{Alg} \to \mathsf{Grp}$ is a group scheme, denoted $\mathbf{SL}_m(R)$. (Formally, $(\mathbf{SL}_m(R))(S) = \mathrm{SL}_m(S)$ for all $S \in R$-Alg.) Indeed, the functoriality is clear, and $\mathrm{SL}_n(S)$ can be naturally identified with the zeroes of the polynomial $\det(x_{ij}) - 1$ in m^2 indeterminates.

(ii) The functor $\mathbf{G}_{\mathbf{m},R} : R\text{-}\mathsf{Alg} \to \mathsf{Grp}$ sending an R-algebra S to its group of invertible elements $S^\times$ is a group scheme. To see this, note that the map $s \mapsto (s, s^{-1}) : S^\times \to S^2$ identifies $S^\times$ with the solutions of the equation $x_1x_2 - 1 = 0$ in S^2 for every $S \in R$-Alg.

(iii) Given a commutative ring R, put $R[i] = R[i \,|\, i^2 = -1]$. Elements of $R[i]$ are formal sums $\alpha + \beta i$ with $\alpha, \beta \in R$, and the product in $R[i]$ is determined by the rule $i^2 = -1$. Write $\sigma_R : R[i] \to R[i]$ for the automorphism sending $\alpha + \beta i$ to $\alpha - \beta i$. For example, if $R = \mathbb{R}$, then $R[i]$ is just $\mathbb{C}$ and $\sigma_{\mathbb{R}}$ is complex conjugation. Given a matrix $a = (\alpha_{ij})_{i,j} \in \mathrm{M}_n(R[i])$, we write a^* for the matrix $(\sigma_R(\alpha_{ji}))_{i,j}$.

Define $\mathrm{SU}_m(R[i]/R) = \{a \in \mathrm{M}_m(R[i]) \,:\, a^*a = 1 \text{ and } \det(a) = 1_{R[i]}\}$; it is a subgroup of $\mathrm{GL}_m(R[i])$. The functor $\mathbf{SU}_m(R[i]/R) : R\text{-}\mathsf{Alg} \to \mathsf{Grp}$ sending a commutative R-algebra S to $\mathrm{SU}_m(S[i]/S)$ is a group scheme. Indeed, we can identify $\mathrm{M}_m(S[i])$ with S^{2m^2} by sending a matrix $a = (\alpha_{j\ell} + i\beta_{j\ell})_{j,\ell}$ to the vector $(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{mm}, \beta_{11}, \beta_{12}, \ldots, \beta_{mm}) \in S^{2m^2}$. The condition $\det(a) = 1_{S[i]}$ can now be rewritten as *two* polynomial equations with coefficients coming from R (or even $\mathbb{Z}$), and the condition $a^*a = 1$ can be rewritten as $2m^2$ polynomial equations.

(iv) We can generalize (iii) by fixing $r_0, r_1 \in R$ and replacing $R[i]$ with $\hat{R} := R[x \,|\, x^2 = r_1x + r_0]$; the R-automorphism $\sigma_R : \hat{R} \to \hat{R}$ then sends x to $r_1 - x$. Moreover, instead of considering matrices $a \in \mathrm{M}_m(\hat{S})$ with $a^*a = 1$, we could fix a matrix $M \in \mathrm{GL}_m(\hat{R})$ with $M^* = M$ and consider the group of matrices $a \in \mathrm{M}_m(S')$ satisfying $a^*Ma = M$ and $\det(a) = 1$. This group is denoted $\mathrm{SU}(f_S)$, where $f_S : \hat{S}^m \times \hat{S}^m \to \hat{S}$ is the σ_S-hermitian form corresponding to M, i.e., $f_S(x, y) = x^*My$ for $x, y \in \hat{S}^n$ (regarded as column vectors). The functor $S \mapsto \mathrm{SU}(f_S) : R\text{-}\mathsf{Alg} \to \mathsf{Grp}$ is a group scheme denoted $\mathbf{SU}(f)$.

[5] In this paper, all group schemes are affine and of finite presentation.

Suppose that G and H are group schemes over R. A morphism from G to H is a natural transformation f from G to H. In particular, the data of f consists of a group homomorphism $f_S : G(S) \to H(S)$ for every $S \in R\text{-}\mathsf{Alg}$. We say that f is a monomorphism if $f_S : G(S) \to H(S)$ is one-to-one for all $S \in R\text{-}\mathsf{Alg}$.

If $R' \in R\text{-}\mathsf{Alg}$, then every R' may be regarded as an R-algebra. This defines a functor $R'\text{-}\mathsf{Alg} \to R\text{-}\mathsf{Alg}$, and its composition with $G : R\text{-}\mathsf{Alg} \to \mathsf{Grp}$ is a group scheme over R', denoted $G_{R'}$. (The polynomial equations defining $G_{R'}$ are the same as those defining G, but we think of the coefficients as living in R' instead of R.)

Let I be an ideal of R (written $I \trianglelefteq R$). Then R/I is an R-algebra, and thus the quotient map $q : R \to R/I$ gives rise to a group homomorphism $Gq : G(R) \to G(R/I)$. We define

$$G(R; I) = \ker(G(R) \to G(R/I))$$

and call $G(R; I)$ a *principal congruence subgroup* of G (or $G(R)$, if G is clear from the context). A subgroup of $G(R)$ containing a principal congruence subgroup is called a *congruence subgroup* of G (or $G(R)$).[6]

Example 13.3. Taking $R = \mathbb{Z}$, $I = \ell\mathbb{Z}$ and $G = \mathbf{SL}_n(\mathbb{Z})$, the group $\mathrm{SL}_m(\mathbb{Z}; I) := G(\mathbb{Z}; I)$ is just the group of $m \times m$ integral matrices which have determinant 1 and are congruent to the identity matrix modulo ℓ.

The group scheme G is called *absolutely almost simple (and) simply connected* if there is a faithfully flat commutative R-algebra R' such that, up to isomorphism, $G_{R'}$ is in the list of *split* absolutely almost simple simply connected group schemes over R' (also called the absolutely almost simple simply connected *Chevalley groups* over R'). When R is a domain, this list consists of 4 infinite families and 5 exceptional groups, denoted A_m ($m \geq 1$), B_m ($m \geq 2$), C_m ($m \geq 3$), D_m ($m \geq 4$), E_6, E_7, E_8, F_4, G_2. The description of these group schemes will not matter to us except for the fact that A_m is the group scheme $\mathbf{SL}_{m+1}(R')$.[7] The *type* of G is the symbol (A_m, B_m, C_m, D_m, E_6, E_7, E_8, F_4 or G_2) used to denote $G_{R'}$. For example, G is absolutely almost simple simply connected of type A_m if $G_{R'} \cong \mathbf{SL}_{m+1}(R')$ for some faithfully flat $R' \in R\text{-}\mathsf{Alg}$. When R is a field, the absolutely almost simple simply connected group schemes of a given type further break into two kinds: *inner* and *outer*.

Suppose that R is a field K and G is absolutely almost simple simply connected. The group scheme G is called *isotropic* if there is a monomorphism from $\mathbf{G}_{\mathbf{m},K}$ to G, and *anisotropic* otherwise. The largest $r \in \mathbb{N} \cup \{0\}$ for which there is a monomorphism $f : (\mathbf{G}_{\mathbf{m},K})^r \to G$ is called the *rank* of G, and denoted $\operatorname{rank} G$. More generally, given a field extension L of K, we say that G is L-anisotropic (resp. L-isotropic) if G_L is anisotropic (resp. isotropic), and define the L-rank of G as $\operatorname{rank}_L G := \operatorname{rank}(G_L)$.

Example 13.4. (i) For all $m > 1$, the group $\mathbf{SL}_m(R)$ is absolutely almost simple simply connected of type A_{m-1}. If R is a field K, then $\mathbf{SL}_m(K)$ is isotropic of rank $m-1$. A monomorphism $f : (\mathbf{G}_{\mathbf{m},K})^{m-1} \to \mathbf{SL}_m(K)$ is given by $f_S(s_1, \dots, s_{m-1}) = \operatorname{diag}(s_1, \dots, s_{m-1}, s_1^{-1} \cdots s_{m-1}^{-1})$ for all $S \in K\text{-}\mathsf{Alg}$.

(ii) Suppose that $2 \in R^\times$. The group scheme $\mathbf{SU}_m(R[i]/R)$ of Example 13.2(iv) is absolutely almost simple simply connected of type A_{m-1}. To see this, suppose first that R contains an

[6]This definition of congruence subgroups is different from the one used in [PR94]. The definitions are nevertheless equivalent by [Fir22].

[7]The groups schemes B_m, C_m, D_m are also not difficult to describe and are $\mathbf{Spin}_{2m+1}(R')$, $\mathbf{Sp}_{2m}(R')$ and $\mathbf{Spin}_{2m}(R')$, respectively.

element $\varepsilon \in R$ with $\varepsilon^2 = -1$. Using this element, one can define an isomorphism of R-algebras $\alpha+\beta i \mapsto (\alpha+\varepsilon\beta, \alpha-\varepsilon\beta) : R[i] \to R\times R$ (this is bijective because $2 \in R^\times$). Under this isomrophism, σ_R corresponds to $\sigma'_R : R\times R \to R\times R$ given by $\sigma'_R(x,y) = (y,x)$. Now, a routine computation shows that the induced R-algebra isomorphism $\mathrm{M}_m(R[i]) \to \mathrm{M}_m(R\times R) \cong \mathrm{M}_m(R)\times \mathrm{M}_m(R)$ maps $\mathrm{SU}_m(R[i]/R)$ to the pairs of matrices $(a,b) \in \mathrm{M}_m(R)\times\mathrm{M}_m(R)$ with $\det(a) = \det(b) = 1_R$ and $ab = 1$, namely, onto $\{(a,a^{-1}) \mid a \in \mathrm{SL}_m(R)\}$. Since a similar computation applies over any commutative R-algebra, we have constructed an isomorphism from $G := \mathbf{SU}_m(R[i]/R)$ to $\mathbf{SL}_m(R)$. If R does not contain a root of -1, then we can simply adjoin one, setting $R' = R[i]$, and get that $G_{R'} = \mathbf{SU}_m(R'[i]/R') \cong \mathbf{SL}_m(R')$.

When R is a field, the algebraic group $\mathbf{SU}_m(R[i]/R)$ is inner if R contains a square root of -1 and outer otherwise.

(iii) The group scheme $\mathbf{SU}(f)$ of Example 13.2(v) is absolutely almost simple simply connected of type A_{m-1} if $r_1^2 + 4r_0 \in R^\times$. Assuming this and that R is a field K, it is inner if and only if $x^2 - r_1x - r_0$ has a root in K.

In the remainder of this section we will use the following general notation:

Notation 13.5.

- K is a global field, e.g., $\mathbb{Q}$ or $\mathbb{F}_p(t)$.
- $\mathcal{V}$ is the set of places of K and $\mathcal{V}_\infty$ is the subset of archimedean places.
- K_ρ is the completion of K at $\rho \in \mathcal{V}$.

If $\rho \in \mathcal{V}$ is a non-archimedean place, we also use ρ to denote the corresponding additive valuation $\rho : K_\rho \to \mathbb{Z}\cup\{\infty\}$ and set

- $\mathcal{O}_\rho = \{x \in K_\nu \,:\, \nu(x) \geq 0\}$,
- $\mathfrak{m}_\rho = \{x \in K_\nu \,:\, \nu(x) > 0\}$ (the maximal ideal of $\mathcal{O}_\rho$),
- $k(\rho) = \mathcal{O}_\rho/\mathfrak{m}_\rho$ (the residue field at ρ),
- $P_\rho = \mathcal{O}\cap\mathfrak{m}_\rho = \{x \in \mathcal{O} \,:\, \rho(x) > 0\}$ (the prime ideal of $\mathcal{O}$ corresponding to ρ).

We further fix the following data:

- S is a nonempty subset of $\mathcal{V}$ containing $\mathcal{V}_\infty$.
- $\mathcal{O} = \mathcal{O}^S$ is the ring of S-integers in K, namely, $\{x \in K \,:\, \rho(x) \geq 0 \text{ for all } \rho \in \mathcal{V} - S\}$. The fraction field of $\mathcal{O}$ is K.
- ν is a fixed non-archimedean place in S.
- $\mathbf{G}$ is a simply connected absolutely almost simple algebraic group over K.
- $\mathcal{G}$ is a group scheme over $\mathcal{O}$ such that $\mathcal{G}_K = \mathbf{G}$.
- $G = \mathbf{G}(K_\nu) = \mathcal{G}(K_\nu)$.
- Y is the affine building attached to $\mathbf{G}_{K_\nu}$. Its dimension is $\operatorname{rank}_{K_\nu}\mathbf{G}$ [Tit79].

Given an ideal $I \trianglelefteq \mathcal{O}$ and $\rho \in \mathcal{V} - S$, we let

- $I_\rho = I \cdot \mathcal{O}_\rho$.
- $\rho(I) = \min\{\nu(x) \mid x \in I\}$; if $I \neq 0$, then this is also the unique $n \in \mathbb{N} \cup \{0\}$ such that $I_\rho = \mathfrak{m}_\rho^n$.

For every $\rho \in \mathcal{V}$, the group $\mathbf{G}(K_\rho)$ inherits a topology from K_ρ, and if $\rho \notin S$, then $\mathcal{G}(\mathcal{O}_\rho)$ is a compact open subgroup of $\mathbf{G}(K_\rho)$.[8] A theorem of Bruhat, Tits and Rousseau (see [Pra82], for instance) states that $\mathbf{G}$ is K_ρ-anisotropic if and only if $\mathbf{G}(K_\rho)$ is compact. We shall also need the following facts.

Proposition 13.6. *With notation as in Notation 13.5, if there is $\theta \in S$ such that $\mathbf{G}$ is K_θ-isotropic, then $\mathcal{G}(\mathcal{O}; I)$ is dense $\prod_{\rho \in \mathcal{V}-S} \mathcal{G}(\mathcal{O}_\rho; I_\rho)$.*

Proof. Let $\mathbb{A}^S = \prod'_{\rho \in \mathcal{V}-S} K_\rho$ denote the adélès away from S. We embed K diagonally in $\mathbb{A}^S$. By the Strong Approximation Theorem ([Pra77], [Mar77]), $\mathbf{G}(K)$ is dense $\mathbf{G}(\mathbb{A}^S)$. Since $U := \prod_{\rho \in \mathcal{V}-S} \mathcal{G}(\mathcal{O}_\rho; I_\rho)$ is open in $\mathbf{G}(\mathbb{A}^S)$, the set $\mathbf{G}(K) \cap U$ is dense in U. As $K \cap \prod_{\rho \in \mathcal{V}-S} I_\rho = I$ inside $\mathbb{A}^S$, we have $\mathbf{G}(K) \cap U = \mathcal{G}(\mathcal{O}; I)$, and the proposition follows. □

Proposition 13.7. *With notation as in Notation 13.5, there is $D \in \mathbb{N}$ such that the following hold: Let R be a commutative $\mathcal{O}$-algebra with trivial $\mathcal{O}$-torsion and let $I \subseteq J$ be ideals of R. Then $\mathcal{G}(R; I)/\mathcal{G}(R; IJ)$ is isomorphic to a subgroup of the additive group $(I/IJ)^D$.*

Proof. We view both R and K as a subrings of $L := R \otimes_{\mathcal{O}} K$. By [Fir22], there is a monomorphism of algebraic groups $f : \mathbf{G} \to \mathbf{SL}_m(K)$ such that $f(\mathcal{G}(R; N)) = f(\mathbf{G}(L)) \cap \mathrm{SL}_m(R; N)$ for all $N \trianglelefteq R$. Thus, f induces a one-to-one group homomorphism $\overline{f} : \mathcal{G}(R; I)/\mathcal{G}(R; IJ) \to \mathrm{SL}_m(R; I)/\mathrm{SL}_m(R; IJ)$. Since $I \subseteq J$, we can define a map $g : \mathrm{SL}_m(R; I)/\mathrm{SL}_m(R; IJ) \to \mathrm{M}_m(I/IJ)$ by sending a matrix $x \in 1 + \mathrm{M}_m(I)$ to the image of $x - 1$ in $\mathrm{M}_m(I/IJ)$. It is straightforward to check that g is a one-to-one group homomorphism (use the fact that $I^2 \subseteq IJ$). Taking $D = m^2$, the lemma follows. □

13.2 Finite Quotients of Buildings

Keeping Notation 13.5, recall that $G = \mathbf{G}(K_\nu)$ acts on Y via simplicial automorphisms. In particular, for any $I \trianglelefteq \mathcal{O}$, the principal congruence subgroup $\Gamma = \mathcal{G}(\mathcal{O}; I)$ also acts on Y. In this subsection we will be concerned with determining when is $\Gamma \backslash Y$ a *finite* simplicial complex covered by Y, and, provided this is so, when does it admit an infinite tower of connected double coverings.

Beware that in general $\Gamma \backslash Y$ is only a partially ordered set relative to the face-inclusion ordering it inherits from Y. When $\Gamma \backslash Y$ is isomorphic to a simplicial complex as a partially ordered set, we will say that $\Gamma \backslash Y$ is a simplicial complex and treat it as one for all purposes. However, even when $\Gamma \backslash Y$ is a simplicial complex, the quotient map $Y \to \Gamma \backslash Y$ may not be a covering map.

Proposition 13.8. *With notation as in Notation 13.5, suppose that $\mathbf{G}$ is K-anisotropic, K_ρ-anisotropic for every $\rho \in S - \{\nu\}$, and K_ν-isotropic. Then:*

(i) $\mathcal{G}(\mathcal{O})$ is a discrete subgroup of G and $\mathcal{G}(\mathcal{O}) \backslash G$ is compact.

(ii) There is a finite subset $U \subseteq \mathcal{G}(\mathcal{O}) - \{1_G\}$ such that if $I \trianglelefteq \mathcal{O}$, $\mathcal{G}(\mathcal{O}; I) \cap U = \emptyset$ and Γ is a finite-index subgroup of $\mathcal{G}(\mathcal{O}; I)$, then Γ acts freely on Y, the quotient $\Gamma \backslash Y$ is a finite simplicial complex, $Y \to \Gamma \backslash Y$ is a covering map, and $\pi_1(\Gamma \backslash Y) \cong \Gamma$.

[8]Indeed, $\mathbf{G}(K_\rho)$ (resp. $\mathcal{G}(\mathcal{O}_\rho)$) may be understood as the solution set of some polynomial equations $f_1 = \cdots = f_r = 0$ in K_ρ^n (resp. $\mathcal{O}_\rho^n$). We give $\mathbf{G}(K_\rho)$ (resp. $\mathcal{G}(\mathcal{O}_\rho)$) the topology induced from K_ρ^n ($\mathcal{O}_\rho^n$). This is independent of how $\mathbf{G}$ (resp. $\mathcal{G}$) is realized as the solutions of polynomial equations.

Proof. (i) As in the proof of Proposition 13.6, let $\mathbb{A}$ denote the adélès ring of K, let $\mathbb{A}^S$ be the adélès away from S and set $\mathbb{A}_S = \prod_{\rho\in S} K_\rho$. Since $\mathbf{G}$ is K-anisotropic, the quotient $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ is compact, see [PR94, Theorem 5.5] and [Har69, Corollary 2.2.7]. Embedding K diagonally in $\mathbb{A}^S \times \mathbb{A}_S$, we have $\mathcal{O} = K \cap ([\prod_{\rho\notin S}\mathcal{O}_\rho] \times \mathbb{A}_S)$. Thus, $\mathcal{G}(\mathcal{O}) = \mathbf{G}(K) \cap (U \times \mathbf{G}(\mathbb{A}_S))$, where $U = \prod_{\rho\notin S}\mathcal{G}(\mathcal{O}_\rho)$ and the intersection is taken in $\mathbf{G}(\mathbb{A})$. It is now routine to check that the map $\mathcal{G}(\mathcal{O})\backslash\mathbf{G}(\mathbb{A}_S) \to \mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})/(U \times \{1_{\mathbf{G}(\mathbb{A}_S)}\})$ given by $\mathcal{G}(\mathcal{O})x \mapsto \mathbf{G}(k)(1_{\mathbf{G}(\mathbb{A}^S)}, x)(U \times \{1_{\mathbf{G}(\mathbb{A}_S)}\})$ is injective. By the Strong Approximation Theorem ([Pra77], [Mar77]), this map is also onto (here we need $\mathbf{G}$ to be simply-connected and K_ν-isotropic). As it is open as well, it is a homeomorphism and we conclude that $\mathcal{G}(\mathcal{O})\backslash\mathbf{G}(\mathbb{A}_S)$ is compact. Since $\mathcal{G}(\mathcal{O})\backslash G$ the image of $\mathcal{G}(\mathcal{O})\backslash\mathbf{G}(\mathbb{A}_S)$ under a continuous map, $\mathcal{G}(\mathcal{O})\backslash G$ is also compact. Finally, note that $\mathcal{O}$ is discrete in $\mathbb{A}_S$, and therefore $\mathcal{G}(\mathcal{O})$ is a discrete subgroup of $\mathbf{G}(\mathbb{A}_S) = \prod_{\rho\in S}\mathbf{G}(k_\rho)$. Since $\mathbf{G}(k_\rho)$ is compact for all $\rho \in S - \{\nu\}$, the image of $\mathcal{G}(\mathcal{O})$ in $G = \mathbf{G}(K_\rho)$ is also discrete.

(ii) The building Y attached to $\mathbf{G}_{K_\nu}$ is constructed so that the stabilizer of every nonempty face in Y is compact and open in G. Let $y_1, \ldots, y_t$ be representatives for the G-orbits in Y and let $K_i = \{g \in G : gy_i = y_i\}$. Then the G-set $\bigsqcup_{i=1}^t G/K_i$ can be identified with Y by mapping gK_i to gy_i for all $g \in G$ and $i \in \{1, \ldots, t\}$. Consequently, $\mathcal{G}(\mathcal{O})\backslash Y$ is in bijection with $\bigsqcup_{i=1}^t \mathcal{G}(\mathcal{O})\backslash G/K_i$, which is compact by (i). Since $\bigsqcup_{i=1}^t \mathcal{G}(\mathcal{O})\backslash G/K_i$ is also discrete (because each K_i is open in G), it must be finite, and it follows that $\mathcal{G}(\mathcal{O})\backslash Y$ is finite. Applying [Fir16, Corollaries 3.11, 3.12] (here we need the fact the $G(\mathcal{O})$ is discrete in $G(K_\nu)$) now gives the set U and the desired conclusions. $\square$

Corollary 13.9. *Suppose that the assumptions of Proposition 13.8 hold. Let $\{I_m\}_{m\in\mathbb{N}}$ be a decreasing sequence of ideals of $\mathcal{O}$ such that $\bigcap_{m\in\mathbb{N}} I_m = \{0\}$, let $\rho \in \mathcal{V} - S$ and let $p = \operatorname{char} k(\rho)$. Then:*

(i) There exists $m_0 \in \mathbb{N}$ such that for every finite-index subgroup Γ of $\mathcal{G}(\mathcal{O}; I_{m_0})$, the action of Γ on Y is free, the quotient $\Gamma\backslash Y$ is a finite simplicial complex, $Y \to \Gamma\backslash Y$ is a covering map, and $\pi_1(\Gamma\backslash Y) \cong \Gamma$.

(ii) If moreover $I_{m_0} \subseteq P_\rho$, then, for every Γ as in (i), the complex $\Gamma\backslash Y$ has an infinite tower of connected C_p-Galois coverings (C_p is the cyclic group order p).

Proof. (i) Let $U \subseteq \mathcal{G}(\mathcal{O})$ be the subset from Proposition 13.8(ii). Our assumptions on the sequence $\{I_m\}_{m\in\mathbb{N}}$ imply that $\bigcap_{m\in\mathbb{N}}\mathcal{G}(\mathcal{O}; I_m) = \{1_G\}$ and $\mathcal{G}(\mathcal{O}; I_1) \supseteq \mathcal{G}(\mathcal{O}; I_2) \supseteq \ldots$. Thus, there exists $m_0 \in \mathbb{N}$ such that $\mathcal{G}(\mathcal{O}; I_{m_0}) \cap U = \emptyset$, and the conclusion follows from Proposition 13.8(ii).

(ii) For every $i \geq 0$, put $\Gamma_i = \Gamma \cap \mathcal{G}(\mathcal{O}; I_{m_0}P_\rho^i)$. Then Γ_i/Γ_{i+1} is isomorphic to a subgroup of $\mathcal{G}(\mathcal{O}; I_{m_0}P_\rho^i)/\mathcal{G}(\mathcal{O}; I_{m_0}P_\rho^{i+1})$, which is an elementary abelian p-group by Proposition 13.7. This means that there are normal subgroups

$$\Gamma_i = \Gamma_{i,1} \supseteq \cdots \supseteq \Gamma_{i,t(i)} \supseteq \Gamma_{i,t(i)+1} = \Gamma_{i+1}$$

such that $|\Gamma_{i,k}/\Gamma_{i,k+1}| = p$ for all $k \in \{1, \ldots, t(i)\}$. Put $X_{i,k} = \Gamma_{i,k}\backslash Y$. Then

$$\cdots \to X_{2,2} \to X_{2,1} \cdots \to X_{1,2} \to X_{1,1} \to \cdots \to X_{0,2} \to X_{0,1} = \Gamma\backslash Y$$

is the required tower of C_p-Galois coverings. $\square$

Remark 13.10. Suppose we are given a global field K, a finite place ν, and an absolutely almost simple simply connected isotropic algebraic group $\mathbf{H}$ over K_ν, and we wish to complete this data to the setting of Notation 13.5 in such a way that $\mathbf{H} = \mathbf{G}_{K_\nu}$ and the assumptions of Proposition 13.8 hold. This is known to be possible if $\operatorname{char} K = 0$, and thus the affine building Y of $\mathbf{H}$ covers infinitely many finite similicial complexes. On the other hand, if $\operatorname{char} K > 0$ and $\operatorname{rank}\mathbf{H} > 1$, then completing the data in this manner is possible only if $\mathbf{H}$ is of type A_m.

13.3 The Congruence Subgroup Property

Keep Notation 13.5. We proceed by recalling the *congruence subgroup property* and using it to bound the the number of group homomorphisms from a principal congruence subgroup $\mathcal{G}(\mathcal{O}; I)$ to the additive group of $\mathbb{F}_p$.

Let $\widehat{\mathcal{G}}_K = \varprojlim \mathcal{G}(K)/U$, where the limit ranges over the finite index subgroups U of $\mathcal{G}(\mathcal{O})$, and let $\overline{\mathcal{G}}_K = \varprojlim \mathcal{G}(K)/\mathcal{G}(\mathcal{O}; I)$, where the limit ranges over the nonzero ideals $I \trianglelefteq \mathcal{O}$. While $\mathcal{G}(K)/U$, resp. $\mathcal{G}(K)/\mathcal{G}(\mathcal{O}; I)$, are not groups, $\widehat{\mathcal{G}}_K$ and $\overline{\mathcal{G}}_K$ are groups, which we topologize by giving $\mathcal{G}(K)/U$, resp. $\mathcal{G}(K)/\mathcal{G}(\mathcal{O}; I)$, the discrete topology and taking the limit topology. There is an evident surjective group homomorphism $\widehat{\mathcal{G}}_K \to \overline{\mathcal{G}}_K$. The kernel of this map, denoted $C^S(\mathbf{G})$, is called the *congruence kernel* of $(\mathbf{G}, S)$, and $(\mathbf{G}, S)$ is said to satisfy the *congruence subgroup property* (CSP) if $C^S(\mathbf{G})$ is finite. For example, $C^S(\mathbf{G})$ is trivial if and only if any finite index subgroup of $\mathcal{G}(\mathcal{O})$ contains a principal congruence subgroup. The question of which pairs $(\mathbf{G}, S)$ have CSP has a long and rich history; we refer the reader to [PR10] for a survey, and to [PR94, §9.5] for an extensive discussion. The main conjecture in this field is due to Serre:

Conjecture 13.11 (Serre)**.** *With notation as in Notation 13.5, $(\mathbf{G}, S)$ has CSP if $\mathbf{G}$ is K_ρ-isotropic for every $\rho \in S - \mathcal{V}_\infty$ and $\sum_{\rho\in S} \operatorname{rank}_{K_\rho} G > 1$. The pair $(\mathbf{G}, S)$ does not have CSP if $\sum_{\rho\in S} \operatorname{rank}_{K_\rho} G = 1$.*

The conjecture is known to hold in many cases. The following two theorems are a culmination of many results due to Borovoi, P. Gille, Platonov, G. Prasad, Raghunathan, A. Rapinchuk, Segev, Seitz, Tomanov and others. See [PR10], [PR94, §9.5] and the references therein.

Theorem 13.12. *With notation as in Notation 13.5, suppose that*

(1) $\mathbf{G}$ is K_ρ-isotropic for all $\rho \in S - \mathcal{V}_\infty$,

(2) $\sum_{\rho\in S} \operatorname{rank}_{K_\rho} \mathbf{G} \geq 2$, and

(3) $\mathbf{G}$ is K-isotropic, or of the types B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 5)$, E_7, E_8, F_4, G_2, or $\mathbf{G} = \mathbf{SU}(f)$ where f is a nondegenerate hermitian form of dimension ≥ 4 over a quadratic Galois extension of K (cf. Example 13.2(iv)).

Then $(\mathbf{G}, S)$ has CSP.

Proof. Theorem 2 in [PR10] states that if (1) and (2) hold, then CSP for $(\mathbf{G}, S)$ follows from the centrality of $C^S(\mathbf{G})$ in $\widehat{\mathcal{G}}_K$. The group $C^S(\mathbf{G})$ is known to be central in $\widehat{\mathcal{G}}_K$ when (3) holds; see [Rag76] and [Rag86] for the case where $\mathbf{G}$ is isotropic (the assumption $G(k) = (k)^+$ for groups of type E_6 that was unknown at the time and was established in [Gil09]), and [PR94, Theorems 9.23, 9.24] for the anisotropic cases. □

Theorem 13.13. *With notation as in Notation 13.5, assume that conditions (1) and (2) of 13.12 hold that $\mathbf{G}$ has CSP (this follows from (1) and (2) if Conjecture 13.11 holds) and that*

(3′) $\mathbf{G}$ is K-isotropic, or of the types B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4$, excluding ${}^{3,6}D_4)$, E_7, E_8, F_4, G_2, or inner of type A_n, or $\mathbf{G} = \mathbf{SU}(f)$ where f is a nondegenerate hermitian form of dimension ≥ 3 over a quadratic Galois extension of K (cf. Example 13.2(iv)).

Then $C^S(\mathbf{G})$ is isomorphic to a subgroup of $\mu(K)$, the group of roots of unity in K. If moreover S contains a non-archimedean place, the $C^S(\mathbf{G})$ is trivial.

Proof. Condition (3′) lists the cases where the *Margulis–Platonov conjecture* is known to hold, see [Rap06] (inner type A_n), [Gil09] (isotropic groups) and [RS01, Appendix A] (all other cases). By [PR10, Theorem 2], if the Margulis–Platonov conjecture holds for $\mathbf{G}$, and $(\mathbf{G}, S)$ has CSP, then $C^S(\mathbf{G})$ is isomorphic to the *metaplectic kernel* $M(S, \mathbf{G})$. G. Prasad and A. Rapinchuk [PR96] (or [PR10, Theorem 3]) showed that the latter is isomorphic to a subgroup of $\mu(K)$, and it is moreover trivial if there is $\rho \in S - \mathcal{V}_\infty$ such that $\mathbf{G}$ is K_ρ-isotropic. $\square$

In the following theorem, we use CSP in order to bound the number of group homomorphism from a principal congruence subgroup to $\mathbb{F}_p$. If A and B are topological groups, we write $\mathrm{Hom}_c(A, B)$ for the set of continuous group homomorphisms from A to B. We give $\mathbb{F}_p$ the discrete topology.

Theorem 13.14. *With notation as in Notation 13.5, suppose that* $\sum_{\rho \in S} \mathrm{rank}_{K_\rho} \mathbf{G} \geq 2$*, that* $\mathbf{G}$ *is* K_ρ*-isotropic for every* $\rho \in S - \mathcal{V}_\infty$ *and that* $(\mathbf{G}, S)$ *has CSP (this is superfluous if Conjecture 13.11 holds). Let* $p \in \mathbb{N}$ *be a prime number, and let* $I \trianglelefteq \mathcal{O}$*. Define the following sets of places:*

- T_1 *is the set of* $\rho \in \mathcal{V} - S$ *such that* $\mathrm{char}\, k(\rho) \neq p$ *and* $I_\rho \neq \mathcal{O}_\rho$*.*
- T_2 *is the set of* $\rho \in \mathcal{V} - S$ *such that* $\mathrm{char}\, k(\rho) \neq p$*,* $I_\rho = \mathcal{O}_\rho$*,* $\mathcal{G}_{\mathcal{O}_\rho}$ *is absolutely almost simple simply connected and* $|k(\rho)| \geq 4$*.*
- T_3 *is the set of* $\rho \in \mathcal{V} - S$ *such that* $\mathrm{char}\, k(\rho) = p$ *and* $\mathcal{G}_{\mathcal{O}_\rho}$ *is a* split *absolutely almost simple simply connected. If* $\mathcal{G}_{\mathcal{O}_\rho}$ *is of type* C_2 *or* G_2*, we further require that* $|k(\rho)| > 2$*.*
- $T_4 = \mathcal{V} - S - T_1 - T_2 - T_3$*.*

Then there are $C, M \in \mathbb{N} \cup \{0\}$*, depending only on* $\mathcal{G}$ *and* K*, such that if* $\mathrm{char}\, K = 0$*, then*

$$\dim_{\mathbb{F}_p} \mathrm{Hom}(\mathcal{G}(\mathcal{O}; I), \mathbb{F}_p) \leq C|T_3| + \sum_{\rho \in T_4} \dim_{\mathbb{F}_p} \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p) + M.$$

and if $\mathrm{char}\, K > 0$*, then*

$$\dim_{\mathbb{F}_p} \mathrm{Hom}(\mathcal{G}(\mathcal{O}; I), \mathbb{F}_p) \leq C \sum_{\rho \in T_3} \rho(I) \dim_{\mathbb{F}_p} k(\rho) + \sum_{\rho \in T_4} \dim_{\mathbb{F}_p} \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p) + M.$$

Each term $\dim_{\mathbb{F}_p} \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p)$ *is finite. Moreover, if* $p \nmid |C^S(\mathbf{G})|$*, then we can take* $M = 0$*.*

The group scheme $\mathcal{G}_{\mathcal{O}_\rho}$ is absolutely almost simple simply connected for all but finitely many $\rho \in \mathcal{V} - S$, see [Rag76, Lemma 1.9]. Thus, if $\mathrm{char}\, K = 0$, then $\mathcal{V} - S - T_2$ is finite.

Proof. Put $\Gamma = \mathcal{G}(\mathcal{O}; I)$, let $\widehat{\Gamma}$ be the profinite completion of Γ, and let $\overline{\Gamma} = \varprojlim \mathcal{G}(\mathcal{O}; I)/\mathcal{G}(\mathcal{O}; I')$ where I' ranges over the nonzero ideals of $\mathcal{O}$ contained in I. Then $\widehat{\Gamma}$ and $\overline{\Gamma}$ are subgroups of $\widehat{\mathcal{G}}_K$ and $\overline{\mathcal{G}}_K$, respectively, and the natural map $\widehat{\mathcal{G}}_K \to \overline{\mathcal{G}}_K$ restricts to a surjective map $\widehat{\Gamma} \to \overline{\Gamma}$. The kernel $H := \ker(\widehat{\Gamma} \to \overline{\Gamma})$ is a subgroup of $C^S(\mathbf{G})$, hence finite by our assumptions.

There is a one-to-one correspondence between group homomorphisms $\varphi : \Gamma \to \mathbb{F}_p$ and continuous group homomorphisms $\hat{\varphi} : \widehat{\Gamma} \to \mathbb{F}_p$ ($\mathbb{F}_p$ is regarded as a discrete tolopgical space). Thus, it is enough to bound the $\mathbb{F}_p$-dimension of $\mathrm{Hom}_c(\widehat{\Gamma}, \mathbb{F}_p)$. The short exact sequence $1 \to H \to \widehat{\Gamma} \to \overline{\Gamma} \to 1$ gives rise to an exact sequence

$$0 \to \mathrm{Hom}_c(\overline{\Gamma}, \mathbb{F}_p) \to \mathrm{Hom}_c(\widehat{\Gamma}, \mathbb{F}_p) \to \mathrm{Hom}_c(H, \mathbb{F}_p),$$

so

$$\dim \mathrm{Hom}(\Gamma, \mathbb{F}_p) \leq \dim \mathrm{Hom}_c(\overline{\Gamma}, \mathbb{F}_p) + \dim \mathrm{Hom}(H, \mathbb{F}_p).$$

We take $M = \max_{H' \le C^S(\mathbf{G})} \dim_{\mathbb{F}_p} \mathrm{Hom}(H', \mathbb{F}_p)$, so that $\dim_{\mathbb{F}_p} \mathrm{Hom}(H, \mathbb{F}_p) \le M$. By strong approximation, the group $\overline{\Gamma}$ is the profinite group $\prod_{\rho \in \mathcal{V}-S} \mathcal{G}(\mathcal{O}_\rho; I_\rho)$, cf. Proposition 13.6. Since the kernel of every continuous homomorphism $\prod_{\rho \in \mathcal{V}-S} \mathcal{G}(\mathcal{O}_\rho; I_\rho) \to \mathbb{F}_p$ is open, we have

$$\mathrm{Hom}_c(\prod_{\rho \in \mathcal{V}-S} \mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p) \cong \bigoplus_{\rho \in \mathcal{V}-S} \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p).$$

Putting everything together, we get

$$\dim \mathrm{Hom}(\Gamma, \mathbb{F}_p) \le \sum_{\rho \in \mathcal{V}-S} \dim \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p) + M.$$

Now, in order to prove the theorem, it remains to bound $\dim \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p)$ according to whether ρ is in T_1, T_2, T_3 or T_4. We split into cases.

Suppose first that $\rho \in T_1$. Then $I_\rho \subseteq \mathfrak{m}_\rho$ and $\ell := \mathrm{char}\, k(\rho) \neq p$. We claim that $\mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p) = 0$. To see this, note that there is some $n \in \mathbb{N}$ such that $I_\rho = \mathfrak{m}_\rho^n$. Let $\varphi : \mathcal{G}(\mathcal{O}_\rho; \mathfrak{m}_\rho^n) \to \mathbb{F}_p$ be a continuous homomorphism. Then $\ker \varphi$ is open and thus contains $\mathcal{G}(\mathcal{O}; \mathfrak{m}_\rho^m)$ for some $m \ge n$. By Proposition 13.7, $\mathcal{G}(\mathcal{O}_\rho, \mathfrak{m}_\rho^i)/\mathcal{G}(\mathcal{O}_\rho, \mathfrak{m}_\rho^{i+1})$ is an elementary abelian ℓ-group for all ℓ and $i \ge 1$. Since $\ell \neq p$, this forces φ to be 0.

Suppose next that $\rho \in T_2$. Then $I_\rho = \mathcal{O}_\rho$, $\ell := \mathrm{char}\, k(\rho) \neq p$, $\mathcal{G}_{\mathcal{O}_\rho}$ is absolutely almost simple simply connected and $|k(\rho)| \ge 4$. We need to show that $\mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p) = \mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho), \mathbb{F}_p) = 0$. Let $\varphi : \mathcal{G}(\mathcal{O}_\rho) \to \mathbb{F}_p$ be a continuous homomorphism. By the previous paragraph, φ vanishes on $\mathcal{G}(\mathcal{O}_\rho; \mathfrak{m}_\rho)$. Since $\mathcal{G}_{\mathcal{O}_\rho}$ is absolutely almost simple simply connected, the same applies to $\mathcal{G}_{k(\rho)}$. Now, theorems of Chevalley, Steinberg and Tits [PR94, Proposition 7.5] tell us that $\mathcal{G}(k(\rho))$ is a perfect group (here we need $|k(\rho)| \ge 4$). Since $\mathcal{G}_{\mathcal{O}_\rho} \to \mathrm{Spec}\, \mathcal{O}_\rho$ is smooth, $\mathcal{G}(\mathcal{O}_\rho)/\mathcal{G}(\mathcal{O}_\rho; \mathfrak{m}_\rho) \cong \mathcal{G}(k(\rho))$. Since $\mathbb{F}_p$ is abelian, this forces φ to be 0, as claimed.

Suppose now that $\rho \in T_3$. Then $\mathrm{char}\, k(\rho) = p$ and $\mathcal{G}_{\mathcal{O}_\rho}$ is a split almost simple simply connected group scheme over $\mathcal{O}_\rho$. Write $R = \mathcal{O}_\rho$, $\mathfrak{m} = \mathfrak{m}_\rho$ and $J = I_\rho$. We will make use of the (absolute) *elementary subgroups* $E(R; J)$ of $\mathcal{G}(R; J)$; see [HVZ13, §2] and the references therein for their definition. In fact, in our case, $E(R; J) = \mathcal{G}(R; J)$ by [AS76, Propositions 2.3, 2.4]. It follows from the definition of the elementary subgroups that $E(R; J)^p \supseteq E(R; pJ)$, hence $\mathcal{G}(R; J)^p \supseteq \mathcal{G}(R; pJ)$. Moreover, by [HVZ13, Lemma 17] and the comment following Lemma 18 in that source, we have $[E(R; J), E(R; J)] \supseteq E(R; J^3)$ (in fact, we can replace $E(R; J^3)$ with $E(R; J^2)$ if $\mathcal{G}$ is not of type C_ℓ), and likewise for $\mathcal{G}(R; J)$. Consequently, any group homomorphism $\varphi : \mathcal{G}(R; J) \to \mathbb{F}_p$ must vanish on $\mathcal{G}(R; pJ)$ and $\mathcal{G}(R; J^3)$. If $J = \mathcal{O}_\rho$ (i.e. $\rho(I) = 0$), then φ must be zero, so assume $J \neq \mathcal{O}_\rho$. In particular, $pR \subseteq \mathfrak{m}$. We now break into subcases.

Suppose first that $\mathrm{char}\, K = 0$. Then $pJ \neq 0$. Let $\overline{\varphi}$ denote the induced map $\mathcal{G}(R; J)/\mathcal{G}(R; pJ) \to \mathbb{F}_p$. We need to bound the number of these maps by p^C for C depending only on $\mathcal{G}$ and K. Write $t = \rho(p) > 0$, $s = \rho(J)$ and $f = \dim_{\mathbb{F}_p} k(\rho)$. Proposition 13.7 tells us that there is a constant D such that $\mathcal{G}(R; \mathfrak{m}^i)/\mathcal{G}(R; \mathfrak{m}^{i+1})$ is an elementary abelian p-group of rank $\le D \cdot f$. Thus, $\mathcal{G}(R; J)/\mathcal{G}(R; pJ) = \mathcal{G}(R; \mathfrak{m}^s)/\mathcal{G}(R; \mathfrak{m}^{t+s})$ is a p-group with at most p^{Dtf} elements, hence it admits at most p^{Dtf} homomorphisms into $\mathbb{F}_p$. Our assumption that $\mathrm{char}\, K = 0$ implies that only finitely many places $\rho \in \mathcal{V} - S$ divide p, so there is a constant $C_1 \in \mathbb{N}$ such that $tf \le C_1$ for all $\rho \in T_3$. We can take $C = DC_1$.

Now suppose that $\mathrm{char}\, K > 0$. Then $\mathrm{char}\, K = \mathrm{char}\, k(\rho) = p$. Let $\overline{\varphi}$ denote the induced map $\mathcal{G}(R; J)/\mathcal{G}(R; J^3) \to \mathbb{F}_p$, and write $J = \mathfrak{m}^s$ (so that $s = \rho(I)$). An argument similar to the previous paragraph shows that the dimension of the $\mathbb{F}_p$-vector space of such $\overline{\varphi}$ is at most $2sD \dim_{\mathbb{F}_p} k(\rho) = 2D\rho(I) \dim_{\mathbb{F}_p} k(\rho)$. Taking $C = 2D$ completes the case $\rho \in T_3$.

Finally, we need to show that $\mathrm{Hom}_c(\mathcal{G}(\mathcal{O}_\rho; I_\rho), \mathbb{F}_p)$ is finite for all $\rho \in \mathcal{V} - S$. To that end, it is enough to prove that $\mathcal{G}(\mathcal{O}_\rho; I_\rho)$ is finitely generated as a profinite group. By Proposition 13.6, $\mathcal{G}(\mathcal{O}; I)$ is dense in $\mathcal{G}(\mathcal{O}_\rho; I_\rho)$. Thus, it is enough to show that $\mathcal{G}(\mathcal{O}; I)$ is finitely generated. This holds by [PR94, Theorem 5.11] if $\mathrm{char}\, K = 0$ and [Beh69] (see also [Beh87]) if $\mathrm{char}\, K > 0$. □

13.4 Proof of Theorem 13.1

We prove Theorem 13.1 by exhibiting an example of a simpicial complex and showing that it satisfies (i) and (ii). The example can be generalized, of course. To that end, we would like to apply Corollary 13.9 and Theorem 13.14 together. Their joint assumptions force the set of places S to be $\mathcal{V}_\infty \cup \{\nu\}$, and moreover, $\mathbf{G}$ has to be K_ν-isotropic and K_ρ-anisotropic for every $\rho \in \mathcal{V}_\infty$.

We first recall the following well-known fact from algebraic topology.

Lemma 13.15. *Let X be a connected (finite) simplicial complex with fundamental group Γ and let p be a prime number. Then $\mathrm{H}^1(X, \mathbb{F}_p) \cong \mathrm{Hom}(\Gamma, \mathbb{F}_p)$ as $\mathbb{F}_p$-vector spaces.*

Construction 13.16. We specialize Notation 13.5 as follows. Take

- $K = \mathbb{Q}$.

The set of places $\mathcal{V}$ can be identified with the set of prime numbers together with ∞. Fix a prime number p with $p \equiv 1 \bmod 4$ and take:

- $S = \{p, \infty\}$,
- $\nu = p$ (ν will also denote the p-adic valuation),

Thus, $\mathcal{O} = \mathbb{Z}[\frac{1}{p}]$. Fix $d \geq 3$ and let

- $\mathcal{G} = \mathbf{SU}_{d+1}(\mathcal{O}[i]/\mathcal{O})$,
- $\mathbf{G} = \mathbf{SU}_{d+1}(\mathbb{Q}[i]/\mathbb{Q})$,

where the notation is as in Example 13.2(iii). Thus, $K_\nu = \mathbb{Q}_p$, and Y is the affine building attached to $\mathbf{G}_{\mathbb{Q}_p}$. Our choice of p implies that $\mathbb{Q}_p$ contains a square root of -1, so by Example 13.4(ii), $\mathbf{G}_{\mathbb{Q}_p} \cong \mathbf{SL}_{d+1}(\mathbb{Q}_p)$. Thus, Y is the the familiar affine building of $\mathbf{SL}_{d+1}(\mathbb{Q}_p)$. In particular, $\dim Y = d$ and Y is $(p+1)$-thick.

Observe that $\mathbf{G}(\mathbb{R})$ is nothing but the group of $(d+1) \times (d+1)$ complex unitary matrices, so it is compact. Thus, $\mathbf{G}$ is $\mathbb{R}$-anisotropic, and therefore $\mathbb{Q}$-anisotropic. On the other hand, $\mathbf{G}$ is $\mathbb{Q}_p$-isotropic by Example 13.4(i). This allows us to apply Corollary 13.9 with the sequence of ideals $I_m = 2^m 3\mathcal{O}$; set $\Gamma = \mathcal{G}(\mathcal{O}; I_{m_0}) = \mathcal{G}(\mathcal{O}; 2^{m_0} 3\mathcal{O})$, where $m_0 \in \mathbb{N}$ is in the corollary. (Explicitly, Γ is the set of matrices in $\mathrm{SU}_{d+1}(\mathbb{Z}[\frac{1}{p}][i]/\mathbb{Z}[\frac{1}{p}])$ which are congruent to the identity matrix modulo $2^{m_0} \cdot 3$.) Put $X := \Gamma \backslash Y$ and $X'_r = \mathcal{G}(\mathcal{O}; 2^{m_0} 3^{r+1}\mathcal{O}) \backslash Y$ for all $r \in \mathbb{N} \cup \{0\}$. Then $X = X'_0, X'_1, X'_2, \dots$ are connected simplicial complexes covered by Y, and the evident quotient maps give rise to a tower of coverings $\dots X'_2 \to X'_1 \to X'_0 = X$.

Proposition 13.17. *The complex X and the tower $\dots X'_2 \to X'_1 \to X'_0 = X$ of Construction 13.16 satisfy conditions (i) and (ii) of Theorem 13.1.*

Proof. Condition (i) follows from Corollary 13.9(ii).

To show (ii), observe that Proposition 13.7 implies that $\mathcal{G}(\mathcal{O}; 2^{m_0} 3^{r+1}\mathcal{O}) / \mathcal{G}(\mathcal{O}; 2^{m_0} 3^{r+2}\mathcal{O})$ is an elementary abelian 3-group for all $r \geq 1$. Thus, $[X'_r : X] = [\mathcal{G}(\mathcal{O}; 2^{m_0} 3\mathcal{O}) : \mathcal{G}(\mathcal{O}; 2^{m_0} 3^{r+1}\mathcal{O})]$ is a power of 3, and in particular odd. (In the equality we used the fact that the $\mathcal{G}(\mathcal{O};2^{m_0}3\mathcal{O})$ acts freely on Y, which we know by Corollary 13.9(i).)

Next, Lemma 13.15 and Corollary 13.9(i) tell us that

$$\dim \mathrm{H}^1(X'_r,\mathbb{F}_2)=\dim_{\mathbb{F}_2}\mathrm{Hom}(\pi_1(X'_r),\mathbb{F}_2)=\dim_{\mathbb{F}_2}\mathrm{Hom}(\mathcal{G}(\mathcal{O};2^{m_0}3^{r+1}\mathcal{O}),\mathbb{F}_2).$$

We bound the right hand side using Theorem 13.14 with $p=2$ and $I=2^{m_0}3^{r+1}\mathcal{O}$. The assumptions of the theorem hold because $\mathrm{rank}_{\mathbb{R}}\,\mathbf{G}=0$ and $\mathrm{rank}_{\mathbb{Q}_p}\,\mathbf{G}=\mathrm{rank}\,\mathbf{SL}_{d+1}(\mathbb{Q}_p)=d>1$ (Example 13.4(i)), and $(\mathbf{G},S)$ has CSP holds by Theorem 13.12 (here we need $d\geq 3$). In the notation of Theorem 13.14, we have $T_1=\{3\}$, $T_2=\mathcal{V}-\{2,3,p,\infty\}$, $T_3=\emptyset$ and $T_4=\{2\}$, so the theorem implies that $\dim_{\mathbb{F}_2}\mathrm{Hom}(\mathcal{G}(\mathcal{O};2^{m_0}3^{r+1}\mathcal{O}),\mathbb{F}_2)$ is bounded by a constant C independent of r. □

Proof of Theorem 13.1. The theorem follows from Proposition 13.17. The thickness requirement on the building Y can be taken care of by choosing p in Construction 13.16 large enough in advance. □

Remark 13.18. If Conjecture 13.11 holds, then the proof of Theorem 13.1 also applies with $d=2$. In fact, the theorem holds when $d=2$ even without assuming Conjecture 13.11. The idea is to use the affine buildings of the (unique) simply connected algebraic group of type G_2 over $\mathbb{Q}_p$. To that end, one could take $\mathcal{G}$ to be the automorphism group scheme of a $\mathbb{Z}$-order in the standard *octonion algebra* over $\mathbb{Q}$ and argue as in the proof of Proposition 13.17. We omit the details.

For later use, we also introduce a variation of Construction 13.16.

Construction 13.19. Let $K=\mathbb{Q}$ and $\mathcal{V}$ be as in Construction 13.16. Let $E=\mathbb{Q}[\sqrt{-7}]$ and $A=\mathbb{Z}[\frac{\sqrt{-7}+1}{2}]\cong\mathbb{Z}[x\,|\,x^2=x-2]$, and let p be an odd prime number such that x^2-x+2 factors into two distinct factors modulo p, or equivalently, p is congruent to 1, 2, or 4 modulo 7. Let

- $S=\{p,\infty\}$ and
- $\nu=p$ (ν will also denote the p-adic valuation).

Then $\mathcal{O}=\mathbb{Z}[\frac{1}{p}]$.

Fix $d\geq 3$ and let $f:A^{d+1}\times A^{d+1}\to A$ denote the hermitian form $f((x_i),(y_i))=\sum_{i=1}^{d+1}\overline{x_i}y_i$, where $\overline{x_i}$ is the complex conjugate of x_i. Now, using the notation of Example 13.2(iv) (with $R=\mathbb{Z}$, $r_0=-2$, $r_1=1$), define

- $\mathcal{G}=\mathbf{SU}(f)$,
- $\mathbf{G}=\mathbf{SU}(f_{\mathbb{Q}})$.

Since the polynomial x^2-x+2 factors into two distinct factors modulo p, we have $E\otimes_{\mathbb{Z}}\mathbb{Q}_p\cong\mathbb{Q}_p\times\mathbb{Q}_p$ and under isomorphism, complex conjugation becomes swapping the coordinates. Thus, as in Example 13.4(ii), we see that $\mathbf{G}_{K_\nu}\cong\mathbf{SL}_{d+1}(\mathbb{Q}_p)$, so Y is the the affine building of $\mathbf{SL}_{d+1}(\mathbb{Q}_p)$.

Since $E\otimes_{\mathbb{Q}}\mathbb{R}=\mathbb{C}$, we again see that $\mathbf{G}(\mathbb{R})$ is the group of $(d+1)\times(d+1)$ complex unitary matrices, so again, $\mathbf{G}$ is $\mathbb{R}$-anisotropic, and therefore $\mathbb{Q}$-anisotropic. On the other hand, $\mathbf{G}$ is $\mathbb{Q}_p$-isotropic by Example 13.4(i). This allows us to apply Corollary 13.9 with the sequence of ideals $I_m=7^m3\mathcal{O}$; set $\Gamma=\mathcal{G}(\mathcal{O};I_{m_0})=\mathcal{G}(\mathcal{O};7^{m_0}3\mathcal{O})$, where $m_0\in\mathbb{N}$ is in the corollary. Put $X:=\Gamma\backslash Y$. Then X is a connected simplicial complexes covered by Y.

Proposition 13.20. *The complex X of Construction 13.19 satisfies* $\mathrm{H}^1(X,\mathbb{F}_2)=0$.

Proof. As in the proof of Proposition 13.17, we reduce into showing $\mathrm{Hom}(\mathcal{G}(\mathbb{Z}; 7^{m_0}3\mathbb{Z}), \mathbb{F}_2) = 0$. We show this by applying Theorem 13.14 with $p = 2$. Note that $x^2 - x + 2$ splits into two distinct factors modulo 2, and thus $A \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so $\mathcal{G}_{\mathbb{Z}_2} \cong \mathbf{SL}_{d+1}(\mathbb{Z}_2)$ (cf. Example 13.4(ii)). This means that, $T_1 = \{7, 3\}$, $T_2 = \mathcal{V} - S - \{2, 3, 7\}$, $T_3 = \{2\}$ and $T_4 = \emptyset$. By Theorem 13.13, $C^S(\mathbf{G}) = 1$, so we conclude that $\mathrm{Hom}(\mathcal{G}(\mathbb{Z}; 7^{m_0}3\mathbb{Z}), \mathbb{F}_2) = 0$. □

14 Sheaves of Large Dimension with Small Cohomology

This final section has to purposes. First, we construct examples of locally cosntant $\mathbb{F}$-sheaves $\mathcal{F}$ of arbitrarily large dimension such that $h^1(\mathcal{F}) \ll \dim \mathcal{F}$. In particular, we shall prove Theorem 12.10 by constructing examples with $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$ and $\dim \mathcal{F} \to \infty$; this uses Theorem 13.1 as a black box. Sheaves with $h^1(\mathcal{F}) \ll \dim \mathcal{F}$ are natural candidates for the iterative modification process discussed in Section 12.

Second, we will show that every two of the conditions (t1)–(t3) of the tower paradigm (Theorem 11.1 with $k = 0$) are satisfied for some sheaved complex $(X, \mathcal{F})$. Otherwise stated, no two of the prerequisites of the tower paradigm are contradictory.

14.1 Proof of Theorem 12.10

We begin with proving the following lemma. Recall that $\mathbb{F}_X$ denotes the constant sheaf $\mathbb{F}$ on a simplicial complex X.

Lemma 14.1. *Let $\mathbb{F}$ be a field, let $u : Y \to X$ be a degree-m covering of connected simplicial complexes, and suppose that $\mathrm{char}\,\mathbb{F} \nmid m$. Define a sheaf morphism $\varphi : u_*\mathbb{F}_Y \to \mathbb{F}_X$ by $\varphi_x(\alpha_1, \ldots, \alpha_m) = \sum_i \alpha_i$ for all $x \in X - \{\emptyset\}$, and put $\mathcal{G} = \ker \varphi$. Then $h^0(\mathcal{G}) = 0$ and $h^1(\mathcal{G}) = \dim \mathrm{H}^1(Y, \mathbb{F}) - \dim \mathrm{H}^1(X, \mathbb{F})$.*

Proof. Note that $u_*\mathbb{F}_Y(x) = \prod_{y \in u^{-1}(x)} \mathbb{F}$ for all $x \in X - \{\emptyset\}$ and we implicitly identified $\prod_{y \in u^{-1}(x)} \mathbb{F}$ with $\mathbb{F}^m$ by numbering the faces in $u^{-1}(x)$. Define $\psi : \mathbb{F}_X \to u_*\mathbb{F}_Y$ by $\psi_x(\alpha) = (m^{-1}\alpha, \ldots, m^{-1}\alpha)$ (m times) for all $x \in X - \{\emptyset\}$ (here we used the assumption $\mathrm{char}\,\mathbb{F} \nmid m$). It is routine to check that ψ is indeed a morphism of sheaves, and moreover, $\varphi \circ \psi = \mathrm{id}_{\mathbb{F}_X}$. This means that $u_*\mathbb{F}_Y$ breaks as a product of the sheaves $\mathrm{im}\,\psi \cong \mathbb{F}_X$ and $\ker \varphi = \mathcal{G}$. Consequently, $h^i(\mathcal{G}) = h^i(u_*\mathbb{F}_Y) - h^i(\mathbb{F}_X)$ for all $i \geq 0$. By Lemma 4.11, we have $h^i(u_*\mathbb{F}_Y) = h^i(\mathbb{F}_Y)$. Since X and Y are connected $h^0(\mathbb{F}_X) = h^0(\mathbb{F}_Y) = 1$, and lemma follows. □

Proof of Theorem 12.10. Recall that we are given $q, d \in \mathbb{N}$ with $d \geq 2$, and we need to construct a q-thick d-dimensional affine building Y covering a finite simplicial complex X and a nonzero locally constant $\mathbb{F}_2$-sheaf $\mathcal{G}$ such that X admits an infinite tower of connected double coverings and $h^0(\mathcal{G}) = h^1(\mathcal{G}) = 0$.

By Theorem 13.1, there exist a q-thick affine building Y covering a simlicial complex X'_0 and a tower of connected coverings $\cdots \to X'_2 \to X'_1 \to X'_0$ such that every connected covering of X'_0 admits an infinite tower of connected double coverings, and $[X'_r : X'_0]$ is odd and bounded by some $C \in \mathbb{N}$ for all $r \in \mathbb{N} \cup \{0\}$. The latter implies that there are $t > s \geq 0$ with $\dim \mathrm{H}^1(X'_t, \mathbb{F}_2) = \dim \mathrm{H}^2(X'_s, \mathbb{F}_2)$. Let u denote the covering map $X'_t \to X'_s$. The degree of u — call it m — is odd and greater than 1, so Lemma 14.1 provides us with a locally constant $\mathbb{F}_2$-sheaf $\mathcal{G}$ on X'_s of dimension $m-1 > 0$ such that $h^0(\mathcal{G}) = 0$ and $h^1(\mathcal{G}) = \dim \mathrm{H}^1(X'_t, \mathbb{F}_2) - \dim \mathrm{H}^2(X'_s, \mathbb{F}_2) = 0$. Taking $X = X'_s$, we have obtained the desired sheaved d-complex $(X, \mathcal{G})$. Alternatively, writing p for the covering map $X'_s \to X'_0$, we can also take $(X'_0, p_*\mathcal{G})$ thanks to Lemma 4.11. □

Remark 14.2. If X admits a tower of coverings $\cdots \to X'_2 \to X'_1 \to X'_0 = X$ such that $\dim \mathrm{H}^1(X'_r, \mathbb{F}_2) \ll [X'_r : X]$ as r grows, then by setting $\mathcal{F}_r = (u_r)_*(\mathbb{F}_2)_{X'_r}$, where u_r is the map $X'_r \to X$, we get a family $\{\mathcal{F}_r\}_{r\in\mathbb{N}}$ of $\mathbb{F}_2$-sheaves on X with $\dim \mathcal{F}_r \to \infty$ and $h^1(\mathcal{F}_r) \ll \dim \mathcal{F}_r$ as $r \to \infty$.

Applying this approach to the Ramanujan complexes of [LSV05a] to construct $\mathbb{F}_2$-sheaves and then applying the modification process of Construction 12.2 to these sheaves gives the explicit example considered in §2.7.

14.2 Satisfying Every Two of The Three Prerequisites of The Tower Paradigm

We finish by demonstrating that every two of the conditions conditions (t1)–(t3) of Theorem 11.1 with $k = 0$ are met for some sheaved complex $(X, \mathcal{F})$. Theorem 13.1 will play a role in all of the constructions.

Example 14.3 (Conditions (t1) and (t2) of Theorem 11.1 can be met)**.** Fix $q, d \geq 3$ and let Y, X be simplicial complexes satisfying condition (i) of Theorem 13.1. Then Y is a q-thick d-dimensional affine building covering X and X has an infinite tower of connected double coverings. The latter implies (t1). Let $\mathcal{F}$ denote the constant sheaf $\mathbb{F}_2$ on X. Choosing q sufficiently large in advance allows us to apply Theorem 9.2(i) to $(X, \mathcal{F})$, thus establishing (t2).

We observed in Proposition 11.4 that (t3) does not hold for this choice of $(X, \mathcal{F})$.

Example 14.4 (Conditions (t1) and (t3) of Theorem 11.1 can be met)**.** Again, fix $d \geq 3$ and let X be a d-complex satisfying condition (i) of Theorem 13.1. Then X has an infinite tower of double coverings, hence (t1) holds.

Define a sheaf $\mathcal{F}$ on X by setting $\mathcal{F}(v) = \mathbb{F}_2^k$ for all $v \in X(0)$, $\mathcal{F}(y) = \mathbb{F}_2$ for all $y \in X - X(0) - X(-1)$, and setting all the restriction maps $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x}$ to be 0. Then $\dim \mathrm{H}^0(X, \mathcal{F}) = k\cdot|X(0)|$ while $\dim \mathrm{H}^1(X, \mathcal{F}) = |X(1)|$, so if we choose $k > \frac{|X(1)|}{|X(0)|}$, then condition (t3) is satisfied.

Of course, $(X_z, \mathcal{F}_z)$ is a poor coboundary expander, in all dimension, for all $z \in X - \{\emptyset\}$, so condition (t2) does not hold for $(X, \mathcal{F})$. More generally, this highlights the difficulty in securing (t2) when the dimensions of the spaces $\{\mathcal{F}(y)\}_{y\in X(1)}$ are significantly smaller than those of $\{\mathcal{F}(x)\}_{x\in X(0)}$, which is the naive approach to making $h^0(\mathcal{F})$ large.

Example 14.5 (Conditions (t2) and (t3) of Theorem 11.1 can be met)**.** Fix $d \geq 3$ and a prime number p that is congruent to 1, 2 or 4 modulo 7. We apply Construction 13.19 with p to get a d-dimensional affine building Y covering a simplicial complex X. Let $\mathcal{F}$ be the constant sheaf $\mathbb{F}_2$ on X. By Proposition 13.20, $\dim_{\mathbb{F}_2} \mathrm{H}^1(X, \mathcal{F}) = 0$, while $\dim_{\mathbb{F}_2} \mathrm{H}^0(X, \mathcal{F}) = 1$, so (t3) holds for $(X, \mathcal{F})$. In addition, choosing p sufficiently large in advance allows us to apply Theorem 9.2(i), which tells us that (t2) holds for $(X, \mathcal{F})$.

However, (t1) fails in this case because X has no double coverings. Indeed, it is well-known that the double coverings of X are classified by $\mathrm{H}^1(X, \mathbb{F}_2)$, which is 0 in our case.

Appendices

A Sheaves on Simplicial Comlexes versus Sheaves on Topological Spaces

In this appendix we explain the relation between the sheaves on simplicial complexes defined in this paper (Section 4) and the well-known sheaves on topological spaces. Notably, we will show that sheaves on simplicial complexes can be realized as sheaves on certain topological spaces in such a way that the cohomologies agree. The comparison will lead to a definition of the pushforward of a sheaf along an arbitrary morphism of simplicial complex, extending the definition given in §4.3 for dimension preserving maps.

We have made the first two subsections of this appendix accessible to readers with no prior knowledge of sheaves. However, the more advanced topics considered in the remaining sections require some familiarity with pushforward, pullback and sheaf cohomology; the relevant background material can be found in [Ive86], for instance.

Throughout, simplicial complexes are allowed to be infinite. We denote the category of sheaves on a simplicial complex X by $\mathrm{Sh}(X)$ (cf. Remark 4.3).

A.1 Sheaves on Topological Spaces: a Quick Introduction

Let Y be a topological space. Recall that a sheaf (of abelian groups) $\mathcal{F}$ on Y consists of

(1) an abelian group $\mathcal{F}(U)$ for every open subset $U \subseteq Y$ and

(2) a group homomorphism $\mathrm{res}^{\mathcal{F}}_{V\leftarrow U} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every $V \subseteq U$ open in Y

such that the following conditions are met:

(S1) $\mathrm{res}^{\mathcal{F}}_{U\leftarrow U} = \mathrm{id}_{\mathcal{F}(U)}$ for every open $U \subseteq Y$.

(S2) $\mathrm{res}^{\mathcal{F}}_{W\leftarrow V} \circ \mathrm{res}^{\mathcal{F}}_{V\leftarrow U} = \mathrm{res}^{\mathcal{F}}_{W\leftarrow U}$ for all $W \subseteq V \subseteq U$ open in Y.

(S3) Given open subsets $\{U_i\}_{i\in I}$ of Y and elements $f_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that, for all $i, j \in I$, we have $\mathrm{res}^{\mathcal{F}}_{U_i\cap U_j\leftarrow U_i} f_i = \mathrm{res}^{\mathcal{F}}_{U_i\cap U_j\leftarrow U_j} f_j$ in $\mathcal{F}(U_i \cap U_j)$, there exists a unique $f \in \mathcal{F}(U)$, where $U = \bigcup_{i\in I} U_i$, such that $f_i = \mathrm{res}^{\mathcal{F}}_{U_i\leftarrow U} f$ for all $i \in I$.

The maps $\mathrm{res}^{\mathcal{F}}_{V\leftarrow U}$ are called *restriction maps* and elements of $\mathcal{F}(U)$ are called *U-sections*, or just *sections*. Elements of $\mathcal{F}(X)$ are called *global sections*. It is common to abbreviate $\mathrm{res}^{\mathcal{F}}_{V\leftarrow U} f$ to $f|_{U\to V}$ or $f|_V$. The abelian group $\mathcal{F}(U)$ is also written $\Gamma(U, \mathcal{F})$.

If $\mathbb{F}$ is a field, then a sheaf of $\mathbb{F}$-vector spaces on Y is defined similarly, by requiring each $\mathcal{F}(U)$ to be an $\mathbb{F}$-vector space and each restriction map to be an $\mathbb{F}$-linear map. In the same manner, one can define sheaves of groups, R-modules, sets (the restriction maps are arbitrary functions), and so on.

Remark A.1. Condition (S3) is also required to hold with $I = \emptyset$, in which case $\{U_i\}_{i\in I}$ is an empty collection and U must be $\emptyset$. This choice of $\{U_i\}_{i\in I}$ tells us that $\mathcal{F}(\emptyset)$ is the trivial group.

The most fundamental example of a sheaf on Y is obtained by setting

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \,:\, f \text{ is continuous}\},$$

with $\mathrm{res}^{\mathcal{F}}_{U\leftarrow V}$ being given by $\mathrm{res}^{\mathcal{F}}_{U\leftarrow V} f = f|_V$ (the right hand side is the restriction of f to a function from V to $\mathbb{R}$). The addition law in $\mathcal{F}(U)$ is point-wise addition. Notice that in this case, $\mathrm{res}^{\mathcal{F}}_{U\leftarrow V}$ is literally the restriction-of-domain operation. Conditions (S1)–(S3) now become to the following simple facts:

(S1′) If U is open in Y and $f : U \to \mathbb{R}$ is continuous, then $f|_U = f$.

(S2′) If $W \subseteq V \subseteq U$ are open in Y and $f : U \to \mathbb{R}$ is continuous, then $(f|_V)|_W = f|_W$.

(S3′) Given open subsets $\{U_i\}_{i\in I}$ of Y and, for each $i \in I$, a continuous function $f_i : U_i \to \mathbb{R}$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then all the f_i glue uniquely to a continuous function $f : \bigcup_{i\in I} U_i \to \mathbb{R}$ such that $f|_{U_i} = f_i$ for all $i \in I$.

The sheaf $\mathcal{F}$ is actually a sheaf of $\mathbb{R}$-vector spaces.

Similarly, given an abelian group A, we could define a sheaf $\mathcal{F}_A$ on Y by setting $\mathcal{F}_A(U)$ to be the abelian group of *all* functions from U to A, and again define the restriction maps by restriction of the domain. If A were a topological group, we could replace "all" with "continuous" and get a sheaf as well; the example from the previous paragraph is the special case $A = \mathbb{R}$.

In light of the previous examples, the concept of a sheaf on Y can be seen as axiomatizing an ensemble of "good" (e.g. continuous) functions from open subsets of Y to some fixed target space, but without specifying what "good" means, or what is the target.

The following example is one reason why elements of $\mathcal{F}(U)$ are called sections.

Example A.2. Let X be another topological space and let $p : X \to Y$ be a continuous function. Recall that a (continuous) section of p is a continuous function $f : Y \to X$ such that $p \circ f = \mathrm{id}_Y$. More generally, given an open subset $U \subseteq Y$, we say that a continuous function $f : U \to Y$ is a section of p if $p \circ f = \mathrm{id}_U$. Denote by $\mathcal{F}_p(U)$ the set of sections $f : U \to Y$ of p. Then $\mathcal{F}_p$ defines a set-sheaf on Y by setting $\mathrm{res}^{\mathcal{F}_p}_{V\leftarrow U} f = f|_V$. Moreover, the U-sections of $\mathcal{F}_p$ are the exactly the sections of p defined on U.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on Y, then a *morphism* φ from $\mathcal{F}$ to $\mathcal{G}$ consists of a group homomorphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subseteq Y$ such that

$$\varphi_V \circ \mathrm{res}^{\mathcal{F}}_{V\leftarrow U} = \mathrm{res}^{\mathcal{G}}_{V\leftarrow U} \circ \varphi_U$$

for all $V \subseteq U$ open in Y. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathbb{F}$-vector spaces (resp. rings, sets, etc.), then we instead require that each φ_U is a linear transformation (resp. ring homomorphism, any function, etc.). The sheaves on Y and the morphisms between them form a category denoted $\mathrm{Sh}(Y)$.

A.2 Sheaves on Simplicial Complexes as Sheaves on Topological Spaces

Let X be a simplicial complex. We say that a subset $U \subseteq X - \{\emptyset\}$ is *simplicially open* (in X) if $x \in U$ implies that $X_{\supseteq x} \subseteq U$. Informally, the set $X_{\supseteq x}$ may be regarded as the smallest simplicial neighborhood of x in X. A subset of U of $X - \{\emptyset\}$ is therefore simplicially open if and only if it contains a simplicial neighborhood of every face in U. The collection of simplicially open sets forms a topology on $X - \{\emptyset\}$, and we denote by X° the resulting topological space. By design, the subcollection $\{X_{\supseteq x} \,|\, x \in X - \{\emptyset\}\}$ is a basis of X°.

Let $\mathcal{G}$ be a sheaf on X° and let $U \subseteq X^\circ$ be an open subset. Condition (S3) in the definition of a sheaf on a topological space implies that we can recover $\mathcal{G}(U)$ (up to isomorphism) by knowing the groups $\{\mathcal{G}(X_{\supseteq x}) \,|\, x \in X^\circ\}$ and the restriction maps between them. More precisely, $\mathcal{G}(U)$ may be naturally identified with the set of ensembles $(g_x)_{x\in U}$ where $g_x \in \mathcal{G}(X_{\supseteq x})$ for all $x \in U$ and such that $g_x|_{X_{\supseteq x}\cap X_{\supseteq y}} = g_y|_{X_{\supseteq x}\cap X_{\supseteq y}}$ for all $x, y \in U$. Indeed, such a collection $(g_x)_{x\in U}$ determines a unique $g \in \mathcal{F}(U)$ with $g|_{X_{\supseteq x}} = g_x$ for all $x \in U$. Note also that $X_{\supseteq x} \cap X_{\supseteq y}$ is $X_{\supseteq x\cup y}$ if $x \cup y$ is a face of X, and $\emptyset$ otherwise. Thus, the condition on the $(g_x)_{x\in X}$ is equivalent to having $g_x|_{X_{\supseteq z}} = g_y|_{X_{\supseteq z}}$ whenever $x, y, z \in U$ and $x, y \subseteq z$. Taking $y = z$ or $x = z$, this is in turn equivalent to having $g_x|_{X_{\supseteq y}} = g_y$ for all $x, y \in U$ with $x \subsetneq y$. Now, abbreviating $\mathcal{G}(X_{\supseteq x})$ to $\mathcal{G}^\triangle(x)$ and $\mathrm{res}^{\mathcal{G}}_{X_{\supseteq y}\leftarrow X_{\supseteq x}}$ to $\mathrm{res}^{\mathcal{G}^\triangle}_{y\leftarrow x}$ for every $x, y \in U$ with $x \subsetneq y$, we find that $\mathcal{G}^\triangle$ is a sheaf on X in the sense of §4.1, and we can recover $\mathcal{G}$ (up to isomorphism) from $\mathcal{G}^\triangle$ via

$$\mathcal{G}(U) \cong \{(g_x)_{x\in U} \in \prod_{x\in U} \mathcal{G}^\triangle(x) \,:\, \mathrm{res}^{\mathcal{G}^\triangle}_{y\leftarrow x}\, g_x = g_y \text{ for all } x, y \in U \text{ with } x \subsetneq y\},$$

where the isomorphism is given by $g \mapsto (g|_{X_{\supseteq x}})_{x\in U}$. To conclude, each sheaf $\mathcal{G}$ on X° determines a sheaf $\mathcal{G}^\triangle$ on X, and we can recover $\mathcal{G}$ from $\mathcal{G}^\triangle$.

Conversely, we may start with a sheaf $\mathcal{F}$ on X in the sense of §4.1 and construct a sheaf $\mathcal{F}^\circ$ on X° as follows: Given an open subset $U \subseteq X^\circ$, let $\mathcal{F}^\circ(U)$ denote the set of $(f_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}(x)$ such that $\mathrm{res}^{\mathcal{F}}_{y\leftarrow x} f_x = f_y$ for all $x, y \in U$ with $x \subsetneq y$. Then, given $V \subseteq U$ open in X°, define $\mathrm{res}^{\mathcal{F}^\circ}_{V\leftarrow U} : \mathcal{F}^\circ(U) \to \mathcal{F}^\circ(V)$ by $(f_x)_{x\in U} \mapsto (f_x)_{x\in V}$. It is routine to check that this defines a sheaf on X°.

As we shall now see, up to sheaf isomorphism, the constructions $\mathcal{G} \mapsto \mathcal{G}^\triangle$ and $\mathcal{F} \mapsto \mathcal{F}^\circ$ are inverse to each other. Thus, sheaves on the topological space X° and sheaves on the simplicial complex X are essentially the same thing. Here is a precise statement:

Theorem A.3. *The assignment $\mathcal{G} \mapsto \mathcal{G}^\triangle$ extends naturally to a functor $\mathrm{Sh}(X^\circ) \to \mathrm{Sh}(X)$, and the assignment $\mathcal{F} \mapsto \mathcal{F}^\circ$ extends naturally to a functor $\mathrm{Sh}(X) \to \mathrm{Sh}(X^\circ)$. These functors are mutual inverses, up to natural isomorphism.*

Proof (sketch). The extension of $\mathcal{G} \mapsto \mathcal{G}^\triangle$ (resp. $\mathcal{F} \mapsto \mathcal{F}^\circ$) to a functor is straightforward, but we include it for the sake of completeness. Given a morphism $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ between two sheaves on X°, define $\varphi^\triangle : {\mathcal{G}_1}^\triangle \to {\mathcal{G}_2}^\triangle$ by $\varphi^\triangle_x = \varphi_{X_{\supseteq x}}$ for all $x \in X - \{\emptyset\}$. Given a morphism $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ between two sheaves on X, define $\psi^\circ : {\mathcal{F}_1}^\circ \to {\mathcal{F}_2}^\circ$ by $\psi^\circ_U((f_x)_{x\in U}) = (\psi_x f_x)_{x\in U}$ for all open $U \subseteq X^\circ$. We leave it to the reader to check that these constructions determine functors $(-) \mapsto (-)^\triangle : \mathrm{Sh}(X^\circ) \to \mathrm{Sh}(X)$ and $(-) \mapsto (-)^\circ : \mathrm{Sh}(X) \to \mathrm{Sh}(X^\circ)$, and proceed with showing that these functors are inverse to each other up to natural isomorphism.

Given a sheaf $\mathcal{F}$ on X and $x \in X - \{\emptyset\}$, observe that

$$(\mathcal{F}^\circ)^\triangle(x) = \mathcal{F}^\circ(X_{\supseteq x}) = \{(f_y)_y \in \prod_{y\in X_{\supseteq x}} \mathcal{F}(y) \,:\, f_x|_y = f_y \text{ for all } y \in X_{\supseteq x} - \{x\}\}.$$

Define $\psi_{\mathcal{F},x} : (\mathcal{F}^\circ)^\triangle(x) \to \mathcal{F}(x)$ by $\psi_{\mathcal{F},x}((f_y)_{y\supseteq x}) = f_x$. It is routine to check that $\psi_{\mathcal{F}} := \{\psi_{\mathcal{F},x}\}_{x\in X-\{\emptyset\}}$ is a sheaf morphism from $(\mathcal{F}^\circ)^\triangle$ to $\mathcal{F}$, with inverse given by $\psi^{-1}_{\mathcal{F},x}(f_x) = (f_x|_y)_{y\in X_{\supseteq x}}$. Moreover, it is straightforward to check that $\psi_{\mathcal{F}} : (\mathcal{F}^\circ)^\triangle \to \mathcal{F}$ is natural in $\mathcal{F}$.

Next, let $\mathcal{G}$ be a sheaf on X°. Then for any open $U \subseteq X^\circ$, we have

$$(\mathcal{G}^\triangle)^\circ(U) = \{(g_x)_x \in \prod_{x\in U} \mathcal{G}(X_{\supseteq x}) \,:\, g_x|_{X_{\supseteq y}} = g_y \text{ for all } y \in X_{\supseteq x}\}.$$

Using this, define $\varphi_{\mathcal{G},U} : \mathcal{G}(U) \to (\mathcal{G}^{\triangle})^{\circ}(U)$ by $\varphi_{\mathcal{G},U}(f) = (f|_{X_{\supseteq x}})_{x\in U}$. It is straightforward to check that $\varphi_{\mathcal{G}} := \{\varphi_{\mathcal{G},U}\}_{U \text{ open in } X^\circ}$ defines a morphism of sheaves from $\mathcal{G}$ to $(\mathcal{G}^{\triangle})^{\circ}$. Moreover, we observed earlier that condition (S3) (and the nullity of $\mathcal{G}(\emptyset)$, see Remark A.1) implies that each $\psi_{\mathcal{G},U}$ is bijective, so $\psi_{\mathcal{G}}$ is a sheaf isomorphism. Checking that $\psi_{\mathcal{G}} : \mathcal{G} \to (\mathcal{G}^{\triangle})^{\circ}$ is natural in $\mathcal{G}$ is routine. This completes the proof. □

Remark A.4. A similar argument shows that categories of sheaves of $\mathbb{F}$-vector spaces (resp. groups, rings, sets, etc.) over X and X° are equivalent.[9]

A.3 Comparing Additional Structure: Pullback, Pushforward and Cohomology

Keep the notation of §A.2. Having identified sheaves on X with sheaves on X°, we turn to show that this identification respects pullback, pushforward and cohomology. Thus, the theory of sheaves on simplicial complexes introduced in Section 4 is really a special case of the theory of sheaves on a topological space, which can be described in a more elementary way using the combinatorics of the simplicial complex at hand.

We begin with the following lemma.

Lemma A.5. *Let $f : Y \to X$ be a morphism of simplicial complexes (see §3.1), and let f° denote the induced map $f : Y - \{\emptyset\} \to X - \{\emptyset\}$. Then $f^\circ : Y^\circ \to X^\circ$ is continuous.*

Proof. Let $x \in X - \{\emptyset\}$. Then $(f^\circ)^{-1}(X_{\supseteq x}) = \cup_{y\in f^{-1}(x)} Y_{\supseteq y}$, which is open in Y°. Since the sets $\{X_{\supseteq x} \,|\, x \in X - \{\emptyset\}\}$ form a basis to X°, this means that f° is continuous. □

Let Y and Y' be topological spaces and let $u : Y' \to Y$ be a continuous map. Given a sheaf $\mathcal{G}'$ on Y', recall that the *pushforward* of $\mathcal{G}'$ along u is the sheaf $u_*\mathcal{G}'$ determined by

$$u_*\mathcal{G}'(U) = \mathcal{G}'(u^{-1}(U)) \qquad \text{and} \qquad \mathrm{res}^{u_*\mathcal{G}'}_{V\leftarrow U} = \mathrm{res}^{\mathcal{G}'}_{u^{-1}(V)\leftarrow u^{-1}(U)}$$

for all $V \subseteq U$ open in Y.[10] The counterpart of this construction is the *pullback*, which takes a sheaf $\mathcal{G}$ on Y and produces a sheaf $u^*\mathcal{G}$ on Y'. In contrast with pullback of sheaves on simplicial complexes (see §4.3), the construction of $u^*\mathcal{G}$ is somewhat more involved and can be found in [Ive86, II.§4] or [Sta20, Tag 008C], for instance. The functor $u^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(Y')$ can be implicitly defined as the *left adjoint* of $u_* : \mathrm{Sh}(Y') \to \mathrm{Sh}(Y)$.

Under the equivalence of Theorem A.3, pushforward and pullback of sheaves on simplicial complexes corresponds to pushforward and pullback of sheaves on the associated topological spaces. Formally:

Theorem A.6. *Let $u : Y \to X$ be a morphism of simplicial complexes, let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a sheaf on Y. Then:*

(i) If u is dimension-preserving (see §3.1), then there is a natural isomorphism $(u_*\mathcal{G})^\circ \cong (u^\circ)_*\mathcal{G}^\circ$.

(ii) There is a natural isomorphism $(u^*\mathcal{F})^\circ \cong (u^\circ)^*\mathcal{F}^\circ$.

[9] In fact, it is enough to show this for set-valued sheaves, since all other types of sheaves can be defined internally within the *topoi* of sheaves on X and X°.

[10] Recommended exercise for beginners: check that $u_*\mathcal{G}'$ is a sheaf on Y.

Proof. (i) We will actually show that $(u_*\mathcal{G})^\circ = (u^\circ)_*\mathcal{G}^\circ$. Let $U \subseteq X^\circ$ be an open subset. Then

$$(u^\circ)_*\mathcal{G}^\circ(U) = \mathcal{G}^\circ(u^{-1}(U)) = \{(g_y)_{y\in u^{-1}(U)} \in \prod_{y\in u^{-1}(U)} \mathcal{G}(y) \,:\, \mathrm{res}^{\mathcal{G}}_{y'\leftarrow y}\, g_y = g_{y'} \text{ whenever } y \subsetneq y'\}.$$

On the other hand,

$$(u_*\mathcal{G})^\circ(U) = \{(\tilde{g}_x)_{x\in U} \in \prod_{x\in U} u_*\mathcal{G}(x) \,:\, \mathrm{res}^{u_*\mathcal{G}}_{x'\leftarrow x}\, \tilde{g}_x = \tilde{g}_{x'} \text{ whenever } x \subsetneq x'\}.$$

Recall from §4.3 that $u^*\mathcal{G}(x) = \prod_{y\in u^{-1}(x)} \mathcal{G}(y)$, so each $\tilde{g}_x$ is a collection $(g_y)_{y\in u^{-1}(x)}$ with $g_y \in \mathcal{G}(y)$ for all y. Moreover, for all $x \subsetneq x'$ in U, we have

$$\mathrm{res}^{u_*\mathcal{G}}_{x'\leftarrow x}\, \tilde{g}_x = \mathrm{res}^{u_*\mathcal{G}}_{x'\leftarrow x}((g_y)_{y\in u^{-1}(x)}) = (\mathrm{res}^{\mathcal{G}}_{y'\leftarrow y'(x)}\, g_{y'(x)})_{y'\in u^{-1}(x')},$$

where $y'(x)$ denotes the unique face of y' mapping to x (it is unique because u is dimension-preserving). Thus, the condition $\mathrm{res}^{u_*\mathcal{G}}_{x'\leftarrow x}\, \tilde{g}_x = \tilde{g}_{x'}$ is equivalent to having $\mathrm{res}^{\mathcal{G}}_{y'\leftarrow y}\, g_y = g_{y'}$ for all $y' \in u^{-1}(x')$ and $y \in u^{-1}(x)$ with $y \subsetneq y'$. Now, identifying $(\tilde{g}_x)_{x\in U} = ((g_y)_{y\in u^{-1}(x)})_{x\in U}$ with $(g_y)_{y\in u^{-1}(U)}$, we see that $(u_*\mathcal{G})^\circ(U) = (u^\circ)_*\mathcal{G}^\circ(U)$.

A similar argument shows that for every $V \subseteq U$ open in X°, we have $\mathrm{res}^{(u_*\mathcal{G})^\circ}_{V\leftarrow U} = \mathrm{res}^{(u^\circ)_*\mathcal{G}^\circ}_{V\leftarrow U}$, so $(u_*\mathcal{G})^\circ = (u^\circ)_*\mathcal{G}^\circ$. That this isomorphism is natural in $\mathcal{G}$ is routine.

(ii) Unfortunately, we shall need to unfold the definition of the pullback $(u^\circ)^*\mathcal{F}^\circ$. We use the definition in [Sta20, Tag 008C], which makes use of *presheaves*[11], *sheafification* and *stalks*; see [Sta20, Tag 006A] for details. Recall that for a sheaf $\mathcal{H}$ on X°, the pullback $(u^\circ)^*\mathcal{H}$ is the sheafification of the presheaf $\mathcal{P}$ on Y given by $\mathcal{P}(U) = \varinjlim_{V\supseteq u(U)} \mathcal{H}(V)$, where V ranges over the open subsets of X° containing $u(U)$. Fortunately, in our situation, every subset $T \subseteq X^\circ$ admits a minimal open subset containing it, namely, $T^\wedge := \bigcup_{x\in T} X_{\supseteq x}$. The definition of the presheaf $\mathcal{P}$ therefore simplifies to $\mathcal{P}(U) = \mathcal{H}(u(U)^\wedge)$. Taking $\mathcal{H} = \mathcal{F}^\circ$, we find that $(u^\circ)^*\mathcal{F}^\circ$ is the sheafification of the presheaf $\mathcal{P}$ on Y given by

$$\mathcal{P}(U) = \mathcal{F}^\circ(u(U)^\wedge) = \{(f_x)_x \in \prod_{x\in u(U)^\wedge} \mathcal{F}(x) \,:\, \mathrm{res}^{\mathcal{F}}_{x'\leftarrow x}\, f_x = f_{x'} \text{ whenever } x \subsetneq x'\}$$

and $\mathrm{res}^{\mathcal{P}}_{V\leftarrow U}((f_x)_{x\in u(U)^\wedge}) = (f_x)_{x\in u(V)^\wedge}$ for all $V \subseteq U$ open in Y°. On the other hand, by unfolding the definitions, we find that

$$(u^*\mathcal{F})^\circ(U) = \{(f_y)_y \in \prod_{y\in U} \mathcal{F}(u(y)) \,:\, \mathrm{res}^{\mathcal{F}}_{u(y')\leftarrow u(y)}\, f_y = f_{y'} \text{ whenever } y \subsetneq y'\}.$$

Define $\varphi_U : \mathcal{P}(U) \to (u^*\mathcal{F})^\circ(U)$ by $\varphi_U((f_x)_{x\in u(U)^\wedge}) = (f_{u(y)})_{y\in U}$. It is routine to check that this is well-defined and that $\varphi = (\varphi_U)_U$ is a morphism of presheaves from $\mathcal{P}$ to $(u^*\mathcal{F})^\circ$.

By the universal property of sheafification [Sta20, Tag 0080], φ determines a sheaf morphism φ^{a} from $(u^\circ)^*\mathcal{F}^\circ$ (the sheafification of $\mathcal{P}$) to $(u^*\mathcal{F})^\circ$. Moreover, in order to show that φ^{a} is an isomorphism, it is enough to check that $\varphi : \mathcal{P} \to (u^*\mathcal{F})^\circ$ induces an isomorphism at the stalks [Sta20, Tags 007Z, 007T]. Recall that if $\mathcal{H}$ is a presheaf on Y, then the stalk of $\mathcal{H}$ at $y \in \mathcal{H}$ is $\mathcal{H}_y := \varinjlim_{V\ni y} \mathcal{H}(V)$ where V ranges over the open sets containing y. In our situation, there is a unique minimal open subset of Y° containing y, namely $Y_{\supseteq y}$, so the stalk $\mathcal{H}_y$ is just $\mathcal{H}(Y_{\supseteq y})$. We are therefore reduced to showing that $\varphi_{Y_{\supseteq y}} : \mathcal{P}(Y_{\supseteq y}) \to (u^*\mathcal{F})^\circ(Y_{\supseteq y})$ is an isomorphism for all $y \in Y$. Write $x = u(y)$. Then $u(Y_{\supseteq y})^\wedge = X_{\supseteq x}$. It is routine to check that $(f_{y'})_{y'\in Y_{\supseteq y}} \mapsto (\mathrm{res}^{\mathcal{F}}_{x'\leftarrow x}\, f_y)_{x'\in X_{\supseteq x}}$ defines an inverse to $\varphi_{Y_{\supseteq y}}$. This shows that $\varphi^{\mathrm{a}} : (u^\circ)^*\mathcal{F}^\circ \to (u^*\mathcal{F})^\circ$ is a sheaf isomorphism.

One readily checks that the formation of $\mathcal{P}$ is funtorial in $\mathcal{F}$ and that $\varphi : \mathcal{P} \to (u^*\mathcal{F})^\circ$ is natural in $\mathcal{F}$, so $\varphi^{\mathrm{a}} : (u^\circ)^*\mathcal{F}^\circ \to (u^*\mathcal{F})^\circ$ is also natural in $\mathcal{F}$. □

[11] Presheaves on a topological space are defined like sheaves, but without the requiring condition (S3).

Remark A.7. Theorem A.6(i) suggests a way to define the pushforward of a sheaf on a simplicial complex along an *arbitrary* morphism of simplicial complexes. Specifically, if $u : Y \to X$ is such a morphism and $\mathcal{G}$ is a sheaf on Y, define $u_*\mathcal{G}$ to be $((u^\circ)_* G^\circ)^\triangle$. (This is conceptually correct because $u_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is a right adjoint of $u^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.) Unfolding this definition, we find that for $x \in X - \{\emptyset\}$, we have

$$u_*\mathcal{G}(x) = \{(f_y)_y \in \prod_{y \in u^{-1}(x)^\wedge} \mathcal{G}(y) \,:\, \mathrm{res}^{\mathcal{G}}_{y' \leftarrow y} f_y = f_{y'} \text{ whenver } y \subsetneq y'\},$$

where $u^{-1}(x)^\wedge = \bigcup_{y \in u^{-1}(x)} Y_{\supseteq y}$, and the restriction maps $\mathrm{res}^{u_*\mathcal{G}}_{x' \leftarrow x} : u_*\mathcal{G}(x) \to u_*\mathcal{G}(x')$ are given by forgetting coordinates. It is not difficult to check that $(f_y)_{y \in u^{-1}(x)^\wedge} \mapsto (f_y)_{y \in u^{-1}(x)}$ defines an isomorphism from $u_*\mathcal{G}(x)$ to

$$\{(f_y)_y \in \prod_{y \in u^{-1}(x)} \mathcal{G}(y) \,:\, \mathrm{res}^{\mathcal{G}}_{y' \leftarrow y} f_y = f_{y'} \text{ whenver } y \subsetneq y'\},$$

so we may take the latter as the definition of $u^*\mathcal{G}(x)$. The restriction maps are then given by $\mathrm{res}^{u_*\mathcal{G}}_{x' \leftarrow x}(f_y)_{y \in \pi^{-1}(x)} = (\mathrm{res}_{y' \leftarrow y'(x)} f_{y'(x)})_{y' \in \pi^{-1}(x')}$, where $y'(x)$ is an arbitrary face of y' mapping to x (its choice is inconsequential). When $u : Y \to X$ is dimension preserving, no face in $u^{-1}(x)$ contains another such face, so $u^*\mathcal{G}(x) = \prod_{y \in u^{-1}(x)} \mathcal{G}(y)$ and we recover the definition of the pullback given in §4.3.

Recall that if Y is a topological space and $\mathcal{G}$ is a sheaf on Y, then we write $\Gamma(Y, \mathcal{G})$ for $Y(\mathcal{G})$, the group of global sections of $\mathcal{G}$. Letting $\mathcal{G}$ vary, the assignment $\Gamma(Y, -)$ defines a left exact functor from $\mathrm{Sh}(Y)$ to abelian groups, and its right derived functors are denoted $\{\mathrm{H}^i(Y, -)\}_{i \geq 0}$. The group $\mathrm{H}^i(Y, \mathcal{G})$ is the *i-th cohomology group* of the sheaf $\mathcal{G}$; see [Ive86, II.§3] for further details. For example, $\mathrm{H}^0(Y, \mathcal{G})$ is just $\Gamma(Y, \mathcal{G}) = \mathcal{G}(Y)$.

We finish this section by showing that the equivalence of Theorem A.3 is compatible with taking cohomology. More precisely:

Theorem A.8. *Let X be a simplicial complex.*

(i) For every sheaf $\mathcal{F}$ on X and $i \geq 0$, there is an isomorphism $\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(X^\circ, \mathcal{F}^\circ)$ natural in $\mathcal{F}$.

(ii) If $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves on X, then $0 \to \mathcal{F}^\circ \to \mathcal{F}'^\circ \to \mathcal{F}''^\circ \to 0$ is a short exact sequence of sheaves on X°, and there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^i(X,\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{H}^i(X,\mathcal{F}') \arrow[r] \arrow[d] & \mathrm{H}^i(X,\mathcal{F}'') \arrow[r] \arrow[d] & \mathrm{H}^{i+1}(X,\mathcal{F}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^i(X^\circ,\mathcal{F}^\circ) \arrow[r] & \mathrm{H}^i(X^\circ,\mathcal{F}'^\circ) \arrow[r] & \mathrm{H}^i(X^\circ,\mathcal{F}''^\circ) \arrow[r] & \mathrm{H}^{i+1}(X^\circ,\mathcal{F}^\circ) \arrow[r] & \cdots
\end{tikzcd}$$

in which the rows are the long cohomology exact sequences associated to $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ (see §4.2) and $0 \to \mathcal{F}^\circ \to \mathcal{F}'^\circ \to \mathcal{F}''^\circ \to 0$, and the vertical maps are isomorphism from (i).

Proof. The categories $\mathrm{Sh}(X)$ and $\mathrm{Sh}(X^\circ)$ are abelian, so the equivalence $\mathcal{F} \mapsto \mathcal{F}^\circ : \mathrm{Sh}(X) \to \mathrm{Sh}(X^\circ)$ of Theorem A.3 is necessarily exact. This shows that the sequence $0 \to \mathcal{F}^\circ \to \mathcal{F}'^\circ \to \mathcal{F}''^\circ \to 0$ in (ii) is exact. The equivalence also implies that the right derived functors of $\mathcal{F} \mapsto \Gamma(X^\circ, \mathcal{F}^\circ)$ from $\mathrm{Sh}(X)$ to Ab — the category of abelian groups — are $\{\mathcal{F} \mapsto \mathrm{H}^i(X^\circ, \mathcal{F}^\circ)\}_{i \geq 0}$. We will show in Appendix B that the functors $\{\mathrm{H}^i(X, -)\}_{i \geq 0}$ are right derived functors of $\mathrm{H}^0(X, -) : \mathrm{Sh}(X) \to \mathrm{Ab}$. Since derived functors are unique up to a natural isomorphism, the theorem will follow if we show that $\mathrm{H}^0(X, \mathcal{F})$ is naturally isomorphic to $\Gamma(X^\circ, \mathcal{F}^\circ) = \mathcal{F}^\circ(X^\circ)$. This is the content of the following lemma. $\square$

Lemma A.9. *Let X be a simplicial complex. There is a natural isomorphism* $\mathrm{H}^0(X,\mathcal{F}) \cong \mathcal{F}^\circ(X^\circ)$.

Proof. Unfolding the definitions, we find that

$$\mathcal{F}^\circ(X^\circ) = \{(f_x)_x \in \prod_{x\in X-\{\emptyset\}} \mathcal{F}(x) \,:\, \mathrm{res}^{\mathcal{F}}_{y\leftarrow x} f_x = f_y \text{ whenever } x \subsetneq y\},$$

whereas

$$\mathrm{H}^0(X,\mathcal{F}) = \{(f_v)_{v\in X(0)} \in \prod_{v\in X(0)} \mathcal{F}(v) \,:\, \mathrm{res}^{\mathcal{F}}_{e\leftarrow v} f_v = \mathrm{res}^{\mathcal{F}}_{e\leftarrow u} f_u \text{ for all } u,v \in X(0) \text{ with } e = u\cup v \in X(1)\}.$$

Define $\varphi_{\mathcal{F}} : \mathcal{F}^\circ(X^\circ) \to \mathrm{H}^0(X,\mathcal{F})$ by $\varphi_{\mathcal{F}}((f_x)_{x\in X-\{\emptyset\}}) = (f_v)_{v\in X(0)}$. It is routine to check that $\varphi_{\mathcal{F}}$ is well-defined and natural in $\mathcal{F}$.

To see that $\varphi_{\mathcal{F}}$ is invertible, observe that if $(f_v)_{v\in X(0)} \in \mathrm{H}^0(X,\mathcal{F})$, $x \in X - \{\emptyset\}$ and u,v are two 0-faces of x, then $f_v|_x = f_u|_x$. Indeed, this is clear if $u = v$, and otherwise, we have $f_v|_{u\cup v} = f_u|_{u\cup v}$ and applying $\mathrm{res}^{\mathcal{F}}_{x\leftarrow u\cup v}$ to both sides gives the desired equality. This allows us to define $\psi_{\mathcal{F}} : \mathrm{H}^0(X,\mathcal{F}) \to \mathcal{F}^\circ(X^\circ)$ by $\psi_{\mathcal{F}}((f_v)_{v\in X(0)}) = (f_{v(x)}|_x)_{x\in X-\{\emptyset\}}$, where $v(x)$ is an arbitrary 0-face of x. It is straightforward to check that $\psi_{\mathcal{F}}$ is defines an inverse to $\varphi_{\mathcal{F}}$, so $\varphi_{\mathcal{F}}$ is a isomorphism. $\square$

A.4 Aside: Augmented Sheaves as Sheaves on Topological Spaces

We finish with explaining how some of the results in §A.2 and §A.3 may be adapted to augmented sheaves.

Let X be a simplicial complex. We let X°_+ denote the set X together with the topology consisting of all simplicially open subsets $U \subseteq X - \{\emptyset\}$ and the set X. As in §A.2, given a sheaf $\mathcal{G}$ on X°_+, we can define an *augmented* sheaf $\mathcal{G}^\triangle$ on X by setting $\mathcal{G}^\triangle(x) = \mathcal{G}(X_{\supseteq x})$ and $\mathrm{res}^{\mathcal{G}^\triangle}_{y\leftarrow x} = \mathrm{res}^{\mathcal{G}}_{X_{\supseteq y}\leftarrow X_{\supseteq x}}$; mind that x is allowed to be the empty face. Conversely, an augmented sheaf $\mathcal{F}$ on X gives rise to a sheaf $\mathcal{F}^\circ$ on X°_+ defined using the same formulas as in the non-augmented sheaf case. The same argument as in the proof of Theorem A.3 shows that $\mathcal{G} \mapsto \mathcal{G}^\triangle$ defines an equivalence of categories from $\mathrm{Sh}(X^\circ_+)$ to the category of augmented sheaves on X, and $\mathcal{F} \mapsto \mathcal{F}^\circ$ is its inverse up to natural isomorphism. Thus, augmented sheaves on the simplicial complex X are essentially the same thing as sheaves on the topological space X°_+.

However, in contrast with the non-augmented sheaf case, the equivalence between augmented sheaves on X and sheaves on X°_+ does not respect cohomology. Rather, the dimensions are shifted by 1, i.e., there is a natural isomorphism $\mathrm{H}^{i-1}(X,\mathcal{F}) \cong \mathrm{H}^i(X^\circ_+,\mathcal{F}^\circ)$ for every augmented sheaf $\mathcal{F}$ and every $i \in \mathbb{N}\cup\{0\}$. This can be shown as in the proof of Theorem A.8, except now one has to establish a natural isomorphism $\mathrm{H}^{-1}(X,\mathcal{F}) \cong \Gamma(X^\circ_+,\mathcal{F}^\circ)$ and show that $\mathrm{H}^i(X,-)$ is the $(i+1)$-th right derived functor of $\mathrm{H}^{-1}(X,-)$.

Finally, while we have not defined in §4.3 the pushforward and pullback of augmented sheaves on simplicial complexes, the equivalence with $\mathrm{Sh}(X^\circ_+)$ suggests a way one might define them. That is, given a morphism of simplicial complexes $u : Y \to X$, an augmented sheaf $\mathcal{F}$ on X and an augmented sheaf $\mathcal{G}$ on Y, let $u^*\mathcal{F} = ((u^\circ_+)^*\mathcal{F}^\circ)^\triangle$ and $u_*\mathcal{G} = ((u^\circ_+)_*\mathcal{G}^\circ)^\triangle$, where u°_+ is just u viewed as a continuous function from Y°_+ to X°_+. We leave it to the reader to work out what $u^*\mathcal{F}$ and $u_*\mathcal{G}$ turn out to be. Beware, however, that these constructions may present exceptional behavior over the empty face. For example, if $u : Y \to X$ is a covering of degree n, and $\mathbb{F}_+$ denotes the constant augmented sheaf on Y associated to a field $\mathbb{F}$, then $u_*(\mathbb{F}_+)(\emptyset) = \mathbb{F}$ while $u_*(\mathbb{F}_+)(x) \cong \mathbb{F}^n$ for all $x \in X - \{\emptyset\}$. (The conceptual reason for this is that $u^\circ_+ : Y^\circ_+ \to X^\circ_+$ is generally not a degree-n covering of topological spaces.)

B Sheaf Cohomology is a Right Derived Functor

Throughout, X is a possibly-infinite simplicial complex. Recall that $\mathrm{Sh}(X)$ denotes the category of sheaves on X and let Ab denote the category of abelian groups. Then $\mathrm{H}^0(X,-)$ defines a left exact functor from $\mathrm{Sh}(X)$ to Ab. The purpose of this appendix is to prove that the higher cohomology groups $\mathrm{H}^i(X,-)$ defined in §4.2 are the *right derived functors* of $\mathrm{H}^0(X,-)$. In particular, the category $\mathrm{Sh}(X)$ has *enough injectives* so that the right derived functors of $\mathrm{H}^0(X,-)$ are defined. The necessary material about derived functors can be found in [Ive86] and [Sta20, Tag 010P], for instance.

We begin by introducing the following construction.

Construction B.1. Let $x \in X - \{\emptyset\}$ and let A be an abelian group. Define a sheaf $A_x = A_{X,x}$ on X by

$$A_x(y) = \begin{cases} A & y \subseteq x \\ 0 & y \nsubseteq x \end{cases}, \qquad \mathrm{res}^{A_x}_{z\leftarrow y} = \begin{cases} \mathrm{id}_A & z \subseteq x \\ 0 & z \nsubseteq x \end{cases}$$

for all $y, z \in X - \{\emptyset\}$ with $y \subsetneq z$.

Remark B.2. Under the equivalence of Theorem A.3, the sheaf A_x corresponds to a *skyscraper* sheaf.

Lemma B.3. *With notation as in Construction B.1, we have* $\mathrm{H}^i(X, A_x) = 0$ *for all* $i \geq 1$.

Proof. Let $d = \dim x$. We many forget about all faces in X not contained in x, and thus assume that X has a single d-face x, and A_x is the constant sheaf A. In this case, the topological realization $|X|$ of X is contractible, so by Corollary 4.6, $\mathrm{H}^i(X, A) = \mathrm{H}^i(|X|, A) = 0$ for $i \geq 1$. ☐

Lemma B.4. *Let X, x, A be as in Construction B.1 and let $\mathcal{F}$ be any sheaf on X. There is a natural (in $\mathcal{F}$ and A) bijection between* $\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, A_x)$ *and* $\mathrm{Hom}_{\mathrm{Ab}}(\mathcal{F}(x), A)$ *given by* $\varphi \mapsto \varphi_x$.

Proof. Let us first show that $\varphi \mapsto \varphi_x : \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, A_x) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathcal{F}(x), A)$ is injective. Let $\psi : \mathcal{F} \to A_x$ be another morphism with $\psi_x = \varphi_x$, and let $y \in X - \{\emptyset\}$. If $y \nsubseteq x$, then we must have $\psi_y = 0 = \varphi_x$, because $A_x(y) = 0$. On the other hand, if $y \subseteq x$, then $\psi_y = \mathrm{res}^{A_x}_{x\leftarrow y} \circ \psi_y = \psi_x \circ \mathrm{res}^{\mathcal{F}}_{x\leftarrow y} = \varphi_x \circ \mathrm{res}^{\mathcal{F}}_{x\leftarrow y} = \mathrm{res}^{A_x}_{x\leftarrow y} \circ \varphi_y = \varphi_y$ (with $\mathrm{res}^{\mathcal{F}}_{x\leftarrow x}$ being $\mathrm{id}_{\mathcal{F}(x)}$). We conclude that $\psi = \varphi$.

Conversely, given an abelian group homomorphism $\varphi_0 : \mathcal{F}(x) \to A$, we can define a morphism $\varphi : \mathcal{F} \to A_x$ by setting $\varphi_y = 0$ if $y \nsubseteq x$ and $\varphi_y = \varphi_0 \circ \mathrm{res}^{\mathcal{F}}_{x\leftarrow y}$ if $y \subseteq x$ (with $\mathrm{res}^{\mathcal{F}}_{x\leftarrow x}$ being $\mathrm{id}_{\mathcal{F}(x)}$). It is routine to check that φ is indeed a sheaf morphism and $\varphi_x = \varphi_0$, so the map in the lemma is onto.

That $\varphi \mapsto \varphi_x$ is natural in $\mathcal{F}$ and A is straightforward. ☐

We can now prove the following key lemma.

Lemma B.5. *Let $\mathcal{F}$ be a sheaf on X. Then there exists a sheaf $\mathcal{G}$ on X and a monomorphism $j : \mathcal{F} \to \mathcal{G}$ such that* $\mathrm{H}^i(X, \mathcal{G}) = 0$ *for all* $i \geq 1$. *Moreover, $\mathcal{G}$ can be taken to be injective in* $\mathrm{Sh}(X)$.

Proof. Let $x \in X - \{\emptyset\}$. By Lemma B.4 (applied with $A = \mathcal{F}(x)$), the identity map $\mathrm{id} : \mathcal{F}(x) \to \mathcal{F}(x)$ gives rise to a sheaf morphism $j^{(x)} : \mathcal{F} \to \mathcal{F}(x)_x$. Let $\mathcal{G} = \prod_{x \in X - \{\emptyset\}} \mathcal{F}(x)_x$ (note that the product may be infinite). Then the morphisms $\{j^{(x)}\}_{x \in X - \{\emptyset\}}$ determine a morphism $j : \mathcal{F} \to \mathcal{G}$ given by $j_y(f) = (j^{(x)}_y(f))_{x \in X - \{\emptyset\}}$. Since the the y-component of $j_y(f)$ is just f, we have $\ker j = 0$. Moreover, by Lemma B.3, $\mathrm{H}^i(X, \mathcal{F}(x)_x) = 0$ for all $i \geq 1$, which means that the cochain complex $C^\bullet(X, \mathcal{F}(x)_x)$ is exact in degrees ≥ 1. Since $C^\bullet(X, \mathcal{G})$ is the product of the cochain complexes

$\{C^\bullet(X, \mathcal{F}(x)_x)\}_{x\in X-\{\emptyset\}}$, it is also exact in degrees ≥ 1, and we conclude that $\mathrm{H}^i(X, \mathcal{G}) = 0$ for $i \geq 1$.

In order to choose $\mathcal{G}$ that is also injective, for every $x \in X - \{\emptyset\}$, choose an embedding $i_x : \mathcal{F}(x) \to E^{(x)}$ of $\mathcal{F}(x)$ into an injective $\mathbb{Z}$-module $E^{(x)}$, and use the i_x to construct the $j^{(x)} : \mathcal{F} \to (E^{(x)})_x$ and $j : \mathcal{F} \to \mathcal{G} := \prod_x (E^{(x)})_x$. Lemma B.4 implies readily that each of the sheaves $(E^{(x)})_x$ is injective, and therefore, so is $\mathcal{G}$. □

Theorem B.6. *Let X be a (possibly infinite) simplicial complex. Then:*

(i) The abelian category $\mathrm{Sh}(X)$ *has enough injectives.*

(ii) The functor $\mathrm{H}^i(X, -) : \mathrm{Sh}(X) \to \mathrm{Ab}$ *of §4.2 is the i-th right derived functor of* $\mathrm{H}^0(X, -)$.

Proof. (i) This is Lemma B.5. Alternatively, by Theorem A.3, $\mathrm{Sh}(X)$ is equivalent to $\mathrm{Sh}(X^\circ)$ and the latter is well-known to have enough injectives [Ive86, II, Theorem 3.1].

(ii) The derived functors of $\mathrm{H}^0(X, -)$ form a universal cohomological δ-functor, and we observed in §4.2 that the functors $\{H^i(X, -)\}_{i\geq 0}$ form a cohomological δ-functor. Since universal δ-functors are unique up to natural isomorphism, it is enough to show that the $\{H^i(X, -)\}_{i\geq 0}$ are universal. By [Sta20, Tag 010T], this will follow if we show that every sheaf $\mathcal{F}$ on X embeds in a sheaf $\mathcal{G}$ with $\mathrm{H}^i(X, \mathcal{G}) = 0$ for all $i \geq 1$, and that is exactly what Lemma B.5 tells us. □

Remark B.7. Let R be a ring, and let $\mathrm{Sh}_R(X)$ denote the category of sheaves of left R-modules. The cohomology groups of a sheaf in $\mathrm{Sh}_R(X)$ defined in §4.2 are naturally left R-modules, so we may regard $\mathrm{H}^i(X, -)$ as a functor from $\mathrm{Sh}_R(X)$ to the category of left R-modules, denoted R-Mod. The same argument as in the proof of Theorem B.6 can be used to show that $\mathrm{H}^i(X, -) : \mathrm{Sh}_R(X) \to R\text{-Mod}$ is the i-th right derived functor of $\mathrm{H}^0(X, -) : \mathrm{Sh}_R(X) \to R\text{-Mod}$.

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