Memory effects in multipartite systems coupled by non-diagonal dephasing mechanisms

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(Dated: September 2, 2022)

The developing of (non-Markovian) memory effects strongly depends on the underlying system-environment dynamics. Here we study this problem in multipartite arrangements where all subsystems are coupled to each other by non-diagonal Markovian (Lindblad) dephasing mechanisms. Taking as system and environment arbitrary sets of complementary subsystems it is shown that both operational and non-operational approaches to quantum non-Markovianity can be characterized in an exact analytical way. Similarly to previous studies about dissipative-entanglement-generation in this kind of dynamics [Seif, Wang, and Clerk, Phys. Rev. Lett. **128**, 070402 (2022)], we found that memory effects can only emerge when a time-reversal symmetry is broken. Nevertheless, it is also found that departures from Markovianity can equivalently be represented through a statistical mixture of Markovian dephasing dynamics, which does not involve any system-environment entanglement. Specific bipartite and multipartite dynamics exemplify the main general results.

I. INTRODUCTION

In the last years remarkable advancements has been achieved in the study and characterization of open quantum systems [1–3]. In particular, the old association of memory effects with time-convoluted contributions in the time-evolution of the system density matrix [4] has been surpassed. Instead, *quantum non-Markovianity* can now be understood from two alternative powerful theoretical main streamlines.

First, in *non-operational approaches*, memory effects are only determined by taking into account the (unperturbed) system density propagator. Markovianity (memoryless regime) is univocally associated to quantum semigroup structures [5]. Thus, deviations in the propagator properties with respect to this reference are used to quantify the magnitude of memory effects [6, 7]. Diverse witnesses have been proposed, such as the trace distance between two initial states [8], the divisibility of the propagator [9], non-Markovianity degree [10], the quantum regression theorem [11, 12], and the sign of the rates in a canonical Lindblad structure [13], just to name a few. Secondly, *operational approaches* have been introduced more recently. Here, the system of interest is subjected to a set of explicit measurement processes. Markovianity is related to the usual concept in terms of probabilities [4]. Thus, memory effects are characterized from the joint probabilities of the measurement outcomes [14–20].

Both operational and non-operational approaches to quantum non-Markovianity provide complementary and valid frames to understand memory effects. Nevertheless, different conclusions can be obtained in some cases. For example, the conditions under which memory effects can be interpreted in terms of an environment-to-system backflow of information strongly differ in both schemes [21–27].

In the operational approach the absence of any (physical) environment-to-system backflow of information was associated to (non-Markovian) casual bystander environments [28], that is, those whose self-dynamics do not depend at all on the system degrees of freedom. A measurement based procedure enables to detecting this condition [29]. In addition, it allows to determine if the environment action, when considering the outcome statistics, can be represented in terms of this kind of "passive environments," such as for example *statistical mixtures of different Markovian evolutions* (unitary [27] or dissipative Lindblad ones). This kind of evolutions, in the unitary case, has also been studied from the perspective of memory effects in non-operational approaches [30]. Interestingly, with a totally different motivation, the possibility of representing an open quantum system dynamics in terms of a statistical mixture (random noisy ensembles) of Markovian evolutions has been associated to the *classicality* of the system-environment interaction [31–36].

All previous issues have been mainly discussed in single open quantum systems. Nevertheless, given that quantum information becomes relevant when implemented in *multipartite arrangements*, there has been a growing interest in the study of this kind of dynamics (from an open system perspective), both from unitary and dissipative (or effective) underlying descriptions [37–47]. The main goal of this work is to contribute to this research line by providing a full characterization of quantum non-Markovianity, jointly with the previous topics, in a class of multipartite dissipative dynamics [47].

In Ref. [47] the authors study a multipartite qubit dynamics, where all subsystems are coupled between them by non-diagonal dephasing mechanisms. Depending on the dimensionality (number of qubits) and coupling parameters the dynamics may lead to the emergence of transient *multipartite entanglement* [48]. This property is read as a signature of the nonclassicality of the evolution. Here, by considering both Hamiltonian and dissipative couplings [see Eqs. (1) and (2)] we show that, for any kind of subsystems (qubits or arbitrary ones), the multipartite dynamics can be diagonalized in an exact analytical way. Consequently, both operational and non-operational approaches to quantum non-Markovianity can be tackled in the same way. Similarly to the study of entanglement generation [47], we find that the break of a time-reversal symmetry plays a fundamental role when considering the emergence of memory effects. In contrast, the possibility of representing the dynamic of an arbitrary set of subsystems in terms of a statistical mixture of Markovian dephasing dynamics is also established.

The paper is outlined as follows. In Sec. II the multipartite dynamics is solved in an exact way. Introducing an arbitrary system-environment splitting, conditions for the emergence of memory effects in non-operational approaches are obtained. In Sec. III we characterize memory effects when considering successive measurement processes performed over the subsystems of interest. In Sec. IV we study bipartite and multipartite specific examples. In Sec. V we provide the Conclusions. Extensions and calculation details are provided in the Appendixes.

II. MULTIPARTITE NON-DIAGONAL DEPHASING DYNAMICS

We consider a multipartite system consisting of an arbitrary set of n subsystems. In general, each one has associated a (possibly different) Hilbert space $\mathcal{H}_i$. Hence, the total Hilbert space is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_n$. By assumption, the total density matrix ρ_t obeys the evolution

$$\frac{d\rho_t}{dt} = -i[H, \rho_t] + \sum_{i,j} \Gamma_{ij} (S^{(i)} \rho_t S^{(j)} - \frac{1}{2}\{S^{(j)} S^{(i)}, \rho_t\}_+). \tag{1}$$

The indexes $i = 1, 2, \cdots n$ and $j = 1, 2, \cdots n$ label the subsystems. In addition, $S^{(i)}$ is an arbitrary Hermitian operator ($S^{(i)} = S^{(i)\dagger}$) acting on each subsystem Hilbert space $\mathcal{H}_i$. $\{A, B\}_+$ denotes an anticommutator operation between two arbitrary operators A and B. Hence, the second term in Eq. (1) is a Lindblad contribution that introduces a dissipative coupling between all pairs of subsystems. For guarantying the complete positive nature of the solution map, the complex (rate) coefficients $\{\Gamma_{ij}\}$ must to constitute a positive definite Hermitian matrix [1]. The total Hamiltonian is assumed to be

$$H = \frac{1}{2} \sum_{i,j} h_{ij} S^{(i)} S^{(j)}, \tag{2}$$

where h_{ij} are real coefficients. They scale a unitary coupling between all subsystems. The model studied in Ref. [47] is recovered by taking all subsystems as qubits with $S^{(i)}$ the z-Pauli matrix in $\mathcal{H}_i$.

A. Density matrix solution

An explicit expression for ρ_t can be obtained by introducing an appropriate basis for the full Hilbert space. Given that each operator $S^{(i)}$ is Hermitian, its eigenvectors $\{|s_i\rangle\}$ provide a natural basis for $\mathcal{H}_i$, where $S^{(i)}|s_i\rangle = s_i|s_i\rangle$. The set $\{s_i\}$ are the corresponding eigenvalues. The basis $\{|\mathbf{s}\rangle\}$ of the full multipartite Hilbert space $\mathcal{H}$ is then taken as

$$|\mathbf{s}\rangle \equiv |s_1 \cdots s_n\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle. \tag{3}$$

With the previous definitions, the dephasing nature of Eq. (1) can explicitly be shown, that is, the matrix elements of ρ_t do not couple to each other. In fact, taking two arbitrary basis states, $|\mathbf{s}\rangle$ and $|\tilde{\mathbf{s}}\rangle$, and using that $S^{(i)}|\mathbf{s}\rangle = s_i|\mathbf{s}\rangle$, from Eq. (1) we get

$$\frac{d}{dt}\langle\tilde{\mathbf{s}}|\rho_t|\mathbf{s}\rangle = -\Phi_{\tilde{\mathbf{s}},\mathbf{s}}\langle\tilde{\mathbf{s}}|\rho_t|\mathbf{s}\rangle. \tag{4}$$

The complex coefficients $\Phi_{\tilde{\mathbf{s}},\mathbf{s}}$ are given by

$$\Phi_{\tilde{\mathbf{s}},\mathbf{s}} = i(\Omega_{\tilde{\mathbf{s}}} - \Omega_{\mathbf{s}}) + \Upsilon_{\tilde{\mathbf{s}},\mathbf{s}}. \tag{5}$$

Here, the "frequencies" $\Omega_{\mathbf{s}}$ are induced by the Hamiltonian contribution (2), being defined as

$$\Omega_{\mathbf{s}} = \frac{1}{2} \sum_{i,j} h_{ij} s_i s_j. \tag{6}$$

The contribution $\Upsilon_{\tilde{\mathbf{s}},\mathbf{s}}$, induced by the non-diagonal Lindblad term in Eq. (1), after a simple algebra, can be written as

$$\Upsilon_{\tilde{\mathbf{s}},\mathbf{s}} = \sum_{i,j} (\tilde{s}_i - s_i) \frac{\Gamma_{ij}}{2} (\tilde{s}_j - s_j) + \sum_{i,j} \frac{\Gamma_{ij}}{2} (\tilde{s}_j s_i - \tilde{s}_i s_j). \tag{7}$$

Notice that the first and second sum contributions depend respectively on the *real and imaginary parts of the coefficients* $\{\Gamma_{ij}\}$. These properties follow straightforwardly from the index interchange $i \leftrightarrow j$.

The matrix element behavior defined by Eq. (4) can be integrated straightforwardly. Consequently, the multipartite state ρ_t can explicitly be written as

$$\rho_t = \sum_{\mathbf{s},\tilde{\mathbf{s}}} |\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_0|\mathbf{s}\rangle\langle\mathbf{s}| \exp[-\Phi_{\tilde{\mathbf{s}},\mathbf{s}} t], \tag{8}$$

where ρ_0 is the initial multipartite state. Notice that populations do not evolve in time, $\langle\mathbf{s}|\rho_t|\mathbf{s}\rangle = \langle\mathbf{s}|\rho_0|\mathbf{s}\rangle$. This property follows from Eqs. (6) and (7), which imply $\Phi_{\mathbf{s},\mathbf{s}} = 0$. The expression (8) allows us to analyze diverse aspects of the dynamics in an explicit analytical way. It is valid for arbitrary operators $\{S^{(i)}\}$ and coupling matrixes $\{h_{ij}\}$ and $\{\Gamma_{ij}\}$. Interestingly, an analytical solution can also be found even when the unitary and dissipative coupling in Eq. (1) are defined by more than two (multipartite) operators (see Appendix A).

B. System-environment splitting

In Eq. (1) all subsystems play the same role. In order to analyze memory effects an arbitrary system-environment splitting must be introduced. Thus, the total Hilbert space is written as $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. We consider that n_S and n_B subsystems, with $n_S + n_B = n$, define the system ($\mathcal{H}_S$) and "bath" ($\mathcal{H}_B$) Hilbert space respectively. When $n_B > 1$ the environment is a multipartite one. In a similar way, without loss of generality, each element of the basis $\{|\mathbf{s}\rangle\}$ [Eq. (3)] is rewritten as

$$|\mathbf{s}\rangle \to |\mathbf{sb}\rangle \equiv |s_1 \cdots s_{n_S}\rangle \otimes |b_1 \cdots b_{n_B}\rangle. \tag{9}$$

Introducing the change of notation $\rho_t \to \rho_t^{se}$, the total density matrix defined by Eq. (8) is re-expressed as

$$\rho_t^{se} = \sum_{\mathbf{s},\tilde{\mathbf{s}},\mathbf{b},\tilde{\mathbf{b}}} |\tilde{\mathbf{s}}\tilde{\mathbf{b}}\rangle\langle\tilde{\mathbf{s}}\tilde{\mathbf{b}}|\rho_0^{se}|\mathbf{sb}\rangle\langle\mathbf{sb}| \exp[-\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}}t]. \tag{10}$$

Here, $\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}}$ follows from Eq. (5) after introducing the splitting $\tilde{\mathbf{s}} \to (\tilde{\mathbf{s}}, \tilde{\mathbf{b}})$ and $\mathbf{s} \to (\mathbf{s}, \mathbf{b})$, that is,

$$\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}} = i(\Omega_{\tilde{\mathbf{s}}\tilde{\mathbf{b}}} - \Omega_{\mathbf{sb}}) + \Upsilon_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}}. \tag{11}$$

The frequency terms associated to the unitary evolution immediately lead to $\Omega_{\mathbf{s}} \to \Omega_{\mathbf{sb}}$, with

$$\Omega_{\mathbf{sb}} = \Omega_{\mathbf{s}} + \Omega_{\mathbf{b}} + \sum_{i\in S, j\in B} \Big(\frac{h_{ij} + h_{ji}}{2}\Big) s_i b_j. \tag{12}$$

The sum indexes $i \in S$ and $j \in B$ run over the subsystems associated to the system and the environment respectively. The "non-coupling" contributions $\Omega_{\mathbf{s}}$ and $\Omega_{\mathbf{b}}$ are given by Eq. (6) but restricting the sum indexes as $(i,j) \in S$ and $(i,j) \in B$ respectively. On the other hand, the contribution $\Upsilon_{\tilde{\mathbf{s}},\mathbf{s}} \to \Upsilon_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}}$ can be written as

$$\Upsilon_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}} = \Upsilon_{\tilde{\mathbf{s}},\mathbf{s}} + \Upsilon_{\tilde{\mathbf{b}},\mathbf{b}} + \chi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}}. \tag{13}$$

The terms $\Upsilon_{\tilde{\mathbf{s}},\mathbf{s}}$ and $\Upsilon_{\tilde{\mathbf{b}},\mathbf{b}}$ have the same structure than Eq. (7) with the restrictions $(i,j) \in S$ and $(i,j) \in B$ respectively. The contribution $\chi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}}$ introduces the system-environment coupling. It reads

$$\begin{aligned}\chi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{sb}} &= \sum_{i\in S, j\in B} (\tilde{s}_i - s_i)\Big(\frac{\Gamma_{ij} + \Gamma_{ji}}{2}\Big)(\tilde{b}_j - b_j) \\ &\quad + \sum_{i\in S, j\in B} \Big(\frac{\Gamma_{ij} - \Gamma_{ji}}{2}\Big)(\tilde{b}_j s_i - \tilde{s}_i b_j). \end{aligned}\tag{14}$$

Notice that the sum terms depend respectively on the real and imaginary parts of the matrix $\{\Gamma_{ij}\}$.

C. System dynamics

Of special interest is to determine the system density matrix, which is obtained by tracing out the environment degrees of freedom, $\rho_t^{(s)} \equiv \mathrm{Tr}_e[\rho_t^{se}]$. Similarly, for the environment $\rho_t^{(e)} \equiv \mathrm{Tr}_s[\rho_t^{se}]$. By taking separable initial conditions $\rho_0^{se} = \rho_0^{(s)} \otimes \rho_0^{(e)}$, from Eq. (10) we get

$$\rho_t^{(s)} = \sum_{\mathbf{s},\tilde{\mathbf{s}}} f_{\tilde{\mathbf{s}}\mathbf{s}}(t) |\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_0^{(s)}|\mathbf{s}\rangle\langle\mathbf{s}|, \tag{15}$$

where the set of functions $\{f_{\tilde{\mathbf{s}}\mathbf{s}}(t)\}$ is given by

$$f_{\tilde{\mathbf{s}}\mathbf{s}}(t) = \sum_{\mathbf{b}} \langle\mathbf{b}|\rho_0^{(e)}|\mathbf{b}\rangle \exp(-t\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{sb}}). \tag{16}$$

From these expressions it is simple to realize that a dephasing mechanism also characterizes the system dynamics, where the decay of the system coherences $\langle\tilde{\mathbf{s}}|\rho_t^{(s)}|\mathbf{s}\rangle$ is defined by the functions $f_{\tilde{\mathbf{s}}\mathbf{s}}(t)$. Consistently, given that $f_{\mathbf{ss}}(t) = \sum_{\mathbf{b}} \langle\mathbf{b}|\rho_0^{(e)}|\mathbf{b}\rangle = 1$, the populations do not change with time, $\langle\mathbf{s}|\rho_t^{(s)}|\mathbf{s}\rangle = \langle\mathbf{s}|\rho_0^{(s)}|\mathbf{s}\rangle$. In Appendix B we explicitly write the environment state.

In contrast to Eq. (8), the coherences behavior defined by $f_{\tilde{\mathbf{s}}\mathbf{s}}(t)$ strongly depart from an (complex) exponential one. This property anticipates the presence of memory effects, which is supported by characterizing the time-evolution of $\rho_t^{(s)}$. The most general time-dependent (dephasing) evolution consistent with Eq. (15) can be written as

$$\frac{d\rho_t^{(s)}}{dt} = \mathcal{L}_t[\rho_t^{(s)}] + \sum_{\tilde{\mathbf{s}}\mathbf{s}} \gamma_t^{\tilde{\mathbf{s}}\mathbf{s}} (\Pi_{\tilde{\mathbf{s}}}\rho_t^{(s)}\Pi_{\mathbf{s}} - \frac{1}{2}\{\Pi_{\mathbf{s}}\Pi_{\tilde{\mathbf{s}}}, \rho_t^{(s)}\}_+), \tag{17}$$

where we have introduced the system projectors $\Pi_{\mathbf{s}} \equiv |\mathbf{s}\rangle\langle\mathbf{s}|$ and $\mathcal{L}_t[\rho_t^{(s)}] \equiv -i[H_t^{(s)}, \rho_t^{(s)}]$, with Hamiltonian

$$H_t^{(s)} = \frac{1}{2}\sum_{\mathbf{s}} \omega_t^{\mathbf{s}} |\mathbf{s}\rangle\langle\mathbf{s}|. \tag{18}$$

The set of (time-dependent) frequencies $\{\omega_t^{\mathbf{s}}\}$ and the Hermitian matrix of (complex) coefficients $\{\gamma_t^{\tilde{\mathbf{s}}\mathbf{s}}\}$ can be determined after knowing the set of functions $\{f_{\tilde{\mathbf{s}}\mathbf{s}}(t)\}$ [Eq. (16)]. From Eq. (17), they are related by the equations ($\tilde{\mathbf{s}} \neq \mathbf{s}$)

$$\frac{df_{\tilde{\mathbf{s}}\mathbf{s}}(t)}{dt} = -\frac{1}{2}[i(\omega_t^{\tilde{\mathbf{s}}} - \omega_t^{\mathbf{s}}) + (\gamma_t^{\tilde{\mathbf{s}}\tilde{\mathbf{s}}} + \gamma_t^{\mathbf{ss}}) - 2\gamma_t^{\tilde{\mathbf{s}}\mathbf{s}}] f_{\tilde{\mathbf{s}}\mathbf{s}}(t). \tag{19}$$

Therefore, the unknown functions $\{\omega_t^{\mathbf{s}}\}$ and $\{\gamma_t^{\tilde{\mathbf{s}}\mathbf{s}}\}$ can be determinated from $\{[1/f_{\tilde{\mathbf{s}}\mathbf{s}}(t)](d/dt)f_{\tilde{\mathbf{s}}\mathbf{s}}(t)\}$.

D. Necessary condition for the development of memory effects

In *non-operational approaches* to quantum non-Markovianity [6, 7], when the matrix $\{\gamma_t^{\tilde{\mathbf{s}}\mathbf{s}}\}$ in Eq. (17) is positive definite the system evolution is classified as Markovian. This kind of general characterization of the matrix $\{\gamma_t^{\tilde{\mathbf{s}}\mathbf{s}}\}$ cannot be established in our case of study. Nevertheless, after providing a specific underlying model [Eq. (1)], it can always be calculated in an exact analytical way.

In spite of the previous limitation, it is possible to establish a *necessary condition* for the developing of memory effects. It terms of the partial diagonal ($\tilde{\mathbf{b}} = \mathbf{b}$) multipartite dephasing rates it reads

$$\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{sb}} \neq \Phi_{\tilde{\mathbf{s}},\mathbf{s}}. \tag{20}$$

In fact, when this condition is not met $[\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}=\Phi_{\tilde{\mathbf{s}},\mathbf{s}}]$ the system coherences behavior Eq. (16), using that $\sum_{\mathbf{b}}\langle\mathbf{b}|\rho_0^{(e)}|\mathbf{b}\rangle=1$, becomes (complex) exponential. Consequently the system density matrix [Eq. (17)] obey a time-independent "Markovian" Lindblad equation. We remark that in non-operational approaches the condition (20) is necessary but in general not sufficient for the developing of memory effects.

From the explicit expression for $\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{s}\mathbf{b}}$ [Eq. (13)], taking $\tilde{\mathbf{b}}=\mathbf{b}$, straightforwardly it follows

$$\begin{aligned}\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}} &= \Phi_{\tilde{\mathbf{s}},\mathbf{s}}+\sum_{i\in S,j\in B} i\Big(\frac{h_{ij}+h_{ji}}{2}\Big)(\tilde{s}_i-s_i)b_j \\ &\quad -\sum_{i\in S,j\in B}\Big(\frac{\Gamma_{ij}-\Gamma_{ji}}{2}\Big)(\tilde{s}_i-s_i)b_j.\end{aligned} \tag{21}$$

The first contribution has the same structure as Eq. (5), $\Phi_{\tilde{\mathbf{s}},\mathbf{s}}=i(\Omega_{\tilde{\mathbf{s}}}-\Omega_{\mathbf{s}})+\Upsilon_{\tilde{\mathbf{s}},\mathbf{s}}$, but here it only involves system degrees of freedom. The two remaining sum contributions lead to memory effects [Eq. (20)].

In Eq. (21), the sum contribution proportional to $(h_{ij}+h_{ji})/2$ corresponds to the system-environment coupling induced by the Hamiltonian term. On the other hand, the dissipative coupling induced by the non-diagonal structure is proportional to the imaginary part $(\Gamma_{ij}-\Gamma_{ji})/2$ of the coupling rates. It is completely independent of the corresponding real part $(\Gamma_{ij}+\Gamma_{ji})$. Thus, *system-environment correlations induced by the real part of* $\{\Gamma_{ij}\}$ *does not lead to memory effects.* In addition, *memory effects can only emerge when the a time-reversal symmetry is broken.* In fact, this symmetry is broken when the matrix $\{\Gamma_{ij}\}$ is a complex one. Interestingly, the same conditions were found in Ref. [47] when considering the production of transient entanglement.

An relevant conclusion can also be obtained from Eq. (21). *While the unitary and dissipative couplings may lead to different system-environment correlations, they may induce exactly the same non-Markovian system dynamics.* In fact, in Eq. (21) the dependence of the sum contributions with respect to the eigenvalues $\{(\tilde{s}_i-s_i)b_j\}$ is exactly the same. Consequently, under the mapping $i(h_{ij}+h_{ji})/2 \leftrightarrow -(\Gamma_{ij}-\Gamma_{ji})/2$, exactly the same system memory effects are induced by the unitary and dissipative couplings respectively [see Eqs. (15) and (16)].

III. OPERATIONAL APPROACH TO QUANTUM NON-MARKOVIANITY

In operational approaches to quantum non-Markovianity the *system of interest* is subjected to a set of measurement processes [14, 15]. The classification of the dynamics relies on determining if the corresponding outcome joint-probability fulfills or does not fulfill a standard Markov definition [4]. Interestingly, a full characterization of this approach can be formulated for the dynamics under study.

We assume that the system [defined by the splitting (9)] is subjected to three successive measurement processes. The goal is to calculate the joint probability $P(z,y,x)$ where the sets $\{x\}$, $\{y\}$, and $\{z\}$ correspond to the outcomes of each measurement, which are performed at times 0, t, and $t+\tau$ respectively. The measurement operators are defined as $\{\Pi_m\}$ with $m=x,y,z$. They fulfill the normalization condition $\sum_m \Pi_m^\dagger\Pi_m=\mathrm{I}_s$, where I_s is the identity operator in the system Hilbert space. The intermediate measurement is assumed to be a projective one. In all cases, the measurements induce the transformation $\rho\to\rho_m$, where the post measurement states are $\rho_m=\Pi_m\rho\Pi_m^\dagger/\mathrm{Tr}[\Pi_m^\dagger\Pi_m\rho]$, each case occurring with probability $P(m)=\mathrm{Tr}[E_m\rho]$. For simplifying the expressions we denote $E_m\equiv\Pi_m^\dagger\Pi_m$, where $m=x,y,z$.

A. Joint probability of measurement outcomes

We maintain the system-environment splitting defined by Eq. (9). Thus, the corresponding propagator is set by Eq. (10). Furthermore, separable initial conditions are assumed $\rho_0^{se}=\rho_0^s\otimes\rho_0^e$. Consequently, the outcome probability for the first measurement is $P(x)=\mathrm{Tr}_s[E_x\rho_0^s]$, while the post-measurement state is

$$\rho_0^{se}\to\rho_x^{se}=\rho_x\otimes\rho_0^e. \tag{22}$$

Afterwards, during a time interval of duration t, the arrange follows the dynamics (10), which induces the transformation $\rho_x^{se}\to\rho_x^{se}(t)$, where

$$\rho_x^{se}(t)=\sum_{\mathbf{s},\tilde{\mathbf{s}},\mathbf{b},\tilde{\mathbf{b}}}|\tilde{\mathbf{s}}\tilde{\mathbf{b}}\rangle\langle\tilde{\mathbf{s}}\tilde{\mathbf{b}}|\rho_x\otimes\rho_0^e|\mathbf{s}\mathbf{b}\rangle\langle\mathbf{s}\mathbf{b}|\exp[-t\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{s}\mathbf{b}}]. \tag{23}$$

The conditional probability for the second measurement outcomes $\{y\}$, given that the first measurement outcome is x, is given by $P(y|x)=\mathrm{Tr}_{se}[E_y\rho_x^{se}(t)]$. It can explicitly be written as

$$P(y|x)=\sum_{\mathbf{s},\tilde{\mathbf{s}},\mathbf{b}}\langle\mathbf{s}|E_y|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_x|\mathbf{s}\rangle\langle\mathbf{b}|\rho_0^e|\mathbf{b}\rangle\exp[-t\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}]. \tag{24}$$

Using that the second measurement is a projective one, the corresponding post-measurement state is

$$\rho_x^{se}(t)\to\rho_{yx}^{se}(t)=\rho_y\otimes\rho_{yx}^e(t), \tag{25}$$

where the environment state is

$$\begin{aligned}\rho_{yx}^e(t) &= \frac{1}{P(y|x)}\sum_{\mathbf{b},\tilde{\mathbf{b}}}|\tilde{\mathbf{b}}\rangle\langle\tilde{\mathbf{b}}|\rho_0^e|\mathbf{b}\rangle\langle\mathbf{b}| \\ &\quad\times\sum_{\mathbf{s},\tilde{\mathbf{s}}}\langle\mathbf{s}|E_y|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_x|\mathbf{s}\rangle\exp[-t\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{s}\mathbf{b}}].\end{aligned} \tag{26}$$

Finally, the arrange evolves during a time interval τ, inducing the transformation $\rho_{yx}^{se}(t)\to\rho_{yx}^{se}(t+\tau)$. Using the propagator defined by Eq. (10) it follows

$$\rho_{yx}^{se}(t+\tau) = \sum_{\mathbf{s},\tilde{\mathbf{s}},\mathbf{b},\tilde{\mathbf{b}}} |\tilde{\mathbf{s}}\tilde{\mathbf{b}}\rangle\langle\tilde{\mathbf{s}}\tilde{\mathbf{b}}|\rho_{yx}^{se}(t)|\mathbf{s}\mathbf{b}\rangle\langle\mathbf{s}\mathbf{b}| \exp[-\tau\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{s}\mathbf{b}}]. \tag{27}$$

The conditional probability for the last measurement outcomes $\{z\}$, given that the previous ones were y and x, is given by $P(z|y,x) = \mathrm{Tr}_{se}[E_z \rho_{yx}^{se}(t+\tau)]$, which yields

$$P(z|y,x) = \sum_{\mathbf{s},\tilde{\mathbf{s}},\mathbf{b}} \langle\mathbf{s}|E_z|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_y|\mathbf{s}\rangle\langle\mathbf{b}|\rho_{yx}^{e}(t)|\mathbf{b}\rangle \exp[-\tau\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}], \tag{28}$$

where $\rho_{yx}^{e}(t)$ is defined by Eq. (26).

The previous calculations steps allows to obtain the joint probability for the set of three measurement outcomes. In fact, from Bayes rule it follows that $P(z,y,x) = P(z|y,x)P(y|x)P(x)$. Using Eqs. (28) and (24) we get

$$\begin{aligned} P(z,y,x) \;=\; \sum_{\mathbf{b}} \Big\{ & \Big[\sum_{\mathbf{s},\tilde{\mathbf{s}}} \langle\mathbf{s}|E_z|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_y|\mathbf{s}\rangle \exp(-\tau\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}) \Big] \\ & \times \Big[\sum_{s,\tilde{s}} \langle\mathbf{s}|E_y|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_x|\mathbf{s}\rangle \exp(-t\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}) \Big] \\ & \times \langle\mathbf{b}|\rho_0^e|\mathbf{b}\rangle \Big\} P(x). \end{aligned} \tag{29}$$

This final result provides an explicit expression for $P(z,y,x)$. It only depends on the chosen measurement processes, the initial conditions, and the characteristic dephasing rates $\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}$ [defined by Eq. (21)].

B. Markovian case

In the operational approach, the dynamics is memoryless if the outcomes joint probability fulfill the Markov property: $P(z,y,x) = P(z|y)P(y|x)P(x)$. This equality must be valid for *arbitrary measurement processes*. In general, the expression (29) does not fulfill this condition, which implies a non-Markovian system dynamics. On the other hand, it is simple to realize that under the condition

$$\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}} = \Phi_{\tilde{\mathbf{s}},\mathbf{s}}, \tag{30}$$

the (operational) Markov property is fulfilled for any election of the measurement processes. In fact, using that $\sum_{\mathbf{b}}\langle\mathbf{b}|\rho_0^e|\mathbf{b}\rangle = 1$, from Eq. (29) we get

$$\begin{aligned} P(z,y,x) \;=\; & \Big[\sum_{\mathbf{s},\tilde{\mathbf{s}}} \langle\mathbf{s}|E_z|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_y|\mathbf{s}\rangle \exp(-\tau\Phi_{\tilde{\mathbf{s}},\mathbf{s}}) \Big] \\ & \times \Big[\sum_{\mathbf{s},\tilde{\mathbf{s}}} \langle\mathbf{s}|E_y|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_x|\mathbf{s}\rangle \exp(-t\Phi_{\tilde{\mathbf{s}},\mathbf{s}}) \Big] P(x). \end{aligned} \tag{31}$$

The first and second sum contributions can be read as $P(z|y)$ and $P(y|x)$ respectively, which implies the validity of the Markov property $P(z,y,x) = P(z|y)P(y|x)P(x)$.

We remark that in the operational approach, Eq. (30) is a *necessary and sufficient condition for Markovianity.* That is, in contrast to the non-operational approach, here the inequality $\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}} \neq \Phi_{\tilde{\mathbf{s}},\mathbf{s}}$ [Eq. (20)] guarantees the presence of memory effects. Taking into account Eq. (21), *a non-vanishing Hamiltonian term* $(h_{ij}+h_{ji})/2$ *or any non-vanishing dissipative imaginary coupling* $(\Gamma_{ij}-\Gamma_{ji})/2$ *guarantee the presence of detectable memory effects.* On the other hand, it is possible that condition (30) is fulfilled but $\Phi_{\tilde{\mathbf{s}}\tilde{\mathbf{b}},\mathbf{s}\mathbf{b}} \neq \Phi_{\tilde{\mathbf{s}},\mathbf{s}}$ ($\tilde{\mathbf{b}} \neq \mathbf{b}$). As before, this case emerges when the non-diagonal rate coefficients are real. In fact, system-environment correlations induced by the real part $(\Gamma_{ij}+\Gamma_{ji})/2$ do not lead to departure from (operational or non-operational) Markovianity.

C. Statistical mixture representation

During the dynamics, system and environment are intrinsically coupled by their mutual interaction and transient quantum entanglement can be produced [47]. Consistently, the environment state and dynamics depend on the system degrees of freedom (see Appendix B). In particular, between the successive measurement processes the environment state is actively modified.

In spite of the previous properties, we notice that the same outcome probability [Eq. (29)] can be obtained from an alternative underlying dynamics. In fact, the expression for $P(z,y,x)$ can be read as a *statistical mixture (random superposition) of different system dephasing Markovian dynamics* [compare with Eq. (31)], each one with dephasing rates $\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}}$, where the statistical weight of each one is given by the population $\langle\mathbf{b}|\rho_0^e|\mathbf{b}\rangle$. Thus, one can obtain the same joint statistics by considering an "environment" whose participation in the developing of memory effects is completely passive, which in turn does not involve any system-environment entanglement. The same affirmation is valid for the system state $\rho_t^{(s)}$ [see Eqs. (15) and (16)].

The reading of the system dynamics in terms of a statistical mixture of Markovian dynamics can be seen as a non-unitary extension of the Hamiltonian ensemble introduced in Ref. [31]. Interestingly, this kind of equivalence can be detected through the measurement scheme. Considering the results of Ref. [29], a *random selection* of the system state after the second measurement should render the statistics Markovian. Explicitly, in Eq. (29), the following two changes are introduced

$$\rho_y \to \rho_{\breve{y}}, \qquad\qquad E_y \to \sum_y E_y = \mathrm{I}_s. \tag{32}$$

The first change implies that after the second measurement, the post-measurement state is randomly chosen $y \to \breve{y}$ over the set $\{\rho_y\}$. This change (performed for example with an unitary transformation) is chosen with an arbitrary conditional probability $\wp(\breve{y}|x)$. As a consequence, the original y-outcome is disregarded, property that lead to the corresponding addition $\sum_y$. Introducing in successive order the changes (32) into Eq. (29) it follows

$$P(z,\breve{y},x) \overset{r}{=} \Big[\sum_{\mathbf{b},\mathbf{s},\tilde{\mathbf{s}}} \langle \mathbf{s}|E_z|\tilde{\mathbf{s}}\rangle\langle\tilde{\mathbf{s}}|\rho_{\breve{y}}|\mathbf{s}\rangle \exp(-\tau\Phi_{\tilde{\mathbf{s}}\mathbf{b},\mathbf{s}\mathbf{b}})\langle \mathbf{b}|\rho_0^e|\mathbf{b}\rangle\Big] \times\wp(\breve{y}|x)P(x), \tag{33}$$

where the symbol $\overset{r}{=}$ implies that this equality is only valid under the steps (32). As expected, this final expression has the structure $P(z,\breve{y},x) \overset{r}{=} P(z|\breve{y})\wp(\breve{y}|x)P(x)$, that is, independently of the measurement process and chosen probability $\wp(\breve{y}|x)$, a Markov property is induced. In general, this Markovian property is not fulfilled. When it applies, it provides an experimental technique [29] for detecting when an environment can be replaced by a passive, or in general, by a casual bystander one [28]. This feature in turn can be read as the absence of any physical environment-to-system backflow of information.

IV. EXAMPLES

In this section we apply the previous general theoretical approach to some specific examples. The properties of memory effects are discussed in detail.

A. Bipartite arrangement

First we consider a bipartite arrangement [Eq. (1) with $n=2$]. Therefore, both the system and the environment consist in one single system. For clarity, their density matrix evolution is explicitly written as

$$\frac{d\rho_t^{se}}{dt} - i[H,\rho_t^{se}] + \mathcal{L}[\rho_t^{se}], \tag{34}$$

where the bipartite Hamiltonian is

$$H = \Omega(S\otimes B), \tag{35}$$

while the non-diagonal Lindblad contribution $\mathcal{L}[\rho_t^{se}]$ is defined by

$$\begin{aligned}\mathcal{L}[\rho_t^{se}] \;=\; &+\gamma\Big(S\rho_t^{se}S - \frac{1}{2}\{S^2,\rho_t^{se}\}_+\Big)\\ &+\beta\Big(B\rho_t^{se}B - \frac{1}{2}\{B^2,\rho_t^{se}\}_+\Big)\\ &+\Big[\chi\Big(S\rho_t^{se}B - \frac{1}{2}\{SB,\rho_t^{se}\}_+\Big) + h.c.\Big].\end{aligned} \tag{36}$$

In these expressions S and B are arbitrary Hermitian operators acting in the system and bath Hilbert spaces respectively. The frequency Ω measure the strength of the unitary system-environment interaction. On the other hand, the Hermitian matrix

$$\{\Gamma_{ij}\} = \begin{pmatrix} \gamma & \chi \\ \chi^* & \beta \end{pmatrix}, \tag{37}$$

sets the dissipative system-environment interaction. Given that $\{\Gamma_{ij}\}$ must be a positive definite matrix, it follows the constraints $\gamma\geq 0$, $\beta\geq 0$, and $\gamma\beta - |\chi|^2\geq 0$.

Introducing the eigenvectors and eigenvalues $S|s\rangle = s|s\rangle$, $B|b\rangle = b|b\rangle$, the dephasing rates (5) under the splitting (9) can be written as [Eq. (11)]

$$\begin{aligned}\Phi_{\tilde{s}\tilde{b},sb} \;=\; & i\Omega(\tilde{s}\tilde{b}-sb) - i\chi_I(\tilde{s}b - s\tilde{b}) + \frac{\gamma}{2}(\tilde{s}-s)^2\\ &+\frac{\beta}{2}(\tilde{b}-b)^2 + \chi_R(\tilde{s}-s)(\tilde{b}-b),\end{aligned} \tag{38}$$

where the real and imaginary parts of the non-diagonal coupling rate χ are denoted as $\chi_R = \mathrm{Re}[\chi]$ and $\chi_I = \mathrm{Im}[\chi]$ respectively.

For the emerging of system memory effects we have to consider the (partial) diagonal contribution $\tilde{b}=b$ [see Eqs. (15) and (16)], which leads to

$$\Phi_{\tilde{s}b,sb} = -i(\chi_I - \Omega)b(\tilde{s}-s) + \frac{\gamma}{2}(\tilde{s}-s)^2. \tag{39}$$

Consequently the parameters β and χ_R does not participate in the developing of memory effects. This result is consistent with the general expression (21). On the other hand, from the point of view of the system dynamics the parameters Ω and χ_I play exactly the same role. In fact, the sign of both parameters is arbitrary. Nevertheless, notice that the underlying coupling processes associated to these two constants, and the system-environment correlations induced by each one, are different in general.

Two qubits

As an specific example we consider that both subsystems are qubits. For simplicity both operators S and B are taken as a z-Pauli matrix (σ_z) in the corresponding Hilbert spaces. Thus, $s=\pm1$ and $b=\pm1$. Using the partial transpose criteria [48] in Eq. (34), it follows that system-environment entanglement can only be induced by the Hamiltonian H [Eq. (35)]. Complementarily, the dissipative non-diagonal coupling is unable to generate entanglement in this case.

The system density matrix [Eq. (15)] reads

$$\rho_t^{(s)} = \begin{pmatrix} p_+ & c_0 f(t) \\ c_0^* f^*(t) & p_- \end{pmatrix}, \tag{40}$$

where $p_\pm \equiv \langle\pm|\rho_0^{(s)}|\pm\rangle$ and $c_0\equiv\langle +|\rho_0^{(s)}|-\rangle$ are respectively the initial populations and coherence of the system in the eigenbase of σ_z. The behavior of the coherences [Eq. (16)] is given by

$$f(t) = e^{-2t\gamma}(q_+ e^{+it2\underline{\chi}_I} + q_- e^{-it2\underline{\chi}_I}), \tag{41}$$

where $q_\pm\equiv\langle\pm|\rho_0^{(e)}|\pm\rangle$ are the initial populations of the environment. In this expression and the following ones, for shortening the expression we introduced the parameter $\underline{\chi}_I \equiv (\chi_I - \Omega)$.

a. Non-operational approach to memory effects From Eq. (17), and consistently with the solution (40), the system density matrix time-evolution can be cast in the form

$$\frac{d\rho_t^{(s)}}{dt} = -i\frac{\omega(t)}{2}[\sigma_z, \rho_t^{(s)}] + \gamma(t)(\sigma_z\rho_t^{(s)}\sigma_z - \rho_t^{(s)}). \quad (42)$$

Using the procedure defined by Eq. (19), the time-dependent frequency is

$$\omega(t) = -\frac{2\underline{\chi}_I(q_+ - q_-)}{q_+^2 + q_-^2 + 2(q_+q_-)\cos(4\underline{\chi}_I t)}, \quad (43)$$

while the time-dependent rate is

$$\gamma(t) = \gamma + \frac{2\underline{\chi}_I(q_+q_-)\sin(4\underline{\chi}_I t)}{q_+^2 + q_-^2 + 2(q_+q_-)\cos(4\underline{\chi}_I t)}. \quad (44)$$

Consistently, both $\omega(t)$ and $\gamma(t)$ only depends on the characteristic rates γ and $\underline{\chi}_I$. On the other hand, the environment populations $\{q_\pm\}$ also govern the emergence of memory effects. In fact, when $q_\pm = 1$, $q_\mp = 0$, it follows $\omega(t) = \mp 2\underline{\chi}_I$, and $\gamma(t) = \gamma$. Hence, the system dynamics is Markovian.

In general, the rate $\gamma(t)$ may assume both positive an negative values, which can be used as a witness of memory effects [13]. In Fig. 1(a) and (b) we plot both the frequency $\omega(t)$ and the rate $\gamma(t)$ for two different values of the (scaled) non-diagonal coupling $\underline{\chi}_I/\gamma$. Depending on its value, a transition from Markovian $[\gamma(t) \geq 0]$ to non-Markovian dynamics $[\gamma(t) \gtrless 0]$ is clearly observed.

b. Operational approach to memory effects For implementing the operational approach, we assume that the three measurements are *projective* ones. They are performed successively in the Bloch directions $\hat{x} - \hat{n} - \hat{x}$, where $\hat{n} = (\cos(\phi), \sin(\phi), 0)$ is an arbitrary direction in $\hat{x}$-$\hat{y}$ plane defined by the angle ϕ. The successive measurement outcomes are $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$. Using the corresponding measurement projectors $\{\Pi_m = |m\rangle\langle m|\}$ associated to each direction [49], from the general expression (29), using Eq. (39), we get

$$\frac{P(z,y,x)}{P(x)} = \frac{1}{4}[1 + yxf_\phi^{(+)}(t) + zyf_\phi^{(-)}(\tau) + zxf_\phi(t,\tau)]. \quad (45)$$

Here, the auxiliary functions are

$$f_\phi^{(\pm)}(t) = e^{-2t\gamma}\{q_+\cos[2t\underline{\chi}_I(\pm)\phi] + q_-\cos[2t\underline{\chi}_I(\mp)\phi]\},$$

while the last one is

$$\begin{aligned} f_\phi(t,\tau) = e^{-2\gamma(t+\tau)}[&q_+\cos(2t\underline{\chi}_I + \phi)\cos(2\tau\underline{\chi}_I - \phi) \\ &+ q_-\cos(2t\underline{\chi}_I - \phi)\cos(2\tau\underline{\chi}_I + \phi)]. \end{aligned}$$

Taking $\underline{\chi}_I = 0$ in Eq. (45), it is simple to show that $P(z,y,x)$ fulfill a Markov property. A simple way of *witnessing* departures of $P(z,y,x)$ from Markovianity is through a conditional past-future correlation [15]. It is defined as

$$C_{pf}(t,\tau)|_y = \sum_{z,x} zx[P(z,x|y) - P(z|y)P(x|y)]. \quad (46)$$

Here, $\{z\}$ and $\{x\}$ represent the possible outcomes in the last (future) and first (past) measurement processes respectively, while the conditional y is an arbitrary outcome of the intermediate (present) measurement. From Bayes rule, the Markov property $P(z,y,x) = P(z|y)P(y|x)P(x)$ can be rephrased as a conditional past-future independence, $P(z,x|y) = P(z|y)P(x|y)$, which lead to $C_{pf}(t,\tau)|_y = 0$. Hence, the condition $C_{pf}(t,\tau)|_y \neq 0$ implies the presence of memory effects.

Using that $P(z,x|y) = P(z,y,x)/P(y)$, where $P(y) = \sum_{z,x} P(z,y,x)$, from Eq. (45) the correlation (46) reads

$$\begin{aligned} C_{pf}(t,\tau)|_y = &-(4q_+q_-)\sin^2(\phi)e^{-2\gamma(t+\tau)} \\ &\times \sin(2t\underline{\chi}_I)\sin[2\tau\underline{\chi}_I]. \end{aligned} \quad (47)$$

For simplicity, this result was derived by assuming system initial conditions such that $P(x) = 1/2$. In Fig. 1(c) and (d) we plot $C_{pf}(t,\tau)|_y$ at equal measurement time-intervals, $\tau = t$. The non-diagonal coupling is in correspondence with Fig. 1(a) and (b) respectively. Consistently with previous general results, for any non-vanishing value of $\underline{\chi}_I/\gamma \neq 0$, in contrast to the negative rate criteria, here the dynamics is non-Markovian, $C_{pf}(t,\tau)|_y \neq 0$.

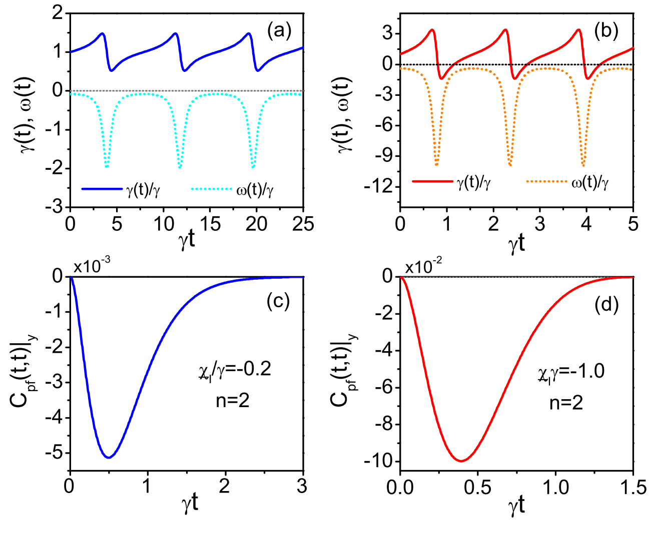

FIG. 1: Time dependence of the frequency $\omega(t)$ and rate $\gamma(t)$, Eqs. (43) and (44) respectively, jointly with the correlation $C_{pf}(t,\tau)|_y$ [Eq. (47)] at equal times, for a system coupled to a single qubit environment, $n = 2$. In (a) and (c) the non-diagonal parameter is $\underline{\chi}_I/\gamma = -0.2$, while in (b) and (d) it is $\underline{\chi}_I/\gamma = -1.0$. In all cases the environment populations are taken as $q_+ = 0.4$, $q_- = 0.6$, while the angle of the intermediate measurement is $\phi = \pi/2$.

B. Multipartite environment

Now we consider a multipartite dynamics [Eq. (1) with $n > 2$]. As in the previous example all subsystems are taken as qubits. The "first qubit" is the taken as the system ($n_S = 1$) and consequently the rest are part of the environment ($n_B = n-1$). Similarly, all coupling operators are taken as the z-Pauli matrix (σ_z) in the corresponding Hilbert spaces. The matrix of rate coefficients $\{\Gamma_{jk}\}$ is taken as

$$\{\Gamma_{jk}\} = \{(\gamma - \chi)\delta_{jk}\} + \chi|f_\lambda\rangle\langle f_\lambda|, \tag{48}$$

where γ and χ are two real parameters. Furthermore, δ_{jk} is the Kronecker delta function. The complex vector is $|f_\lambda\rangle \equiv \sum_{j=1}^{n} e^{2\pi i(j-1)\lambda/n}|\mathbf{e}_j\rangle$, where $\{|\mathbf{e}_j\rangle\}_{j=1}^{n}$ is the standard basis of a vectorial space of dimension n, while λ is an arbitrary dimensionless real parameter. It is simple to check that γ and χ scale the diagonal and non-diagonal elements of $\{\Gamma_{jk}\}$ respectively. The structure of $\{\Gamma_{jk}\}$ introduced in Ref. [47] is recovered when $\chi = \gamma$.

While the developed results allows to characterizing the dynamics in an exact analytical way, simple expressions are only obtained for special values of the free parameter λ. From now on we take $\lambda = n/4$. Hence, Eq. (48) becomes $\Gamma_{jk} = (\gamma - \chi)\delta_{jk} + \chi\ (i)^{j-1}(-i)^{k-1}$, that is,

$$\{\Gamma_{jk}\} = \begin{pmatrix} \gamma & -i\chi & -\chi & +i\chi & +\chi \\ +i\chi & \gamma & -i\chi & -\chi & \ddots \\ -\chi & +i\chi & \gamma & -i\chi & \ddots \\ -i\chi & -\chi & +i\chi & \gamma & \ddots \\ +\chi & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{49}$$

The diagonal elements are equal to γ while the non-diagonal couplings alternatively change between imaginary ($\pm i\chi$) and real ($\pm\chi$) values. The positive definite character of $\{\Gamma_{jk}\}$ [Eq. (49)], which guarantees that the full evolution is a completely positive one, implies that $\gamma > 0$ and the inequalities

$$-\frac{\gamma}{(n-1)} \leq \chi \leq \gamma, \tag{50}$$

where $n \geq 2$. In addition, in Eq. (1) we assume $H = 0$.

1. Entanglement generation

The generation of entanglement is dephasing dynamics has been characterized in unitary dynamics [50]. For the multipartite non-diagonal dissipative dynamics [Eq. (1)] the corresponding analysis has been presented previously [47]. The basic procedure is to calculate the matrixes $(\tilde{h}_{ij}, \tilde{\Gamma}_{ij})$, which define the evolution of the total density matrix after transposing the environment degrees of freedom, $\rho_t \to \rho_t^{\mathsf{T}}$ (see Eq. (4) in Ref. [47]). Using

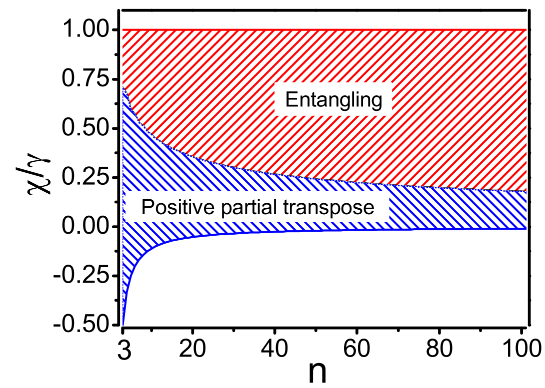

FIG. 2: Parameter values χ/γ as a function of the arrangement size n that lead to system-environment entanglement. The matrix of rate coefficients is given by Eq. (49). The upper and lower boundaries of χ/γ are defined by Eq. (50). The frontier between the entangling (negative partial transpose) and the positive partial transpose is determine numerically (see text).

the partial transpose criterion [48], it is possible to conclude that when $\{\tilde{\Gamma}_{ij}\}$ has negative eigenvalues the dynamics generates transient entanglement. When $\{\tilde{\Gamma}_{ij}\}$ has positive eigenvalues, the partial transpose state ρ_t^{T} is positive definite and entanglement generation is not granted [47, 48].

By determining $\{\tilde{\Gamma}_{ij}\}$ from Eq. (49), and by calculating its eigenvalues for each n (total number of qubits) it is possible to determinate (numerically) the minimal value of the parameter χ/γ that guarantees entanglement generation. In Fig. 2, we plot the regions where entanglement generation is granted and where complementarily ρ_t^{T} is positive definite. Only for $n \geq 3$ there is entanglement generation. Furthermore, positive values of χ/γ are necessary, which in turn decreases with n. Both regions are limited by the constraints defined by Eq. (50). Beyond these frontiers, the dynamics must be implemented with Hamiltonians contributions.

2. System memory effects

Independently of the value of the parameter χ/γ, the system state $\rho_t^{(s)}$ and its evolution can be written as in Eqs. (40) and (42) respectively. For simplicity, we assume that all subsystems of the environment begin in an (multipartite) uncorrelated state, each subsystem having equal upper and lower populations. Thus, $\langle\mathbf{b}|\rho_0^{(e)}|\mathbf{b}\rangle = (1/2)^{n-1}$. The coherence behavior $f(t)$ from Eq. (16), after some algebra, is given by

$$f(t) = e^{-2\gamma t}[\cos(2\chi t)]^{\bar{n}}, \tag{51}$$

where $\bar{n} \equiv \mathrm{Int}[n/2]$ is the integer part of $(n/2)$. The dependence on $\bar{n}$ emerges because the non-diagonal elements of $\{\Gamma_{jk}\}$ alternate between real and imaginary values [see Eq. (49)]. The time-evolution of $\rho_t^{(s)}$ [Eq. (42)] is defined with

$$\omega(t) = 0, \qquad \gamma(t) = \gamma + \bar{n}\chi \tan(2\chi t)]. \tag{52}$$

The absence of a Hamiltonian contribution [$\omega(t) = 0$] follows from the equality of the upper and lower populations of each subsystem associated to the environment. From $\gamma(t)$, we deduce that in *non-operational approach* to memory effects, the system dynamics is non-Markovian whenever $\chi \neq 0$. Interestingly, this kind of "trigonometric eternal non-Markovianity" with periodic divergences was also found in a different kind of underlying multipartite dynamics [44].

For the *operational approach* we choose the same set of measurements than in the previous bipartite case, $\hat{x}-\hat{n}-\hat{x}$, where the intermediate one is defined by the angle ϕ. The joint probability of measurement outcomes $P(z, y, x)$ can be written with the structure (45). From Eq. (29) it follows

$$f_\phi^{(\pm)}(t) = f_\phi(t) = f(t)\cos(\phi), \tag{53}$$

where $f(t)$ is given by Eq. (51), while

$$f_\phi(t,\tau) = \frac{1}{2} e^{-2\gamma(t+\tau)}\{\cos^{\bar{n}}[2\chi(t+\tau)] + \cos(2\phi)\cos^{\bar{n}}[2\chi(t-\tau)]\}. \tag{54}$$

Consequently, it is simple to check that $P(z, y, x)$ fulfill the Markov condition only when $\chi = 0$. This property is corroborated by the conditional past-future correlation [Eq. (46)], which here can be written as

$$C_{pf}(t,\tau)|_y = f_\phi(t,\tau) - f_\phi(t) f_\phi(\tau). \tag{55}$$

As before, this result was derived by assuming system initial conditions such that $P(x) = 1/2$.

In Fig. 3(a) and (b) we plot the coherence decay (51) and the conditional past-future correlation (55) with $n = 6$ and taking different values of χ/γ. Consistently with their analytical expressions, the developing of entanglement (see Fig. 2) does not lead to any significant change in these two objects. This independence follows from the previous general analysis. In fact, both the system dynamics [Eqs. (15) and (16)] and the outcome statistics [Eq. (29)] can equivalently be obtained from a random superposition of Markovian dephasing dynamics without involving any multipartite entanglement.

In Fig. 3(c) and (d) we plot the coherence decay (51) and the conditional past-future correlation (55) for different number n of qubits. Given that $\chi/\gamma = 1$, transient entanglement is granted in all cases. When increasing n the decoherence function $f(t)$ decay in a faster way and in addition $C_{pf}(t,\tau)|_y$ assume higher values, which can consistently be read as an increasing of system memory effects.

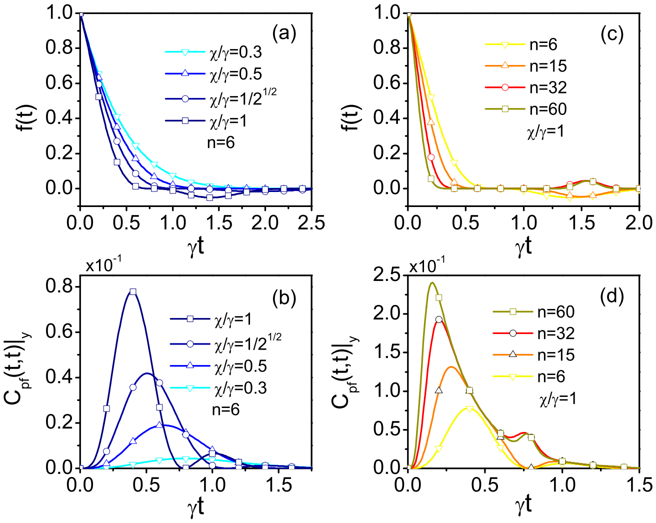

FIG. 3: Coherence decay $f(t)$ [Eq. (51)] and correlation $C_{pf}(t,\tau)|_y$ at equal times [Eq. (55)] for a system coupled to a multipartite environment. The non-diagonal coupling rate χ/γ and the total number of qubits n is indicated in each plot. In all cases, the qubits of the environment begin with equal upper and lower populations. The angle of the intermediate measurement is $\phi = 0$.

3. *Infinite bath size*

The system dynamics can also be characterized in the limit in which the number of subsystems of the environment become infinite. Nevertheless, for getting a smooth system coherence decay [Eq. (51)], the non-diagonal dissipative coupling χ in Eq. (49) must be scaled with the arrangement size n. We assume

$$\chi \longrightarrow \chi_n = g\sqrt{\frac{2}{n}}, \tag{56}$$

where g is an arbitrary scaling constant. It is simple to proof that

$$\lim_{n\to\infty} [\cos(2\chi_n t)]^{\bar{n}} = e^{-2g^2t^2}. \tag{57}$$

Therefore, for increasing n, the system coherence decay can be fit as

$$\lim_{n\to\infty} f(t) = e^{-2\gamma t}\, e^{-2g^2t^2}. \tag{58}$$

While the diagonal contribution lead to an exponential decay with rate γ, the non-diagonal coupling lead to a Gaussian decay behavior. The time-dependent rate [Eq. (52)] becomes $\lim_{n\to\infty} \gamma(t) = \gamma + 2g^2 t$. Remarkably, a similar Gaussian behaviors can also be obtained from unitary system-environment dynamics [15].

V. SUMMARY AND CONCLUSIONS

We studied the emergence and properties of memory effects in a class of multipartite arrangements where all subsystems are coupled to each other by non-diagonal Lindblad dephasing mechanisms [Eq. (1)]. By choosing an appropriate basis for the total Hilbert space, the multipartite density matrix was obtained in an exact analytical way [Eq. (8)]. An arbitrary number of subsystems are associated to the system of interest, while the rest define its environment. This splitting [Eq. (9)] provided the basis for characterizing in an exact way both non-operational and operational approaches to quantum non-Markovianity.

In non-operational approaches to quantum non-Markovianity, memory effects are determined from the properties of the system density matrix evolution. We showed that its general structure can be written as a non-diagonal time-dependent dephasing evolution [Eq. (17)]. Its characteristic parameters are set by the corresponding system coherence behaviors [Eq. (19)]. A necessary condition for the emergence of memory effects can be cast in terms of the multipartite dephasing rates [Eq. (20)]. Explicitly, memory effects may be induced by Hamiltonian couplings or when the dissipative coupling breaks a time-reversal symmetry, that is, the non-diagonal coupling rates must be complex ones. In these dynamics, these conditions are also necessary for the development of transient entanglement [47].

In operational approaches to quantum non-Markovianity, memory effects are determine from a set of measurement processes performed over the system of interest. We calculated in an explicit analytical way the joint probability of measurement outcomes [Eq. (29)]. In this case the previous conditions for the emergence of memory effects become sufficient, that is, any non-vanishing unitary or dissipative coupling consistent with the break of time-reversal symmetry lead to departures from Markovianity.

While the multipartite dynamics lead to entanglement generation, we concluded that this feature is not relevant when considering the properties of system memory effects. In fact, both the density matrix dynamics and the statistics of measurement outcomes [Eqs. (15) and (29)] can alternatively be obtained from a statistical mixture of Markovian dephasing evolutions. This equivalent representation does not involve any entanglement. In addition, in the operational approach, this property imply that memory effects can be obtained without the occurrence of any physical environment-to-system backflow of information.

As examples we studied bipartite and multipartite dynamics [with coupling rates given by Eqs. (37) and (49)], where each subsystem is a qubit. The properties of the corresponding memory effects support the previous main results (Figs. 1 to 3).

Understanding the role of system-environment correlations in the developing of memory effects is a central problem in open quantum system theory. The present analysis shed light on possible memory features that can emerge in systems embedded in multipartite dissipative arrangements. Their validity can in principle be checked in optical setups where this kind of dynamics can be implemented [47].

Acknowledgments

The author thanks to Mariano Bonifacio for a critical reading of the manuscript. This paper was supported by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

Appendix A: Generalization to multipartite couplings

In the evolution defined by Eq. (1) the coupling between the subsystems are bipartite ones, that is, it only involves the action of two operators: $S^{(i)}$ and $S^{(j)}$. Multipartite coupling mechanisms can also be considered, where more than two subsystems are involved. In this situation, the density matrix can be written as

$$\frac{d\rho_t}{dt} = -i[\underline{\mathrm{H}}, \rho_t] + \sum_{\mu,\nu} \Gamma_{\mu,\nu}(S_\mu \rho_t S_\nu - \frac{1}{2}\{S_\nu S_\mu, \rho_t\}_+), \tag{A1}$$

where the indexes are $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$. The operators $\{S_\mu\}$ are defined by the product

$$S_\mu = S^{(1)}_{\mu_1} \otimes \cdots \otimes S^{(n)}_{\mu_N}, \tag{A2}$$

where each operator $S^{(i)}_{\mu_i}$ [defined in $\mathcal{H}_i$] depend on the subindex μ_i. It is defined as

$$S^{(i)}_{\mu_i} \equiv \begin{cases} S^{(i)} & if \quad \mu_i = 1 \\ \mathrm{I}^{(i)} & if \quad \mu_i = 0 \end{cases}, \tag{A3}$$

where $\mathrm{I}^{(i)}$ is the identity operator in the Hilbert space $\mathcal{H}_i$. Thus, it is simple to realize that, in contrast to Eq. (1), arbitrary multipartite coupling mechanisms are associated to the coupling rates $\Gamma_{\mu,\nu}$. Similarly, the Hamiltonian is taken as $\underline{\mathrm{H}}= (1/2)\sum_\mu h_\mu S_\mu$, where h_μ are real coefficients.

In this general situation, by writing $S^{(i)}_{\mu_i} = \mu_i S^{(i)} + (1-\mu_i)\mathrm{I}^{(i)}$ it is simple to check that $S_\mu|\mathbf{s}\rangle = \lambda^\mu_{\mathbf{s}}|\mathbf{s}\rangle$, where the eigenvalue is given by $\lambda^\mu_{\mathbf{s}} = \prod_{i=1}^n[\mu_i s_i + (1-\mu_i)]$ and $|\mathbf{s}\rangle$ is defined by Eq. (3). After similar calculations steps, the density matrix can also be written as in Eq. (8). Here, the frequencies are defined by

$$\omega_{\mathbf{s}} = \frac{1}{2}\sum_\mu \lambda^\mu_{\mathbf{s}} h_\mu, \tag{A4}$$

while the multipartite dissipative couplings lead to

$$\Upsilon_{\tilde{\mathbf{s}},\mathbf{s}} = \sum_{\mu,\nu} \Gamma_{\mu,\nu}(\lambda^\mu_{\tilde{\mathbf{s}}}\lambda^\nu_{\mathbf{s}} - \frac{1}{2}\lambda^\mu_{\tilde{\mathbf{s}}}\lambda^\nu_{\tilde{\mathbf{s}}} - \frac{1}{2}\lambda^\mu_{\mathbf{s}}\lambda^\nu_{\mathbf{s}}). \tag{A5}$$

By adding and subtracting appropriates terms, this result can be cast with the same structure than as Eq. (7).

Appendix B: Environment dynamics

A relevant aspect when characterizing memory effects is the environment dynamics. The system dynamics depends on the environment degrees of freedom [Eqs. (15) and (16)]. Given that the system-environment splitting is arbitrary, a similar property must be valid for the environment. Specifically, during the dynamics the environment depends on the system degrees of freedom. In fact, from Eq. (10) it follows

$$\rho_t^{(e)} = \mathrm{Tr}_s[\rho_t^{se}] = \sum_{\mathbf{b},\tilde{\mathbf{b}}} F_{\tilde{\mathbf{b}}\mathbf{b}}(t)|\tilde{\mathbf{b}}\rangle\langle\tilde{\mathbf{b}}|\rho_0^{(e)}|\mathbf{b}\rangle\langle\mathbf{b}|, \qquad \text{(B1)}$$

where we have introduced the functions

$$F_{\tilde{\mathbf{b}}\mathbf{b}}(t) = \sum_{\mathbf{s}} \langle\mathbf{s}|\rho_0^{(s)}|\mathbf{s}\rangle \exp[-t\Phi_{\mathbf{s}\tilde{\mathbf{b}},\mathbf{s}\mathbf{b}}]. \qquad \text{(B2)}$$

Similarly to Eq. (16), here the behavior of the environment coherences $\{F_{\tilde{\mathbf{b}}\mathbf{b}}(t)\}$ is time-dependent and depend on the system degrees of freedom. Thus, independently of the specific system-environment splitting [Eq. (9)] the environment is not a casual bystander one [28], that is, it dynamically participates in the generation and developing of system memory effects. Only when the initial environment state $\rho_0^{(e)}$ is diagonal in the basis $\{|\mathbf{b}\rangle\}$, using that $\Phi_{\mathbf{sb},\mathbf{sb}} = 0$, the bath dynamics become independent of the system, $\rho_t^{(e)} = \rho_0^{(e)}$.

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