ON BASES OF CERTAIN GROTHENDIECK GROUPS

G. Lusztig

Introduction

0.1. Let $\mathcal{S}$ be the set of (isomorphism classes of irreducible) unipotent representations of

(a) a symplectic or odd special orthogonal group over a finite field, or

(b) an even split full orthogonal group over a finite field, or

(c) an even non-split full orthogonal group over a finite field.

(In cases (b),(c) we say that an irreducible representation is unipotent if its restriction to the corresponding special orthogonal group is unipotent; we further assume that this restriction is irreducible.)

Let W be the corresponding Weyl group and let $Ce(W)$ be the set of two-sided cells of W, which in cases (b),(c) are stable under the graph automorphism of W induced by an element in the full orthogonal group which is not in the special orthogonal group.

From [L84] (or a slight extension) one deduces a natural partition

$\mathcal{S} = \sqcup_{c\in Ce(W)}\mathcal{S}_c$.

Thus, the Grothendieck group of $\mathcal{S}$ is a direct sum $\oplus_{c\in Ce(W)}\mathbf{Z}[\mathcal{S}_c]$ where for a finite set Y, $\mathbf{Z}[Y]$ denotes the free abelian group with basis Y.

We now fix $c \in Ce(W)$. In [L20, L22] a new basis of $\mathbf{Z}[\mathcal{S}_c]$ with strong positivity properties with respect to Fourier transform was defined in case (a). Here we shall call it the "second basis". In case (b) there were two versions of such a basis in [L20] and the second one was adopted in [L22] under the name "second basis".

In this paper we give a somewhat different presentation and refinements of the results of [L20, L22] and extend them to include case (c).

From [L84] (or a slight extension), to c one can attach a pair of finite subsets $U' \subset U$ of $\mathbf{N}$ with $|U-U'|$ odd in case (a) and even in cases (b),(c), so that $\mathcal{S}_c$ is identified with the set $Sy(U',U)$ of "symbols", that is ordered pairs (S,T) where S,T are finite subsets of U such that $S\cup T = U, S\cap T = U', 0 \notin U'$ and

(d) $|S|-|T|$ is in $2\mathbf{Z}+1$ in case (a), in $4\mathbf{Z}$ in case (b), in $4\mathbf{Z}+2$ in case (c).

(The cardinal of a finite set Y is denoted by $|Y|$.) We can identify $Sy(U',U)$ with $Sy(\emptyset,[1,D+1])$ where $D+1 = |U-U'|$. (For i,j in $\mathbf{Z}$ we set

Supported by NSF grant DMS-2153741

$[i,j]=\{z\in\mathbf{Z}, i\le z\le j\}$.)
Namely, to $(S,T)\in Sy(U',U)$ we associate $(S',T')\in Sy(\emptyset,[1,D+1])$ where S',T' are the images of $S-U',T-U'$ under the unique order preserving bijection $U-U'\to[1,D+1]$. In this way $\mathbf{Z}[\mathcal{S}_c]$ becomes $\mathbf{Z}[Sy(\emptyset,[1,D+1])]$ and our task becomes that of defining a second basis of $\mathbf{Z}[Sy(\emptyset,[1,D+1])]$.

We shall write Sy_D (resp. Sy^+_D, Sy^-_D) instead of $Sy(\emptyset,[1,D+1])$ in case (a) (resp. (b),(c)). Thus,

Sy_D is the set of ordered pairs (S,T) of disjoint subsets of $[1,D+1]$ such that $S\cup T=[1,D+1]$, $|S|-|T|\in 2\mathbf{Z}+1$, D even ≥ 0;

Sy^+_D is the set of ordered pairs (S,T) of disjoint subsets of $[1,D+1]$ such that $S\cup T=[1,D+1]$, $|S|-|T|\in 4\mathbf{Z}$, D odd ≥ 1;

Sy^-_D is the set of ordered pairs (S,T) of disjoint subsets of $[1,D+1]$ such that $S\cup T=[1,D+1]$, $|S|-|T|\in 4\mathbf{Z}+2$, D odd ≥ 1.

Under our identification, the partition of $\mathcal{S}_c$ according to Harish-Chandra series corresponds to the partition

$Sy_D=\sqcup_{s\in 2\mathbf{N}+1}Sy^s_D$, $Sy^s_D=\{(S,T)\in Sy_D; abs(|S|-|T|)=s\}$ in case (a);

$Sy^+_D=\sqcup_{s\in 4\mathbf{Z}}Sy^s_D$, $Sy^s_D=\{(S,T)\in Sy^+_D; |S|-|T|=s\}$ in case (b);

$Sy^-_D=\sqcup_{s\in 4\mathbf{Z}+2}Sy^s_D$, $Sy^s_D=\{(S,T)\in Sy^-_D; |S|-|T|=s\}$ in case (c).

Here, the absolute value of an integer z is denoted by $abs(z)$.

1. The second basis

1.1. Let F be the field consisting of two elements. Let $D\ge 0$ be an integer. We set $N=D+1$ if D is even, $N=D+2$ if D is odd. Thus N is odd. When $N\ge 3$, for any $k\in[1,D]$ we define an (injective) map $\iota_k:[1,N-2]\to[1,N]$ by $\iota_k(i)=i$ for $i\in[1,k-1]$, $\iota_k(i)=i+2$ for $i\in[k,N-2]$. Note that

$\iota_k(1)<\iota_k(2)<\dots<\iota_k(N-2)$ and $\iota_k(i)=i \mod 2$ for all $i\in[1,N-2]$.

Let $\tilde E_N$ be the set of subsets of $[1,N]$ viewed as an F-vector space in which the sum of A,B is $(A\cup B)-(A\cap B)$. Let $E_N=\{X\in\tilde E_N; |X|=0 \mod 2\}$, a codimension 1 subspace of $\tilde E_N$. Let ${}^2E_N$ be the set of all 2 element subsets of E_N. When $N\ge 3$, $k\in[1,D]$, we define an F-linear map $\tilde E_{N-2}\to\tilde E_N$ by $\{i\}\mapsto\{\iota_k(i)\}$ for all $i\in[1,N-2]$; this map is denoted again by ι_k. It restricts to an F-linear map $E_{N-2}\to E_N$.

A subset $\{i,j\}\in{}^2E_N$ will be often written as ij if

$i<j$ and $i-j=1 \mod 2$ (we then say that $ij\in{}^2E'_N$) or if

$i>j$ and $i-j=0 \mod 2$ (we then say that $ij\in{}^2E''_N$).

In this way, ${}^2E_N={}^2E'_N\sqcup{}^2E''_N$ is identified with a subset of $E_N\times E_N$. For $ij\in{}^2E_N$ we set

$\lfloor i,j\rfloor=[i,j]$ if $i<j$, $i-j=1 \mod 2$ and

$\lfloor i,j\rfloor=[i,N]\sqcup[1,j]$ if $i>j$, $i=j \mod 2$.

We have

$|\lfloor i,j\rfloor|=j-i+1$ if $i<j$, $|\lfloor i,j\rfloor|=N-i+1+j$ if $i>j$.

Thus, $|\lfloor i,j\rfloor|=0 \mod 2$ that is, $\lfloor i,j\rfloor\in E_N$.

When $N\ge 3$, $k\in[1,D]$ and $ij\in{}^2E_{N-2}$ we have, using the definitions, in E_N:

(a) $\lfloor \iota_k(i), \iota_k(j) \rfloor = \iota_k(\lfloor i, j \rfloor) + c\{k, k+1\}$ where $c \in \{0, 1\}$

1.2. Let $\mathcal{P}_N$ be the set of all unordered sequences $X_1, X_2, \dots, X_t$ in ${}^2E_N$ such that $X_a \cap X_b = \emptyset$ for any $a \neq b$. (We have necessarily $t \le (N-1)/2$.) For $B \in \mathcal{P}_N$ let $\text{supp}(B) = \cup_{ij \in B}\{i, j\} \subset [1, N]$ (this is a disjoint union) and $B^1 = B \cap {}^2E'_N$, $B^0 = B \cap {}^2E''_N$; if $B^0 \neq \emptyset$ we denote by i_B the largest number i such that $ij \in {}^2E''_N$ for some j.

We define a subset Pr_D of $\mathcal{P}_N$ as follows: if $D = 0$, Pr_D consists of $Q^0_D = \emptyset$. If D is even ≥ 2, Pr_D consists of $Q^0_D = \emptyset$ and of

$B = Q^t_D = \{\{D+1, 1\}, \{D, 2\}, \dots, \{D+2-t, t\}\}$ with $t \in [2, D/2]$, t even (we have $|B^0| = t$),

$B = Q^{-t}_D = \{\{D+1, 1\}, \{D, 2\}, \dots, \{D+3-t, t-1\}\}$ with $t \in [2, (D+2)/2]$, t even (we have $|B^0| = t-1$).

If D is odd, Pr_D consists of $Q^{0,+}_D = \emptyset$, $Q^{0,-}_D = \{D+1, D+2\}$ and of

$B = Q^{t,+}_D = \{\{D+1, 2\}, \{D, 3\}, \{D-1, 4\}, \dots, \{D+3-t, t\}\}$ with t even, $t \in [2, (D+1)/2]$ (we have $|B^0| = t-1$, $i_B \in 2\mathbf{N}$, $N \notin \text{supp}(B)$),

$B = Q^{-t,+}_D = \{\{D, 1\}, \{D-1, 2\}, \{D-2, 3\}, \dots, \{D+2-t, t-1\}\}$ with t even, $t \in [2, (D+1)/2]$ (we have $|B^0| = t-1$, $i_B \in 2\mathbf{N}+1$, $N \notin \text{supp}(B)$),

$B = Q^{-t,-}_D = \{\{D+2, 1\}, \{D+1, 2\}, \{D, 3\}, \dots, \{D+4-t, t-1\}\}$ with t even, $t \in [2, (D+3)/2]$, (we have $|B^0| = t-1$, $i_B = N$),

$B = Q^{t,-}_D = \{\{D+1, D+2\}, \{D, 1\}, \{D-1, 2\}, \dots, \{D+1-t, t\}\}$ with t even, $t \in [2, (D-1)/2]$ (we have $|B^0| = t$, $i_B < N$, $N \in \text{supp}(B)$).

For example,

Pr_2 consists of $\emptyset$ and $\{31\}$;

Pr_4 consists of $\emptyset$ and $\{51\}, \{51, 42\}$;

Pr_6 consists of $\emptyset$ and $\{71\}, \{71, 62\}, \{71, 62, 53\}$;

Pr_8 consists of $\emptyset$ and $\{91\}, \{91, 82\}, \{91, 82, 73\}, \{91, 82, 73, 64\}$;

Pr_1 consists of $\emptyset$ and $\{31\}, \{23\}$;

Pr_3 consists of $\emptyset$ and $\{42\}, \{31\}, \{51\}, \{45\}$;

Pr_5 consists of $\emptyset$ and $\{62\}, \{51\}, \{71\}, \{71, 62, 53\}, \{67\}, \{67, 51, 42\}$;

Pr_7 consists of $\emptyset$ and

$\{82\}, \{82, 73, 64\}, \{71\}, \{71, 62, 53\}, \{91\}, \{91, 82, 73\}, \{89\}, \{89, 71, 62\}$.

When $N \geq 3$ and $k \in [1, D]$ we define a map $I_k : \mathcal{P}_{N-2} \to \mathcal{P}_N$ by

$$(i_1j_1, i_2j_2, \dots, i_tj_t) \mapsto (\iota_k(i_1)\iota_k(j_1), \iota_k(i_2)\iota_k(j_2), \dots, \iota_k(i_t)\iota_k(j_t), \{k, k+1\}).$$

(This is well defined since $\iota_k : [1, N-2] \to [1, N]$ is injective with image not containing $k, k+1$.)

We define a subset $\mathcal{X}_D$ of $\mathcal{P}_N$ by induction on D as follows. We set $\mathcal{X}_0 = \mathcal{P}_1$; it consists of $\emptyset$. We set $\mathcal{X}_1 = Pr_1 \sqcup \{12\}$. Assume now that $D \geq 2$ so that $N \geq 3$. Let $B \in \mathcal{P}_N$. We say that $B \in \mathcal{X}_D$ if either $B \in Pr_D$ or if there exists $B' \in \mathcal{X}_{D-2}$ and $k \in [1, D]$ such that $B = I_k(B')$. We see that $Pr_D \subset \mathcal{X}_D$ and that $I_k(\mathcal{X}_{D-2}) \subset \mathcal{X}_D$ for $D \geq 2, k \in [1, D]$.

In the remainder of this subsection we fix $B \in \mathcal{P}_N$ and $\mathcal{J} \subset [1,N]$ such that either $\mathcal{J} = [i,j]$ for some $i \le j$ or that $\mathcal{J} = \emptyset$; define $e \in \{0,1\}$ by $|\mathcal{J}| = e \mod 2$. We say that $\mathcal{J}$ is *e-covered by B* if

(a) *there exists a sequence $a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$ in $[1,N]$ such that $a_1b_1, a_2b_2, \dots, a_mb_m$ are in B^1 and we have*
$\mathcal{J} = [a_1,b_1] \sqcup [a_2,b_2] \sqcup \dots \sqcup [a_m,b_m] \sqcup \mathcal{J}_0$
where $\mathcal{J}_0 \subset \mathcal{J}$ satisfies $|\mathcal{J}_0| = e$.

1.3. For $B \in \mathcal{P}_N$ we consider the following property:

$(P1)$ *There exists a sequence $i_1 < i_2 < \dots < i_{2s}$ in $[1,N]$ such that $B^0 = \{i_{2s}i_1, i_{2s-1}i_2, \dots, i_{s+1}i_s\}$ (it is automatically unique).*

1.4. For $B \in \mathcal{P}_N$ we define $\eta_B \in \{0,1\}$ as follows. If D is even or if D is odd and $|B^0| = 0$ we have $|B^0| = \eta_B \mod 2$. If D is odd and $|B^0| \ne 0$ we set $\eta_B = 0$ if $N \in \text{supp}(B)$ and $i_B \ne N$ and $\eta_B = 1$ if $N \notin \text{supp}(B)$ or $i_B = N$. We consider the following property:

$(P2)$ *We have $|B^0| = \eta_B \mod 2$.*

1.5. In this subsection we fix $B \in \mathcal{P}_N$ such that $(P1)$ holds for B. Let $i_1 < \dots < i_{2s}$ be as in $(P1)$. We consider the following property:

$(P3)$ *The following subsets of $[1,N]$ are 0-covered by B:*
$[i+1, j-1]$ *for any* $ij \in B^1$,
$[i_1+1, i_2-1], [i_2+1, i_3-1], \dots, [i_{s-1}+1, i_s-1], [i_{s+1}+1, i_{s+2}-1],$
$[i_{s+2}+1, i_{s+3}-1], \dots, [i_{2s-1}+1, i_{2s}-1]$ *(if $s \ge 1$).*
(In particular any two consecutive terms of $i_1, i_2, \dots, i_s$ have different parities, hence any two consecutive terms of $i_{s+1}, i_{s+2}, \dots, i_{2s}$ have different parities.) In addition, $[1, i_1-1], [i_{2s}+1, N]$ are 0-covered by B if D is odd and $N \in \text{supp}(B)$ or if D is even; if D is odd and $N \notin \text{supp}(B)$, $i_B \in 2\mathbf{N}+1$, then $[1, i_1-1]$ is 0-covered by B and $[i_{2s}+1, N-1]$ is 1-covered by B; if D is odd and $N \notin \text{supp}(B)$, $i_B \in 2\mathbf{N}$, then $[1, i_1-1]$ is 1-covered by B and $[i_{2s}+1, N-1]$ is 0-covered by B (again, if $s \ge 1$).

1.6. Let $\tilde{\mathcal{X}}_D$ be the set of all $B \in \mathcal{P}_N$ that satisfy $(P1), (P2), (P3)$. We show:

(a) *If $B \in \mathcal{X}_D$ then $B \in \tilde{\mathcal{X}}_D$.*

We argue by induction on D. If $D \in \{0,1\}$ or if $D \ge 2$ and $B \in Pr_D$, the result is obvious. Thus we can assume that $D \ge 2$ and $B = I_k(B')$ for some $k \in [1,D]$ and some $B' \in \mathcal{X}_{D-2}$. Then $(P1)$ for B follows immediately from the analogous statement for B'.

To prove $(P2)$ for B we can assume that D is odd and $|B^0| \ne 0$. Assume first that $|B^0| \in \{2,4,6,\dots\}$. We have $|B^0| = |B'^0|$ so that $|B'^0| \in \{2,4,6,\dots\}$. From the induction hypothesis we see that $N-2 \in \text{supp}(B')$ and $i_{B'} \ne N-2$. It follows that $N \in \text{supp}(B)$ and $i_B \ne N$ as desired. Next we assume that $|B^0| \in \{1,3,5,\dots\}$. We have $|B^0| = |B'^0|$ so that $|B'^0| \in \{1,3,5,\dots\}$. From the induction hypothesis we see that $N-2 \notin \text{supp}(B')$ or $i_{B'} = N-2$. It follows that $N \notin \text{supp}(B)$ or $i_B = N$ as desired. We see that $(P2)$ holds for B. It is easy to verify that if $(P3)$ holds for B' then it holds for B. This completes the proof of (a).

1.7. We show:

(a) *If $B \in \tilde{\mathcal{X}}_D$ then $B \in \mathcal{X}_D$.*

We argue by induction on D. If $D \le 1$, (a) is easily verified. Now assume that $D \ge 2$. Let ${}^*B$ be B^0 if D is even and $B^0 \cup \{N-1, N\}$ if D is odd. In the first part of the proof we assume that $B = {}^*B$.

If $B^0 = \emptyset$ then B is either $\emptyset$ or D is odd and $B = \{N-1, N\}$; in both cases we have $B \in Pr_D$ and we are done. Thus we can assume that $B^0 \ne \emptyset$. Let $i_1 < \cdots < i_{2s}$ be as in $(P1)$; note that $s \ge 1$. If $r \in \{1, 2, \dots, s-1, s+1, s+2, \dots, 2s\}$ and $i_{r+1} - i_r > 1$, then by $(P3)$, $[i_r+1, i_{r+1}-1]$ is 0-covered by B and is nonempty, so that there exists $ab \in B^1$ such that $[a, b] \subset [i_r+1, i_{r+1}-1]$. We have $ab \in B - {}^*B$, contradicting $B = {}^*B$. We see that $i_{r+1} = i_r + 1$ and

$$(i_1, i_2, \dots, i_s, i_{s+1}, i_{s+2}, \dots, i_{2s})$$
$$= (i_1, i_1+1, \dots, i_1+s-1, i_{2s}-s+1, \dots, i_{2s}-1, i_{2s}).$$

Assume now that D is even. If $i_1 \ge 2$ then by $(P3)$, $[1, i_1-1]$ is 0-covered by B and is nonempty so that there exists $ab \in B^1$ with $[a, b] \subset [1, i_1-1]$. This contradicts $B = {}^*B$. We see that $i_1 = 1$. If $N - i_{2s} \ge 2$ then by $(P3)$, $[i_{2s}+1, N]$ is 0-covered by B and is nonempty so that there exists $ab \in B^1$ with $[a, b] \subset [i_{2s}+1, N]$. This contradicts $B = {}^*B$. We see that $i_{2s} = N-1$ hence

$(i_1, i_2, \dots, i_s, i_{s+1}, i_{s+2}, \dots, i_{2s}) = (1, 2, \dots, s, N-s, \dots, N-2, N-1)$,

so that $B \in Pr_D$.

We now assume that D is odd and $N \in \mathrm{supp}(B)$ or $N \notin \mathrm{supp}(B), i_B \in 2\mathbf{N}+1$. If $i_1 \ge 2$ then by $(P3)$, $[1, i_1-1]$ is 0-covered by B and is nonempty so that there exists $ab \in B^1$ with $[a, b] \subset [1, i_1-1]$. We have $ab \in B - {}^*B$, contradicting $B = {}^*B$. We see that $i_1 = 1$.

We now assume that D is odd and $N \notin \mathrm{supp}(B)$, $i_B \in 2\mathbf{N}$. If $i_1 \ge 3$ then by $(P3)$, $[1, i_1-1]$ is 1-covered by B and has at least 2 elements, so that there exists $ab \in B^1$ with $[a, b] \subset [1, i_1-1]$. We have $ab \in B - {}^*B$, contradicting $B = {}^*B$. We see that $i_1 \le 2$. Since $i_1 = i_{2s} \mod 2$ and $i_{2s} \in 2\mathbf{N}$ we see that $i_1 = 2$.

We now assume that D is odd and $N \notin \mathrm{supp}(B)$, $i_B \in 2\mathbf{N}$. If $N - i_{2s} \ge 3$, then by $(P3)$, $[i_{2s}+1, N-1]$ is 0-covered by B and is nonempty so that there exists $ab \in B^1$ with $[a, b] \subset [i_{2s}+1, N-1]$. We have $ab \in B - {}^*B$, contradicting $B = {}^*B$. We see that $i_{2s} \ge N-2$ hence $i_{2s} = N-1$. Thus we have

$(i_1, i_2, \dots, i_s, i_{s+1}, i_{s+2}, \dots, i_{2s}) = (2, \dots, s+1, N-s, \dots, N-2, N-1)$,

with s odd (see $(P2)$), so that $B \in Pr_D$.

We now assume that D is odd and $N \notin \mathrm{supp}(B)$, $i_B \in 2\mathbf{N}+1$. If $N - i_{2s} \ge 3$, then by $(P3)$, $[i_{2s}+1, N-1]$ is 1-covered by B and has at least 2 elements (since $i_{2s}+1 = N-1 \mod 2$) so that there exists $ab \in B^1$ with $[a, b] \subset [i_{2s}+1, N-1]$. We have $ab \in B - {}^*B$, contradicting $B = {}^*B$. We see that $i_{2s} \ge N-2$; since $i_{2s} = 1 \mod 2, i_{2s} \ne N$, we must have $i_{2s} = N-2$ in this case. Thus we have

$(i_1, i_2, \dots, i_s, i_{s+1}, i_{s+2}, \dots, i_{2s}) = (1, 2, \dots, s, N-s-1, \dots, N-3, N-2)$, with s odd (see $(P2)$), so that $B \in Pr_D$.

We now assume that D is odd and $N \in \text{supp}(B)$, $i_B \ne N$. If $N - i_{2s} \ge 3$, then by $(P3)$, $[i_{2s}+1, N]$ is 0-covered by B and contains at least three elements so that there exists $ab \in B^1$ other than $\{N-1, N\}$ with $[a,b] \subset [i_{2s}+1, N]$. We have $ab \in B - {}^*B$, contradicting $B = {}^*B$. We see that $i_{2s} \ge N-2$. As we have seen earlier, in this case we have $i_1 = 1$. Since $i_{2s} = i_1 \mod 2$ we see that i_{2s} is odd. Since $i_{2s} \ge N-2$ and $i_{2s} < N$ we see that $i_{2s} = N-2$. Thus we have

$(i_1, i_2, \dots, i_s, i_{s+1}, i_{s+2}, \dots, i_{2s}) = (1, 2, \dots, s, N-s-1, \dots, N-3, N-2)$, with s even (see $(P2)$). By $(P3)$, $[i_{2s}+1, N] = [N-1, N]$ is 0-covered by B. It follows that $\{N-1, N\} \in B$. We see that $B \in Pr_D$.

We now assume that D is odd and $i_B = N$ hence $N \in \text{supp}(B)$. In this case we have

$(i_1, i_2, \dots, i_s, i_{s+1}, i_{s+2}, \dots, i_{2s}) = (1, 2, \dots, s, N-s+1, \dots, N-1, N)$, with s odd (see $(P2)$) so that $B \in Pr_D$.

We have thus proved that if $B = {}^*B$ then $B \in Pr_D$; in particular we have $B \in \mathcal{X}_D$. We see that it is enough to prove (a) assuming that $B \ne {}^*B$. Then we can find $ab \in B^1$ such that when D is odd we have $\{a,b\} \ne \{N-1, N\}$. We can assume in addition that $b-a$ is minimum possible. If $b-a > 1$ then $b-a \ge 3$ and by $(P3)$, $[a+1, b-1]$ is 0-covered by B and is nonempty so that we can find $a'b' \in B^1$ with $[a', b'] \subset [a+1, b-1]$; if D is odd we have automatically $\{a', b'\} \ne \{N-1, N\}$. This contradicts the minimality of $b-a$. We see that $b-a = 1$, Thus there exists $k \in [1, D]$ such $\{k, k+1\} \in B$. If $cd \in B - \{k, k+1\}$ then $\{c,d\} \cap \{k, k+1\} = \emptyset$ (since $B \in \mathcal{P}_N$). Hence there are unique c', d' in $[1, N-2]$ such that $c = \iota_k(c'), d = \iota_k(d')$. We have $c'd' \in {}^2E_{N-2}$. Let $B' \in \mathcal{P}_{N-2}$ be the set consisting of all $c'd'$ associated as above to the various $cd \in B - \{k, k+1\}$. Note that $B = I_k(B')$. One can verify that $(P1), (P2), (P3)$ hold for B' since they hold for B. By the induction hypothesis we have $B' \in \mathcal{X}_{D-2}$. It follows that $B \in \mathcal{X}_D$. This proves (a).

1.8. For $B \in \mathcal{X}_D$ we show:

(a) *If $ab \in B, cd \in B$ are distinct (so that $\{a,b\} \cap \{c,d\} = \emptyset$) then $\lfloor a,b \rfloor \cap \lfloor c,d \rfloor = \emptyset$ or $\lfloor a,b \rfloor \subset \lfloor c,d \rfloor$ or $\lfloor c,d \rfloor \subset \lfloor a,b \rfloor$.*

We argue by induction on D. If $D \in \{0,1\}$ or if $D \ge 2$ and $B \in Pr_D$, the result is obvious. Thus we can assume that $D \ge 2$ and $B = I_k(B')$ for some $k \in [1, D]$ and some $B' \in \mathcal{X}_{D-2}$. Then (a) for B follows immediately from the analogous statement for B'.

Let $B \in \mathcal{P}_N$ and let $\mathcal{J} \subset [1, N]$ with $\mathcal{J} = [i, j]$ for some $i \le j$ be such that $\mathcal{J}$ is e-covered by B (where $e = |\mathcal{J}| \mod 2, e \in \{0,1\}$) and let $a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$ be as in 1.2(a). We show that:

(b) *the sequence $a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$ is unique.*

We argue by induction on $|\mathcal{J}|$. If $|\mathcal{J}| \le 1$, the result is obvious. Now assume that $|\mathcal{J}| \ge 2$ so that $\mathcal{J} = [i, j]$ for some $i < j$. Let $a'_1 < b'_1 < a'_2 < b'_2 < \dots < a'_{m'} < b'_{m'}$

be a sequence with the same properties as $a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m$. Now i equals a_1 or a_1+1 so that $i \in [a_1, b_1]$. Similarly, $i \in [a'_1, b'_1]$. Using (a) we see that $[a_1, b_1] = [a'_1, b'_1]$. Let $\mathcal{J}' = \mathcal{J} - [a_1, b_1]$. We have $|\mathcal{J}'| < |\mathcal{J}|$. The induction hypothesis is applicable to $\mathcal{J}'$ instead of $\mathcal{J}$; we see that (b) holds for $\mathcal{J}$.

In the case where $e = 1$, the unique element in $\mathcal{J}$ which is not in $[a_1, b_1] \sqcup [a_2, b_2] \sqcup \ldots \sqcup [a_m, b_m]$ is said to be the *distinguished element* of $\mathcal{J}$.

1.9. Assume first that D is even. We have a partition $\mathcal{X}_D = \sqcup_{t \in 2\mathbf{Z}} \mathcal{X}_D^t$ where $\mathcal{X}_D^t$ consists of all $B \in \mathcal{X}_D$ such that $|B^0| = t$ (if $t \geq 0$), $|B^0| = -t-1$ (if $t < 0$). The subsets $\mathcal{X}_D^t$ are said to be the *pieces* of $\mathcal{X}_D$. Note that when $t \in 2\mathbf{Z}$, Q_D^t is the unique element in $Pr_D \cap \mathcal{X}_D^t$.

In the remainder of this subsection we assume that D is odd. We have a partition $\mathcal{X}_D = \mathcal{X}_D^+ \sqcup \mathcal{X}_D^-$ where

$\mathcal{X}_D^+ = \{B \in \mathcal{X}_D; N \notin \mathrm{supp}(B)\}$, $\mathcal{X}_D^- = \{B \in \mathcal{X}_D; N \in \mathrm{supp}(B)\}$.

For $t \in 2\mathbf{Z}$ we define a subset $\mathcal{X}_D^{t,+}$ of $\mathcal{X}_D^+$ to be

$\{B \in \mathcal{X}_D^+; |B^0| = 0\}$ if $t = 0$,

$\{B \in \mathcal{X}_D^+; |B^0| = t-1, i_B \in 2\mathbf{N}\}$ if $t \in \{2, 4, 6, \ldots\}$,

$\{B \in \mathcal{X}_D^+; |B^0| = -t-1, i_B \in 2\mathbf{N}+1\}$ if $t \in \{-2, -4, -6, \ldots\}$.

For $t \in 2\mathbf{Z}$ we define a subset $\mathcal{X}_D^{t,-}$ of $\mathcal{X}_D^-$ to be

$\{B \in \mathcal{X}_D^-; |B^0| = 0\}$ if $t = 0$,

$\{B \in \mathcal{X}_D^-; |B^0| = t, i_B \neq N\}$ if $t \in \{2, 4, 6, \ldots\}$,

$\{B \in \mathcal{X}_D^-; |B^0| = -t-1, i_B = N\}$ if $t \in \{-2, -4, -6, \ldots\}$.

Note that when $t \in 2\mathbf{Z}$, $Q_D^{t,+}$ (resp. $Q_D^{t,-}$) is the unique element of $Pr_D \cap \mathcal{X}_D^{t,+}$ (resp. $Pr_D \cap \mathcal{X}_D^{t,-}$). From $(P3)$ we see that the subsets $\mathcal{X}^{t,+}, \mathcal{X}^{t,-}$ of $\mathcal{X}_D$ (with $t \in 2\mathbf{Z}$) form a partition of $\mathcal{X}_D$; these subsets are said to be the *pieces* of $\mathcal{X}_D$.

1.10. In this subsection we asssume that D is odd and that $B \in \mathcal{X}_D$ is such that $|B^0| > 0$ and $N \notin \mathrm{supp}(B)$. Let $i_1 < \cdots < i_{2s}$ be as in (P1); here $s \geq 1$. If $i_{2s} \in 2\mathbf{N}+1$ (so that $i_{2s}+1 = N-1 \mod 2$) then by $(P3)$, $[i_{2s}+1, N-1]$ is 1-covered by B; we denote by u_B the distinguished element of $[i_{2s}+1, N-1]$. Note that $[i_{2s}+1, u_B-1]$ and $[u_B+1, N-1]$ are 0-covered by B. If $i_{2s} \in 2\mathbf{N}$ (so that $i_1 - 1 = 1$) then by $(P3)$, $[1, i_1-1]$ is 1-covered by B; we denote by u_B the distinguished element of $[1, i_1-1]$. Note that $[1, u_B-1]$ and $[u_B+1, i_1-1]$ are 0-covered by B.

1.11. For any $B \in \mathcal{X}_D$ we define $\underline{B} \in E_N$ as follows. If D is even we set $\underline{B} = \emptyset$. If D is odd and $B \notin \cup_{t \in 2\mathbf{Z}-\{0\}} \mathcal{X}_D^{t,+}$, we set $\underline{B} = \emptyset$. If D is odd and $B \in \mathcal{X}_D^{t,+}$ for some $t \in \{-2, -4, -6, \ldots\}$, we set $\underline{B} = \lfloor u_B, N \rfloor = [u_B, N]$; if D is odd and $B \in \mathcal{X}_D^{t,+}$ for some $t \in \{2, 4, 6, \ldots\}$, we set $\underline{B} = \lfloor N, u_B \rfloor = \{N\} \cup [1, u_B]$.

For any $B \in \mathcal{X}_D$ we define $\epsilon(B) \in E_N$ by

(a) $\epsilon(B) = \sum_{ij \in B} \lfloor i, j \rfloor + \underline{B}$.

(Sum in E_N.) We show:

(b) *If* $N \geq 3, k \in [1, D]$, $B' \in \mathcal{X}_{D-2}$ *then* $\epsilon(I_k(B')) = \iota_k(\epsilon(B')) + c\{k, k+1\}$ *for some* $c \in F$.

We have:

$\epsilon(I_k(B')) = \sum_{ij\in B'} \lfloor \iota_k(i), \iota_k(j) \rfloor + \{k, k+1\} + \underline{I_k(B')}$,

$\iota_k(\epsilon(B')) = \sum_{ij\in B'} \iota_k(\lfloor i, j \rfloor) + \iota_k(\underline{B'})$.

Using 1.1(a), we see that it is enough to show that

$\underline{I_k(B')} = \iota_k(\underline{B'}) + c_1\{k, k+1\}$

for some $c_1 \in F$. If D is even, both sides are zero (and $c_1 = 0$). Thus we can assume that D is odd. From the definitions we have $u_B = \iota_k(u_{B'})$, $N = \iota_k(N-2)$ so that the desired equality follows again from 1.1(a). This proves (b).

1.12. Define $\gamma : E_N \to \mathbf{Z}$ by $\gamma(X) = |X \cap (2\mathbf{Z})| - |X \cap (2\mathbf{Z}+1)|$. Note that the image of γ is contained in $2\mathbf{Z}$. We show:

(a) *If $N \geq 3, k \in [1, D]$, $B' \in \mathcal{X}_{D-2}$ then $\gamma(\epsilon(I_k(B'))) = \gamma(\epsilon(B'))$.*

Using 1.11(b), we see that it is enough to prove that for any $c \in F$ we have

$\gamma(\iota_k(\epsilon(B')) + c\{k, k+1\}) = \gamma(\epsilon(B'))$.

It is also enough to show that for any $X \in E_{N-2}$ we have $\gamma(\iota_k(X) + c\{k, k+1\}) = \gamma(X)$. From the definitions we have $\gamma(\iota_k(X)) = \gamma(X)$ for any $X \in E_{N-2}$. It is then enough to show that $\gamma(\iota_k(X) + \{k, k+1\}) = \gamma(X)$. From the definition we have $\iota_k(X) \cap \{k, k+1\} = \emptyset$ hence $\gamma(\iota_k(X) + \{k, k+1\}) = \gamma(\iota_k(X)) + \gamma(\{k, k+1\}) = \gamma(X) + \gamma(\{k, k+1\}) = \gamma(X)$, since $\gamma(\{k, k+1\}) = 0$. This proves (a).

1.13. We now describe $\epsilon(B)$ and $\gamma(\epsilon(B)$ when $B \in Pr_D$. If $B = \emptyset$, then $\epsilon(B) = \emptyset \in E_N$, $\gamma(\epsilon(B)) = 0$.

If D is even and $B = Q_D^t$ with $t \in [2, D/2]$, t even, we have

$\epsilon(B) = \{2, 4, 6, \dots, t, D+2-t, D+4-t, \dots, D\}$, $\gamma(\epsilon(B)) = t$.

If D is even and $B = Q_D^{-t}$ with $t \in [2, (D+2)/2]$, t even, we have

$\epsilon(B) = \{1, 3, \dots, t-1, D+3-t, D+5-t, \dots, D+1\}$, $\gamma(\epsilon(B)) = -t$.

We now assume that D is odd. If $B = Q_D^{0,-}$ then $\epsilon(B) = \{D+1, D+2\}$ and $\gamma(\epsilon(B)) = 0$.

If $B = Q_D^{t,+}$ with t even, $t \in [2, (D+1)/2]$, we have

$\epsilon(B) = \{2, 4, 6, \dots, t, D+3-t, D+5-t, \dots, D+1\}$, $\gamma(\epsilon(B)) = t$.

If $B = Q_D^{-t,+}$ with t even, $t \in [2, (D+1)/2]$, then

$\epsilon(B) = \{1, 3, \dots, t-1, D+2-t, D+4-t, \dots, D\}$, $\gamma(\epsilon(B)) = -t$.

If $B = Q_D^{-t,-}$ with t even, $t \in [2, (D+3)/2]$, then

$\epsilon(B) = \{1, 3, \dots, t-1, D+4-t, D+6-t, \dots, D+2\}$, $\gamma(\epsilon(B)) = -t$.

If $B = Q_D^{t,-}$ with t even, $t \in [2, (D-1)/2]$, then

$\epsilon(B) = \{2, 4, 6, \dots, t, D+1-t, D+3-t, \dots, D-1, D+1, D+2\}$, $\gamma(\epsilon(B)) = t$.

1.14. We show:

(a) *If $N \geq 3, k \in [1, D]$, D odd, $B' \in \mathcal{X}_{D-2}$, $B = I_k(B') \in \mathcal{X}_D$, then $N \in \epsilon(B)$ if and only if $N-2 \in \epsilon(B')$.*

Recall from 1.11 that $\epsilon(B) = \iota_k(\epsilon(B')) + c\{k, k+1\}$ for some $c \in \{0, 1\}$. Note that $N \notin \{k, k+1\}$ so that we have $N \in \epsilon(B)$ if and only if $N \in \iota_k(\epsilon(B'))$ and this happens if and only if $N-2 \in \epsilon(B')$. (From the definitions we see that for $X \subset [1, N-2]$ we have $N \in_k^{-1} (X)$ if and only if $N-2 \in X$.) This proves (a).

1.15. When D is even we have a partition $E_N = \sqcup_{t\in 2\mathbf{Z}}\mathcal{E}_D^t$ where
$\mathcal{E}_D^t = \{X \in E_N; \gamma(X) = t\}$.
Now assume that D is odd. We define a partition $E_N = \mathcal{E}_D^+ \sqcup \mathcal{E}_D^-$ by
$\mathcal{E}_D^+ = \{X \in E_N; N \notin X\}$, $\mathcal{E}_D^- = \{X \in E_N; N \in X\}$.
We have $\mathcal{E}_D^+ = \sqcup_{t\in 2\mathbf{Z}}\mathcal{E}_D^{t,+}$, $\mathcal{E}_D^- = \sqcup_{t\in 2\mathbf{Z}}\mathcal{E}_D^{t,}$ where
$\mathcal{E}_D^{t,+} = \{X \in \mathcal{E}_D^+; \gamma(X) = t\}$, $\mathcal{E}_D^{t,-} = \{X \in \mathcal{E}_D^-; \gamma(X) = t\}$.
We show:

(a) If D is even and $B \in \mathcal{X}_D^t$, $t \in 2\mathbf{Z}$, then $\gamma(\epsilon(B)) = t$.

(b) If D is odd and $B \in \mathcal{X}_D^{t,+}$, $t \in 2\mathbf{Z}$, then $\gamma(\epsilon(B)) = t$ and $N \notin \epsilon(B)$.

(c) If D is odd and $B \in \mathcal{X}_D^{t,-}$, $t \in 2\mathbf{Z}$, then $\gamma(\epsilon(B)) = t$ and $N \in \epsilon(B)$.

We argue by induction on D. For $D \in \{0, 1\}$, the result is obvious. When $B \in Pr_D$ this follows from 1.13. Assume now that $D \geq 2$ and $B \notin Pr_D$. We can find $B' \in \mathcal{X}_{D-2}, k \in [1, D]$ such that $B = I_k(B')$. From the definitions we have $B' \in \mathcal{X}_{D-2}^t$ (in (a)), $B' \in \mathcal{X}_{D-2}^{t,+}$ (in (b)), $B' \in \mathcal{X}_{D-2}^{t,-}$ (in (c)). By the induction hypothesis we have $\gamma(\epsilon(B')) = t$ (in case (a),(b),(c)), $N \notin \epsilon(B')$ (in case (b)), $N \in \epsilon(B')$ (in case (c)). Using 1.14(a) we deduce that $\gamma(\epsilon(B)) = t$ (in case (a),(b),(c)). Using 1.11(b) we see that $N \notin \epsilon(B)$ in case (b) and $N \in \epsilon(B)$ in case (c). This completes the proof of (a),(b),(c).

1.16. Let $t \in 2\mathbf{Z}$. From 1.15(a),(b),(c) we see that $\epsilon : \mathcal{X}_D \to E_N$ restricts to a map

(a) $\mathcal{X}_D^t \to \mathcal{E}_D^t$

if D is even, and to maps

(b) $\mathcal{X}_D^{t,+} \to \mathcal{E}_D^{t,+}$, $\mathcal{X}_D^{t,-} \to \mathcal{E}_D^{t,-}$

if D is odd. Hence it restricts to maps

(c) $\mathcal{X}_D^+ \to \mathcal{E}_D^+$, $\mathcal{X}_D^- \to \mathcal{E}_D^-$.

1.17. For $B \in \mathcal{X}_D$ let

(a) $\langle B\rangle$ be the subspace of E_N spanned by the vectors $ij \in B$ (viewed as elements of E_N).

We show:

(b) $\epsilon(B) \in \langle B\rangle$.

We argue by induction on D. If $D \in \{0, 1\}$ the result is obvious. We now assume that $D \geq 2$. If $B \in Pr_D$ the result follows from 1.13. Thus we can assume that $B \notin Pr_D$ so that we can find $B' \in \mathcal{X}_{D-2}, k \in [1, D]$ such that $B = I_k(B')$. By 1.11(b) we have $\epsilon(B) = \iota_k(\epsilon(B')) + c\{k, k+1\}$ for some $c \in F$. From the definition of I_k we see that $\{k, k+1\} \subset \langle B\rangle$ and by the induction hypothesis we have $\epsilon(B') \in \langle B'\rangle$. Thus it is enough to prove that $\iota_k(\langle B'\rangle) \subset \langle B\rangle$ or that for any $ij \in B'$ we have $\iota_k(\{i, j\}) \in \langle B\rangle$; this follows from $\{\iota_k(i), \iota_k(j)\} \in B$.

Let $\leq_D$ be the transitive relation on E_N generated by the relation for which X, X' in E_N are related if $X \in \langle\epsilon^{-1}(X')\rangle$.

Theorem 1.18. *(a) There is a unique bijection $\epsilon' : \mathcal{X}_D \to E_N$ such that for any $B \in \mathcal{X}_D$ we have $\epsilon'(B) \in \langle B\rangle$.*

(b) We have $\epsilon'(B)=\epsilon(B)$ for any $B\in\mathcal{X}_D$.
(c) The relation $\le_D$ is a partial order on E_N.
(d) The maps 1.16(a), 1.16(b), 1.16(c) are bijections.

When D is even, (a),(b),(c) can be deduced from the results of [L20, L22], see §3. The formula for ϵ' given by (a) is simpler than the one in [L20, L22]; the equivalence of the two formulas is proved in §3. The proof of (a),(b),(c) for odd D can be given along similar lines. Now (d) follows from (a),(b).

1.19. For $X\in E_N$ we set $\mathcal{C}(X)=[1,N]-X$ and
$X^*=(X\cap(2\mathbf{Z}+1))\cup(\mathcal{C}(X)\cap(2\mathbf{Z}))\subset[1,N]$,
$X^{**}=(X\cap(2\mathbf{Z}))\cup(\mathcal{C}(X)\cap(2\mathbf{Z}+1))=\mathcal{C}(X^*)\subset[1,N]$.
We have
$|X^*|=|X\cap(2\mathbf{Z}+1)|+|(2\mathbf{Z})\cap[1,N]|-|X\cap(2\mathbf{Z})|=(N-1)/2-\gamma(X)$,
$|X^{**}|=(N+1)/2+\gamma(X)$. Hence $|X^{**}|-|X^*|=2\gamma(X)+1$.

Assume now that D is even. The assignment $X\mapsto(X^*,X^{**})$ defines a bijection $E_N\to Sy_D$ (notation of 0.1); it restricts for any $t\in 2\mathbf{Z}$ to a bijection $\mathcal{E}_D^t\to Sy_D^{abs(2t+1)}$.

For X,X' in E_N we set $M_{X,X'}=1$ if $X\in\langle\epsilon^{-1}(X')\rangle$ and $M_{X,X'}=0$, otherwise. From 1.18 we see that $(M_{X,X'})$ is an upper triangular matrix with entries in $\{0,1\}$ and with 1 on diagonal. It follows that the elements

(a) $\sum_{X\in E_N}M_{X,X'}X'\in\mathbf{Z}[E_N]$ (for various $X'\in E_N$)

form a $\mathbf{Z}$-basis of $\mathbf{Z}[E_N]$, said to be the second basis. (This basis appears in [L20] where it is called the new basis.)

Using the bijection $E_N\to Sy_D$ we see that the second basis of $\mathbf{Z}[E_N]$ can be viewed as a $\mathbf{Z}$-basis of $\mathbf{Z}[Sy_D]$ which is also called the second basis.

We now assume that D is odd.

If $X\in\mathcal{E}_D^+$ then $(X^{**}-\{N\},X^*)\in Sy_D^+$ and $|X^{**}-\{N\}|-|X^*|=2\gamma(X)$; the assignment $X\mapsto(X^{**}-\{N\},X^*)$ defines a bijection $\mathcal{E}_D^+\to Sy_D^+$; it restricts for any $t\in 2\mathbf{Z}$ to a bijection $\mathcal{E}_D^{t,+}\to Sy_D^{2t}$.

For X,X' in $\mathcal{E}_D^+$ we set $M^+_{X,X'}=1$ if $X\in\langle\epsilon^{-1}(X')\rangle$ and $M^+_{X,X'}=0$, otherwise. From 1.18 we see that $(M^+_{X,X'})$ is an upper triangular matrix with entries in $\{0,1\}$ and with 1 on diagonal. It follows that the elements

(b) $\sum_{X\in\mathcal{E}_D^+}M_{X,X'}X'\in\mathbf{Z}[\mathcal{E}_D^+]$ (for various $X'\in\mathcal{E}_D^+$)

form a $\mathbf{Z}$-basis of $\mathbf{Z}[\mathcal{E}_D^+]$, said to be the second basis. (This basis appears in [L22].) Using the bijection $\mathcal{E}_D^+\to Sy_D^+$, we see that the second basis of $\mathbf{Z}[\mathcal{E}_D^+]$ can be viewed as a $\mathbf{Z}$-basis of $\mathbf{Z}[Sy_D^+]$ which is also called the second basis.

If $X\in\mathcal{E}_D^-$ then $(X^{**},X^*-\{N\})\in Sy_D^-$ and $|X^{**}|-|X^*-\{N\}|=2\gamma(X)+2$; the assignment $X\mapsto(X^{**},X^*-\{N\})$ defines a bijection $\mathcal{E}_D^-\to Sy_D^-$; it restricts for any $t\in 2\mathbf{Z}$ to a bijection $\mathcal{E}_D^{t,-}\to Sy_D^{2t+2}$.

For X,X' in $\mathcal{E}_D^-$ we set $M^-_{X,X'}=1$ if $X\in\langle\epsilon^{-1}(X')\rangle$ and $M^-_{X,X'}=0$, otherwise. From 1.18 we see that $(M^-_{X,X'})$ is an upper triangular matrix with entries in $\{0,1\}$ and with 1 on diagonal. It follows that the elements

(c) $\sum_{X\in\mathcal{E}_D^-} M_{X,X'}X' \in \mathbf{Z}[\mathcal{E}_D^-]$ (for various $X' \in \mathcal{E}_D^-$)
form a $\mathbf{Z}$-basis of $\mathbf{Z}[\mathcal{E}_D^-]$, said to be the second basis.

Using the bijection $\mathcal{E}_D^- \to Sy_D^-$, we see that the second basis of $\mathbf{Z}[\mathcal{E}_D^-]$ can be viewed as a $\mathbf{Z}$-basis of $\mathbf{Z}[Sy_D^-]$ which is also called the second basis.

1.20. We have
$|Sy_D^s| = \binom{N}{(s+N)/2}$ if D is even, $s \in 2\mathbf{N}+1$,
$|Sy_D^s| = \binom{N-1}{(s+N-1)/2}$, if D is odd, $s \in 2\mathbf{Z}$.
Here $\binom{a}{b}$ is defined to be 0 if $b<0$ or if $b>a$. It follows that
$|\mathcal{E}_D^t| = \binom{N}{(abs(2t+1)+N)/2}$ for D even, $t \in 2\mathbf{Z}$,
$|\mathcal{E}_D^{t,+}| = \binom{N-1}{(2t+N-1)/2}$ for D odd, $t \in 2\mathbf{Z}$,
$|\mathcal{E}_D^{t,-}| = \binom{N-1}{(2t+N+1)/2}$ for D odd, $t \in 2\mathbf{Z}$.

2. Tables for $\mathcal{X}_D$

2.1. In this section we give tables describing $\mathcal{X}_D$ and the map $\epsilon : \mathcal{X}_D \to E_N$ for $D = 1,2,3,4,5,6,7$. The table for $\mathcal{X}_D$ is given by a list of elements of the various pieces of $\mathcal{X}_D$; each such element B is written in the form $(?,?,\dots,?,[?,?,\dots,?])$ where each ? stands for an element of B and the ? inside the bracket $[,]$ are such that their sum in E_N is equal to $\epsilon(B)$. The elements of $\mathcal{X}_D$ are written in an order in which $B \in \mathcal{X}_D$ preceeds $B' \in \mathcal{X}_D$ whenever $\epsilon(B) \le_D \epsilon(B')$.

2.2. Table for $\mathcal{X}_2$.
$\mathcal{X}_2^0$:
$([\emptyset]), ([12]), ([23])$
$\mathcal{X}_2^{-2}$:
$([13]$

2.3. Table for $\mathcal{X}_4$.
$\mathcal{X}_4^0$:
$([\emptyset]), ([12]), ([23]), ([34]), ([45]), ([12,34]), ([12,45]),$
$([23,45]), (23,[14]), (34,[25])$

$\mathcal{X}_4^{-2}$:
$([51]), ([34,51]), ([23,51]), (45,[31]), (12,[53])$

$\mathcal{X}_4^2$:
$(51,[42])$

2.4. Table for $\mathcal{X}_6$.
$\mathcal{X}_6^0$:
$([\emptyset]), ([12]), ([23]), ([34]), ([45]), ([56]), ([67]), ([12,45]),$
$([12,67]), ([23,45]), ([23,56]), ([23,67]), ([34,67]), ([45,67]), ([12,34]),$
$([12,56]), ([34,56]), (23,[14]), (34,[25]), (45,[36]),$
$(56,[47]), ([12,34,56]), ([12,34,67]), (45,[12,36]), ([12,45,67]), ([23,45,67]),$

$(56,[12,47]),(56,[23,47]),(23,[56,14]),(23,[67,14]),(23,45,[16]),(25,[34,16]),$
$(34,[67,25]),(34,56,[27]),(36,[45,27])$

$\mathcal{X}_6^{-2}$:
$([71]),([56,71]),([45,71]),([34,71]),([23,71]),(67,[51]),$
$(12,[73]),([23,56,71]),([34,56,71]),([23,45,71]),(67,[34,51]),$
$(12,[45,73]),(67,[23,51]),(12,[56,73]),(34,[25,71]),(45,[36,71]),$
$(45,67,[31]),(12,34,[75]),(12,67,[53]),(47,[56,31]),(14,[23,75])$

$\mathcal{X}_6^{2}$:
$(71,[62]),(71,[34,62]),(71,[45,62]),(71,23,[64]),(71,56,[42]),$
$(73,[12,64]),(51,[67,42])$

$\mathcal{X}_6^{-4}$:
$(62,[71,53])$

2.5. Table for $\mathcal{X}_1$.
$\mathcal{X}_1^{0,+}$:
$([\emptyset]),([12])$

$\mathcal{X}_1^{0,-}$:
$([23])$

$\mathcal{X}_1^{-2,-}$:
$[31]$

2.6. Table for $\mathcal{X}_3$.
$\mathcal{X}_3^{0,+}$:
$([\emptyset]),([12]),([23]),$
$([34]),([12,34]),(23,[14])$

$\mathcal{X}_3^{-2,+}$:
$([31])$

$\mathcal{X}_3^{2,+}$:
$([42])$

$\mathcal{X}_3^{0,-}$:
$([45]),([12,45]),([23,45]),(34,[25])$

$\mathcal{X}_3^{-2,-}$:
$([51]),([51,34]),([51,23]),(12,[53])$

2.7. Table for $\mathcal{X}_5$.
$\mathcal{X}_5^{0,+}$:
$([\emptyset]),([12]),([23]),([34]),([45]),([12,34]),$
$([12,45]),([23,45]),(23,[14]),(34,[25]),([56]),([12,56]),([34,56]),$
$([12,34,56]),([23,56]),(23,[14,56]),(45,[36]),(45,[12,36]),$
$(23,45,[16]),(25,[34,16])$

$\mathcal{X}_5^{-2,+}$:
$([51]), ([34, 51]), ([23, 51]), (45, [31]), (12, [53]), ([31, 56])$

$\mathcal{X}_5^{2,+}$:
$([62]), ([34, 62]), ([45, 62]), (56, [42]), (23, [64]), ([12, 64])$

$\mathcal{X}_5^{0,-}$:
$([67]), ([12, 67]), ([23, 67]), ([34, 67]), ([45, 67]), (56, [47]),$
$([12, 34, 67]), ([12, 45, 67]), ([23, 45, 67]), (23, [14, 67]), (34, [25, 67]),$
$(56, [12, 47]), (56, [23, 47]), (34, 56, [27]), (36, [45, 27])$

$\mathcal{X}_5^{-2,-}$:
$([71]), ([23, 71]), ([34, 71]), ([45, 71]), ([56, 71]),$
$([23, 45, 71]), ([23, 56, 71]), ([34, 56, 71]), (34, [25, 71]), (45, [36, 71]),$
$(12, [73]), (12, [45, 73]), (12, [56, 73]), (12, 34, [75]), (14, [23, 75])$

$\mathcal{X}_5^{2,-}$:
$(51, [42, 67])$

$\mathcal{X}_5^{-4,-}$:
$(62, [53, 71])$

2.8. Table for $\mathcal{X}_7$.

$\mathcal{X}_7^{0,+}$:
$([\emptyset]), ([12]), ([23]), ([34]), ([45]), ([56]), ([67]),$
$([12, 45]), ([12, 67]), ([23, 45]), ([23, 56]), ([23, 67]), ([34, 67]), ([45, 67]),$
$([12, 34]), ([12, 56]), ([34, 56]), (23, [14]), (34, [25]), (45, [36]),$
$(56, [47]), ([12, 34, 56]), ([12, 34, 67]), (45, [12, 36]), ([12, 45, 67]),$
$([23, 45, 67]), (56, [12, 47]), (56, [23, 47]), (23, [14, 56]), (23, [14, 67]),$
$(23, 45, [16]), (25, [16, 34]), (34, [25, 67]), (34, 56, [27]), (36, [27, 45]),$
$([78]), ([12, 78]), ([23, 78]), ([34, 78]), ([45, 78]), ([56, 78]), ([12, 34, 78]),$
$([12, 45, 78]), ([12, 56, 78]), ([23, 45, 78]), ([23, 56, 78]), ([34, 56, 78]),$
$(67, [58]), (23, [14, 78]), (34, [25, 78]), (45, [36, 78]), (67, [12, 58]),$
$(67, [23, 58]), (67, [34, 58]), ([12, 34, 56, 78]), (23, [14, 56, 78]), (45, [12, 36, 78]),$
$(67, [12, 34, 58]), (45, 67, [38]), (47, [38, 56]), (23, 45, [16, 78]), (25, [16, 34, 78]),$
$(45, 67, [12, 38]), (23, 67, [14, 58], (23, 45, 67, [18]), (47, [12, 38, 56]),$
$(25, 67, [18, 34]), (23, 47, [18, 56]), (27, [18, 34, 56]), (27, 45, [18, 36])$

$\mathcal{X}_7^{-2,+}$:
$([71]), ([56, 71]), ([45, 71]), ([34, 71]), ([23, 71]),$
$(67, [51]), (12, [73]), ([23, 71, 56]), ([34, 71, 56]), ([23, 71, 45]), (67, [34, 51]),$
$(12, [73, 45]), (67, [23, 51]), (12, [73, 56]), (34, [71, 25]), (45, [71, 36]),$
$(45, 67, [31]), (12, 34, [75]), (12, 67, [53]), (47, [31, 56]), (14, [75, 23]),$
$([51, 78]), ([34, 51, 78]), (45, [31, 78]), ([23, 51, 78]), ([31, 56, 78]),$
$(12, [53, 78]), (67, [31, 58])$

$\mathcal{X}_7^{2,+}$:
$([82]), ([82, 34]), ([82, 45]), ([82, 56]), ([82, 67]),$
$([82, 34, 56]), ([82, 34, 67]), ([82, 45, 67]), (23, [84]), ([84, 12]), (23, [84, 56]),$
$([84, 12, 56]), (23, [84, 67]), ([84, 12, 67]), (45, [82, 36]), (23, 45, [86]),$
$(45, [86, 12]), (25, [86, 34]), ([86, 12, 34]), (23, [86, 14]), (56, [82, 47]),$
$(78, [62]), (78, [62, 34]), (78, [62, 45]), (78, 56, [42]), (78, 23, [64]),$
$(78, [64, 12]), (58, [42, 67])$

$\mathcal{X}_7^{-4,+}$:
$(62, [71, 53])$

$\mathcal{X}_7^{4,+}$:
$(73, [82, 64])$

$\mathcal{X}_7^{0,-}$:
$([89]), ([23, 89]), ([34, 89]), ([45, 89]), ([56, 89]),$
$([67, 89]), ([78, 69]), ([12, 34, 89]), ([12, 45, 89]), ([12, 56, 89]),$
$([12, 67, 89]), ([23, 45, 89]), ([23, 56, 89]), ([23, 67, 89]), ([34, 56, 89]),$
$([34, 67, 89]), ([45, 67, 89]), (23, [14, 89]), (78, [12, 69]), (78, [23, 69]),$
$(78, [34, 69]), (78, [45, 69]), (34, [25, 89]), (45, [36, 89]), (56, [47, 89]),$
$(56, 78, [49]), (58, [67, 49]), ([12, 34, 56, 89]), ([12, 34, 67, 89]),$
$([12, 45, 67, 89]), ([23, 45, 67, 89]), (45, [12, 36, 89]), (78, [12, 34, 69]),$
$(78, [12, 45, 69]), (78, [23, 45, 69]), (56, [12, 47, 89]), (56, [23, 47, 89]),$
$(23, [14, 56, 89]), (23, [14, 67, 89]), (23, 45, [16, 89]), (34, 56, [27, 89]),$
$(23, 78, [14, 69]), (34, 78, [25, 69]), (25, [34, 16, 89]), (56, 78, [12, 49]),$
$(56, 78, [23, 49]), (36, [45, 27, 89]), (58, [12, 67, 49]), (58, [23, 67, 49]),$
$(34, 56, 78, [29]), (34, 58, [67, 29]), (36, 78, [45, 29]), (38, [45, 67, 29]),$
$(56, 38, [47, 29])$

$\mathcal{X}_7^{-2,-}$:
$([91]), ([91, 23]), ([91, 34]), ([91, 45]), ([91, 56]),$
$([91, 67]), ([91, 78]), (12, [93]), ([91, 23, 45]), ([91, 23, 56]), ([91, 23, 67]),$
$([91, 23, 78]), ([91, 34, 56]), ([91, 34, 67]), ([91, 34, 78]), ([91, 45, 67]),$
$([91, 45, 78]), ([91, 56, 78]), ([91, 23, 45, 67]), ([91, 23, 45, 78]), ([91, 23, 56, 78]),$
$([91, 34, 56, 78]), (34, [91, 25]), (45, [91, 36]), (56, [91, 47]), (67, [91, 58]),$
$(56, [91, 23, 47]), (67, [91, 23, 58]), (34, [91, 25, 67]), (34, [91, 25, 78]),$
$(67, [91, 34, 58]), (45, [91, 36, 78]), (34, 56, [91, 27]), (45, 67, [91, 38]),$
$(36, [91, 27, 45]), (47, [91, 38, 56]), (12, [93, 45]), (12, [93, 56]), (12, [93, 67]),$
$(12, [93, 78]), (12, [93, 45, 67]), (12, [93, 45, 78]), (12, [93, 56, 78]),$
$(12, 56, [93, 47]), (12, 67, [93, 58]), (12, 34, [95]), (14, [23, 95]), (12, 34, [95, 67]),$
$(12, 34, [95, 78]), (14, [23, 95, 67]), (14, [23, 95, 78]), (12, 34, 56, [97]),$
$(14, 56, [23, 97]), (12, 36, [45, 97]), (16, [23, 45, 97]), (16, 34, [25, 97])$

$\mathcal{X}_7^{2,-}$:
$(71,[62,89]),(71,[62,89,45]),(71,[62,89,34]),$
$(71,23,[64,89]),(73,[64,89,12]),(71,56,[42,89]),(51,[42,67,89]),$
$(51,78,[42,69])$

$\mathcal{X}_7^{-4,-}$:
$(82,[73,91]),(82,[73,91,45]),(34,82,[75,91]),$
$(84,[75,91,23]),(12,84,[75,93]),(82,[73,91,56]),(67,82,[53,91]),$
$(62,[53,91,78])$

3. Comparison with [L22]

3.1. In this section we assume that D is even so that $N=D+1$. The vectors $e_1=\{1,2\},e_2=\{2,3\},\dots,e_D=\{D,D+1\}$ form a basis of E_N. When $N\ge 3$ we denote by $e'_1,e'_2,\dots,e'_{D-2}$ the analogous vectors in E_{N-2}; for $k\in[1,D],j\in[1,D-2]$ we have $\iota_k(e'_j)=e_j$ if $j+1\le k-1$, $\iota_k(e'_j)=e_j+e_{j+1}+e_{j+2}$ if $j=k-1$, $\iota_k(e'_j)=e_{j+2}$ if $k\le j,j+1\le N-2$. Let $\mathbf{S}_D^*$ be as in [21, 3.1] where we take $\mathbf{a}=N$. We have a unique bijection $\tau:\mathcal{X}_D\to\mathbf{S}_D^*$ which maps ij to $\lfloor i,j\rfloor\in\mathbf{S}_D^*$ for any $ij\in B$. The bijection $\mathbf{S}_D^*\to E_N$ in [L22, 3.3(a)] becomes via τ a bijection $\tilde\epsilon:\mathcal{X}_D\to E_N$ which, by [L20, L22], satisfies the requirement of 1.18(a) hence is equal to ϵ' in 1.18(a). We show that for any $B\in\mathcal{X}_D$ we have

(a) $\tilde\epsilon(B)=\epsilon(B)$,

so that 1.18(b) holds (for our D). From [L22, 3.2] we have

$$\tilde\epsilon(B)=\sum_{k\in[1,D]}[n_k]\{k,k+1\}+[n_{D+1}]\{1,N\}$$

(sum in E_N) where $n_k=|\{ij\in B;\{k,k+1\}\subset\lfloor i,j\rfloor\}|$ for $k\in[1,D]$ and $n_{D+1}=|B^0|$ (for $n\in\mathbf{N}$ we set $[n]=n(n+1)/2$); we view $[n_k]$ as integers $\mod 2$.

From 1.11(a) we have

$$\epsilon(B)=\sum_{k\in[1,N]}m_k\{k\}$$

(sum in E_N) where $m_k=|\{ij\in B;k\in\lfloor i,j\rfloor\}|$; we view m_k as an integer $\mod 2$.

For any $k\in[1,D+1]$ we set $k'=D+1,k''=2$ (if $k=1$), $k'=k-1,k''=k+1$ (if $k\in[2,D]$), $k'=D,k''=1$ (if $k=D+1$). To prove (a) we must show that $[n_k]+[n_{k'}]=m_k$ for $k\in[1,D+1]$ (equalities in F).

Let $Z_k=\{ij\in B;\{k',k,k''\}\subset\lfloor i,j\rfloor\}$, $Z'_k=\{ij\in B;\{k',k\}\subset\lfloor i,j\rfloor,k''\notin\lfloor i,j\rfloor\}$, $Z''_k=\{ij\in B;\{k,k''\}\subset\lfloor i,j\rfloor,k'\notin\lfloor i,j\rfloor\}$. We have $n_k=|Z_k|+|Z''_k|,n_{k'}=|Z_k|+|Z'_k|,m_k=|Z_k|+|Z'_k|+|Z''_k|$. From 1.8 we see that $|Z'_k|\in\{0,1\},|Z''_k|\in\{0,1\}$ and that at least one of $|Z'_k|,|Z''_k|$ must be zero. From 1.8 we see also that if $|Z_k|>0$ then there is a unique $i_0j_0\in Z_k$ such that $\lfloor i_0,j_0\rfloor\subset\lfloor i,j\rfloor$ for any $ij\in Z_k$. Using $(P3)$ we see that

(b) if $k\in[2,D]$ then there exists $\tilde i\tilde j\in B^1$ such that $k\in[\tilde i,\tilde j]$ and $[\tilde i,\tilde j]\subsetneqq\lfloor i_0,j_0\rfloor$;

(c) if $k=D+1$ then there exists $\tilde{i}\tilde{j}\in B^1$ such that $\tilde{j}=k$ and $[\tilde{i},\tilde{j}]\subsetneqq\lfloor i_0,j_0\rfloor$ (we use that $i_{2s}<D+1$, with notation of $(P1)$);

(d) if $k=1$ then there exists $\tilde{i}\tilde{j}\in B^1$ such that $\tilde{i}=1$ and $[\tilde{i},\tilde{j}]\subsetneqq\lfloor i_0,j_0\rfloor$ (we use that $i_1>1$, with notation of $(P1)$).

In case (b), by the minimality of $\lfloor i_0,j_0\rfloor$ we have $[\tilde{i},\tilde{j}]\notin Z_k$; since $k\in[\tilde{i},\tilde{j}]$ we must have either $[\tilde{i},\tilde{j}]\in Z'_k$ or $[\tilde{i},\tilde{j}]\in Z''_k$. In case (c) we have $[\tilde{i},\tilde{j}]\in Z'_k$; in case (d) we have $[\tilde{i},\tilde{j}]\in Z''_k$. Setting $A=|Z_k|,A'=|Z'_k|,A''=|Z''_k|$, we see that we have either

(i) $A>0,A'=1,A''=0$, or

(ii) $A>0,A'=0,A''=1$, or

(iii) $A=0,A'\in\{0,1\},A''\in\{0,1\}$.

The equality to be proved is

$(A+A')(A+A'+1)/2+(A+A'')(A+A''+1)/2=A+A'+A''$ (in F).

In case (i) this is the same as

$(A+1)(A+2)/2+A(A+1)/2=A+1$,

that is $A^2+2A+1=A+1$ (in F). This is obvious. Now case (ii) is completely similar. In case (iii) we must show that

$A'(A'+1)/2+A''(A''+1)/2=A'+A''$ (in F);

this is obvious when $A'\in\{0,1\},A''\in\{0,1\}$.

This completes the proof of (a).

4. Even special orthogonal groups

4.1. In this section we assume that we are in case 0.1(b) or 0.1(c). In this case, $\mathcal{S}_c$ (see 0.1) admits a fixed point free involution whose orbits are the pairs of unipotent representations in $\mathcal{S}_c$ which have isomorphic restrictions to the corresponding even special orthogonal group. As in 0.1 we identify $\mathcal{S}_c$ with Sy_D^+ (in case 0.1(b)) or with Sy_D^- (in case 0.1(c)); here D is an odd integer ≥ 1. The involution of $\mathcal{S}_c$ becomes the fixed point free involution $(S,T)\mapsto(T,S)$ of Sy_D^+ or Sy_D^-. The set of orbits of this involution can be identified with the set $\mathcal{S}'_c$ of unipotent representations of the special orthogonal group attached to 0.1(b) or 0.1(c).

4.2. Via the bijection $\mathcal{E}_D^+\to Sy_D^+$ (resp. $\mathcal{E}_D^-\to Sy_D^-$) in 1.19, the involution of Sy_D^+ (resp. Sy_D^-) in 4.1 becomes the fixed point free involution $X\mapsto X^!=X+[1,D+1]$ of $\mathcal{E}_D^+$, interchanging $\mathcal{E}_D^{t,+},\mathcal{E}_D^{-t,+}$ for any $t\in 2\mathbf{Z}$, (resp. of $\mathcal{E}_D^-$, interchanging $\mathcal{X}_D^{t,-},\mathcal{X}_D^{-t-2,-}$ for any $t\in 2\mathbf{Z}$). Via the bijections 1.16(c) this becomes a fixed point free involution $B\mapsto B^!$ of $\mathcal{X}_D^+$, interchanging $\mathcal{X}_D^{t,+},\mathcal{X}_D^{-t,+}$ for any $t\in 2\mathbf{Z}$, (resp. of $\mathcal{X}_D^-$, interchanging $\mathcal{X}_D^{t,-},\mathcal{X}_D^{-t-2,-}$ for any $t\in 2\mathbf{Z}$).

4.3. We define a partition $\mathcal{E}_D^{0,+}={}'\mathcal{E}_D^{0,+}\sqcup{}''\mathcal{E}_D^{0,+}$ by

$$'\mathcal{E}_D^{0,+}=\{X\in\mathcal{E}_D^{0,+};D+1\notin X\},$$

$$''\mathcal{E}_D^{0,+}=\{X\in\mathcal{E}_D^{0,+};D+1\in X\}.$$

Note that the involution $X \mapsto X^!$ interchanges ${}'\mathcal{E}_D^{0,+}, {}''\mathcal{E}_D^{0,+}$.

We define a partition $\mathcal{X}_D^{0,+} = {}'\mathcal{X}_D^{0,+} \sqcup {}''\mathcal{X}_D^{0,+}$ by

$${}'\mathcal{X}_D^{0,+} = \{B \in \mathcal{X}_D^{0,+}; D+1 \notin \mathrm{supp}(B)\},$$

$${}''\mathcal{X}_D^{0,+} = \{B \in \mathcal{X}_D^{0,+}; D+1 \in \mathrm{supp}(B)\}.$$

We show:

(a) *If $B \in {}'\mathcal{X}_D^{0,+}$ then $\epsilon(B) \in {}'\mathcal{E}_D^{0,+}$.*

Indeed, since $\epsilon(B) \in \langle B\rangle$ (see 1.17(b)), we see that $\epsilon(B)$ is the sum in E_N of certain elements $ij \in B$; now each such ij satisfies $i \notin N-1, j \notin N-1$ so that $N-1 \notin \epsilon(B)$, proving (a). We show:

(b) *If $B \in {}''\mathcal{X}_D^{0,+}$ then $\epsilon(B) \in {}''\mathcal{E}_D^{0,+}$.*

By assumption there exists $ij \in B$ such that $i = N-1$ or $j = N-1$; the first possibility does not occur since $B^0 = \emptyset$ and $N \notin \mathrm{supp}(B)$. Thus we have $\{h, N-1\} \in B$ for some $h \in [1, N-2]$; since $B \in \mathcal{P}_N$, such h is in fact unique. If $ab \in B - \{h, N-1\}$ then we have $ab \in B^!$ since $B^0 = \emptyset$ and $a < b < N-1$. We have $\epsilon(B) = [h, N-1] + \sum_{ab \in B - \{h,N-1\}} [a, b]$ where the last sum does not involve $N-1$; thus $N-1$ appears with coefficient 1 in $\epsilon(B)$ so that $N-1 \in \epsilon(B)$, proving (b).

From (a),(b) we see that the bijection $\mathcal{X}_D^{0,+} \to \mathcal{E}_D^{0,+}$ in 1.16(b) restricts to bijections ${}'\mathcal{X}_D^{0,+} \to {}'\mathcal{E}_D^{0,+}$, ${}''\mathcal{X}_D^{0,+} \to {}''\mathcal{E}_D^{0,+}$. It follows that the involution $B \mapsto B^!$ interchanges ${}'\mathcal{X}_D^{0,+}, {}''\mathcal{X}_D^{0,+}$.

4.4. One can verify that the following properties of $\le_D$ hold.

If $X \in \mathcal{E}_D^{t,+}, X' \in \mathcal{E}_D^{t',+}$ are such that $X \le_D X'$ (with $t \in 2\mathbf{Z}, t' \in 2\mathbf{Z}$), then we have $t = t'$ or $\max(t, -t) < \max(t', -t')$; if in addition $X' \in {}'\mathcal{E}_D^{0,+}$, then $X \in {}'\mathcal{E}_D^{0,+}$.

If $X \in \mathcal{E}_D^{t,-}, X' \in \mathcal{E}_D^{t',-}$ are such that $X \le_D X'$ (with $t \in 2\mathbf{Z}, t' \in 2\mathbf{Z}$), then we have $t = t'$ or $\max(t, -t-2) < \max(t', -t'-2)$.

4.5. Let

$\mathcal{E}_D^{++} = \sqcup_{t \in 2\mathbf{Z}, t>0} \mathcal{E}_D^{t,+} \sqcup {}'\mathcal{E}_D^{0,+}$,

$\mathcal{E}_D^{+-} = \sqcup_{t \in 2\mathbf{Z}, t<0} \mathcal{E}_D^{t,+} \sqcup {}'\mathcal{E}_D^{0,+}$,

$\mathcal{E}_D^{-+} = \sqcup_{t \in 2\mathbf{Z}, t\ge 0} \mathcal{E}_D^{t,-}$,

$\mathcal{E}_D^{--} = \sqcup_{t \in 2\mathbf{Z}, t<0} \mathcal{E}_D^{t,-}$.

Note that each of $\mathcal{E}_D^{++}, \mathcal{E}_D^{+-}$ is a set of representatives for the orbits of the involution $X \mapsto X^!$ of $\mathcal{E}_D^{+}$ and that each of $\mathcal{E}_D^{-+}, \mathcal{E}_D^{--}$ is a set of representatives for the orbits of the involution $X \mapsto X^!$ of $\mathcal{E}_D^{-}$.

For X, X' in $\mathcal{E}_D^{++}$ let $M_{X,X'}^{++} = |\{Z \in \{X, X^!\}; M_{Z,X'}^{+} = 1\}|$.

For X, X' in $\mathcal{E}_D^{+-}$ let $M_{X,X'}^{+-} = |\{Z \in \{X, X^!\}; M_{Z,X'}^{+} = 1\}|$.

For X, X' in $\mathcal{E}_D^{-+}$ let $M_{X,X'}^{-+} = |\{Z \in \{X, X^!\}; M_{Z,X'}^{-} = 1\}|$.

For X, X' in $\mathcal{E}_D^{--}$ let $M_{X,X'}^{--} = |\{Z \in \{X, X^!\}; M_{Z,X'}^{-} = 1\}|$.

From 4.4 we see that $(M^{++}_{X,X'})$, $(M^{+-}_{X,X'})$, $(M^{-+}_{X,X'})$, $(M^{--}_{X,X'})$ are upper triangular matrices with entries in $\{0,1,2\}$ and with 1 on diagonal. It follows that

(a) $\sum_{X\in\mathcal{E}_D^{++}} M^{++}_{X,X'}X' \in \mathbf{Z}[\mathcal{E}_D^{++}]$ (for various $X'\in\mathcal{E}_D^{++}$) form a $\mathbf{Z}$-basis of $\mathbf{Z}[\mathcal{E}_D^{++}]$ and

(b) $\sum_{X\in\mathcal{E}_D^{+-}} M^{+-}_{X,X'}X' \in \mathbf{Z}[\mathcal{E}_D^{+-}]$ (for various $X'\in\mathcal{E}_D^{+-}$) form a $\mathbf{Z}$-basis of $\mathbf{Z}[\mathcal{E}_D^{+-}]$.

It also follows that

(c) $\sum_{X\in\mathcal{E}_D^{-+}} M^{-+}_{X,X'}X' \in \mathbf{Z}[\mathcal{E}_D^{-+}]$ (for various $X'\in\mathcal{E}_D^{-+}$) form a $\mathbf{Z}$-basis of $\mathbf{Z}[\mathcal{E}_D^{-+}]$ and

(d) $\sum_{X\in\mathcal{E}_D^{--}} M^{--}_{X,X'}X' \in \mathbf{Z}[\mathcal{E}_D^{--}]$ (for various $X'\in\mathcal{E}_D^{--}$) form a $\mathbf{Z}$-basis of $\mathbf{Z}[\mathcal{E}_D^{--}]$.

4.6. Let $\overline{\mathcal{E}}_D^+$ (resp. $\overline{\mathcal{E}}_D^-$) be the set of orbits of the fixed point free involution $X\mapsto X+[1,D+1]$ of $\mathcal{E}_D^+$ (resp. $\mathcal{E}_D^-$). The orbit maps $\mathcal{E}_D^+\to\overline{\mathcal{E}}_D^+$, $\mathcal{E}_D^-\to\overline{\mathcal{E}}_D^-$ define bijections $\mathcal{E}_D^{++}\to\overline{\mathcal{E}}_D^+$, $\mathcal{E}_D^{+-}\to\overline{\mathcal{E}}_D^+$, $\mathcal{E}_D^{-+}\to\overline{\mathcal{E}}_D^-$, $\mathcal{E}_D^{--}\to\overline{\mathcal{E}}_D^-$ from which we get bijections $\mathcal{E}_D^{++}\to\mathcal{S}'_c$, $\mathcal{E}_D^{+-}\to\mathcal{S}'_c$ (in case 0.1(b)) and $\mathcal{E}_D^{-+}\to\mathcal{S}'_c$, $\mathcal{E}_D^{--}\to\mathcal{S}'_c$ (in case 0.1(c)).

Hence 4.5(a),(b) can be viewed as bases of the Grothendieck group $\mathbf{Z}[\mathcal{S}'_c]$ (in case 0.1(b)) and 4.5(c),(d) can be viewed as bases of the Grothendieck group $\mathbf{Z}[\mathcal{S}'_c]$ (in case 0.1(c)).

References

[L84] G. Lusztig, *Characters of reductive groups over a finite field*, Ann.Math.Studies 107, Princeton U.Press 1984.

[L20] G. Lusztig, *The Grothendieck group of unipotent representations: a new basis*, Represent.Th. **24** (2020), 178-209.

[L22] G. Lusztig, *A parametrization of unipotent representations*, Bull. Inst. Math. Acad. Sinica (2022) (to appear).

Department of Mathematics, M.I.T., Cambridge, MA 02139