

On the interaction of a magnetic quadrupole moment with an electric field in a rotating frame

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Abstract

We discuss results obtained recently for a quantum-mechanical model given by a neutral particle with a magnetic quadrupole moment in a radial electric field and a scalar potential proportional to the radial variable in cylindrical coordinates that also includes the noninertial effects of a rotating reference frame. We show that the conjectured allowed values of the cyclotron frequency are a mere artifact of the truncation of the power series used to solve the radial eigenvalue equation. Our analysis proves that the analytical expression for the eigenvalues are far from correct.

1 Introduction

In a paper published recently Fonseca and Bakke [1] discussed a neutral particle with a magnetic quadrupole moment in a radial electric field and a scalar potential proportional to the radial variable in cylindrical coordinates. They also considered the noninertial effects of a rotating reference frame. The Schrödinger equation for this quantum-mechanical model is separable in cylindrical coordinates and the authors solved the radial eigenvalue equation by means of the

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Frobenius (power-series) method. Upon forcing the truncation of the series they derived exact analytical polynomial expressions for the radial eigenfunctions as well as an exact analytical formula for the energy levels. They concluded that there are some permitted cyclotron frequencies determined by the angular velocity of the rotating frame, the parameter associated to the scalar potential and the quantum numbers. In this paper we test the validity of those results and conclusions.

In section 2 we discuss the application of the Frobenius method to the radial eigenvalue equation and in section 3 we summarize the main results and draw conclusions.

2 The Frobenius method

The starting point of our discussion is the eigenvalue equation

$$\begin{aligned} F''(r) + \frac{1}{r}F'(r) - \frac{l^2}{r^2}F(r) - r^2F(r) - \nu rF(r) + WF(r) &= 0, \\ W &= \frac{4}{\alpha} \left(E + \frac{1}{2}\vartheta l + l\varpi \right), \quad \alpha^2 = \vartheta^2 + 4\varpi\vartheta, \\ \nu &= \frac{2^{5/2}a}{\sqrt{m\alpha^3}}, \end{aligned} \tag{1}$$

where m is the mass of the particle, E the energy, $l = 0, \pm 1, \pm 2, \dots$ the rotational quantum number, a a constant in the scalar potential, ϑ the cyclotron frequency and ϖ the angular velocity of the rotating frame [1]. We can draw some straightforward conclusions from this equation. Since the behaviour of $F(r)$ at $r \rightarrow 0$ and $r \rightarrow \infty$ is determined by the terms l^2/r^2 and r^2 , respectively, we conclude that there are square integrable solutions for all values of ν . Therefore, there is no restriction on the values of the cyclotron frequency ϑ , contrary to what the authors stated.

There are square integrable solutions

$$\int_0^\infty |F(r)|^2 r dr < \infty, \tag{2}$$

for particular values of $W = W_{i,l}(\nu)$, $i = 0, 1, \dots$, that are continuous functions

of ν . Besides, from the Hellmann-Feynman theorem (HFT) [2, 3]

$$\frac{\partial W}{\partial \nu} = \langle r \rangle > 0, \quad (3)$$

we conclude that each $W_{i,l}(\nu)$ is an increasing function of ν .

Fonseca and Bakke [1] focused on polynomial solutions to equation (1) that we discuss in what follows. If we look for a solution of the form

$$F(r) = r^s \exp\left(-\frac{r^2}{2} - \frac{\nu r}{2}\right) \sum_{j=0}^{\infty} c_j r^j, \quad s = |l|, \quad (4)$$

we conclude that the expansion coefficients should satisfy the three-term recurrence relation

$$\begin{aligned} c_{j+2} &= A_j c_{j+1} + B_j c_j, \quad j = -1, 0, 1, \dots, \quad c_{-1} = 0, \\ A_j &= \frac{\nu(2j+2s+3)}{2(j+2)[j+2(s+1)]}, \\ B_j &= -\frac{4W - 8j + \nu^2 - 8(s+1)}{4(j+2)[j+2(s+1)]}. \end{aligned} \quad (5)$$

In order to obtain a polynomial solution of order n , $n = 0, 1, \dots$, we require that $c_n \neq 0$, $c_{n+1} = 0$ and $c_{n+2} = 0$ that leads to $B_n = 0$. From the last equality we obtain

$$W = W_l^{(n)} = 2(n+s+1) - \frac{\nu^2}{4}, \quad (6)$$

that leads to

$$B_j = B_{j,n} = \frac{2(j-n)}{(j+2)[j+2(s+1)]}. \quad (7)$$

We immediately realize that something is amiss because the eigenvalues $W_l^{(n)}$ do not satisfy the HFT (3).

Since c_{n+1} is a polynomial function of ν of degree $n+1$, the second condition $c_{n+1} = 0$ leads to $n+1$ particular values of ν , $\nu_{n,i,l}$, $i = 1, 2, \dots, n+1$. From these roots Fonseca and Bakke [1] concluded that “only specific values of the cyclotron frequency ϑ are permitted”. However, they did not attempt to investigate the actual meaning of these values of the model parameter ν . For convenience, here we organize them in decreasing order $\nu_{n,i,l} > \nu_{n,i+1,l}$. We appreciate that for each value of n the resulting eigenvalues (6)

$$W_l^{(n,i)} = 2(n+s+1) - \frac{\nu_{n,i,l}^2}{4}, \quad (8)$$

are located on an inverted parabola. When $n + 1$ is odd there is a root $\nu = 0$ that leads to the exact eigenvalues of the harmonic oscillator.

The truncation of the series in equation (4) leads to particular polynomial solutions of the form

$$F_l^{(n,i)}(r) = r^{|l|} \exp\left(-\frac{r^2}{2} - \frac{\nu_{n,i,l}r}{2}\right) \sum_{j=0}^n c_{j,n,l} r^j. \quad (9)$$

They are square integrable but there are other solutions that satisfy the condition (2) that do not have polynomial factors. Since Fonseca and Bakke overlooked the latter, they drew the wrong conclusion mentioned above.

This kind of problems is commonly called quasi-exactly solvable or conditionally solvable because one obtains eigenvalues W only for particular values of ν . They have been studied by several authors (see, for example, the review by Turbiner [4] and the references therein). Fonseca and Bakke [1] seemed to be unaware of this fact and appeared to believe that the only quadratically integrable solutions to equation (1) are the polynomial ones (9). For this reason they concluded, wrongly, that there are allowed values of the cyclotron frequency ϑ . The true fact is that the allowed energies $E_{n,l}$ are continuous functions of this model parameter.

There is no doubt that $W_l^{(n,i)}$ and $F_l^{(n,i)}(r)$ are eigenvalues and eigenfunctions, respectively, of the differential equation (1). A question now arises about the connection between the eigenvalues $W_l^{(n,i)}$ of such polynomial solutions and the actual eigenvalues $W_{j,l}(\nu)$ mentioned above. Taking into account the HFT (3) and the convenient ordering of the roots $\nu_{n,i,l}$ chosen above, we conclude that $W_l^{(n,i)}(\nu_{n,i,l}) = W_{i-1,l}(\nu_{n,i,l})$; in other words, $W_l^{(n,i)}(\nu_{n,i,l})$ is a particular point on the continuous curve $W_{i-1,l}(\nu)$.

Figure 1 shows some selected points $W_0^{(n,i)}(\nu_{n,i,0})$, $n \leq 22$, $i \leq \min(n+1, 3)$, connected by continuous lines that draw the curves $W_{j,0}(\nu)$, $j = 0, 1, 2$. The inverted parabola $W_0^{(10)}(\nu) = 22 - \nu^2/4$ connects some of the solutions $W_0^{(10,i)}$ given by equation (8). The intersections of the vertical dashed line with the curves $W_{j,0}(\nu)$ are the actual eigenvalues of the quantum model with a given value of the parameter ν . Such a vertical line passes through, at most, one value

of $W_0^{(n,i)}(\nu_{n,i,0})$ as shown in figure 1.

In order to obtain $W_{j-,l}(\nu)$ for $\nu_{n,j,l} < \nu < \nu_{n+1,j,l}$, we simply resort to any suitable interpolation method. For example, from least squares we obtain

$$\begin{aligned} W_{0,0}(\nu) &= 2 + 0.8523002844\nu - 0.02975046592\nu^2 + 0.0008706577439\nu^3, \\ W_{1,0}(\nu) &= 6 + 1.547791990\nu - 0.04202730246\nu^2 + 0.001218822726\nu^3, \\ W_{2,0}(\nu) &= 10 + 2.010156364\nu - 0.04562156939\nu^2 + 0.001269456909\nu^3, \end{aligned} \tag{10}$$

that are sufficiently accurate in the range of ν values shown in figure 1.

3 Conclusions

The eigenvalues $W_{i,l}(\nu)$ of the differential equation (1) are continuous functions of the model parameter ν . Consequently, the energy eigenvalues $E_{n,l}$ are continuous functions of the cyclotron frequency ϑ that can take any physically acceptable value. The truncation of the series in equation (4) only yields eigenvalues $W_l^{(n,i)}$ for particular values $\nu_{n,i,l}$. From this particular values of ν Fonseca and Bakke [1] concluded, wrongly, that there are allowed or permitted values of the cyclotron frequency ϑ . The eigenvalues $\mathcal{E}_{n,l}$ in equation (17) of the paper by Fonseca and Bakke are meaningless because the model parameters change with the quantum numbers n and l . More precisely, $\mathcal{E}_{n,l}$ and $\mathcal{E}_{n',l'}$ are energy eigenvalues of different physical problems.

References

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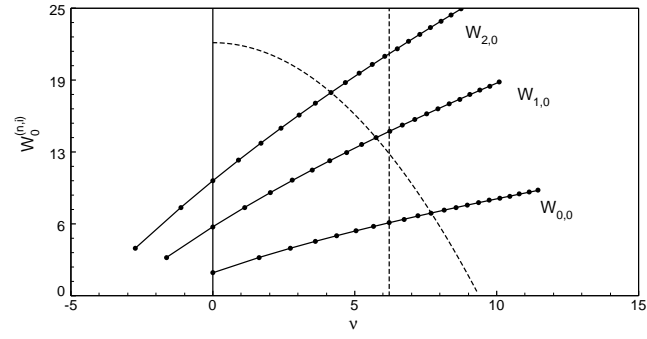


Figure 1: Eigenvalues $W_0^{(n,i)}$ from the truncation method and actual eigenvalues $W_{n,0}(\nu)$ of the differential equation (1)