CODERIVED AND CONTRADERIVED CATEGORIES OF LOCALLY PRESENTABLE ABELIAN DG-CATEGORIES

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Abstract. The concept of an abelian DG-category, introduced by the first-named author in [52], unites the notions of abelian categories and (curved) DG-modules in a common framework. In this paper we consider coderived and contraderived categories in the sense of Becker. Generalizing some constructions and results from the preceding papers by Becker [5] and by the present authors [55], we define the contraderived category of a locally presentable abelian DG-category $\mathbf{B}$ with enough projective objects and the coderived category of a Grothendieck abelian DG-category $\mathbf{A}$. We construct the related abelian model category structures and show that the resulting exotic derived categories are well-generated. Then we specialize to the case of a locally coherent Grothendieck abelian DG-category $\mathbf{A}$, and prove that its coderived category is compactly generated by the absolute derived category of finitely presentable objects of $\mathbf{A}$, thus generalizing a result from the second-named author's preprint [62]. In particular, the homotopy category of graded-injective left DG-modules over a DG-ring with a left coherent underlying graded ring is compactly generated by the absolute derived category of DG-modules with finitely presentable underlying graded modules. We also describe compact generators of the coderived categories of quasi-coherent matrix factorizations over coherent schemes.

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Introduction

To any abelian category E, one can assign its unbounded derived category $\mathsf{D}(\mathsf{E})$. To any DG-ring $\boldsymbol{R}^\bullet = (R^*, d)$, one can assign the derived category $\mathsf{D}(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ of

DG-modules over $\boldsymbol{R}^\bullet$. Is there a natural common generalization of these two constructions? In this paper, following a preceding paper [52], we suggest a framework providing an answer to this question, but there is a caveat. We consider *derived categories of the second kind* instead of the conventional derived categories (the terminology is inspired by related constructions of derived functors in [26]).

Simply put, this means that the homotopy category $\mathsf{H}^0(\mathbf{C}(\mathsf{E}_{\mathsf{inj}}))$ of unbounded complexes of injective objects in E and the homotopy category $\mathsf{H}^0(\mathbf{C}(\mathsf{E}_{\mathsf{proj}}))$ of unbounded complexes of projective objects in E are our two versions of the derived category of E. Similarly, the homotopy category $\mathsf{H}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}_{\mathbf{inj}})$ of DG-modules $\boldsymbol{J}^\bullet = (J^*, d_J)$ over $\boldsymbol{R}^\bullet$ with injective underlying graded R^*-modules J^* and the homotopy category $\mathsf{H}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}_{\mathbf{proj}})$ of DG-modules $\boldsymbol{P}^\bullet = (P^*, d_P)$ over $\boldsymbol{R}^\bullet$ with projective underlying graded R^*-modules P^* are our two versions of the derived category of DG-modules over $\boldsymbol{R}^\bullet$.

Derived categories of the second kind were, in fact, recently studied in numerous contexts under various names. In connection with representation theory of finite groups and commutative algebra, they were studied by Jørgensen [28] and Krause [33]. They also naturally appear in the context of Grothendieck duality in the work of Krause and Iyengar [27] and Gaitsgory and Rozenblyum [17, 18] (the derived ind-completion of the bounded derived category of coherent sheaves on a nice enough scheme X is equivalent to $\mathsf{H}^0(\mathbf{C}(\mathsf{E}_{\mathsf{inj}}))$ for $\mathsf{E} = X\text{–}\mathsf{qcoh}$, essentially by [33]). Lurie studied their universal properties under the name "unseparated derived category" [39, Appendix C.5]. They play an essential role in understanding the Koszul duality (see e. g. [36, 48, 53]). Last but not least, Becker [5] significantly contributed to the theory of derived categories of the second kind for curved DG-modules, motivated by the special case of categories of matrix factorizations (whose theory, going back to [14], requires derived categories of the second kind for its full development, as shown in [45, 13]).

We will use the terminology from the paper by Becker [5] and name the homotopy category of unbounded complexes of injective objects or the homotopy category of graded-injective DG-modules the *coderived category in the sense of Becker*. Dually, we call the homotopy category of unbounded complexes of projective objects or the homotopy category of graded-projective DG-modules the *contraderived category in the sense of Becker*. To be precise, the definitions of Becker's contraderived and coderived categories in Sections 6.3 and 7.3 are spelled out in a more fancy way: these derived categories of the second kind are defined as certain Verdier quotient categories of the homotopy category of arbitrary complexes or (curved) DG-modules, rather than subcategories. But then Corollaries 6.14 and 7.10 tell that, in particular, in the case of DG-modules, the Becker contraderived and coderived categories as defined in Sections 6.3 and 7.3 agree with the homotopy categories of graded-projective and graded-injective DG-modules, respectively.

Let us now explain what we believe is a main contribution of the present paper. The second-named author proved the following fact in [62] (which was one of the main results there):

Theorem 0.1 ([62, Corollary 6.13]). *Let A be a locally coherent Grothendieck abelian category and $\mathsf{H}^0(\mathbf{C}(\mathsf{A}_{\mathsf{inj}}))$ be the homotopy category of unbounded complexes of injective objects in A. Then $\mathsf{H}^0(\mathbf{C}(\mathsf{A}_{\mathsf{inj}}))$ is a compactly generated triangulated category. The full subcategory of compact objects in $\mathsf{H}^0(\mathbf{C}(\mathsf{A}_{\mathsf{inj}}))$ is equivalent to the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{A}_{\mathsf{fp}})$ of the abelian category of finitely presentable objects in A.*

This was already a substantial generalization of Krause's [33, Theorem 1.1(2)], where the same result was obtained for A locally Noetherian. Here we go further and prove the following generalization of Theorem 0.1, where by a locally coherent abelian DG-category, we mean a DG-category $\mathbf{A}$ with finite direct sums, shifts and cones whose underlying additive category $\mathsf{Z}^0(\mathbf{A})$ is a locally coherent Grothendieck category.

Theorem 0.2 (Theorem 8.19 below). *Let $\mathbf{A}$ be a locally coherent abelian DG-category and $\mathsf{D}^{\mathsf{bco}}(\mathbf{A}) = \mathsf{H}^0(\mathbf{A}_{\mathbf{inj}})$ be its Becker's coderived category, that is, the homotopy category of graded-injective objects in $\mathbf{A}$. Then $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is a compactly generated triangulated category. Up to adjoining direct summands, the full subcategory of compact objects in $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is equivalent to the absolute derived category $\mathsf{D}^{\mathsf{abs}}(\mathbf{A}_{\mathbf{fp}})$ of the abelian DG-category of finitely presentable objects in $\mathbf{A}$.*

We will explain details in the following paragraphs. At the moment, we just point out that when specializing to the abelian DG-categories of curved DG-modules, we obtain the following corollary, which seems to be new as well. (The reader would not lose much by assuming further that $h = 0$, in which case $\boldsymbol{R}^\bullet = (R^*, d)$ is a DG-ring and the curved DG-modules mentioned in the corollary are simply the DG-modules; but curved DG-modules are a more natural context for derived categories of the second kind.)

Corollary 0.3 (Corollary 8.20 below). *Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a curved DG-ring whose underlying graded ring R^* is graded left coherent (i. e., all finitely generated homogeneous left ideals in R^* are finitely presentable). Then the homotopy category $\mathsf{H}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}_{\mathbf{inj}})$ of left curved DG-modules over $\boldsymbol{R}^\bullet$ with injective underlying graded R^*-modules is compactly generated. Up to adjoining direct summands, the full subcategory of compact objects in $\mathsf{H}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}_{\mathbf{inj}})$ is equivalent to the absolute derived category $\mathsf{D}^{\mathsf{abs}}(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}_{\mathbf{fp}})$ of left curved DG-modules over $\boldsymbol{R}^\bullet$ with finitely presentable underlying graded R^*-modules.*

We also work out the following application of Theorem 0.2 to quasi-coherent matrix factorizations on coherent schemes, generalizing the previously known result for Noetherian schemes [13, Proposition 1.5(d) and Corollary 2.3(l)]. If X is affine and regular and $L = O_X$, the absolute derived category below is none other than the homotopy category of classical matrix factorizations with projective components (there, compact generators were studied in detail in special cases e.g. in [11]).

Corollary 0.4 (Corollary 9.4 below). *Let X be coherent scheme (i. e., X is quasi-compact, quasi-separated, and locally coherent). Let L be a line bundle (invertible sheaf) on X and $w \in L(X)$ be a global section. Then the homotopy category*

$\mathsf{H}^0(\mathbf{F}_{\mathrm{qc}}(X, L, w)_{\mathbf{inj}})$ *of injective quasi-coherent factorizations of the potential* w *on* X *is compactly generated. Up to adjoining direct summands, the full subcategory of compact objects in* $\mathsf{H}^0(\mathbf{F}_{\mathrm{qc}}(X, L, w)_{\mathbf{inj}})$ *is equivalent to the absolute derived category of coherent factorizations* $\mathsf{D}^{\mathsf{abs}}(\mathbf{F}_{\mathrm{coh}}(X, L, w))$.

Along with developing general theory of the derived categories of the second kind for abelian DG-categories, we also clarify certain rather basic aspects of their construction. For example, the ordinary derived category $\mathsf{D}(\mathsf{E})$ of an abelian category E is usually constructed as a Verdier quotient $\mathsf{H}^0(\mathbf{C}(\mathsf{E}))/\mathsf{H}^0(\mathbf{C}(\mathsf{E}))_{\mathsf{ac}}$ of the homotopy category of complexes modulo the full subcategory of acyclic complexes. It has been known for some time [20, 55] that one can construct Becker's coderived category of a Grothendieck abelian category A in the same way as $\mathsf{H}^0(\mathbf{C}(\mathsf{A}))/\mathsf{H}^0(\mathbf{C}(\mathsf{A}))^{\mathsf{bco}}_{\mathsf{ac}}$. However, the corresponding subcategory $\mathsf{H}^0(\mathbf{C}(\mathsf{A}))^{\mathsf{bco}}_{\mathsf{ac}}$ of the so-called *Becker-coacyclic objects* did not have an easy description. Here we prove in Corollary 7.17 (in fact, in a much more general setting) that the class of Becker-coacyclic objects viewed in the Grothendieck category of complexes $\mathsf{Z}^0(\mathbf{C}(\mathsf{A}))$ is given precisely as the closure of the class of contractible complexes under extensions and directed colimits. If we turn back to the locally coherent case, we obtain the following consequence.

Corollary 0.5. *Let* A *be a locally coherent Grothendieck abelian category. Then the class of Becker-coacyclic objects coincides in* $\mathsf{Z}^0(\mathbf{C}(\mathsf{A}))$ *with the direct limit closure of the subcategory of acyclic bounded complexes of finitely presentable objects.*

This is an amazingly simple description, and the general form for locally coherent abelian DG-categories (see Proposition 8.13), which in particular includes the case of (curved) DG-modules studied by Becker [5], is equally nice. Again, this appears to be completely new.

Now we are going to introduce our aims and the our setting more in detail.

0.1. **The starting point.** Speaking of DG-modules, one observes that the category of DG-rings is a (nonfull) subcategory in a larger category of so-called *curved DG-rings* (*CDG-rings*), and the assignment of the DG-category of DG-modules to a DG-ring extends naturally to an assignment of the DG-category of *CDG-modules* to any curved DG-ring [46, 48]. As mentioned, CDG-rings and CDG-modules, rather than DG-rings and DG-modules, form a natural context for the homotopy categories of graded-injective and graded-projective differential modules. The theory of the homotopy categories of graded-injective and graded-projective CDG-modules over a CDG-ring, together with the related complete cotorsion pairs and abelian model structures, was developed in Becker's paper [5]. The relevant result in [5] is [5, Proposition 1.3.6].

Speaking of abelian categories, a natural generality for the derived categories of the second kind in the sense of Becker is achieved in certain classes of locally presentable abelian categories. The natural context for the homotopy category of unbounded complexes of injective objects $\mathsf{H}^0(\mathbf{C}(\mathsf{A}_{\mathsf{inj}}))$ seems to be that of Grothendieck abelian categories A. Dually, the natural generality level for the homotopy category of unbounded complexes of projective objects $\mathsf{H}^0(\mathbf{C}(\mathsf{B}_{\mathsf{proj}}))$ is that of locally presentable

$$\begin{tikzcd} \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) \arrow[d] \\ R^*\text{–}\mathsf{Mod} \arrow[u, bend left=60, "G^+"] \arrow[u, bend right=60, "G^-=G^+[1]"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathsf{Z}^0(\mathbf{E}) \arrow[d, "\Phi_{\mathbf{E}}"] \\ \mathsf{Z}^0(\mathbf{E}^\natural) \arrow[u, bend left=60, "\Psi^+_{\mathbf{E}}"] \arrow[u, bend right=60, "\Psi^-_{\mathbf{E}}=\Psi^+_{\mathbf{E}}[1]"'] \end{tikzcd}$$

FIGURE 1. Two "underlying" additive categories of a DG-category.

abelian categories B with enough projective objects. This is the point of view suggested in the present authors' paper [55].

The coderived abelian model structure on the category of complexes in a Grothendieck abelian category A was constructed by Gillespie in [20, Theorem 4.2]; another exposition can be found in [55, Section 9]. Inverting the weak equivalences in this model category structure produces the homotopy category of unbounded complexes of injectives in A. For discussions of the homotopy category of unbounded complexes of injective objects in a Grothendieck category *not* based on the notions of cotorsion pairs and abelian model structures, see [43, 35]. The contraderived abelian model structure on the category of complexes in a locally presentable abelian category B with enough projective objects was constructed in [55, Section 7]. Inverting the weak equivalences in this model category structure produces the homotopy category of unbounded complexes of projectives in B.

0.2. **Abelian DG-categories.** The unifying framework of *abelian* (and *exact*) *DG-categories* was suggested in the first-named author's paper [52]. A simple definition of an abelian DG-category $\mathbf{E}$ is that $\mathbf{E}$ is a DG-category with finite direct sums, shifts, and cones such that the additive category $\mathsf{Z}^0(\mathbf{E})$ of closed morphisms of degree 0 in $\mathbf{E}$ is abelian.

A more substantial approach is based on the idea that there are *two* abelian categories naturally associated with an abelian DG-category. For example, to the DG-category $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ of DG-modules over a DG-ring $\boldsymbol{R}^\bullet = (R^*, d)$ one can assign the abelian category $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ of DG-modules over $\boldsymbol{R}^\bullet$ and closed morphisms of degree 0 between them, and also the abelian category R^*–Mod of graded R^*-modules and homogeneous morphisms of degree 0 between them. The proper theory of $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ as an abelian DG-category must involve both the abelian categories $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ and R^*–Mod, as well as naturally defined functors acting between these abelian categories. Such functors include not only the forgetful functor $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) \longrightarrow R^*\text{–}\mathsf{Mod}$ assigning to a DG-module its underlying graded module, but also the two functors G^+ and $G^-\colon R^*\text{–}\mathsf{Mod} \longrightarrow \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ left and right adjoint to the forgetful functor, which happen to differ by a shift: $G^- = G^+[1]$ (see the left-hand side of Figure 1).

Given a DG-category $\mathbf{E}$ (with finite direct sums), one would like to construct an additive category which would serve as "the category of underlying graded objects" for the objects of $\mathbf{E}$. A naïve approach would be to consider the additive category

$\mathbf{E}^0$ whose objects are the objects of $\mathbf{E}$ and whose morphisms are the arbitrary (not necessarily closed) morphisms of degree 0 in $\mathbf{E}$. The problem with this construction is that, taking $\mathbf{E} = \boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ to be the DG-category of DG-modules over a DG-ring $\boldsymbol{R}^\bullet = (R^*, d)$, the additive category $\mathbf{E}^0$ is usually *not* abelian, and generally not well-behaved. The reason is that, while the morphisms in $\mathbf{E}^0$ are what the morphisms in the category of underlying graded objects are supposed to be, there are too many "missing objects" in $\mathbf{E}^0$. Simply put, an arbitrary graded R^*-module does *not* admit a DG-module structure over (R^*, d) in general.

The main construction of the paper [52] (first suggested in the memoir [48, Section 3.2]) assigns to a DG-category $\mathbf{E}$ another DG-category $\mathbf{E}^\natural$. Assuming, as we will always do, that the DG-category $\mathbf{E}$ has finite direct sums, shifts, and cones, the DG-category $\mathbf{E}^\natural$ comes endowed with an additive functor $\Phi_\mathbf{E}\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ acting between the additive categories of closed morphisms of degree 0 in $\mathbf{E}$ and $\mathbf{E}^\natural$. The functor $\Phi_\mathbf{E}$ has adjoint functors on both sides, denoted by $\Psi^+_\mathbf{E}$ and $\Psi^-_\mathbf{E}\colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$. The two functors Ψ^+ and Ψ^- only differ by a shift: one has $\Psi^-(X) = \Psi^+(X)[1]$ for all $X \in \mathbf{E}^\natural$ (see the right-hand side of Figure 1; so the functor $\Phi_\mathbf{E}$ is actually a part of a doubly infinite ladder of adjoint functors acting between $\mathsf{Z}^0(\mathbf{E})$ and $\mathsf{Z}^0(\mathbf{E}^\natural)$). Furthermore, both the functors Φ and Ψ^+ can be naturally extended to fully faithful functors acting from the additive categories of arbitrary morphisms of degree 0 in $\mathbf{E}$ and $\mathbf{E}^\natural$; so there are fully faithful functors $\widetilde{\Phi}_\mathbf{E}\colon \mathbf{E}^0 \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ and $\widetilde{\Psi}^+_\mathbf{E}\colon (\mathbf{E}^\natural)^0 \longrightarrow \mathsf{Z}^0(\mathbf{E})$.

The existence of a fully faithful functor $\widetilde{\Phi}_\mathbf{E}$ is one way of saying that the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$ is a candidate for the role of the category of underlying graded objects for $\mathbf{E}$, having the correct groups of morphisms between objects coming from $\mathbf{E}$ but also some "missing objects" added. The functor $\Phi_\mathbf{E}$ plays the role of the forgetful functor assigning to a (curved) DG-module its underlying graded module. This in fact works very well in the case of abelian DG-categories, where $\mathsf{Z}^0(\mathbf{E})$ being abelian automatically implies that also $\mathsf{Z}^0(\mathbf{E}^\natural)$ is abelian. If one wishes to generalize the theory from abelian to exact DG-categories (which we are not going to do here), the category $\mathsf{Z}^0(\mathbf{E}^\natural)$ may turn out be too big and one may want to consider so-called "exact DG-pairs" rather than just exact DG-categories (we refer to [52] for details).

The construction assigning the DG-category $\mathbf{E}^\natural$ to a DG-category $\mathbf{E}$ is an "almost involution". For any DG-category $\mathbf{E}$ with finite direct sums, shifts, and cones, there is a fully faithful DG-functor $\natural\natural\colon \mathbf{E} \longrightarrow \mathbf{E}^{\natural\natural}$. The passage from a DG-category $\mathbf{E}$ to the DG-category $\mathbf{E}^{\natural\natural}$ adjoins all twists of objects with respect to Maurer–Cartan cochains in their complexes of endomorphisms (and some direct summands of such twists). In fact, for any DG-category $\mathbf{E}$, all twists exist in the DG-category $\mathbf{E}^\natural$. For an idempotent-complete DG-category $\mathbf{E}$ with finite direct sums, shifts, and twists, the DG-functor $\natural\natural\colon \mathbf{E} \longrightarrow \mathbf{E}^{\natural\natural}$ is an equivalence of DG-categories.

In particular, any abelian DG-category $\mathbf{E}$ is idempotent-complete and has twists. In other words, for any abelian DG-category $\mathbf{E}$, the DG-functor $\natural\natural\colon \mathbf{E} \longrightarrow \mathbf{E}^{\natural\natural}$ is an equivalence of DG-categories. So the passage from $\mathbf{E}$ to $\mathbf{E}^\natural$ is indeed an involution on abelian DG-categories. The reader should be warned that the assignment of $\mathbf{E}^\natural$ to

$\mathbf{E}$ does *not* preserve quasi-equivalences of DG-categories, however. A DG-category being abelian is also a property of this DG-category considered up to equivalence, but *not* up to quasi-equivalence.

0.3. **Derived categories of the second kind.** This is a common name for several constructions, most notably the coderived, contraderived, and *absolute derived* categories. The constructions of derived categories of the second kind are well-defined for abelian (and exact) DG-categories, which seem to form their natural context. Once again, we should however warn the reader that the construction of the conventional derived category does *not* seem to make sense for a general abelian DG-category.

In particular, for any DG-ring $\boldsymbol{R}^\bullet = (R^*, d)$, and in fact for any CDG-ring $\boldsymbol{R}^\bullet = (R^*, d, h)$, one can easily construct an *acyclic* DG-ring $\widehat{\widehat{\boldsymbol{R}}}^\bullet$ such that the DG-category of DG-modules over $\widehat{\widehat{\boldsymbol{R}}}^\bullet$ is equivalent to the DG-category of (C)DG-modules over $\boldsymbol{R}^\bullet$. In this sense, one can say that any CDG-ring, and in particular any DG-ring, is Morita equivalent to an acyclic DG-ring. This kind of Morita equivalence preserves the DG-category of (C)DG-modules, and in fact it preserves the coderived, contraderived, and absolute derived categories. But, of course, the conventional derived category of DG-modules over a DG-ring vanishes if and only if the DG-ring is acyclic.

The time has come for us to say explicitly that the coderived and contraderived categories in the sense of Becker have to be distinguished from the coderived and contraderived category in the sense of the first-named author of this paper, as defined in the monograph [47], the memoir [48], and subsequent publications such as [13, 49]. In fact, it is an open question whether derived categories of the second kind in the sense of Becker are *ever* different from the ones in the sense of Positselski within their common domain of definition. But, at least, as long as the question is open, the distinction has to be made. The coderived and contraderived categories in the sense of Positselski tend to be defined in a greater generality than the ones in the sense of Becker, but the latter may have better properties in the more narrow contexts for which they are suited. We refer to [55, Remark 9.2] or [53, Section 7] for a discussion of the history and philosophy of derived categories of the second kind.

For abelian (and exact) DG-categories, derived categories of the second kind in the sense of Positselski are defined and their theory is developed in the paper [52]. The aim of the present paper is to define and study the coderived and contraderived categories in the sense of Becker for abelian DG-categories. The notion of the absolute derived category (which has no known separate Becker's version, but only one in the sense of the first-named author of the present paper) turns out to play an important role in this study, as we have seen in Theorem 0.2.

0.4. **Locally presentable and Grothendieck DG-categories.** Let us say a few words about locally presentable and Grothendieck DG-categories. Given a DG-category $\mathbf{B}$ with shifts and cones such that all colimits exist in the additive category $\mathsf{Z}^0(\mathbf{B})$, one can show that the additive category $\mathsf{Z}^0(\mathbf{B})$ is locally λ-presentable (for a fixed regular cardinal λ) if and only if the additive category $\mathsf{Z}^0(\mathbf{B}^\natural)$ is locally λ-presentable. If the DG-category $\mathbf{B}$ is abelian, then the abelian category $\mathsf{Z}^0(\mathbf{B})$

has enough projective objects if and only if the abelian category $\mathsf{Z}^0(\mathbf{B}^\natural)$ has enough projective objects. Hence the class of "locally presentable abelian DG-categories $\mathbf{B}$ with enough projective objects", for which the contraderived category in the sense of Becker is well-defined.

Given an abelian DG-category $\mathbf{A}$ with infinite coproducts, the abelian category $\mathsf{Z}^0(\mathbf{A})$ is Grothendieck if and only if the abelian category $\mathsf{Z}^0(\mathbf{A}^\natural)$ is Grothendieck. Hence the class of "Grothendieck abelian DG-categories", for which the coderived category in the sense of Becker is well-behaved. These are the contraderived and coderived categories which we study in the present paper, constructing the contraderived abelian model structure on the abelian category $\mathsf{Z}^0(\mathbf{B})$ in the former case and the coderived abelian model structure on the abelian category $\mathsf{Z}^0(\mathbf{A})$ in the latter case. In particular, we obtain the semiorthogonal decompositions of the homotopy categories $\mathsf{H}^0(\mathbf{B})$ and $\mathsf{H}^0(\mathbf{A})$ associated with the Becker contraderived and coderived categories, and prove that such derived categories of the second kind $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B}) = \mathsf{H}^0(\mathbf{B}_{\mathbf{proj}})$ and $\mathsf{D}^{\mathsf{bco}}(\mathbf{A}) = \mathsf{H}^0(\mathbf{A}_{\mathbf{inj}})$ are well-generated triangulated categories.

Let us reiterate the warning that, similarly to [52], all the theory of DG-categories developed in this paper is a *strict* theory. We work with complexes of morphisms in DG-categories up to (natural) isomorphism, *not* up to quasi-isomorphism; and accordingly consider DG-categories up to equivalence, *not* up to quasi-equivalence. Similarly, a fully faithful DG-functor for us is a DG-functor inducing isomorphisms and *not* just quasi-isomorphisms of the complexes of morphisms.

0.5. **Structure of the paper.** This paper is based on an abundance of preliminary material. The theory of abelian DG-categories, as developed in the paper [52], is summarized for our purposes in Sections 1–3. A summary of the theory of abelian model structures, following the exposition in [55] (which in turn is based on the results of [25, 5, 54]), occupies Sections 4–5.

A construction of the contraderived abelian model structure for a locally presentable abelian DG-category $\mathbf{B}$ with enough projective objects is the main result of Section 6. In Section 7 we not only construct the coderived model structure for a Grothendieck abelian DG-category $\mathbf{A}$, but also show that the class of all Becker-coacyclic objects is closed under directed colimits in $\mathsf{Z}^0(\mathbf{A})$.

Various descriptions of the class of all Becker-coacyclic objects in $\mathbf{A}$ and (in the case of a locally coherent DG-category $\mathbf{A}$) of the class of all absolutely acyclic objects in $\mathbf{A}_{\mathbf{fp}}$ are obtained in Sections 7 and 8. The main results about compact generators of Becker's coderived category in the locally coherent case, formulated above as Theorem 0.2 and Corollary 0.3, are proved at the end of Section 8. The application to algebraic geometry (matrix factorizations on coherent schemes) is discussed in the last Section 9, with Corollary 0.4 proved at the end of the paper.

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1. Preliminaries on DG-Categories

In this paper, we will presume all our graded objects and complexes to be graded by the group of integers $\mathbb{Z}$. The differentials on complexes raise the degree by 1. This follows the convention in [55] and constitutes a restriction of generality as compared to the paper [52], where a grading group Γ is considered. In practice, this is more of a notational convention than an actual change of setting, because all the definitions and arguments can be generalized to a grading group Γ in a straightforward way.

The material below in this section is mostly an extraction from [52, Sections 1–2]. The reader can consult with [48, Section 1.2] and/or [52, Sections 1–2] for details.

1.1. **Enriched categories.** Let C be an (associative, unital) monoidal category with the tensor product operation $\otimes\colon \mathsf{C}\times\mathsf{C}\longrightarrow\mathsf{C}$ and the unit object $\mathbb{I}\in\mathsf{C}$. Then a C-*enriched category* $\mathcal{K}$ consists of a class of objects, an object $\mathcal{H}om_{\mathcal{K}}(X,Y)\in\mathsf{C}$ defined for every pair of objects X, $Y\in\mathcal{K}$, a multiplication (also known as composition) morphism $\mathcal{H}om_{\mathcal{K}}(Y,Z)\otimes\mathcal{H}om_{\mathcal{K}}(X,Y)\longrightarrow\mathcal{H}om_{\mathcal{K}}(X,Z)$ in C defined for every triple of objects X, Y, $Z\in\mathcal{K}$, and a unit morphism $\mathbb{I}\longrightarrow\mathrm{Hom}_{\mathcal{K}}(X,X)$ in C defined for every object $X\in\mathcal{K}$. The usual associativity and unitality axioms are imposed [30, Section 1.2].

The functor $\mathrm{Hom}_{\mathsf{C}}(\mathbb{I},-)$ is a lax monoidal functor from C to the monoidal category of sets Sets (with the Cartesian product in the role of the tensor product in Sets). Consequently, the functor $\mathrm{Hom}_{\mathsf{C}}(\mathbb{I},-)$ takes monoids in C to monoids in Sets, and C-enriched categories to Sets-enriched categories, which means the usual categories. In other words, the *underlying category* $\mathsf{K}=\mathcal{K}_0$ of a C-enriched category $\mathcal{K}$ is defined by the rule $\mathrm{Hom}_{\mathsf{K}}(X,Y)=\mathrm{Hom}_{\mathsf{C}}(\mathbb{I},\mathcal{H}om_{\mathcal{K}}(X,Y))$ (where the class of objects of K coincides with the class of objects of $\mathcal{K}$) [30, Section 1.3].

1.2. **Graded categories.** A *preadditive category* E is a category enriched in (the monoidal category of) abelian groups, with the operation of tensor product of abelian groups $\otimes_{\mathbb{Z}}$ defining the monoidal structure.

A *graded category* $\mathcal{E}$ is a category enriched in (the monoidal category of) graded abelian groups. So, for every pair of objects X, $Y\in\mathcal{E}$, a graded abelian group $\mathrm{Hom}^*_{\mathcal{E}}(X,Y)=\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}^n_{\mathcal{E}}(X,Y)$ is defined, together with the related multiplication maps $\mathrm{Hom}^*_{\mathcal{E}}(Y,Z)\otimes_{\mathbb{Z}}\mathrm{Hom}^*_{\mathcal{E}}(X,Y)\longrightarrow\mathrm{Hom}^*_{\mathcal{E}}(X,Z)$ for all objects X, Y, $Z\in\mathcal{E}$ and unit elements $\mathrm{id}_X\in\mathrm{Hom}^0_{\mathcal{E}}(X,X)$ for all objects $X\in\mathcal{E}$.

The underlying preadditive category $\mathsf{E}=\mathcal{E}^0$ of a graded category $\mathcal{E}$ is defined by the rule $\mathrm{Hom}_{\mathcal{E}^0}(X,Y)=\mathrm{Hom}^0_{\mathcal{E}}(X,Y)$. This is the definition one obtains by specializing to graded categories the construction of the underlying category of a C-enriched category explained above in Section 1.1.

Given an integer $i\in\mathbb{Z}$ and an object $X\in\mathcal{E}$, the *shift* $Y=X[i]$ of the object X is an object of $\mathcal{E}$ endowed with a pair of morphisms $f\in\mathrm{Hom}^{-i}_{\mathcal{E}}(X,Y)$ and $g\in\mathrm{Hom}^i_{\mathcal{E}}(Y,X)$ such that $gf=\mathrm{id}_X$ and $fg=\mathrm{id}_Y$. Given a finite collection of objects X_1, …, $X_n\in\mathcal{E}$, the *direct sum* $\bigoplus_{i=1}^n X_i\in\mathcal{E}$ can be simply defined as the direct sum of the same objects in the preadditive category $\mathcal{E}^0$. A graded category $\mathcal{E}$ is called *additive* if all finite direct sums exist in $\mathcal{E}$, or equivalently, if the preadditive category $\mathcal{E}^0$ is additive (i. e., has finite direct sums).

Let X_α be a (possibly infinite) collection of objects in $\mathcal{E}$. Then the *coproduct* $Y = \coprod_\alpha X_\alpha$ is an object of $\mathcal{E}$ such that a natural isomorphism of graded abelian groups $\mathrm{Hom}^*_{\mathcal{E}}(Y, Z) \simeq \prod_\alpha \mathrm{Hom}^*_{\mathcal{E}}(X_\alpha, Z)$ is defined for all objects $Z \in \mathcal{E}$ in a way functorial with respect to all morphisms $f \in \mathrm{Hom}^*_{\mathcal{E}}(Z, Z')$ in $\mathcal{E}$. Here $\prod_\alpha$ denotes the product functor in the category of graded abelian groups. Dually, the *product* $W = \prod_\alpha X_\alpha$ is an object of $\mathcal{E}$ such that a functorial isomorphism of graded abelian groups $\mathrm{Hom}^*_{\mathcal{E}}(Z, W) \simeq \prod_\alpha \mathrm{Hom}^*_{\mathcal{E}}(Z, X_\alpha)$ is defined for all objects $Z \in \mathcal{E}$.

1.3. **DG-categories and related graded/preadditive categories.** A *DG-category* $\mathbf{E}$ is a category enriched in (the monoidal category of) complexes of abelian groups. So, for every pair of objects X, $Y \in \mathbf{E}$, a complex of abelian groups $\mathrm{Hom}^\bullet_{\mathbf{E}}(X, Y)$ with the components $\mathrm{Hom}^n_{\mathbf{E}}(X, Y)$, $n \in \mathbb{Z}$, and a differential $d\colon \mathrm{Hom}^n_{\mathbf{E}}(X, Y) \longrightarrow \mathrm{Hom}^{n+1}_{\mathbf{E}}(X, Y)$ is defined, together with the related multiplication maps $\mathrm{Hom}^\bullet_{\mathbf{E}}(Y, Z) \otimes_{\mathbb{Z}} \mathrm{Hom}^\bullet_{\mathbf{E}}(X, Y) \longrightarrow \mathrm{Hom}^\bullet_{\mathbf{E}}(X, Z)$, which must be morphisms of complexes of abelian groups for all objects X, Y, $Z \in \mathbf{E}$. The unit elements (identity morphisms) $\mathrm{id}_X \in \mathrm{Hom}^0_{\mathbf{E}}(X, X)$ must be closed morphisms of degree 0, i. e., $d(\mathrm{id}_X) = 0$ for all objects $X \in \mathbf{E}$.

There are three natural functors from the category of complexes of abelian groups to the category of graded abelian groups: to any complex of abelian groups, one can assign its underlying graded abelian group, or its graded subgroup of cocycles, or its graded abelian group of cohomology. The former functor is monoidal, while the latter two functors are lax monoidal; so all the three functors, applied to the complexes of morphisms, transform DG-categories into graded categories.

The underlying graded category $\mathcal{E} = \mathbf{E}^*$ of a DG-category $\mathbf{E}$ can be naïvely defined by the rule that the objects of $\mathcal{E}$ are the objects of $\mathbf{E}$ and $\mathrm{Hom}^*_{\mathcal{E}}(X, Y)$ is the underlying graded abelian group of the complex $\mathrm{Hom}^\bullet_{\mathbf{E}}(X, Y)$ for all objects X and Y. For the proper (non-naïve) construction of the underlying graded category of a DG-category, see Section 2. In particular, the underlying preadditive category $\mathbf{E}^0$ of a DG-category $\mathbf{E}$ is naïvely defined as $\mathbf{E}^0 = (\mathbf{E}^*)^0$; so $\mathrm{Hom}_{\mathbf{E}^0}(X, Y) = \mathrm{Hom}^0_{\mathbf{E}}(X, Y)$.

A morphism $f \in \mathrm{Hom}^n_{\mathbf{E}}(X, Y)$ is said to be *closed* if $d(f) = 0$. The graded category whose objects are the objects of $\mathbf{E}$ and whose graded abelian group of morphisms $X \longrightarrow Y$ is the homogeneous subgroup of closed elements in $\mathrm{Hom}^*_{\mathcal{E}}(X, Y)$ is denoted by $\mathcal{Z}(\mathbf{E})$. In particular, $\mathsf{Z}^0(\mathbf{E}) = (\mathcal{Z}(\mathbf{E}))^0$ is the preadditive category whose objects are the objects of $\mathbf{E}$ and whose morphisms are the closed morphisms of degree 0 in $\mathbf{E}$. The construction of the underlying category of a C-enriched category from Section 1.1 assigns to a DG-category $\mathbf{E}$ the preadditive category $\mathsf{Z}^0(\mathbf{E})$.

The graded category whose objects are the objects of $\mathbf{E}$ and whose graded abelian group of morphisms $X \longrightarrow Y$ is the graded group of cohomology of the complex $\mathrm{Hom}^\bullet_{\mathbf{E}}(X, Y)$ is denoted by $\mathcal{H}(\mathbf{E})$. In particular, $\mathsf{H}^0(\mathbf{E}) = (\mathcal{H}(\mathbf{E}))^0$ is the preadditive category whose objects are the objects of $\mathbf{E}$ and whose morphisms are the cochain homotopy classes of degree 0 morphisms in $\mathbf{E}$. The preadditive category $\mathsf{H}^0(\mathbf{E})$ is known as the *homotopy category* of a DG-category $\mathbf{E}$. Morphisms in $\mathsf{Z}^0(\mathbf{E})$ whose images in $\mathsf{H}^0(\mathbf{E})$ agree are called *homotopic*, and isomorphisms in $\mathsf{H}^0(\mathbf{E})$ are called *homotopy equivalences* of objects of $\mathbf{E}$.

1.4. **Shifts, direct sums, products, and coproducts in DG-categories.** Two objects X and Y in a DG-category $\mathbf{E}$ are said to be *isomorphic* if they are isomorphic in $\mathsf{Z}^0(\mathbf{E})$. This is the strict notion of isomorphism in DG-categories.

The *shifts* and *finite direct sums* in a DG-category $\mathbf{E}$ can be simply defined as the shifts in the graded category $\mathcal{Z}(\mathbf{E})$ and the finite direct sums in the preadditive category $\mathsf{Z}^0(\mathbf{E})$, respectively. In particular, given an integer $i \in \mathbb{Z}$ and an object $X \in \mathbf{E}$, the *shift* $Y = X[i]$ of the object X is an object of $\mathbf{E}$ endowed with a pair of closed morphisms $f \in \operatorname{Hom}_{\mathbf{E}}^{-i}(X,Y)$ and $g \in \operatorname{Hom}_{\mathbf{E}}^{i}(Y,X)$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$. So a DG-category $\mathbf{E}$ has shifts if and only if the graded category $\mathcal{Z}(\mathbf{E})$ does. In this case, the graded categories $\mathbf{E}^*$ and $\mathcal{H}(\mathbf{E})$ also have shifts.

The definition of the direct sum of a finite collection of objects in $\mathbf{E}$ can be spelled out similarly. So the direct sum of a finite collection of objects in $\mathbf{E}$ is their direct sum in $\mathsf{Z}^0(\mathbf{E})$. A DG-category $\mathbf{E}$ is called *additive* if all finite direct sums exists in $\mathbf{E}$, or equivalently, if the preadditive category $\mathsf{Z}^0(\mathbf{E})$ is additive. In this case, the graded categories $\mathcal{Z}(\mathbf{E})$, $\mathbf{E}^*$, $\mathcal{H}(\mathbf{E})$ and the preadditive categories $\mathbf{E}^0$, $\mathsf{H}^0(\mathbf{E})$ are also additive.

Let X_α be an (infinite) collection of objects in $\mathbf{E}$. Then the *coproduct* $Y = \coprod_\alpha X_\alpha$ is an object of $\mathbf{E}$ such that a natural isomorphism of complexes of abelian groups $\operatorname{Hom}_{\mathbf{E}}^{\bullet}(Y,Z) \simeq \prod_\alpha \operatorname{Hom}_{\mathbf{E}}^{\bullet}(X_\alpha,Z)$ is defined for all objects $Z \in \mathbf{E}$ in a way functorial with respect to all morphisms $f \in \operatorname{Hom}_{\mathbf{E}}^{n}(Z,Z')$, $n \in \mathbb{Z}$. Here $\prod_\alpha$ denotes the product functor in the category of complexes of abelian groups. Dually, the *product* $W = \prod_\alpha X_\alpha$ is an object of $\mathbf{E}$ such that a functorial isomorphism of complexes of abelian groups $\operatorname{Hom}_{\mathbf{E}}^{\bullet}(Z,W) \simeq \prod_\alpha \operatorname{Hom}_{\mathbf{E}}^{\bullet}(Z,X_\alpha)$ is defined for all objects $Z \in \mathbf{E}$. The products and coproducts in $\mathbf{E}$, when they exist, are defined uniquely up to a unique isomorphism (i. e., closed isomorphism of degree 0). Under mild assumptions on $\mathbf{E}$, the existence of infinite products and coproducts in $\mathbf{E}$ is again equivalent to the existence of such products and coproducts, respectively, in $\mathsf{Z}^0(\mathbf{E})$. This will be made precise in Lemma 6.5.

Notice that the functors assigning to a complex of abelian groups its grading components, groups of cocycles, and groups of cohomology commute with infinite products in the respective categories. Therefore, any (co)product of a family of objects in a DG-category $\mathbf{E}$ is also their (co)product in the graded categories $\mathcal{Z}(\mathbf{E})$, $\mathbf{E}^*$, $\mathcal{H}(\mathbf{E})$ and the preadditive categories $\mathsf{Z}^0(\mathbf{E})$, $\mathbf{E}^0$, $\mathsf{H}^0(\mathbf{E})$.

1.5. **Twists and cones in DG-categories.** A *Maurer–Cartan cochain* in the complex of endomorphisms of an object $X \in \mathbf{E}$ is an element $a \in \operatorname{Hom}_{\mathbf{E}}^{1}(X,X)$ satisfying the equation $d(a) + a^2 = 0$ in $\operatorname{Hom}_{\mathbf{E}}^{2}(X,X)$. The *twist* $Y = X(a)$ of the object X by a Maurer–Cartan cochain $a \in \operatorname{Hom}_{\mathbf{E}}^{1}(X,X)$ is an object of $\mathbf{E}$ endowed with a pair of morphisms $f \in \operatorname{Hom}_{\mathbf{E}}^{0}(X,Y)$ and $g \in \operatorname{Hom}_{\mathbf{E}}^{0}(Y,X)$ such that $gf = \mathrm{id}_X$, $fg = \mathrm{id}_Y$, and $d(f) = fa$. Assuming the former two equations, the latter one is equivalent to $d(g) = -ag$ (therefore, $Y = X(a)$ implies $X = Y(-fag)$).

An object $Y \in \mathbf{E}$ is a twist of an object $X \in \mathbf{E}$ (by some Maurer–Cartan cochain in the complex of endomorphisms) if and only if the objects X and Y are isomorphic in the preadditive category $\mathbf{E}^0$. For any pair of mutually inverse isomorphisms $f\colon X \longrightarrow Y$ and $Y \longrightarrow X$ in $\mathbf{E}^0$, the elements $a = gd(f) = -d(g)f \in \operatorname{Hom}^1_{\mathbf{E}}(X, X)$ and $-fag = fd(g) = -d(f)g \in \operatorname{Hom}^1_{\mathbf{E}}(Y, Y)$ are Maurer–Cartan cochains. The twist of a given object in $\mathbf{E}$ by a given Maurer–Cartain cochain in its complex of endomorphisms, when it exists, is defined uniquely up to a unique closed isomorphism of degree 0.

An object $C \in \mathbf{E}$ is said to be the *cone* of a closed morphism $f\colon X \longrightarrow Y$ of degree 0 in $\mathbf{E}$ if an isomorphism between the complex of abelian groups $\operatorname{Hom}^\bullet_{\mathbf{E}}(Z, C)$ and the cone of the morphism of complexes $\operatorname{Hom}^\bullet_{\mathbf{E}}(Z, f)\colon \operatorname{Hom}^\bullet_{\mathbf{E}}(Z, X) \longrightarrow \operatorname{Hom}^\bullet_{\mathbf{E}}(Z, Y)$ is specified for all objects $Z \in \mathbf{E}$, in a way functorial with respect to all morphisms $g \in \operatorname{Hom}^n_{\mathbf{E}}(Z', Z)$, $n \in \mathbb{Z}$. Equivalently, C is the cone of f if an isomorphism between the complex of abelian groups $\operatorname{Hom}^\bullet_{\mathbf{E}}(C[-1], Z)$ and the cone of the morphism of complexes $\operatorname{Hom}^\bullet_{\mathbf{E}}(f, Z)\colon \operatorname{Hom}^\bullet_{\mathbf{E}}(Y, Z) \longrightarrow \operatorname{Hom}^\bullet_{\mathbf{E}}(X, Z)$ is specified for all objects $Z \in \mathbf{E}$, in a way functorial with respect to all morphisms $g \in \operatorname{Hom}^n_{\mathbf{E}}(Z, Z')$. The object $C[-1]$ is called the *cocone* of f. The (co)cone of a closed morphism in $\mathbf{E}$, if it exists, is defined uniquely up to a unique closed isomorphism of degree 0.

If the shift $X[1]$ and the direct sum $Y \oplus X[1]$ exist in $\mathbf{E}$, then the cone $C = \operatorname{cone}(f)$ can be constructed as the twist $(Y \oplus X[1])(a_f)$ of the object $Y \oplus X[1]$ by a suitable Maurer–Cartan cochain a_f produced from the morphism f. Consequenly, any additive DG-category with shifts and twists has cones. Conversely, any DG-category with shifts, cones, and a zero object is additive.

Whenever the shift $X[1]$ and the cone $C = \operatorname{cone}(f)$ exist in $\mathbf{E}$, there is a natural short sequence $0 \longrightarrow Y \longrightarrow C \longrightarrow X[1] \longrightarrow 0$ in $\mathsf{Z}^0(\mathbf{E})$ which is split exact in $\mathbf{E}^0$. Conversely, if $\mathbf{E}$ is a DG-category with shifts, then any short sequence $0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$ of closed morphisms of degree 0 in $\mathbf{E}$ which is split exact in $\mathbf{E}^0$ arises from a closed morphism $f\colon A \longrightarrow B[1]$; so $C = \operatorname{cone}(f)[-1]$.

For any additive DG-category $\mathbf{E}$ with shifts and cones, the homotopy category $\mathsf{H}^0(\mathbf{E})$ has a natural structure of triangulated category.

1.6. **Totalizations of complexes in DG-categories.** Let $X^\bullet$ be a complex in the preadditive category $\mathsf{Z}^0(\mathbf{E})$. So, for every $n \in \mathbb{Z}$, the differential $d_n\colon X^n \longrightarrow X^{n+1}$ is a closed morphism of degree 0 in $\mathbf{E}$, and the composition $X^{n-1} \longrightarrow X^n \longrightarrow X^{n+1}$ vanishes as a closed morphism in $\mathbf{E}$.

Then an object $T \in \mathbf{E}$ is said to be the *product totalization* of the complex $X^\bullet$ and denoted by $T = \operatorname{Tot}^\sqcap(X^\bullet)$ if an isomorphism between the complex of abelian groups $\operatorname{Hom}^\bullet_{\mathbf{E}}(Z, T)$ and the totalization of the bicomplex of abelian groups $\operatorname{Hom}^\bullet_{\mathbf{E}}(Z, X^\bullet)$, constructed by taking the countable products of abelian groups along the diagonals, is specified for all objects $Z \in \mathbf{E}$, in a way functorial with respect to all morphisms $g \in \operatorname{Hom}^i_{\mathbf{E}}(Z', Z)$, $i \in \mathbb{Z}$. Dually, an object $T \in \mathbf{E}$ is said to be the *coproduct totalization* of the complex $X^\bullet$ and denoted by $T = \operatorname{Tot}^\sqcup(X^\bullet)$ if an isomorphism between the complex of abelian groups $\operatorname{Hom}^\bullet_{\mathbf{E}}(T, Z)$ and the totalization of the bicomplex of abelian groups $\operatorname{Hom}^\bullet_{\mathbf{E}}(X^\bullet, Z)$, constructed by taking the countable products of abelian groups along the diagonals, is specified for all objects $Z \in \mathbf{E}$, in a way functorial with respect to all morphisms $g \in \mathrm{Hom}^i_{\mathbf{E}}(Z, Z')$.

For a finite complex $X^\bullet$ in $\mathbf{E}$, there is no difference between the product and the coproduct totalization; so we will write simply $T = \mathrm{Tot}(X^\bullet)$. The cone of a morphism $f\colon X \longrightarrow Y$ in $\mathsf{Z}^0(\mathbf{E})$ is the totalization of the two-term complex $X \longrightarrow Y$. Conversely, in a DG-category $\mathbf{E}$ with shifts and cones, the totalization of any finite complex can be expressed as a finitely iterated cone.

1.7. **DG-functors.** A *DG-functor* is a functor of categories enriched in complexes of abelian groups. More explicitly, given two DG-categories $\mathbf{A}$ and $\mathbf{B}$, a DG-functor $F\colon \mathbf{B} \longrightarrow \mathbf{A}$ is a rule assigning to every object $X \in \mathbf{B}$ an object $F(X) \in \mathbf{A}$ and to every pair of objects X, $Y \in \mathbf{B}$ a morphism of complexes $F_{X,Y}\colon \mathrm{Hom}^\bullet_{\mathbf{B}}(X,Y) \longrightarrow \mathrm{Hom}^\bullet_{\mathbf{A}}(F(X),F(Y))$ in such a way that the compositions of morphisms and the identity morphisms are preserved.

Any DG-functor preserves finite direct sums and shifts of objects, twists of objects by Maurer–Cartan cochains, and cones of closed morphisms of degree 0. In other words, one can say that these operations are examples of "absolute weighted (co)limits" in DG-categories, in the sense of [44, Section 5].

A DG-functor $F\colon \mathbf{B} \longrightarrow \mathbf{A}$ is said to be *fully faithful* if the map $F_{X,Y}$ is an isomorphism of complexes of abelian groups for all objects X, $Y \in \mathbf{B}$. A DG-functor F is said to be an *equivalence of DG-categories* if it is fully faithful and essentially surjective. The latter condition means that for every object $W \in \mathbf{A}$ there exists an object $X \in \mathbf{B}$ together with a closed isomorphism $F(X) \simeq W$ of degree 0 in $\mathbf{A}$. The datum of a fully faithful DG-functor $\mathbf{B} \longrightarrow \mathbf{A}$ means that, up to an equivalence, $\mathbf{B}$ can be viewed as a *full DG-subcategory* in $\mathbf{A}$.

Given two additive DG-categories $\mathbf{A}$ and $\mathbf{B}$ with shifts and cones, any DG-functor $F\colon \mathbf{B} \longrightarrow \mathbf{A}$ induces a triangulated functor $\mathsf{H}^0(F)\colon \mathsf{H}^0(\mathbf{B}) \longrightarrow \mathsf{H}^0(\mathbf{A})$.

1.8. **Example: DG-category of complexes.** Let E be a preadditive category. Then the preadditive category $\mathsf{G}(\mathsf{E})$ of graded objects in E is constructed as follows. The objects of $\mathsf{G}(\mathsf{E})$ are collections of objects $X^* = (X^n \in \mathsf{E})_{n\in\mathbb{Z}}$ in the category E indexed by the integers n. The abelian group of morphisms $\mathrm{Hom}_{\mathsf{G}(\mathsf{E})}(X^*, Y^*)$ is defined as the product $\prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{E}}(X^n, Y^n)$. The composition of morphisms and the identity morphisms in $\mathsf{G}(\mathsf{E})$ are constructed in the obvious way. So $\mathsf{G}(\mathsf{E}) = \mathsf{E}^{\mathbb{Z}}$ is simply the Cartesian product of $\mathbb{Z}$ copies of E.

The shift functor on $\mathsf{G}(\mathsf{E})$ is defined by the rule $X^*[i]^n = X^{n+i}$ for all i, $n \in \mathbb{Z}$. The graded category $\mathcal{G}(\mathsf{E})$ is constructed as follows. The objects of $\mathcal{G}(\mathsf{E})$ are the objects of $\mathsf{G}(\mathsf{E})$, and the graded abelian group of morphisms $\mathrm{Hom}^*_{\mathcal{G}(\mathsf{E})}(X^*, Y^*)$ has the grading components $\mathrm{Hom}^i_{\mathcal{G}(\mathsf{E})}(X^*, Y^*) = \mathrm{Hom}_{\mathsf{G}(\mathsf{E})}(X^*, Y^*[i])$ for all $i \in \mathbb{Z}$. Once again, we omit the obvious construction of the composition of morphisms and identity morphisms in $\mathcal{G}(\mathsf{E})$. By the definition, one has $(\mathcal{G}(\mathsf{E}))^0 = \mathsf{G}(\mathsf{E})$.

The DG-category $\mathbf{C}(\mathsf{E})$ of complexes in E is constructed as follows. The objects of $\mathbf{C}(\mathsf{E})$ are complexes $X^\bullet$ with the components $X^n \in \mathsf{E}$; the differentials $d_{X,n}\colon X^n \longrightarrow X^{n+1}$ are morphisms in E. Let us denote by $X^* \in \mathcal{G}(\mathsf{E})$ the underlying graded object of $X^\bullet$. Given two complexes $X^\bullet$ and $Y^\bullet \in \mathbf{C}(\mathsf{E})$, the underlying graded abelian group of the complex of morphisms $\mathrm{Hom}^\bullet_{\mathbf{C}(\mathsf{E})}(X^\bullet, Y^\bullet)$ is the graded abelian group $\mathrm{Hom}^*_{\mathcal{G}(\mathsf{E})}(X^*, Y^*)$. The differential d in the complex $\mathrm{Hom}^\bullet_{\mathbf{C}(\mathsf{E})}(X^\bullet, Y^\bullet)$ is defined by the usual formula $d(f) = d_Y \circ f - (-1)^{|f|} f \circ d_X$, where $f \in \mathrm{Hom}^{|f|}_{\mathbf{C}(\mathsf{E})}(X^\bullet, Y^\bullet)$.

By the definition, the composition of morphisms and the identity morphisms in the DG-category $\mathbf{C}(\mathsf{E})$ agree with those in the graded category $\mathcal{G}(\mathsf{E})$. Notice that any graded object in E admits a differential making it a complex in E (e. g., the zero differential). So one has $(\mathbf{C}(\mathsf{E}))^* = \mathcal{G}(\mathsf{E})$ and $(\mathbf{C}(\mathsf{E}))^0 = \mathsf{G}(\mathsf{E})$.

All shifts exist in the DG-category $\mathbf{C}(\mathsf{E})$; the shift $X^\bullet[i]$ of a complex $X^\bullet \in \mathbf{C}(\mathsf{E})$ is constructed by the well-known rules $X^\bullet[i]^n = X^{n+i}$ and $d_{X[i],n} = (-1)^i d_{X,n+i}$. Furthermore, all twists exist in the DG-category $\mathbf{C}(\mathsf{E})$. When the category E is additive, so is the DG-category $\mathbf{C}(\mathsf{E})$. In this case, the DG-category $\mathbf{C}(\mathsf{E})$ also has cones. When the category E has infinite (co)products, so does the DG-category $\mathbf{C}(\mathsf{E})$.

The notation for the categories of closed morphisms in $\mathbf{C}(\mathsf{E})$ is $\mathcal{C}(\mathsf{E}) = \mathcal{Z}(\mathbf{C}(\mathsf{E}))$ and $\mathsf{C}(\mathsf{E}) = \mathsf{Z}^0(\mathbf{C}(\mathsf{E}))$. These are the (respectively, graded and preadditive) *categories of complexes* in E, i. e., the categories of complexes and closed morphisms between them. The usual notation for the homotopy category is $\mathsf{K}(\mathsf{E}) = \mathsf{H}^0(\mathbf{C}(\mathsf{E}))$.

1.9. **Example: DG-category of CDG-modules.** The concept of a *CDG-algebra* goes back to the paper [46]. A more advanced discussion of CDG-rings can be found in [48, Section 3], and of CDG-coalgebras, in [48, Section 4].

By the definition, a *CDG-ring* $\boldsymbol{R}^\bullet = (R^*, d, h)$ is the following set of data:

- $R^* = \bigoplus_{n\in\mathbb{Z}} R^n$ is a graded ring;
- d is an odd derivation of degree 1 on R^*, i. e., an additive map $d_n \colon R^n \longrightarrow R^{n+1}$ defined for all $n \in \mathbb{Z}$ and satisfying the Leibniz rule with signs $d(rs) = d(r)s + (-1)^{|r|} r d(s)$ for all $r \in R^{|r|}$ and $s \in R^{|s|}$;
- $h \in R^2$ is an element of degree 2.

Two equations involving d and h must be satisfied:

- $d^2(r) = hr - rh$ for all $r \in R^*$;
- $d(h) = 0$.

So $\boldsymbol{R}^\bullet = (R^*, d, h)$ is not a complex: the square of the differential d is not equal to zero, but rather to the commutator with h. The element $h \in R^2$ is called the *curvature element* of a CDG-ring (R^*, d, h). We refer to [50, Section 3.2], [53, Section 6], or [52, Sections 2.2 and 2.4] for further details on CDG-rings, including the (nontrivial!) definition of a morphism of CDG-rings.

A *left CDG-module* $\boldsymbol{M}^\bullet = (M^*, d_M)$ over a CDG-ring $\boldsymbol{R}^\bullet$ is the following set of data:

- $M^* = \bigoplus_{n\in\mathbb{Z}} M^n$ is a graded left R^*-module;
- d_M is an odd derivation of degree 1 on the graded left R^*-module M^* compatible with the derivation d on the graded ring R^*, i. e., an additive map $d_{M,n} \colon M^n \longrightarrow M^{n+1}$ is defined for all $n \in \mathbb{Z}$ and satisfies the Leibniz rule with signs $d_M(rx) = d(r)x + (-1)^{|r|} r d_M(x)$ for all $r \in R^{|r|}$ and $x \in M^{|x|}$.

The following equation describing the square of the differential d_M must be satisfied:

- $d_M^2(x) = hx$ for all $x \in M^*$.

Similarly, a *right CDG-module* $\boldsymbol{N}^\bullet = (N^*, d_N)$ over a CDG-ring $\boldsymbol{R}^\bullet$ is the following set of data:

- $N^* = \bigoplus_{n\in\mathbb{Z}} N^n$ is a graded right R^*-module;
- an additive map $d_{N,n}\colon N^n \longrightarrow N^{n+1}$ is defined for all $n \in \mathbb{Z}$ and satisfies the Leibniz rule with signs $d_N(yr) = d_N(y)r + (-1)^{|y|}yd(r)$ for all $r \in R^{|r|}$ and $y \in N^{|y|}$.

The next equation describing the square of the differential d_N must be satisfied:

- $d_N^2(y) = -yh$ for all $y \in N^*$.

A *DG-ring* $\boldsymbol{R}^\bullet = (R^*, d)$ is a CDG-ring with vanishing curvature element, $h = 0$. DG-modules over a DG-ring (R^*, d) are the same things as CDG-modules over the CDG-ring $(R^*, d, 0)$. The definitions below show that the rather familiar construction of the DG-category $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ of DG-modules over a DG-ring $\boldsymbol{R}^\bullet$ can be extended to a construction of the DG-category of CDG-modules $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ over an arbitrary CDG-ring $\boldsymbol{R}^\bullet$ in a quite natural way.

Let R^* be a graded ring. We denote by R^*–Mod the abelian category of graded left R^*-modules (and homogeneous morphisms of degree zero between them). Let us define the graded category R^*–$\mathcal{M}od$ of graded left R^*-modules.

The objects of R^*–$\mathcal{M}od$ are the graded left R^*-modules. Given two objects L^*, $M^* \in R^*$–$\mathcal{M}od$, the graded abelian group of morphisms $\mathrm{Hom}^*_{R^*\text{–}\mathcal{M}od}(L^*, M^*) = \mathrm{Hom}^*_{R^*}(L^*, M^*)$ is constructed by the following rule. For every $i \in \mathbb{Z}$, the degree i component $\mathrm{Hom}^i_{R^*}(L^*, M^*)$ is the group of all homogeneous maps of graded abelian groups $f\colon L^* \longrightarrow M^*$ of degree i which are compatible with left R^*-module structures in the sense of the sign rule $f(rz) = (-1)^{i|r|}rf(z)$ for all $r \in R^{|r|}$ and $z \in L^{|z|}$. The composition of morphisms and the unit morphisms in the graded category R^*–$\mathcal{M}od$ are defined in the obvious way.

Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a CDG-ring. The DG-category $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ of left CDG-modules over $\boldsymbol{R}^\bullet$ is constructed as follows. The objects of $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ are the left CDG-modules over $\boldsymbol{R}^\bullet$. Given two left CDG-modules $\boldsymbol{L}^\bullet = (L^*, d_L)$ and $\boldsymbol{M}^\bullet = (M^*, d_M) \in \boldsymbol{R}^\bullet$–$\mathbf{Mod}$, the underlying graded abelian group of the complex of morphisms $\mathrm{Hom}^\bullet_{\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}}(\boldsymbol{L}^\bullet, \boldsymbol{M}^\bullet) = \mathrm{Hom}^\bullet_{\boldsymbol{R}^\bullet}(\boldsymbol{L}^\bullet, \boldsymbol{M}^\bullet)$ is the graded abelian group $\mathrm{Hom}^*_{R^*}(L^*, M^*)$. The differential d in the complex $\mathrm{Hom}^\bullet_{\boldsymbol{R}^\bullet}(\boldsymbol{L}^\bullet, \boldsymbol{M}^\bullet)$ is defined by the usual formula $d(f)(z) = d_M(f(z)) - (-1)^{|f|}f(d_L(z))$ for all $f \in \mathrm{Hom}^{|f|}_{R^*}(L^*, M^*)$ and $z \in L^{|z|}$.

By the definition, the composition of morphisms and the identity morphisms in the DG-category $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ agree with those in the graded category R^*–$\mathcal{M}od$. So $(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^*$ is the full graded subcategory in R^*–$\mathcal{M}od$ whose objects are all the graded left R^*-modules *that admit a structure of CDG-module over* $\boldsymbol{R}^\bullet$, and similarly, $(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^0$ is the full additive subcategory in R^*–Mod whose objects are all the graded R^*-modules *that admit a structure of CDG-module over* $\boldsymbol{R}^\bullet$. Not all the graded R^*-modules can be endowed with such a structure in general; see counterexamples in [52, Examples 3.2 and 3.3].

All shifts, twists, and infinite products and coproducts (hence also all finite direct sums and cones) exist in the DG-category $\boldsymbol{R}^\bullet$–**Mod**. The category $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ of CDG-modules over $\boldsymbol{R}^\bullet$ with closed morphisms of degree 0 is abelian. In fact, $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ is the abelian category of graded modules over a graded ring denoted by $\widehat{\boldsymbol{R}}^*$, as we will discuss below in Section 2.6.

1.10. **Example: DG-category of factorizations.** The concept of a *matrix factorization* of a polynomial, or more generally, of a section of a line bundle on a scheme goes back to the paper [14]; see [45, 13] for the nonaffine case. The following abstract category-theoretic version was suggested in the papers [12, Section 6 and Appendix A] and [3] (see [52, Section 2.5] for an even more general approach).

We follow [52, Remark 2.7] (restricting ourselves to the case of the grading group $\Gamma = \mathbb{Z}$). Let E be a preadditive category and $\Delta\colon \mathsf{E} \longrightarrow \mathsf{E}$ be an autoequivalence. Then the preadditive category $\mathsf{P}(\mathsf{E},\Delta)$ of 2-Δ-*periodic objects* in E is constructed as follows. An object $X^\circ \in \mathsf{P}(\mathsf{E},\Delta)$ is a collection of objects $(X^n \in \mathsf{E})_{n\in\mathbb{Z}}$ endowed with isomorphisms $\delta_X^{n+2,n}\colon \Delta(X^n) \xrightarrow{\simeq} X^{n+2}$ defined for all $n \in \mathbb{Z}$. The collection of all objects $(X^n \in \mathsf{E})_{n\in\mathbb{Z}}$ with the periodicity isomorphisms $\delta_X^{n+2,n}$ forgotten defines the underlying graded object $X^* \in \mathsf{G}(\mathsf{E})$ of a 2-Δ-periodic object $X^\circ \in \mathsf{P}(\mathsf{E},\Delta)$.

For any two objects X° and $Y^\circ \in \mathsf{P}(\mathsf{E},\Delta)$, the abelian group $\mathrm{Hom}_{\mathsf{P}(\mathsf{E},\Delta)}(X^\circ,Y^\circ)$ is defined as the subgroup in $\mathrm{Hom}_{\mathsf{G}(\mathsf{E})}(X^*,Y^*)$ consisting of all the morphisms $(f_n\colon X^n \to Y^n)_{n\in\mathbb{Z}}$ satisfying the equations $\delta_Y^{n+2,n}\circ\Delta(f_n) = f_{n+2}\circ\delta_X^{n+2,n}$ for all $n \in \mathbb{Z}$. The category $\mathsf{P}(\mathsf{E},\Delta)$ is naturally equivalent to the Cartesian product $\mathsf{E}\times\mathsf{E}$ of two copies of the category E, the equivalence being provided by the functor taking a 2-Δ-periodic object $X^\circ \in \mathsf{P}(\mathsf{E},\Delta)$ to the pair of objects $(X^0,X^1) \in \mathsf{E}\times\mathsf{E}$.

The shift functor on $\mathsf{P}(\mathsf{E},\Delta)$ is defined by the rule $X^\circ[i]^n = X^{n+i}$ and $\delta_{X[i]}^{n,n+2} = \delta_X^{n+i,n+i+2}$ for all i, $n \in \mathbb{Z}$. The graded category $\mathcal{P}(\mathsf{E},\Delta)$ is constructed as follows. The objects of $\mathcal{P}(\mathsf{E},\Delta)$ are the objects of $\mathsf{P}(\mathsf{E},\Delta)$, i. e., the 2-$\Delta$-periodic objects in E. The graded abelian group of morphisms $\mathrm{Hom}^*_{\mathcal{P}(\mathsf{E},\Delta)}(X^\circ,Y^\circ)$ has the grading components $\mathrm{Hom}^i_{\mathcal{P}(\mathsf{E},\Delta)}(X^\circ,Y^\circ) = \mathrm{Hom}_{\mathsf{P}(\mathsf{E},\Delta)}(X^\circ,Y^\circ[i])$ for all $i \in \mathbb{Z}$. By the definition, one has $(\mathcal{P}(\mathsf{E},\Delta))^0 = \mathsf{P}(\mathsf{E},\Delta)$. The composition of morphisms in the graded category $\mathcal{P}(\mathsf{E},\Delta)$ agrees with the one in the graded category $\mathcal{G}(\mathsf{E})$. *Unlike* the preadditive category $\mathsf{P}(\mathsf{E},\Delta)$, the graded category $\mathcal{P}(\mathsf{E},\Delta)$ is not determined by the category E alone; it depends on the autoequivalence Δ.

A *potential* v for an autoequivalence $\Delta\colon \mathsf{E} \longrightarrow \mathsf{E}$ is a natural transformation $v\colon \mathrm{Id}_\mathsf{E} \longrightarrow \Delta$ satisfying the equation $v_{\Delta(E)} = \Delta(v_E)$ for all $E \in \mathsf{E}$. Given a potential v, the DG-category $\mathbf{F}(\mathsf{E},\Delta,v)$ of *factorizations of* v in E is constructed as follows. A factorization $\boldsymbol{N}^\bullet = (N^\circ,d_N)$ (in the sense of [3]) is a 2-Δ-periodic object endowed with a homogeneous endomorphism $d_N \in \mathrm{Hom}^1_{\mathcal{P}(\mathsf{E},\Delta)}$ such that $d_N^2 = \delta_N v_N$, that is, for every $n \in \mathbb{Z}$, the composition $d_{N,n+1}d_{N,n}\colon N \longrightarrow N^{n+1} \longrightarrow N^{n+2}$ is equal to the composition $\delta_N^{n,n+2}v_{N^n}\colon N^n \longrightarrow \Delta(N^n) \longrightarrow N^{n+2}$. Given two factorizations $L^\bullet = (L^\circ,d_L)$ and $M^\bullet = (M^\circ,d_M)$ of the same potential v, the underlying graded abelian group of the complex $\mathrm{Hom}^\bullet_{\mathbf{F}(\mathsf{E},\Delta,v)}(L^\blacklozenge, M^\blacklozenge)$ is the graded abelian group $\mathrm{Hom}^*_{\mathcal{P}(\mathsf{E},\Delta)}(L^\circ, M^\circ)$. The differential d in the complex $\mathrm{Hom}^\bullet_{\mathbf{F}(\mathsf{E},\Delta,v)}(L^\blacklozenge, M^\blacklozenge)$ is given by the usual formula $d(f) = d_M \circ f - (-1)^{|f|} f \circ d_L$. One can check that $d^2 = 0$.

By the definition, the composition of morphisms and the identity morphisms in the DG-category $\mathbf{F}(\mathsf{E},\Delta,v)$ agree with those in the graded category $\mathcal{P}(\mathsf{E},\Delta)$. So $\mathbf{F}(\mathsf{E},\Delta,v)^*$ is the full graded subcategory in $\mathcal{P}(\mathsf{E},\Delta)$ whose objects are all the 2-Δ-periodic objects in E *that admit a differential defining a structure of factorization of* v, and similarly, $\mathbf{F}(\mathsf{E},\Delta,v)^0$ is the full preadditive subcategory in $\mathsf{P}(\mathsf{E},\Delta)$ whose objects are all the 2-Δ-periodic objects in E *that admit a differential defining a structure of factorization of* v. Not all the 2-Δ-periodic objects in E can be endowed with such a structure in general, as illustrated by [52, Examples 3.2] interpreted in the context of factorizations; see [52, Example 3.19] for a discussion.

All shifts and twists exist in the DG-category $\mathbf{F}(\mathsf{E},\Delta,v)$. When the category E is additive, so is the DG-category $\mathbf{F}(\mathsf{E},\Delta,v)$; hence this DG-category also has cones. When the category E has infinite (co)products, so does the DG-category $\mathbf{F}(\mathsf{E},\Delta,v)$.

The notation for the category of closed morphisms in $\mathbf{F}(\mathsf{E},\Delta,v)$ is $\mathsf{F}(\mathsf{E},\Delta,v) = \mathsf{Z}^0(\mathbf{F}(\mathsf{E},\Delta,v))$. This is the preadditive category of factorizations of v in E. The notation for the homotopy category is $\mathsf{K}(\mathsf{E},\Delta,v) = \mathsf{H}^0(\mathbf{F}(\mathsf{E},\Delta,v))$.

2. The Almost Involution on DG-Categories

This section is an extraction from [52, Section 3]. Most proofs are omitted and replaced with references to [52].

2.1. **The DG-category $\mathbf{E}^\natural$.** Let $\mathbf{E}$ be a DG-category. The DG-category $\mathbf{E}^\natural$ is constructed as follows. The objects of $\mathbf{E}^\natural$ are pairs $X^\natural = (X, \sigma_X)$, where X is an object of $\mathbf{E}$ and $\sigma_X \in \mathrm{Hom}^{-1}_{\mathbf{E}}(X,X)$ is an endomorphism of degree -1 such that $d(\sigma_X) = \mathrm{id}_X$ and $\sigma_X^2 = 0$. In other words, σ_X is a contracting homotopy for the object $X \in \mathbf{E}$ satisfying the additional condition that the square of σ_X vanishes in $\mathrm{Hom}^{-2}_{\mathbf{E}}(X,X)$.

The construction of the complex of morphisms $\mathrm{Hom}^\bullet_{\mathbf{E}^\natural}(X^\natural, Y^\natural)$ for a pair of objects $X^\natural$, $Y^\natural \in \mathbf{E}^\natural$ involves a change of the sign of the cohomological grading. For every $n \in \mathbb{Z}$, the group $\mathrm{Hom}^n_{\mathbf{E}^\natural}(X^\natural, Y^\natural)$ consists of all the *cocycles* in the group $\mathrm{Hom}^{-n}_{\mathbf{E}}(X,Y)$, i. e., all elements $f \in \mathrm{Hom}^{-n}_{\mathbf{E}}(X,Y)$ annihilated by the differential $d_{-n}\colon \mathrm{Hom}^{-n}_{\mathbf{E}}(X,Y) \longrightarrow \mathrm{Hom}^{-n+1}_{\mathbf{E}}(X,Y)$. The differential $d^\natural$ in the complex $\mathrm{Hom}^\bullet_{\mathbf{E}^\natural}(X^\natural, Y^\natural)$ is defined as the commutator with the contracting homotopies σ, that is, $d^\natural(f) = \sigma_Y f - (-1)^n f \sigma_X$. The composition of morphisms in the DG-category $\mathbf{E}^\natural$ is induced by the composition of morphisms in $\mathbf{E}$ in the obvious way, and the identity morphisms in $\mathbf{E}^\natural$ are the identity morphisms in $\mathbf{E}$, i. e., $\mathrm{id}_{X^\natural} = \mathrm{id}_X$.

All twists exist in the DG-category $\mathbf{E}^\natural$. Given an object $X^\natural = (X, \sigma_X) \in \mathbf{E}^\natural$ and a Maurer–Cartan cochain $a \in \mathrm{Hom}^1_{\mathbf{E}^\natural}(X^\natural, X^\natural) \subset \mathrm{Hom}^{-1}_{\mathbf{E}}(X,X)$, the twisted object $X^\natural(a)$ is constructed as $X^\natural(a) = (X, \sigma_X + a)$. Finite direct sums, as well as infinite products and/or coproducts, exist in the DG-category $\mathbf{E}^\natural$ whenever they exist in the DG-category $\mathbf{E}$, and are constructed in the obvious way. Shifts exists in $\mathbf{E}^\natural$ whenever they exist in $\mathbf{E}$; here one has to observe the grading sign change: given an object $X^\natural = (X, \sigma_X) \in \mathbf{E}^\natural$, the object $X^\natural[1]$ is given by the rule $X^\natural[1] = (X[-1], -\sigma_X[-1])$. Consequently, the DG-category $\mathbf{E}^\natural$ has cones whenever the DG-category $\mathbf{E}$ has shifts and finite direct sums [52, Section 3.2].

To be more precise, let us spell out the sign rule (or lack thereof) in the definition of the morphism $\sigma_X[-1] \in \operatorname{Hom}^{-1}_{\mathbf{E}}(X[-1], X[-1])$ in the formula for $X^\natural[1]$ above. Given an object $A \in \mathbf{E}$, the object $A[-1] \in \mathbf{E}$ comes together with a pair of closed morphisms $s_A \in \operatorname{Hom}^1_{\mathbf{E}}(A, A[-1])$ and $t_A \in \operatorname{Hom}^{-1}_{\mathbf{E}}(A[-1], A)$ such that $t_A s_A = \mathrm{id}_A$ and $s_A t_A = \mathrm{id}_{A[-1]}$. For any morphism $f \in \operatorname{Hom}^n_{\mathbf{E}}(A, B)$ in $\mathbf{E}$, $n \in \mathbb{Z}$, we put $f[-1] = s_B f t_A \in \operatorname{Hom}^n_{\mathbf{E}}(A[-1], B([-1])$.

2.2. **The functors Φ and $\Psi^\pm$.** There is a doubly infinite ladder of adjoint functors acting between the additive categories $\mathsf{Z}^0(\mathbf{E})$ and $\mathsf{Z}^0(\mathbf{E}^\natural)$ of closed morphisms of degree 0 in the DG-categories $\mathbf{E}$ and $\mathbf{E}^\natural$.

The functor $\Phi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ is *interpreted* as the forgetful functor assigning to a complex, CDG-module, or factorization its underlying graded object/module, while the functors $\Psi^+_{\mathbf{E}}$ and $\Psi^-_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ left and right adjoint to $\Phi_{\mathbf{E}}$ are interpreted as assigning to a graded module the CDG-module (co)freely (co)generated by it (see the discussion of examples in Sections 2.5–2.7 below). The constructions of the three functors do not immediately suggest this somewhat counterintuitive interpretation, as it is the functor $\Psi^+_{\mathbf{E}}$ that is *constructed* as a forgetful functor, while the construction of the functor $\Phi_{\mathbf{E}}$ is more complicated.

In fact, up to adjoining twists and direct summands to the DG-categories, the roles of the three functors are completely symmetric, as we will see in Section 2.4.

Let $\mathbf{E}$ be a DG-category with shifts and cones. Let $A \in \mathbf{E}$ be an object, and let L be the cone of the identity endomorphism of the object $A[-1] \in \mathbf{E}$. The object L is defined in terms of its structure morphisms $\iota \in \operatorname{Hom}^1_{\mathbf{E}}(A, L)$, $\pi \in \operatorname{Hom}^0_{\mathbf{E}}(L, A)$, $\pi' \in \operatorname{Hom}^{-1}_{\mathbf{E}}(L, A)$, and $\iota' \in \operatorname{Hom}^0_{\mathbf{E}}(A, L)$ satisfying the equations

$$\pi'\iota' = 0 = \pi\iota, \quad \pi'\iota = \mathrm{id}_A = \pi\iota', \quad \iota\pi' + \iota'\pi = \mathrm{id}_L,$$
$$d(\iota) = 0 = d(\pi), \quad d(\pi') = \pi, \quad d(\iota') = \iota.$$

Put $\sigma_L = \iota'\pi' \in \operatorname{Hom}^{-1}_{\mathbf{E}}(L, L)$. Then $L^\natural = (L, \sigma_L)$ is an object of the DG-category $\mathbf{E}^\natural$. We put $\Phi_{\mathbf{E}}(A) = L^\natural$. The action of the functor $\Phi_{\mathbf{E}}$ on morphisms in $\mathsf{Z}^0(\mathbf{E})$ (i. e., on the closed morphisms of degree 0 in $\mathbf{E}$) is defined in the obvious way.

The functors $\Psi^+_{\mathbf{E}}$ and $\Psi^-_{\mathbf{E}}$ are defined by the rules $\Psi^+(X^\natural) = X$ and $\Psi^-(X^\natural) = X[1]$ for all objects $X^\natural = (X, \sigma_X) \in \mathbf{E}^\natural$. It is explained in [52, proof of Lemma 3.4] that the functor Ψ^+ is left adjoint to Φ, while the functor Ψ^- is right adjoint to Φ.

All the three functors Ψ^+, Ψ^-, and Φ are faithful. They are also conservative [52, Lemma 3.12]. The functors Φ and $\Psi^\pm$ transform the shift functors $[n]$, $n \in \mathbb{Z}$ on the DG-categories $\mathbf{E}$ and $\mathbf{E}^\natural$ into the inverse shift functors [52, Lemma 3.11]:

$$\Phi \circ [n] = [-n] \circ \Phi, \quad \Psi^+ \circ [n] = [-n] \circ \Psi^+, \quad \Psi^- \circ [n] = [-n] \circ \Psi^-.$$

The compositions of the functors Φ and $\Psi^\pm$ are computable as follows. For any DG-category $\mathbf{E}$ with shifts and cones, denote by $\Xi = \Xi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ the additive functor taking an object $A \in \mathbf{E}$ to the object $\Xi(A) = \operatorname{cone}(\mathrm{id}_A[-1])$ (and acting on the morphisms in the obvious way). Then there are natural isomorphisms of additive functors [52, Lemma 3.8]

$$\Psi^+_{\mathbf{E}} \circ \Phi_{\mathbf{E}} = \Xi_{\mathbf{E}}$$

and

$$\Phi_{\mathbf{E}} \circ \Psi^-_{\mathbf{E}} = \Xi_{\mathbf{E}^\natural}.$$

2.3. **The functors $\widetilde{\Phi}$ and $\widetilde{\Psi}^{\pm}$.** The additive functor $\Psi^+\colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ can be naturally extended to a fully faithful additive functor

$$\widetilde{\Psi}^+\colon (\mathbf{E}^\natural)^0 \longrightarrow \mathsf{Z}^0(\mathbf{E}).$$

The action of the functor $\widetilde{\Psi}^+$ on the objects is given by the obvious rule $\widetilde{\Psi}^+(X^\natural) = X$ for any $X^\natural = (X, \sigma_X) \in \mathbf{E}^\natural$. The functor $\widetilde{\Psi}^+$ acts on the morphisms by the natural isomorphism

$$\operatorname{Hom}_{(\mathbf{E}^\natural)^0}(X^\natural, Y^\natural) = \operatorname{Hom}_{\mathsf{Z}^0(\mathbf{E})}(X, Y) \quad \text{for all } X^\natural = (X, \sigma_X) \text{ and } Y^\natural = (Y, \sigma_Y) \in \mathbf{E}^\natural,$$

which is a part of the definition of the DG-category $\mathbf{E}^\natural$.

Similarly, the additive functor $\Psi^-\colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ can be naturally extended to a fully faithful additive functor

$$\widetilde{\Psi}^-\colon (\mathbf{E}^\natural)^0 \longrightarrow \mathsf{Z}^0(\mathbf{E}).$$

The functor $\widetilde{\Psi}^-$ is constructed as $\widetilde{\Psi}^- = \widetilde{\Psi}^+[1]$.

Moreover, the additive functor $\Phi\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ can be naturally extended to a fully faithful additive functor

$$\widetilde{\Phi}\colon \mathbf{E}^0 \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural).$$

On objects, the functor $\widetilde{\Phi}$ is defined by the obvious rule $\widetilde{\Phi}(A) = \Phi(A) = \operatorname{cone}(\mathrm{id}_A[-1])$ for all $A \in \mathbf{E}$. To construct the action of the functor $\widetilde{\Phi}$ on morphisms, suppose given a morphism $f \in \operatorname{Hom}^0_{\mathbf{E}}(A, B)$ for some objects A, $B \in \mathbf{E}$. Put $L = \operatorname{cone}(\mathrm{id}_A[-1])$ and $M = \operatorname{cone}(\mathrm{id}_B[-1])$. Let ι_A, π_A, ι'_A, π'_A and ι_B, π_B, ι'_B, π'_B be the related morphisms (as in Section 2.2). Then we put

$$\widetilde{\Phi}(f) = g = \iota'_B f \pi_A + \iota_B f \pi'_A + \iota'_B d(f) \pi'_A \in \operatorname{Hom}_{\mathsf{Z}^0(\mathbf{E}^\natural)}(L^\natural, M^\natural) \subset \operatorname{Hom}^0_{\mathbf{E}}(L, M),$$

where $L^\natural = (L, \sigma_L) = \Phi(A)$ and $M^\natural = (M, \sigma_M) = \Phi(B)$. One has to check that $d(g) = 0$, $d^\natural(g) = 0$, $\widetilde{\Phi}(\mathrm{id}_A) = \mathrm{id}_{\Phi(A)}$, and $\widetilde{\Phi}(f_1 \circ f_2) = \widetilde{\Phi}(f_1) \circ \widetilde{\Phi}(f_2)$ for any pair of composable morphisms f_1, f_2 in $\mathbf{E}^0$. Then one also has to check that the assignment $f \longmapsto g$ is an isomorphism of the Hom groups

$$\operatorname{Hom}^0_{\mathbf{E}}(A, B) \simeq \operatorname{Hom}_{\mathsf{Z}^0(\mathbf{E})}(L^\natural, M^\natural).$$

This is the result of [52, Lemma 3.9].

It follows from the existence of the functors $\widetilde{\Phi}$ and $\widetilde{\Psi}^{\pm}$ that the functors Φ and $\Psi^{\pm}$ transform twists into isomorphisms [52, Lemma 3.11].

Notice that the fully faithful functors $\widetilde{\Phi}$ and $\widetilde{\Psi}^{\pm}$ usually do *not* have either left or right adjoints. Indeed, these functors are embeddings of pretty badly behaved full subcategories, in general [52, Examples 3.2 and 3.3].

2.4. **The DG-functor $\natural\natural$.** For any DG-category $\mathbf{E}$ with shifts and cones, there is a naturally defined fully faithful DG-functor $\natural\natural\colon \mathbf{E} \longrightarrow \mathbf{E}^{\natural\natural}$. In order to construct the DG-functor $\natural\natural$, let us first describe the DG-category $\mathbf{E}^{\natural\natural}$ more explicitly.

The objects of $\mathbf{E}^{\natural\natural}$ are triples $W^{\natural\natural} = (W, \sigma, \tau)$, where W is an object of $\mathbf{E}$ endowed with two endomorphisms $\sigma \in \mathrm{Hom}_{\mathbf{E}}^{-1}(W, W)$ and $\tau \in \mathrm{Hom}_{\mathbf{E}}^{1}(W, W)$. The morphisms σ and τ must satisfy the equations

$$\sigma^2 = 0 = \tau^2, \quad \sigma\tau + \tau\sigma = \mathrm{id}_W, \quad d(\sigma) = \mathrm{id}_W, \quad d(\tau) = 0.$$

Here $W^{\natural} = (W, \sigma)$ is an object of $\mathbf{E}^{\natural}$, and $\tau \in \mathrm{Hom}_{\mathbf{E}^{\natural}}^{-1}(W^{\natural}, W^{\natural})$ is an endomorphism with $d^{\natural}(\tau) = \sigma\tau + \tau\sigma = \mathrm{id}_W$ and $\tau^2 = 0$.

Given two objects $U^{\natural\natural} = (U, \sigma_U, \tau_U)$ and $V^{\natural\natural} = (V, \sigma_V, \tau_U) \in \mathbf{E}^{\natural\natural}$, the complex of morphisms $\mathrm{Hom}_{\mathbf{E}^{\natural\natural}}^{\bullet}(U^{\natural\natural}, V^{\natural\natural})$ is described as follows. For every $n \in \mathbb{Z}$, the group $\mathrm{Hom}_{\mathbf{E}^{\natural\natural}}^{n}(U^{\natural\natural}, V^{\natural\natural})$ is a subgroup in $\mathrm{Hom}_{\mathbf{E}}^{n}(U, V)$ consisting of all the morphisms $f\colon U \longrightarrow V$ of degree n in $\mathbf{E}$ such that $d(f) = 0$ and $d^{\natural}(f) = \sigma_V f - (-1)^n f\sigma_U = 0$. The differential $d^{\natural\natural}$ on $\mathrm{Hom}_{\mathbf{E}^{\natural\natural}}^{\bullet}(U^{\natural\natural}, V^{\natural\natural})$ is given by the rule $d^{\natural\natural}(f) = \tau_V f - (-1)^n f\tau_U$. The composition of morphisms in $\mathbf{E}^{\natural\natural}$ is induced by the composition of morphisms in $\mathbf{E}$ in the obvious way.

The DG-functor $\natural\natural$ assigns to an object $A \in \mathbf{E}$ the object $L = \mathrm{cone}(\mathrm{id}_A[-1])$ endowed with the endomorphisms $\sigma_L = \iota'\pi' \in \mathrm{Hom}_{\mathbf{E}}^{-1}(L, L)$ and $\tau_L = \iota\pi \in \mathrm{Hom}_{\mathbf{E}}^{1}(L, L)$, in the notation from Section 2.2. So $\natural\natural(A) = (L, \sigma_L, \tau_L)$.

To define the action of the DG-functor $\natural\natural$ on morphisms, consider two objects A and $B \in \mathbf{E}$. Put $L = \mathrm{cone}(\mathrm{id}_A[-1])$ and $M = \mathrm{cone}(\mathrm{id}_B[-1])$, and denote by ι_A, π_A, ι'_A, π'_A and ι_B, π_B, ι'_B, π'_B the related morphisms as in Sections 2.2–2.3. Then the DG-functor $\natural\natural$ assigns to a morphism $f \in \mathrm{Hom}_{\mathbf{E}}^{n}(A, B)$ the morphism $g = \natural\natural(f) \in \mathrm{Hom}_{\mathbf{E}^{\natural\natural}}^{n}(\natural\natural(A), \natural\natural(B)) \subset \mathrm{Hom}_{\mathbf{E}}^{n}(L, M)$ given by the formula

$$\natural\natural(f) = g = (-1)^n \iota'_B f \pi_A + \iota_B f \pi'_A + \iota'_B d(f) \pi'_A.$$

One can check that $d(g) = 0$, $d^{\natural}(g) = 0$, $d^{\natural\natural}(g) = \natural\natural(df)$, and $\natural\natural(f_1 \circ f_2) = \natural\natural(f_1) \circ \natural\natural(f_2)$ for any composable pair of morphisms f_1, f_2 in $\mathbf{E}$. It is a result of [52, Proposition 3.5] that the DG-functor $\natural\natural$ is fully faithful.

An additive DG-category $\mathbf{E}$ is said to be *idempotent-complete* if the additive category $\mathsf{Z}^0(\mathbf{E})$ is idempotent-complete (i. e., any idempotent endomorphism in $\mathsf{Z}^0(\mathbf{E})$ arises from a direct sum decomposition). Given an additive DG-category $\mathbf{E}$, the additive DG-category $\mathbf{E}^{\natural}$ is idempotent-complete whenever the additive DG-category $\mathbf{E}$ is. If $\mathbf{E}$ is an idempotent-complete additive DG-category with twists, then the DG-functor $\natural\natural$ is an equivalence of DG-categories. Generally speaking, the DG-category $\mathbf{E}^{\natural\natural}$ is obtained from the DG-category $\mathbf{E}$ by adjoining all twists and some of their direct summands [52, Proposition 3.14].

For any DG-category $\mathbf{E}$ with shifts and cones, there are natural isomorphisms of additive functors

$$\Psi^+_{\mathbf{E}^\natural} \circ \mathsf{Z}^0(\natural\natural) \simeq \Phi_{\mathbf{E}}$$

and

$$\mathsf{Z}^0(\natural\natural) \circ \Psi^-_{\mathbf{E}} \simeq \Phi_{\mathbf{E}^\natural}$$

showing that the "almost involution" $\mathbf{E} \longmapsto \mathbf{E}^{\natural\natural}$ interchanges the roles of the functors Φ and $\Psi^\pm$ [52, Lemmas 3.6 and 3.7]. Moreover, one has [52, Remark 3.10]

$$\widetilde{\Psi}^+_{\mathbf{E}^\natural} \circ (\natural\natural)^0 \simeq \widetilde{\Phi}_{\mathbf{E}}$$

and

$$\mathsf{Z}^0(\natural\natural) \circ \widetilde{\Psi}^-_{\mathbf{E}} \simeq \widetilde{\Phi}_{\mathbf{E}^\natural}.$$

2.5. **Example: DG-category of complexes.** Let E be an additive category. Then the DG-category $\mathbf{C}(\mathsf{E})$ of complexes in E (see Section 1.8) is an additive DG-category with shifts, twists, and cones. Recall the notation $\mathsf{G}(\mathsf{E}) = \mathsf{E}^{\mathbb{Z}}$ for the additive category of graded objects in E and $\mathsf{C}(\mathsf{E}) = \mathsf{Z}^0(\mathbf{C}(\mathsf{E}))$ for the additive category of complexes in E.

The forgetful functor $X^\bullet \longmapsto X^{\bullet\#} \colon \mathsf{C}(\mathsf{E}) \longrightarrow \mathsf{G}(\mathsf{E})$, assigning to a complex $X^\bullet$ in E its underlying graded object $X^* = X^{\bullet\#}$, has adjoints on both sides. The functor $G^+ \colon \mathsf{G}(\mathsf{E}) \longrightarrow \mathsf{C}(\mathsf{E})$ left adjoint to $X^\bullet \longmapsto X^{\bullet\#}$ assigns to a graded object $E^* \in \mathsf{G}(\mathsf{E})$ the complex $G^+(E^*)$ freely generated by E^*. Explicitly, one has $G^+(E^*)^n = E^n \oplus E^{n-1}$ for all $n \in \mathbb{Z}$, and the differential $d_{G^+,n} \colon E^n \oplus E^{n-1} \longrightarrow E^{n+1} \oplus E^n$ is given by the 2×2 matrix of morphisms whose only nonzero entry is the identity map $E^n \longrightarrow E^n$.

The functor $G^- \colon \mathsf{G}(\mathsf{E}) \longrightarrow \mathsf{C}(\mathsf{E})$ left adjoint to $X^\bullet \longmapsto X^{\bullet\#}$ assigns to a graded object $E^* \in \mathsf{G}(\mathsf{E})$ the complex $G^-(E^*)$ cofreely cogenerated by E^*. Explicitly, $G^-(E^*)^n = E^n \oplus E^{n+1}$ for all $n \in \mathbb{Z}$, and the differential $d_{G^-,n} \colon E^n \oplus E^{n+1} \longrightarrow E^{n+1} \oplus E^{n+2}$ is given by the 2×2 matrix of morphisms whose only nonzero entry is the identity map $E^{n+1} \longrightarrow E^{n+1}$. So one has $G^-(E^*) \simeq G^+(E^*)[1]$.

Let us assume that the additive category E is idempotent-complete; then so is the DG-category $\mathbf{C}(\mathsf{E})$. In this case, there is a natural equivalence of additive categories

$$\Upsilon = \Upsilon_{\mathsf{E}} \colon \mathsf{G}(\mathsf{E}) \simeq \mathsf{Z}^0(\mathbf{C}(\mathsf{E})^\natural)$$

[52, Example 3.16]. The equivalence of categories Υ forms the following commutative diagrams of additive functors with the functors $\#$, Φ, $\widetilde{\Phi}$, $G^\pm$, and $\Psi^\pm$:

(1)
$$\begin{tikzcd}
\mathsf{C}(\mathsf{E}) \arrow[r, equal] \arrow[d, "\#"'] & \mathsf{Z}^0(\mathbf{C}(\mathsf{E})) \arrow[d, "\Phi_{\mathbf{C}(\mathsf{E})}"] \\
\mathsf{G}(\mathsf{E}) \arrow[r, Rightarrow, "\Upsilon_{\mathsf{E}}"'] & \mathsf{Z}^0(\mathbf{C}(\mathsf{E})^\natural)
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \mathbf{C}(\mathsf{E})^0 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow, "\widetilde{\Phi}_{\mathbf{C}(\mathsf{E})}"] & \\
\mathsf{G}(\mathsf{E}) \arrow[rr, Rightarrow, "\Upsilon_{\mathsf{E}}"'] & & \mathsf{Z}^0(\mathbf{C}(\mathsf{E})^\natural)
\end{tikzcd}
\tag{2}
$$

and

$$
\begin{tikzcd}
\mathsf{G}(\mathsf{E}) \arrow[r, Rightarrow, "\Upsilon_{\mathsf{E}}"] \arrow[d, "G^\pm"'] & \mathsf{Z}^0(\mathbf{C}(\mathsf{E})^\natural) \arrow[d, "\Psi^\pm_{\mathbf{C}(\mathsf{E})}"] \\
\mathsf{C}(\mathsf{E}) \arrow[r, equal] & \mathsf{Z}^0(\mathbf{C}(\mathsf{E}))
\end{tikzcd}
\tag{3}
$$

Here the upper horizontal double line in (1) and the lower horizontal double line in (3) is essentially the definition of the additive category of complexes $\mathsf{C}(\mathsf{E})$. The leftmost diagonal double arrow in (2) is the obvious equivalence of additive categories mentioned in Section 1.8. The leftmost vertical arrow in (1) is the forgetful functor $X^\bullet \longmapsto X^{\bullet\#} = X^*$. There are, actually, two commutative diagrams depicted in (3): one has to choose either G^+ for the leftmost vertical arrow and Ψ^+ for the rightmost one, or G^- for the leftmost vertical arrow and Ψ^- for the rightmost one. All the double lines and double arrows are category equivalences; all the ordinary arrows are faithful functors [52, Example 3.16].

All objects of the DG-category $\mathbf{C}(\mathsf{E})^\natural$ are contractible [52, Example 3.16]; so the DG-category $\mathbf{C}(\mathsf{E})^\natural$ is quasi-equivalent to the zero DG-category. Still the DG-category $\mathbf{C}(\mathsf{E})^{\natural\natural}$ is equivalent to the DG-category $\mathbf{C}(\mathsf{E})$. So the passage from a DG-category $\mathbf{E}$ to the DG-category $\mathbf{E}^\natural$ does *not* preserve quasi-equivalences.

2.6. **Example: DG-category of CDG-modules.** Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a CDG-ring. Then the DG-category $\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ of left CDG-modules over $\boldsymbol{R}^\bullet$ (see Section 1.9) is an additive DG-category with shifts, twists, and cones (as well as infinite products and coproducts). Recall the notation $R^*\text{–}\mathsf{Mod}$ for the abelian category of graded left modules over the graded ring R^*.

The forgetful functor $\boldsymbol{M}^\bullet \longmapsto \boldsymbol{M}^{\bullet\#} \colon \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) \longrightarrow R^*\text{–}\mathsf{Mod}$, assigning to a CDG-module $\boldsymbol{M}^\bullet = (M^*, d_M)$ over $\boldsymbol{R}^\bullet$ its underlying graded R^*-module $M^* = \boldsymbol{M}^{\bullet\#}$, has adjoints on both sides. The functor $G^+ \colon R^*\text{–}\mathsf{Mod} \longrightarrow \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ left adjoint to $\boldsymbol{M}^\bullet \longmapsto \boldsymbol{M}^{\bullet\#}$ assigns to a graded R^*-module M^* the CDG-module $G^+(M^*)$ freely generated by M^*. The functor $G^- \colon R^*\text{–}\mathsf{Mod} \longrightarrow \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ right adjoint to $\boldsymbol{M}^\bullet \longmapsto \boldsymbol{M}^{\bullet\#}$ assigns to a graded R^*-module M^* the CDG-module $G^-(M^*)$ cofreely cogenerated by M^*.

Explicit constructions of the functors G^+ and G^- for CDG-modules over a CDG-ring can be found in [48, proof of Theorem 3.6] or [52, Proposition 3.1] (see also [5, Proposition 1.3.2]). For any graded left R^*-module M^*, there is a natural isomorphism of CDG-modules $G^-(M) \simeq G^+(M)[1]$ [52, Proposition 3.1(c)].

The abelian category $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ of left CDG-modules over a CDG-ring $\boldsymbol{R}^\bullet = (R^*, d, h)$ can be described as the category of graded modules over the following graded ring $R^*[\delta]$. The ring $R^*[\delta]$ is obtained by adjoining to the graded ring R^* a new element δ, which is homogeneous of degree 1 and subject to the relations

- $\delta r - (-1)^{|r|} r\delta = d(r)$ for all $r \in R^{|r|}$, $|r| \in \mathbb{Z}$;
- $\delta^2 = h$.

The ring R^* is a graded subring in $R^*[\delta]$. Viewed as either a graded left R^*-module or a graded right R^*-module, the graded ring $R^*[\delta]$ is a free graded R^*-module with two generators 1 and δ. We refer to [52, Section 3.1] or [50, Section 4.2] for a further discussion of this construction. Given a left CDG-module $\boldsymbol{M}^\bullet = (M^*, d_M)$ over $\boldsymbol{R}^\bullet$, the action of the graded ring R^* in M^* is extended to an action of the graded ring $R^*[\delta]$ by the obvious rule $\delta \cdot x = d_M(x)$ for all $x \in M^*$.

The graded ring $R^*[\delta]$ is endowed with an odd derivation $\partial = \partial/\partial\delta$ of degree -1 defined by the rules $\partial(r) = 0$ for all $r \in R^*$ and $\partial(\delta) = 1$. In order to make ∂ a cohomological differential, we follow the notation in [52] and denote by $\widehat{\boldsymbol{R}}^*$ the graded ring $R^*[\delta]$ with the sign of the grading changed: $\widehat{\boldsymbol{R}}^n = R^*[\delta]^{-n}$ for all $n \in \mathbb{Z}$. Then the graded ring $\widehat{\boldsymbol{R}}^*$ with the differential ∂ becomes a DG-ring, which we will denote by $\widehat{\boldsymbol{R}}^\bullet = (\widehat{\boldsymbol{R}}^*, \partial)$. Notice that $\widehat{\boldsymbol{R}}^\bullet$ is an *acyclic* DG-ring: one has $H_\partial(\widehat{\boldsymbol{R}}^\bullet) = 0$, since the unit element vanishes in the cohomology ring $H_\partial(\widehat{\boldsymbol{R}}^\bullet)$.

The DG-category $(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural$ is naturally equivalent to the DG-category $\widehat{\boldsymbol{R}}^\bullet\text{–}\mathbf{Mod}$ of DG-modules over the DG-ring $\widehat{\boldsymbol{R}}^\bullet$. Furthermore, there is a natural equivalence of abelian categories

$$\Upsilon = \Upsilon_{\boldsymbol{R}^\bullet} \colon R^*\text{–}\mathsf{Mod} \simeq \mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural)$$

[52, Example 3.17]. The equivalence of categories $\Upsilon_{\boldsymbol{R}^\bullet}$ transforms the functor $\#$ into the functor Φ, the functors $G^\pm$ into the functors $\Psi^\pm$, and the natural inclusion $(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^0 \longrightarrow R^*\text{–}\mathsf{Mod}$ into the functor $\widetilde{\Phi}$. In other words, there are the following commutative diagrams of additive functors:

$$\begin{tikzcd}
 & \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) \arrow[dl, "\#"'] \arrow[dr, "\Phi_{\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}}"] & \\
R^*\text{–}\mathsf{Mod} \arrow[rr, Rightarrow, "\Upsilon_{\boldsymbol{R}^\bullet}"'] & & \mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural)
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
 & (\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^0 \arrow[dl, tail] \arrow[dr, tail, "\widetilde{\Phi}_{\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}}"] & \\
R^*\text{–}\mathsf{Mod} \arrow[rr, Rightarrow, "\Upsilon_{\boldsymbol{R}^\bullet}"'] & & \mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural)
\end{tikzcd} \tag{5}$$

and

$$
(6)\qquad
\begin{tikzcd}
R^*\text{–}\mathsf{Mod} \arrow[rr, Rightarrow, "\Upsilon_{\boldsymbol{R}^\bullet}"] \arrow[dr, "G^\pm"'] & & \mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural) \arrow[dl, tail, "\Psi^\pm_{\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}}"] \\
& \mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) &
\end{tikzcd}
$$

Here the leftmost diagonal arrow in (4) is the forgetful functor $\boldsymbol{M}^\bullet \longmapsto \boldsymbol{M}^{\bullet\#} = M^*$. The leftmost diagonal arrow in (5) is the obvious fully faithful inclusion of additive categories mentioned in Section 1.9. There are, actually, two commutative diagrams depicted in (6): one has to choose either G^+ for the leftmost diagonal arrow and Ψ^+ for the rightmost one, or G^- for the leftmost diagonal arrow and Ψ^- for the rightmost one. The horizontal double arrow (which is the same on all the three diagrams) is an abelian category equivalence. The arrows with tails are fully faithful functors; the ordinary arrows are faithful exact functors [52, Example 3.17].

Iterating the passage from $\boldsymbol{R}^\bullet$ to $\widehat{\boldsymbol{R}}^\bullet$, we obtain an acyclic DG-ring $\widehat{\widehat{\boldsymbol{R}}}^\bullet$. Iterating the assertion above, we conclude that the DG-category $(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^{\natural\natural}$ is naturally equivalent to $\widehat{\widehat{\boldsymbol{R}}}^\bullet\text{–}\mathbf{Mod}$. On the other hand, $\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ is an idempotent-complete additive DG-category with twists (indeed, the additive category $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ is abelian, hence idempotent-complete). So Section 2.4 tells that the DG-functor $\natural\natural\colon \boldsymbol{R}^\bullet\text{–}\mathbf{Mod} \longrightarrow (\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^{\natural\natural}$ is an equivalence of DG-categories. Thus we obtain an equivalence of DG-categories

$$\boldsymbol{R}^\bullet\text{–}\mathbf{Mod} \simeq \widehat{\widehat{\boldsymbol{R}}}^\bullet\text{–}\mathbf{Mod}$$

showing that the DG-category of CDG-modules over any CDG-ring $\boldsymbol{R}^\bullet = (R^*, d, h)$ is equivalent to the DG-category of DG-modules over an acyclic DG-ring $\widehat{\widehat{\boldsymbol{R}}}^\bullet$.

2.7. **Example: DG-category of factorizations.** Let E be an additive category, $\Delta\colon \mathsf{E} \longrightarrow \mathsf{E}$ be an auto-equivalence, and $v\colon \mathrm{Id}_{\mathsf{E}} \longrightarrow \Delta$ be a potential (see Section 1.10). Then the DG-category $\mathbf{F}(\mathsf{E}, \Delta, v)$ of factorizations of v in E is an additive DG-category with shifts, twists, and cones. Recall the notation $\mathsf{P}(\mathsf{E}, \Delta) \simeq \mathsf{E} \times \mathsf{E}$ for the additive category of 2-Δ-periodic objects in E and $\mathsf{F}(\mathsf{E}, \Delta, v) = \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v))$ for the additive category of factorizations of v.

The forgetful functor $X^\bullet \longmapsto X^{\bullet\#}\colon \mathsf{F}(\mathsf{E}, \Delta, v) \longrightarrow \mathsf{P}(\mathsf{E}, \Delta)$, assigning to a factorization $X^\bullet$ its underlying 2-Δ-periodic object $X^\circ = X^{\bullet\#}$, has adjoints on both sides. The functor $G^+\colon \mathsf{P}(\mathsf{E}, \Delta) \longrightarrow \mathsf{F}(\mathsf{E}, \Delta, v)$ left adjoint to $X^\bullet \longmapsto X^{\bullet\#}$ assigns to a 2-Δ-periodic object $E^\circ \in \mathsf{P}(\mathsf{E}, \Delta)$ the factorization $G^+(E^\circ)$ freely generated by E°. Explicitly, one has $G^+(E^\circ)^n = E^n \oplus E^{n-1}$ for all $n \in \mathbb{Z}$, and the differential $d_{G^+,n}\colon E^n \oplus E^{n-1} \longrightarrow E^{n+1} \oplus E^n$ is given by the 2×2 matrix of morphisms whose only nonzero entries are the identity morphism $E^n \longrightarrow E^n$ and the composition $\delta_E^{n+1,n-1} \circ v_{E^{n-1}}\colon E^{n-1} \longrightarrow \Delta(E^{n-1}) \longrightarrow E^{n+1}$. The functor $G^-\colon \mathsf{P}(\mathsf{E}, \Delta) \longrightarrow \mathsf{F}(\mathsf{E}, \Delta, v)$ right adjoint to $X^\bullet \longmapsto X^{\bullet\#}$ assigns to a 2-Δ-periodic object $E^\circ \in \mathsf{P}(\mathsf{E}, \Delta)$ the factorization $G^-(E^\circ)$ cofreely cogenerated by E°. Explicitly, $G^-(E^\circ)^n = E^n \oplus E^{n+1}$ for all $n \in \mathbb{Z}$, and $G^-(E^\circ) \simeq G^+(E^\circ)[1]$.

Assume that the additive category E is idempotent-complete; then so is the DG-category $\mathbf{F}(\mathsf{E}, \Delta, v)$. In this case, there is a natural equivalence of additive categories

$$\Upsilon = \Upsilon_{\mathsf{E},\Delta,v} \colon \mathsf{P}(\mathsf{E}, \Delta) \simeq \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)^\natural)$$

(cf. [52, Example 3.19 and Remark 2.7]). The equivalence of categories Υ transforms the functor $\#$ into the functor Φ, the functors $G^\pm$ into the functors $\Psi^\pm$, and the natural inclusion $\mathbf{F}(\mathsf{E}, \Delta, v)^0 \longrightarrow \mathsf{P}(\mathsf{E}, \Delta)$ into the functor $\widetilde{\Phi}$. In other words, there are the following commutative diagrams of additive functors:

(7)
$$\begin{tikzcd}
\mathsf{F}(\mathsf{E}, \Delta, v) \arrow[r, equal] \arrow[d, "\#"'] & \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)) \arrow[d, "\Phi_{\mathbf{F}(\mathsf{E},\Delta,v)}"] \\
\mathsf{P}(\mathsf{E}, \Delta) \arrow[r, Rightarrow, "\Upsilon_{\mathsf{E},\Delta,v}"'] & \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)^\natural)
\end{tikzcd}$$

(8)
$$\begin{tikzcd}
 & \mathbf{F}(\mathsf{E}, \Delta, v)^0 \arrow[dl, tail] \arrow[dr, tail, "\widetilde{\Phi}_{\mathbf{F}(\mathsf{E},\Delta,v)}"] & \\
\mathsf{P}(\mathsf{E}, \Delta) \arrow[rr, Rightarrow, "\Upsilon_{\mathsf{E},\Delta,v}"'] & & \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)^\natural)
\end{tikzcd}$$

and

(9)
$$\begin{tikzcd}
\mathsf{P}(\mathsf{E}, \Delta) \arrow[r, Rightarrow, "\Upsilon_{\mathsf{E},\Delta,v}"] \arrow[d, "G^\pm"'] & \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)^\natural) \arrow[d, "\Psi^\pm_{\mathbf{F}(\mathsf{E},\Delta,v)}"] \\
\mathsf{F}(\mathsf{E}, \Delta, v) \arrow[r, equal] & \mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v))
\end{tikzcd}$$

Here the upper horizontal double line in (7) and the lower horizontal double line in (9) is the definition of the additive category of factorizations $\mathsf{F}(\mathsf{E}, \Delta, v)$. The leftmost diagonal double arrow in (8) is the obvious fully faithful inclusion of additive categories mentioned in Section 1.10. The leftmost vertical arrow in (7) is the forgetful functor $X^\bullet \longmapsto X^{\bullet\#} = X^\circ$. There are, actually, two commutative diagrams depicted in (9): one has to choose either G^+ for the leftmost vertical arrow and Ψ^+ for the rightmost one, or G^- for the leftmost vertical arrow and Ψ^- for the rightmost one. All the double lines and double arrows are category equivalences; the arrows with tails are fully faithful functors; the ordinary arrows are faithful functors.

3. Abelian DG-Categories

This section is mostly an extraction from [52, Sections 4.1, 4.3, 4.5, 4.6, and 9.3].

3.1. **Exactness properties of the functors Ξ and Ψ.** Let $\mathbf{E}$ be a DG-category and $f\colon A \longrightarrow B$ be a closed morphism of degree 0 in $\mathbf{E}$ such that the shift $A[1]$ and the cone $\mathrm{cone}(f)$ exist in $\mathbf{E}$. Consider the pair of natural closed morphisms $B \longrightarrow \mathrm{cone}(f) \longrightarrow A[1]$ of degree 0 in $\mathbf{E}$.

Lemma 3.1. *In the pair of natural morphisms $B \longrightarrow \mathrm{cone}(f) \longrightarrow A[1]$ in the preadditive category $\mathsf{Z}^0(\mathbf{E})$, the former morphism is a kernel of the latter one and the latter morphism is a cokernel of the former one.*

Proof. See [52, proof of Lemma 4.2]. □

Let $\mathbf{E}$ be an additive DG-category with shifts and cones. As in Section 2.2, we denote by $\Xi = \Xi_{\mathbf{E}}\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ the additive functor $\Xi(A) = \mathrm{cone}(\mathrm{id}_A[-1])$. The endofunctor $\Xi\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ is left and right adjoint to its own shift $\Xi[1]\colon A \longmapsto \mathrm{cone}(\mathrm{id}_A)$; hence the functor Ξ preserves all limits and colimits (in particular, all kernels and cokernels) that exist in the additive category $\mathsf{Z}^0(\mathbf{E})$ [52, Lemma 4.1]. The endofunctor $\Xi\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ is faithful and conservative [52, Lemma 4.3]. The endofunctor Ξ can be naturally extended to a faithful additive functor $\widetilde{\Xi} = \widetilde{\Xi}_{\mathbf{E}}\colon \mathbf{E}^0 \longrightarrow \mathsf{Z}^0(\mathbf{E})$ [52, Lemma 4.5].

Corollary 3.2. *Let $\mathbf{E}$ be an additive DG-category with shifts and cones. Then*

(a) *for any object $A \in \mathbf{E}$, there is a pair of natural morphisms $A[-1] \rightarrowtail \Xi(A) \twoheadrightarrow A$ in $\mathsf{Z}^0(\mathbf{E})$ in which the former morphism is a kernel of the latter one and the latter morphism is a cokernel of the former one;*

(b) *any object $A \in \mathbf{E}$ is the cokernel of a natural morphism $\Xi(A)[-1] \longrightarrow \Xi(A)$ in $\mathsf{Z}^0(\mathbf{E})$.*

Proof. Part (a) is a particular case of Lemma 3.1. To deduce part (b) from part (a), consider the composition $\Xi(A)[-1] \twoheadrightarrow A[-1] \rightarrowtail \Xi(A)$ and observe that its cokernel agrees with the cokernel of the morphism $A[-1] \rightarrowtail \Xi(A)$, since the morphism $\Xi(A)[-1] \twoheadrightarrow A[-1]$ is an epimorphism. □

The following lemma is trivial but useful.

Lemma 3.3. *Let $\mathbf{E}$ be a DG-category with shifts and cones. Then any contractible object in $\mathbf{E}$ is a direct summand of an object $\Xi(A)$, for a suitable object $A \in \mathbf{E}$.*

Proof. For any two homotopic closed morphisms f', $f''\colon A \longrightarrow B$ of degree 0 in $\mathbf{E}$, the cones of f' and f'' are isomorphic in $\mathsf{Z}^0(\mathbf{E})$. In particular, if an object $A \in \mathbf{E}$ is contractible, then the object $\Xi(A) = \mathrm{cone}(\mathrm{id}_A[-1])$ is isomorphic to the cone of zero morphism $\mathrm{cone}(0\colon A[-1] \to A[-1]) \simeq A \oplus A[-1]$ as objects of $\mathsf{Z}^0(\mathbf{E})$. Consequently, A is a direct summand of $\Xi(A)$. □

Lemma 3.4. *Let $\mathbf{E}$ be an additive DG-category with shifts and cones, and let $A \xrightarrow{f} B \xrightarrow{g} C$ be a composable pair of morphisms in the additive category $\mathsf{Z}^0(\mathbf{E})$. In this context:*

(a) *if the morphism $\Xi(f)$ is a kernel of the morphism $\Xi(g)$, then the morphism f is a kernel of the morphism g in $\mathsf{Z}^0(\mathbf{E})$;*

(b) *if the morphism* $\Xi(g)$ *is a cokernel of the morphism* $\Xi(f)$*, then the morphism* g *is a cokernel of the morphism* f *in* $\mathsf{Z}^0(\mathbf{E})$.

Proof. This is [52, Lemma 4.4]. □

The definition of an *idempotent-complete* additive DG-category was given in Section 2.4.

Lemma 3.5. *Let* $\mathbf{E}$ *be an idempotent-complete additive DG-category with shifts and twists. In this setting:*

(a) *if* g *is a morphism in* $\mathsf{Z}^0(\mathbf{E})$ *and the morphism* $\Xi(g)$ *has a kernel in* $\mathsf{Z}^0(\mathbf{E})$*, then the morphism* g *also has a kernel in* $\mathsf{Z}^0(\mathbf{E})$*;*

(b) *if* f *is a morphism in* $\mathsf{Z}^0(\mathbf{E})$ *and the morphism* $\Xi(f)$ *has a cokernel in* $\mathsf{Z}^0(\mathbf{E})$*, then the morphism* f *also has a cokernel in* $\mathsf{Z}^0(\mathbf{E})$.

Proof. This is [52, Lemma 4.6]. □

The DG-category $\mathbf{E}^\natural$ was constructed in Section 2.1 and the additive functor $\Psi^+\colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ was defined in Section 2.2. The functor Ψ^+ has adjoints on both sides, so it preserves all limits and colimits (in particular, all kernels and cokernels) that exist in $\mathsf{Z}^0(\mathbf{E}^\natural)$.

Lemma 3.6. *Let* $\mathbf{E}$ *be an additive DG-category with shifts and cones. In this setting:*

(a) *if* g *is a morphism in* $\mathsf{Z}^0(\mathbf{E}^\natural)$ *and the morphism* $\Psi^+(g)$ *has a kernel in* $\mathsf{Z}^0(\mathbf{E})$*, then the morphism* g *has a kernel in* $\mathsf{Z}^0(\mathbf{E}^\natural)$*;*

(b) *if* f *is a morphism in* $\mathsf{Z}^0(\mathbf{E}^\natural)$ *and the morphism* $\Psi^+(f)$ *has a cokernel in* $\mathsf{Z}^0(\mathbf{E})$*, then the morphism* f *has a cokernel in* $\mathsf{Z}^0(\mathbf{E}^\natural)$.

Proof. This is [52, Lemma 4.16]. □

3.2. **Equivalent characterizations of abelian DG-categories.** An additive DG-category $\mathbf{E}$ with shifts and cones is called *abelian* if both the additive categories $\mathsf{Z}^0(\mathbf{E})$ and $\mathsf{Z}^0(\mathbf{E}^\natural)$ are abelian [52, Section 4.5]. The following lemma is helpful.

Lemma 3.7. *Let* A *be an additive category with kernels and cokernels, let* B *be an abelian category, and let* $\Theta\colon \mathsf{A} \longrightarrow \mathsf{B}$ *be a conservative additive functor preserving kernels and cokernels. Then* A *is also an abelian category.*

Proof. This is [52, Lemma 4.31(a)]. For any morphism f in A, consider the natural morphism g from the coimage to the image of f. Then the morphism $\Theta(g)$ is an isomorphism, since the functor Θ preserves kernels and cokernels (hence images and coimages), and the category B is abelian. Since the functor Θ is conservative by assumption, we can conclude that g is an isomorphism. □

The following proposition, which is a simplified restatement of [52, Corollary 4.37], lists equivalent characterizations of abelian DG-categories.

Proposition 3.8. *Let* $\mathbf{E}$ *be an additive DG-category with shifts and cones. Then the following conditions are equivalent:*

(1) *the DG-category* $\mathbf{E}$ *is abelian;*

(2) *the additive category* $\mathsf{Z}^0(\mathbf{E})$ *is abelian;*
(3) *all kernels and cokernels exist in the additive category* $\mathsf{Z}^0(\mathbf{E})$*, and the additive category* $\mathsf{Z}^0(\mathbf{E}^\natural)$ *is abelian;*
(4) *the DG-category* $\mathbf{E}$ *is idempotent-complete and has all twists, and the additive category* $\mathsf{Z}^0(\mathbf{E}^\natural)$ *is abelian.*

Proof. The implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are obvious.

(2) $\Longrightarrow$ (1) To prove that the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$ is abelian, apply Lemma 3.7 to the additive functor $\Psi^+\colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ and use Lemma 3.6.

(3) $\Longrightarrow$ (1) To prove that the additive category $\mathsf{Z}^0(\mathbf{E})$ is abelian, apply Lemma 3.7 to the additive functor $\Phi\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$, which preserves all limits and colimits (hence all kernels and cokernels) because it has adjoints on both sides. The functor Φ is conservative by [52, Lemma 3.12], as mentioned in Section 2.2.

(4) $\Longrightarrow$ (3) We need to prove the existence of kernels and cokernels in the additive category $\mathsf{Z}^0(\mathbf{E})$. Let f be a morphism in $\mathsf{Z}^0(\mathbf{E})$. Then the morphism $\Phi(f)$ has a kernel in $\mathsf{Z}^0(\mathbf{E}^\natural)$, since the category $\mathsf{Z}^0(\mathbf{E}^\natural)$ is abelian by assumption. Hence the morphism $\Psi^+\Phi(f)$ has a kernel in $\mathsf{Z}^0(\mathbf{E})$, since the functor Ψ^+ preserves kernels. By [52, Lemma 3.8], the morphism $\Psi^+\Phi(f)$ is isomorphic to the morphism $\Xi(f)$ (see the end of Section 2.2). So the morphism $\Xi(f)$ has a kernel in $\mathsf{Z}^0(\mathbf{E})$, and it remains to apply Lemma 3.5 in order to conclude that the morphism f has a kernel in $\mathsf{Z}^0(\mathbf{E})$. The existence of cokernels is provable similarly.

(1) $\Longrightarrow$ (4) The additive category $\mathsf{Z}^0(\mathbf{E})$ is abelian, hence idempotent-complete, which means that the DG-category $\mathbf{E}$ is idempotent-complete. The assertion about existence of twists is so important that we formulate it as a separate proposition. $\square$

Proposition 3.9. *All twists exist in any abelian DG-category.*

Proof. This is [52, Proposition 4.35]. The point is that, for any object A in an abelian DG-category $\mathbf{E}$, all twists of A can be produced as the cokernels of suitable morphisms $\Xi(A)[-1] \longrightarrow \Xi(A)$ (cf. Corollary 3.2(b)). Let us empasize that the functor Ξ takes twists to isomorphisms [52, Lemma 3.11 or 4.5]. The discussion in [44, Section 7] sheds some additional light on this argument. $\square$

Corollary 3.10. *For any abelian DG-category* $\mathbf{E}$*, the DG-functor* $\natural\natural\colon \mathbf{E} \longrightarrow \mathbf{E}^{\natural\natural}$ *is an equivalence of DG-categories.*

Proof. This is [52, Corollary 4.36 or proof of Proposition 4.35]. Indeed, any abelian DG-category is obviously idempotent-complete, so the assertion follows from Proposition 3.9 and the discussion in Section 2.4. $\square$

3.3. **Exactly embedded full abelian DG-subcategories.** Let A be an abelian category and $\mathsf{B} \subset \mathsf{A}$ be a full subcategory. We will say that B is an *exactly embedded full abelian subcategory* in A if B is an abelian category and the inclusion $\mathsf{B} \rightarrowtail \mathsf{A}$ is an exact functor of abelian categories. Equivalently, a strictly full subcategory $\mathsf{B} \subset \mathsf{A}$ is an exactly embedded full abelian subcategory if and only if it is closed under finite direct sums, kernels, and cokernels in A.

Let $\mathbf{A}$ be a DG-category and $\mathbf{B} \subset \mathbf{A}$ be a (strictly) full DG-subcategory. Then the DG-category $\mathbf{B}^\natural$ can be naturally viewed as a (strictly) full DG-subcategory in $\mathbf{A}^\natural$. Assuming that the DG-category $\mathbf{A}$ has shifts and cones and the full subcategory $\mathbf{B}$ is closed under shifts and cones in $\mathbf{A}$, the functors $\Phi_\mathbf{B}$ and $\Psi^\pm_\mathbf{B}$ agree with the functors $\Phi_\mathbf{A}$ and $\Psi^\pm_\mathbf{A}$, respectively. Similarly, the DG-functor $\natural\natural_\mathbf{B} \colon \mathbf{B} \longrightarrow \mathbf{B}^{\natural\natural}$ agrees with the DG-functor $\natural\natural_\mathbf{A} \colon \mathbf{A} \longrightarrow \mathbf{A}^{\natural\natural}$.

Let $\mathbf{A}$ be an abelian DG-category and $\mathbf{B} \subset \mathbf{A}$ be a full DG-subcategory closed under finite direct sums, shifts, and cones. We will say that $\mathbf{B}$ is an *exactly embedded full abelian DG-subcategory* in $\mathbf{A}$ if $\mathbf{B}$ is an abelian DG-category and both the fully faithful inclusions $\mathsf{Z}^0(\mathbf{B}) \rightarrowtail \mathsf{Z}^0(\mathbf{A})$ and $\mathsf{Z}^0(\mathbf{B}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{A}^\natural)$ are exact functors of abelian categories [52, Section 9.3].

Proposition 3.11. *Let $\mathbf{A}$ be an abelian DG-category and $\mathbf{B} \subset \mathbf{A}$ be a (strictly) full DG-subcategory closed under finite direct sums, shifts, and cones. Then the following conditions are equivalent:*

(1) $\mathbf{B}$ *is an exactly embedded full abelian DG-subcategory in* $\mathbf{A}$*;*
(2) $\mathsf{Z}^0(\mathbf{B})$ *is an exactly embedded full abelian subcategory in* $\mathsf{Z}^0(\mathbf{A})$*;*
(3) *the full DG-subcategory $\mathbf{B} \subset \mathbf{A}$ is closed under twists and direct summands, and $\mathsf{Z}^0(\mathbf{B}^\natural)$ is an exactly embedded full abelian subcategory in $\mathsf{Z}^0(\mathbf{A}^\natural)$.*

Proof. The implication (1) $\Longrightarrow$ (2) is obvious.

(1) $\Longrightarrow$ (3) For any abelian DG-category $\mathbf{B}$, the additive category $\mathsf{Z}^0(\mathbf{B})$ is abelian, hence idempotent-complete, hence closed under direct summands in any ambient additive category $\mathsf{Z}^0(\mathbf{A})$. Similarly, any abelian DG-category $\mathbf{B}$ has all twists by Proposition 3.9, hence $\mathbf{B}$ is closed under twists in any ambient DG-category $\mathbf{A}$.

(2) $\Longrightarrow$ (1) We need to prove that the full subcategory $\mathsf{Z}^0(\mathbf{B}^\natural)$ is closed under kernels and cokernels in $\mathsf{Z}^0(\mathbf{A}^\natural)$. For this purpose we observe that, given an object $K \in \mathsf{Z}^0(\mathbf{A}^\natural)$, one has $K \in \mathsf{Z}^0(\mathbf{B}^\natural)$ whenever $\Psi^+(K) \in \mathsf{Z}^0(\mathbf{B})$. Let $f \colon X \longrightarrow Y$ be a morphism in $\mathsf{Z}^0(\mathbf{B}^\natural)$ and K be the kernel of f computed in $\mathsf{Z}^0(\mathbf{A}^\natural)$. Then $\Psi^+(K)$ is the kernel of $\Psi^+(f)$ in $\mathsf{Z}^0(\mathbf{A})$. Since the full subcategory $\mathsf{Z}^0(\mathbf{B})$ is closed under kernels in $\mathsf{Z}^0(\mathbf{A})$, it follows that $\Psi^+(K) \in \mathsf{Z}^0(\mathbf{B})$, hence $K \in \mathsf{Z}^0(\mathbf{B}^\natural)$. The proof of the closure under cokernels is similar.

(3) $\Longrightarrow$ (1) We need to prove that the full subcategory $\mathsf{Z}^0(\mathbf{B})$ is closed under kernels and cokernels in $\mathsf{Z}^0(\mathbf{A})$. Let $f \colon M \longrightarrow N$ be a morphism in $\mathsf{Z}^0(\mathbf{B})$. Then $\Phi(f) \colon \Phi(M) \longrightarrow \Phi(N)$ is a morphism in $\mathsf{Z}^0(\mathbf{B}^\natural)$. Since the full subcategory $\mathsf{Z}^0(\mathbf{B}^\natural)$ is closed under kernels and cokernels in the abelian category $\mathsf{Z}^0(\mathbf{A}^\natural)$, the morphism $\Phi(f)$ has the same kernel K in both $\mathsf{Z}^0(\mathbf{A}^\natural)$ and $\mathsf{Z}^0(\mathbf{B}^\natural)$. Hence the object $\Psi^+(K) \in \mathsf{Z}^0(\mathbf{B})$ is the kernel of the morphism $\Psi^+\Phi(f)$ in both $\mathsf{Z}^0(\mathbf{A})$ and $\mathsf{Z}^0(\mathbf{B})$.

Let L be a kernel of the morphism f in the abelian category $\mathsf{Z}^0(\mathbf{A})$. Then the object $\Xi(L)$ is a kernel of the morphism $\Xi(f)$ in $\mathsf{Z}^0(\mathbf{A})$. As the morphism $\Xi(f)$ is isomorphic to $\Psi^+\Phi(f)$, we see that $\Xi(L) \simeq \Psi^+(K) \in \mathsf{Z}^0(\mathbf{B})$. It remains to observe that the object L is a direct summand of $L \oplus L[-1]$, and the latter is a twist of $\Xi(L)$ in $\mathbf{A}$, in order to conclude that $L \in \mathbf{B}$ (since the full DG-subcategory $\mathbf{B}$ is closed under twists and direct summands in $\mathbf{A}$). The argument for cokernels is similar. $\square$

Example 3.12. The following counterexample [52, Example 4.30] shows that a full DG-subcategory $\mathbf{B}$ closed under finite direct sums, shifts, and cones in an abelian DG-category $\mathbf{A}$ need *not* be an abelian DG-category even when the additive category $\mathsf{Z}^0(\mathbf{B}^\natural)$ is abelian. In particular, an additive DG-category $\mathbf{E}$ with shifts and cones need *not* be abelian when the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$ is abelian.

Let $\mathbf{A}$ be the DG-category of complexes of vector spaces over a field k and $\mathbf{B} \subset \mathbf{A}$ be the full DG-subcategory of acyclic complexes. Then the full DG-subcategory $\mathbf{B}^\natural$ coincides with the whole ambient DG-category $\mathbf{A}^\natural$, and $\mathsf{Z}^0(\mathbf{B}^\natural) = \mathsf{Z}^0(\mathbf{A}^\natural)$ is an abelian category equivalent to the category of graded k-vector spaces. But the additive category of acyclic complexes of k-vector spaces $\mathsf{Z}^0(\mathbf{B})$ is not abelian.

3.4. **Examples.** The following three classes of abelian DG-categories are the main intended examples in this paper.

Example 3.13. Let E be an abelian category. Then the DG-category $\mathbf{C}(\mathsf{E})$ of complexes in E has finite direct sums, shifts, twists, and cones (see Section 1.8). The additive category $\mathsf{Z}^0(\mathbf{C}(\mathsf{E}))$ of complexes in E and closed morphisms of degree 0 between them is well-known to be abelian. According to Section 2.5, the additive category $\mathsf{Z}^0(\mathbf{C}(\mathsf{E})^\natural)$ is equivalent to the category $\mathsf{G}(\mathsf{E})$ of graded objects in E, which is also abelian. Thus $\mathbf{C}(\mathsf{E})$ is an abelian DG-category.

Example 3.14. Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a CDG-ring. Then the DG-category $\boldsymbol{R}^\bullet$–**Mod** of left CDG-modules over $\boldsymbol{R}^\bullet$ has finite direct sums (as well as infinite products and coproducts), shifts, twists, and cones (see Section 1.9). According to Section 2.6, the additive category $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ of left CDG-modules over $\boldsymbol{R}^\bullet$ and closed morphisms of degree 0 between them is abelian (in fact, it is the category of graded modules over a graded ring $\widehat{\boldsymbol{R}}^*$). Furthermore, the additive category $\mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural)$ is equivalent to the category R^*–**Mod** of graded modules over the graded ring R^*, so it is also abelian. Therefore, $\boldsymbol{R}^\bullet$–**Mod** is an abelian DG-category.

Example 3.15. Let E be an abelian category, $\Delta\colon \mathsf{E} \longrightarrow \mathsf{E}$ be an autoequivalence, and $v\colon \mathrm{Id}_\mathsf{E} \longrightarrow \Delta$ be a potential (see Section 1.10). Then the DG-category of factorizations $\mathbf{F}(\mathsf{E}, \Delta, v)$ has finite direct sums, shifts, twists, and cones. One can easily check that the additive category $\mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v))$ is abelian. According to Section 2.7, the additive category $\mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)^\natural)$ is equivalent to the category $\mathsf{P}(\mathsf{E}, \Delta)$ of 2-Δ-periodic objects in E, which is also abelian (being, in turn, equivalent to $\mathsf{E} \times \mathsf{E}$). Thus $\mathbf{F}(\mathsf{E}, \Delta, v)$ is an abelian DG-category.

For further examples of abelian DG-categories we refer the reader to [52, Example 4.41]. Notice that Example 3.15 is a particular case of [52, Example 4.42], as per [52, Remark 2.7].

4. Cotorsion Pairs

The concept of a complete cotorsion pair goes back to Salce [60]. The material of this section is an accumulated result of the work of many people over the decades. The paper by Eklof and Trlifaj [15] was the main development. The exposition below is an extraction from [55, Sections 1 and 3], which in turn is based on [54, Sections 3–4]. All proofs are omitted and replaced with references.

4.1. **Complete cotorsion pairs.** Let E be an abelian category and L, $\mathsf{R} \subset \mathsf{E}$ be two classes of objects in E. We will denote by $\mathsf{L}^{\perp_1} \subset \mathsf{E}$ the class of all objects $X \in \mathsf{E}$ such that $\mathrm{Ext}^1_{\mathsf{E}}(L, X) = 0$ for all $L \in \mathsf{L}$. Similarly, ${}^{\perp_1}\mathsf{R} \subset \mathsf{E}$ denotes the class of all objects $Y \in \mathsf{E}$ such that $\mathrm{Ext}^1_{\mathsf{E}}(Y, R) = 0$ for all $R \in \mathsf{R}$.

A pair of classes of objects (L, R) in E is called a *cotorsion pair* if $\mathsf{R} = \mathsf{L}^{\perp_1}$ and $\mathsf{L} = {}^{\perp_1}\mathsf{R} \subset \mathsf{E}$. A cotorsion pair (L, R) in E is said to be *complete* if for every object $E \in \mathsf{E}$ there exist short exact sequences

$$0 \longrightarrow R' \longrightarrow L \longrightarrow E \longrightarrow 0, \tag{10}$$

$$0 \longrightarrow E \longrightarrow R \longrightarrow L' \longrightarrow 0 \tag{11}$$

in E with L, $L' \in \mathsf{L}$ and R, $R' \in \mathsf{R}$.

Given a cotorsion pair (L, R) in E, the short exact sequence (10) is called a *special precover sequence*, and an epimorphism $L \longrightarrow E$ in E with $L \in \mathsf{L}$ and a kernel in R is called a *special* L*-precover* of an object $E \in \mathsf{E}$. The short exact sequence (11) is called a *special preenvelope sequence*, and a monomorphism $E \longrightarrow R$ in E with $R \in \mathsf{R}$ and a cokernel in L is called a *special* R*-preenvelope* of an object $E \in \mathsf{E}$. The exact sequences (10–11) are collectively referred to as the *approximation sequences*.

A class of objects $\mathsf{L} \subset \mathsf{E}$ is said to be *generating* if for every object $E \in \mathsf{E}$ there exists an epimorphism $L \twoheadrightarrow E$ in E with $L \in \mathsf{L}$. A class of objects $\mathsf{R} \subset \mathsf{E}$ is said to be *cogenerating* if for every object $E \in \mathsf{E}$ there exists a monomorphism $E \rightarrowtail R$ in E with $R \in \mathsf{R}$. Clearly, in a complete cotorsion pair (L, R), the class L must be generating and the class R must be cogenerating. But in a noncomplete cotorsion pair this need not be the case, unless it is assumed additionally that the abelian category E has enough projective and/or injective objects [55, Remark 3.5 and Example 3.6].

Lemma 4.1. *Let* (L, R) *be a cotorsion pair in an abelian category* E*.*

(a) *If the class* L *is generating in* E *and every object of* E *admits a special preenvelope sequence* (11)*, then every object of* E *also admits a special precover sequence* (10)*. So the cotorsion pair* (L, R) *is complete in this case.*

(b) *If the class* R *is cogenerating in* E *and every object of* E *admits a special precover sequence* (10)*, then every object of* E *also admits a special preenvelope sequence* (11)*. So the cotorsion pair* (L, R) *is complete in this case.*

Proof. This a generalization of the classical Salce lemmas [60]. A proof in the stated generality can be found in [55, Lemma 1.1]. ☐

Lemma 4.2. *Let* (L, R) *be a cotorsion pair in an abelian category* E*. Assume that the class* L *is generating and the class* R *is cogenerating in* E*. Then the following conditions are equivalent:*

(1) *the class* L *is closed under kernels of epimorphisms in* E*;*
(2) *the class* R *is closed under cokernels of monomorphisms in* E*;*

(3) $\mathrm{Ext}^2_{\mathsf{E}}(L,R) = 0$ *for all* $L \in \mathsf{L}$ *and* $R \in \mathsf{R}$*;*
(4) $\mathrm{Ext}^n_{\mathsf{E}}(L,R) = 0$ *for all* $L \in \mathsf{L}$*,* $R \in \mathsf{R}$*, and* $n \geq 1$*.*

Proof. See [55, Lemma 1.4] for a brief discussion with references. □

A cotorsion pair satisfying the equivalent conditions of Lemma 4.2 is said to be *hereditary*.

Given any class of objects S in an abelian category E, one can construct a cotorsion pair (L, R) in E by setting $\mathsf{R} = \mathsf{S}^{\perp_1}$ and $\mathsf{L} = {}^{\perp_1}\mathsf{R}$. The resulting cotorsion pair (L, R) is said to be *generated* by the class S. Obviously, one has $\mathsf{S} \subset \mathsf{L}$.

4.2. **Transfinitely iterated extensions.** Let E be an abelian category and α be an ordinal. Let $(F_i \to F_j)_{i<j\leq\alpha}$ be a direct system of objects in E indexed by the ordinal $\alpha + 1$. One says that the direct system $(F_i)_{i\leq\alpha}$ is a *smooth* (or *continuous*) chain if $F_j = \varinjlim_{i<j} F_i$ for all limit ordinals $j \leq \alpha$.

Let $(F_i)_{i\leq\alpha}$ be a direct system in E which is a smooth chain. Assume that $F_0 = 0$ and the morphism $F_i \longrightarrow F_{i+1}$ is a monomorphism for every ordinal $i < \alpha$, and let S_i be a cokernel of $F_i \longrightarrow F_{i+1}$ in E. Then the object $F = F_\alpha$ is said to be *filtered by* the objects S_i, $0 \leq i < \alpha$. In this case, the object $F \in \mathsf{E}$ is also said to be a *transfinitely iterated extension* (*in the sense of the directed colimit*) of the objects S_i.

Let S be a class of objects in E. Then the class of all objects in E filtered by (objects isomorphic to) objects from S is denoted by $\mathsf{Fil}(\mathsf{S})$. Given a class of objects $\mathsf{F} \subset \mathsf{E}$, we will denote by $\mathsf{F}^{\oplus} \subset \mathsf{E}$ the class of all direct summands of objects from F.

Lemma 4.3. *Let* E *be an abelian category. Then, for any class of objects* $\mathsf{R} \subset \mathsf{E}$*, the class of objects* ${}^{\perp_1}\mathsf{R}$ *is closed under transfinitely iterated extensions (in the sense of the directed colimit) and direct summands in* E*. In other words,* ${}^{\perp_1}\mathsf{R} = \mathsf{Fil}({}^{\perp_1}\mathsf{R})^{\oplus}$*.*

Proof. This result is known as the *Eklof lemma* [15, Lemma 1]. In the stated generality, a proof can be found in [54, Lemma 4.5]. We refer to [55, Proposition 1.3 and the preceding paragraph] for a further discussion and historical references. □

4.3. **Small object argument.** We start with a brief discussion of presentable objects and locally presentable categories.

Let λ be an infinite regular cardinal. A poset I is said to be λ-*directed* if any subset of I of the cardinality $< \lambda$ has an upper bound in I. A λ-*directed colimit* is a colimit of a diagram indexed by a λ-directed poset.

Let K be a category where λ-directed colimits exist. An object $K \in \mathsf{K}$ is said to be λ-*presentable* if the functor $\mathrm{Hom}_{\mathsf{K}}(K, -)\colon \mathsf{K} \longrightarrow \mathsf{Sets}$ (where Sets denotes the category of sets) preserves λ-directed colimits. The category K is said to be λ-*accessible* if it has a set of λ-presentable objects such that every object of K is a λ-directed colimit of objects from this set. The category K is said to be *locally* λ-*presentable* if it is cocomplete and λ-accessible [1, Definition 1.17].

Equivalently, a category is locally λ-presentable if and only if it is cocomplete and has a strong generating set consisting of λ-presentable objects [1, Theorem 1.20]. In this paper, we are interested in abelian categories, and any generating set of objects in an abelian category is a strong generating set; so we do not recall the details of the definition of a strong generating set here, referring the reader to [1, Section 0.6] instead. To summarize it briefly, one can say that a set $(K_i)_{i\in I}$ of objects in a category K is generating if and only if the related functor $(\mathrm{Hom}_{\mathsf{K}}(K_i,-))_{i\in I}$ from K to the category of I-sorted sets Sets^I is faithful; a generating set is strong if and only if this functor is also conservative.

A category is called *locally presentable* if it is locally λ-presentable for some infinite regular cardinal λ. In particular, locally ω-presentable categories (where ω is the minimal infinite cardinal) are referred to as *locally finitely presentable*, and ω-presentable objects are called *finitely presentable* [1, Definitions 1.1 and 1.9, Theorem 1.11].

The following theorem summarizes the application of Quillen's classical "small object argument" to cotorsion pairs. It forms a generalization of the Eklof–Trlifaj theorem [15, Theorems 2 and 10] from module categories to locally presentable abelian categories.

Theorem 4.4. *Let E be a locally presentable abelian category, $\mathsf{S}\subset\mathsf{E}$ be a set of objects, and (L,R) be the cotorsion pair generated by S in E.*

(a) *If the class L is generating and the class R is cogenerating in E, then the cotorsion pair (L,R) is complete.*

(b) *If the class of objects $\mathsf{Fil}(\mathsf{S})$ is generating in E, then one has $\mathsf{L}=\mathsf{Fil}(\mathsf{S})^{\oplus}$.*

Proof. A proof in the stated generality can be found in [54, Corollary 3.6 and Theorem 4.8] or [55, Theorems 3.3 and 3.4]. We refer to [55, Section 3] for a historical discussion with references. For counterexamples showing that the additional assumptions in part (a) cannot be dropped, see [55, Remark 3.5 and Examples 3.6].

The argument in [54] and [55] is based on the notion of a *weak factorization system*, which we will recall below in Sections 5.1–5.3. See Proposition 5.2 for a formulation of the small object argument in the relevant generality and Proposition 5.3 for the connection between cotorsion pairs and weak factorization systems. $\square$

5. Abelian Model Structures

The notion of an abelian model structure and the connection with complete cotorsion pairs is due to Hovey [25]. The exposition in this section is an extraction from [55, Sections 2 and 4], which in turn is based on the papers [25] and [5]. All the proofs are omitted and replaced with references.

5.1. **Weak factorization systems.** A weak factorization system is "a half of a model structure". We refer to [55, Section 2] for a terminological and historical discussion with references.

Let E be a category, and let $l\colon A\longrightarrow B$ and $r\colon C\longrightarrow D$ be two morphisms. One says that r has the *right lifting property* with respect to l, or equivalently, l has the *left lifting property* with respect to r if for any commutative square as on the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

a diagonal arrow exists making both the triangles commutative. (The word "weak" in the expression "weak factorization system", which is defined below, refers to the fact that the diagonal filling need not be unique.)

Let $\mathcal{L}$ and $\mathcal{R}$ be two classes of morphisms in E. We will denote by $\mathcal{L}^\square$ the class of all morphisms x in E having the right lifting property with respect to all morphisms $l \in \mathcal{L}$. Similarly, ${}^\square\mathcal{R}$ is the class of all morphisms y in E having the left lifting property with respect to all morphisms $r \in \mathcal{R}$.

A pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ in a category E is said to be a *weak factorization system* if $\mathcal{L} = {}^\square\mathcal{R}$, $\mathcal{R} = \mathcal{L}^\square$, and every morphism f in E can be factorized as $f = rl$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$.

5.2. **Transfinite compositions.** Let α be an ordinal, and let $(E_i \to E_j)_{i<j\le\alpha}$ be a direct system of objects in E indexed by the ordinal $\alpha + 1$. Assume that the direct system $(E_i)_{i\le\alpha}$ is a smooth chain (as defined in Section 4.2). Then we will say that the morphism $E_0 \longrightarrow E_\alpha$ in the direct system is a *transfinite composition* of the morphisms $E_i \longrightarrow E_{i+1}$, $0 \le i < \alpha$ (*in the sense of the directed colimit*).

Let $A \longrightarrow B$ be a morphism in E. Then a morphism $X \longrightarrow Y$ in E is said to be a *pushout* of the morphism $A \longrightarrow B$ if there exists a pair of morphisms $A \longrightarrow X$ and $B \longrightarrow Y$ such that $A \longrightarrow B \longrightarrow Y$, $A \longrightarrow X \longrightarrow Y$ is a cocartesian square (otherwise known as a pushout square) in E.

One says that an object X is a *retract* of an object $A \in \mathsf{E}$ if there exist morphisms $i\colon X \longrightarrow A$ and $p\colon A \longrightarrow X$ such that $pi = \mathrm{id}_X$ in E. A morphism z is said to be a retract of a morphism f in E if z is a retract of f in the category $\mathsf{E}^\to$ of morphisms in E (with commutative squares in E being morphisms in $\mathsf{E}^\to$).

Given a class of morphisms $\mathcal{S}$ in E, one denotes by $\mathcal{C}ell(\mathcal{S})$ the closure of $\mathcal{S}$ under pushouts and transfinite compositions, and by $\mathcal{C}of(\mathcal{S})$ the closure of $\mathcal{S}$ under pushouts, transfinite compositions, and retracts. (See [55, Section 2] for a discussion of the basic properties of these constructions.)

The following very basic lemma is a "morphism version" of Lemma 4.3.

Lemma 5.1. *For any class of morphisms $\mathcal{R}$ in E, the class of morphisms ${}^\square\mathcal{R}$ is closed under transfinite compositions (in the sense of the directed colimit), pushouts, and retracts. In other words, ${}^\square\mathcal{R} = \mathcal{C}of({}^\square\mathcal{R})$.* $\square$

The following theorem is a formulation of Quillen's small object argument suitable for deducing Theorem 4.4.

Proposition 5.2. *Let E be a locally presentable category and $\mathcal{S}$ be a set of morphisms in E. Then any morphism f in E can be factorized as $f = rl$ with $r \in \mathcal{S}^\square$ and $l \in \mathcal{C}ell(\mathcal{S})$. Moreover, this factorization can be chosen for all morphisms f in E*

in such a way that it depends functorially on f. The pair of classes of morphisms $\mathcal{L} = \mathcal{C}of(\mathcal{S})$ and $\mathcal{R} = \mathcal{S}^{\square}$ is a weak factorization system in E.

Proof. A proof can be found in [6, Proposition 1.3]. We refer to [55, Theorem 3.1] for a discussion with references (see also [55, Lemma 2.1]). $\square$

5.3. **Abelian weak factorization systems.** Let E be an abelian category, and let L and R be two classes of objects E. By an L-*monomorphism* we mean a monomorphism in E with a cokernel belonging to L. Dually, an R-*epimorphism* is an epimorphism in E with a kernel belonging to R. The class of all L-monomorphisms in E is denoted by L-$\mathcal{M}ono$, and the class of all R-epimorphisms in E is denoted by R-$\mathcal{E}pi$.

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in E is said to be *abelian* if there exists a pair of classes of objects L and R in E such that $\mathcal{L} = \mathsf{L}$-$\mathcal{M}ono$ and $\mathcal{R} = \mathsf{R}$-$\mathcal{E}pi$.

Proposition 5.3. *Let L and R be two classes of objects in an abelian category E. Then the pair of classes of morphisms $\mathcal{L} = \mathsf{L}$-$\mathcal{M}ono$ and $\mathcal{R} = \mathsf{R}$-$\mathcal{E}pi$ forms a weak factorization system $(\mathcal{L}, \mathcal{R})$ in E if and only if (L, R) is a complete cotorsion pair in E. So there is a bijection between abelian weak factorization systems and complete cotorsion pairs in any abelian category E.*

Proof. This result is essentially due to Hovey [25]; see also [55, Theorem 2.4]. $\square$

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category E is said to be *cofibrantly generated* if there exists a set of morphisms $\mathcal{S}$ in E such that $\mathcal{R} = \mathcal{S}^{\square}$.

Lemma 5.4. *Let E be a locally presentable abelian category and $(\mathcal{L}, \mathcal{R})$ be an abelian weak factorization system in E corresponding to a complete cotorsion pair (L, R), as in Proposition 5.3. Then the weak factorization system $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated if and only if the cotorsion pair (L, R) is generated by some set of objects.*

Proof. This is [55, Lemma 3.7]. $\square$

5.4. **Model categories.** The concept of a model category is due to Quillen [57]. There are several modern expositions available; see, e. g., [24]. A discussion penned by the present authors can be found in [55, Section 4]. In this section, we restrict ourselves to a brief sketch.

A *model structure* on a category E is the datum of three classes of morphisms $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{W}$ satisfying the following conditions:

- the pair of classes of morphisms $\mathcal{L}$ and $\mathcal{R} \cap \mathcal{W}$ is a weak factorization system in E;
- the pair of classes of morphisms $\mathcal{L} \cap \mathcal{W}$ and $\mathcal{R}$ is a weak factorization system in E;
- the class $\mathcal{W}$ is closed under retracts and satisfies the 2-out-of-3 property: given a composable pair of morphisms f and g in E, if two of the three morphisms f, g, and fg belong to $\mathcal{W}$, then the third one also does.

The morphisms from the classes $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{W}$ are called *cofibrations*, *fibrations*, and *weak equivalences*, respectively. The morphisms from the class $\mathcal{L} \cap \mathcal{W}$ are called *trivial cofibrations*. The morphisms from the class $\mathcal{R} \cap \mathcal{W}$ are called *trivial fibrations*.

A *model category* is a complete, cocomplete category with a model structure. The *homotopy category* $\mathsf{E}[\mathcal{W}^{-1}]$ of a category E with a model structure is obtained by adjoining the formal inverse morphisms for all weak equivalences.

Let E be a category with a model structure. Assume that E has an initial object $\varnothing$ and a terminal object $*$. Then an object $L \in \mathsf{E}$ is said to be *cofibrant* if the morphism $\varnothing \longrightarrow L$ is a cofibration. Dually, an object $R \in \mathsf{E}$ is said to be *fibrant* if the morphism $R \longrightarrow *$ is a fibration.

Assume further that E is a pointed category, i. e., it has an initial and a terminal object and they coincide, $\varnothing = *$. The initial-terminal object of a pointed category is called the *zero object* and denoted by 0. Given a pointed category E with a model structure, an object $W \in \mathsf{E}$ is called *weakly trivial* if the morphism $0 \longrightarrow W$ is a weak equivalence, or equivalently, the morphism $W \longrightarrow 0$ is a weak equivalence. Weakly trivial cofibrant objects are called *trivially cofibrant*, and weakly trivial fibrant objects are called *trivially fibrant*.

5.5. **Abelian model structures.** The following definition of an *abelian model structure* is due to Hovey [25] (a generalization to exact categories in the sense of Quillen can be found in [19]). Let E be an abelian category. A model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$ on E is said to be *abelian* if the class of all cofibrations $\mathcal{L}$ is the class of all monomorphisms with cofibrant cokernels and the class of all fibrations $\mathcal{R}$ is the class of all epimorphisms with fibrant kernels. A model category is called *abelian* if its underlying category is abelian and the model structure is abelian.

Let E be an abelian category. The following lemma tells that a model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$ on E is abelian if and only if both the weak factorization systems $(\mathcal{L}, \mathcal{R}\cap\mathcal{W})$ and $(\mathcal{L}\cap\mathcal{W},\ \mathcal{R})$ are abelian.

Lemma 5.5. *In an abelian model structure, the class of all trivial cofibrations $\mathcal{L}\cap\mathcal{W}$ is the class of all monomorphisms with trivially cofibrant cokernels and the class of all trivial fibrations $\mathcal{R}\cap\mathcal{W}$ is the class of all epimorphisms with trivially fibrant kernels.*

Proof. This is a part of [25, Proposition 4.2]; see also [55, Lemma 4.1]. $\square$

A class of objects W in an abelian category E is said to be *thick* if it is closed under direct summands and satisfies the following version of 2-out-of-3 property: for any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in E, if two of the three objects A, B, C belong to W, then the third one also does.

Theorem 5.6. *Let E be an abelian category. Then there is a bijection between abelian model structures on E and triples of classes of objects $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ in E such that*

- $(\mathsf{L},\ \mathsf{R}\cap\mathsf{W})$ *is a complete cotorsion pair in* E*;*
- $(\mathsf{L}\cap\mathsf{W},\ \mathsf{R})$ *is a complete cotorsion pair in* E*;*
- W *is a thick class of objects in* E*.*

To an abelian model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$, the triple of classes of objects $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ is assigned, where L is the class of all cofibrant objects, R is the class of all fibrant objects, and W is the class of all weakly trivial objects. To a triple of classes of objects $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ satisfying the conditions above, the abelian model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$ is *assigned, where* $\mathcal{L} = \mathsf{L}$-$\mathcal{M}ono$ *and* $\mathcal{R} = \mathsf{R}$-$\mathcal{E}pi$. *The classes of all trivial cofibrations and trivial fibrations are* $\mathcal{L}\cap\mathcal{W} = (\mathsf{L}\cap\mathsf{W})$-$\mathcal{M}ono$ *and* $\mathcal{R}\cap\mathcal{W} = (\mathsf{R}\cap\mathsf{W})$-$\mathcal{E}pi$. *The class of all weak equivalences* $\mathcal{W}$ *consists of all morphisms* w *in* E *that can be factorized as* $w = rl$ *with* $r \in \mathcal{R}\cap\mathcal{W}$ *and* $l \in \mathcal{L}\cap\mathcal{W}$.

Proof. This is a part of [25, Theorem 2.2]; see also [55, Theorem 4.2]. A generalization to Quillen exact categories can be found in [19, Theorem 3.3]. $\square$

In the rest of this paper, we will identify abelian model structures with the triples of classes of objects $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ satisfying the conditions of Theorem 5.6, and write simply "an abelian model structure $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ on an abelian category E".

A model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$ is called *cofibrantly generated* if both the weak factorization systems $(\mathcal{L},\ \mathcal{R}\cap\mathcal{W})$ and $(\mathcal{L}\cap\mathcal{W},\ \mathcal{R})$ are cofibrantly generated.

Lemma 5.7. *An abelian model structure* $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ *on a locally presentable abelian category* E *is cofibrantly generated if and only if both the cotorsion pairs* $(\mathsf{L},\ \mathsf{R}\cap\mathsf{W})$ *and* $(\mathsf{L}\cap\mathsf{W},\ \mathsf{R})$ *are generated by some sets of objects in* E.

Proof. This is [55, Corollary 4.3]. The assertion follows from Lemma 5.4. $\square$

5.6. **Projective and injective abelian model structures.** Given an abelian category E, we denote by $\mathsf{E}_{\mathsf{proj}} \subset \mathsf{E}$ the class (or the full subcategory) of all projective objects in E and by $\mathsf{E}_{\mathsf{inj}} \subset \mathsf{E}$ the class/full subcategory of all injective objects in E.

A model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$ on a category B is called *projective* if all the objects of B are fibrant. For an abelian model structure $(\mathsf{L}, \mathsf{W}, \mathsf{R})$, this means that $\mathsf{R} = \mathsf{B}$, or equivalently, $\mathsf{L}\cap\mathsf{W}$ is the class of all projective objects in B, that is $\mathsf{L}\cap\mathsf{W} = \mathsf{B}_{\mathsf{proj}}$. So an abelian model structure is projective if and only if the trivial cofibrations are the monomorphisms with projective cokernel.

Dually, a model structure $(\mathcal{L}, \mathcal{W}, \mathcal{R})$ on a category A is called *injective* if all the objects of A are cofibrant. For an abelian model structure $(\mathsf{L}, \mathsf{W}, \mathsf{R})$, this means that $\mathsf{L} = \mathsf{A}$, or equivalently, $\mathsf{R}\cap\mathsf{W}$ is the class of all injective objects in A, that is $\mathsf{R}\cap\mathsf{W} = \mathsf{A}_{\mathsf{inj}}$. So an abelian model structure is injective if and only if the trivial fibrations are the epimorphisms with injective kernel.

Lemma 5.8. *Let* $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ *be an abelian model structure on an abelian category* E. *Then the cotorsion pair* $(\mathsf{L},\ \mathsf{R}\cap\mathsf{W})$ *is hereditary if and only if the cotorsion pair* $(\mathsf{L}\cap\mathsf{W},\ \mathsf{R})$ *is hereditary in* E.

Proof. This standard observation can be found in [55, Lemma 4.4]. The assertion follows easily from Lemma 4.2 above. $\square$

An abelian model structure satisfying the equivalent conditions of Lemma 5.8 is called *hereditary*. In particular, all projective and all injective abelian model structures are hereditary [5, Corollary 1.1.12] (see also [55, paragraph before Lemma 4.5]).

Lemma 5.9. (a) *A pair of classes of objects* (L, W) *in an abelian category* B *defines a projective abelian model structure* $(\mathsf{L}, \mathsf{W}, \mathsf{B})$ *on* B *if and only if* B *has enough projective objects,* (L, W) *is a complete cotorsion pair in* B *with* $\mathsf{L}\cap\mathsf{W} = \mathsf{B}_{\mathsf{proj}}$, *and the class* $\mathsf{W} \subset \mathsf{B}$ *is thick.*

(b) *A pair of classes of objects* (W,R) *in an abelian category* A *defines an injective abelian model structure* $(\mathsf{A},\mathsf{W},\mathsf{R})$ *on* A *if and only if* A *has enough injective objects,* (W,R) *is a complete cotorsion pair in* A *with* $\mathsf{R}\cap\mathsf{W}=\mathsf{A}_{\mathsf{inj}}$*, and the class* $\mathsf{W}\subset\mathsf{A}$ *is thick.*

Proof. This is [5, Corollary 1.1.9]; see also [55, Lemma 4.5]. □

Lemma 5.10. (a) *Let* (L,W) *be a hereditary complete cotorsion pair in an abelian category* B *such that* $\mathsf{L}\cap\mathsf{W}=\mathsf{B}_{\mathsf{proj}}$*. Then the class* W *is thick in* B*.*

(b) *Let* (W,R) *be a hereditary complete cotorsion pair in an abelian category* A *such that* $\mathsf{R}\cap\mathsf{W}=\mathsf{A}_{\mathsf{inj}}$*. Then the class* W *is thick in* A*.*

Proof. This is [5, Lemma 1.1.10]; see also [55, Lemma 4.6]. □

5.7. **Stable combinatorial abelian model structures.** Let E be a finitely complete, finitely cocomplete category (i. e., all finite limits and finite colimits exist in E). Assume further that E is a pointed category (as defined in Section 5.4). In particular, all abelian categories satisfy these conditions.

Let $(\mathcal{L},\mathcal{W},\mathcal{R})$ be a model structure on E. In this context, Quillen [57, Theorem I.2.2] constructed a pair of adjoint endofunctors Σ and $\Omega\colon \mathsf{E}[\mathcal{W}^{-1}]\longrightarrow\mathsf{E}[\mathcal{W}^{-1}]$ on the homotopy category $\mathsf{E}[\mathcal{W}^{-1}]$. The functor Σ is called the *suspension functor* and the functor Ω is the *loop functor*.

Under stricter assumptions that E is complete and cocomplete and that factorizations of morphisms in the weak factorization systems $(\mathcal{L},\ \mathcal{R}\cap\mathcal{W})$ and $(\mathcal{L}\cap\mathcal{W},\ \mathcal{R})$ can be chosen to be functorial, Hovey [24, Chapter 5] constructs a natural closed action of the homotopy category of pointed simplicial sets on $\mathsf{E}[\mathcal{W}^{-1}]$ and uses this action to define the suspension and loop functors in [24, Definition 6.1.1]. Then [24, Lemma 6.1.2] tells that the functor Σ is left adjoint to Ω.

The model structure $(\mathcal{L},\mathcal{W},\mathcal{R})$ is called *stable* if the functor Σ is an auto-equivalence of the category $\mathsf{E}[\mathcal{W}^{-1}]$, or equivalently, the functor Ω is an auto-equivalence of $\mathsf{E}[\mathcal{W}^{-1}]$. It is well-known that, for any stable model structure $(\mathcal{L},\mathcal{W},\mathcal{R})$ on a finitely complete, finitely cocomplete, pointed category E, the homotopy category $\mathsf{E}[\mathcal{W}^{-1}]$ is triangulated (see, e. g., [24, Proposition 7.1.6]). The suspension functor Σ plays the role of the shift functor $X\longmapsto X[1]$ on $\mathsf{E}[\mathcal{W}^{-1}]$.

Lemma 5.11. *Any hereditary abelian model structure is stable.*

Proof. See [5, Corollary 1.1.15 and the preceding discussion]. □

We refer to [42] or [32, 34] for the definition of a *well-generated triangulated category* (see [55, Section 4] for a brief discussion). A *combinatorial model category* E is a locally presentable category endowed with a cofibrantly generated model structure. (Notice that any locally presentable category is complete and cocomplete, so our terminology here is consistent with the one in Section 5.4.)

Proposition 5.12. *For any stable combinatorial model category* E*, the homotopy category* $\mathsf{E}[\mathcal{W}^{-1}]$ *is a well-generated triangulated category.*

Proof. This is [59, Proposition 6.10] or [8, Theorem 3.1, Definition 3.2, and Theorem 3.9]. ☐

6. Contraderived Model Structure

The aim of this is section is to work out a common generalization of the contraderived model structure on the abelian category of CDG-modules over a CDG-ring [5, Proposition 1.3.6(1)] and the contraderived model structure on the category of complexes over a locally presentable abelian category with enough projective objects [55, Section 7]. The locally presentable abelian DG-categories with enough projective objects (to be defined in this section) form a suitable context.

6.1. **Projective objects in abelian DG-categories.** We start with a general lemma connecting the groups Ext^1 in the abelian category $\mathsf{Z}^0(\mathbf{E})$ with the groups Hom in the triangulated category $\mathsf{H}^0(\mathbf{E})$ for an abelian DG-category $\mathbf{E}$.

Lemma 6.1. *Let $\mathbf{E}$ be an abelian DG-category and X, $Y \in \mathbf{E}$ be two objects. Then the kernel of the abelian group homomorphism*

$$\Phi\colon \mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{E})}(X,Y) \longrightarrow \mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{E}^\natural)}(\Phi(X),\Phi(Y))$$

induced by the exact functor $\Phi_{\mathbf{E}}\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ is naturally isomorphic to the group $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{E})}(X,Y[1])$. In particular, if either $\Phi(X)$ is a projective object in $\mathsf{Z}^0(\mathbf{E}^\natural)$, or $\Phi(Y)$ is an injective object in $\mathsf{Z}^0(\mathbf{E}^\natural)$, then

$$\mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{E})}(X,Y) \simeq \mathrm{Hom}_{\mathsf{H}^0(\mathbf{E})}(X,Y[1]).$$

Proof. This is a generalization of a well-known [55, Lemma 5.1] and a particular case of [52, Lemma 9.41]. Following Section 2.3, the additive functor $\Phi\colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ factorizes as $\mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathbf{E}^0 \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$, where $\mathsf{Z}^0(\mathbf{E}) \rightarrowtail \mathbf{E}^0$ is the natural inclusion (bijective on the objects) and $\widetilde{\Phi}\colon \mathbf{E}^0 \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ is a fully faithful functor. Consequently, a short exact sequence in $\mathsf{Z}^0(\mathbf{E})$ splits after applying the functor Φ if and only if it splits in $\mathbf{E}^0$.

It remains to refer to the (well-known and easy) description of short sequences $0 \longrightarrow Y \longrightarrow W \longrightarrow X \longrightarrow 0$ in $\mathsf{Z}^0(\mathbf{E})$ that are split exact in $\mathbf{E}^0$ in terms of the cones of closed morphisms $f\colon X \longrightarrow Y[1]$ of degree 0 in $\mathbf{E}$ (see Section 1.5). Notice that, for any such closed morphism f, the related cone sequence $0 \longrightarrow Y \longrightarrow \mathrm{cone}(f)[-1] \longrightarrow X \longrightarrow 0$ is exact in $\mathsf{Z}^0(\mathbf{E})$ by Lemma 3.1. One also needs to observe that two closed morphisms f' and $f''\colon X \rightrightarrows Y[1]$ of degree 0 in $\mathbf{E}$ lead to equivalent extensions if and only if they are homotopic. ☐

Lemma 6.2. *Let $\mathbf{B}$ be an abelian DG-category. Then the abelian category $\mathsf{Z}^0(\mathbf{B})$ has enough projective objects if and only if the abelian category $\mathsf{Z}^0(\mathbf{B}^\natural)$ has enough projective objects.*

Proof. The functors $\Phi_{\mathbf{B}}\colon \mathsf{Z}^0(\mathbf{B}) \longrightarrow \mathsf{Z}^0(\mathbf{B}^\natural)$ and $\Psi^+_{\mathbf{B}}\colon \mathsf{Z}^0(\mathbf{B}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{B})$ take projectives to projectives, because they are left adjoint to exact functors. Furthermore, the roles of the functors Φ and $\Psi^\pm$ are completely symmetric in view of Corollary 3.10 and the last paragraph of Section 2.4. Assume that the abelian category $\mathsf{Z}^0(\mathbf{B}^\natural)$ has enough projective objects, and let $B \in \mathsf{Z}^0(\mathbf{B})$ be an object. Choose a projective object $P \in \mathsf{Z}^0(\mathbf{B}^\natural)$ together with an epimorphism $p\colon P \longrightarrow \Phi(B)$ in $\mathsf{Z}^0(\mathbf{B}^\natural)$. By adjunction, we have the corresponding morphism $q\colon \Psi^+(P) \longrightarrow B$ in $\mathsf{Z}^0(\mathbf{B})$. Let us show that q is an epimorphism.

Indeed, consider the morphism $\Psi^+(p)\colon \Psi^+(P) \longrightarrow \Psi^+\Phi(B)$ in $\mathsf{Z}^0(\mathbf{B})$. The functor Ψ^+ is exact, since it has adjoints on both sides; so in particular $\Psi^+(p)$ is an epimorphism. The morphism q is equal to the composition $\Psi^+(P) \longrightarrow \Psi^+\Phi(B) \longrightarrow B$, where $\Psi^+\Phi(B) \longrightarrow B$ is the adjunction morphism. According to the last paragraph of Section 2.2, we have a natural isomorphism $\Psi^+\Phi(B) \simeq \Xi(B)$ in $\mathsf{Z}^0(\mathbf{B})$. It remains to recall that the natural morphism $\Xi(B) \longrightarrow B$ is an epimorphism in $\mathsf{Z}^0(\mathbf{B})$ by Corollary 3.2(a). (Cf. [52, Lemma 5.7(b)].) $\square$

An abelian DG-category satisfying the equivalent conditions of Lemma 6.2 is said to *have enough projective objects*.

Lemma 6.3. *Let* $\mathbf{B}$ *be an abelian DG-category with enough projective objects. Then*

(a) *an object* $Q \in \mathsf{Z}^0(\mathbf{B})$ *is projective if and only if* Q *is contractible in* $\mathbf{B}$ and *the object* $\Phi(Q) \in \mathsf{Z}^0(\mathbf{B}^\natural)$ *is projective;*

(b) *an object* $P \in \mathsf{Z}^0(\mathbf{B}^\natural)$ *is projective if and only if* P *is contractible in* $\mathbf{B}^\natural$ and *the object* $\Psi^+(P) \in \mathsf{Z}^0(\mathbf{B})$ *is projective.*

Proof. This is a common generalization of [5, Lemma 1.3.3] (which is the CDG-module version) and of the long well known [55, Lemma 5.2(b)] (which is the version for complexes in abelian categories). Let us prove part (a).

"Only if": let Q be a projective object in $\mathsf{Z}^0(\mathbf{B})$. As explained in the proof of Lemma 6.2, the functor Φ preserves projectivity; so $\Phi(Q) \in \mathsf{Z}^0(\mathbf{B}^\natural)$ is a projective object. Furthermore, following the same proof, there are enough projective objects in $\mathsf{Z}^0(\mathbf{B})$ of the form $\Psi^+(P)$, where P ranges over the projective objects of $\mathsf{Z}^0(\mathbf{B}^\natural)$. Therefore, Q is a direct summand of $\Psi^+(P)$ in $\mathsf{Z}^0(\mathbf{B})$ for some projective $P \in \mathsf{Z}^0(\mathbf{B}^\natural)$. It remains to observe that the essential image of the functor Ψ^+ consists of contractible objects (by construction) and direct summands of contractible objects are contractible in order to conclude that the object $Q \in \mathbf{B}$ is contractible. (Similarly, in view of a natural isomorphism in the last paragraph of Section 2.4, the essential image of the functor Φ consists of contractible objects.)

"If": let $Q \in \mathbf{B}$ be a contractible object such that the object $\Phi(Q) \in \mathsf{Z}^0(\mathbf{B}^\natural)$ is projective. For any object $Y \in \mathsf{Z}^0(\mathbf{B})$, we have $\operatorname{Ext}^1_{\mathsf{Z}^0(\mathbf{B})}(Q,Y) \simeq \operatorname{Hom}_{\mathsf{H}^0(\mathbf{B})}(Q,Y[1])$ by Lemma 6.1. Since the object Q vanishes in $\mathsf{H}^0(\mathbf{B})$, it follows that $\operatorname{Ext}^1_{\mathsf{Z}^0(\mathbf{B})}(Q,Y) = 0$ for all $Y \in \mathsf{Z}^0(\mathbf{B})$, hence $Q \in \mathsf{Z}^0(\mathbf{B})$ is a projective object. $\square$

Let $\mathbf{B}$ be an abelian DG-category. We will say that an object $Q \in \mathbf{B}$ is *graded-projective* if the object $\Phi(Q)$ is projective in the abelian category $\mathsf{Z}^0(\mathbf{B}^\natural)$. The full

DG-subcategory formed by the graded-projective objects in $\mathbf{B}$ is denoted by $\mathbf{B_{proj}} \subset \mathbf{B}$. So the notation $\mathsf{Z}^0(\mathbf{B_{proj}}) \subset \mathsf{Z}^0(\mathbf{B})$ stands for the full subcategory of all graded-projective objects in $\mathsf{Z}^0(\mathbf{B})$, while $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}} \subset \mathsf{Z}^0(\mathbf{B})$ is the (smaller) full subcategory of all projective objects in $\mathsf{Z}^0(\mathbf{B})$. Since the functor Φ is additive and transforms shifts into inverse shifts and twists into isomorphisms, the full DG-subcategory $\mathbf{B_{proj}}$ is closed under finite direct sums, shifts, twists, and cones in $\mathbf{B}$.

Lemma 6.4. *Let $\mathbf{B}$ be an abelian DG-category with enough projective objects, and let (L, W) be a cotorsion pair in the abelian category $\mathsf{Z}^0(\mathbf{B})$ such that $\mathsf{L} \subset \mathsf{Z}^0(\mathbf{B_{proj}})$. Assume that the cotorsion pair (L, W) is preserved by the shift: $\mathsf{W} = \mathsf{W}[1]$, or equivalently, $\mathsf{L} = \mathsf{L}[1]$. Then $\mathsf{L} \cap \mathsf{W} = \mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$. If, moreover, the cotorsion pair (L, W) is complete, then it is hereditary and the class W is thick in $\mathsf{Z}^0(\mathbf{B})$.*

Proof. This is a common generalization of [5, Lemma 1.3.4(1)] (the CDG-module version) and [55, Lemma 5.3(b)] (the version for complexes in abelian categories).

The inclusion $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}} \subset {}^{\perp_1}\mathsf{W} = \mathsf{L}$ is obvious. The inclusion $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}} \subset \mathsf{Z}^0(\mathbf{B_{proj}})^{\perp_1}$ holds because $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$ is the full subcategory of projective-injective objects in the Frobenius exact category $\mathsf{Z}^0(\mathbf{B_{proj}})$ (with the exact structure inherited from the abelian exact structure of $\mathsf{Z}^0(\mathbf{B})$). Specifically, for any $P \in \mathsf{Z}^0(\mathbf{B_{proj}})$ and $Q \in \mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$ we have, by Lemmas 6.1 and 6.3(a), $\mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{B})}(P, Q) \simeq \mathrm{Hom}_{\mathsf{H}^0(\mathbf{B})}(P, Q[1]) = 0$ since the object $Q \in \mathbf{B}$ is contractible. Hence $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}} \subset \mathsf{L}^{\perp_1} = \mathsf{W}$.

To prove the inclusion $\mathsf{L} \cap \mathsf{W} \subset \mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$, it suffices to show that every object $Q \in \mathsf{L} \cap \mathsf{W}$ is contractible in $\mathbf{B}$ (as we already know that $\mathsf{L} \cap \mathsf{W} \subset \mathsf{L} \subset \mathsf{Z}^0(\mathbf{B_{proj}})$, and all contractible objects in $\mathsf{Z}^0(\mathbf{B_{proj}})$ belong to $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$ by Lemma 6.3(a)). Indeed, by Lemma 6.1, $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{B})}(Q, Q) = \mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{B})}(Q, Q[-1]) = 0$.

To prove the remaining assertions of the lemma, we observe that the class L is closed under syzygies in $\mathsf{Z}^0(\mathbf{B})$. Indeed, for each $P \in \mathsf{L}$ we have a short exact sequence $0 \longrightarrow P[-1] \longrightarrow \Xi(P) \longrightarrow P \longrightarrow 0$ in $\mathsf{Z}^0(\mathbf{B})$ by Corollary 3.2(a) and an isomorphism $\Xi(P) \simeq \Psi^+\Phi(P)$ by the last paragraph of Section 2.2. Now $\Phi(P)$ is a projective object in $\mathsf{Z}^0(\mathbf{B}^\natural)$, hence $\Psi^+\Phi(P) \in \mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$ as explained in the proof of Lemma 6.2. It follows by a standard dimension shifting argument that the cotorsion pair (L, W) is hereditary. In order to conclude that the class W is thick, it remains to apply Lemma 5.10(a). $\square$

6.2. **Locally presentable DG-categories.** This section is mostly an extraction from [52, Sections 9.1 and 9.2].

Let λ be an infinite regular cardinal. The definitions of λ-presentable objects and locally λ-presentable categories (based on the exposition in [1]) have already appeared in Section 4.3.

Lemma 6.5. *For any additive DG-category $\mathbf{E}$ with shifts and cones, coproducts in the additive category $\mathsf{Z}^0(\mathbf{E})$ are the same things as coproducts in the DG-category $\mathbf{E}$. (Infinite) coproducts exist in the DG-category $\mathbf{E}$ if and only if they exist in the additive category $\mathsf{Z}^0(\mathbf{E})$. If this is the case, then coproducts also exist in the DG-category $\mathbf{E}^\natural$ and in the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$, and the functors $\Phi_{\mathbf{E}}$ and $\Psi^\pm_{\mathbf{E}}$ preserve them.*

Proof. The functors Φ and $\Psi^{\pm}$ always preserve limits and colimits, since they have adjoint functors on both sides. The assertion that coproducts in $\mathbf{E}$ (as defined above in Section 1.4) are the same things as coproducts in $\mathsf{Z}^0(\mathbf{E})$ when $\mathbf{E}$ has shifts and cones is not immediately obvious; it is [52, Lemma 9.2]. The key observation is that for each $n \in \mathbb{Z}$ and $X, Y \in \mathbf{E}$, one has a natural isomorphism $\mathrm{Hom}^n_{\mathbf{E}}(X, Y) \simeq \mathsf{Z}^0 \mathrm{Hom}^\bullet_{\mathbf{E}}(X, \mathrm{cone}(\mathrm{id}_Y)[n])$. The coproducts in $\mathbf{E}^\natural$ are easily constructed in terms of those in $\mathbf{E}$ when the latter exist. $\square$

Lemma 6.6. *Let $\mathbf{E}$ be an additive DG-category with shifts and cones such that all colimits exist in the additive category $\mathsf{Z}^0(\mathbf{E})$. Then the additive functor $\Xi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E})$* reflects *$\lambda$-presentability of objects. In other words, if $A \in \mathbf{E}$ is an object such that the object $\Xi(A)$ is λ-presentable in $\mathsf{Z}^0(\mathbf{E})$, then the object A is λ-presentable in $\mathsf{Z}^0(\mathbf{E})$.*

Proof. The class of all λ-presentable objects is closed under λ-small colimits by [1, Proposition 1.16]; in particular, it is closed under cokernels. Thus the assertion follows from Corollary 3.2(b). $\square$

Lemma 6.7. *Let $\mathbf{E}$ be an additive DG-category with shifts and cones such that all colimits exist in the additive category $\mathsf{Z}^0(\mathbf{E})$. Then the additive functors $\Phi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ and $\Psi^+_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ preserve* and *reflect λ-presentability of objects.*

Proof. This is explained in [52, Lemmas 9.6 and 9.7]. Under the assumptions of the lemma, the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$ has all coproducts by Lemma 6.5 and all cokernels by Lemma 3.6. Therefore, all colimits exist in $\mathsf{Z}^0(\mathbf{E}^\natural)$. The functors $\Phi_{\mathbf{E}}$ and $\Psi^+_{\mathbf{E}}$ preserve λ-presentability of objects, since they are left adjoint to functors preserving λ-directed colimits. Since the compositions $\Xi_{\mathbf{E}} \simeq \Psi^+_{\mathbf{E}} \circ \Phi_{\mathbf{E}}$ and $\Xi_{\mathbf{E}^\natural} \simeq \Phi_{\mathbf{E}} \circ \Psi^-_{\mathbf{E}}$ (see the last paragraph of Section 2.2) reflect λ-presentability by Lemma 6.6, it follows that the functors $\Phi_{\mathbf{E}}$ and $\Psi^-_{\mathbf{E}}$ also reflect λ-presentability. $\square$

Lemma 6.8. *Let $\mathbf{E}$ be an additive DG-category with shifts and cones. Then the additive functors $\Phi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ and $\Psi^+_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ take (strongly) generating sets of objects to (strongly) generating sets of objects.*

Proof. This is [52, Lemma 9.1]. The assertion about generating sets of objects holds because the functors Φ and Ψ^+ have faithful right adjoints. The assertion about strongly generating sets of objects is valid because the right adjoint functors to Φ and Ψ^+ are also conservative (but recall also the discussion of generators and strong generators in Section 4.3). $\square$

Proposition 6.9. *Let $\mathbf{E}$ be an additive DG-category with shifts and cones such that the additive category $\mathsf{Z}^0(\mathbf{E})$ has all colimits. Then the additive category $\mathsf{Z}^0(\mathbf{E})$ is locally λ-presentable if and only if the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$ is locally λ-presentable.*

Proof. Under the assumptions of the proposition, the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$ also has all colimits, as it was explained in the proof of Lemma 6.7. Now applying the functor Φ to any strong generating set of λ-presentable objects in $\mathsf{Z}^0(\mathbf{E})$ produces a strong generating set of λ-presentable objects in $\mathsf{Z}^0(\mathbf{E}^\natural)$, and conversely with the functor Ψ^+ (by Lemmas 6.7 and 6.8). $\square$

We will say that an additive DG-category $\mathbf{E}$ with shifts and cones is *locally λ-presentable* if both the additive categories $\mathsf{Z}^0(\mathbf{E})$ and $\mathsf{Z}^0(\mathbf{E}^\natural)$ are locally λ-presentable. In other words, this means that $\mathbf{E}$ satisfies the assumptions and the equivalent conditions of Proposition 6.9. A DG-category is called *locally presentable* if it is locally λ-presentable for some infinite regular cardinal λ.

In particular, locally ω-presentable DG-categories (where ω is the minimal infinite cardinal) are called *locally finitely presentable.*

Examples 6.10. (1) Let E be a locally λ-presentable additive category. Then it is clear that the additive category of graded objects $\mathsf{G}(\mathsf{E}) = \mathsf{E}^{\mathbb{Z}}$ is locally λ-presentable. For an uncountable regular cardinal λ, an object of $\mathsf{G}(\mathsf{E})$ is λ-presentable if and only if all its grading components are λ-presentable (see Example 8.8(1) below for a discussion of the case $\lambda = \omega$). The additive category of complexes $\mathsf{C}(\mathsf{E}) = \mathsf{Z}^0(\mathbf{C}(\mathsf{E}))$ is locally λ-presentable by [55, Lemma 6.3]. We recall that the additive category $\mathsf{Z}^0(\mathbf{C}(\mathsf{E})^\natural)$ is equivalent to $\mathsf{G}(\mathsf{E})$ according to Section 2.5. Hence the DG-category $\mathbf{C}(\mathsf{E})$ of complexes in E is locally λ-presentable.

(2) Let B be a locally presentable abelian category with enough projective objects. Then it is obvious that the abelian category of graded objects $\mathsf{G}(\mathsf{B}) = \mathsf{B}^{\mathbb{Z}}$ also has enough projectives. Following Example 3.13, the DG-category of complexes $\mathbf{C}(\mathsf{B})$ is an abelian DG-category. Thus $\mathbf{C}(\mathsf{B})$ is a locally presentable abelian DG-category with enough projective objects.

Example 6.11. Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a CDG-ring. Then the DG-category $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ of left CDG-modules over $\boldsymbol{R}^\bullet$ is an abelian DG-category (see Example 3.14). Moreover, following Section 2.6, both the abelian categories $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ and $\mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural)$ are graded module categories, $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) = \widehat{\boldsymbol{R}}^*\text{–}\mathbf{Mod}$ and $\mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural) \simeq R^*\text{–}\mathbf{Mod}$. The abelian category of graded modules over any graded ring is locally finitely presentable with enough projective objects; hence $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ is a locally finitely presentable abelian DG-category with enough projective objects.

Examples 6.12. (1) Let E be a locally λ-presentable additive category and $\Delta\colon \mathsf{E} \longrightarrow \mathsf{E}$ be an autoequivalence. Then it is clear that the additive category of 2-Δ-periodic objects $\mathsf{P}(\mathsf{E}, \Delta) \simeq \mathsf{E} \times \mathsf{E}$ is locally λ-presentable. For any potential $v\colon \mathrm{Id}_{\mathsf{E}} \longrightarrow \Delta$, the additive category of factorizations $\mathsf{F}(\mathsf{E}, \Delta, v)$ has all colimits (since the category E does). Recall that the additive category $\mathsf{Z}^0(\mathbf{F}(\mathsf{E}, \Delta, v)^\natural)$ is equivalent to $\mathsf{P}(\mathsf{E}, \Delta)$ according to Section 2.7. Thus the DG-category $\mathbf{F}(\mathsf{E}, \Delta, v)$ of factorizations of v in E is locally λ-presentable by Proposition 6.9.

(2) Let B be a locally presentable abelian category with enough projective objects, $\Delta\colon \mathsf{B} \longrightarrow \mathsf{B}$ be an autoequivalence, and $v\colon \mathrm{Id}_{\mathsf{B}} \longrightarrow \Delta$ be a potential. Then it is obvious that the abelian category of 2-Δ-periodic objects $\mathsf{P}(\mathsf{B}, \Delta) \simeq \mathsf{B} \times \mathsf{B}$ also has enough projectives. Following Example 3.15, the DG-category of factorizations $\mathbf{F}(\mathsf{B}, \Delta, v)$ is an abelian DG-category. Thus $\mathbf{F}(\mathsf{B}, \Delta, v)$ is a locally presentable abelian DG-category with enough projective objects.

6.3. **Becker's contraderived category.** In the rest of Section 6, we will be mostly working with a locally presentable abelian DG-category $\mathbf{B}$ with enough projective objects. Here the definition of an abelian DG-category was given in Section 3.2, abelian DG-categories with enough projective objects were defined in Section 6.1, and the definition of a locally presentable DG-category can be found in Section 6.2.

In this paper we are interested in the contraderived category *in the sense of Becker* [28, 5, 55], which we denote by $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$. In well-behaved cases, it is equivalent to the homotopy category of graded-projective objects, $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B}) \simeq \mathsf{H}^0(\mathbf{B}_{\mathbf{proj}})$.

The contraderived category in the sense of Becker has to be distinguished from the contraderived category in the sense of books and papers [47, 48, 49, 50]. The two definitions of a contraderived category are known to be equivalent under certain assumptions [48, Section 3.8], [51, Corollary 4.9], [52, Theorem 5.10(b)], but it is an open question whether they are equivalent for the category of modules over an arbitrary ring (see [49, Example 2.6(3)], [55, Remark 9.2], and [53, Section 7.9] for a discussion).

Notice that, in any locally presentable category, all limits exists, and in particular all products exist, by [1, Corollary 2.47]. So the dual version of Lemma 6.5 tells that infinite products exist in any locally presentable DG-category $\mathbf{B}$, and consequently they also exist in the homotopy category $\mathsf{H}^0(\mathbf{B})$. Furthermore, infinite products are exact in any abelian category with enough projective objects (see [52, Remark 5.2] for a more general assertion concerning exact categories.)

Let $\mathbf{B}$ be an abelian DG-category. An object $X \in \mathbf{B}$ is said to be *contraacyclic* (*in the sense of Becker*) if $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{B})}(Q, X) = 0$ for all graded-projective objects $Q \in \mathbf{B}_{\mathbf{proj}}$. We will denote the full subcategory of Becker-contraacyclic objects by $\mathsf{H}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}} \subset \mathsf{H}^0(\mathbf{B})$. Clearly, $\mathsf{H}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ is a triangulated (and even thick) subcategory in the homotopy category $\mathsf{H}^0(\mathbf{B})$. The triangulated Verdier quotient category $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B}) = \mathsf{H}^0(\mathbf{B})/\mathsf{H}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ is called the *contraderived category of* $\mathbf{B}$ (in the sense of Becker).

Lemma 6.13. (a) *For any short exact sequence* $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ *in the abelian category* $\mathsf{Z}^0(\mathbf{B})$*, the total object* $\mathrm{Tot}(K \to L \to M) \in \mathbf{B}$ *(as defined in Section 1.6) belongs to* $\mathsf{H}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$.

(b) *The full subcategory of contraacyclic objects* $\mathsf{H}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ *is closed under infinite products in* $\mathsf{H}^0(\mathbf{B})$.

Proof. This is a common generalization of [55, Lemma 7.1] (which is the version for complexes in abelian categories) and [51, Proposition 4.2] (the CDG-module version). A proof of an even more general assertion can be found in [52, Theorem 5.5(b)]. $\square$

The following result, generalizing [55, Corollary 7.4] and a similar unstated corollary of [5, Proposition 1.3.6(1)], will be deduced below in Section 6.5.

Corollary 6.14. *Let* $\mathbf{B}$ *be a locally presentable abelian DG-category with enough projective objects. Then the composition of the fully faithful inclusion of triangulated categories* $\mathsf{H}^0(\mathbf{B}_{\mathbf{proj}}) \longrightarrow \mathsf{H}^0(\mathbf{B})$ *with the Verdier quotient functor* $\mathsf{H}^0(\mathbf{B}) \longrightarrow \mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$ *is a triangulated equivalence* $\mathsf{H}^0(\mathbf{B}_{\mathbf{proj}}) \simeq \mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$.

6.4. **Deconstructibility of graded-projectives.** Let E be a cocomplete abelian category. A class of objects $\mathsf{L} \subset \mathsf{E}$ is said to be *deconstructible* if there exists a set of objects $\mathsf{S} \subset \mathsf{L}$ such that $\mathsf{L} = \mathsf{Fil}(\mathsf{S})$ (see Section 4.2 for the notation). Another useful property, which does not seem to have a convenient name in the literature, is existence of a set of objects $\mathsf{S} \subset \mathsf{L}$ such that $\mathsf{L} = \mathsf{Fil}(\mathsf{S})^{\oplus}$.

Proposition 6.15. *Let* $\mathbf{B}$ *be a locally presentable abelian DG-category with enough projective objects. Then there exists a set of graded-projective objects* $\mathsf{S} \subset \mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})$ *such that* $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}}) = \mathsf{Fil}(\mathsf{S})^{\oplus}$ *in the abelian category* $\mathsf{Z}^0(\mathbf{B})$.

This important technical assertion is a generalization of [55, Proposition 7.2]. Similarly to the exposition in [55], we suggest two proofs.

First proof. We restrict ourselves to a sketch of this abstract category-theoretic argument based on [38, Proposition A.1.5.12], [40, Corollary 3.6], and [55, Proposition 2.5].

Let $\mathbf{E}$ be an additive DG-category with shifts and cones. Then we have a pair of adjoint functors $\Psi^-_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ and $\Phi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$. Consequently, the composition $\Xi_{\mathbf{E}}[1] = \Psi^-_{\mathbf{E}} \circ \Phi_{\mathbf{E}}$, $A \longmapsto \mathrm{cone}(\mathrm{id}_A)$ for all $A \in \mathbf{E}$, is a monad on the additive category $\mathsf{Z}^0(\mathbf{E})$. The claim is that the functor $\Psi^-_{\mathbf{E}}$ is monadic, i. e., the comparison functor from $\mathsf{Z}^0(\mathbf{E}^\natural)$ to the category of algebras over the monad $\Xi_{\mathbf{E}}[1]$ on the category $\mathsf{Z}^0(\mathbf{E})$ is a category equivalence. This can be established by examining the construction of the DG-category $\mathbf{E}^\natural$. (Dually, the functor $\Xi_{\mathbf{E}} \colon A \longmapsto \mathrm{cone}(\mathrm{id}_A[-1])$ is a comonad on $\mathsf{Z}^0(\mathbf{E})$ and the functor $\Psi^+_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ is comonadic, so this functor is in fact both monadic and comonadic; cf. [44, Section 2] for a discussion of a particular case.)

Notice that the monad $\Xi_{\mathbf{E}}[1] \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E})$ preserves all limits and colimits, as does its left and right adjoint comonad $\Xi_{\mathbf{E}}$ (see Section 3.1).

Now let $\mathbf{E}$ be an abelian DG-category. Then, according to Corollary 3.10, we have $\mathbf{E} \simeq \mathbf{E}^{\natural\natural}$. Taking into account the isomorphisms of functors from the last paragraph of Section 2.4, it follows that the functor $\Phi_{\mathbf{E}} \colon \mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{Z}^0(\mathbf{E}^\natural)$ is monadic (as well as comonadic), i. e., the additive category $\mathsf{Z}^0(\mathbf{E})$ is equivalent to the category of algebras over the monad $\Xi_{\mathbf{E}^\natural}[1] = \Phi_{\mathbf{E}} \circ \Psi^+_{\mathbf{E}}$ on the additive category $\mathsf{Z}^0(\mathbf{E}^\natural)$. This sets the stage for our intended application of [40, Corollary 3.6].

The rest is essentially explained in [55, first proof of Proposition 7.2]. Following [55], we say that a class of objects L in an abelian category E is *transmonic* if any transfinite composition of L-monomorphisms is a monomorphism. Then it follows from [38, Proposition A.1.5.12] in view of [55, Proposition 2.5] that, for any transmonic set of objects S in a locally presentable abelian category E, the class of objects $\mathsf{Fil}(\mathsf{S})^{\oplus}$ is deconstructible, i. e., there exists a set of objects $\mathsf{S}' \subset \mathsf{E}$ such that $\mathsf{Fil}(\mathsf{S})^{\oplus} = \mathsf{Fil}(\mathsf{S}')$.

Furthermore, for any locally presentable abelian category E with a colimit-preserving monad $M \colon \mathsf{E} \longrightarrow \mathsf{E}$ and a transmonic deconstructible class of objects $\mathsf{L} \subset \mathsf{E}$, consider the category F of M-algebras in E and the forgetful functor $U \colon \mathsf{F} \longrightarrow \mathsf{E}$. Then it follows from [40, Corollary 3.6] in view of [55, Proposition 2.5] that the class $U^{-1}(\mathsf{L}) \subset \mathsf{F}$ is (transmonic and) deconstructible. Notice that the category F is abelian (with an exact functor U) by [50, Lemma 5.9] and F is locally presentable by [1, Theorem and Remark 2.78]. It is also easy to see that the functor U preserves all colimits.

In particular, for any locally presentable abelian category B with a projective generator P, one has $\mathsf{B}_{\mathsf{proj}} = \mathsf{Fil}(\{P\})^{\oplus}$ (see [55, Lemmas 6.1 and 6.2]). As the class $\mathsf{Fil}(\{P\})$ is transmonic and deconstructible in B, it follows that the class of all projective objects $\mathsf{B}_{\mathsf{proj}}$ is deconstructible.

Returning to the situation at hand with a locally presentable abelian DG-category $\mathbf{B}$ with enough projective objects, it remains to apply the observations above to the abelian categories $\mathsf{B} = \mathsf{E} = \mathsf{Z}^0(\mathbf{B}^\natural)$ and $\mathsf{F} = \mathsf{Z}^0(\mathbf{B})$ and the class of objects $\mathsf{L} = \mathsf{B}_{\mathsf{proj}}$ in order to conclude that the class $\mathsf{Z}^0(\mathbf{B_{proj}})$ is deconstructible in $\mathsf{Z}^0(\mathbf{B})$. So there exists a set of objects $\mathsf{S}'' \subset \mathsf{Z}^0(\mathbf{B_{proj}})$ such that $\mathsf{Z}^0(\mathbf{B_{proj}}) = \mathsf{Fil}(\mathsf{S}'')$ in $\mathsf{Z}^0(\mathbf{B})$. As one obviously has $\mathsf{Z}^0(\mathbf{B_{proj}}) = \mathsf{Z}^0(\mathbf{B_{proj}})^{\oplus}$, we have obtained an even stronger result than claimed in the proposition. $\square$

Second proof. Here is a direct proof of the assertion stated in Proposition 6.15. Notice that the functor $\Phi\colon \mathsf{Z}^0(\mathbf{B}) \longrightarrow \mathsf{Z}^0(\mathbf{B}^\natural)$ preserves extensions and directed colimits; hence it also preserves transfinitely iterated extensions (in the sense of the directed colimit). Since the class of all projective objects in $\mathsf{Z}^0(\mathbf{B}^\natural)$ is closed under transfinitely iterated extensions and direct summands [55, Lemma 6.2], it follows that the class $\mathsf{Z}^0(\mathbf{B_{proj}})$ is also closed under these operations in $\mathsf{Z}^0(\mathbf{B})$, i. e., $\mathsf{Fil}(\mathsf{Z}^0(\mathbf{B_{proj}}))^{\oplus} \subset \mathsf{Z}^0(\mathbf{B_{proj}})$. So it suffices to find a set of graded-projective objects $\mathsf{S} \subset \mathsf{Z}^0(\mathbf{B_{proj}})$ such that $\mathsf{Z}^0(\mathbf{B_{proj}}) \subset \mathsf{Fil}(\mathsf{S})^{\oplus}$.

Fix a single projective generator P of the abelian category $\mathsf{Z}^0(\mathbf{B}^\natural)$ (cf. [55, Lemma 6.1]). Replacing if needed P by $\coprod_{n\in\mathbb{Z}} P[n]$, we can assume without loss of generality that P is invariant under the shift, $P \simeq P[1]$. According to Section 2.2, we have $\Phi\Psi^+(P) \simeq \Xi_{\mathbf{B}^\natural}(P)[1]$, and by Corollary 3.2(a) there is a short exact sequence $0 \longrightarrow P \longrightarrow \Xi_{\mathbf{B}^\natural}(P) \longrightarrow P[1] \longrightarrow 0$ in $\mathsf{Z}^0(\mathbf{B}^\natural)$, which is split because the object $P[1]$ is projective. So we have $\Phi\Psi^+(P) \simeq P \oplus P[1] \simeq P \oplus P$.

Let us show that any graded-projective object $Q' \in \mathsf{Z}^0(\mathbf{B_{proj}})$ is a direct summand of an object $Q \in \mathsf{Z}^0(\mathbf{B_{proj}})$ such that $\Phi(Q)$ is a coproduct of copies of P. Indeed, let μ be a cardinal such that $\Phi(Q')$ is a direct summand of $P^{(\mu)}$, i. e. $\Phi(Q') \oplus P' \simeq P^{(\mu)}$ for some $P' \in \mathsf{Z}^0(\mathbf{B}^\natural)$. Put $Q = Q' \oplus \Psi^+(P^{(\mu\times\omega)})$. Then $\Phi(Q) = \Phi(Q') \oplus \Phi\Psi^+(P^{(\mu\times\omega)}) \simeq \Phi(Q') \oplus P^{(\mu\times 2\omega)} \simeq \Phi(Q') \oplus P^{(\mu\times\omega)}$. Now $\Phi(Q') \oplus P^{(\mu\times\omega)} \simeq \Phi(Q') \oplus (\Phi(Q') \oplus P')^{(\omega)} \simeq (\Phi(Q') \oplus P')^{(\omega)} \simeq P^{(\mu\times\omega)}$ by the cancellation trick.

Let κ be an uncountable regular cardinal such that the object $P \in \mathsf{Z}^0(\mathbf{B}^\natural)$ is κ-presentable (then the category $\mathsf{Z}^0(\mathbf{B}^\natural)$ is locally κ-presentable). Let S be the set of (representatives of isomorphism classes) of objects $S \in \mathsf{Z}^0(\mathbf{B})$ such that $\Phi(S)$ is a coproduct of less than κ copies of P. Notice that the object $\Phi(S) \in \mathsf{Z}^0(\mathbf{B}^\natural)$ is κ-presentable in this case, and by Proposition 6.7 it follows that the object $S \in \mathsf{Z}^0(\mathbf{B})$ is also κ-presentable. Since there is only a set of κ-presentable objects in a locally presentable category (up to isomorphism) [1, Remarks 1.19 and 1.20], we can see that S is indeed a set. Clearly, $\mathsf{S} \subset \mathsf{Z}^0(\mathbf{B_{proj}})$. We claim that any object $Q \in \mathsf{Z}^0(\mathbf{B})$ such that $\Phi(Q)$ is a coproduct of copies of P belongs to $\mathsf{Fil}(\mathsf{S})$.

Indeed, assume that $\Phi(Q) = P^{(X)}$ for some set X. Let α be the successor cardinal of the cardinality of X. Proceeding by transfinite induction on ordinals $0 \le \beta \le \alpha$, we will construct a smooth chain of subobjects $R_\beta \subset Q$ and a smooth chain of subsets $Y_\beta \subset X$ such that $\Phi(R_\beta) = P^{(Y_\beta)}$ as a subobject in $\Phi(Q) = P^{(X)}$ for all $0 \le \beta \le \alpha$, $Y_0 = \varnothing$ (hence $R_0 = 0$) and $Y_\alpha = X$ (hence $R_\alpha = Q$), and the cardinality of $Y_{\beta+1} \setminus Y_\beta$ is smaller than κ for all $0 \le \beta < \alpha$. Then it will follow that $\Phi(R_{\beta+1}/R_\beta) \simeq P^{(Y_{\beta+1})}/P^{(Y_\beta)} \simeq P^{(Y_{\beta+1} \setminus Y_\beta)}$, since the functor Φ preserves cokernels; hence $R_{\beta+1}/R_\beta \in \mathsf{S}$, as desired.

Suppose that the subsets $Y_\gamma \subset X$ and the subobjects $R_\gamma \subset Q$ have been already constructed for all $\gamma < \beta$. For a limit ordinal β, we put $Y_\beta = \bigcup_{\gamma<\beta} Y_\gamma$ and $R_\beta = \varinjlim_{\gamma<\beta} R_\gamma$; then Y_β is a subset in X and there is the induced morphism $R_\beta \longrightarrow Q$ in $\mathsf{Z}^0(\mathbf{B})$. We have $\Phi(R_\beta) = \varinjlim_{\gamma<\beta} \Phi(R_\gamma) = \varinjlim_{\gamma<\beta} P^{(Y_\gamma)} = P^{(Y_\beta)}$, since the functor Φ preserves colimits. So, applying the functor Φ to the morphism $R_\beta \longrightarrow Q$, we obtain the (split) monomorphism $P^{(Y_\beta)} \longrightarrow P^{(X)}$. Since the functor Φ is faithful and preserves kernels, it follows that the morphism $R_\beta \longrightarrow Q$ is a monomorphism. This finishes the construction in the case of a limit ordinal β. To deal with the case of a successor ordinal, we need some preparatory work.

For every element $x \in X$, let $\iota_x \colon P \longrightarrow P^{(X)} = \Phi(Q)$ be the direct summand inclusion corresponding to the element x. By adjunction, we have the corresponding morphism $\Psi^+(P) \longrightarrow Q$ in the category $\mathsf{Z}^0(\mathbf{B})$. Applying the functor Φ, we obtain a morphism $\tilde\iota_x \colon \Phi\Psi^+(P) \longrightarrow \Phi(Q) = P^{(X)}$ in $\mathsf{Z}^0(\mathbf{B}^\natural)$. We know that $\Phi\Psi^+(P) \simeq P \oplus P$; so the object $\Phi\Psi^+(P)$ is κ-presentable in $\mathsf{Z}^0(\mathbf{B}^\natural)$. Therefore, there exists a subset $V_x \subset X$ of the cardinality smaller than κ such that the morphism $\tilde\iota_x$ factorizes as $\Phi\Psi^+(P) \longrightarrow P^{(V_x)} \rightarrowtail P^{(X)}$. Proceeding by induction in $n \in \omega$, define subsets $W_x^n \subset X$ by the rules $W_x^0 = \{x\}$ and $W_x^{n+1} = \bigcup_{w \in W_x^n} V_w$. Put $W_x = \bigcup_{n\in\omega} W_x^n$. Then W_x is a subset in X of the cardinality smaller than κ.

Similarly to the notation in the previous paragraph, given a subset $W \subset X$, let us denote by $\iota_W \colon P^{(W)} \longrightarrow P^{(X)} = \Phi(Q)$ the related split monomorphism. By adjunction, we have the corresponding morphism $\Psi^+(P^{(W)}) \longrightarrow Q$ in $\mathsf{Z}^0(\mathbf{B})$; applying the functor Φ, we obtain a morphism $\tilde\iota_W \colon \Phi\Psi^+(P^{(W)}) \longrightarrow \Phi(Q) = P^{(X)}$. There is also the natural adjunction morphism $P^{(W)} \longrightarrow \Phi\Psi^+(P^{(W)})$, whose composition with the morphism $\tilde\iota_W$ is the morphism ι_W. Notice that $\Phi\Psi^+(P^{(W)}) = (\Phi\Psi^+(P))^{(W)}$. For any element $x \in X$, the subset $W_x \subset X$ was constructed in such a way that the morphism $\tilde\iota_{W_x}$ factorizes through the split monomorphism ι_{W_x}:

$$P^{(W_x)} \longrightarrow \Phi\Psi^+(P^{(W_x)}) \longrightarrow P^{(W_x)} \rightarrowtail P^{(X)} = \Phi(Q). \tag{12}$$

Denote by L_x the image of the morphism $\Psi^+(P^{(W_x)}) \longrightarrow Q$ in the abelian category $\mathsf{Z}^0(\mathbf{B})$. Applying the exact functor Φ, we see that the object $\Phi(L_x)$ is the image of the morphism $\tilde\iota_W$ in the abelian category $\mathsf{Z}^0(\mathbf{B}^\natural)$:

$$\Phi\Psi^+(P^{(W_x)}) \twoheadrightarrow \Phi(L_x) \rightarrowtail \Phi(Q). \tag{13}$$

Comparing (12) with (13), we conclude that $\Phi(L_x)$ and $P^{(W_x)}$ is one and the same subobject in $\Phi(Q) = P^{(X)}$.

Now we can return to our successor ordinal $\beta = \gamma + 1 \le \alpha$. If $Y_\gamma = X$, then $\Phi(Q/R_\gamma) = 0$ and it follows that $R_\gamma = Q$ (since the functor Φ is faithful); so we put $Y_\beta = X$ and $R_\beta = Q$ as well. Otherwise, choose an element $x \in X \setminus Y_\gamma$. Put $Y_\beta = Y_\gamma \cup W_x \subset X$ and $R_\beta = R_\gamma + L_x \subset Q$. ☐

6.5. **Contraderived abelian model structure.** The results of this section form a common generalization of [5, Proposition 1.3.6(1)] (the CDG-module case) and [55, Section 7] (the case of complexes in abelian categories). The references to preceding results in the literature can be found in [55].

Let $\mathbf{B}$ be an abelian DG-category. Introduce the notation $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ for the full subcategory of Becker-contraacyclic objects in $\mathsf{Z}^0(\mathbf{B})$. So $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}} \subset \mathsf{Z}^0(\mathbf{B})$ is the full preimage of $\mathsf{H}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}} \subset \mathsf{H}^0(\mathbf{B})$ under the obvious functor $\mathsf{Z}^0(\mathbf{B}) \longrightarrow \mathsf{H}^0(\mathbf{B})$.

Theorem 6.16. *Let $\mathbf{B}$ be a locally presentable abelian DG-category with enough projective objects. Then the pair of classes of objects $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})$ and $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ is a hereditary complete cotorsion pair in the abelian category $\mathsf{Z}^0(\mathbf{B})$.*

Proof. This is a generalization of [55, Theorem 7.3].

Let $\mathsf{S} \subset \mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})$ be a set of graded-projective objects in $\mathbf{B}$ such that $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}}) = \mathsf{Fil}(\mathsf{S})^\oplus$ in $\mathsf{Z}^0(\mathbf{B})$, as in Proposition 6.15. The claim is that the set $\mathsf{S} \subset \mathsf{Z}^0(\mathbf{B})$ generates the desired cotorsion pair.

Indeed, for any objects $Q \in \mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})$ and $B \in \mathsf{Z}^0(\mathbf{B})$ we have $\mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{B})}(Q, B) \simeq \mathrm{Hom}_{\mathsf{H}^0(\mathbf{B})}(Q, B[1])$ by Lemma 6.1. Hence $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})^{\perp_1} = \mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}} \subset \mathsf{Z}^0(\mathbf{B})$. By Lemma 4.3, it follows that $\mathsf{S}^{\perp_1} = \mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$.

Furthermore, the class $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}}) = \mathsf{Fil}(\mathsf{S})^\oplus$ is generating in $\mathsf{Z}^0(\mathbf{B})$ (e. g., because $\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}} \subset \mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})$ by Lemma 6.3(a) and there are enough projective objects in the abelian category $\mathsf{Z}^0(\mathbf{B})$ by assumption). Applying Theorem 4.4(b), we can conclude that $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}}) = {}^{\perp_1}(\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}})$.

Here is an alternative elementary argument proving the latter equality. Notice that in the pair of adjoint functors between abelian categories $\Phi\colon \mathsf{Z}^0(\mathbf{B}) \longrightarrow \mathsf{Z}^0(\mathbf{B}^\natural)$ and $\Psi^-\colon \mathsf{Z}^0(\mathbf{B}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{B})$, both the functors are exact. Therefore, for any objects $B \in \mathsf{Z}^0(\mathbf{B})$ and $E \in \mathsf{Z}^0(\mathbf{B}^\natural)$ one has $\mathrm{Ext}^n_{\mathsf{Z}^0(\mathbf{B})}(B, \Psi^-(E)) \simeq \mathrm{Ext}^n_{\mathsf{Z}^0(\mathbf{B}^\natural)}(\Phi(B), E)$ for all $n \ge 0$ (see, e. g., [52, Lemma 9.34]). In particular, this isomorphism holds for $n = 1$. Since the object $\Psi^-(E)$ is contractible (by the construction of the functor Ψ^-), and consequently belongs to $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$, it follows that ${}^{\perp_1}(\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}) \subset \mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})$. The converse inclusion holds since we already know that $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}})^{\perp_1} = \mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$.

Any object $B \in \mathsf{Z}^0(\mathbf{B})$ is a subobject of the contractible object $\Xi(B)[1] = \mathrm{cone}(\mathrm{id}_B)$ by Corollary 3.2(a), so the class $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ is cogenerating in $\mathsf{Z}^0(\mathbf{B})$. Hence Theorem 4.4(a) is applicable, and we can conclude that our cotorsion pair is complete. The cotorsion pair is hereditary, since the class $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ is closed under the cokernels of monomorphisms in $\mathsf{Z}^0(\mathbf{B})$, as one can see from Lemma 6.13(a). It is also clear that the class $\mathsf{Z}^0(\mathbf{B}_{\mathbf{proj}}) \subset \mathsf{Z}^0(\mathbf{B})$ is closed under the kernels of epimorphisms, since the functor Φ preserves kernels. ☐

Now we are ready to prove the corollary formulated in Section 6.3.

Proof of Corollary 6.14. It is clear from the definitions that the triangulated functor $\mathsf{Z}^0(\mathbf{B_{proj}}) \longrightarrow \mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$ is fully faithful (cf. [55, Proposition 6.5]). In order to prove the corollary, it remains to find, for any object $B \in \mathsf{Z}^0(\mathbf{B})$, a graded-projective object $Q \in \mathsf{Z}^0(\mathbf{B_{proj}})$ together with a morphism $Q \longrightarrow B$ in $\mathsf{Z}^0(\mathbf{B})$ whose cone belongs to $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$.

For this purpose, consider a special precover short exact sequence $0 \longrightarrow X \longrightarrow Q \longrightarrow B \longrightarrow 0$ in the complete cotorsion pair of Theorem 6.16. So we have $X \in \mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$ and $Q \in \mathsf{Z}^0(\mathbf{B_{proj}})$. By Lemma 6.13(a), the totalization $\mathrm{Tot}(X \to Q \to C)$ of the short exact sequence $0 \longrightarrow X \longrightarrow Q \longrightarrow C \longrightarrow 0$ in $\mathsf{Z}^0(\mathbf{B})$ is a Becker-contraacyclic object. Since the object X is Becker-contraacyclic, it follows that the cone of the closed morphism $Q \longrightarrow C$ is Becker-contraacyclic, too. ☐

Theorem 6.17. *Let $\mathbf{B}$ be a locally presentable abelian DG-category with enough projective objects. Then the triple of classes of objects $\mathsf{L} = \mathsf{Z}^0(\mathbf{B_{proj}})$, $\mathsf{W} = \mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$, and $\mathsf{R} = \mathsf{Z}^0(\mathbf{B})$ is a cofibrantly generated hereditary abelian model structure on the abelian category $\mathsf{Z}^0(\mathbf{B})$.*

Proof. This is a common generalization of [5, Proposition 1.3.6(1)] (which is the CDG-module case) and [55, Theorem 7.5] (which is the case of complexes in abelian categories). The pair of classes (L, W) is a hereditary complete cotorsion pair in $\mathsf{Z}^0(\mathbf{B})$ by Theorem 6.16. According to Lemma 6.4, it follows that $\mathsf{L} \cap \mathsf{W} = \mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}$ is the class of all projective objects in $\mathsf{Z}^0(\mathbf{B})$ and the class of all Becker-contraacyclic objects W is thick in $\mathsf{Z}^0(\mathbf{B})$ (see also Lemma 6.13(a)). By Lemma 5.9(a), the triple $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ is a projective abelian model structure on the abelian category $\mathsf{Z}^0(\mathbf{B})$.

Finally, it was shown in the proof of Theorem 6.16 that the cotorsion pair (L, W) in $\mathsf{Z}^0(\mathbf{B})$ is generated by a set of objects. The cotorsion pair $(\mathsf{Z}^0(\mathbf{B})_{\mathsf{proj}}, \mathsf{Z}^0(\mathbf{B}))$ is generated by the empty set of objects (or by the single zero object, or by any chosen single projective generator) in $\mathsf{Z}^0(\mathbf{B})$. According to Lemma 5.7, this means that our abelian model structure is cofibrantly generated. ☐

The abelian model structure $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ defined in Theorem 6.17 is called the *contraderived model structure* on the category $\mathsf{Z}^0(\mathbf{B})$.

Lemma 6.18. *For any locally presentable abelian DG-category $\mathbf{B}$ with enough projective objects, the class $\mathcal{W}$ of all weak equivalences in the contraderived model structure on $\mathsf{Z}^0(\mathbf{B})$ coincides with the class of all closed morphisms of degree 0 in $\mathbf{B}$ with the cones belonging to $\mathsf{Z}^0(\mathbf{B})^{\mathsf{bctr}}_{\mathsf{ac}}$.*

Proof. [25, Lemma 5.8] tells that a monomorphism in an abelian model category is a weak equivalence if and only if its cokernel is weakly trivial. Dually, an epimorphism is a weak equivalence if and only if its kernel is weakly trivial.

In the contraderived model structure on $\mathsf{Z}^0(\mathbf{B})$, the class of weakly trivial objects W is the class of all Becker-contraacyclic objects. Any morphism f in $\mathsf{Z}^0(\mathbf{B})$ can be factorized as $f = rl$, where l is, say, a trivial cofibration, and r is a fibration. Now l is a monomorphism with a Becker-contraacyclic cokernel, hence in view of Lemma 6.13(a) also with a Becker-contraacyclic cone; at the same time l is a weak equivalence. On the other hand, r is an epimorphism.

Notice that both the class of weak equivalences and the class of morphisms with a Becker-contraacyclic cone satisfy the 2-out-of-3 property. If f is a weak equivalence, then so is r; then the kernel of r is Becker-contraacyclic, hence by Lemma 6.13(a) the cone of r is Becker-contraacyclic as well; it follows that the cone of f is Becker-contraacyclic. If f has a Becker-contraacyclic cone, then so does r; hence by Lemma 6.13(a) the kernel of r is Becker-contraacyclic; thus r is a weak equivalence, and it follows that f is a weak equivalence. Alternatively, one could use the fact that any morphism f is the composition of a cofibration l and a trivial fibration r and analogous reasoning in that case. □

The following corollary presumes existence of infinite coproducts in Becker's contraderived category $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$. Such coproducts can be simply constructed as the coproducts in the homotopy category of graded-projective objects $\mathsf{H}^0(\mathbf{B}_{\mathbf{proj}})$, which is equivalent to $\mathsf{B}^{\mathsf{bctr}}(\mathbf{B})$ by Corollary 6.14. Notice that coproducts of graded-projective objects in $\mathbf{B}$ are graded-projective, since the coproducts of projectives in $\mathsf{Z}^0(\mathbf{B}^\natural)$ are projective and the functor Φ preserves coproducts.

Corollary 6.19. *For any locally presentable abelian DG-category* $\mathbf{B}$ *with enough projective objects, Becker's contraderived category* $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$ *is a well-generated triangulated category.*

Proof. This is a generalization of [55, Corollary 7.7]. Let us show that the contraderived category $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$ can be equivalently defined as the homotopy category $\mathsf{Z}^0(\mathbf{B})[\mathcal{W}^{-1}]$ of the contraderived model structure on $\mathsf{Z}^0(\mathbf{B})$.

The key observation, generalizing the proof in [55], is that, for any DG-category $\mathbf{E}$ with shifts and cones, inverting all the homotopy equivalences in $\mathsf{Z}^0(\mathbf{E})$ produces the homotopy category $\mathsf{H}^0(\mathbf{E})$. In other words, homotopic closed morphisms of degree 0 become equal after inverting the homotopy equivalences. Indeed, for any pair of homotopic closed morphisms of degree zero f', $f''\colon A \longrightarrow B$ in $\mathbf{E}$, there exist a morphism $h\colon A \oplus \operatorname{cone}(\mathrm{id}_A) \longrightarrow B$ and two morphisms ι', $\iota''\colon A \longrightarrow A \oplus \operatorname{cone}(\mathrm{id}_A)$ in $\mathsf{Z}^0(\mathbf{E})$ such that $f' = h\iota'$ and $f'' = h\iota''$, while $\pi\iota' = \mathrm{id}_A = \pi\iota''$, where $\pi\colon A \oplus \operatorname{cone}(\mathrm{id}_A) \longrightarrow A$ is the direct summand projection. The morphism π is a homotopy equivalence in $\mathbf{E}$, since the object $\operatorname{cone}(\mathrm{id}_A)$ is contractible. Inverting π identifies ι' with ι'' and consequently f' with f''.

In the situation at hand, it follows that inverting all the morphisms with Becker-contraacyclic cones in $\mathsf{Z}^0(\mathbf{B})$ produces Becker's contraderived category $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B})$. It remains to use Lemma 6.18 in order to conclude that $\mathsf{D}^{\mathsf{bctr}}(\mathbf{B}) = \mathsf{Z}^0(\mathbf{B})[\mathcal{W}^{-1}]$.

By Theorem 6.17, the contraderived model structure on $\mathsf{Z}^0(\mathbf{B})$ is hereditary abelian and cofibrantly generated. By Lemma 5.11, this model structure is also stable. So Proposition 5.12 can be applied, finishing the proof. □

7. Coderived Model Structure

In this section we work out a common generalization of the coderived model structure on the abelian category of CDG-modules over a CDG-ring [5, Proposition 1.3.6(2)] and the coderived model structure on the category of complexes over a Grothendieck abelian category [20, Section 4.1], [55, Section 9]. The Grothendieck abelian DG-categories $\mathbf{A}$ (as defined in [52, Section 9.1]) form a suitable context.

The main results of the section, however, are Corollary 7.17, claiming that the class of all Becker-coacyclic objects is closed under directed colimits, and Theorem 7.18, providing a sufficient condition for a set of objects to generate Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ as a triangulated category with coproducts.

7.1. **Injective objects in abelian DG-categories.** In this section we briefly list the dual assertions to the ones in Section 6.1 and fix the related terminology/notation.

Lemma 7.1. *Let $\mathbf{A}$ be an abelian DG-category. Then the abelian category $\mathsf{Z}^0(\mathbf{A})$ has enough injective objects if and only if the abelian category $\mathsf{Z}^0(\mathbf{A}^\natural)$ has enough injective objects.*

Proof. This is the dual version of Lemma 6.2. □

An abelian DG-category is said to *have enough injective objects* if it satisfies the equivalent conditions of Lemma 7.1.

Lemma 7.2. *Let $\mathbf{A}$ be an abelian DG-category with enough injective objects. Then*

(a) *an object $J \in \mathsf{Z}^0(\mathbf{A})$ is injective if and only if J is contractible in $\mathbf{A}$* and *the object $\Phi(J) \in \mathsf{Z}^0(\mathbf{A}^\natural)$ is injective;*

(b) *an object $I \in \mathsf{Z}^0(\mathbf{A}^\natural)$ is injective if and only if I is contractible in $\mathbf{A}^\natural$* and *the object $\Psi^-(I) \in \mathsf{Z}^0(\mathbf{A})$ is injective.*

Proof. This is the dual version of Lemma 6.3. □

Let $\mathbf{A}$ be an abelian DG-category. We will say that an object $J \in \mathbf{A}$ is *graded-injective* if the object $\Phi(J)$ is injective in the abelian category $\mathsf{Z}^0(\mathbf{A}^\natural)$. The full DG-subcategory formed by the graded-injective objects in $\mathbf{A}$ is denoted by $\mathbf{A}_{\mathbf{inj}} \subset \mathbf{A}$. So the notation $\mathsf{Z}^0(\mathbf{A}_{\mathbf{inj}}) \subset \mathsf{Z}^0(\mathbf{A})$ stands for the full subcategory of all graded-injective objects in $\mathsf{Z}^0(\mathbf{A})$, while $\mathsf{Z}^0(\mathbf{A})_{\mathsf{inj}} \subset \mathsf{Z}^0(\mathbf{A})$ is the (smaller) full subcategory of all injective objects in $\mathsf{Z}^0(\mathbf{A})$. Similarly to the case of graded-projectives discussed in Section 6.1, the full DG-subcategory of graded-injective objects $\mathbf{A}_{\mathbf{inj}}$ is closed under finite direct sums, shifts, twists, and cones in $\mathbf{A}$.

Lemma 7.3. *Let $\mathbf{A}$ be an abelian DG-category with enough injective objects, and let (W, R) be a cotorsion pair in the abelian category $\mathsf{Z}^0(\mathbf{A})$ such that $\mathsf{R} \subset \mathsf{Z}^0(\mathbf{A}_{\mathbf{inj}})$. Assume that the cotorsion pair (W, R) is preserved by the shift: $\mathsf{W} = \mathsf{W}[1]$, or equivalently, $\mathsf{R} = \mathsf{R}[1]$. Then $\mathsf{R} \cap \mathsf{W} = \mathsf{Z}^0(\mathbf{A})_{\mathsf{inj}}$. If, moreover, the cotorsion pair (W, R) is complete, then it is hereditary and the class W is thick in $\mathsf{Z}^0(\mathbf{A})$.*

Proof. This is the dual version of Lemma 6.4. □

7.2. **Grothendieck abelian DG-categories.** This section is mostly an extraction from [52, Section 9.1].

Lemma 7.4. *Let* $\mathbf{A}$ *be an abelian DG-category with infinite coproducts. Then the coproduct functors are exact in the abelian category* $\mathsf{Z}^0(\mathbf{A})$ *if and only if they are exact in the abelian category* $\mathsf{Z}^0(\mathbf{A}^\natural)$*. Moreover, the directed colimits are exact in* $\mathsf{Z}^0(\mathbf{A})$ *if and only if they are exact in* $\mathsf{Z}^0(\mathbf{A}^\natural)$*.*

Proof. This is [52, Lemma 9.3]. The point is that both the functors $\Phi\colon \mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{Z}^0(\mathbf{A}^\natural)$ and $\Psi^+\colon \mathsf{Z}^0(\mathbf{A}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{A})$ are exact and faithful, and preserve directed colimits. So any instance of nonexactness of a directed coproduct/colimit in one of the two abelian categories would be taken by the respective functor to an instance of nonexactness of a similar coproduct/colimit in the other category. $\square$

A *Grothendieck category* is an abelian category with a (single) generator, infinite coproducts, and exact directed colimits.

Proposition 7.5. *Let* $\mathbf{A}$ *be an abelian DG-category with infinite coproducts. Then the abelian category* $\mathsf{Z}^0(\mathbf{A})$ *is Grothendieck if and only if the abelian category* $\mathsf{Z}^0(\mathbf{A}^\natural)$ *is Grothendieck.*

Proof. This is [52, Proposition 9.4]. The assertion follows from Lemmas 6.8 and 7.4 above. $\square$

An abelian DG-category $\mathbf{A}$ with infinite coproducts is said to be *Grothendieck* if both the abelian categories $\mathsf{Z}^0(\mathbf{A})$ and $\mathsf{Z}^0(\mathbf{A}^\natural)$ are Grothendieck. In other words, this means that $\mathbf{A}$ satisfies assumptions and the equivalent conditions of Proposition 7.5.

Notice that any locally finitely presentable abelian category is Grothendieck [1, Proposition 1.59]. Consequently, any locally finitely presentable abelian DG-category is Grothendieck. On the other hand, all Grothendieck abelian categories are locally presentable [35, Corollary 5.2]. Therefore, any Grothendieck abelian DG-category is locally presentable. Furthermore, it is well-known that all Grothendieck abelian categories have enough injective objects. Consequently, all Grothendieck abelian DG-categories have enough injective objects.

Example 7.6. Let A be a Grothendieck abelian category. Then the DG-category $\mathbf{C}(\mathsf{A})$ of complexes in A is abelian, as explained in Example 3.13. It is clear that the abelian category of graded objects $\mathsf{G}(\mathsf{A}) = \mathsf{A}^{\mathbb{Z}}$ is Grothendieck, and it is also easy to see that the abelian category of complexes $\mathsf{C}(\mathsf{A}) = \mathsf{Z}^0(\mathbf{C}(\mathsf{A}))$ is Grothendieck. According to Section 2.5, the abelian category $\mathsf{Z}^0(\mathbf{C}(\mathsf{A})^\natural)$ is equivalent to $\mathsf{G}(\mathsf{A})$. Hence $\mathbf{C}(\mathsf{A})$ is a Grothendieck abelian DG-category.

Example 7.7. Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a CDG-ring. Then, according to Example 6.11, the DG-category $\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ of left CDG-modules over $\boldsymbol{R}^\bullet$ is a locally finitely presentable abelian DG-category. Consequently, $\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ is also a Grothendieck DG-category.

Example 7.8. Let A be a Grothendieck abelian category, $\Delta\colon \mathsf{A} \longrightarrow \mathsf{A}$ be an autoequivalence, and $v\colon \mathrm{Id}_\mathsf{A} \longrightarrow \mathsf{A}$ be a potential. Then the DG-category $\mathbf{F}(\mathsf{A}, \Delta, v)$ of factorizations of v in A is abelian, as per Example 3.15. It is clear that the abelian category of 2-Δ-periodic objects $\mathsf{P}(\mathsf{A},\Delta) \simeq \mathsf{A}\times\mathsf{A}$ is Grothendieck and the abelian category of factorizations $\mathsf{F}(\mathsf{A},\Delta,v)$ has infinite coproducts. According to Section 2.7, the abelian category $\mathsf{Z}^0(\mathbf{F}(\mathsf{A},\Delta,v)^\natural)$ is equivalent to $\mathsf{P}(\mathsf{A},\Delta)$. Hence $\mathbf{F}(\mathsf{A},\Delta,v)$ is a Grothendieck abelian DG-category.

7.3. **Becker's coderived category.** In the rest of Section 7, we will be mostly working with a Grothendieck abelian DG-category $\mathbf{A}$.

In this paper we are interested in the coderived category *in the sense of Becker* [33, 5, 62, 55], which we denote by $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$. In well-behaved cases, it is equivalent to the homotopy category of graded-injective objects, $\mathsf{D}^{\mathsf{bco}}(\mathbf{A}) \simeq \mathsf{H}^0(\mathbf{A_{inj}})$.

The coderived category in the sense of Becker has to be distinguished from the coderived category in the sense of books and papers [47, 48, 13, 49, 50]. The two definitions of a coderived category are known to be equivalent under certain assumptions [48, Section 3.7], [51, Corollary 4.18], [52, Theorem 5.10(a)], but it is an open question whether they are equivalent for the category of modules over an arbitrary ring (see [49, Example 2.5(3)], [55, Remark 9.2], and [53, Section 7.9] for a discussion).

Let $\mathbf{A}$ be an abelian DG-category. An object $Y\in\mathbf{A}$ is said to be *coacyclic* (*in the sense of Becker*) if $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{A})}(Y,J)=0$ for all graded-injective objects $J\in\mathbf{A_{inj}}$. We will denote the full subcategory of Becker-coacyclic objects by $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \subset \mathsf{H}^0(\mathbf{A})$ and its preimage under the obvious functor $\mathsf{Z}^0(\mathbf{A})\longrightarrow\mathsf{H}^0(\mathbf{A})$ by $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}\subset\mathsf{Z}^0(\mathbf{A})$. Clearly, $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is a triangulated (and even thick) subcategory in the homotopy category $\mathsf{H}^0(\mathbf{A})$. The triangulated Verdier quotient category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})=\mathsf{H}^0(\mathbf{A})/\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is called the *coderived category of* $\mathbf{A}$ (in the sense of Becker).

Lemma 7.9. *The class of all Becker-coacyclic objects $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under transfinitely iterated extensions (in the sense of the directed colimit) in the abelian category $\mathsf{Z}^0(\mathbf{A})$. In particular,*

(a) *for any short exact sequence $0\longrightarrow K\longrightarrow L\longrightarrow M\longrightarrow 0$ in the abelian category $\mathsf{Z}^0(\mathbf{A})$, the total object $\mathrm{Tot}(K\to L\to M)\in\mathbf{A}$ (as defined in Section 1.6) belongs to $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$.*

(b) *The full subcategory of coacyclic objects $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under infinite coproducts in $\mathsf{H}^0(\mathbf{A})$.*

Proof. For any object $Y\in\mathbf{A}$ and any graded-injective object $J\in\mathbf{A_{inj}}$, there is an isomorphism of abelian groups $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{A})}(Y,J)\simeq\mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{A})}(Y,J[-1])$ provided by Lemma 6.1. Therefore, the full subcategory $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}\subset\mathsf{Z}^0(\mathbf{A})$ consists precisely of all the objects $Y\in\mathbf{A}$ such that $\mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{A})}(Y,J)=0$ for all $J\in\mathbf{A_{inj}}$. Now Lemma 4.3 implies that the class $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under transfinitely iterated extensions in $\mathsf{Z}^0(\mathbf{A})$.

Both parts (a) and (b) are particular cases of the assertion about transfinitely iterated extensions (assuming that coproducts exist in $\mathbf{A}$). Indeed, the object $\mathrm{Tot}(K\to L\to M)$ fits into a short exact sequence

$$0\longrightarrow\mathrm{cone}(\mathrm{id}_K)\longrightarrow\mathrm{Tot}(K\to L\to M)\longrightarrow\mathrm{cone}(\mathrm{id}_M)[-1]\longrightarrow 0$$

in $\mathsf{Z}^0(\mathbf{A})$ and (the suitable shifts of) the cones of identity endomorphisms of the objects K and M are contractible and therefore coacyclic. If coproducts exist in $\mathbf{A}$, then the coproducts in $\mathsf{H}^0(\mathbf{A})$ agree with those in $\mathsf{Z}^0(\mathbf{A})$, and the latter can be interpreted as transfinitely iterated extensions. Alternatively, both parts (a) and (b) are the dual versions of the respective parts of Lemma 6.13. $\square$

The following result, generalizing [55, Corollary 9.5] and a similar unstated corollary of [5, Proposition 1.3.6(2)], will be deduced below in Section 7.4. Other relevant references include [43, Theorem 2.13], [35, Corollary 5.13], and [20, Theorem 4.2].

Corollary 7.10. *Let $\mathbf{A}$ be a Grothendieck abelian DG-category. Then the composition of the fully faithful inclusion of triangulated categories $\mathsf{H}^0(\mathbf{A_{inj}}) \longrightarrow \mathsf{H}^0(\mathbf{A})$ with the Verdier quotient functor $\mathsf{H}^0(\mathbf{A}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is a triangulated equivalence $\mathsf{H}^0(\mathbf{A_{inj}}) \simeq \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$.*

7.4. **Coderived abelian model structure.** The results of this section form a common generalization of [5, Proposition 1.3.6(2)] (the CDG-module case) and [55, Section 9] (the case of complexes in abelian categories). The references to preceding results in the literature can be found two paragraphs above or in [55].

Theorem 7.11. *Let $\mathbf{A}$ be a Grothendieck abelian DG-category. Then the pair of classes of objects $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ and $\mathsf{Z}^0(\mathbf{A_{inj}})$ is a hereditary complete cotorsion pair in the abelian category $\mathsf{Z}^0(\mathbf{A})$.*

Proof. This is a generalization of [55, Theorem 9.3].

In any Grothendieck abelian category A, there exists a set of objects S_0 such that $\mathsf{A} = \mathsf{Fil}(\mathsf{S}_0)$ [5, Example 1.2.6(1)], [55, Lemma 8.2]. Let S_0 be such a set of objects in the Grothendieck abelian category $\mathsf{A} = \mathsf{Z}^0(\mathbf{A}^\natural)$. The claim is that the desired cotorsion pair is generated by the set of objects $\mathsf{S} = \{\Psi^+(S) \mid S \in \mathsf{S}_0\} \subset \mathsf{Z}^0(\mathbf{A})$.

Indeed, for any objects $E \in \mathsf{Z}^0(\mathbf{A}^\natural)$ and $A \in \mathsf{Z}^0(\mathbf{A})$ we have $\mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{A})}(\Psi^+(E), A) \simeq \mathrm{Ext}^1_{\mathsf{Z}^0(\mathbf{A}^\natural)}(E, \Phi(A))$ by [52, Lemma 9.34], because both functors in the adjoint pair $\Psi^+ \colon \mathsf{Z}^0(\mathbf{A}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{A})$ and $\Phi \colon \mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{Z}^0(\mathbf{A}^\natural)$ are exact (cf. the proof of Theorem 6.16). The functor Ψ^+ is exact and preserves colimits, so it also preserves transfinitely iterated extensions. In view of Lemma 4.3, it follows that $\mathsf{S}^{\perp_1} = \Psi^+(\mathsf{Z}^0(\mathbf{A}^\natural))^{\perp_1} = \mathsf{Z}^0(\mathbf{A_{inj}}) \subset \mathsf{Z}^0(\mathbf{A})$.

Furthermore, $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} = {}^{\perp_1}\mathsf{Z}^0(\mathbf{A_{inj}}) \subset \mathsf{Z}^0(\mathbf{A})$, as we have already seen in the proof of Lemma 7.9. Hence our pair of classes of objects is indeed the cotorsion pair generated by the set S in $\mathsf{Z}^0(\mathbf{A})$.

Any object $A \in \mathsf{Z}^0(\mathbf{A})$ is a quotient object of the contractible object $\Xi(A) = \mathrm{cone}(\mathrm{id}_A[-1])$ by Corollary 3.2. All contractible objects are Becker-coacyclic; so the class $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is generating in $\mathsf{Z}^0(\mathbf{A})$. Any object in $\mathsf{Z}^0(\mathbf{A})$ is also a subobject of an injective object, and all injective objects are graded-injective by Lemma 7.2(a); hence the class $\mathsf{Z}^0(\mathbf{A_{inj}})$ is cogenerating in $\mathsf{Z}^0(\mathbf{A})$. Applying Theorem 4.4(a), we conclude that our cotorsion pair is complete.

The cotorsion pair is hereditary, because the class $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under the kernels of epimorphisms in $\mathsf{Z}^0(\mathbf{A})$, as one can see from Lemma 7.9(a). It is also clear that the class $\mathsf{Z}^0(\mathbf{A_{inj}}) \subset \mathsf{Z}^0(\mathbf{A})$ is closed under the cokernels of monomorphisms, since the functor Φ preserves cokernels. $\square$

The proof of Theorem 7.11 implies the following description of the class of all Becker-coacyclic objects.

Corollary 7.12. *Let $\mathbf{A}$ be a Grothendieck abelian DG-category. Then the Becker-coacyclic objects in $\mathsf{Z}^0(\mathbf{A})$ are precisely the direct summands of objects filtered by contractible ones. In fact, if $\mathsf{S}_0 \subset \mathsf{Z}^0(\mathbf{A}^\natural)$ is a set of objects such that $\mathsf{Z}^0(\mathbf{A}^\natural) = \mathsf{Fil}(\mathsf{S}_0)$ and $\mathsf{S} = \{\Psi^+(S) \mid S \in \mathsf{S}_0\}$ as above, then $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} = \mathsf{Fil}(\mathsf{S})^\oplus$.*

Proof. This is a generalization of [55, Corollary 9.4].

Let $E \in \mathsf{Z}^0(\mathbf{A}^\natural)$ be an arbitrary object. Since $E \in \mathsf{Fil}(\mathsf{S}_0)$ and the functor Ψ^+ preserves transfinitely iterated extensions, we have $\Psi^+(E) \in \mathsf{Fil}(\mathsf{S})$. As any object $A \in \mathsf{A}$ is a quotient object of the object $\Xi(A) \simeq \Psi^+\Phi(A)$, we see that the class $\mathsf{Fil}(\mathsf{S})$ is generating in $\mathsf{Z}^0(\mathbf{A})$. Applying Theorem 4.4(b), we can conclude that $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} = \mathsf{Fil}(\mathsf{S})^\oplus$. Since every object in S is contractible, any contractible object is Becker-coacyclic, and the class of all Becker-coacyclic objects is closed under transfinitely iterated extensions, it also follows that $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is the class of all direct summands of transfinitely iterated extensions of contractible objects. $\square$

Proof of Corollary 7.10. It is clear from the definitions that the functor $\mathsf{H}^0(\mathbf{A_{inj}}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is fully faithful (cf. [55, Corollary 9.5]). In order to prove the corollary, it remains to find, for any object $A \in \mathsf{Z}^0(\mathbf{A})$, a graded-injective object $J \in \mathsf{Z}^0(\mathbf{A_{inj}})$ together with a morphism $A \longrightarrow J$ in $\mathsf{Z}^0(\mathbf{A})$ whose cone belongs to $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$.

For this purpose, consider a special preenvelope short exact sequence $0 \longrightarrow A \longrightarrow J \longrightarrow Y \longrightarrow 0$ in the complete cotorsion pair of Theorem 7.11. So we have $J \in \mathsf{Z}^0(\mathbf{A_{inj}})$ and $Y \in \mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$. By Lemma 7.9(a), the totalization $\mathrm{Tot}(A \to J \to Y)$ of the short exact sequence $0 \longrightarrow A \longrightarrow J \longrightarrow Y \longrightarrow 0$ is a Becker-coacyclic object. Since the object Y is Becker-coacyclic, it follows that the cone of the morphism $A \longrightarrow J$ is Becker-coacyclic, too. $\square$

Theorem 7.13. *Let $\mathbf{A}$ be a Grothendieck abelian DG-category. Then the triple of classes of objects $\mathsf{L} = \mathsf{Z}^0(\mathbf{A})$, $\mathsf{W} = \mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$, and $\mathsf{R} = \mathsf{Z}^0(\mathbf{A_{inj}})$ is a cofibrantly generated hereditary abelian model structure on the abelian category $\mathsf{Z}^0(\mathbf{A})$.*

Proof. This is a common generalization of [5, Proposition 1.3.6(2)] (which is the CDG-module case) and [20, Theorem 4.2] or [55, Theorem 9.6] (which is the case of complexes in abelian categories).

The pair of classes (W, R) is a complete cotorsion pair in $\mathsf{Z}^0(\mathbf{A})$ by Theorem 7.11. According to Lemma 7.3, it follows that $\mathsf{R} \cap \mathsf{W} = \mathsf{Z}^0(\mathbf{A})_{\mathsf{inj}}$. It follows from Lemma 7.9(a) that the class of Becker-coacyclic complexes W is thick in $\mathsf{Z}^0(\mathbf{A})$ (see also Lemma 5.10(b) or 7.3). By Lemma 5.9(b), the triple $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ is an injective abelian model structure on the category $\mathsf{Z}^0(\mathbf{A})$.

Finally, it was shown in the proof of Theorem 7.11 that the cotorsion pair (W, R) in $\mathsf{Z}^0(\mathbf{A})$ is generated by a set of objects. The cotorsion pair $(\mathsf{Z}^0(\mathbf{A}), \mathsf{Z}^0(\mathbf{A})_{\mathsf{inj}})$ is generated by any set of objects $\mathsf{S}_0 \subset \mathsf{Z}^0(\mathbf{A})$ such that $\mathsf{Z}^0(\mathbf{A}) = \mathsf{Fil}(\mathsf{S}_0)$, as in [5, Example 1.2.6(1)] or [55, Lemma 8.2] applied to the category $\mathsf{Z}^0(\mathbf{A})$. By Lemma 5.7, this means that our abelian model structure is cofibrantly generated. □

The abelian model structure $(\mathsf{L}, \mathsf{W}, \mathsf{R})$ defined in Theorem 7.13 is called the *coderived model structure* on the category $\mathsf{Z}^0(\mathbf{A})$.

Lemma 7.14. *For any Grothendieck abelian DG-category $\mathbf{A}$, the class $\mathcal{W}$ of all weak equivalences in the coderived model structure on $\mathsf{Z}^0(\mathbf{A})$ coincides with the class of all closed morphisms of degree 0 in $\mathbf{A}$ with the cones belonging to $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$.*

Proof. Similar to Lemma 6.18. □

The next corollary presumes existence of infinite coproducts in Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$. Here we point out that, since the thick subcategory of Becker-coacyclic complexes $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \subset \mathsf{H}^0(\mathbf{A})$ is closed under infinite coproducts by Lemma 7.9(b), the coproducts in Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ are induced by those in the homotopy category $\mathsf{H}^0(\mathbf{A})$. In other words, the Verdier quotient functor $\mathsf{H}^0(\mathbf{A}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ preserves coproducts [42, Lemma 3.2.10].

Corollary 7.15. *For any Grothendieck abelian DG-category $\mathbf{A}$, Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is a well-generated triangulated category.*

Proof. This is a generalization of the well-known result for the categories of complexes in Grothendieck abelian categories (which can be found in [43, Theorem 3.13], [35, Theorem 5.12], [20, Theorem 4.2], or [55, Corollary 9.8]).

The proof is similar to that of Corollary 6.19. One observes that Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ can be equivalently defined as the homotopy category $\mathsf{Z}^0(\mathbf{A})[\mathcal{W}^{-1}]$ of the coderived model structure on $\mathsf{Z}^0(\mathbf{A})$, and uses Theorem 7.13 together with Lemma 5.11 and Proposition 5.12. □

7.5. **Directed colimits of Becker-coacyclic objects.** Varions descriptions of the class of all Becker-coacyclic objects play a key role in the rest of this paper. One such description was already obtained in Corollary 7.12. The aim of this section is to provide another one. Under more restrictive assumptions, yet another description of the class $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ will be given in Section 8.4.

Proposition 7.16. *Let A be an abelian category with coproducts and exact functors of directed colimit, and let $\mathsf{C} \subset \mathsf{A}$ be a class of objects. Then the following conditions are equivalent:*

1. *the class C is closed under the cokernels of monomorphisms and transfinitely iterated extensions (in the sense of the directed colimit) in A;*
2. *the class C is closed under the cokernels of monomorphisms, extensions, and directed colimits of smooth well-ordered chains of monomorphisms in A;*
3. *the class C is closed under the cokernels of monomorphisms, extensions, and directed colimits in A.*

Proof. (3) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (1) It suffices to observe that a transfinitely iterated extension in a category with exact functors of directed colimit is by the definition built from extensions and directed colimits of smooth chains of monomorphisms.

(1) $\Longrightarrow$ (2) Let $(C_i \to C_j)_{i<j\le\alpha}$ be a smooth well-ordered chain of objects in A, indexed by a limit ordinal α, such that $C_i \in \mathsf{C}$ for all $i < \alpha$ and the morphism $C_i \longrightarrow C_{i+1}$ is a monomorphism in A for all $i < \alpha$. Without loss of generality we can assume that $C_0 = 0$. By assumption, the cokernel S_i of the morphism $C_i \longrightarrow C_{i+1}$ belongs to C for all $i < \alpha$. The object $C = C_\alpha$ is a transfinitely iterated extension of the objects S_i, $i < \alpha$; hence it also belongs to C, as desired.

(2) $\Longrightarrow$ (3) It is well-known that a class of objects in a cocomplete category is closed under directed colimits if and only if it is closed under directed colimits of smooth well-ordered chains [1, Lemma 1.6, Corollary 1.7, and Remark 1.7].

Let $(f_{ji}\colon D_i \to D_j)_{i<j\le\alpha}$ be a smooth well-ordered chain of objects in A, indexed by a limit ordinal α, such that $D_i \in \mathsf{C}$ for all $i < \alpha$. Under the assumptions of (2), we have to show that $D_\alpha \in \mathsf{C}$. Following [21, Lemma 2.1] or [4, Proposition 4.1], we consider for every ordinal $\beta \le \alpha$ the short exact sequence

$$0 \longrightarrow K_\beta \longrightarrow \coprod\nolimits_{i<\beta} D_i \longrightarrow \varinjlim\nolimits_{i<\beta} D_i \longrightarrow 0 \tag{14}$$

of canonical presentation of the directed colimit $\varinjlim_{i<\beta} D_i$. In particular, for any limit ordinal $\beta \le \alpha$, the sequence (14) takes the form

$$0 \longrightarrow K_\beta \longrightarrow \coprod\nolimits_{i<\beta} D_i \longrightarrow D_\beta \longrightarrow 0, \tag{15}$$

since $D_\beta = \varinjlim_{i<\beta} D_i$ by assumption.

For any successor ordinal $\beta = \gamma + 1 < \alpha$, we have $\varinjlim_{i<\beta} D_i = \varinjlim_{i\le\gamma} D_i = D_\gamma$. In this case, the sequence (14) is split and takes the form

$$0 \longrightarrow \coprod\nolimits_{i<\gamma} D_i \longrightarrow \coprod\nolimits_{i\le\gamma} D_i \longrightarrow D_\gamma \longrightarrow 0, \tag{16}$$

Here the components of the map $\coprod_{i<\beta} D_i \longrightarrow D_\gamma$ are precisely the morphisms $f_{\gamma i}\colon D_i \longrightarrow D_\gamma$, $i < \gamma$, and $\mathrm{id}_{D_\gamma}\colon D_\gamma \longrightarrow D_\gamma$. In this case, we have a natural isomorphism $K_\beta \simeq \coprod_{i<\gamma} D_i$.

The short exact sequences (14) form a smooth chain (in the category) of short exact sequences in A. Proceeding by the transfinite induction, we can now prove that $K_\beta \in \mathsf{C}$ for all ordinals $\beta \le \alpha$. Indeed, the assertion holds for successor ordinals β, since the class C is closed under coproducts (which can be easily expressed in terms of extensions and directed colimits of smooth well-ordered chains of monomorphisms). For a limit ordinal $\beta \le \alpha$, the object K_β is the directed colimit of the smooth well-ordered chain of monomorphisms $(K_i \to K_j)_{i<j<\beta}$.

We have shown that $K_\alpha \in \mathsf{C}$. Finally, consider the short exact sequence (15) for $\beta = \alpha$,

$$0 \longrightarrow K_\alpha \longrightarrow \coprod\nolimits_{i<\alpha} D_i \longrightarrow D_\alpha \longrightarrow 0, \tag{17}$$

and observe that $\coprod_{i<\alpha} D_i \in \mathsf{C}$ and $K_\alpha \in \mathsf{C}$. Since the class C is closed under the cokernels of monomorphisms, we can conclude that $D_\alpha \in \mathsf{C}$. $\square$

Corollary 7.17. *Let $\mathbf{A}$ be a Grothendieck abelian DG-category. Then the class of all Becker-coacyclic objects in $\mathsf{Z}^0(\mathbf{A})$ is closed under directed colimits. Moreover, the class $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is precisely the closure of the class of all contractible objects under extensions and directed colimits. In fact, if $\mathsf{S}_0 \subset \mathsf{Z}^0(\mathbf{A}^\natural)$ is a set of objects such that $\mathsf{Z}^0(\mathbf{A}^\natural) = \mathsf{Fil}(\mathsf{S}_0)$ and $\mathsf{S} = \{\Psi^+(S) \mid S \in \mathsf{S}_0\} \subset \mathsf{Z}^0(\mathbf{A})$, then $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is the closure of S under extensions and directed colimits in $\mathsf{Z}^0(\mathbf{A})$.*

Proof. The class of all Becker-coacyclic objects $\mathsf{C} = \mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ in the Grothendieck abelian category $\mathsf{A} = \mathsf{Z}^0(\mathbf{A})$ is closed under the cokernels of monomorphisms by Lemma 7.9(a), and under transfinitely iterated extensions by the first assertion of the same lemma. So the class C satisfies condition (1) of Proposition 7.16. Therefore, the equivalent condition (3) of the same proposition is also satisfied, and we can conclude that the class $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under directed colimits in $\mathsf{Z}^0(\mathbf{A})$. Furthermore, all contractible objects obviously belong to $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$; so the closure of the class of contractible objects under extensions and directed colimits is contained in $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$.

Conversely, by Corollary 7.12, all the Becker-coacyclic objects in $\mathbf{A}$ can be obtained as direct summands of transfinitely iterated extensions of objects from S in the abelian category $\mathsf{Z}^0(\mathbf{A})$. A transfinitely iterated extension is built from extensions and directed colimits; and a direct summand of an object can be expressed as a countable directed colimit of copies of that object. $\square$

7.6. **Generation theorem.** Let T be a triangulated category. A class of objects $\mathsf{S} \subset \mathsf{T}$ is said to *weakly generate* T if any object $X \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{T}}(S, X[n]) = 0$ for all $S \in \mathsf{S}$ and $n \in \mathbb{Z}$ vanishes in T, that is $X = 0$.

Let T be a triangulated category with (infinite) coproducts. Then a class of objects $\mathsf{S} \subset \mathsf{T}$ is said to *generate* T (as a triangulated category with coproducts) if the minimal full triangulated subcategory of T containing S and closed under coproducts coincides with T.

Notice that, for any object $X \in \mathsf{T}$, the full subcategory of all objects $Y \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{T}}(Y, X[n]) = 0$ for all $n \in \mathbb{Z}$ is a full triangulated subcategory closed under coproducts. Therefore, the terminology above is consistent: if S generates T, then S also weakly generates T.

The converse implication holds for well-generated triangulated categories T and *sets* of objects $\mathsf{S} \subset \mathsf{T}$. If T is a well-generated triangulated category and S is a weakly generating set of objects in T, then S is also a generating set of objects for T [34, Theorems 7.2.1 and 5.1.1].

Theorem 7.18. *Let $\mathbf{A}$ be a Grothendieck abelian DG-category and $\mathsf{S} \subset \mathsf{Z}^0(\mathbf{A})$ be a set of objects such that the whole abelian category $\mathsf{Z}^0(\mathbf{A})$ is the closure of S under shifts, kernels of epimorphisms, cokernels of monomorphisms, extensions, and directed colimits. Then the set of objects S generates the triangulated category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$.*

Proof. Corollary 7.15 tells that Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is well-generated. According to the discussion above, it suffices to show that the set S weakly generates the triangulated category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$.

According to Corollary 7.10, the coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is equivalent to the homotopy category of graded-injective objects $\mathsf{H}^0(\mathbf{A_{inj}})$. It is clear from the definition of the class of Becker-coacyclic objects $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ that the natural map of abelian groups $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{A})}(Y,J) \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{bco}}(\mathbf{A})}(Y,J)$ is an isomorphism for all $Y \in \mathbf{A}$ and $J \in \mathbf{A_{inj}}$. Let $X \in \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ be an object such that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{bco}}(\mathbf{A})}(S, X[n]) = 0$ for all $S \in \mathsf{S}$ and $n \in \mathbb{Z}$. Choose a graded-injective object $J \in \mathbf{A_{inj}}$ isomorphic to X in $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$. Then we have $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{A})}(S, J[n]) = 0$ for all $S \in \mathsf{S}$ and $n \in \mathbb{Z}$.

Denote by $\mathsf{Y} \subset \mathsf{Z}^0(\mathbf{A})$ the class of all objects Y such that $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{A})}(Y, J[n]) = 0$ for all $n \in \mathbb{Z}$. Clearly, the class of objects Y is closed under shifts and cones. Furthermore, by the definition, all Becker-coacyclic objects in $\mathbf{A}$ belong to Y, that is $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \subset \mathsf{Y}$. In particular, all the totalizations of short exact sequences in $\mathsf{Z}^0(\mathbf{A})$ belong to Y by Lemma 7.9(a). It follows that the class Y is closed under the kernels of epimorphisms, cokernels of monomorphisms, and extensions in $\mathsf{Z}^0(\mathbf{A})$.

Finally, by Lemma 6.1, the class Y can be described as $\mathsf{Y} = {}^{\perp_1}\{J[n] \mid n \in \mathbb{Z}\} \subset \mathsf{Z}^0(\mathbf{A})$. By Lemma 4.3, the class Y is closed under transfinitely iterated extensions in $\mathsf{Z}^0(\mathbf{A})$. Now Proposition 7.16 (1) $\Rightarrow$ (3) tells that the class Y is closed under directed colimits in $\mathsf{Z}^0(\mathsf{Y})$. Since $\mathsf{S} \subset \mathsf{Y}$ by assumption, we can conclude that the class Y coincides with the whole abelian category $\mathsf{Z}^0(\mathbf{A})$. Thus $J \in \mathsf{Y}$, and it follows that the object J is contractible. Hence $X = 0$ in $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$. ☐

Let us formulate explicitly the natural particular case of Theorem 7.18 for locally finitely presentable abelian DG-categories (as defined in Section 6.2).

Corollary 7.19. *Let $\mathbf{A}$ be a locally finitely presentable abelian DG-category and S be the set of all (representatives of isomorphism classes of) finitely presentable objects in the locally finitely presentable abelian category $\mathsf{Z}^0(\mathbf{A})$. Then the set of objects S generates Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$.*

First proof. We recall that any locally finitely presentable abelian DG-category is Grothendieck (see Section 7.2). All the objects of the locally finitely presentable category $\mathsf{Z}^0(\mathbf{A})$ can be obtained as directed colimits of finitely presentable objects, so Theorem 7.18 is applicable. ☐

Another proof of Corollary 7.19, working in the special case of a locally coherent abelian DG-category $\mathbf{A}$, will be given below in Section 8.6.

8. Locally Coherent DG-Categories

The aim of this section is to prove that Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is compactly generated, and describe (up to direct summands) its full subcategory of compact objects, for a locally coherent abelian DG-category $\mathbf{A}$. In particular, this result applies to the coderived category $\mathsf{D}^{\mathsf{bco}}(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$ of CDG-modules over a CDG-ring (R^*, d, h) whose underlying graded ring R^* is graded left coherent. The main results are Theorem 8.19 and Corollary 8.20.

8.1. **Locally finitely presentable abelian categories.** The definitions of a finitely presentable object and a locally finitely presentable category were given in Section 4.3 (based on the book [1], which we use as the standard reference). Other important references include the papers [37, 10, 31].

Notice that the categories called "locally finitely presented" in [10, 31] are called "finitely accessible" in the terminology of [1]. These categories need not have finite colimits or finite limits, but only directed colimits. So the general setting in [10, 31] is more general than ours in this section.

Before stating the following lemma, we recall that an object $E \in \mathsf{A}$ is said to be *finitely generated* if the functor $\mathrm{Hom}_{\mathsf{A}}(E, -)\colon \mathsf{A} \longrightarrow \mathsf{Sets}$ preserves the colimits of directed diagrams of monomorphisms [1, Section 1.E]. An object $E \in \mathsf{A}$ is finitely generated if and only if it cannot be represented as the union of an infinite directed set of its proper subobjects [61, Proposition V.3.2].

Lemma 8.1. *In any locally finitely presentable abelian category, the class of all finitely presentable objects is closed under extensions.*

Proof. Let A be a locally finitely presentable (abelian) category. The following properties are straightforward to prove:

- the class of all finitely generated objects is closed under quotients and extensions in A [61, Lemma V.3.1];
- an object in A is finitely generated if and only if it is a quotient of a finitely presentable object;
- a finitely generated object $F \in \mathsf{A}$ is finitely presentable if and only if, for any epimorphism $f\colon E \longrightarrow F$ onto F from a finitely generated object $E \in \mathsf{A}$, the kernel of f is finitely generated [61, Proposition V.3.4].

Suppose now that $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is a short exact sequence in A with X and Z finitely presentable and that $f\colon E \longrightarrow Y$ is an epimorphism from a finitely generated object E. Then we can form a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, two heads, "g"'] & E \arrow[r] \arrow[d, two heads, "f"'] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

where D is the pullback of $X \longrightarrow Y \longleftarrow E$. Since D is the kernel of $E \longrightarrow Z$ and Z is finitely presentable, it follows that D is finitely generated. Consequently, the kernel of g, which coincides with the kernel of f, is finitely generated since X is finitely presentable. Since this holds for any f as above, it follows that Y is finitely presentable. $\square$

Given a category A with directed colimits and a class of objects $\mathsf{C} \subset \mathsf{A}$, we denote by $\varinjlim \mathsf{C} \subset \mathsf{A}$ the class of all colimits of directed diagrams of objects from C in A. For a locally finitely presentable category A, we denote by $\mathsf{A}_{\mathsf{fp}} \subset \mathsf{A}$ the full subcategory of all finitely presentable objects.

Proposition 8.2. *Let A be a locally finitely presentable additive category and $\mathsf{C} \subset \mathsf{A}_{\mathsf{fp}}$ be a full subcategory closed under finite direct sums. Then the class of objects $\varinjlim \mathsf{C} \subset \mathsf{A}$ is closed under coproducts and directed colimits in A. An object $B \in \mathsf{A}$ belongs to $\varinjlim \mathsf{C}$ if and if, for any object $A \in \mathsf{A}_{\mathsf{fp}}$, any morphism $A \longrightarrow B$ in A factorizes through an object from C.*

Proof. This is [37, Proposition 2.1], [10, Section 4.1], or [31, Proposition 5.11]. □

8.2. **Locally coherent abelian categories.** Let A be a locally finitely presentable abelian category. A finitely presentable object $F \in \mathsf{A}$ is said to be *coherent* if any finitely generated subobject of F (as defined in Section 8.1) is finitely presentable. The category A is said to be *locally coherent* [58, Section 2] if it has a generating set consisting of coherent objects. Equivalently, A is locally coherent if and only if the kernel of any (epi)morphism of finitely presentable objects in A is finitely presentable.

In a locally coherent abelian category A, an object is coherent if and only if it is finitely presentable, and the full subcategory of all coherent objects A_{fp} is closed under kernels, cokernels, and extensions in A. So, A_{fp} is an abelian category; in the terminology of Section 3.3, A_{fp} is an exactly embedded full abelian subcategory of A. We refer to [52, Section 9.5] for a further discussion.

In the following lemma, we denote by Ab the category of abelian groups.

Lemma 8.3. *Let A be a locally coherent abelian category and $E \in \mathsf{A}_{\mathsf{fp}}$ be a finitely presentable object. Then, for every $n \geq 0$, the functor $\operatorname{Ext}^n_{\mathsf{A}}(E, -)\colon \mathsf{A} \longrightarrow \mathsf{Ab}$ preserves directed colimits.*

Proof. In any abelian category A, the functor $\operatorname{Ext}^n_{\mathsf{A}}(X, Y)$ can be computed as the filtered colimit of the cohomology groups $H^n \operatorname{Hom}_{\mathsf{A}}(R_\bullet, Y)$, taken over the (large) filtered category of all exact complexes $\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow X \longrightarrow 0$ in A. Here the morphisms in the category of such arbitrary resolutions $R_\bullet \longrightarrow X$ are the usual closed morphisms of complexes acting by the identity morphisms on the object X and viewed up to the cochain homotopy.

In the situation at hand with A locally coherent, the full subcategory $\mathsf{A}_{\mathsf{fp}} \subset \mathsf{A}$ has the property of [29, Section 12], or in other words, it is "self-resolving" in the sense of [52, Section 7.1]. This means that for any epimorphism $A \longrightarrow F$ with $A \in \mathsf{A}$ and $F \in \mathsf{A}_{\mathsf{fp}}$ there exists an epimorphism $G \longrightarrow F$ with $G \in \mathsf{A}_{\mathsf{fp}}$ and a morphism $G \longrightarrow A$ in A such that the triangle diagram $G \longrightarrow A \longrightarrow F$ is commutative in A. Furthermore, the kernel of the morphism $G \longrightarrow F$ also belongs to A_{fp}.

Consequently, the full subcategory of resolutions $\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$ with $F_i \in \mathsf{A}_{\mathsf{fp}}$ is cofinal in the category of all resolutions $\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow E \longrightarrow 0$ with $R_i \in \mathsf{A}$ (when $E \in \mathsf{A}_{\mathsf{fp}}$). So one can compute the group $\operatorname{Ext}^n_{\mathsf{A}}(E, Y)$ as the filtered colimit of $H^n \operatorname{Hom}_{\mathsf{A}}(F_\bullet, Y)$ taken over all the resolutions $F_\bullet \longrightarrow E$ with $F_i \in \mathsf{A}_{\mathsf{fp}}$. Now the functor $\operatorname{Hom}_{\mathsf{A}}(F, -)\colon \mathsf{A} \longrightarrow \mathsf{Ab}$, by the definition, preserves directed colimits when $F \in \mathsf{A}_{\mathsf{fp}}$. The functors of cohomology of a complex of abelian groups also preserve directed colimits, and the filtered colimit over the category of all resolutions $F_\bullet$ commutes with directed colimits. $\square$

A version of the following proposition for module categories can be found in [2, Theorem 2.3].

Proposition 8.4. *Let* A *be a locally coherent abelian category and* $\mathsf{C} \subset \mathsf{A}_{\mathsf{fp}}$ *be a full subcategory closed under extensions. Then the full subcategory* $\varinjlim \mathsf{C} \subset \mathsf{A}$ *is also closed under extensions in* A.

Proof. Given an abelian category A and two classes of objects X, $\mathsf{Y} \subset \mathsf{A}$, let us denote by $\mathsf{X}*\mathsf{Y}$ the class of of all objects $Z \in \mathsf{A}$ for which there exists a short exact sequence $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ in A. In the situation at hand, the proposition claims that $\varinjlim \mathsf{C} * \varinjlim \mathsf{C} \subset \varinjlim \mathsf{C}$. We will prove three inclusions

$$\varinjlim \mathsf{C} * \varinjlim \mathsf{C} \subset \varinjlim(\varinjlim \mathsf{C} * \mathsf{C}) \subset \varinjlim \varinjlim(\mathsf{C} * \mathsf{C}) \subset \varinjlim(\mathsf{C} * \mathsf{C})$$

for any class of objects $\mathsf{C} \subset \mathsf{A}_{\mathsf{fp}}$.

Firstly, given a cocomplete abelian category A with exact directed colimits and two classes of objects D, $\mathsf{E} \subset \mathsf{A}$, we claim that $\mathsf{D} * \varinjlim \mathsf{E} \subset \varinjlim(\mathsf{D} * \mathsf{E})$. Let $(E_i)_{i\in I}$ be a directed system in A, indexed by a directed poset I, with $E_i \in \mathsf{E}$, and let

$$0 \longrightarrow D \longrightarrow G \longrightarrow \varinjlim\nolimits_{i\in I} E_i \longrightarrow 0 \tag{18}$$

be a short exact sequence in A with $D \in \mathsf{D}$. Taking the pullback of (18) with respect to the natural morphism $E_j \longrightarrow \varinjlim_{i\in I} E_i$, for every $j \in I$, we obtain a short exact sequence

$$0 \longrightarrow D \longrightarrow G_j \longrightarrow E_j \longrightarrow 0. \tag{19}$$

As $j \in I$ varies, the short exact sequences (19) form a directed system whose directed colimit is the original short exact sequence (18). Thus $G_j \in \mathsf{D} * \mathsf{E}$ and $G = \varinjlim_{j\in I} G_j$.

Secondly, for a locally coherent abelian category A and two classes of objects $\mathsf{C} \subset \mathsf{A}$ and $\mathsf{D} \subset \mathsf{A}_{\mathsf{fp}}$, we assert that $(\varinjlim \mathsf{C}) * \mathsf{D} \subset \varinjlim(\mathsf{C} * \mathsf{D})$. Let $(C_i)_{i\in I}$ be a directed system in A, indexed by a directed poset I, with $C_i \in \mathsf{C}$, and let

$$0 \longrightarrow \varinjlim\nolimits_{i\in I} C_i \longrightarrow H \longrightarrow D \longrightarrow 0 \tag{20}$$

be a short exact sequence in A with $D \in \mathsf{D}$. By Lemma 8.3 (for $n = 1$), the related class in $\operatorname{Ext}^1_{\mathsf{A}}(D, \varinjlim_{i\in I} C_i)$ comes from an element of $\operatorname{Ext}^1_{\mathsf{A}}(D, C_k)$ for some index $k \in I$. Consider the related short exact sequence

$$0 \longrightarrow C_k \longrightarrow H_k \longrightarrow D \longrightarrow 0 \tag{21}$$

For every $j \in I$, $j \geq k$, take the pushout of the short exact sequence (21) with respect to the morphism $C_k \longrightarrow C_j$:

$$0 \longrightarrow C_j \longrightarrow H_j \longrightarrow D \longrightarrow 0. \tag{22}$$

As the index $j \in I$, $j \geq k$ varies, the short exact sequences (22) form a directed system whose directed colimit is the original short exact sequence (20). Therefore, $H_j \in \mathsf{C} * \mathsf{D}$ and $H = \varinjlim^{j\geq k}_{j\in I} H_j$.

Finally, for any class of objects $\mathsf{C} \subset \mathsf{A}_{\mathsf{fp}}$ we have $\mathsf{C} * \mathsf{C} \subset \mathsf{A}_{\mathsf{fp}}$ by Lemma 8.1 and $\varinjlim \varinjlim \mathsf{C} \subset \varinjlim \mathsf{C}$ by Proposition 8.2. $\square$

8.3. **Locally coherent abelian DG-categories.** In this section, which is largely an extraction from [52, Section 9.5], we prove the (locally) coherent versions of the results of Sections 6.2 and 7.2.

Lemma 8.5. *Let $\mathbf{A}$ be a locally finitely presentable abelian DG-category. Then the additive functor $\Xi_{\mathbf{A}}\colon \mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{Z}^0(\mathbf{A})$ preserves coherence of objects.*

Proof. Using the properties listed in the proof of Lemma 8.1, one can easily show that the class of all coherent objects in a locally finitely presentable abelian category is closed under extensions. Hence the assertion of the lemma follows from Corollary 3.2(a). $\square$

Lemma 8.6. *Let $\mathbf{A}$ be a locally finitely presentable abelian DG-category. Then the additive functors $\Phi_{\mathbf{A}}\colon \mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{Z}^0(\mathbf{A}^\natural)$ and $\Psi^+_{\mathbf{A}}\colon \mathsf{Z}^0(\mathbf{A}^\natural) \longrightarrow \mathsf{Z}^0(\mathbf{A})$ preserve and reflect coherence of objects.*

Proof. This is our version of [52, Lemma 9.13]. Let us show that the functor $\Phi_{\mathbf{A}}$ reflects coherence. Let $E \in \mathbf{A}$ be an object for which the object $\Phi(E)$ is coherent in $\mathsf{Z}^0(\mathbf{A}^\natural)$. By Lemma 6.7, the object E is finitely presentable in $\mathsf{Z}^0(\mathbf{A})$.

Let $F \subset E$ be a finitely generated subobject. It is easily provable similarly to Lemma 6.7 (cf. [52, Lemma 9.6]) that the functor Φ preserves finite generatedness of objects; so the object $\Phi(F)$ is finitely generated. Therefore, $\Phi(F)$ is a finitely generated subobject in $\Phi(E)$; by assumption, it follows that the object $\Phi(F)$ is finitely presentable in $\mathsf{Z}^0(\mathbf{A}^\natural)$. It remains to invoke Lemma 6.7 again in order to conclude that the object $F \in \mathsf{Z}^0(\mathbf{A})$ is finitely presentable.

Similarly one shows that the functor $\Psi^+_{\mathbf{A}}$ reflects coherence. Since the compositions $\Xi_{\mathbf{A}} \simeq \Psi^+_{\mathbf{A}} \circ \Phi_{\mathbf{A}}$ and $\Xi_{\mathbf{A}^\natural} \simeq \Phi_{\mathbf{A}} \circ \Psi^-_{\mathbf{A}}$ (see the last paragraph of Section 2.2) preserve coherence by Lemma 8.5, and the functors $\Psi^+_{\mathbf{A}}$ and $\Phi_{\mathbf{A}}$ reflect coherence, it follows that the functors $\Phi_{\mathbf{A}}$ and $\Psi^-_{\mathbf{A}}$ preserve coherence. $\square$

Proposition 8.7. *Let $\mathbf{A}$ be a locally finitely presentable abelian DG-category. Then the abelian category $\mathsf{Z}^0(\mathbf{A})$ is locally coherent if and only if the abelian category $\mathsf{Z}^0(\mathbf{A}^\natural)$ is locally coherent.*

Proof. This is [52, Proposition 9.14]. The assertion follows from Lemmas 6.8 and 8.6 above. $\square$

We will say that an abelian DG-category $\mathbf{A}$ is *locally coherent* if both the abelian categories $\mathsf{Z}^0(\mathbf{A})$ and $\mathsf{Z}^0(\mathbf{A}^\natural)$ are locally coherent. In other words, this means that $\mathbf{A}$ satisfies the assumptions and the equivalent conditions of Proposition 8.7.

Given a locally finitely presentable abelian DG-category $\mathbf{A}$, we denote by $\mathbf{A}_{\mathbf{fp}} \subset \mathbf{A}$ the full DG-subcategory whose objects are all the objects of $\mathbf{A}$ that are finitely presentable as objects of the category $\mathsf{Z}^0(\mathbf{A})$. So, by the definition, we have $\mathsf{Z}^0(\mathbf{A}_{\mathbf{fp}}) = \mathsf{Z}^0(\mathbf{A})_{\mathsf{fp}}$. The full DG-subcategory $\mathbf{A}_{\mathbf{fp}} \subset \mathbf{A}$ is obviously closed under shifts and finite direct sums; in view of Lemmas 3.1 and 8.1, it is also closed under cones. Lemma 6.7

implies that the full subcategory $\mathsf{Z}^0((\mathbf{A_{fp}})^\natural) \subset \mathsf{Z}^0(\mathbf{A}^\natural)$ consists of all the finitely presentable objects in $\mathsf{Z}^0(\mathbf{A}^\natural)$, that is $\mathsf{Z}^0((\mathbf{A_{fp}})^\natural) = \mathsf{Z}^0(\mathbf{A}^\natural)_{\mathsf{fp}}$.

When $\mathbf{A}$ is a locally coherent abelian DG-category, the full DG-subcategory $\mathbf{A_{fp}} \subset \mathbf{A}$ is an exactly embedded full abelian DG-subcategory in $\mathbf{A}$ in the sense of Section 3.3. In particular, $\mathbf{A_{fp}}$ is an abelian DG-category.

Examples 8.8. (1) Let A be a locally finitely presentable abelian category. Consider the abelian DG-category $\mathbf{C}(\mathsf{A})$ of complexes in A, as per Example 3.13. According to Sections 1.8 and 2.5, we have $\mathsf{Z}^0(\mathbf{C}(\mathsf{A})^\natural) \simeq \mathsf{G}(\mathsf{A}) = \mathsf{A}^{\mathbb{Z}}$. By Example 6.10(1), $\mathbf{C}(\mathsf{A})$ is a locally finitely presentable abelian DG-category.

One can easily see that a graded object in A is finitely presentable if and only if it is *bounded* with all but a finite number of grading components vanishing *and* all the grading components finitely presentable. By Lemma 6.7, the finitely presentable objects of the abelian category $\mathsf{C}(\mathsf{A}) = \mathsf{Z}^0(\mathbf{C}(\mathsf{A}))$ of complexes in A are precisely all the bounded complexes of finitely presentable objects in A.

(2) Now if A is a locally coherent abelian category, then so is $\mathsf{G}(\mathsf{A}) = \mathsf{A}^{\mathbb{Z}}$. By Proposition 8.7, it follows that $\mathbf{C}(\mathsf{A})$ is a locally coherent abelian DG-category.

A graded ring R^* is said to be *graded left coherent* if all finitely generated homogeneous left ideals in R^* are finitely presentable (as graded left R^*-modules). In other words, a graded ring R^* is graded left coherent if and only if the abelian category of graded left R^*-modules $R^*\text{–}\mathsf{Mod}$ is locally coherent.

Corollary 8.9. *Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a CDG-ring and $\boldsymbol{M}^\bullet = (M^*, d_M)$ be a left CDG-module over $\boldsymbol{R}^\bullet$. Then*

(a) *M^* is finitely presentable as a graded left $R^*[\delta]$-module (i. e., as an object of the category $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) \simeq R^*[\delta]\text{–}\mathsf{Mod}$) if and only if it is finitely presentable as a graded left R^*-module;*

(b) *the graded ring $R^*[\delta]$ is graded left coherent if and only if the graded ring R^* is graded left coherent, and if and only if the locally finitely presentable abelian DG-category $\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ is locally coherent.*

Proof. The equivalences of abelian categories $\mathsf{Z}^0(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}) \simeq R^*[\delta]\text{–}\mathsf{Mod}$ and $\mathsf{Z}^0((\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})^\natural) \simeq R^*\text{–}\mathsf{Mod}$ were discussed in Section 2.6. In view of these category equivalences, part (a) is a particular case of Lemma 6.7, and part (b) is a particular case of Proposition 8.7. We recall that the DG-category $\boldsymbol{R}^\bullet\text{–}\mathbf{Mod}$ is always locally finitely presentable by Example 6.11. (See also [52, Example 9.15].) $\square$

Example 8.10. (1) Let A be a locally finitely presentable abelian category, $\Delta\colon \mathsf{A} \longrightarrow \mathsf{A}$ be an autoequivalence, and $v\colon \mathrm{Id}_{\mathsf{A}} \longrightarrow \Delta$ be a potential. Consider the abelian DG-category $\mathbf{F}(\mathsf{A}, \Delta, v)$ of factorizations of v in A, as per Example 3.15. According to Sections 1.10 and 2.7, we have $\mathsf{Z}^0(\mathbf{F}(\mathsf{A}, \Delta, v)^\natural) \simeq \mathsf{P}(\mathsf{A}, \Delta) \simeq \mathsf{A} \times \mathsf{A}$. By Example 6.12(1), $\mathbf{F}(\mathsf{A}, \Delta, v)$ is a locally finitely presentable abelian DG-category.

Obviously, a 2-Δ-periodic object in A is finitely presentable if and only if all its grading components are finitely presentable in A (there are essentially only two such grading componets). By Lemma 6.7, the finitely presentable objects of the abelian category of factorizations $\mathsf{F}(\mathsf{A},\Delta,v) = \mathsf{Z}^0(\mathbf{F}(\mathsf{A},\Delta,v))$ are precisely all the factorizations of v with finitely presentable grading components in A.

(2) Now if A is a locally coherent abelian category, then so is $\mathsf{P}(\mathsf{A},\Delta) \simeq \mathsf{A}\times\mathsf{A}$. By Proposition 8.7, it follows that $\mathbf{F}(\mathsf{A},\Delta,v)$ is a locally coherent abelian DG-category.

8.4. **Absolute derived category.** The following definition is taken from [52, Section 5.1], but it goes back to [47, Section 2.1] and [48, Sections 3.3 and 4.2].

Let $\mathbf{E}$ be an abelian DG-category. An object $X \in \mathbf{E}$ is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category $\mathsf{H}^0(\mathbf{E})$ containing the totalizations $\mathrm{Tot}(K \to L \to M)$ of all short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathsf{Z}^0(\mathbf{E})$. The thick subcategory of absolutely acyclic objects is denoted by $\mathsf{H}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}} \subset \mathsf{H}^0(\mathbf{E})$, and its full preimage under the obvious functor $\mathsf{Z}^0(\mathbf{E}) \longrightarrow \mathsf{H}^0(\mathbf{E})$ is denoted by $\mathsf{Z}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}} \subset \mathsf{Z}^0(\mathbf{E})$.

The triangulated Verdier quotient category $\mathsf{D}^{\mathsf{abs}}(\mathbf{E}) = \mathsf{H}^0(\mathbf{E})/\mathsf{H}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}}$ of the homotopy category $\mathsf{H}^0(\mathbf{E})$ by its thick subcategory of absolutely acyclic objects is called the *absolute derived category* of an abelian DG-category $\mathbf{E}$.

Lemma 8.11. *The full subcategory of absolutely acyclic objects $\mathsf{Z}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}}$ is closed under the kernels of epimorphisms, the cokernels of monomorphisms, extensions, and direct summands in the abelian category $\mathsf{Z}^0(\mathbf{E})$ (i. e. it is thick there in the sense of Section 5.5).*

Proof. By construction, the full subcategory of absolutely acyclic objects is closed under shifts, cones, and direct summands in $\mathsf{H}^0(\mathbf{E})$, hence also in $\mathsf{Z}^0(\mathbf{E})$. Since the full subcategory $\mathsf{Z}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}}$ contains the totalizations of short exact sequences, it follows that it is closed under the kernels of epimorphisms, the cokernels of monomorphisms, and extensions in $\mathsf{Z}^0(\mathbf{E})$. ☐

Proposition 8.12. *For any abelian DG-category $\mathbf{E}$, the full subcategory of absolutely acyclic objects $\mathsf{Z}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}}$ is precisely the closure of the class of all contractible objects in $\mathbf{E}$ under extensions and direct summands in the abelian category $\mathsf{Z}^0(\mathbf{E})$.*

Proof. Denote by $\mathsf{C} \subset \mathsf{Z}^0(\mathbf{E})$ the closure of the class of all contractible objects under extensions and direct summands. Since all contractible objects are absolutely acyclic, it follows from Lemma 8.11 that $\mathsf{C} \subset \mathsf{Z}^0(\mathbf{E})^{\mathsf{abs}}_{\mathsf{ac}}$.

To prove the converse inclusion, we have to show that the class C contains the totalizations of short exact sequences in $\mathsf{Z}^0(\mathbf{E})$ and is closed under shifts, cones, homotopy equivalences, and direct summands in the homotopy category $\mathsf{H}^0(\mathbf{E})$. Indeed, the totalizations of short exact sequences can be obtained as extensions of contractible objects, as explained in the proof of Lemma 7.9. The class C is closed under shifts, because the class of all contractible objects is closed under shifts and the shift is an auto-equivalence of the abelian category $\mathsf{Z}^0(\mathbf{E})$. The cone of a closed morphism of degree 0 in $\mathsf{Z}^0(\mathbf{E})$ is a particular case of an extension (see Lemma 3.1).

It remains to explain that the class C is closed under homotopy equivalences and homotopy direct summands, and it suffices to consider the homotopy direct summands. Let $C \in \mathsf{C}$ and $X \in \mathsf{Z}^0(\mathbf{E})$ be two objects and $X \xrightarrow{f} C \xrightarrow{g} X$ be two morphisms in $\mathsf{Z}^0(\mathbf{E})$ such that the composition gf is homotopic to the identity endomorphism of the object X. Then the difference $\mathrm{id}_X - gf\colon X \longrightarrow X$ is homotopic to zero in $\mathbf{E}$. It follows that the morphism $\mathrm{id}_X - gf$ in the category $\mathsf{Z}^0(\mathbf{E})$ factorizes through the natural morphism $X \longrightarrow \mathrm{cone}(\mathrm{id}_X)$. Thus the identity endomorphism of the object X in $\mathsf{Z}^0(\mathbf{E})$ factorizes as $X \longrightarrow C \oplus \mathrm{cone}(\mathrm{id}_X) \longrightarrow X$, so X is a direct summand of the object $C \oplus \mathrm{cone}(\mathrm{id}_X)$ in the abelian category $\mathsf{Z}^0(\mathbf{E})$. Obviously, $C \oplus \mathrm{cone}(\mathrm{id}_X) \in \mathsf{C}$, hence $X \in \mathsf{C}$, and we are done. $\square$

Proposition 8.13. *Let $\mathbf{A}$ be a locally coherent abelian DG-category. Then the class of all Becker-coacyclic objects $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ consists precisely of all the directed colimits, taken in the abelian category $\mathsf{Z}^0(\mathbf{A})$, of (directed diagrams of) objects absolutely acyclic with respect to the abelian DG-category $\mathbf{A}_{\mathbf{fp}}$. In other words, $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} = \varinjlim\bigl(\mathsf{Z}^0(\mathbf{A}_{\mathbf{fp}})^{\mathsf{abs}}_{\mathsf{ac}}\bigr) \subset \mathsf{Z}^0(\mathbf{A})$.*

Proof. Put $\mathsf{C} = \mathsf{Z}^0(\mathbf{A}_{\mathbf{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$ for brevity. According to Proposition 8.12 applied to the abelian DG-category $\mathbf{E} = \mathbf{A}_{\mathbf{fp}}$, the class C is the closure of the class of all contractible objects in $\mathbf{A}_{\mathbf{fp}}$ under extensions and direct summands in the abelian category $\mathsf{Z}^0(\mathbf{A}_{\mathbf{fp}})$.

By Proposition 8.2, the class $\varinjlim \mathsf{C} \subset \mathsf{Z}^0(\mathbf{A})$ is closed under directed colimits. By Proposition 8.4, the class $\varinjlim \mathsf{C}$ is also closed under extensions in $\mathsf{Z}^0(\mathbf{A})$. So $\varinjlim \mathsf{C}$ is the closure of the class of all contractible objects in $\mathbf{A}_{\mathbf{fp}}$ under extensions and directed colimits in $\mathsf{Z}^0(\mathbf{A})$. (It is helpful to keep in mind that direct summands can be obtained as countable directed colimits.)

Furthermore, by Lemma 3.3, the contractible objects of $\mathbf{A}$ are the direct summands of the objects of the form $\Xi_{\mathbf{A}}(A)$ with $A \in \mathbf{A}$. The object A is a directed colimit of finitely presentable objects in $\mathsf{Z}^0(\mathbf{A})$, and the functor Ξ preserves directed colimits. So all the contractible objects of $\mathbf{A}$ are direct summands of some directed colimits of contractible objects of $\mathbf{A}_{\mathbf{fp}}$ in the category $\mathsf{Z}^0(\mathbf{A})$.

Hence the class $\varinjlim \mathsf{C} \subset \mathsf{Z}^0(\mathbf{A})$ contains all the contractible objects of $\mathbf{A}$. Therefore, it can be described as the closure of the class of all contractible objects of $\mathbf{A}$ under extensions and directed colimits in $\mathsf{Z}^0(\mathbf{A})$. Glancing into Corollary 7.17, we conclude that $\varinjlim \mathsf{C} = \mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$, as desired. $\square$

A different proof of Proposition 8.13 will be indicated in Remark 8.17 in the next Section 8.5.

8.5. **Approachability of Becker-coacyclic objects.** The following definition is taken from [52, Section 7.2].

Let T be a triangulated category and S, $\mathsf{Y} \subset \mathsf{T}$ be two full subcategories. An object $X \in \mathsf{T}$ is said to be *approachable from* S *via* Y if every morphism $S \longrightarrow X$ in T with $S \in \mathsf{S}$ factorizes through an object of Y.

Equivalently, an object X is approachable from S via Y if and only if, for every morphism $S \longrightarrow X$ as above there exists an object $S' \in \mathsf{T}$ and a morphism $S' \longrightarrow S$ with a cone belonging to Y such that the composition $S' \longrightarrow S \longrightarrow X$ vanishes in T. When (as it will be the case in our applications) S is a full triangulated subcategory in T and $\mathsf{Y} \subset \mathsf{S}$, the conditions in this criterion imply that $S' \in \mathsf{S}$.

Lemma 8.14. *Let* T *be a triangulated category and* S, $\mathsf{Y} \subset \mathsf{T}$ *be full triangulated subcategories such that* $\mathsf{Y} \subset \mathsf{S}$. *Then the full subcategory of all objects approachable from* S *via* Y *is a thick subcategory in* T, *i. e., it is closed under shifts, cones, and direct summands in* T.

Proof. This is [52, Lemmas 7.3 and 7.5]. $\square$

If $\mathsf{S} = \mathsf{H}^0(\mathbf{A_{fp}})$, approachability is very closely related to the closure under directed colimits as discussed in Section 8.1.

Lemma 8.15. *Let* $\mathbf{A}$ *be a locally coherent abelian DG-category and* $\mathsf{Y} \subset \mathsf{H}^0(\mathbf{A_{fp}})$ *be a full subcategory closed under finite direct sums. Then* $X \in \mathsf{H}^0(\mathbf{A})$ *is approachable from* $\mathsf{H}^0(\mathbf{A_{fp}})$ *via* Y *if and only if* $X \in \varinjlim \widetilde{\mathsf{Y}}$ *in* $\mathsf{Z}^0(\mathbf{A})$, *where* $\widetilde{\mathsf{Y}}$ *stands for the preimage of* Y *under the obvious functor* $\mathsf{Z}^0(\mathbf{A_{fp}}) \longrightarrow \mathsf{H}^0(\mathbf{A_{fp}})$.

Proof. The proof relies on Proposition 8.2. On the one hand, any morphism from a finitely presentable object to an object $X \in \varinjlim \widetilde{\mathsf{Y}}$ factorizes through an object of $\widetilde{\mathsf{Y}}$ in the abelian category $\mathsf{Z}^0(\mathbf{A})$; hence it factorizes through an object of Y in the homotopy category $\mathsf{H}^0(\mathbf{A})$. In particular, any object $X \in \varinjlim \widetilde{\mathsf{Y}}$ is approachable.

On the other hand, suppose that X is approachable from $\mathsf{H}^0(\mathbf{A_{fp}})$ via Y. In view of Proposition 8.2, it suffices to show that any morphism $f\colon E \longrightarrow X$ in the category $\mathsf{Z}^0(\mathbf{A})$ from a finitely presentable object $E \in \mathsf{Z}^0(\mathbf{A_{fp}})$ factorizes through an object in $\widetilde{\mathsf{Y}}$. However, by approachability such a factorization exists in the homotopy category $\mathsf{H}^0(\mathbf{A})$. So there is an object $Y \in \widetilde{\mathsf{Y}}$ and two morphisms $g\colon E \longrightarrow Y$ and $h\colon Y \longrightarrow X$ in $\mathsf{Z}^0(\mathbf{A})$ such that the morphism f is homotopic to hg. Since the morphism $f - hg\colon E \longrightarrow X$ is homotopic to zero in $\mathbf{A}$, it follows that the morphism $f - hg$ in the category $\mathsf{Z}^0(\mathbf{A})$ factorizes through the natural morphism $E \longrightarrow \operatorname{cone}(\mathrm{id}_E)$. Thus the morphism f in the abelian category $\mathsf{Z}^0(\mathbf{A})$ factorizes as $E \longrightarrow Y \oplus \operatorname{cone}(\mathrm{id}_E) \longrightarrow X$, and it remains to point out that the contractible object $\operatorname{cone}(\mathrm{id}_E)$ as well as the direct sum $Y \oplus \operatorname{cone}(\mathrm{id}_E)$ belong to $\widetilde{\mathsf{Y}}$. $\square$

As a consequence, we obtain the following proposition which is the key technical result of Section 8.

Proposition 8.16. *Let* $\mathbf{A}$ *be a locally coherent abelian DG-category. Then the class of Becker-coacyclic objects of* $\mathbf{A}$ *coincides precisely with the class of objects of* $\mathsf{H}^0(\mathbf{A})$ *which are approachable from the class* $\mathsf{H}^0(\mathbf{A_{fp}})$ *of finitely presentable objects of* $\mathbf{A}$ *via the class* $\mathsf{H}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$ *of absolutely acyclic objects with respect to the abelian DG-category of finitely presentable objects in* $\mathbf{A}$.

Proof. The argument is based on Proposition 8.13, whose proof uses the theory of Sections 7.5 and 8.1–8.2. Indeed, by Proposition 8.13, we have $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} = \varinjlim\bigl(\mathsf{Z}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}\bigr) \subset \mathsf{Z}^0(\mathbf{A})$, and the conclusion follows from Lemma 8.15. $\square$

Remark 8.17. One implication of the latter proposition, that each Becker-coacyclic object is approachable from the finitely presentable objects via absolutely acyclic finitely presentable objects, can be proved by other means using the the theory of approachability developed in [52, Sections 7.2–7.4].

Consider the full subcategory X in the homotopy category $\mathsf{H}^0(\mathbf{A})$ formed by all the objects approachable from $\mathsf{H}^0(\mathbf{A_{fp}})$ via $\mathsf{H}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$. By Lemma 8.14, X is a thick subcategory in the triangulated category $\mathsf{H}^0(\mathbf{A})$. Denote by $\widetilde{\mathsf{X}} \subset \mathsf{Z}^0(\mathbf{A})$ the full preimage of X under the obvious functor $\mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{H}^0(\mathbf{A})$. By Lemma 8.15, $\widetilde{\mathsf{X}} = \varinjlim \mathsf{Z}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$ in $\mathsf{Z}^0(\mathbf{A})$, so $\widetilde{\mathsf{X}}$ is closed under directed colimits by Proposition 8.2. In fact, it is easy to prove the last fact directly from the definition of approachability.

There is a rather general technical result, [52, Proposition 7.10(a)], implying that, in the homotopy category $\mathsf{H}^0(\mathbf{A})$, all the totalizations of short exact sequences in $\mathsf{Z}^0(\mathbf{A})$ are approachable from $\mathsf{H}^0(\mathbf{A_{fp}})$ via totalizations of short exact sequences in $\mathsf{Z}^0(\mathbf{A_{fp}})$ (since the full subcategory $\mathsf{Z}^0((\mathbf{A_{fp}})^\natural) = \mathsf{Z}^0(\mathbf{A}^\natural)_{\mathsf{fp}}$ is self-resolving in the abelian category $\mathsf{Z}^0(\mathbf{A}^\natural)$, in the sense of [52, Section 7.1]). In the situation at hand, one could arrive at the same conclusion by showing that all the short exact sequences in $\mathsf{Z}^0(\mathbf{A})$ are directed colimits of short exact sequences in $\mathsf{Z}^0(\mathbf{A_{fp}}) = \mathsf{Z}^0(\mathbf{A})_{\mathsf{fp}}$. Hence all the totalizations of short exact sequences in $\mathsf{Z}^0(\mathbf{A})$ belong to $\widetilde{\mathsf{X}}$.

It follows that the full subcategory $\widetilde{\mathsf{X}}$ is closed under extensions in $\mathsf{Z}^0(\mathbf{A})$ (as well as under kernels of epimorphisms and cokernels of monomorphisms). Since we know that $\widetilde{\mathsf{X}}$ is closed under directed colimits in $\mathsf{Z}^0(\mathbf{A})$, it is also closed under transfinitely iterated extensions (in the sense of the directed colimit). As obviously, all the contractible objects of $\mathbf{A}$ belong to X, and consequently to $\widetilde{\mathsf{X}}$, we have by Corollary 7.12, that $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \subset \widetilde{\mathsf{X}}$; hence $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \subset \mathsf{X}$.

The latter argument for a half of Proposition 8.16 is independent of Proposition 8.13 and, in fact, we can deduce Proposition 8.13 back from it. Indeed, we already know that $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \subset \widetilde{\mathsf{X}} = \varinjlim \mathsf{Z}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$. On the other hand, $\mathsf{Z}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}} \subset \mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ and $\mathsf{Z}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under directed colimits in $\mathsf{Z}^0(\mathbf{A})$ by Corollary 7.17 and Proposition 8.12; hence the other inclusion.

8.6. **Full-and-faithfulness and compactness.** We start with presenting an alternative proof of the theorem about triangulated generation of the coderived category.

Second proof of Corollary 7.19. This argument is applicable in the particular case of a locally coherent DG-category $\mathbf{A}$.

Let $A \in \mathbf{A}$ be an object such that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{bco}}(\mathbf{A})}(E, A) = 0$ for all $E \in \mathbf{A_{fp}}$. By the general property of the construction of the triangulated Verdier quotient category, this means that any morphism $E \longrightarrow A$ in $\mathsf{H}^0(\mathbf{A})$ factorizes as $E \longrightarrow X \longrightarrow A$ for some Becker-coacyclic object $X \in \mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$. According to Proposition 8.16, the morphism $E \longrightarrow X$ in $\mathsf{H}^0(\mathbf{A})$ factorizes as $E \longrightarrow Y \longrightarrow X$ for some object $Y \in \mathsf{H}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$. This, however, means that A itself is approachable from $\mathsf{H}^0(\mathbf{A_{fp}})$ via $\mathsf{H}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$ in $\mathsf{H}^0(\mathbf{A})$. Thus $A \in \mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ by Proposition 8.16, as desired. □

Let T be a triangulated category with infinite coproducts. We recall than an object $S \in \mathsf{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathsf{T}}(S, -) \colon \mathsf{T} \longrightarrow \mathsf{Ab}$ preserves coproducts. It is well-known [41, Theorems 4.1 and 2.1(2)] that a set of compact objects generates

T as a triangulated category with coproducts if and only if it weakly generates T (in the terminology of Section 7.6). A triangulated category T is said to be *compactly generated* if it has a set of generators consisting of compact objects.

The following lemma describes our technique for proving full-and-faithfulness of the triangulated functor $\mathsf{D}^{\mathsf{abs}}(\mathbf{A_{fp}}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ and compactness of objects in its essential image.

Lemma 8.18. *Let T be a triangulated category with infinite coproducts, let $\mathsf{S} \subset \mathsf{T}$ be a full triangulated subcategory, and let $\mathsf{X} \subset \mathsf{T}$ be a strictly full triangulated subcategory closed under coproducts. Let $\mathsf{Y} \subset \mathsf{S} \cap \mathsf{X}$ be a full triangulated subcategory in the intersection. Assume that all the objects of S are compact in T and all the objects of X are approachable from S via Y in T. Then the induced triangulated functor between the Verdier quotient categories*

$$\mathsf{S}/\mathsf{Y} \longrightarrow \mathsf{T}/\mathsf{X}$$

is fully faithful, and the objects in its essential image are compact in T/X.

Proof. This is [52, Lemma 9.20]. □

The following theorem is one of the main results of this paper. Its idea goes back to Krause [33, Proposition 2.3], whose paper covers the case of the (DG-category of) complexes in a locally Noetherian abelian category. For complexes in a locally coherent abelian category, this result was obtained by the second-named author of the present paper in the preprint [62, Corollary 6.13].

Theorem 8.19. *Let $\mathbf{A}$ be a locally coherent abelian DG-category and $\mathbf{A_{fp}} \subset \mathbf{A}$ be its (exactly embedded, full) abelian DG-subcategory of finitely presentable objects. Then the triangulated functor*

$$\mathsf{D}^{\mathsf{abs}}(\mathbf{A_{fp}}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$$

induced by the inclusion of abelian DG-categories $\mathbf{A_{fp}} \rightarrowtail \mathbf{A}$ is fully faithful. The objects in its image are compact in $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$, and form a set of compact generators for Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$.

Proof. It was mentioned already before Corollary 7.15 that infinite coproducts exist in Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ and the Verdier quotient functor $\mathsf{H}^0(\mathbf{A}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ preserves coproducts. Concerning the more elementary question of existence of coproducts in the homotopy category $\mathsf{H}^0(\mathbf{A})$, see Section 1.4.

To prove the theorem, we apply Lemma 8.18 to the triangulated category $\mathsf{T} = \mathsf{H}^0(\mathbf{A})$ with its full triangulated subcategories $\mathsf{X} = \mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}} \supset \mathsf{Y} = \mathsf{H}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}} \subset \mathsf{S} = \mathsf{H}^0(\mathbf{A_{fp}})$. The full subcategory of Becker-coacyclic objects $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ is closed under coproducts in $\mathsf{H}^0(\mathbf{A})$ by Lemma 7.9(b). All objects of $\mathsf{H}^0(\mathbf{A})^{\mathsf{bco}}_{\mathsf{ac}}$ are approachable from $\mathsf{H}^0(\mathbf{A_{fp}})$ via $\mathsf{H}^0(\mathbf{A_{fp}})^{\mathsf{abs}}_{\mathsf{ac}}$ by Proposition 8.16.

In order to finish verifying the assumptions of Lemma 8.18, it remains to mention the essentially trivial observation that all finitely presentable objects are compact in the homotopy category $\mathsf{H}^0(\mathbf{A})$. This can be explained in many ways; e. g., one can observe that, for any $E \in \mathbf{A_{fp}}$, the DG-functor $\mathrm{Hom}^\bullet_{\mathbf{A}}(E,-)\colon \mathbf{A} \longrightarrow \mathbf{C}(\mathsf{Ab})$

preserves coproducts (cf. [52, Lemma 9.18]), and the passage to the cohomology of a complex of abelian groups preserves coproducts. Alternatively, the fact that the functor $\mathrm{Hom}_{\mathsf{H}^0(\mathbf{A})}(E,-)\colon \mathsf{H}^0(\mathbf{A}) \longrightarrow \mathsf{Ab}$ preserves coproducts is easily deduced directly from the facts that the functor $\mathrm{Hom}_{\mathsf{Z}^0(\mathbf{A})}(E,-)\colon \mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{Ab}$ does and the natural functor $\mathsf{Z}^0(\mathbf{A}) \longrightarrow \mathsf{H}^0(\mathbf{A})$ is surjective on objects and morphisms.

Now Lemma 8.18 tells that the triangulated functor $\mathsf{D}^{\mathsf{abs}}(\mathbf{A_{fp}}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{A})$ is fully faithful and all the objects in its image are compact. The assertion that the image of this functor generates the target category is provided by Corollary 7.19. ☐

In the case of CDG-modules over a graded Noetherian CDG-ring, the following result was worked out in the first-named author's memoir [48, Section 3.11] (the main part of the argument is due to Arinkin). For a CDG-ring $\boldsymbol{R}^\bullet$ whose underlying graded ring R^* is left coherent and all homogeneous left ideals in R^* have at most $\aleph_n$ generators (where n is a fixed integer), a proof can be found in [52, Theorem 9.39 and Corollary 9.42]. One reason why the additional assumption was needed is that the coderived categories in the sense of Positselski, rather than in the sense of Becker, are considered in [52]; see [52, Theorem 9.38] for a comparison theorem.

Corollary 8.20. *Let $\boldsymbol{R}^\bullet = (R^*, d, h)$ be a curved DG-ring such that the graded ring R^* is graded left coherent. Let $\boldsymbol{R}^\bullet$–$\mathbf{Mod}$ be the locally coherent abelian DG-category of left CDG-modules over $\boldsymbol{R}^\bullet$, and let $\boldsymbol{R}^\bullet$–$\mathbf{Mod_{fp}} \subset \boldsymbol{R}^\bullet$–$\mathbf{Mod}$ be the (exactly embedded full abelian) DG-subcategory of all CDG-modules with finitely presentable underlying graded left R^*-modules. Then the induced triangulated functor from the absolute derived category to Becker's coderived category*

$$\mathsf{D}^{\mathsf{abs}}(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod_{fp}}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\boldsymbol{R}^\bullet\text{–}\mathbf{Mod})$$

is fully faithful. The objects in its image are compact in $\mathsf{D}^{\mathsf{bco}}(\boldsymbol{R}^\bullet$–$\mathbf{Mod})$, and form a set of compact generators for $\mathsf{D}^{\mathsf{bco}}(\boldsymbol{R}^\bullet$–$\mathbf{Mod})$.

Proof. This is the particular case of Theorem 8.19 for $\mathbf{A} = \boldsymbol{R}^\bullet$–$\mathbf{Mod}$. It is clear from Corollary 8.9 that Theorem 8.19 is applicable. ☐

9. Coherent Schemes and Matrix Factorizations

In this section we apply the results of Section 8 to the locally coherent abelian DG-categories of complexes of quasi-coherent sheaves, or more generally, quasi-coherent matrix factorizations on coherent schemes. The main result is a generalization of the existence and description of compact generators in the coderived category from the case of a Noetherian scheme treated in [48, Section 3.11], [13, Proposition 1.5(d) and Corollary 2.3(l)] to that of a coherent scheme.

9.1. **Coherent schemes and coherent sheaves.** The following lemma establishes the key fact of Zariski locality of the coherence property for commutative rings.

Lemma 9.1. (a) *Let U be an an affine scheme and $V \subset U$ be an affine open subscheme in U. Assume that the ring $O(U)$ is coherent. Then the ring $O(V)$ is coherent, too.*

(b) *Let U be an affine scheme and $U = \bigcup_{\alpha=1}^{d} V_\alpha$ be a finite affine open covering of U. Assume that all the rings $O(V_\alpha)$ are coherent. Then the ring $O(U)$ is coherent as well.*

Proof. Part (a): notice that the restriction map $O(U) \longrightarrow O(V)$ is a flat epimorphism of commutative rings (in the sense of [61, Sections XI.1–2]). For any flat epimorphism of commutative rings $R \longrightarrow S$, coherence of the ring R implies coherence of the ring S [9, Proposition 3.7].

Part (b): notice that the restriction map $O(U) \longrightarrow \bigoplus_{\alpha=1}^{d} O(V_\alpha)$ is a faithfully flat homomorphism of commutative rings. For any faithfully flat homomorphism of commutative rings $R \longrightarrow S$, coherence of the ring S implies coherence of the ring R [23, Corollary 2.1], [7, Propositions I.5.9 and I.6.11]. ☐

Given a scheme X, we denote by X–Qcoh the abelian category of quasi-coherent sheaves on X. A scheme X is said to be *locally coherent* [9, 16] if, for every affine open subscheme $U \subset X$, the commutative ring $O(U)$ is coherent. It follows from Lemma 9.1 that it suffices to check this property for affine open subschemes $U_\alpha \subset X$ ranging over any given affine open covering of a scheme X.

We will say that a scheme X is *coherent* if X is locally coherent, quasi-compact, and quasi-separated.

Proposition 9.2. *Let $X = \bigcup_\alpha U_\alpha$ be an affine open covering of a coherent scheme X. Then the abelian category X–Qcoh is locally coherent. An object $M \in X$–Qcoh is coherent (equivalently, finitely presentable) if and only if, for every index α, the $O(U_\alpha)$-module $M(U_\alpha)$ is coherent (equivalently, finitely presentable).*

Proof. We refer to [7, Proposition I.6.11] or [56, Lemma 2.1] for a discussion of Zariski locality of finite presentability of modules over commutative rings. For any quasi-compact quasi-separated scheme X, the category of quasi-coherent sheaves X–Qcoh is locally finitely presentable, and an object $M \in X$–Qcoh is finitely presentable if and only if the $O(U_\alpha)$-module $M(U_\alpha)$ is finitely presentable for every index α of an affine open covering $X = \bigcup_\alpha U_\alpha$ [22, 0.5.2.5 and Corollaire I.6.9.12]. Now the condition that the kernel of any epimorphism of finitely presentable quasi-coherent sheaves on X is finitely presentable can be checked locally, and it holds for a coherent scheme X because it holds for a coherent affine scheme. Finally, in a locally coherent abelian category an object is coherent if and only if it is finitely presentable. ☐

A quasi-coherent sheaf M on a coherent scheme X is said to be *coherent* if it satisfies the equivalent conditions of the second assertion of Proposition 9.2. We denote the full subcategory of coherent sheaves by $X\text{–coh} = (X\text{–Qcoh})_{\mathsf{fp}} \subset X\text{–Qcoh}$. According to Proposition 9.2, X–coh is an abelian category for any coherent scheme X.

Corollary 9.3. *Let X be a coherent scheme and $\mathsf{H}^0(\mathbf{C}(X\text{–Qcoh}_{\mathsf{inj}}))$ be the homotopy category of unbounded complexes of injective quasi-coherent sheaves on X. Then $\mathsf{H}^0(\mathbf{C}(X\text{–Qcoh}_{\mathsf{inj}}))$ is a compactly generated triangulated category. The full subcategory of compact objects in $\mathsf{H}^0(\mathbf{C}(X\text{–Qcoh}_{\mathsf{inj}}))$ is equivalent to the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–coh})$ of the abelian category of coherent sheaves on X.*

Proof. This is a particular case of [62, Corollary 6.13], which is applicable in view of Proposition 9.2; cf. Theorem 0.1 from the introduction. Disregarding the direct summand closure issue, this is also a particular case of Theorem 8.19 applied to the locally coherent abelian DG-category $\mathbf{A} = \mathbf{C}(X\text{–Qcoh})$ of complexes in X–Qcoh (as per Examples 8.8); cf. Theorem 0.2. $\square$

9.2. **Quasi-coherent and coherent factorizations.** The setting in this section is a common particular case of [52, Example 4.41] and [52, Example 4.42 with Remark 2.7].

Let X be a scheme, L be a line bundle (invertible quasi-coherent sheaf) on X, and $w \in L(X)$ be a global section. In the context of Example 7.8, put $\mathsf{A} = X\text{–Qcoh}$, and let $\Delta\colon \mathsf{A} \longrightarrow \mathsf{A}$ be the functor of twisting with L, that is $\Delta(M) = L \otimes_{O_X} M$ for all $M \in X\text{–Qcoh}$. Let $v\colon \mathrm{Id}_{\mathsf{A}} \longrightarrow \Delta$ be the natural transformation of multiplication with w, i. e., the map $v_M(U)\colon M(U) \longrightarrow (L \otimes_{O_X} M)(U)$ takes a local section $s \in M(U)$ to the local section $w|_U \otimes s \in (L \otimes_{O_X} M)(U)$ for all $M \in X\text{–Qcoh}$ and all open subschemes $U \subset X$.

Then the objects of the DG-category $\mathbf{F}_{\mathrm{qc}}(X, L, w) = \mathbf{F}(\mathsf{A}, \Delta, v)$ are called *quasi-coherent (matrix) factorizations* of the potential $w \in L(X)$ on the scheme X. Explicitly, a quasi-coherent factorization $N^{\bullet}$ on X is a sequence of quasi-coherent sheaves $N^n \in X\text{–Qcoh}$, $n \in \mathbb{Z}$, together with a sequence of periodicity isomorphisms $\delta_N^{n+2,n}\colon L \otimes_{O_X} N^n \simeq N^{n+2}$ and a sequence of differentials $d_{N,n}\colon N^n \longrightarrow N^{n+1}$ such that the composition $d_{N,n+1} \circ d_{N,n}\colon N^n \longrightarrow N^{n+2}$ is equal to the composition of the multiplication map $w\colon N^n \longrightarrow L \otimes_{O_X} N^n$ with the isomorphism $\delta_N^{n+2,n}$ for every $n \in \mathbb{Z}$. Example 7.8 tells that the DG-category of quasi-coherent factorizations $\mathbf{F}_{\mathrm{qc}}(X, L, w)$ is a Grothendieck abelian DG-category.

Let us mention that, following the discussion in Section 2.7 (commutative diagram (7)), the full DG-subcategory of graded-injective objects $\mathbf{F}_{\mathrm{qc}}(X, L, w)_{\mathbf{inj}} \subset \mathbf{F}_{\mathrm{qc}}(X, L, w)$ consists of all the *injective quasi-coherent factorizations*. The latter term means quasi-coherent factorizations $N^{\bullet}$ of the potential w on X such that the quasi-coherent sheaf N^n is injective in X–Qcoh for every $n \in \mathbb{Z}$. So Becker's coderived category $\mathsf{D}^{\mathsf{bco}}(\mathbf{F}_{\mathrm{qc}}(X, L, w)) \simeq \mathsf{H}^0(\mathbf{F}_{\mathrm{qc}}(X, L, w)_{\mathbf{inj}})$ is the homotopy category of injective quasi-coherent factorizations.

Now let us assume that X is a coherent scheme (as defined in Section 9.1). Then the full DG-subcategory of *coherent factorizations* $\mathbf{F}_{\mathrm{coh}}(X, L, w) \subset \mathbf{F}_{\mathrm{qc}}(X, L, w)$ consists of all the factorizations $N^{\bullet}$ such that N^n is a coherent sheaf on X for every $n \in \mathbb{Z}$. Following Example 3.15 applied to the abelian category $\mathsf{E} = X\text{–coh}$, the DG-category $\mathbf{F}_{\mathrm{coh}}(X, L, w)$ is abelian; in fact, it is an exactly embedded full abelian DG-subcategory in $\mathbf{F}_{\mathrm{qc}}(X, L, w)$ in the sense of Section 3.3.

Moreover, Example 8.10 together with Proposition 9.2 tell that the DG-category of quasi-coherent factorizations $\mathbf{F}_{\mathrm{qc}}(X, L, w)$ on a coherent scheme X is a locally coherent abelian DG-category, and the full DG-subcategory of coherent factorizations it its full DG-subcategory of finitely presentable/coherent objects, $\mathbf{F}_{\mathrm{coh}}(X, L, w) = \mathbf{F}_{\mathrm{qc}}(X, L, w)_{\mathbf{fp}}$.

The following corollary generalizes [13, Corollary 2.3(l)] from the case of a Noetherian scheme to that of a coherent one.

Corollary 9.4. *Let X be a coherent scheme, L be a line bundle on X, and $w \in L(X)$ be a global section. Let $\mathbf{F}_{\mathrm{qc}}(X, L, w)$ be the locally coherent DG-category of quasi-coherent factorizations of the potential $w \in L(X)$ on X, and let $\mathbf{F}_{\mathrm{coh}}(X, L, w) \subset \mathbf{F}_{\mathrm{qc}}(X, L, w)$ be the (exactly embedded full abelian) DG-subcategory of coherent factorizations. Then the induced triangulated functor from the absolute derived category to Becker's coderived category*

$$\mathsf{D}^{\mathsf{abs}}(\mathbf{F}_{\mathrm{coh}}(X, L, w)) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathbf{F}_{\mathrm{qc}}(X, L, w))$$

is fully faithful. The objects in its image are compact in $\mathsf{D}^{\mathsf{bco}}(\mathbf{F}_{\mathrm{qc}}(X, L, w))$, and form a set of compact generators for $\mathsf{D}^{\mathsf{bco}}(\mathbf{F}_{\mathrm{qc}}(X, L, w))$.

Proof. This is the particular case of Theorem 8.19 for $\mathbf{A} = \mathbf{F}_{\mathrm{qc}}(X, L, w)$. Example 8.10 with Proposition 9.2 imply that Theorem 8.19 is applicable. ☐

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