

# A Polynomial-time Algorithm for the Large Scale of Airplane Refueling Problem

Jinchuan Cui<sup>1\*</sup> and Xiaoya Li<sup>2</sup>

<sup>1\*</sup>Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55 Zhongguancun East Road, Beijing, 100190, China.

<sup>2</sup>Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55 Zhongguancun East Road, Beijing, 100190, China.

\*Corresponding author(s). E-mail(s): [cjc@amss.ac.cn](mailto:cjc@amss.ac.cn);  
Contributing authors: [xyli@amss.ac.cn](mailto:xyli@amss.ac.cn);

## Abstract

Airplane refueling problem is a nonlinear unconstrained optimization problem with  $n!$  feasible solutions. Given a fleet of  $n$  airplanes with mid-air refueling technique, the question is to find the best refueling policy to make the last remaining airplane travels the farthest. In order to deal with the large scale of airplanes refueling instances, we proposed the definition of sequential feasible solution by employing the refueling properties of data structure. We proved that if an airplanes refueling instance has feasible solutions, it must have the sequential feasible solutions; and the optimal feasible solution must be the optimal sequential feasible solution. Then we proposed the sequential search algorithm which consists of two steps. The first step of the sequential search algorithm aims to seek out all of the sequential feasible solutions. When the input size of  $n$  is greater than an index number, we proved that the number of the sequential feasible solutions will change to grow at a polynomial rate. The second step of the sequential search algorithm aims to search for the maximal sequential feasible solution by bubble sorting all of the sequential feasible solutions. Moreover, we built an efficient computability scheme, according to which we could forecast within a polynomial time the computational complexity of the sequential search algorithm that runs on any given airplanes refueling instance. Thus we could provide a computational strategy for decision makers or algorithm users by considering with their available computing resources.

**Keywords:** Airplane refueling problem (ARP), Sequential search algorithm (SSA), Polynomial-time algorithm, Efficient computability scheme

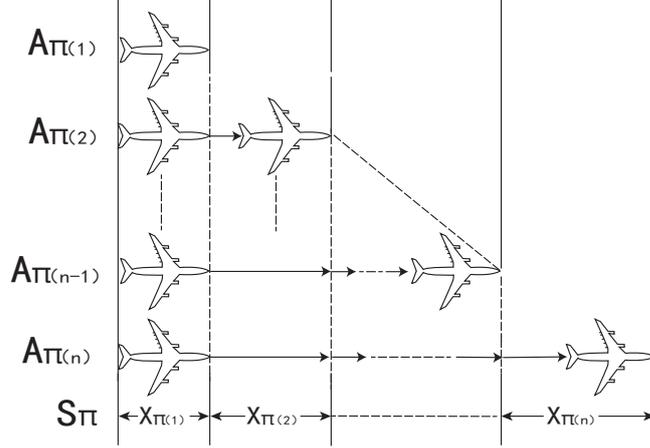
# 1 Introduction

The Airplane Refueling Problem (ARP) was raised by Woeginger [1] from a math puzzle problem [2]. Suppose there are  $n$  airplanes referred to  $A_1, \dots, A_n$ , each  $A_i$  can carry  $v_i$  tanks of fuel, and consumes  $c_i$  tanks of fuel per kilometers for  $1 \leq i \leq n$ . The fleet starts to fly together to a same target at a same rate without getting fuel from outside, but each airplane can refuel to other airplanes instantaneously during the trip and then be dropped out. The goal is to determine a drop out permutation  $\pi = (\pi(1), \dots, \pi(n))$  that maximize the traveled distance by the last remaining airplane. Previous research on ARP centralized on its complexity analysis and its exact algorithm [3–6]. Related research also focused on equivalent problem of ARP such as the  $n$ -vehicle exploration problem [7–9]. Höhn [3] pointed out that ARP is a complexity form of the single machine scheduling problem, and regarded the complexity of ARP as an open problem. Vásquez [4] studied the structural properties of ARP such as connections between local precedence and global precedence. Iftah and Danny [5] proposed a fast and easy-to-implement pseudo polynomial time algorithm that attained a polynomial-time approximation scheme for ARP. Li et al. [6] put forward a fast exact algorithm for the first time by searching for all the feasible solutions satisfied with some necessary conditions. They run the algorithm on some large scale of ARP instances and got more efficient results than previous algorithms using pruning technique. However, the authors did not investigate the theoretical computational complexity the fast exact algorithm will perform on worst case, and did not prove that how fast and to what degree will the fast algorithm be when perform it on larger scale of ARP instances. Is it an exponential-time algorithm, or is it probably a polynomial-time algorithm?

The above questions are investigated and addressed in this paper. We are mainly interested to the performance of an algorithm on instances with large scale, and we mainly care about that how long will a given algorithm take to solve real-life instances. What we mainly concern is to explore the possibility of a fast exact algorithm which is efficient on larger size of ARP instances, since a superior algorithm is able to handle the large size problem in any reasonable amount of time [10]. According to the theoretical analysis on the algorithmic upper bound, we explained why there will exist an "inflection point" for the algorithmic complexity. Besides, we proved that the algorithmic complexity grows at a polynomial rate when the input size of  $n$  is getting to be sufficiently large. We also proposed an efficient computability scheme to predict the sequential search algorithmic complexity for any given ARP instance. The idea of efficient computability is inspired by the following research. In book [11], the authors mentioned that it is possible to quantify some precise senses in which an instance may be easier than the worst case, and to take advantage of these situations when they occur. They also pointed out that some "size" parameters has an enormous effect on the running time because an input with "special structure" can help us avoid many of the difficulties that can make the worst case intractable. In [12], the authors provided a worst-case explanation for the phenomenon that why some real-life maximum clique instances are not intractable. They claimed that real-life instances of maximum clique problem often have a small clique-core gap, however real hard instances are still hard no matter the input size is large or small.

## 2 Preliminary

We consider a permutation order  $\pi$  and its related sequence  $A_{\pi(1)} \Rightarrow A_{\pi(2)} \Rightarrow \dots \Rightarrow A_{\pi(n)}$ , where  $A_{\pi(i)}$  refuels to  $A_{\pi(j)}$  for any  $i < j$ . In Figure 1, let  $S_\pi = \sum_{i=1}^n x_{\pi(i)}$  denotes the total distance that the  $n$  airplanes can approach, and  $x_{\pi(i)}$  denotes the distance that  $A_{\pi(i)}$  travels farther than  $A_{\pi(i-1)}$ , which is also the flight distance that  $A_{\pi(i)}$  contributes to  $S_\pi$  separately.



**Fig. 1** Description of ARP with order  $\pi$ .

Then ARP is transformed into the following scheduling problem: Given  $n$  airplanes with  $v_1, v_2, \dots, v_n$  as fuel capacities, and  $c_1, c_2, \dots, c_n$  as fuel consumption rates respectively. The goal of ARP is to find a drop out permutation  $\pi = (\pi(1), \pi(2), \dots, \pi(n))$  that maximizes  $S_\pi$ .

$$S_\pi = \frac{v_{\pi(1)}}{c_{\pi(1)} \cdots + c_{\pi(n)}} + \cdots + \frac{v_{\pi(n)}}{c_{\pi(n)}} \quad (1)$$

We use  $C$  to denote the cumulative sum of fuel consumption rates. For an order  $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ , assume  $C_{\pi(n)} = 0$ , and  $C_{\pi(l)} = \sum_{t=l+1}^n c_{\pi(t)}$  for any  $l < n$ . For general case, we use  $C_o$  to denote the current sum of fuel consumption rates. Particularly, for  $\pi$ ,  $C_o$  could be  $C_{\pi(i)}$  for any  $1 \leq i \leq n$ .

Since each  $S_\pi$  corresponds to a permutation order of  $n$  airplanes, there are totally  $n!$  of  $S_\pi$  needs to be compared to get the optimal solution.

A special case of  $n$ -vehicle exploration problem [7] is extended to ARP as follows.

**Theorem 1.** *In an ARP instance, if*

$$\frac{v_1}{c_1^2} \leq \frac{v_2}{c_2^2} \leq \dots \leq \frac{v_{n-1}}{c_{n-1}^2} \leq \frac{v_n}{c_n^2}$$

and

$$\frac{v_1}{c_1} \leq \frac{v_2}{c_2} \leq \dots \leq \frac{v_{n-1}}{c_{n-1}} \leq \frac{v_n}{c_n},$$

then the optimal sequence is  $A_1 \Rightarrow A_2 \Rightarrow \dots \Rightarrow A_n$ .

For general case of ARP instances, we randomly choose two neighbored airplanes  $A_i$  and  $A_j$  from  $\pi$  and suppose  $C_o$  is the cumulative sum of fuel consumption rates of all the airplanes that take precedence over  $A_i$  and  $A_j$ .

If  $A_j \Rightarrow A_i$ , then

$$S_{(j,i)} = x_{\pi(j)} + x_{\pi(i)} = \frac{v_j}{c_j + c_i + C_o} + \frac{v_i}{c_i + C_o},$$

otherwise,

$$S_{(i,j)} = x_{\pi(i)} + x_{\pi(j)} = \frac{v_i}{c_i + c_j + C_o} + \frac{v_j}{c_j + C_o}.$$

Consequently, if  $\frac{v_i}{c_i \times (c_i + C_o)} > \frac{v_j}{c_j \times (c_j + C_o)}$ , then  $S_{(j,i)} > S_{(i,j)}$ . Otherwise, if  $\frac{v_i}{c_i \times (c_i + C_o)} \leq \frac{v_j}{c_j \times (c_j + C_o)}$ , then  $S_{(j,i)} \leq S_{(i,j)}$ .

**Definition 1.** Given current cumulative sum of fuel consumption rates  $C_o$ , the relative distance factor of  $A_i$  is defined as  $\varphi(A_i, C_o) = \frac{v_i}{c_i \times (c_i + C_o)}$ .

### 3 Overall strategy

We mainly focused on seeking for the polynomial-time algorithm to solve the large scale of ARP instances. At first, we proposed the definition of the sequential feasible solution. Then we put forward the sequential search algorithm (SSA) by bubble sorting all of the sequential feasible solutions. The computational complexity of the SSA depends on the number of sequential feasible solutions for any given ARP instance. In studying the complexity of an algorithm, instances with larger inputs are of more priorities than other kinds because the larger inputs determine the applicability of the algorithm [13]. It is common that an algorithm runs in exponential time on small size of instances, it is still exponential-time on larger size of instances, especially on worst case of difficulty problems. For this reason we mainly centralized on the worst case of ARP instances, and to investigate the corresponding algorithmic complexity.

We focused on two major challenges. One is that will the running time of the SSA decline down to polynomial-time if the input size of  $n$  gets to be sufficiently large? The other challenge is that how can we predict the specific computational complexity of any given ARP instance in polynomial time before we running the SSA on it? The answer to the first question is positive. Different from previous work, the SSA runs in exponential time only in small inputs size, but when it comes to instances with larger inputs, the SSA will change to run in polynomial time. To be specific, the number of the sequential feasible solutions is upper bounded by  $2^{n-2}$  when  $n$  is in small size, and changes to be upper bounded by  $\frac{m^2}{n} C_n^m$  when  $n$  is greater than  $2m$ , which is referred to an "inflection point". To be noticed, here the index number  $m$  can be obtained in polynomial time, and the  $m$  does not depend on  $n$ . Thus we proved that for large scale of ARP, the SSA changes to be a polynomial-time algorithm. The key point to answer

the second question is to predict the index number  $m$  for any given ARP instance in polynomial time. We proposed an algorithm to estimate the  $m$  for the worst case, and we improved the algorithm for general case by using heuristic method to attain the computational complexity in the design of the efficient computability scheme.

In section 4, we proposed the definition of sequential feasible solution, and constructed the SSA by bubble sorting all of the sequential feasible solutions. We sharpened the upper bound of the number of sequential feasible solutions to  $2^{n-2}$ . In section 5, by exploring the computational complexity of ARP instances from a dynamic perspective, we found that the computational complexity of the SSA grows at a slowing down rate when the input size of  $n$  gets to be greater than an "inflection point". In section 6, our efforts were devoted to construct the efficient computability scheme. We proposed a heuristic algorithm to quantify an estimated  $m$  for general case of APR instances. Then we explained how to use the efficient computability scheme to predict computational complexity before we choose a proper algorithm for any given ARP instance.

## 4 Sequential feasible solutions and the complexity upper bound

**Definition 2.** A sequence  $\pi = (\pi(1), \pi(2), \dots, \pi(n))$  is called a **sequential feasible solution**, if for each pair of airplanes  $A_{\pi(i)}$  and  $A_{\pi(j)}$  for  $i < j$  in  $\pi$ , at least one of the following two inequalities holds.

$$\frac{v_{\pi(i)}}{c_{\pi(i)}(c_{\pi(i)} + C_{\pi(j)})} \leq \frac{v_{\pi(j)}}{c_{\pi(j)}(c_{\pi(j)} + C_{\pi(j)})} \quad (2)$$

$$\frac{v_{\pi(i)}}{c_{\pi(i)}(c_{\pi(i)} + C_{\pi(i)})} \leq \frac{v_{\pi(j)}}{c_{\pi(j)}(c_{\pi(j)} + C_{\pi(i)})} \quad (3)$$

Where  $C_{\pi(n)} = 0$ , and  $C_{\pi(l)} = \sum_{t=l+1}^n c_{\pi(t)}$  for any  $l < n$ .

**Lemma 1.** For each pair of airplanes  $A_i$  and  $A_j$  with  $v_i/c_i^2 > v_j/c_j^2$  and  $v_i/c_i < v_j/c_j$ .

- (1)  $v_i < v_j$ , and  $c_i < c_j$ .
- (2) For any  $C_o > 0$ ,  $\frac{v_i}{c_i + C_o} < \frac{v_j}{c_j + C_o}$ .

*Proof*(1) Since  $\frac{v_i}{v_j} < \frac{c_i}{c_j} < \frac{v_i/c_i}{v_j/c_j} < 1$ , it follows that  $v_i < v_j$  and  $c_i < c_j$ .

- (2) Since  $c_i < c_j$ , we have  $\frac{v_i}{v_j} < \frac{c_i}{c_j} < \frac{c_i + C_o}{c_j + C_o} < 1$ , then  $\frac{v_i}{c_i + C} < \frac{v_j}{c_j + C}$ .

Therefore for any  $C_o > 0$ ,  $\frac{v_i}{c_i + C_o} < \frac{v_j}{c_j + C_o}$ . □

**Lemma 2.** For each pair of airplanes  $A_i$  and  $A_j$  with  $v_i/c_i^2 > v_j/c_j^2$  and  $v_i/c_i < v_j/c_j$ . If there exists a sum of fuel consumption rates  $C_o$ , such that  $\frac{v_i}{c_i(c_i + C_o)} < \frac{v_j}{c_j(c_j + C_o)}$ . Then, for any  $C > C_o$ , it follows that  $\frac{v_i}{c_i(c_i + C)} < \frac{v_j}{c_j(c_j + C)}$ .

*Proof.* Since  $\frac{v_i c_j}{v_j c_i} < \frac{c_i + C_o}{c_j + C_o} < \frac{c_i + C}{c_j + C}$ , it follows that  $\frac{v_i}{c_i(c_i + C)} < \frac{v_j}{c_j(c_j + C)}$ . □

**Theorem 2.** Given an ARP instance:

- (1) If it has feasible solutions, it must have sequential feasible solutions;  
(2) The optimal feasible solution must be the optimal sequential feasible solution.

*Proof*(1) Given  $n$  airplanes  $A_1, \dots, A_n$ , we can obtain a sequential feasible solution by running Algorithm 1. The computational complexity of Algorithm 1 is  $O(n^2)$ .

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**Algorithm 1** *Find – SequentialFS(A)*

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**Require:**  $\mathcal{A} = \{A_1, \dots, A_n\}$

**Ensure:**  $\pi = (\pi(1), \dots, \pi(n))$

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1:  $k \leftarrow n, C_o \leftarrow 0$ 
2: while  $k \geq 1$  do
3:    $\varphi_o \leftarrow \varphi(A_1, C_o)$ , see Definition 1
4:    $k_o \leftarrow 1$ 
5:   for  $i := 2$  to  $k$  do
6:     if  $\varphi(A_i, C_o) > \varphi_o$  then
7:        $k_o \leftarrow i$ 
8:        $\varphi_o \leftarrow \varphi(A_i, C_o)$ 
9:     end if
10:  end for
11:   $C_o \leftarrow C_o + c_{k_o}$ 
12:   $\mathcal{A} \leftarrow \mathcal{A} \setminus \{A_{k_o}\}$ 
13:  if  $k < n$  then
14:     $count \leftarrow 0$ 
15:    for  $i := k + 1$  to  $n$  do
16:      if  $k_o \geq \pi(i)$  then
17:         $count \leftarrow count + 1$ 
18:      end if
19:    end for
20:  end if
21:   $k_o \leftarrow k_o + count$ 
22:   $\pi(k) \leftarrow k_o$ 
23:   $k \leftarrow k - 1$ 
24: end while

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- (2) Given an  $n$ -airplane instance, we will prove that its optimal feasible solution must be a sequential feasible solution. Suppose the optimal feasible solution  $\pi^*$  is not a sequential feasible solution, then there exist two airplanes  $A_{\pi^*(i)}$  and  $A_{\pi^*(j)}$  that do not satisfy the requirements of sequential feasible solution. Suppose  $A_{\pi^*(i)} \Rightarrow A_{\pi^*(A)} \Rightarrow A_{\pi^*(j)}$ , here  $A_{\pi^*(A)}$  denotes the set of airplanes travel between  $A_{\pi^*(i)}$  and  $A_{\pi^*(j)}$ . Suppose  $C_{\pi^*(j)} = \sum_{t=j+1}^n c_{\pi^*(t)}$  and  $C_{\pi^*(i)} = \sum_{t=i+1}^n c_{\pi^*(t)}$ . Then it follows that

$$\frac{v_{\pi^*(i)}}{c_{\pi^*(i)}(c_{\pi^*(i)} + C_{\pi^*(j)})} > \frac{v_{\pi^*(j)}}{c_{\pi^*(j)}(c_{\pi^*(j)} + C_{\pi^*(j)})}$$

and

$$\frac{v_{\pi^*(i)}}{c_{\pi^*(i)}(c_{\pi^*(i)} + C_{\pi^*(i)})} > \frac{v_{\pi^*(j)}}{c_{\pi^*(j)}(c_{\pi^*(j)} + C_{\pi^*(i)})}.$$

According to Theorem 1, if  $v_{\pi^*(i)}/c_{\pi^*(i)}^2 \leq v_{\pi^*(j)}/c_{\pi^*(j)}^2$ , then  $v_{\pi^*(i)}/c_{\pi^*(i)} > v_{\pi^*(j)}/c_{\pi^*(j)}$ . According to Lemma 2, for any  $C \geq C_{\pi^*(j)}$ ,  $A_{\pi^*(j)}$  must refuel to  $A_{\pi^*(i)}$  in a sequential feasible solution, which is contrary to the assumption that  $\pi^*$  is the optimal solution. If  $v_{\pi^*(i)}/c_{\pi^*(i)}^2 > v_{\pi^*(j)}/c_{\pi^*(j)}^2$ , then it follows that  $v_{\pi^*(i)}/c_{\pi^*(i)} < v_{\pi^*(j)}/c_{\pi^*(j)}$  and  $c_{\pi^*(i)} < c_{\pi^*(j)}$ . If we switch the order of  $A_{\pi^*(i)}$  and  $A_{\pi^*(j)}$  to attain a new ordering  $\hat{\pi}$  and its related sequence is  $A_{\pi^*(j)} \Rightarrow A_{\pi^*(A)} \Rightarrow A_{\pi^*(i)}$ . It follows that

$$S_{\hat{\pi}} = \frac{v_{\pi^*(i)}}{c_{\pi^*(i)} + C_{\pi^*(j)}} + \frac{v_{\pi^*(A)}}{c_{\pi^*(i)} + C_{\pi^*(j)} + C_{\pi^*(A)}} + \frac{v_{\pi^*(j)}}{c_{\pi^*(i)} + C_{\pi^*(i)}}$$

and

$$S_{\pi^*} = \frac{v_{\pi^*(j)}}{c_{\pi^*(j)} + C_{\pi^*(j)}} + \frac{v_{\pi^*(A)}}{c_{\pi^*(j)} + C_{\pi^*(j)} + C_{\pi^*(A)}} + \frac{v_{\pi^*(i)}}{c_{\pi^*(j)} + C_{\pi^*(i)}}.$$

Since  $c_{\pi^*(i)} < c_{\pi^*(j)}$ , then it follows that  $S_{\hat{\pi}} > S_{\pi^*}$ , which is contrary to the assumption that  $\pi^*$  is the optimal feasible solution.

Therefore the optimal feasible solution of ARP must be the optimal sequential feasible solution. □

**Definition 3.** An ARP instance is called a **worst case** when it has the greatest number of sequential feasible solutions for given input size of  $n$ .

**Definition 4.** An airplane refueling instance is called a **complete reverse order sequence**, if

$$\frac{v_1}{c_1^2} > \frac{v_2}{c_2^2} > \dots > \frac{v_n}{c_n^2}$$

and

$$\frac{v_1}{c_1} < \frac{v_2}{c_2} < \dots < \frac{v_n}{c_n}.$$

In [8], the authors introduced cluster as a tool to model the computational complexity of  $n$ -vehicle exploration problem and they claimed that the complete reverse order instances have greater computational complexity than the other kinds of instances. According to which, we pose the following Lemma 3.

**Lemma 3.** The worst case must be a complete reverse order sequence for any given input  $n \geq 3$ .

**Theorem 3.** For a worst case of  $n$ -airplane instances with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ , let  $Q_n$  represents the number of its sequential feasible solutions, then  $Q_n \leq 2^{n-2}$  for  $n \geq 2$ .

*Proof.* Given a worst case of ARP with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ , we will prove that  $Q_n$  is upper bounded by  $2^{n-2}$ . To reduce the search space from  $n!$  to  $2^{n-2}$  is not an obvious work, so we introduce combination into the proof because of  $2^{n-2} = C_{n-2}^0 + C_{n-2}^1 + \dots + C_{n-2}^{n-2}$ . Here  $C_{n-2}^p$  relates to the number of the sequential

feasible solutions when  $p$  airplanes are chosen that lie between  $A_{\pi(n)}$  and  $A_n$  for  $0 \leq p \leq n-2$ , here  $\pi(n)$  is the farthest position in an order  $\pi$ .

When  $n=2$ , we've shown that  $Q_2 = 1 = 2^{2-2}$ . We use  $C_0^0$  to calculate  $Q_2$ , which means, from combination point of view, that no airplane is chosen between  $A_{\pi(2)}$  and  $A_2$ , whose combination is account for increasing the number of sequential feasible solutions.

$$C_0^0 : A_2 \Rightarrow A_1.$$

When  $n=3$ , we have  $Q_3 \leq 2 = 2^{3-2}$ . When  $A_1$  is  $A_{\pi(3)}$ ,  $A_2$  is available between  $A_1$  and  $A_3$  that possible leads to  $C_1^1$  sequential feasible solution. When  $A_2$  is  $A_{\pi(3)}$ , there is another sequential feasible solution if and only if  $\frac{v_3}{c_3 \times (c_3 + c_2)} > \frac{v_1}{c_1 \times (c_1 + c_2)}$ . No other airplane is possible between  $A_2$  and  $A_3$ , which leads to  $C_1^0$  sequential feasible solution. There is no possibility when  $A_3$  takes the farthest position in a sequential feasible solution. By running through all the possible situations, it follows that  $Q_3 \leq C_1^0 + C_1^1$ .

$$C_1^1 : A_3 \Rightarrow A_2 \Rightarrow A_1, \text{ or } A_2 \Rightarrow A_3 \Rightarrow A_1,$$

$$C_1^0 : A_1 \Rightarrow A_3 \Rightarrow A_2.$$

When  $n=4$ , let  $A_1$  is  $A_{\pi(4)}$ , then the rest of 3 airplanes correspond to at most 2 sequential feasible solutions, when it satisfies eq. (4). So the rest 3 airplanes correspond to at most  $C_1^0 + C_1^1 = 2$  sequential feasible solutions.

$$\frac{v_4}{c_4(c_4 + c_1 + c_3)} > \frac{v_2}{c_2(c_2 + c_1 + c_3)} \quad (4)$$

If  $A_2$  is  $A_{\pi(4)}$ , then it follows that:

$$\frac{v_2}{c_2(c_2 + c_2 + c_3)} > \frac{v_1}{c_1(c_1 + c_2 + c_3)}. \quad (5)$$

Combining eq. (4) with eq. (5), and by Lemma 2, it follows that:

$$\frac{v_4}{c_4(c_4 + c_2 + c_3)} > \frac{v_2}{c_2(c_2 + c_2 + c_3)} > \frac{v_1}{c_1(c_1 + c_2 + c_3)}. \quad (6)$$

According to eq. (6), there is at most  $C_1^1 = 1$  sequential feasible solution as  $A_1 \Rightarrow A_4 \Rightarrow A_3 \Rightarrow A_2$ .

If  $A_3$  is  $A_{\pi(4)}$ , there is at most  $C_1^0$  possible sequential feasible solution.

By summing up all the above cases, we have  $Q_4 \leq C_1^0 + C_1^1 + C_1^1 + C_1^0 = C_2^0 + C_2^1 + C_2^2$ .

$$C_2^2 : A_4 \Rightarrow A_3 \Rightarrow A_2 \Rightarrow A_1, \text{ or } A_3 \Rightarrow A_4 \Rightarrow A_2 \Rightarrow A_1$$

$$C_2^1 : A_2 \Rightarrow A_4 \Rightarrow A_3 \Rightarrow A_1$$

$$A_1 \Rightarrow A_4 \Rightarrow A_3 \Rightarrow A_2$$

$$C_2^0 : A_1 \Rightarrow A_2 \Rightarrow A_4 \Rightarrow A_3$$

When  $n = k$ , suppose there are at most  $2^{k-2}$  sequential feasible solutions. Moreover, there are at most  $C_{k-2}^p$  sequential feasible solutions when  $p$  airplanes are chosen between  $A_{\pi(k)}$  and  $A_k$ .

When  $n = k+1$ , suppose  $v_k/c_k^2 > v_{k+1}/c_{k+1}^2$  and  $v_k/c_k < v_{k+1}/c_{k+1}$ .  $Q_{k+1}$  consists of  $k$  parts.

- (1) There is at most  $C_{k-1}^0$  sequential feasible solution when  $A_k$  takes the farthest position and  $A_{k+1}$  takes the second farthest position.  
...
- ( $p+1$ ) When  $p$  airplanes lie between  $A_{\pi(k+1)}$  and  $A_{k+1}$ , the number of sequential feasible solutions consists of two parts. The first part refers to  $C_{k-2}^{p-1}$  sequential feasible solutions when  $A_1$  is  $A_{\pi(k+1)}$  and the rest of  $k$  airplanes form a  $k$ -airplane worst case; The other part refers to  $C_{k-2}^p$  sequential feasible solutions when  $A_1$  does not take the farthest position. Totally there are  $C_{k-2}^{p-1} + C_{k-2}^p = C_{k-1}^p$  sequential feasible solutions in this case. Thus for each  $C_{k-2}^p$ , when we add  $A_{k+1}$  into the new sequence, from the view of combination, the number of sequential feasible solution changes to be  $C_{k-2}^p + C_{k-2}^{p-1} = C_{k-1}^p$ .  
...
- ( $k$ ) There is at most  $C_{k-1}^{k-1}$  sequential feasible solution when  $(k-1)$  airplanes lie between  $A_{\pi(k+1)}$  and  $A_{k+1}$ .

$$\begin{aligned}
Q_{k+1} &\leq C_{k-1}^0 + (C_{k-2}^0 + C_{k-2}^1) + \cdots + (C_{k-2}^{k-3} + C_{k-2}^{k-2}) + C_{k-1}^{k-1} \\
&= C_{k-1}^0 + C_{k-1}^1 + \cdots + C_{k-1}^{k-2} + C_{k-1}^{k-1} \\
&= 2^{k-1}
\end{aligned} \tag{7}$$

Therefore,  $Q_n$  is upper bounded by  $2^{n-2}$ .  $\square$

**Corollary 1.** *For the worst case of ARP instance with  $v_1/c_1^2 > \cdots > v_n/c_n^2$  and  $v_1/c_1 < \cdots < v_n/c_n$ . Suppose we have determined all of the airplanes that take precedence over  $A_n$ , then there is only one dropout order for the rest of airplanes in a sequential feasible solution.*

## 5 The SSA is a polynomial-time algorithm to solve the large scale of ARP

Given an ARP instance with  $v_1/c_1^2 > \cdots > v_n/c_n^2$  and  $v_1/c_1 < \cdots < v_n/c_n$ ,  $\varphi(A_n, C) \geq \varphi(A_{n-1}, C)$  for any  $C \geq \mathcal{C}_{n-1, n}$ .

$$\mathcal{C}_{n-1, n} = \frac{v_{n-1}c_n^2 - v_n c_{n-1}^2}{v_n c_{n-1} - v_{n-1} c_n} \tag{8}$$

**Lemma 4.** *For the worst case of ARP instances with  $v_1/c_1^2 > \cdots > v_n/c_n^2$  and  $v_1/c_1 < \cdots < v_n/c_n$ . Suppose  $0 < \mathcal{C}_{n-1, n} < \sum_{k=1}^n c_k$ , then there exists an index number  $m$  associated with a cumulative sum of fuel consumption rates as  $\mathcal{C}_m = \sum_{k=1}^m c_k$ , such that  $\varphi(A_n, C) > \varphi(A_{n-1}, C)$  for any  $C > \mathcal{C}_m$ .*

*Proof.* Since  $c_1 < \dots < c_n$ , we could determine  $m$  by iteratively adding  $c_i$  to  $C_m = \sum_{k=1}^m c_k$  for  $i = 1, \dots, m$ , such that  $C_m \geq C_{n-1,n}$ . The existence of  $m$  is evident.

Given the index number  $m$  and its related  $C_m$ , according to Lemma 2, if  $\varphi(A_n, C_m) > \varphi(A_{n-1}, C_m)$ , then  $\varphi(A_n, C) > \varphi(A_{n-1}, C)$  for any  $C > C_m$ .  $\square$

**Lemma 5.** *For the worst case of ARP instances with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ . If we have determined the index number  $m$  according to Lemma 4, then  $Q_n < \frac{m^2}{n} C_n^m$  when  $n > 2m$ .*

*Proof.* In the proof of Theorem 3 we introduce combination number to calculate the number of sequential feasible solutions. For a given worst case of ARP instance,  $Q_n$  is upper bounded by  $2^{n-2}$ , which is equal to  $\sum_{p=0}^{n-2} C_{n-2}^p$ . Hence  $Q_n$  is divided into  $(n-1)$  parts, and each part has at most  $C_{n-2}^k$  sequential feasible solutions for  $0 \leq p \leq n-2$ . Here,  $C_{n-2}^p$  means that only  $p$  airplanes are chosen from  $(n-2)$  airplanes to travel between  $A_{\pi(n)}$  and  $A_n$  in a sequential feasible solution. Thus there are  $(p+1)$  airplanes have the possibility to take precedence over  $A_n$  and to increase the number of sequential feasible solutions.

According to Lemma 4, given a worst case of ARP instance, we can determine the index number  $m$ , such that for large scale of worst case of ARP instances with  $n > 2m$ , it follows that at most  $(m-1)$  airplanes are available to be chosen to locate between  $A_{\pi(n)}$  and  $A_n$ , which means that  $Q_n$  is composed by at most  $m$  parts such as  $C_{n-2}^0, C_{n-2}^1, \dots, C_{n-2}^{m-1}$ .

$$Q_n = C_{n-2}^0 + C_{n-2}^1 + \dots + C_{n-2}^{m-1} \quad (9)$$

We will show that  $Q_n$  in eq. (9) is less than  $\frac{m^2}{n} C_n^m$  when  $n$  is bigger than  $2m$ . It follows that eq. (10) and eq. (11) are true when  $n > 2m$ .

$$C_{n-2}^0 < C_{n-2}^1 < \dots < C_{n-2}^{m-1} \quad (10)$$

$$\begin{aligned} Q_n &< m C_{n-2}^{m-1} \\ &= m \times \frac{(n-2) \times (n-3) \times \dots \times (n-m)}{(m-1) \times (m-2) \times \dots \times 1} \\ &= \frac{m^2 \times (n-m)}{n \times (n-1)} \times \frac{n \times (n-1) \times (n-2) \times \dots \times (n-m+1)}{m \times (m-1) \times (m-2) \times \dots \times 1} \\ &< \frac{m^2}{n} \times \frac{n \times (n-1) \times (n-2) \times \dots \times (n-m+1)}{m \times (m-1) \times \dots \times 1} \\ &= \frac{m^2}{n} \times C_n^m \end{aligned} \quad (11)$$

Thus it follows that  $Q_n < \frac{m^2}{n} C_n^m$  when  $n > 2m$ .  $\square$

**Theorem 4.** *For large scale of the worst case of ARP instances with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ . Suppose we have found the index number  $m$  by Lemma 4 and  $n > 2m$ . Then there exists an  $N = 2m$  such that  $Q_n < \frac{m^2}{n} C_n^m$  for each  $n > N$ . In addition, when  $n > N$ ,  $Q_n$  is upper bounded by  $n^m$ .*

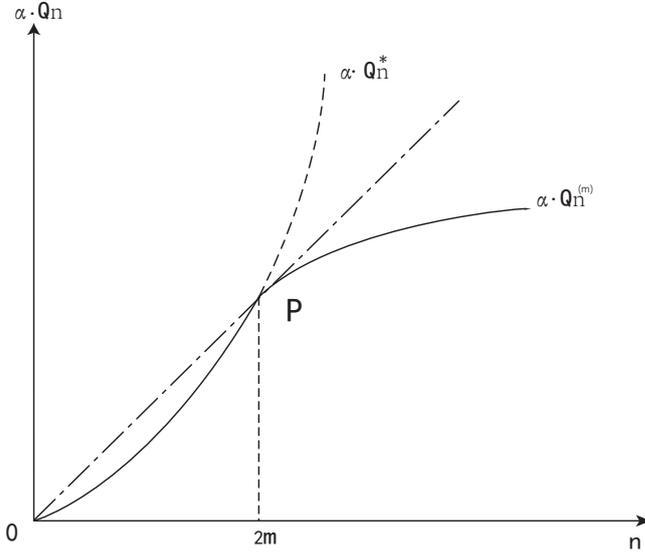
*Proof.* According to Lemma 5,  $Q_n$  is less than  $\frac{m^2}{n}C_n^m$  when  $n > 2m$ . According to Lemma 4, it points out that there exists an index number  $m$  and  $N = 2m$ , such that  $Q_n$  is upper bounded by  $n^m$  when  $n > N$ .

$$Q_n < \frac{m^2}{n}C_n^m = \frac{m^2 \times n \times (n-1) \times \cdots \times (n-m+1)}{n \times m \times (m-1) \times \cdots \times 1} = n^m < n^m \quad (12)$$

□

The SSA similar with the fast exact algorithm [6] is proposed in Algorithm 2 by bubble sorting all of the sequential feasible solutions to get access to the optimal feasible solution. The computational complexity of Algorithm 2 is  $O(n^2Q_n^2)$ . By Theorem 3,  $Q_n \leq 2^{n-2}$  for small size inputs instances, and by Theorem 4,  $Q_n < n^m$  when the input size  $n$  is greater than  $2m$ . The computational complexity of Algorithm 2 is presented as follows.

$$O(n^2Q_n^2) = \begin{cases} O(n^22^{2n-4}), & 2 \leq n \leq 2m \\ O(n^{2m+2}), & n > 2m \end{cases} \quad (13)$$



**Fig. 2** A scaler measure of the  $Q_n$ , here the scaler  $\alpha$  equals to  $2m/2^{2m-2}$ .

According to Theorem 4, a turn toward polynomial running time of Algorithm 2 occurs when the input size  $n$  is greater than  $2m$ . From another viewpoint, given a large set of airplanes with an index number  $m$ , if  $n$  is greater than  $2m$ , then the SSA will run in polynomial time. As is shown in Figure 2, the number of sequential feasible solutions has experienced an exponential rapid rise and then its growth rate is under the trend of slowing down. Such a dynamic pattern of  $Q_n$  looks quite similar with the curve of production function [14] if we add a scaler  $\alpha$  to the complexity measurement

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**Algorithm 2** *Sequential – Search – Algorithm*( $\mathcal{A}$ )

---

**Require:**  $\mathcal{A}$ **Ensure:**  $\pi^*$  and  $S^*$ 

```
1: Step 1: Run Search – all – Sequential( $\mathcal{A}, 0, 0$ )
2: Define Function:  $\Pi \leftarrow \text{Search – all – Sequential}(\mathcal{A}, \varphi_o, C_o)$ 
3: Sort  $A_1, A_2, \dots, A_n$  in decreasing order of  $\varphi(A_i, C_o)$  (see Definition 1) to get a
   new sequence  $sa(1), \dots, sa(n)$ 
4: if  $n = 1$  then
5:   if  $\varphi(A_{sa(1)}, C_o) < \varphi_o$  or  $\varphi_o = 0$  then
6:      $\pi \leftarrow (sa(1))$ 
7:      $\Pi \leftarrow \{\pi\}$ 
8:   end if
9: end if
10: if  $n = 2$  then
11:   if  $\varphi(A_{sa(1)}, C_o) < \varphi_o$  or  $\varphi_o = 0$  then
12:      $\pi \leftarrow (sa(2), sa(1))$ 
13:      $\Pi \leftarrow \{\pi\}$ 
14:   end if
15: end if
16: if  $n > 2$  then
17:   for  $i := 1$  to  $n - 1$  do
18:     if  $\varphi(A_{sa(i)}, C_o) < \varphi_o$  or  $\varphi_o = 0$  then
19:       Let  $A_{sa(i)}$  run the farthest position in the current sequence
20:       for  $j := i + 1$  to  $n$  do
21:         if  $\varphi(A_{sa(i)}, C_o) \geq \varphi(A_{sa(j)}, C_o)$  then
22:            $\mathcal{A} \leftarrow \mathcal{A} \setminus \{A_{\pi(sa(i))}\}$ 
23:            $\varphi_o \leftarrow \varphi(A_{sa(i)}, C_o + c_{sa(i)})$ 
24:            $\Pi^j \leftarrow \text{Search – all – sequential}(\mathcal{A}, \varphi_o, C_o + c_{sa(i)})$ 
25:           for Each element  $k$  in  $\Pi^j$  do
26:             if  $k \geq sa(i)$  then
27:                $k \leftarrow k + 1$ 
28:             end if
29:           end for
30:            $\Pi \leftarrow \{(\Pi^j, sa(i))\}$ 
31:         end if
32:       end for
33:     end if
34:   end for
35: end if
36: Step 2: Run Bubble – sorting – Sequential( $\mathcal{A}, \Pi$ )
37: Define Function:  $(\pi^*, S^*) \leftarrow \text{Bubble – sorting – Sequential}(\mathcal{A}, \Pi)$ 
38:  $m \leftarrow$  the number of rows in  $\Pi$ 
39:  $S^* \leftarrow 0$ 
40: for  $i := 1$  to  $m$  do
41:    $S^o \leftarrow S_{\Pi(i)}$ , see eq. (1)
42:   if  $S^o > S^*$  then
43:      $S^* \leftarrow S_{\Pi(i)}$ 
44:      $\pi^* \leftarrow \Pi(i)$ 
45:   end if
46: end for
```

---

which adjust the slope of line  $OP$  equals to 1 in Figure 2. The growth rate of  $Q_n$  attains an utmost at point  $P$  when  $n$  equals to  $2m$ . The turning point of  $2m$  is regarded as an "inflection point": when  $n$  is less than  $2m$ , the upper bound of  $Q_n$  has risen sharply but when  $n$  is greater than  $2m$ , although the amount of possible sequential feasible solutions has been rising steadily, its growth rate is getting slowing down.

**Corollary 2.** *For a large scale of the worst case of APR instance with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ . There exists an input size of  $N = 2m$ , such that when  $n > N$ , Algorithm 2 runs in polynomial time.*

**Corollary 3.** *Suppose  $\mathcal{A}$  is a set of  $n$  airplanes with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $S \leq v_1/c_1 < \dots < v_n/c_n \leq L$ . There is an upper bound of index number  $\tilde{m}$  related to  $\mathcal{A}$ . For any ARP instance chosen from  $\mathcal{A}$ , and each airplane's flight distance is limited in the interval  $[S, L]$ , then its index number  $m$  must be no greater than  $\tilde{m}$ .*

*Proof.* Since  $c_1 \leq \dots \leq c_n$ , the cumulative sum of fuel consumption rates  $\mathcal{C}_m = \sum_{k=1}^m c_k$  must be a monotone increase function of  $m$ . Airplane  $A_n$  is the airplane with the largest  $v_n/c_n$  in the interval  $[S, L]$ .  $A_n$  takes precedence over airplane  $A_{n-1}$  when the cumulative sum of fuel consumption rates is greater than  $\mathcal{C}_{n-1,n}$ . Then Corollary 3 is verified if we set  $\tilde{m} = \mathcal{C}_{n-1,n}/c_1$ .  $\square$

The following Algorithm 3 is proposed to estimate the  $m$  and  $Q_n$ .

---

**Algorithm 3** *Estimate – SSA – Complexity*( $\mathcal{A}$ )

---

**Require:**  $\mathcal{A}$

**Ensure:**  $m$ , and  $Q_n$

- 1: Sort  $\mathcal{A}$  in decreasing order of  $\varphi(A_i, 0)$ , see Definition 1
  - 2:  $m \leftarrow 1$
  - 3:  $\mathcal{C}_m \leftarrow c_1$
  - 4:  $\mathcal{A} \leftarrow \{A_2, \dots, A_{n-1}\}$
  - 5: **while**  $m < n - 1$  and  $\varphi(A_n, \mathcal{C}_m) < \max\{\varphi(A_i, \mathcal{C}_m) \text{ and } A_i \in \mathcal{A}\}$  **do**
  - 6:      $m \leftarrow m + 1$
  - 7:      $\mathcal{C}_m \leftarrow \mathcal{C}_m + c_m$
  - 8:      $\mathcal{A} \leftarrow \mathcal{A} \setminus \{A_m, \dots, A_{n-1}\}$
  - 9:      $Q_n \leftarrow \frac{m^2}{n} \mathcal{C}_m^m$
  - 10: **end while**
- 

## 6 Efficient computability scheme for ARP

Given an instance of ARP, we could acknowledge within polynomial time how much time the SSA runs on it, according to which we choose a proper algorithm considering available running time. In Theorem 3, we claimed that the upper bound of  $Q_n$  is  $2^{n-2}$ . However, in Theorem 4 we proved that an instance is polynomial-time solvable when its input size of  $n$  is greater than  $2m$ . Here the index number  $m$  is regarded as an "inflection point" of the number of possible sequential feasible solutions: the

computational complexity of given instance grows exponentially for  $m < n \leq 2m$ , but grows polynomially for  $n > 2m$ . For this reason, to predict  $Q_n$ , we shall just find the index number  $m$ .

The main idea of efficient computability is that, given an ARP instance, especially for large scale of ARP instances, we shall forecast in polynomial time the particular computational complexity that the SSA runs on it, which provides useful information to decision makers or algorithm users before they try to solve it.

According to eq. (9) in section 5,  $Q_n$  is an aggregation of  $m$  parts, and each part is upper bounded by  $C_{n-2}^{i-1}$  for  $1 \leq i \leq m$ , here  $C_{n-2}^{i-1}$  is the maximal amount of potential sequential feasible solutions when there are  $i$  airplanes take precedence over the airplane with the greatest  $v_j/c_j$  for  $1 \leq j \leq n$ . We propose the following Algorithm 4 to get an approximate value of  $m$  by using heuristic method.

---

**Algorithm 4** *Heuristic – SSA – Complexity( $\mathcal{A}$ )*

---

**Require:**  $\mathcal{A}$

**Ensure:**  $m'$  and  $Q_n$

- 1: Sort  $\mathcal{A}$  in decreasing order of  $\varphi(A_i, 0)$ , see Definition 1
  - 2: **for**  $i := 1$  to  $n - 1$  **do**
  - 3:     Let  $A_i$  takes the farthest position
  - 4:     Running Algorithm 1 to get a sequence  $\pi^i$
  - 5:     Calculate  $S_{\pi^i}$  according to eq. (1)
  - 6: **end for**
  - 7:  $\pi^* \leftarrow \arg \max\{S_{\pi^i}, 1 \leq i < n\}$
  - 8: Find  $m'$  such that  $\pi^*(m' + 1) = \arg \max\{\frac{v_{\pi^*(i)}}{c_{\pi^*(i)}}\}$
  - 9:  $Q_n \leftarrow \frac{m'^2}{n} C_n^{m'}$
- 

We consider some possible situations in addition to pose a comprehensive mechanism of efficient computability:

- (1) Special case in Theorem 1 with  $v_1/c_1^2 \leq \dots \leq v_n/c_n^2$ , and  $v_1/c_1 \leq \dots \leq v_n/c_n$ . Such instances are easy to solve.
- (2) Not complete reverse order sequence. Such instances can be transformed into lower dimensional form which are relatively easy to solve.
- (3) Complete reverse order sequence including the following three categories:
  - (3-1) When  $m \ll n$ , such instances are easy to solve;
  - (3-2) When  $m \approx n/2$ , such instances are similar with worst case. But once  $m$  is fixed, and when  $n$  gets sufficiently large, the number of sequential feasible solutions changes to be less than  $\frac{m^2}{n} C_n^m$ ;
  - (3-3) When  $m \approx n$ , such instances are easy to solve at this case. But if we increase the input scale of  $n$  to  $2m$ , it changes to be above two cases.

## 7 Numerical illustration

### 7.1 Example 1

We build an ARP instance contains 60 airplanes with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ . The data of example 1 is partly displayed in Table 1.

**Table 1** Data of Example 1

$\mathcal{A}$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$	$A_{10}$	$A_{20}$	$A_{30}$	$A_{40}$	$A_{50}$	$A_{60}$
$v_i$	4	6.27	8.73	11.36	14.18	17.18	31	78.27	143.73	227.36	329.18	449.18
$c_i$	2	3	4	5	6	7	11	21	31	41	51	61

The index number of Example 1 is  $m = 6$  by running Algorithm 3. Suppose we choose different fleet of airplanes from the 60 airplanes in example 1 to get various ARP instances with different input size of  $n$ , we can calculate their number of sequential feasible solutions of  $Q_n$  according to part 1 in Algorithm 2. The different  $Q_n$  is displayed in Table 2.

**Table 2** Corresponding  $Q_n$  of Example 1

$n$	6	12	24	36	48	60
$Q_n$	1	399	2,323	4,374	6,355	8,406
$\frac{m^2}{2^{n-2}} C_n^m$	6	2,772	$2.02 \times 10^5$	$1.95 \times 10^6$	$9.20 \times 10^6$	$3.00 \times 10^7$
$2^{n-2}$	16	1024	$4.19 \times 10^6$	$1.72 \times 10^{10}$	$7.04 \times 10^{13}$	$2.88 \times 10^{17}$

### 7.2 Example 2

We build an ARP instance contains 1000 airplanes with  $v_1/c_1^2 > \dots > v_n/c_n^2$  and  $v_1/c_1 < \dots < v_n/c_n$ . The data of example 2 is partly displayed in Table 3.

**Table 3** Data of Example 2

$\mathcal{A}$	$A_1$	$A_{200}$	$A_{400}$	$A_{600}$	$A_{800}$	$A_{1000}$
$v_i$	1	11,899	39,809	83,711	143,610	219,490
$c_i$	1	3,981	7,981	11,981	15,981	19,981

By running Algorithm 3, we get  $m = 16$ . Similar with the curves in Figure 2, there are two types of growth rate of the computational complexity, one is in exponential way that is upper bounded by  $2^{n-2}$ , and the other is in polynomial way that is less than  $\frac{m^2}{n} C_n^m$ . To show the efficient computability scheme, we select the first  $2m, 10m, \dots$ ,

60m airplanes to compose an instance respectively, and calculate the index number  $m'$  by Algorithm 4. Numerical results are displayed in Table 4.

**Table 4** Comparison between different  $Q_n$ .

$n$	$m'$	$\frac{m'^2}{n} C_n^{m'}$	$\frac{m^2}{n} C_n^m$	$2^{n-2}$
2m	13	$1.83 \times 10^9$	$4.81 \times 10^9$	$1.07 \times 10^9$
10m	10	$1.42 \times 10^{15}$	$6.50 \times 10^{21}$	$3.65 \times 10^{47}$
20m	10	$8.41 \times 10^{17}$	$3.16 \times 10^{26}$	$5.34 \times 10^{95}$
30m	10	$3.39 \times 10^{19}$	$1.57 \times 10^{29}$	$7.80 \times 10^{143}$
40m	10	$4.63 \times 10^{20}$	$1.25 \times 10^{31}$	$1.14 \times 10^{192}$
50m	10	$3.50 \times 10^{21}$	$3.70 \times 10^{32}$	$1.67 \times 10^{240}$
60m	10	$1.82 \times 10^{22}$	$5.85 \times 10^{33}$	$2.44 \times 10^{288}$
1,000	10	$2.63 \times 10^{22}$	$1.08 \times 10^{34}$	$2.68 \times 10^{300}$

A further explanation of Theorem 4 and Corollary 3 is that given a relatively large set of airplanes, if we have found the index  $\tilde{m}$ , then for any  $n$ -airplane refueling instance drawn from the set of airplanes, the "inflection point" must be less than  $2\tilde{m}$ , and the computational complexity is less than  $\frac{\tilde{m}^2}{n} C_n^{\tilde{m}}$ . For example, we choose 500 airplanes from the 1000 airplanes, and we get  $m = 10$  (which is less than  $\tilde{m} = 16$ ). The computational complexity comparison results are presented in Table 5.

**Table 5** Complexity analysis for random instance

$n = 500$	upper bound of $Q_n$
$Q_n \leq 2^{n-2}$	$8.18 \times 10^{149}$
$Q_n < \frac{m^2}{n} C_n^m$ for $m = 16$	$2.93 \times 10^{29}$
$Q_n < \frac{m^2}{n} C_n^m$ for $m = 10$	$4.92 \times 10^{19}$
$Q_n < \frac{m'^2}{n} C_n^{m'}$ for $m' = 6$	$1.17 \times 10^{16}$

## 8 Conclusion

We proposed the SSA to solve the large scale of ARP instances, and posed an efficient computability scheme. We found that the computational complexity of the SSA running on ARP will decline from an exponential level to a polynomial level with the increasing input size of  $n$ . Thus we proved that for any ARP instance the running time of the SSA will grow at a polynomial rate when the input size of  $n$  overpasses the "inflection point". Moreover, we constructed an efficient computability scheme to forecast in polynomial time the particular running time of the SSA running on any given ARP instance, according to which we could provide computational strategy for decision makers and algorithm users.

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