Strange Correlation Functions for Average Symmetry-Protected Topological Phases

Jian-Hao Zhang,[1] Yang Qi,[2,3,4,*] and Zhen Bi[1,†]

[1] *Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, USA*
[2] *State Key Laboratory of Surface Physics, Fudan University, Shanghai 200433, China*
[3] *Center for Field Theory and Particle Physics, Department of Physics, Fudan University, Shanghai 200433, China*
[4] *Collaborative Innovation Center of Advanced Microstructures, Nanjing 210093, China*

Average symmetry-protected topological (ASPT) phase is a generalization of symmetry-protected topological phases to disordered systems or open quantum systems. We devise a "strange correlator" in one and two dimensions to detect nontrivial ASPT states. We demonstrate that for a nontrivial ASPT phase this strange correlator exhibits long-range or power-law behavior. We explore the connection between the strange correlators and correlation functions in two-dimensional loop models with quantum corrections, leading to the exact scaling exponents of the strange correlators.

arXiv:2210.17485v2 [cond-mat.str-el] 9 Apr 2024

Introduction – Symmetry-protected topological (SPT) phases host nontrivial short-range entanglement (SRE) which cannot be destroyed in the presence of symmetries [1–31]. Recently, it has been shown that symmetry-protected SRE can still prevail even if part of the protecting symmetry is broken locally by quenched disorder but restored upon ensemble averaging [32–38], which defines a new class of SPT phases dubbed average SPT (ASPT). The notion of ASPT also generalizes to mixed states arising naturally from the coupling between the system and the environment. Provided that symmetry-breaking disorders or quantum decoherence are unavoidable in experiments, the investigation of properties of ASPTs is of both theoretical and practical significance.

The nontrivial features of SPT phases often manifest on the physical boundaries which often show symmetry-protected gapless spectrum, while the bulk correlation functions of local observables all decay exponentially with distance due to the spectral gap. Therefore, the bulk detection of an SPT state is a nontrivial task if only a wavefunction without boundaries is available. One powerful tool is the *strange correlator* [39–41] defined for a given wavefunction $|\Psi\rangle$ as $C(r,r')=\frac{\langle\Psi|O(r)O(r')|\Psi_0\rangle}{\langle\Psi|\Psi_0\rangle}$. Here $|\Psi_0\rangle$ is a symmetric trivial state serving as a reference, and O's are certain local operators. It has been demonstrated that, given a nontrivial SPT wavefunction, the strange correlation is generically *long-range* or *power-law* in the long-distance limit. The strange correlator has been successfully applied to identify nontrivial SPT wavefunctions in numerical simulations [42–49].

This letter provides a generalization of the strange correlator to ASPT states. We highlight that the fidelity between two density matrices is a natural extension of wavefunction overlap from pure states to mixed states, which allows us to define a basis-independent form of strange correlator for ASPTs. For the rest of the paper, we briefly discuss the notion of ASPTs and define the strange correlator for ASPTs. Then we showcase the power of strange correlators with examples in $1d$ and $2d$. In $1d$, we show the strange correlator of the average cluster state with $\mathbb{Z}_2\times\mathbb{Z}_2^A$ symmetry ("A" denotes an average symmetry throughout this paper) is long-range ordered. In $2d$, we demonstrate two examples of bosonic ASPTs with the so-called $0d$-decoration and $1d$-decoration and one example of fermionic ASPT with $1d$-decoration. We uncover an intriguing connection between the strange correlator and so-called watermelon correlators in $O(n)$ loop models with quantum corrections coming from the decorated domain wall structure of the ASPT states, granting us exact scaling exponents for the strange correlators.

ASPT states – In the ASPT setting [32, 38], we consider the topological properties of a density matrix, which can be either the result of quantum decoherence on a pure state or the ensemble of ground states of disordered Hamiltonians. A mixed state can host two distinct types of symmetries. The exact symmetry K is a symmetry for each individual quantum trajectory or disordered Hamiltonian, while the average G symmetry is only a statistical symmetry of the ensemble. Mathematically, an exact symmetry acts on the density matrix as $U_k\rho=e^{i\theta_k}\rho$, while an average symmetry action is $U_a\rho U_a^\dagger=\rho$. For ASPT states, we demand the density matrix does not break these symmetries spontaneously. Therefore, each state in ρ is a short-range entangled state preserving K symmetry, and the correlation function of G-charged operators are short-ranged, namely $\mathrm{Tr}(\rho\phi_G^\dagger(r)\phi_G(r'))\sim e^{-|r-r'|/\xi}$, indicting the average symmetry is preserved in ρ.

A nontrivial ASPT refers to cases where the SRE properties of the ensemble of states cannot be removed without breaking the exact and average symmetries. Such nontrivial SRE in an ASPT can be captured by the decorated domain wall picture[50]. Essentially, give an n-dimensional symmetry defect of average symmetry G, one can decorate an n-dimensional SPT wavefunction of the exact K symmetry, which we refer to as nd-decoration pattern. The nontrivial decoration dictates that it is impossible to adiabatically connect the state to a trivial state without breaking the K symmetry.

Note that the exact symmetry K is always a normal subgroup of the full symmetry group. Namely, K and G fit into a short exact sequence, $1\to K\to\mathcal{G}\to G\to 1$. For a trivial group extension, the ASPT state can be shown to have a corresponding pure state SPT wavefunction with exact $K \times G$ symmetry. However, if the group extension is nontrivial, there can be ASPT states that cannot have a pure state SPT correspondence, a class of states dubbed *intrinsic ASPT phase* [38]. We stress that the strange correlator defined below can detect nontrivial ASPTs of both classes.

Strange correlator – The strange correlator for a density matrix ρ is defined as

$$C(r,r') = \frac{F\left(\rho, \phi(r)\phi(r')\rho_0\phi(r)\phi(r')\right)}{F(\rho,\rho_0)}, \tag{1}$$

where $F(\rho,\rho_0) = \mathrm{Tr}(\sqrt{\sqrt{\rho}\rho_0\sqrt{\rho}})$ is the fidelity of two density matrices ρ and ρ_0. ρ_0 here is a reference trivial state preserving the exact and average symmetry. It is easy to show that if ρ is a pure-state SPT, the strange correlator we defined comes back to the ordinary strange correlator for pure-state SPTs.

Note our definition of strange correlator requires no specific choice of basis. However, for a generic density matrix ρ with an exact symmetry K and an average symmetry G, it is convenient to use a G-breaking basis: $\rho = \sum_{\mathcal{D}} p_{\mathcal{D}} \left|\Psi_{\mathcal{D}}\right\rangle \left\langle \Psi_{\mathcal{D}}\right|$. Each $\left|\Psi_{\mathcal{D}}\right\rangle$ is a K symmetric but G-breaking wavefunction and the sum over $\mathcal{D}$ takes into account of all possible G symmetry breaking patterns. Physically, each $\left|\Psi_{\mathcal{D}}\right\rangle$ can be regarded as a particular quantum trajectory of decoherence processes or the ground state of a specific disordered Hamiltonians. For a nontrivial ASPT, each $\left|\Psi_{\mathcal{D}}\right\rangle$ has nontrivial K-SPT decorations on the symmetry defects of G. In this basis, we can choose the reference state as $\rho_0 = \sum_{\mathcal{D}} p_{\mathcal{D}} \left|\Phi_{\mathcal{D}}\right\rangle \left\langle \Phi_{\mathcal{D}}\right|$ where we use the same probability distribution for simplicity and each $\left|\Phi_{\mathcal{D}}\right\rangle$ is a trivial product state preserving the K symmetry. Now the strange correlator can be reformulated in the following form,

$$C(r,r') = \frac{\sum_{\mathcal{D}} p_{\mathcal{D}} \left| \left\langle \Psi_{\mathcal{D}}\right| \phi(r)\phi(r') \left|\Phi_{\mathcal{D}}\right\rangle \right|}{\sum_{\mathcal{D}} p_{\mathcal{D}} \left| \left\langle \Psi_{\mathcal{D}} \right| \Phi_{\mathcal{D}}\rangle \right|}. \tag{2}$$

In the following, we will show with examples that the strange correlator has long-ranged or power-law behavior at long distances if ρ describes a nontrivial ASPT.

1d example – The first example is the $1d$ averaged cluster state protected by $\mathbb{Z}_2 \times \mathbb{Z}_2^A$ symmetry. This ASPT can be obtained by decoherence on a pure state SPT with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The pure state SPT is stablized by the cluster Hamiltonian $H = -\sum_{j=1}^{2N} Z_{j-1}X_jZ_{j+1}$, which builds in the decorated domain wall structure. The two $\mathbb{Z}_2$ symmetries are defined on the odd and even sites, respectively, $\mathbb{Z}_2^{\mathrm{odd}} = \prod_{j\in \mathrm{odd}} X_j$, $\mathbb{Z}_2^{\mathrm{even}} = \prod_{j\in \mathrm{even}} X_j$. For simplicity, we consider strong measurements in the Z-basis on the even sites which breaks $\mathbb{Z}_2^{\mathrm{even}}$ down to average. In this case, we can write down the explicit density matrix states, $\rho = \sum_{\mathcal{D}} \frac{1}{2^N} |\Psi_{\mathcal{D}}\rangle\langle\Psi_{\mathcal{D}}|$, where $|\Psi_{\mathcal{D}}\rangle = \bigotimes_{j=1}^{N} |Z_{2j} = \sigma_{2j}^{\mathcal{D}}\rangle \otimes |X_{2j+1} = \sigma_{2j}^{\mathcal{D}}\sigma_{2j+2}^{\mathcal{D}}\rangle$, which follows the decorated domain wall picture. Taking the reference state with $|\Phi_{\mathcal{D}}\rangle = \bigotimes_{j=1}^{N} |Z_{2j} = \sigma_{2j}^{\mathcal{D}}\rangle \otimes |X_{2j+1} = 1\rangle$, where no charge is attached to the domain wall, we explicitly find the strange correlator of Z_i operators on even sites long-range correlated, namely $C_{ZZ}(i,j) = 1$[51]. In the supplementary[52], we show that the long-ranged behavior goes beyond the strong-measurement limit as long as the G symmetry is preserved on average.

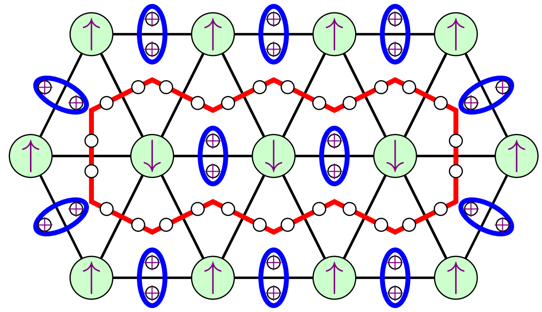


FIG. 1. Decorated domain wall states for bosonic ASPT with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^A$ symmetry and fermionic ASPT with $\mathbb{Z}_2^f \times \mathbb{Z}_2^A$ symmetry. Ising spins associated with $\mathbb{Z}_2^A$ symmetry are on the triangular lattice sites (green). For the boson case, each link hosts two spin-1/2's. A $1d$ cluster state (red) is attached to each Ising domain wall, while in a blue circle, the spins are polarized to $|+\rangle$ states. For fermionic ASPT, the link spins are replaced by Majorana fermions, and the decorated states are replaced by $1d$ Kitaev chains.

2d examples – We consider a $2d$ bosonic ASPT with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^A$ symmetry. The ASPT is described by decorating the domain wall of the average $\mathbb{Z}_2^A$ symmetry with a $1d$ cluster state protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ exact symmetry as pictorially shown in Fig. 1. This ASPT exists for both decohered and disordered systems.

We first consider the strange correlator of the Z operators of spins on the links, namely

$$C_K(r,r') = \frac{F\left(\rho, Z_rZ_{r'}\rho_0Z_rZ_{r'}\right)}{F(\rho,\rho_0)} = \frac{\sum_{\mathcal{D}} p_{\mathcal{D}} |\langle\Psi_{\mathcal{D}}|Z_rZ_{r'}|\Phi_{\mathcal{D}}\rangle|}{\sum_{\mathcal{D}} p_{\mathcal{D}} |\langle\Psi_{\mathcal{D}}|\Phi_{\mathcal{D}}\rangle|}. \tag{3}$$

The subscript K denotes that we consider operators that transform nontrivially under K. The denominator of Eq. (3) is a sum over configurations of the Ising spins, labeled by $\mathcal{D}$. Without loss of generality, we assume the probability $p_{\mathcal{D}}$ the form of the Boltzmann weight of a classical Ising model, or equivalently $p_{\mathcal{D}} = 2\prod_{l\in\mathcal{D}} x^{-L(l)}$, where $x = e^{-2\beta}$ [53]. Here, l labels a domain wall in the Ising configurations and $L(l)$ its length. For the ensemble to be in a $\mathbb{Z}_2^A$ symmetric phase, we need $x > x_c^{\mathrm{Ising}} = 1/\sqrt{3}$ for a triangular lattice. In each configuration, we have the factor of wavefunction overlap which can be decomposed into the product of overlap between a cluster state and a trivial state on each domain wall,

$$\langle\Psi_{\mathcal{D}}|\Phi_{\mathcal{D}}\rangle = \prod_{l\in\mathcal{D}} \langle\psi_{\mathrm{cluster}}|\psi_0\rangle(l) = \prod_{l\in\mathcal{D}} 2\times 2^{-L(l)}. \tag{4}$$

Here the overlap is calculated with the fixed point wave-

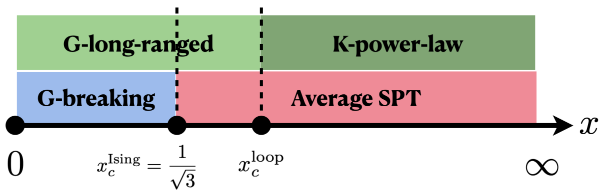


FIG. 2. Phase diagram of the strange correlator. The ASPT is well defined in $x > x_c^{\text{Ising}}$. For $x < x_c^{\text{Ising}}$, the G symmetry is spontaneously broken.

function of the cluster state and trivial state [52]. Each overlap factor decays with the length of the domain wall. The decay rate generally is not universal, while the factor of 2 in front of the exponential decay is *universal* – it is related to the degeneracy of the boundary modes of the decorated $1d$ SPT. We show in the supplementary materials[52] that this factor stays the same for SPT wavefunction away from the fixed point using both field theory and matrix product representations for $1d$ SPT. This factor is crucial for the behavior of strange correlators. Eventually, the denominator can be written as

$$F(\rho,\rho_0) = 2\sum_{\mathcal{D}} \tilde{x}^{-L(\mathcal{D})} 2^{n(\mathcal{D})}, \tag{5}$$

where $L(\mathcal{D})$ is the total length of domain wall, $n(\mathcal{D})$ is the number of domain walls in configuration $\mathcal{D}$, and $\tilde{x} = x/2$ is the renormalized loop tension. Eq. (5) resembles the partition function of an $O(n)$ loop model with loop fugacity $n = 2$ [52]. We emphasize that both the loop fugacity and tension have received nontrivial *quantum corrections* from the decorated domain wall states.

For the numerator, the crucial observation is that $\langle \Psi_{\mathcal{D}} | Z_r Z_{r'} | \Phi_{\mathcal{D}} \rangle$ is non-zero only if the two measured spins reside on the same domain wall. The non-zero value is precisely the strange correlator of the $1d$ cluster state times the factor of wavefunction overlaps. Therefore, the strange correlator, in the end, can be written as

$$C_K(r,r') = \frac{\sum_{\mathcal{D}'} \langle Z_r Z_{r'} \rangle_S \tilde{x}^{-L(\mathcal{D}')} 2^{n(\mathcal{D}')}}{\sum_{\mathcal{D}} \tilde{x}^{-L(\mathcal{D})} 2^{n(\mathcal{D})}}, \tag{6}$$

conditioned on that in every configurations of $\mathcal{D}'$ there must be a domain wall connecting r and r'. The factor $\langle Z_r Z_{r'} \rangle_S$ is a non-zero constant as the strange correlator of the $1d$ cluster state. This quantity maps to the 2-leg watermelon correlator in the $O(2)$ loop model [54] [55]. For the loop tension $\tilde{x} > x_c^{n=2} = 1/\sqrt{2}$, the loop model will be in the dense loop phases, which indicates the strange correlator has a power-law behavior:

$$C_K(r,r') \sim |r - r'|^{-2\Delta_2}, \tag{7}$$

where Δ_2 is known as the 2-leg exponent whose value in the $O(2)$ loop model is $\Delta_2 = 1/2$.

A careful reader may notice that, if we start from the bare loop tension $x \in [1/\sqrt{3}, \sqrt{2}]$, then the renormalized loop tension $\tilde{x}$ is not in the dense loop phase. In fact, loop tension in this regime will flow to 0 in the infrared and the system prefers to have no loops, also known as the dilute loop phase. Therefore, the strange correlator defined in Eq. (3) decays exponentially with distance. However, this does not indicate that the strange correlator fails to detect the ASPT order because we also need to include the strange correlator associated with operators that transform nontrivially under G symmetry, namely

$$C_G(r,r') = \frac{F(\rho, \sigma_r \sigma_{r'} \rho_0 \sigma_r \sigma_{r'})}{F(\rho,\rho_0)}, \tag{8}$$

where σ's are the Ising spins on the sites of the triangular lattice. Note if we were to put in the original ASPT density matrix here to calculate the correlator, then it would be short-ranged per the definition of ASPT. However, it is no longer the case since we are dealing with the strange density matrix which is quantum-corrected to an $O(2)$ loop model. The strange correlator in fact measures the probability of two points sitting in the same domain. In the dilute loop phase where loops are suppressed, this correlator is actually long-ranged,

$$C_G(r,r') \sim \text{const}, \quad \text{for} \quad \tilde{x} < 1/\sqrt{2}. \tag{9}$$

Therefore, strange correlators are either long-ranged or power-law in the whole regime where ASPT is well defined, as shown in the phase diagram in Fig. 2 [56].

We also construct an example of fermionic ASPT in $2d$ with $1d$-decoration. The example is the averaged version of $2d$ fermionic SPT with unitary $\mathbb{Z}_2$ symmetry [57]. The decorated domain wall wavefunction is similar to the boson case shown in Fig. 1. On the links of the triangular lattice, we put 2 Majorana modes, labeled χ_A and χ_B, forming a complex fermion $c = (\chi_A + i\chi_B)/2$. For Majorana modes on an Ising domain wall, they form a $1d$ topological superconductor[58–61]. Otherwise, they pair up locally to even parity states.

Consider the strange correlator of the c fermions,

$$C_K(r,r') = \frac{F(\rho, c(r)c(r')\rho_0 c^\dagger(r')c^\dagger(r))}{F(\rho,\rho_0)}. \tag{10}$$

The essential difference from the previous case is the quantum correction of the loop fugacity from the SPT-trivial wavefunction overlap on the domain walls. Indeed, we can show for the fixed point wavefunction, the overlap has the form $\langle \Psi_{\mathcal{D}} | \Phi_{\mathcal{D}} \rangle = \prod_{l \in \mathcal{D}} \langle \psi_{\text{Majorana}} | \psi_0 \rangle (l) = \prod_{l \in \mathcal{D}} \sqrt{2} \times \sqrt{2}^{-l}$ [52]. This means that the loop model now has loop fugacity $n = \sqrt{2}$. For the numerator, similar to the previous case, the only configurations that are non-zero need to have one domain wall going through the two fermion positions. Therefore, the strange correlator reduces to

$$C_K(r,r') = \frac{\sum_{\mathcal{D}'} \langle c_r c_{r'} \rangle_S \tilde{x}^{-L(\mathcal{D}')} \sqrt{2}^{n(\mathcal{D}')}}{\sum_{\mathcal{D}} \tilde{x}^{-L(\mathcal{D})} \sqrt{2}^{n(\mathcal{D})}}, \tag{11}$$

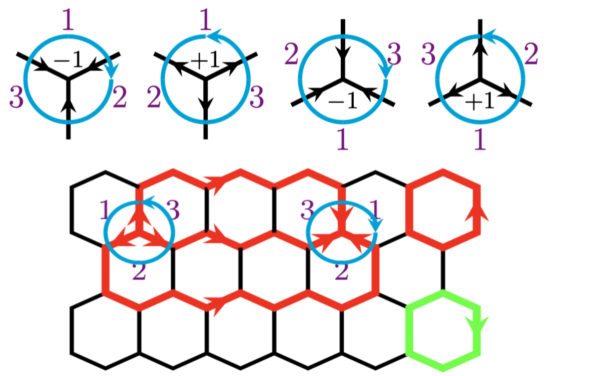


FIG. 3. Top: the charge decoration rule for the $\mathbb{Z}_3 \times \mathbb{Z}_3^A$ symmetry. A $\pm 2\pi$ vortex of $\mathbb{Z}_3^A$ is decorated with ± 1 charge of $\mathbb{Z}_3$. Bottom: The allowed configuration in the numerator of the strange correlator in Eq. (12).

where $\langle c_r c_{r'} \rangle_S$=const as the strange correlator of the $1d$ Kitaev chain. This strange correlator maps to the 2-leg watermelon correlator in the $O(n = \sqrt{2})$ model. The behavior of such a correlator depends on the value of $\tilde{x} = x/\sqrt{2}$. For $\tilde{x} > x_c^{n=\sqrt{2}} \cong 0.601$, the loop model falls into the dense loop fixed point, where the correlator has a similar power-law behavior as in Eq. (7) with an exponent $\Delta_2 = 1/3$. For $\tilde{x} = x_c^{n=\sqrt{2}}$, the loop model is at the dilute fixed point, where the exponent becomes $\Delta_2 = 3/5$. For $\tilde{x} < x_c^{n=\sqrt{2}}$, the loop model is no longer critical and the K-strange correlator becomes short-ranged.

In the regime $\tilde{x} < x_c^{n=\sqrt{2}}$ or $x < \sqrt{2} x_c^{n=\sqrt{2}}$, we can consider the G-strange correlator defined as Eq. (8). Since in this regime, the loop model is in a dilute loop phase, the G-strange correlator is again long-range ordered.

After considering $2d$ cases with $1d$-decoration, we give an example of $2d$ decohered bosonic ASPT with $0d$-decoration with $\mathbb{Z}_3 \times \mathbb{Z}_3^A$ symmetry. This ASPT has a structure in that a nontrivial $\mathbb{Z}_3$ is attached on each vortex of the $\mathbb{Z}_3^A$ order parameter as shown in Fig. 3. The ensemble of states consistent with the decoration rule defines a $\mathbb{Z}_3 \times \mathbb{Z}_3^A$ ASPT.

We consider the strange correlation of creation and annihilation operators of $\mathbb{Z}_3$ charges, namely,

$$C_K(r, r') = \frac{F(\rho, a^\dagger(r) a(r') \rho_0 a^\dagger(r') a(r))}{F(\rho, \rho_0)}. \tag{12}$$

We specify that the reference state has 0 charges decorated on the $\mathbb{Z}_3^A$ vortices. $\mathcal{D}$ again represents all different configurations of $\mathbb{Z}_3^A$ order parameter. In particular, there will be domain wall and vortex configurations. However, if $\mathcal{D}$ has vortices, then the overlap function between the trivial and SPT wavefunction in the denominator will be identically zero due to the nontrivial $\mathbb{Z}_3$ global charge decorated on the vortex. Therefore, all the vortex configurations are killed in the summation and we end up again with a loop model. Note that there are two flavors of loops since there are two different kinds of domain walls in a $\mathbb{Z}_3$ model. Here, without loss of generality, we assume the probability distribution is given by a thermal weight of a $\mathbb{Z}_3$ clock model [62]. As a result, the loop tensions for the two kinds of domain walls are the same. The denominator can be written as

$$F(\rho, \rho_0) = 3 \sum_{\mathcal{D}'} x^{-L(\mathcal{D}')} 2^{n(\mathcal{D}')}, \tag{13}$$

where $\mathcal{D}'$ only contain loop configurations and the overall factor of 3 comes from the $\mathbb{Z}_3$ symmetry. Eq. (13) again maps to the partition function of an $O(2)$ loop model.

By the same logic, the summation in the numerator of Eq. (12) should also contain only loop configurations except there should be a test vortex and a test anti-vortex right at position r and r' respectively due to the charge creation and annihilation operators in the definition of Eq. (12). We demonstrate examples of allowed configuration in Fig. 3. In this case, the strange correlator maps to the 3-leg watermelon correlation of the $O(2)$ model, which gives

$$C_K(r, r') \sim |r - r'|^{-2\Delta_3}, \tag{14}$$

where $\Delta_3 = 9/8$ for $x > x_c^{n=2} = 1/\sqrt{2}$.

Again, the same argument as before shows that, if the loop tension is $x < 1/\sqrt{2}$, then the loop model is in the dilute loop phase and we can measure the analogous G-strange correlator to find long-range order.

Conclusion and Discussion – In this work, we devise a bulk diagnosis that shows long-range or power-law behavior for nontrivial ASPT phases. In $2d$, all the strange correlators considered here map to correlation functions in certain $2d$ loop models. Remarkably, the decorated domain wall states play a peculiar role in determining the loop tension and loop fugacity. The nontrivial role of decorated domain wall states in strange correlators was noticed in Ref. [41] at a more fine-tuned point of a clean SPT wavefunction. For intrinsic ASPTs, the structure of the strange correlator and the mapping to the statistical models are essentially the same. Thus, we expect our strange correlator can detect intrinsic ASPT as well.

The generalization of the strange correlator to higher dimensional ASPTs is an exciting future direction. The map from the strange correlator to the statistical model is not limited to $2d$. For $3d$ ASPT with $1d$-decoration, the resulting statistical model will be $3d$ loop models for which some analytical results are also known [63–65]. For $2d$-decoration, for example, a $3d$ ASPT phase with $K = \mathbb{Z}_2$ and $G = \mathbb{Z}_2^A$ from decorating a 2D Levin-Gu state on the codimension-1 $\mathbb{Z}_2^A$ domain wall, we imagine the strange correlator can be mapped to a correlation function in $3d$ membrane models. However, how exactly the nature of the decorated $2d$ state affects the resulting membrane model is not clear.

Acknowledgements – We thank Yimu Bao, Jing-Yu Zhao, Shuo Yang, Chong Wang, Adrian Po, Meng Cheng, and Carolyn Zhang for stimulating discussions, and especially Adam Nahum for discussing the loop model and their critical behaviors. JHZ and ZB are supported by startup funds from the Pennsylvania State University. Y.Q. acknowledges support from the National Natural Science Foundation of China (Grant No. 12174068).

Note Added: While finishing up this work, we became aware of an independent work [66] which considers other generalizations of strange correlators for ASPT states.

---

* qiyang@fudan.edu.cn
† zjb5184@psu.edu

[1] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[2] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[3] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[4] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[5] T. Senthil, Annu. Rev. Condens. Matter Phys. **6**, 299 (2015).
[6] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[7] Q.-R. Wang and Z.-C. Gu, Phys. Rev. X **10**, 031055 (2020), arXiv:1811.00536 [cond-mat.str-el].
[8] A. Kapustin and R. Thorngren, Journal of High Energy Physics **2017**, 80 (2017).
[9] M. Cheng, Z. Bi, Y.-Z. You, and Z.-C. Gu, Phys. Rev. B **97**, 205109 (2018).
[10] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[11] M. Cheng, N. Tantivasadakarn, and C. Wang, Phys. Rev. X **8**, 011054 (2018).
[12] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[13] C. Wang and T. Senthil, Phys. Rev. B **87**, 235122 (2013).
[14] C. Wang, A. C. Potter, and T. Senthil, Phys. Rev. B **88**, 115137 (2013).
[15] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
[16] C. Wang and T. Senthil, Phys. Rev. B **89**, 195124 (2014).
[17] L. Fu, Phys. Rev. Lett. **106**, 106802 (2011).
[18] T. H. Hsieh, H. Lin, J. Liu, W. Duan, A. Bansil, and L. Fu, Nat. Commun. **3**, 982 (2012).
[19] H. Isobe and L. Fu, Phys. Rev. B **92**, 081304(R) (2015).
[20] H. Song, S.-J. Huang, L. Fu, and M. Hermele, Phys. Rev. X **7**, 011020 (2017).
[21] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, Phys. Rev. B **96**, 205106 (2017).
[22] R. Thorngren and D. V. Else, Phys. Rev. X **8**, 011040 (2018).
[23] H. C. Po, A. Vishwanath, and H. Watanabe, Nature Communications **8**, 50 (2017).
[24] M. Cheng and C. Wang, arXiv:1810.12308 [cond-mat.str-el].
[25] J.-H. Zhang, Q.-R. Wang, S. Yang, Y. Qi, and Z.-C. Gu, Phys. Rev. B **101**, 100501(R) (2020).
[26] J.-H. Zhang, S. Yang, Y. Qi, and Z.-C. Gu, Phys. Rev. Research 4, 033081 (2022) (2020), 10.1103/PhysRevResearch.4.033081, arXiv:2012.15657 [cond-mat.str-el].
[27] F. Tang, H. C. Po, A. Vishwanath, and X. Wan, Nature **566**, 486 (2019).
[28] J.-H. Zhang, Y. Qi, and Z.-C. Gu, (2022), arXiv:2204.13558 [cond-mat.str-el].
[29] J.-H. Zhang, Physical Review B **106**, l020503 (2022).
[30] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, Physical Review X **7**, 041069 (2017).
[31] R.-J. Slager, A. Mesaros, V. Juričić, and J. Zaanen, Nature Physics **9**, 98 (2012).
[32] R. Ma and C. Wang, (2022), arXiv:2209.02723 [cond-mat.str-el].
[33] Z. Ringel, Y. E. Kraus, and A. Stern, Physical Review B **86**, 045102 (2012).
[34] R. S. K. Mong, J. H. Bardarson, and J. E. Moore, Physical Review Letters **108**, 076804 (2012).
[35] I. C. Fulga, B. van Heck, J. M. Edge, and A. R. Akhmerov, Physical Review B **89**, 155424 (2014).
[36] Y.-Z. Chou, R. M. Nandkishore, and L. Radzihovsky, Physical Review B **98**, 054205 (2018).
[37] Y.-Z. Chou and R. M. Nandkishore, Physical Review B **103**, 075120 (2021).
[38] R. Ma, J.-H. Zhang, Z. Bi, M. Cheng, and C. Wang, (2023), arXiv:2305.16399 [cond-mat.str-el].
[39] Y.-Z. You, Z. Bi, A. Rasmussen, K. Slagle, and C. Xu, Physical Review Letters **112**, 247202 (2014).
[40] L. Lepori, M. Burrello, A. Trombettoni, and S. Paganelli, (2022), arXiv:2209.04283 [cond-mat.str-el].
[41] T. Scaffidi and Z. Ringel, Physical Review B **93**, 115105 (2016).
[42] H.-Q. Wu, Y.-Y. He, Y.-Z. You, C. Xu, Z. Y. Meng, and Z.-Y. Lu, Physical Review B **92**, 165123 (2015).
[43] R. Vanhove, M. Bal, D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete, Physical Review Letters **121**, 177203 (2018).
[44] C. Zhou, M.-Y. Li, Z. Yan, P. Ye, and Z. Y. Meng, (2022), arXiv:2209.12917 [cond-mat.str-el].
[45] Y.-Y. He, H.-Q. Wu, Y.-Z. You, C. Xu, Z. Y. Meng, and Z.-Y. Lu, Physical Review B **93**, 115150 (2016).
[46] K. Wierschem and P. Sengupta, Physical Review Letters **112**, 247203 (2014).
[47] R. Vanhove, L. Lootens, H.-H. Tu, and F. Verstraete, Journal of Physics A: Mathematical and Theoretical **55**, 235002 (2022).
[48] K. Wierschem and K. Beach, Physical Review B **93**, 245141 (2016).
[49] K. Wierschem and P. Sengupta, Physical Review B **90**, 115157 (2014).
[50] X. Chen, Y.-M. Lu, and A. Vishwanath, Nature Communications **5** (2014), 10.1038/ncomms4507.
[51] The strange correlator of the Z operators on the odd sites is also long-range correlated as shown in the supplementary [52]. We also make connection between the strange correlator and the string order parameter in the supplementary [52].
[52] see Supplementary Materials for more details.
[53] The most natural way to achieve a $\mathbb{Z}_2$ average symmetry is the thermalization of the Ising degrees of freedom, i.e., the probability distribution of Ising degrees of freedom is chosen to be by a Boltzmann distribution in the paramagnetic phase of the 2d classical Ising model. In general, for a $2d$ classical Ising model in the paramagnetic phase, we can define a "loop tension" x that is the energy per unit length of the Ising domain wall. With the loop ten-

sion, the probability for an Ising domain wall with length L should be x^{-L}.
[54] B. Duplantier, Physics Reports **184**, 229 (1989).
[55] The K strange correlator measures the probability that r and r' are connected through a loop of $O(2)$ loop model. This is exactly the 2-leg watermelon correlator in the $O(2)$ loop model.
[56] We note that the long-range behavior of the G strange correlator should be distinguished from the G symmetry-breaking state or a true topological order. For $x < x_c^{\text{Ising}}$ which corresponds to the G symmetry-breaking long-range order, both the ordinary and strange correlator of G degrees of freedom have long-range behavior; for $x_c^{\text{Ising}} < x < x_c^{\text{loop}}$ which corresponds to the ASPT phase, the ordinary correlator of G should be short-ranged but the strange correlator of G is long-ranged.
[57] A. Kitaev, Annals of Physics **321**, 2 (2006), january Special Issue.
[58] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011).
[59] Z. Wang, S.-Q. Ning, and X. Chen, Phys. Rev. B **98**, 094502 (2018).
[60] B. Ware, J. H. Son, M. Cheng, R. V. Mishmash, J. Alicea, and B. Bauer, Phys. Rev. B **94**, 115127 (2016).
[61] N. Tarantino and L. Fidkowski, Phys. Rev. B **94**, 115115 (2016).
[62] Similar to the $\mathbb{Z}_2$ domain wall, the choice of the probability distribution of $\mathbb{Z}_3$ domain walls is not unique. However, different choices have different domain wall tension x that can be unified to Eq. (13).
[63] J. Cardy, Annals of Physics **318**, 81 (2005).
[64] A. Nahum, J. T. Chalker, P. Serna, M. Ortuño, and A. M. Somoza, Physical Review Letters **107**, 110601 (2011).
[65] A. Nahum, *Critical Phenomena in Loop Models* (Springer International Publishing, 2015).
[66] J. Y. Lee, Y.-Z. You, and C. Xu, (2022), arXiv:2210.16323 [cond-mat.str-el].
[67] B. Nienhuis, *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, London, 1987).
[68] C. Xu and T. Senthil, Physical Review B **87**, 174412 (2013).

Supplemental Materials for "Strange Correlation Function for Average Symmetry-Protected Topological Phases"

Strange correlator of 1d average cluster state

In this section, we explicitly calculate the strange correlator of the 1d average cluster state. The stabilizer for a pure 1d cluster state is

$$H = -\sum_{j=1}^{2N} Z_{j-1} X_j Z_{j+1}, \tag{S1}$$

where Z and X are Pauli matrices that are anti-commute. There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry that is defined on the odd and even sites, respectively, with the generators as

$$\mathbb{Z}_2^{\text{odd}} = \prod_{j\in\text{odd}} X_j,\ \mathbb{Z}_2^{\text{even}} = \prod_{j\in\text{even}} X_j. \tag{S2}$$

Then we perform strong Z measurements on the even sites which can be done by coupling each qubit on an even site with an ancilla qubit through a controlled-Z gate and then tracing out the ancilla. By recording all measurement outcomes, the $\mathbb{Z}_2^{\text{even}}$ symmetry is broken to an average symmetry $\mathbb{Z}_2^A$ in the resulting mixed state. The mixed state is an ensemble of states, each of which is given by the form

$$|\Psi_{\mathcal{D}}\rangle = \bigotimes_{j=1}^{N} |Z_{2j} = \sigma_{2j}^{\mathcal{D}}\rangle \otimes |X_{2j+1} = \sigma_{2j}^{\mathcal{D}} \sigma_{2j+2}^{\mathcal{D}}\rangle, \tag{S3}$$

where $\{\sigma_{2j}\}$ are measurement outcomes. We can see that for each specific measurement outcome, the corresponding wavefunction has an explicitly decorated domain wall structure. The trivial reference wavefunctions are chosen as following

$$|\Phi_{\mathcal{D}}\rangle = \bigotimes_{j=1}^{N} |Z_{2j} = \sigma_{2j}^{\mathcal{D}}\rangle \otimes |X_{2j+1} = 1\rangle, \tag{S4}$$

where there is no charge decorated to the domain wall.

In the strong measurement limit, we have equal probability for each state. Here, we can consider a generalized probability distribution, $p_{\mathcal{D}} \sim e^{-n_{\text{DW}}\beta J}$, where n_{DW} is the number of domain walls of even spins, and J can be thought of as the energy cost of creating a domain wall. This is a Boltzmann factor of a 1d classical Ising model at finite temperature. As long as $\beta < \infty$, the Ising model is in a disordered phase, thus, the ensemble will preserve the average $\mathbb{Z}_2$ symmetry. With this general probability distribution, we can calculate the strange correlator of the Z operators on the even sites,

$$C_{ZZ}(i,j) = \frac{F(\rho, Z_i Z_j \rho_0 Z_i Z_j)}{F(\rho, \rho_0)}, \tag{S5}$$

Firstly we consider the denominator, which is composed of wavefunction overlaps as $F(\rho,\rho_0) = \sum_{\mathcal{D}} p_{\mathcal{D}} |\langle \Psi_{\mathcal{D}} | \Phi_{\mathcal{D}}\rangle|$ in the decorated domain wall basis. We note that according to the explicit expressions in Eqs. (S3) and (S4), the wavefunction overlap $\langle \Psi_{\mathcal{D}} | \Phi_{\mathcal{D}}\rangle$ is nonzero if and only if $\sigma_{2j}^{\mathcal{D}} \sigma_{2j+2}^{\mathcal{D}} = 1$, $\forall j$. It turns out that the denominator of the strange correlator should be

$$F(\rho,\rho_0) = 2e^{-\beta E_0}, \tag{S6}$$

where E_0 is the ground state energy of the 1d Ising model of the qubits on even sites.

Then we focus on the numerator of the strange correlator. We first consider Z_i and Z_j operators on the even sites. The Z's will extract the measurement outcomes $\sigma_i^{\mathcal{D}}$ and $\sigma_j^{\mathcal{D}}$ on the sites i and j. It turns out that the numerator of the strange correlator should be

$$F(\rho, Z_i Z_j \rho_0 Z_i Z_j) = 2e^{-\beta E_0}. \tag{S7}$$

Therefore, the strange correlator $C_{ZZ}(i,j) = 1$ for general $\beta < \infty$, which implies that the decohered average 1d cluster state is a nontrivial ASPT state.

We can also derive the nontrivial strange correlator from the string order parameter. Suppose $i = 2i'$ and $j = 2j'$, we consider the string order parameter to be defined as

$$S_1 = Z_{2i'} \left(\prod_{k=i'}^{j'-1} X_{2k+1} \right) Z_{2j'}, \tag{S8}$$

where the operator in the bracket is the truncated exact symmetry operator. For any $\beta < \infty$, it is easy to check the expectation value of the above string order parameter is equal to 1,

$$\text{Tr}\left(\rho S_1\right) = 1. \tag{S9}$$

This is because that each wavefunction $|\Psi_{\mathcal{D}}\rangle$ in ρ is invariant by acting the string order parameter. Now consider each term in the numerator of the strange correlator

$$\langle \Psi_{\mathcal{D}} | Z_{2i'} Z_{2j'} | \Phi_{\mathcal{D}} \rangle = \langle \Psi_{\mathcal{D}} | Z_{2i'} \left(\prod_{k=i'}^{j'-1} X_{2k+1} \right) Z_{2j'} | \Phi_{\mathcal{D}} \rangle, \tag{S10}$$

because $|\Phi_{\mathcal{D}}\rangle$ is invariant under the truncated symmetry operator. Furthermore, each wavefunction $|\Psi_{\mathcal{D}}\rangle$ is invariant under the string operator. Therefore, we have

$$\langle \Psi_{\mathcal{D}} | Z_{2i'} Z_{2j'} | \Phi_{\mathcal{D}} \rangle = \langle \Psi_{\mathcal{D}} | \Phi_{\mathcal{D}} \rangle. \tag{S11}$$

With this, we can see the strange correlator equals 1 for any $\beta < \infty$.

Now we consider the strange correlator of Z_{2i-1} and Z_{2j+1}. We can show explicitly the strange correlator is long-range correlated in this case as well. We will make a connection with a fidelity version of the string order parameter. The denominator of the strange correlator is the same as in Eq. (S6). The numerator of (S5) can be calculated explicitly: Z_{2i-1}/Z_{2j+1} flips the X eigenvalues at the site-$(2i-1)/(2j+1)$, therefore, the wavefunction overlap $\langle\Psi_{\mathcal{D}}|Z_{2i-1}Z_{2j+1}|\Phi_{\mathcal{D}}\rangle$ is nonzero if and only if there were exactly two domain walls, $\sigma_{2i-2}\sigma_{2i} = -1$ and $\sigma_{2j}\sigma_{2j+2} = -1$. It turns out that the numerator of (S5) should be

$$F(\rho, Z_{2i-1}Z_{2j+1}\rho_0 Z_{2i-1}Z_{2j+1}) = 2e^{-\beta(E_0+2J)}, \quad \text{(S12)}$$

and thus the strange correlator $C_{ZZ}(2i-1, 2j+1) = e^{-2\beta J}$ is finite at any $\beta < \infty$. In particular, at $\beta = 0$, the strange correlator is 1. Therefore, we can determine a nontrivial ASPT density matrix by the strange correlator of both the average and exact degrees of freedom.

Next we consider the connection with the string order parameter associated with the average symmetry. We know that the usual string order parameter defined by $\mathrm{Tr}\,(\rho S_2)$, with

$$S_2 = Z_{2i-1}\left(\prod_{k=i}^{j} X_{2k}\right) Z_{2j+1}, \quad \text{(S13)}$$

decays exponentially with distance[32]. However, we can consider a fidelity version of the string order parameter, defined as

$$F\left(\rho, S_2\rho S_2\right). \quad \text{(S14)}$$

This string order parameter can be shown to be exactly equal to 1 when we consider the fixed-point density matrix. In particular, we can show if the string order parameter is 1, the strange correlator has to be 1 as well. We can see from the following argument. In the decorated domain wall basis, it is easy to show that the explicit form of Eq. (S14) is expressed as

$$F(\rho, S_2\rho S_2) = \sum_{\mathcal{D}} \sqrt{p_{\mathcal{D}} p_{\mathcal{D}+2}}, \quad \text{(S15)}$$

where $\mathcal{D}+2$ and $\mathcal{D}$ are different by two domain walls at $2i$ and $2j$. Furthermore, by utilizing the mean inequalities,

$$\sum_{\mathcal{D}} \sqrt{p_{\mathcal{D}} p_{\mathcal{D}+2}} \leq \sum_{\mathcal{D}} \frac{p_{\mathcal{D}} + p_{\mathcal{D}+2}}{2} = 1, \quad \text{(S16)}$$

where the equality is taken if and only if $p_{\mathcal{D}+2} = p_{\mathcal{D}}$ for $\forall\mathcal{D}$. On the other hand, the strange correlator $C(2i-1, 2j+1) = \sqrt{p_{\mathcal{D}_0+2}/p_{\mathcal{D}_0}}$ where $\mathcal{D}_0$ is the spin configuration with no domain wall. Hence the strange correlator is equal to 1 when the string order parameter is 1.

Strange correlators of ASPT and clean SPT

In this section, we show that the strange correlator of ASPT phases is equivalent to that of SPT in the pure state assuming the ASPT phase has a clean limit.

As mentioned in the introduction, for clean SPT the strange correlator is defined as

$$C(r, r') = \frac{\langle\Omega|\phi(r)\phi(r')|\Psi\rangle}{\langle\Omega|\Psi\rangle}, \quad \text{(S17)}$$

where $\phi(r)$ is some local operator at the position r, $|\Psi\rangle$ is the wavefunction of the SPT phase, and $|\Omega\rangle$ is the wavefunction of a trivial product state. For ASPT the strange correlator is defined as

$$C(r, r') = \frac{F\left(\rho, \phi(r)\phi(r')\rho_0\phi^\dagger(r')\phi^\dagger(r)\right)}{F(\rho, \rho_0)}. \quad \text{(S18)}$$

where ρ_0 is the reference trivial density matrix with the same symmetry class, and $F(\rho, \rho_0) = \mathrm{Tr}\sqrt{\sqrt{\rho}\rho_0\sqrt{\rho}}$ is the fidelity between two density matrices ρ and ρ_0. In particular, if both ρ and ρ_0 are pure states, the fidelity is collapsed to the modulo of wavefunction overlap.

Assuming the ASPT has a clean limit, we can always obtain a purified clean SPT wavefunction from the mixed-state density matrix that defines the ASPT state. In particular, in the decorated domain wall basis, we can have

$$\rho = \sum_{\mathcal{D}} p_{\mathcal{D}}|\Psi_{\mathcal{D}}\rangle\langle\Psi_{\mathcal{D}}| \longrightarrow |\Psi\rangle = \sum_{\mathcal{D}} \sqrt{p_{\mathcal{D}}} e^{i\theta_{\mathcal{D}}}|\Psi_{\mathcal{D}}\rangle, \quad \text{(S19)}$$

where the $\theta_{\mathcal{D}}$ is a Berry phase that should be self-consistently determined using the F-move of G-defects. For an ASPT with a clean limit, this procedure can always be done and the resulting wavefunction is guaranteed to be short-range entangled. However, if the ASPT does not have a clean limit, namely an intrinsic ASPT, one cannot find a consistent assignment of the Berry phases. It is easy to come up with a trivial reference state

$$|\Omega\rangle = \sum_{\mathcal{D}} \sqrt{p_{\mathcal{D}}}|\Phi_{\mathcal{D}}\rangle, \quad \text{(S20)}$$

where $|\Phi_{\mathcal{D}}\rangle$ are just decorating trivial product states on the G symmetry defects.

The numerator of the strange correlator in Eq. (S18) is

$$F(\rho, \phi(r)\phi(r')\rho_0\phi^\dagger(r')\phi^\dagger(r)) = \sum_{\mathcal{D}} p_{\mathcal{D}}|\langle\Phi_{\mathcal{D}}|\phi(r)\phi(r')|\Psi_{\mathcal{D}}\rangle|. \quad \text{(S21)}$$

On the other hand, the numerator of the strange correlator of the clean SPT phases (S17) from the purified wavefunctions (S19) and (S20) is

$$\langle\Omega|\phi(r)\phi(r')|\Psi\rangle = \sum_{\mathcal{D}} p_{\mathcal{D}}\langle\Phi_{\mathcal{D}}|\phi(r)\phi(r')|\Psi_{\mathcal{D}}\rangle e^{i\theta_{\mathcal{D}}}. \quad \text{(S22)}$$

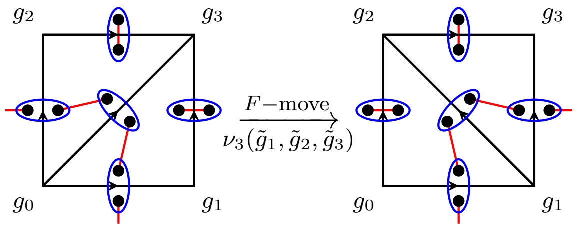


FIG. S1. F-move of the G-defect decorated with $1d$ K-SPT decoration. Black solid dots depict the projective representations of K symmetry, and the red links depict their entanglements.

Each term of the numerators of the strange correlators (S18) and (S17) is equivalent up to a phase factor that is the Berry phase acquired from the F-move of G-defect. Following similar calculations, each term of the denominators of (S18) and (S17) is also equivalent up to this phase.

We demonstrate that if there is no extension between K and G, the phase factor $e^{i\theta_{\mathcal{D}}}$ in the F-symbol illustrated in Fig. S1 is always trivial, namely it forms a coboundary of $G \times K$ group (for fermion SPT phases, the fermion parity $\mathbb{Z}_2^f$ should be included in K). Nontrivial ASPTs with pure state correspondence are characterized by nontrivial decoration pattern, not the Berry phases, as the Berry phases become invisible in a density matrix.

We argue this phase is trivial for the $2d$ SPT phases from decorating $1d$ K-SPT on the codimension-1 G-defect. From Künneth formula, this type of $2d$ SPT phases are labeled by the following 3-cocycle

$$\nu_3(h_1,h_2,h_3) \in \mathcal{H}^1\left(G, \mathcal{H}^2\left[K, U(1)\right]\right), \tag{S23}$$

where $h_j = (g_j, k_j) \in G \times K$. Due to the structure in (S23), by some gauge transformations (attaching 3-coboundaries) we can further write this 3-cocycle in the standard form as

$$\nu_3(h_1,h_2,h_3) = [\omega_2(k_1,k_2)](g_3), \tag{S24}$$

where $\omega_2(k_1,k_2)$ only depends on k_1 and k_2 and gives the classification of $1d$ K-SPT, and evaluating it on g_3 gives the 3-cocycle.

Now we consider the F-move of the G-defect, see Fig. S1. The F-move of the G-defect changes the pattern of domain wall configuration of G. We consider the Berry phase $\nu_3(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3) = \nu_3(g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3)$ that only depends on the group elements of G. Due to Eq. (S24), we know the 3-cocycle $\nu_3(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3)$ can be deformed into $\nu_3(1_k,1_k,\tilde{g}_3)$, where $k_1 = k_2 = 1_k$ are the identity element of the group K, by some gauge transformations. Since the $\nu_3(1_k,1_k,\tilde{g}_3)$ now only depends on one group element $\tilde{g}_3$, one can further apply gauge transformation to gauge this phase away as well. As a consequence, all the Berry phase factors can be gauged away for the symmetry group $\mathcal{G} = G \times K$. Therefore, the strange correlators of ASPT (S18) and clean SPT (S17) are identical for $2d$ SPT from decorating the codimension-1 G-defects by $1d$ K-SPT. It is easy to generalize our argument to the ASPT phases in higher dimensions with any kind of domain wall decorations, with $\mathcal{G} = G \times K$.

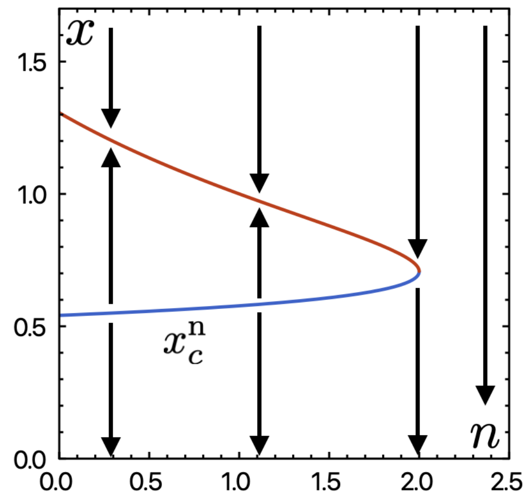


FIG. S2. Phase diagram for 2D $O(n)$ loop models.

A review of $O(n)$ loop model and correlation function

In the main text, we have mapped the strange correlator of $2d$ ASPT phases to a certain correlation function in the self-avoiding $O(n)$ loop model on the honeycomb lattice. Here let us review some of the useful facts on the loop models. The partition function of the loop model is a sum over all possible self-avoiding loop configurations $\mathcal{C}$, weighted by a loop tension x for the total length of loop configurations and a loop fugacity n for the number of loops $|\mathcal{C}|$,

$$Z = \sum_{\mathcal{C}} x^{\text{length}} n^{|\mathcal{C}|}. \tag{S25}$$

It is called the $O(n)$ loop model because the partition function of a honeycomb lattice of $O(n)$ spins can be transformed into the above form.

The phase diagram of the loop model is shown in Fig. S2. For a fixed value of $n \in [-2, 2]$, there is a critical point $x_c = \left[2 + \sqrt{2-n}\right]^{-1/2}$ which separates the so-called dense loop phase and dilute loop phase. For $x > x_c/x < x_c$, we call the corresponding phase the "dense/dilute" loop phase, and x_c remarks their transition termed as the *dilute fixed point* [54].

For the $O(n)$ loop model with $-2 \le n \le 2$, the L-leg *watermelon* correlation function $C_L(\boldsymbol{x}-\boldsymbol{y})$ measures the probability that L non-intersecting lines have a common source point $\boldsymbol{x}$ and shrink at the certain endpoint $\boldsymbol{y}$. At the dense phase $x > x_c$ and the dilute critical point x_c, the L-leg watermelon correlation function has power-law decay behavior as

$$C_L(\boldsymbol{x}-\boldsymbol{y}) \sim |\boldsymbol{x}-\boldsymbol{y}|^{-2\Delta_L}, \tag{S26}$$

while $C_L(\boldsymbol{x}-\boldsymbol{y})$ decays exponentially in the dilute loop phase ($x < x_c$). The watermelon correlation is also power law right at the dilute fixed point but with a different exponent for $n < 2$.

Let us investigate the critical exponent Δ_L. By the two-dimensional Coulomb gas technique [67], the $O(n)$ model can be transformed into a solid-to-solid (SOS) model by orienting the loops in the continuum limit, which is a Gaussian model with the following action

$$S_{\text{Gaussian}} = \frac{g}{4\pi}\int \mathrm{d}^2x(\nabla\phi)^2, \tag{S27}$$

where g is the coupling constant of the Coulomb gas, such that

$$n = -2\cos(\pi g), \tag{S28}$$

where $g \in [1,2]$ for the dilute critical point x_c and $g \in [0,1]$ for the dense loop phase ($x > x_c$). The critical exponent Δ_L of the watermelon correlation function (S26) is determined by

$$\Delta_L = \frac{g}{8}L^2 - \frac{(1-g)^2}{2g}, \tag{S29}$$

for the $O(n)$ model where n is parameterized by Eq. (S28). In the main text, we have utilized Eq. (S26) to calculate various critical exponents of watermelon correlation functions: $L = 2$ and $L = 3$ for $n = 2$ model, and $L = 2$ for $n = \sqrt{2}$ model.

Wavefunction overlap of $1d$ dimerized topological states

In the main text, we emphasize that the wavefunction overlap of the decorated $1d$ SPT phase plays an essential role in the strange correlator (S18). Therefore, in this section, we explicitly calculate the wavefunction overlap of $1d$ cluster states with trivial symmetric product state as well as the overlap of $1d$ Kitaev chain with trivial superconductor in systems with finite size using the fixed point wavefunctions.

For the $1d$ cluster state protected by $\mathbb{Z}_2\times\mathbb{Z}_2$, the wavefunction $|\psi\rangle$ of the $1d$ cluster state in Z-basis is explicitly written as

$$\langle\cdots s_{j-1}s_js_{j+1}\cdots|\psi\rangle = (-1)^{\sum_j s_js_{j+1}}/\sqrt{2^{2L}} \tag{S30}$$

where $s_j = 0, 1$ and $2L$ is the number of qubits. Physically, there is a qubit with $X = -1$ at each domain wall of Z (i.e., $Z_{j-1}Z_{j+1} = -1$) and a qubit with $X = 1$ at the site away from the Z domain wall. Then consider the trivial state $|\psi_0\rangle$ as the ground state of the disentangled Hamiltonian $H_0 = -\sum_j X_j$, with the following explicit form

$$|\psi_0\rangle = \bigotimes_{j=1}^{2L}\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_j = \bigotimes_{j=1}^{2L}|X_j = 1\rangle \tag{S31}$$

Hence the overlap of $1d$ cluster state (S30) and trivial product state (S31) is

$$\langle\psi_0|\psi\rangle = \frac{1}{2^{L-1}} = 2\times 2^{-L} \tag{S32}$$

As we have emphasized in the main text, prefactor 2 is related to the quantum dimension of the boundary of the $1d$ SPT state.

Next, we consider the Majorana chain, with the following Hamiltonian

$$H = -\sum_j c_j^\dagger c_{j+1} + h.c. - \sum_j c_jc_{j+1} + h.c. \tag{S33}$$

The ground states wavefunctions of the Majorana chain are

$$\begin{aligned} |\Psi_0^{\text{even}}\rangle &= \frac{1}{\sqrt{2}}\left(|\Psi_0^+\rangle + |\Psi_0^-\rangle\right) \\ |\Psi_0^{\text{odd}}\rangle &= \frac{1}{\sqrt{2}}\left(|\Psi_0^+\rangle - |\Psi_0^-\rangle\right) \end{aligned} \tag{S34}$$

where the superscript even/odd represents the even/odd fermion parity of the ground state wavefunctions, and

$$|\Psi_0^\pm\rangle = \frac{1}{2^{L/2}}e^{\pm c_1^\dagger}e^{\pm c_2^\dagger}\cdots e^{\pm c_L^\dagger}|0\rangle \tag{S35}$$

It is well-known that the ground state of the Majorana chain with periodic/anti-periodic boundary condition (PBC/anti-PBC) has odd/even fermion parity, hence $|\Psi_0^{\text{even}}/|\Psi_0^{\text{odd}}\rangle$ is the ground state wavefunction of the Majorana chain with anti-PBC/PBC. Then consider the atomic insulator as the $1d$ trivial product state, with the following disentangled Hamiltonian

$$H_0 = \mu\sum_j c_j^\dagger c_j,\ \ \mu > 0 \tag{S36}$$

whose ground state wavefunction is the unoccupied vacuum state $|0\rangle$, with even fermion parity. Hence the wavefunction overlap between $|0\rangle$ and $|\Psi_0^{\text{odd}}\rangle$ vanishes because of the different fermion parity, and the wavefunction overlap between $|0\rangle$ and $|\Psi_0^{\text{even}}\rangle$ is

$$\langle 0|\Psi_0^{\text{even}}\rangle = \left(\frac{1}{\sqrt{2}}\right)^{L-1} = \sqrt{2}\times 2^{-L/2}, \tag{S37}$$

where the prefactor $\sqrt{2}$ also coincides with the quantum dimension of the boundary Majorana zero modes.

Quantum correction of the loop fugacity

In the main text, we proved that the strange correlator of $2d$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^A$ ASPT state is precisely mapped to the loop correlation function in the $O(2)$ loop model. Intuitively, there is only one kind of domain wall of the 2D classical Ising model, hence the Ising domain wall should be described by an $O(1)$ model if we decorate nothing on the Ising domain wall. Here we argue the universal nature of this factor away from the fixed point wavefunction using a field-theoretical representation of the SPT wavefunction. It is known that the wavefunction of a 1d SPT state can be written as $O(3)$ non-linear sigma model (NLσM) with a Wess–Zumino–Witten term at level-1. The overlap between an SPT wavefunction and a trivial state can be represented as [68],

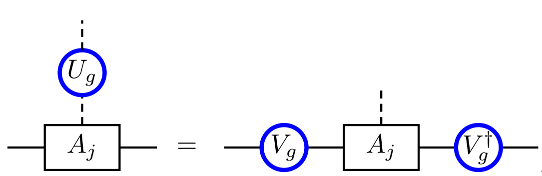
$$\langle\Psi|\Phi\rangle = \int_{\mathcal{D}[\vec{n}(x)]} \exp\left(-\int_{x=0}^{L} \mathcal{L}[\vec{n}] + i\mathrm{WZW}_1[\vec{n}(x)]\right), \tag{S38}$$

where $\mathcal{L}[\vec{n}]$ contains the kinetic term of the NLσM and possible anisotropic terms that break the $SO(3)$ symmetry. Doing a wick rotation, we can equivalently view this overlap as the thermal partition function at temperature $\beta = L$ of a quantum mechanical model of a charged particle moving on a sphere with a 2π magnetic monopole at the center. This Landau level problem has a robust 2-fold degeneracy of its ground state with energy $\frac{1}{2}\hbar\omega_c$ where ω_c depends on the details of the kinetic terms. The thermal partition function at large $\beta = L$ is given by $\langle\Psi|\Phi\rangle \cong 2 \times e^{-L\hbar\omega_c/2}$, where the prefactor 2 comes from the 2-fold ground state degeneracy. We also note that this overlap can be viewed as the boundary partition function of the SPT state. Therefore, the degeneracy is also the boundary degeneracy of the SPT that is decorated on the domain wall.

We can also provide a more rigorous argument based on the matrix product state (MPS) representation of $1d$ SPT phases [2]. A K-symmetric $1d$ SPT state can be described by an injective MPS,

$$|\Psi_{\mathrm{SPT}}\rangle = \sum_{i_1,\cdots i_N} \mathrm{Tr}\left[A_{i_1}\cdots A_{i_N}\right]|i_1\cdots i_N\rangle, \tag{S39}$$

which has the following symmetry property [2]



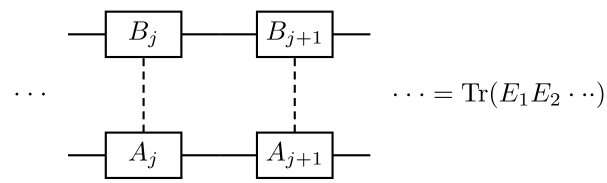
, (S40)

where $g \in K$, U_g, and V_g are local unitary operators acting on the physical indices (dashed lines) and virtual indices (solid lines), respectively. For $g, h \in K$, the unitary operators V_g and V_h have the following property

$$V_g V_h = \omega(g,h) V_{gh}, \tag{S41}$$

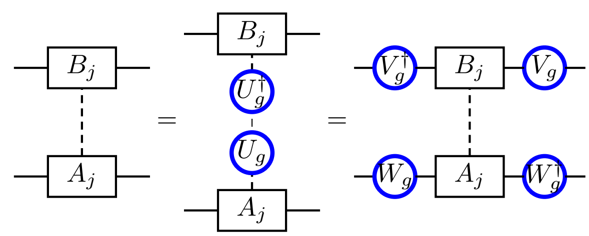
where $\omega(g,h) \in \mathcal{H}^2[K, U(1)]$ which implies that V_g is a projective representation of the group G.

Then we consider two MPSs of two topologically distinct $1d$ SPT phases A and B, which are depicted by two 2-cocycles ω_1 and ω_2 in $\mathcal{H}^2[K, U(1)]$. Their overlap can be represented graphically as



(S42)

where E_j is the transfer matrix of two $1d$ MPSs. Furthermore, by acting U_g and $U_g^\dagger$ to the physical indices of two MPSs, we have



(S43)

i.e., $X_g E_j X_g^\dagger = E_j$, where $X_g = V_g^\dagger \otimes W_g$ that also satisfies the condition of projective representation of K as

$$X_g X_h = \frac{\omega_2(g,h)}{\omega_1(g,h)} X_{gh} \tag{S44}$$

Therefore, the transfer matrix E_j has a K symmetry and transforms projectively under K. For translational invariant MPSs, the overlap of these MPSs is $\mathrm{Tr}(E^L)$ (where L is the system size), which is determined by the largest eigenvalue of E.

For the $\mathbb{Z}_2$-classified $1d$ SPT phases, the largest eigenvalue of the transfer matrix will always have 2-fold degeneracy from the projectively imposed symmetry K, which leads to an overall factor 2 of the wavefunction overlap of the trivial and nontrivial $1d$ SPT states. We call this universal factor coming from the decorated $1d$ SPT as *quantum correction of the loop fugacity.*