

# EXPONENTIAL TIME-DECAY FOR A ONE DIMENSIONAL WAVE EQUATION WITH COEFFICIENTS OF BOUNDED VARIATION

KIRIL DATCHEV AND JACOB SHAPIRO

**ABSTRACT.** We consider the initial-value problem for a one-dimensional wave equation with coefficients that are positive, constant outside of an interval, and have bounded variation (BV). Under the assumption of compact support of the initial data, we prove that the local energy decays exponentially fast in time, and provide the explicit constant to which the solution converges. The key ingredient of the proof is a high frequency resolvent estimate for an associated Helmholtz operator with a BV potential.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

This paper establishes exponential local energy decay for the solution of the following one dimensional wave equation, with compactly supported initial data:

$$\begin{cases} \beta(x)\partial_t^2 w(x, t) - \partial_x(\alpha(x)\partial_x w(x, t)) = 0, & (x, t) \in \mathbb{R} \times (0, \infty), \\ w(x, 0) = w_0(x), \\ \partial_t w(x, 0) = w_1(x), \\ \text{supp } w_0, \text{supp } w_1 \subseteq (-R, R), & R > 0. \end{cases} \quad (1.1)$$

Here, the coefficients  $\alpha, \beta : \mathbb{R} \rightarrow (0, \infty)$  have bounded variation (BV). We suppose also

$$\inf_{\mathbb{R}} \alpha, \inf_{\mathbb{R}} \beta > 0, \quad (1.2)$$

and that there exist  $R_0, \alpha_0, \beta_0 > 0$ , so that

$$\alpha(x) = \alpha_0, \beta(x) = \beta_0, \quad |x| \geq R_0. \quad (1.3)$$

To begin, we address the well-posedness of (1.1) via the spectral theorem for self-adjoint operators. Let  $\mathcal{H}$  be the Hilbert space  $L^2(\mathbb{R}; \beta(x)dx)$  equipped with the inner product

$$\langle u, v \rangle_{\mathcal{H}} := \int_{\mathbb{R}} \overline{u}(x)v(x)\beta(x)dx.$$

(Note that  $L^2(\mathbb{R}; \beta(x)dx) = L^2(\mathbb{R}; dx)$  as sets, and their respective norms generate the same topology, since  $\beta$  has positive upper and lower bounds.) Define the symmetric, nonnegative differential operator

$$Hu := -\beta^{-1}\partial_x(\alpha\partial_x u), \quad (1.4)$$

with domain  $\mathcal{D}(H) := \{u \in L^2(\mathbb{R}) : u, \partial_x u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \text{ and } \partial_x(\alpha\partial_x u) \in L^2(\mathbb{R})\}$ . We will see from Lemma 3.1 in Section 3 that  $H$  is self-adjoint with respect to  $\mathcal{D}(H)$ . It is also conveniently the case that  $D(H^{1/2})$  coincides with the Sobolev space  $H^1(\mathbb{R})$  [Re22b]. For completeness, we prove this fact in Appendix A.

---

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN, 47907-2067, USA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DAYTON, DAYTON, OH 45469-2316, USA

*E-mail addresses:* [kdatchev@purdue.edu](mailto:kdatchev@purdue.edu), [jshapiro1@udayton.edu](mailto:jshapiro1@udayton.edu).

*Key words and phrases.* resolvent estimate, Schrödinger operator, wave decay.

Second author is the corresponding author.

Thus, for initial conditions  $w_0 \in \mathcal{D}(H)$ ,  $w_1 \in \mathcal{D}(H^{1/2})$ ,

$$w(t) = w(\cdot, t) = \cos(tH^{1/2})w_0 + \frac{\sin(tH^{1/2})}{H^{1/2}}w_1. \quad (1.5)$$

is the unique function  $w \in C^2((0, \infty), \mathcal{H})$  with  $w(0) = w_0$ ,  $\partial_t w(0) = w_1$ , and for all  $t > 0$ ,  $w(t) \in \mathcal{D}(H)$  and  $\partial_t^2 w(t) + Hw(t) = 0$ .

**Theorem 1.1.** *Let  $\alpha, \beta : \mathbb{R} \rightarrow (0, \infty)$  have bounded variation and satisfy (1.2) and (1.3). Suppose  $w_0 \in \mathcal{D}(H)$ ,  $w_1 \in \mathcal{D}(H^{1/2})$ , and  $\text{supp } w_0, \text{supp } w_1 \subseteq (-R, R)$  for some  $R > 0$ . Let  $w(t)$  be given by (1.5). For any  $R_1 > 0$ , there exist  $C, c > 0$  so that*

$$\begin{aligned} & \|w(\cdot, t) - w_\infty\|_{H^1(-R_1, R_1)} + \|\partial_t w(\cdot, t)\|_{L^2(-R_1, R_1)} \\ & \leq Ce^{-ct}(\|w_0\|_{H^1(\mathbb{R})} + \|w_1\|_{L^2(\mathbb{R})}), \quad t > 0, \end{aligned} \quad (1.6)$$

where

$$w_\infty := \frac{1}{2(\alpha_0\beta_0)^{1/2}} \int_{\mathbb{R}} w_1(x)\beta(x)dx. \quad (1.7)$$

Theorem 1.1 is motivated by the recent article [AGPP22]. There, the authors prove (1.6), with an explicit constant  $c$  depending on  $\alpha$  and  $\beta$ , provided that  $\alpha$  and  $\beta$  are Lipschitz continuous, bounded from above and below by positive constants, and satisfy (1.3). Our result includes natural examples such as cases where  $\alpha$  and  $\beta$  are piecewise constant and it is easy to see that the exponential decay rate in (1.6) cannot in general be improved to any superexponential rate. See [BIZ16] for dispersive and Strichartz estimates for one dimensional wave equations with BV coefficients.

To prove Theorem 1.1, it suffices to show (1.6) and (1.7) in the special case

$$\alpha(x) = \beta(x) = 1, \quad |x| \geq R_0. \quad (1.8)$$

Indeed, if  $w(x, t)$  solves (1.1) for initial conditions  $w_0, w_1$  and general  $\alpha$  and  $\beta$ , then the function  $u(x, t) := w(\sqrt{\alpha_0}x, \sqrt{\beta_0}t)$  solves  $(\beta(\sqrt{\alpha_0}x)/\beta_0)\partial_t^2 u - \partial_x((\alpha(\sqrt{\alpha_0}x)/\alpha_0)\partial_x u) = 0$  with initial conditions  $u(x, 0) = w_0(\sqrt{\alpha_0}x)$ ,  $\partial_t u(x, 0) = \sqrt{\beta_0}w_1(\sqrt{\alpha_0}x)$ . Then (1.8) applies, giving that  $u$  decays according to (1.6) and (1.7). The asserted decay for  $w$  follows by a change of variables.

For the wave equation with *constant* coefficients and compactly supported initial conditions, it follows readily from D'Alembert's formula that solution to (1.1) converges to  $w_\infty$  in finite time. However, for variable coefficients, exponential decay is a typical scenario. This occurs in the setting of reflection and transmission, e.g., when  $\alpha \equiv 1$  and  $\beta$  assumes precisely two values.

In dimensions two and higher, the recent works [ChIk20, Sh18] treat local energy decay for wave equations with Lipschitz coefficients. Though in higher dimensions, logarithmic, rather than exponential decay, is optimal in general. The study of energy decay more broadly has a long history, going back to the foundational work of Morawetz, Lax-Phillips, and Vainberg [Mo61, LMP62, LaPh89, Va89], which we will not attempt to review here. The reader may consult [Bu98, HiZw17, Sh18, DyZw19] for more historical background and references.

We prove Theorem 1.1 by analyzing  $H$  as a *black box Hamiltonian* in the sense of Sjöstrand and Zworski [SjZw91]. In particular, (1.8) implies that for any  $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$  that is identically one near  $[-R_0, R_0]$ , the cutoff resolvent

$$\chi R(\lambda)\chi := \chi(H - \lambda^2)^{-1}\chi : \mathcal{H} \rightarrow \mathcal{D}(H) \quad (1.9)$$

continues meromorphically from  $\text{Im } \lambda > 0$  to the complex plane. (Here, we equip  $\mathcal{D}(H)$  with the graph norm  $u \mapsto (\|u\|_{\mathcal{H}}^2 + \|Hu\|_{\mathcal{H}}^2)^{1/2}$ .) In particular, we establish the following high frequency bound.

**Theorem 1.2.** *Suppose  $\alpha, \beta : \mathbb{R} \rightarrow (0, \infty)$  have bounded variation and obey (1.2) and (1.8). For any  $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$  that is identically one near  $[-R_0, R_0]$ , there exists  $C, \lambda_0, \varepsilon_0 > 0$  so that*

$$\|\chi R(\lambda)\chi\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq C|\text{Re } \lambda|^{-1}, \quad (1.10)$$

whenever  $|\operatorname{Re} \lambda| \geq \lambda_0$ , and  $|\operatorname{Im} \lambda| \leq \varepsilon_0$ .

In Section 4, we achieve (1.10) by rescaling  $H - \lambda^2$  semiclassically, see (4.2), and apply a resolvent estimate for a Schrödinger operator with a BV potential, namely Theorem 3.2 in Section 3. The proof of Theorem 3.2 uses a positive commutator argument that relies on some basic calculus facts for BV functions. We collect these facts in Section 2, and prove them in Appendix B. Finally, in Section 5, we prove (1.6) by combining (1.10) with an argument involving Plancherel's theorem and contour deformation. A similar strategy appears in [Vo99, Section 3].

Our methods should apply directly to some more general operators, such as the wave operator  $\beta(x)\partial_t^2 - \partial_x(\alpha(x)\partial_x) + V(x)$ , where  $V$  is real-valued, compactly supported, and has BV. In that case, however, the residual  $w_\infty$  in (1.6) may be more complicated, as there may or may not be a resonance at zero, and there may also be discrete negative spectrum. See [DyZw19, Theorem 2.9] for instance, which treats the case  $V \not\equiv 0$  and  $\alpha, \beta \equiv 1$ .

## 2. REVIEW OF BV

To keep the notation concise, for the rest of the article, we use “prime” notation to denote differentiation with respect to  $x$ , e.g.,  $u' := \partial_x u$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a function of locally bounded variation. For all  $x \in \mathbb{R}$ , put

$$f^L(x) := \lim_{\delta \rightarrow 0^+} f(x - \delta), \quad f^R(x) := \lim_{\delta \rightarrow 0^+} f(x + \delta), \quad f^A(x) := (f^L(x) + f^R(x))/2, \quad (2.1)$$

where the limits exist because both the real and imaginary parts of  $f$  are a difference of two increasing functions. Recall that  $f$  is differentiable Lebesgue almost everywhere, so  $f(x) = f^L(x) = f^R(x) = f^A(x)$  for almost all  $x \in \mathbb{R}$ .

We may decompose  $f$  as

$$f = f_{r,+} - f_{r,-} + i(f_{i,+} - f_{i,-}), \quad (2.2)$$

where the  $f_{\sigma,\pm}$ ,  $\sigma \in \{r, i\}$ , are increasing functions on  $\mathbb{R}$ . Each  $f_{\sigma,\pm}^R$  uniquely determines a regular Borel measure  $\mu_{\sigma,\pm}$  on  $\mathbb{R}$  satisfying  $\mu_{\sigma,\pm}(x_1, x_2] = f_{\sigma,\pm}^R(x_2) - f_{\sigma,\pm}^R(x_1)$ , see [Fo07, Theorem 1.16]. We put

$$df := \mu_{r,+} - \mu_{r,-} + i(\mu_{i,+} - \mu_{i,-}), \quad (2.3)$$

which is a complex measure when restricted to any bounded Borel subset. For any  $a < b$ ,

$$\begin{aligned} \int_{(a,b]} df &= f^R(b) - f^R(a), \\ \int_{(a,b)} df &= f^L(b) - f^L(a). \end{aligned} \quad (2.4)$$

We collect several properties of functions of bounded variation, which are well known, and which we use to prove Theorem 3.2 in Section 3. Their proofs are deferred to the appendix.

**Proposition 2.1** (integration by parts). *Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  have locally bounded variation. For any  $a < b$ , and any continuous  $\varphi$ , with  $\varphi'$  piecewise continuous and  $\varphi(a) = \varphi(b) = 0$ ,*

$$\int_{(a,b]} \varphi df = - \int_{(a,b]} \varphi' f dx. \quad (2.5)$$

**Proposition 2.2** (product rule). *Let  $f, g : \mathbb{R} \rightarrow \mathbb{C}$  be functions of locally bounded variation. Then*

$$d(fg) = f^A dg + g^A df \quad (2.6)$$

*as measures on a bounded Borel subset of  $\mathbb{R}$ .*

**Remark:** We note that if  $f$  is continuous, then inductively applying (2.6) yields  $df^n = n f^{n-1} df$ .

**Proposition 2.3** (chain rules). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and have locally bounded variation. Then, as measures on a bounded Borel set of  $\mathbb{R}$ ,*

$$d(e^f) = e^f df. \quad (2.7)$$

On the other hand, let  $x_1, \dots, x_N, r_0, r_1, \dots, r_N \in \mathbb{R}$ , and consider the function

$$g(x) = r_0 \mathbf{1}_{(-\infty, x_1]} + \sum_{j=1}^{N-1} r_j \mathbf{1}_{(x_j, x_{j+1}]} + r_N \mathbf{1}_{(x_N, \infty)}.$$

Then

$$d(e^g) = \sum_{j=1}^N (e^{r_j} - e^{r_{j-1}}) \delta_{x_j}, \quad (2.8)$$

where  $\delta_{x_j}$  denotes the dirac measure at  $x_j$ .

The need to treat separately the case of jump discontinuities in Proposition 2.3 was brought to the authors' attention by [Pi22, Re22a].

### 3. WEIGHTED RESOLVENT ESTIMATE

The purpose of this Section is to prove a weighted resolvent estimate for the semiclassical Schrödinger operator

$$P = P(h) := -h\partial_x(\alpha(x)h\partial_x) + V(x) - E : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad E, h > 0, \quad (3.1)$$

which is the key ingredient in the proof of Theorem 1.2 in Section 4. We suppose  $\alpha$  and  $V$  are real-valued functions of bounded variation on  $\mathbb{R}$ , and

$$\inf_{\mathbb{R}} \alpha > 0. \quad (3.2)$$

Specifically, we show

**Lemma 3.1.** *The operator  $P : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is self adjoint with respect to the domain*

$$\mathcal{D} := \{u \in L^2(\mathbb{R}) : u, u' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \text{ and } Pu \in L^2(\mathbb{R})\}, \quad (3.3)$$

and prove the following resolvent bound, for  $h$  small, and uniformly down to  $[E_{\min}, E_{\max}] \subseteq (0, \infty)$ .

**Theorem 3.2.** *Fix  $[E_{\min}, E_{\max}] \subseteq (0, \infty)$  and  $\delta > 0$ . Assume  $\alpha, V : \mathbb{R} \rightarrow \mathbb{R}$  have bounded variation,  $\alpha$  obeys (3.2), and*

$$\sup_{\mathbb{R}} V < E_{\min}. \quad (3.4)$$

Then there exist  $C, h_0 > 0$ , so that for all  $E \in [E_{\min}, E_{\max}]$ ,  $h \in (0, h_0]$ , and  $\varepsilon > 0$ ,

$$\|(|x| + 1)^{-\frac{1+\delta}{2}} (P(h) - i\varepsilon)^{-1} (|x| + 1)^{-\frac{1+\delta}{2}}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq Ch^{-1}. \quad (3.5)$$

Since  $V$  has limited regularity, we have replaced a more typical nontrapping condition, concerning the escape of trajectories  $\dot{x} = 2\xi$ ,  $\dot{\xi} = -\partial_x V$  that obey  $|\xi|^2 + V(x) = E$ , with the simpler condition (3.4). Indeed, as  $\alpha$  and  $V$  have only bounded variation, the bicharacteristic flow is not necessarily well defined. Moreover, in Section 4, we shall see that (3.4) is a natural assumption, given that the coefficients of the operator  $H$  obey (1.2).

To prove Theorem 3.2, we employ a positive commutator-style argument in the context of the spherical energy method. This strategy has long been used to prove semiclassical resolvent estimates [CaVo02, Da14, KlVo19, DaSh20, GaSh22]. In fact, as we are in one dimension, we just use the pointwise energy

$$F(x) = F[u](x) := \alpha(x)|h\partial_x u(x)|^2 + (E - V(x))|u(x)|^2, \quad u \in \mathcal{D}. \quad (3.6)$$

The goal is to construct a suitable weight function  $w(x)$  so that the derivative of  $wF$ , in the sense of distributions, has a favorable sign. From (3.24) below, we see that  $w$  ought to be designed

so that  $(w(E - V))'$  has a positive lower bound. If  $V$  only has bounded variation, this derivative must be interpreted as a measure, and extra care is needed to control the point masses arising from the discontinuities of  $V$  (see (3.18)).

We first give our attention to Lemma 3.1, which is essentially well known. Our present proof is adapted from [DaSh20, Section 2].

*Proof of Lemma 3.1.* Let

$$\mathcal{D}_{\max} := \{u \in L^2(\mathbb{R}) : u, \alpha u' \text{ are locally absolutely continuous and } Pu \in L^2(\mathbb{R})\},$$

By, [Ze05, Lemma 10.3.1],  $\mathcal{D}_{\max}$  is dense in  $L^2(\mathbb{R})$ . We begin by proving

$$\mathcal{D}_{\max} = \mathcal{D}. \quad (3.7)$$

Indeed, for any  $a > 0$  and  $u \in \mathcal{D}_{\max}$ , by integration by parts and Cauchy–Schwarz,

$$\begin{aligned} \inf \alpha \int_{-a}^a |u'|^2 &\leq \int_{-a}^a \alpha u' \bar{u}' = \alpha u' \bar{u}|_{-a}^a + h^{-2} \int_{-a}^a Pu \bar{u} - h^{-2} \int_{-a}^a V u \bar{u} \\ &\leq 2 \sup \alpha \sup_{[-a,a]} |u'| \sup_{[-a,a]} |u| + h^{-2} \sup |V| \|u\|_{L^2}^2 + h^{-2} \|Pu\|_{L^2} \|u\|_{L^2}, \\ \sup_{[-a,a]} |u|^2 &= \sup_{x \in [-a,a]} \left( |u(0)|^2 + 2 \operatorname{Re} \int_0^x u' \bar{u} \right) \leq |u(0)|^2 + 2 \left( \int_{-a}^a |u'|^2 \right)^{1/2} \|u\|_{L^2}, \\ (\inf \alpha)^2 \sup_{[-a,a]} |u'|^2 &\leq \sup_{[-a,a]} |\alpha u'|^2 = \sup_{x \in [-a,a]} \left( |(\alpha u')(0)|^2 + 2 \operatorname{Re} \int_0^x (\alpha u')' \alpha \bar{u}' \right) \\ &\leq |(\alpha u')(0)|^2 + 2h^{-2} (\sup(\alpha |V|) \|u\|_{L^2} + \sup \alpha \|Pu\|_{L^2}) \left( \int_{-a}^a |u'|^2 \right)^{1/2}. \end{aligned}$$

This is a system of inequalities of the form  $x^2 \leq A + B y z$ ,  $y^2 \leq C + D x$ ,  $z^2 \leq E + F x$ . Thus, for any  $\gamma > 0$ ,

$$\begin{aligned} x^2 &\leq A + \frac{B}{2\gamma} + \gamma(yz)^2 \leq A + \frac{B}{2\gamma} + \gamma(C + Dx)(E + Fx) \\ &\leq A + \frac{B}{2\gamma} + \gamma CE + \gamma \frac{(CF)^2 + (DE)^2}{2} + (\gamma^2 + \gamma DF)x^2. \end{aligned} \quad (3.8)$$

Choosing  $\gamma$  small enough allows one to absorb all the terms involving  $x^2$  on the right side of (3.8), into the left side. Hence  $x$ ,  $y$  and  $z$  are all bounded independently of  $a$ . Letting  $a \rightarrow \infty$ , we conclude that  $u' \in L^2(\mathbb{R})$  and  $u, u' \in L^\infty(\mathbb{R})$ . Hence  $\mathcal{D}_{\max} \subseteq \mathcal{D}$ . The inclusion  $\mathcal{D} \subseteq \mathcal{D}_{\max}$  follows because  $Pu \in L^2(\mathbb{R})$  implies  $(\alpha u')' \in L^2(\mathbb{R})$ , which in turns gives that  $\alpha u'$  is locally absolutely continuous.

Equip  $P$  with the domain  $\mathcal{D}_{\max} = \mathcal{D} \subseteq L^2(\mathbb{R})$ . By integration by parts,  $P \subseteq P^*$ . But, by Sturm–Liouville theory,  $P^* \subseteq P$ ; see [Ze05, Equation 10.3.2]. Hence  $P = P^*$ .  $\square$

We now prove Theorem 3.2, with the argument proceeding in two steps. First, as described above, we build a weight  $w$  so that,  $d(wF)$  has a desirable lower bound in the sense of measures—see (3.24). This yields the Carleman estimate (3.27), which implies the resolvent estimate (3.29).

*Proof of Theorem 3.2.* Decompose

$$\begin{aligned} dV &= dV^d + dV^c, \\ d\alpha &= d\alpha^d + d\alpha^c, \end{aligned}$$

into their discrete and continuous parts. Let  $J_V$ , respectively  $J_\alpha$  be the sets of “positive jumps” of  $V$ ,  $\alpha$  respectively. That is  $J_V$  is the set of  $x$ -values such that  $(V^R - V^L)(x) > 0$ , and similarly for

$J_\alpha$ . Since  $V$  and  $\alpha$  have bounded variation, both  $J_V$  and  $J_\alpha$  are at most countable. We denote by  $\{x_j\}_j$  an enumeration of  $J_V \cup J_\alpha$ . Additionally, let

$$\begin{aligned} dV^c &= dV_+^c - dV_-^c, \\ d\alpha^c &= d\alpha_+^c - d\alpha_-^c, \end{aligned}$$

be Jordan decompositions for  $dV^c$ ,  $d\alpha^c$  respectively.

For each  $N \in \mathbb{N}$ , let  $x_{1,N}, x_{2,N}, \dots, x_{N,N}$  be the elements of  $\{x_j\}_{j=1}^N$  relabeled in increasing order. Define the function  $q_{1,N}$  by

$$q_{1,N}(x) := r_{0,N} \mathbf{1}_{(-\infty, x_{1,N}]} + \sum_{j=1}^{N-1} r_{j,N} \mathbf{1}_{(x_{j,N}, x_{j+1,N}]} + r_{N,N} \mathbf{1}_{(x_{N,N}, \infty)}, \quad (3.9)$$

where the numbers  $\{r_{j,N}\}_{j=0}^N$  are defined recursively as follows:

$$r_{0,N} = 0, \quad r_{j,N} = r_{j-1,N} + \log \max \left\{ 1 + \frac{2A_{j,N}}{1 - A_{j,N}}, 1 + \frac{2B_{j,N}}{1 - B_{j,N}} \right\}, \quad (3.10)$$

$$A_{j,N} := \frac{(V^R - V^L)(x_{j,N})}{2(E - V)^A(x_{j,N})} \in [0, 1), \quad B_{j,N} := \frac{(\alpha^R - \alpha^L)(x_{j,N})}{2\alpha^A(x_{j,N})} \in [0, 1). \quad (3.11)$$

When  $N = 1$ , we omit the summation from (3.9). Moreover, if  $\{x_j\}_j$  is a finite set, we work only with a single function  $q_{1,N_1}$ , where  $x_1 < \dots < x_{N_1}$  is the ordering of  $J_V \cup J_\alpha$ .

Since  $V$  and  $\alpha$  have bounded variation,

$$\sum_j \max\{(V^R - V^L)(x_j), (\alpha^R - \alpha^L)(x_j)\} < \infty. \quad (3.12)$$

Thus  $\max q_{1,N} = r_{N,N}$  is bounded uniformly in  $N$ , by

$$\begin{aligned} r_{N,N} &= \sum_{j=1}^N r_{j,N} - r_{j-1,N} \\ &= \sum_{j=1}^N \log \max \left\{ 1 + \frac{2A_{j,N}}{1 - A_{j,N}}, 1 + \frac{2B_{j,N}}{1 - B_{j,N}} \right\} \\ &\leq \sum_{j=1}^N \max \left\{ \frac{2A_{j,N}}{1 - A_{j,N}}, \frac{2B_{j,N}}{1 - B_{j,N}} \right\} \\ &\leq \sum_{j=1}^N \max \left\{ \frac{(V^R - V^L)(x_{j,N})}{(E - V)^A(x_{j,N}) - \frac{1}{2}(V^R - V^L)(x_{j,N})}, \frac{(\alpha^R - \alpha^L)(x_{j,N})}{\alpha^A(x_{j,N}) - \frac{1}{2}(\alpha^R - \alpha^L)(x_{j,N})} \right\} < \infty. \end{aligned} \quad (3.13)$$

Next, we put

$$q_2(x) := \int_{-\infty}^x \left[ k dV_+^c + \frac{2}{\inf \alpha} d\alpha_+^c + (|x'| + 1)^{-1-\delta} dx' \right], \quad (3.14)$$

where  $k > 0$  is chosen large enough so that

$$k \left( E_{\min} - \sup_{\mathbb{R}} V \right) \geq 1. \quad (3.15)$$

To implement the energy method outlined in Section 1, we will in fact use a family of weight functions depending on  $N$ ,

$$w(x) = w_N(x) = e^{q_{1,N}(x) + q_2(x)}, \quad N \in \mathbb{N}. \quad (3.16)$$

According to (2.7) and (2.8),

$$dw(x) = \sum_{j=1}^N e^{q_2} (e^{r_{j,N}} - e^{r_{j-1,N}}) \delta_{x_{j,N}} + w^A \left( \frac{2}{\inf \alpha} d\alpha_+^c + k dV_+^c + (|x| + 1)^{-1-\delta} \right). \quad (3.17)$$

We now establish lower bounds on the measures  $d(w(E - V))$  and  $dw - (\alpha^A)^{-1} w^A d\alpha$ , which we need in the estimate (3.24) below. For  $d(w(E - V))$ , we have, by (2.6), (3.15) and (3.17),

$$\begin{aligned} d(w(E - V)) &\geq (E - V)^A dw - w^A (dV^d + dV_+^c) \\ &\geq \sum_{j=1}^N e^{q_2} \left( (E - V)^A (e^{r_{j,N}} - e^{r_{j-1,N}}) - (V^R - V^L) \left( \frac{1}{2} e^{r_{j,N}} + \frac{1}{2} e^{r_{j-1,N}} \right) \right) \delta_{x_{j,N}} \\ &\quad - \sum_{x \in J_V \setminus \{x_{j,N}\}_{j=1}^N} w^A (V^R - V^L) \delta_x \\ &\quad + w^A (k(E_{\min} - V)^A - 1) dV_+^c + w^A (E - V)^A (|x| + 1)^{-1-\delta}. \end{aligned} \quad (3.18)$$

with the inequalities holding in the sense of measures. As for  $dw - (\alpha^A)^{-1} w^A d\alpha$ ,

$$\begin{aligned} dw - (\alpha^A)^{-1} w^A d\alpha &\geq dw - (\alpha^A)^{-1} w^A (d\alpha^d + d\alpha_+^c) \\ &\geq \sum_{j=1}^N e^{q_2} \left( (e^{r_{j,N}} - e^{r_{j-1,N}}) - \frac{(\alpha^R - \alpha^L)}{\alpha^A} \left( \frac{1}{2} e^{r_{j,N}} + \frac{1}{2} e^{r_{j-1,N}} \right) \right) \delta_{x_{j,N}} \\ &\quad - \sum_{x \in J_\alpha \setminus \{x_{j,N}\}_{j=1}^N} (\alpha^A)^{-1} w^A (\alpha^R - \alpha^L) \delta_x \\ &\quad + w^A \left( \frac{2}{\inf \alpha} - \frac{1}{\alpha^A} \right) d\alpha_+^c + w^A (|x| + 1)^{-1-\delta}. \end{aligned} \quad (3.19)$$

The first term in line five of (3.18) is nonnegative by (3.15); the first term of line four of (3.19) is nonnegative since  $\inf \alpha < 2\alpha^A$ . Furthermore, the third line of (3.18) and the third line of (3.19), are nonnegative by (3.10) and (3.11).

Thus we conclude

$$\begin{aligned} d(w(E - V)) &\geq w^A (E_{\min} - V)^A (|x| + 1)^{-1-\delta} - \sum_{x \in J_V \setminus \{x_{j,N}\}_{j=1}^N} w^A (V^R - V^L) \delta_x, \\ dw - (\alpha^A)^{-1} w^A d\alpha &\geq w^A (|x| + 1)^{-1-\delta} - \sum_{x \in J_\alpha \setminus \{x_{j,N}\}_{j=1}^N} (\alpha^A)^{-1} w^A (\alpha^R - \alpha^L) \delta_x, \end{aligned} \quad (3.20)$$

which are the lower bounds we shall employ in (3.24).

Next, define the pointwise energy

$$F(x) = F[u](x) := \alpha(x) |hu'(x)|^2 + (E - V(x)) |u(x)|^2, \quad x \in \mathbb{R}, \quad (3.21)$$

with

$$u = (P(h) - i\varepsilon)^{-1} (|x| + 1)^{-\frac{1+\delta}{2}} f \in \mathcal{D}, \quad \varepsilon > 0, f \in L^2(\mathbb{R}). \quad (3.22)$$

By (3.3),  $u, u' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , and  $(\alpha u')' \in L^2(\mathbb{R})$ . Moreover, in the calculations to follow, we work with fixed representatives of  $u$  and  $u'$ , such that both  $u$  and  $\alpha u'$  are locally absolutely continuous. This is justified by (3.7).

From (2.6), we see that  $dF$  is given by

$$dF = h^2 (\alpha u') d(\overline{u'}) + h^2 (\overline{u'})^A (\alpha u')' - |u|^2 dV + 2(E - V)^A \operatorname{Re}(u \overline{u'}).$$

Using

$$(\alpha u')' = (u')^A d\alpha + \alpha^A d(u') \implies d(u') = \frac{(\alpha u')'}{\alpha^A} - \frac{(u')^A}{\alpha^A} d\alpha,$$

we arrive at

$$dF = \frac{h^2}{\alpha^A} (\alpha u') (\alpha \bar{u}')' + h^2 (\bar{u}')^A (\alpha u')' - \frac{h^2}{\alpha^A} (\alpha u') (\bar{u}')^A d\alpha - |u|^2 dV + 2(E - V)^A \operatorname{Re}(u \bar{u}'). \quad (3.23)$$

We now multiply (3.21) by  $w$  and compute  $d(wF)$ :

$$\begin{aligned} d(wF) &= F^A dw + w^A dF \\ &= h^2 (\alpha u') (\bar{u}')^A dw + (E - V)^A |u|^2 dw \\ &\quad + \frac{h^2}{\alpha^A} w^A (\alpha u') (\alpha \bar{u}')' + h^2 w^A (\bar{u}')^A (\alpha u')' - \frac{h^2}{\alpha^A} w^A (\alpha u') (\bar{u}')^A d\alpha \\ &\quad - w^A |u|^2 dV + 2w^A (E - V)^A \operatorname{Re}(u \bar{u}'). \\ &= -w^A \left( -\frac{h^2}{\alpha^A} (\alpha u') (\alpha \bar{u}')' - h^2 (\bar{u}')^A (\alpha u')' + 2(V - E)^A \operatorname{Re}(u \bar{u}') - 2 \operatorname{Re}(i \varepsilon u \bar{u}') \right) \\ &\quad + 2\varepsilon w^A \operatorname{Im}(u \bar{u}') + |u|^2 d(w(E - V)) + h^2 (\alpha u') (\bar{u}')^A \left( dw - w^A \frac{d\alpha}{\alpha^A} \right) \\ &\geq -w^A \left( -\frac{h^2}{\alpha^A} (\alpha u') (\alpha \bar{u}')' - h^2 (\bar{u}')^A (\alpha u')' + 2(V - E)^A \operatorname{Re}(u \bar{u}') - 2 \operatorname{Re}(i \varepsilon u \bar{u}') \right) \\ &\quad + 2\varepsilon w^A \operatorname{Im}(u \bar{u}') + (|x| + 1)^{-1-\delta} ((E_{\min} - \sup_{\mathbb{R}} V) |u|^2 + h^2 (\alpha u') (\bar{u}')^A) \\ &\quad - \sum_{x \in J_V \setminus \{x_{j,N}\}_{j=1}^N} w^A |u|^2 (V^R - V^L) \delta_x - \sum_{x \in J_\alpha \setminus \{x_{j,N}\}_{j=1}^N} (\alpha^A)^{-1} h^2 w^A (\alpha u') (\bar{u}')^A (\alpha^R - \alpha^L) \delta_x. \end{aligned} \quad (3.24)$$

To get lines seven and eight we plugged in (3.20) and used  $w^A \geq 1$ .

We now integrate both sides of (3.24) over all of  $\mathbb{R}$ . Since  $F \in L^1(\mathbb{R})$  and is continuous off of a countable set,  $F(x)$  tends to zero along a sequence of  $x$ -values tending to  $+\infty$ , and at which  $F(x) = F^R(x) = F^L(x)$ . Similarly,  $F(x) = F^R(x) = F^L(x) \rightarrow 0$  along a sequence of  $x$ -values tending to  $-\infty$ . Thus (2.4) gives  $\int_{\mathbb{R}} d(wF) = 0$ . Since the average values of functions that appear are equal to the functions themselves Lebesgue almost-everywhere, for each  $N$ , we arrive at,

$$\begin{aligned} &(1/\max w) \int (|x| + 1)^{-1-\delta} ((E_{\min} - \sup_{\mathbb{R}} V) |u|^2 + \inf \alpha |hu'|^2) \\ &\leq \int 2|(P(h) - i\varepsilon)u \bar{u}'| + 2\varepsilon |uu'| \\ &\quad + \sum_{x \in J_V \setminus \{x_{j,N}\}_{j=1}^N} |u|^2 (V^R - V^L) \delta_x + \sum_{x \in J_\alpha \setminus \{x_{j,N}\}_{j=1}^N} (\alpha^A)^{-1} h^2 (\alpha u') (\bar{u}')^A (\alpha^R - \alpha^L) \delta_x. \end{aligned} \quad (3.25)$$

Sending  $N \rightarrow \infty$ , recalling (3.12) (which gives  $\sup_N (\max w) < \infty$  via (3.13)), (3.22), and  $u, u' \in L^\infty(\mathbb{R})$ , and using Young's inequality, we find

$$\begin{aligned} &\int (|x| + 1)^{-1-\delta} (|u|^2 + |hu'|^2) \\ &\leq C \int \frac{1}{\gamma h^2} |f|^2 + \gamma (|x| + 1)^{-1-\delta} |hu'|^2 + 2\varepsilon |uu'| \quad h, \gamma > 0. \end{aligned} \quad (3.26)$$

Here and below,  $C > 0$  is a constant that may change from line to line, but it is always independent of  $u$ ,  $\varepsilon$ , and  $h$ .



The second term on the right side of (3.26) can be absorbed into the left side by selecting  $\gamma$  small enough. As for the term involving  $\varepsilon$ , by Young's inequality,

$$\int |u\overline{u'}| \leq \frac{1}{2h \inf \alpha} \int |u|^2 + \frac{1}{2h} \int \alpha |hu'|^2, \quad h > 0.$$

Then

$$\begin{aligned} \int \alpha |hu'|^2 &= \operatorname{Re} \int -h^2 (\alpha u')' \overline{u} \\ &= \operatorname{Re} \int ((P(h) - i\varepsilon) - V + E) u \overline{u} \\ &\leq \frac{1}{2} \int (|x| + 1)^{-\frac{1+\delta}{2}} |f|^2 + \left( \frac{1}{2} + \|E_{\max} - V\|_{L^\infty} \right) \int |u|^2. \end{aligned}$$

Substituting these observations and calculations into (3.26) gives, for  $\varepsilon, h > 0$ ,

$$\int (|x| + 1)^{-1-\delta} (|u|^2 + |hu'|^2) \leq \frac{C}{h^2} \int |f|^2 + \frac{C\varepsilon}{h} \int |u|^2. \quad (3.27)$$

To finish, we rewrite  $\varepsilon \int |u|^2$  and estimate, for any  $\gamma > 0$ ,

$$\begin{aligned} \varepsilon \int |u|^2 &= -\operatorname{Im} \int (P(h) - i\varepsilon) u \overline{u} \\ &\leq \frac{1}{\gamma} \int |f|^2 + \gamma \int (|x| + 1)^{-1-\delta} |u|^2. \end{aligned} \quad (3.28)$$

If we now take  $\gamma$  sufficiently small (depending on  $C$  and  $h$ ), we may absorb the integral of  $(|x| + 1)^{-1-\delta} |u|^2$  in (3.28) into the left side of (3.27) to achieve

$$\int (|x| + 1)^{-1-\delta} (|u|^2 + |hu'|^2) \leq \frac{C}{h^2} \int |f|^2, \quad \varepsilon > 0, h \in (0, 1]. \quad (3.29)$$

This completes the proof of (3.5). □

#### 4. HIGH FREQUENCY BOUND ON THE CUTOFF RESOLVENT

In this Section, we prove Theorem 1.2 as an application of Theorem 3.2. We return to working with the operator  $H : \mathcal{D}(H) \rightarrow \mathcal{H}$  as defined by (1.4), where  $\alpha, \beta : \mathbb{R} \rightarrow (0, \infty)$  are BV functions obeying (1.2) and (1.8).

In that situation,  $H$  is a *black box Hamiltonian* in the sense of Sjöstrand and Zworski [SjZw91], as defined in [DyZw19, Definition 4.1]. More precisely, in our setting this means the following. First, if  $u \in \mathcal{D}(H)$ , then  $u|_{\mathbb{R} \setminus [-R_0, R_0]} \in H^2(\mathbb{R} \setminus [-R_0, R_0])$ . Second, for any  $u \in \mathcal{D}(H)$ , we have  $(Hu)|_{\mathbb{R} \setminus [-R_0, R_0]} = -u''|_{\mathbb{R} \setminus [-R_0, R_0]}$ . Third, any  $u \in H^2(\mathbb{R})$  which vanishes on a neighborhood of  $[-R_0, R_0]$  is also in  $\mathcal{D}(H)$ . Fourth,  $\mathbf{1}_{[-R_0, R_0]}(H + i)^{-1}$  is compact on  $\mathcal{H}$ ; this last condition follows from the fact that  $\mathcal{D}(H) \subseteq H^1(\mathbb{R})$ .

Then, by [DyZw19, Theorem 4.4], for any  $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$  that is identically one near  $[-R_0, R_0]$ , the cutoff resolvent (1.9) continues meromorphically  $\mathcal{H} \rightarrow \mathcal{D}(H)$  from  $\operatorname{Im} \lambda > 0$  to the complex plane. The poles of this continuation are precisely at those values  $\lambda$  for which there is a solution  $u$  to  $Hu = \lambda^2 u$  having  $u, u', Hu \in L_{\text{loc}}^2(\mathbb{R})$  in the sense of distributions, and which is outgoing, i.e. obeys

$$\pm x \geq R_0 \quad \implies \quad u(x) = c_\pm e^{\pm i\lambda x}, \quad (4.1)$$

for some nonzero constants  $c_\pm$ .

Observe that  $\lambda = 0$  is such a pole because we may take  $u(x) = 1$  for all  $x$ . Observe further that this is the only pole in the closed half plane  $\operatorname{Im} \lambda \geq 0$ . Indeed, if  $u$  satisfying (4.1) solves  $Hu = \lambda^2 u$  with  $\operatorname{Im} \lambda > 0$ , then  $u \in \mathcal{D}(H)$  and we have  $\lambda^2 \|u\|_{\mathcal{H}}^2 = \langle Hu, u \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \alpha |u'|^2 \geq 0$ , which implies

$\|u\|_{\mathcal{H}} = 0$  since  $\lambda^2 \geq 0$  is impossible when  $\text{Im } \lambda > 0$ . For  $\lambda \in \mathbb{R} \setminus \{0\}$  this follows as in the proof of [DyZw19, (2.2.12)].

*Proof of Theorem 1.2.* Set  $V_\beta := 1 - \beta$  and  $\mathcal{O} := \{\lambda \in \mathbb{C} : \text{Re } \lambda \neq 0, \text{Im } \lambda > 0\}$ . Note that  $\text{supp } V_\beta \subseteq [-R_0, R_0]$ . Define on  $\mathcal{O}$  the following families of operators  $\mathcal{H} \rightarrow \mathcal{H}$  with domain  $\mathcal{D}(H)$ ,

$$\begin{aligned} A(\lambda) &:= (\text{Re } \lambda)^{-2} \beta (H - \lambda^2) \\ &= -(\text{Re } \lambda)^{-2} \partial_x \alpha \partial_x + V_\beta + (\text{Im } \lambda)^2 (\text{Re } \lambda)^{-2} \beta - i2 \text{Im } \lambda (\text{Re } \lambda)^{-1} \beta - 1, \\ B(\lambda) &:= -(\text{Re } \lambda)^{-2} \partial_x \alpha \partial_x + V_\beta + (\text{Im } \lambda)^2 (\text{Re } \lambda)^{-2} - 1 - i2 \text{Im } \lambda (\text{Re } \lambda)^{-1}, \end{aligned} \quad (4.2)$$

Furthermore, define on  $\mathcal{O}$  the family  $\mathcal{H} \rightarrow \mathcal{H}$ ,

$$D(\lambda) := (\text{Im } \lambda)^2 (\text{Re } \lambda)^{-2} V_\beta - i2 \text{Im } \lambda (\text{Re } \lambda)^{-1} V_\beta.$$

We have,

$$B(\lambda) - A(\lambda) = D(\lambda).$$

Composing with inverses gives

$$A(\lambda)^{-1} - B(\lambda)^{-1} = B(\lambda)^{-1} D(\lambda) A(\lambda)^{-1} \implies (I - B(\lambda)^{-1} D(\lambda)) A(\lambda)^{-1} = B(\lambda)^{-1},$$

Multiplying on the left and right by  $\chi$  and noticing that  $D(\lambda) = \chi D(\lambda) \chi$ , we arrive at

$$(I - \chi B(\lambda)^{-1} \chi D(\lambda)) \chi A(\lambda)^{-1} \chi = \chi B(\lambda)^{-1} \chi, \quad \lambda \in \mathcal{O}. \quad (4.3)$$

Next, choose  $\lambda_0, \varepsilon_0 > 0$  so that  $\sup_{\mathbb{R}} V_\beta < 1 - \varepsilon_0^2 \lambda_0^{-2}$ . Identifying  $E_{\min} := 1 - \varepsilon_0^2 \lambda_0^{-2}$ ,  $E_{\max} = 1$ , and  $h := |\text{Re } \lambda|^{-1}$ , we see that Theorem 3.2 applies to  $B(\lambda)^{-1}$ . So for some  $C > 0$  and a possibly larger  $\lambda_0$ , we have

$$\|\chi B(\lambda)^{-1} \chi\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq C |\text{Re } \lambda|, \quad |\text{Re } \lambda| \geq \lambda_0, 0 < \text{Im } \lambda \leq \varepsilon_0. \quad (4.4)$$

Moreover,

$$\|D(\lambda)\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq \varepsilon_0 \|V_\beta\|_{L^\infty} \left( \frac{1}{\lambda_0^2} + \frac{2}{\lambda_0} \right), \quad |\text{Re } \lambda| \geq \lambda_0, 0 < \text{Im } \lambda \leq \varepsilon_0. \quad (4.5)$$

Thus, increasing  $\lambda_0$  again if needed, we can invert  $(I - \chi B(\lambda)^{-1} \chi D(\lambda))$  by a Neumann series when  $|\text{Re } \lambda| \geq \lambda_0, 0 < \text{Im } \lambda \leq \varepsilon_0$ . From (4.3), (4.4), and (4.5), we find

$$\chi A(\lambda)^{-1} \chi = \left( \sum_{k=0}^{\infty} (\chi B(\lambda)^{-1} \chi D(\lambda))^k \right) \chi B(\lambda)^{-1} \chi, \quad |\text{Re } \lambda| \geq \lambda_0, 0 < \text{Im } \lambda \leq \varepsilon_0. \quad (4.6)$$

Since

$$\chi R(\lambda) \chi = (\text{Re } \lambda)^{-2} \chi A(\lambda)^{-1} \chi \beta, \quad \lambda \in \mathcal{O},$$

(1.10) follows from (4.4), (4.5), and (4.6), at least when  $|\text{Re } \lambda| \geq \lambda_0, 0 \leq \text{Im } \lambda \leq \varepsilon_0$ . To get (1.10) for  $|\text{Re } \lambda| \geq \lambda_0, |\text{Im } \lambda| \leq \varepsilon_0$ , we appeal to a resolvent identity argument due to Vodev [Vo14, Theorem 1.5], which was adapted to the non-semiclassical (see, for instance, [Sh18, Lemma 5.1]). It yields, for possibly smaller  $\varepsilon_0$ , holomorphicity of  $\chi R(\lambda) \chi$  in  $|\text{Re } \lambda| \geq \lambda_0, -\varepsilon_0 \leq \text{Im } \lambda \leq 0$ , along with a bound of the form (1.10) there.  $\square$

To conclude this section, we consider the two by two matrix operator

$$G := -i \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix} : \mathcal{D}(H) \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H},$$

which arises naturally from rewriting (1.1) as a first order system. A short computation yields,

$$(G + \lambda)^{-1} = \begin{pmatrix} -\lambda R(\lambda) & -iR(\lambda) \\ i\lambda^2 R(\lambda) + i & -\lambda R(\lambda) \end{pmatrix}, \quad \text{Im } \lambda > 0. \quad (4.7)$$

The following Corollary of Theorem 1.2 is essentially well-known, and is an important input to the proof of Theorem 1.1 in Section 5. We give the proof by recalling several results from [Bu03, Vo14, DyZw19].

**Corollary 4.1.** *Let  $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$  be identically one near  $[-R_0, R_0]$ . The operator*

$$\chi(G + \lambda)^{-1}\chi := \begin{pmatrix} -\lambda\chi R(\lambda)\chi & -i\chi R(\lambda)\chi \\ i\lambda^2\chi R(\lambda)\chi + i\chi^2 & -\lambda\chi R(\lambda)\chi \end{pmatrix} : H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \quad (4.8)$$

*continues meromorphically from  $\text{Im } \lambda > 0$  to  $\mathbb{C}$ . It has no poles on  $\mathbb{R} \setminus \{0\}$  and at  $\lambda = 0$  it has a simple pole: more precisely, if  $w_0 \in H^1(\mathbb{R})$  and  $w_1 \in L^2(\mathbb{R})$ , then*

$$\lim_{\lambda \rightarrow 0} \lambda \chi(G + \lambda)^{-1} \chi \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} -i \lim_{\lambda \rightarrow 0} \lambda \chi R(\lambda) \chi w_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \langle \chi, w_1 \rangle_{\mathcal{H}} \chi \\ 0 \end{pmatrix}. \quad (4.9)$$

*Furthermore, there exist  $C, \lambda_0, \varepsilon_0 > 0$  so that*

$$\|\chi(G + \lambda)^{-1}\chi\|_{H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \oplus L^2(\mathbb{R})} \leq C, \quad (4.10)$$

*whenever  $|\text{Re } \lambda| \geq \lambda_0$ , and  $|\text{Im } \lambda| \leq \varepsilon_0$ .*

*Proof.* As described above, by [DyZw19, Theorem 4.4] and the proof of [DyZw19, (2.2.12)], the operator  $\chi R(\lambda)\chi : L^2(\mathbb{R}) \rightarrow \mathcal{D}(H)$  continues meromorphically from  $\text{Im } \lambda > 0$  to  $\mathbb{C}$ , and has no poles in  $\mathbb{R} \setminus \{0\}$ . This implies that each entry of (4.8) continues meromorphically as an operator between the appropriate spaces, again without poles in  $\mathbb{R} \setminus \{0\}$ .

Next, as in the proof of [DyZw19, Theorem 2.7], (1.8) implies that near  $\lambda = 0$ ,

$$\chi R(\lambda)\chi w_1 = \frac{i}{2\lambda} \langle \chi, w_1 \rangle_{\mathcal{H}} \chi + A(\lambda)w_1, \quad (4.11)$$

where  $A(\lambda) : \mathcal{H} \rightarrow \mathcal{D}(H)$  is holomorphic near zero, and hence we have (4.9).

With (1.10) already in hand, to establish (4.10), it suffices to supply  $\lambda_0, \varepsilon_0 > 0$  so that

$$\lambda^2 \chi R(\lambda)\chi + \chi^2 = \chi H R(\lambda)\chi : H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (4.12)$$

$$\lambda \chi R(\lambda)\chi : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}), \quad (4.13)$$

are uniformly bounded for  $|\text{Re } \lambda| \geq \lambda_0$  and  $|\text{Im } \lambda| \leq \varepsilon_0$ . When  $|\text{Re } \lambda| \geq \lambda_0$  and  $0 < \text{Im } \lambda \leq \varepsilon_0$  this follows from the proof of [Bu03, Proposition 2.4], see in particular [Bu03, (2.14), (2.17), and (2.19)]. To extend these bounds to strips below the real axis, we use once more Vodev's resolvent identity ([Vo14, Theorem 1.5] and [Sh18, Lemma 5.1]).

□

## 5. WAVE DECAY

*Proof of Theorem 1.1.* This section follows part of Section 3 of [Vo99].

Recall that we use  $w(t)$  to denote the solution (1.5) to (1.1), with initial data  $w_0 \in \mathcal{D}(H)$  and  $w_1 \in \mathcal{D}(H^{1/2})$ . We have  $\text{supp } w_0, \text{supp } w_1 \subseteq (-R, R)$ , and the coefficients of (1.1) obey (1.2) and (1.8). We want to show that the local energy  $\|w(\cdot, t) - w_\infty\|_{H^1(-R_1, R_1)} + \|\partial_t w(\cdot, t)\|_{L^2(-R_1, R_1)}$  decays exponentially, for a suitable constant  $w_\infty$ .

Choose  $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$  such that  $\chi = 1$  near  $[-R_1, R_1] \cup [-R, R] \cup [R_0, R_0]$  ( $R_0$  given as in (1.8)). Recall from Corollary 4.1 that there exist  $C, \lambda_0, \varepsilon_0 > 0$  such that

$$\|\chi(G + \lambda)^{-1}\chi f\| \leq C\|f\|,$$

whenever  $|\text{Re } \lambda| \geq \lambda_0$  and  $|\text{Im } \lambda| \leq \varepsilon_0$ , where here and for the rest of this section all norms are  $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$  unless otherwise specified.

We have

$$\begin{aligned} w(t) &= \cos(tH^{1/2})w_0 + \sin(tH^{1/2})H^{-1/2}w_1, \\ \partial_t w(t) &= -\sin(tH^{1/2})H^{1/2}w_0 + \cos(tH^{1/2})w_1, \\ \partial_t^2 w(t) &= -Hw(t). \end{aligned}$$

Consequently, after defining

$$f := \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad U(t)f := \begin{pmatrix} w(t) \\ \partial_t w(t) \end{pmatrix},$$

we have

$$\|U(t)f\| \leq C\|f\|, \quad \partial_t U(t)f = iGU(t)f, \quad U(t)U(s)f = U(t+s)f, \quad (5.1)$$

for all real  $t$  and  $s$ , for some  $C > 0$  independent of  $t$  and  $f$ . (Note that  $U(t)f$  is still defined even if only  $w_0 \in \mathcal{D}(H^{1/2})$ ,  $w_1 \in \mathcal{H}$ .)

Take  $\varphi \in C^\infty(\mathbb{R}; [0, 1])$  which is 0 on  $(-\infty, 1]$  and 1 on  $[2, \infty)$  and put

$$W(t)f := \varphi(t)U(t)f = \int_{\operatorname{Im} \lambda = \varepsilon} e^{-it\lambda} \check{W}(\lambda) d\lambda, \quad \check{W}(\lambda) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{is\lambda} W(s)f ds.$$

Since  $\partial_t W(t)f = \varphi'(t)U(t)f + iGW(t)f$  we get

$$W(t)f = \int_{\operatorname{Im} \lambda = \varepsilon} e^{-it\lambda} (G + \lambda)^{-1} (i\varphi'Uf)^\sim(\lambda) d\lambda.$$

Since  $\operatorname{supp} w_0, \operatorname{supp} w_1 \subseteq (-R, R)$ , by finite speed of propagation for the wave equation, and increasing  $R > 0$  if necessary, we have that,  $x \mapsto U(t)f$  is supported in  $(-R, R)$  for all  $t \in [0, 2]$ . By continuity of integration, the same is true of  $x \mapsto (i\varphi'Uf)^\sim(\lambda)$  for every  $\lambda$ . Hence  $\lambda \mapsto (i\varphi'Uf)^\sim(\lambda)$  is entire and rapidly decaying as  $|\operatorname{Re} \lambda| \rightarrow \infty$  with  $|\operatorname{Im} \lambda|$  remaining bounded and further  $(i\varphi'Uf)^\sim(\lambda) = \chi(i\varphi'Uf)^\sim(\lambda)$ . Take  $\varepsilon \in (0, \varepsilon_0)$  small enough that  $\lambda = 0$  is the only pole of  $\chi R(\lambda)\chi$  (and hence also of  $\chi(G + \lambda)^{-1}\chi$  by (4.7)) in the half plane  $\operatorname{Im} \lambda \geq -\varepsilon$ . By deformation of contour,

$$\chi W(t)f = \lim_{\lambda \rightarrow 0} \lambda \chi(G + \lambda)^{-1} \chi \int_{\mathbb{R}} \varphi'(s)U(s)f ds + \int_{\operatorname{Im} \lambda = -\varepsilon} e^{-it\lambda} \chi(G + \lambda)^{-1} \chi (i\varphi'Uf)^\sim(\lambda) d\lambda.$$

To simplify this, use (4.9) and put

$$W_1(t)f := \int_{-\infty}^{\infty} e^{-it\lambda} (G + \lambda - i\varepsilon)^{-1} (i\varphi'Uf)^\sim(\lambda - i\varepsilon) d\lambda,$$

to obtain

$$\chi W(t)f = \left( \frac{1}{2} \chi \int_{\mathbb{R}} \int_0^2 \beta(x) \chi(x) \varphi'(s) \partial_s w(s, x) ds dx \right)_0 + e^{-\varepsilon t} \chi W_1(t)f.$$

To simplify the first term, we integrate by parts in  $s$ , using  $\varphi' = -(1 - \varphi)'$ , to obtain

$$\int_{\mathbb{R}} \int_0^2 \beta(x) \chi(x) \varphi'(s) \partial_s w(s, x) ds dx = \int_{\mathbb{R}} \beta \chi w_1 + \int_{\mathbb{R}} \int_0^2 \beta(x) \chi(x) (1 - \varphi(s)) \partial_s^2 w(s, x) ds dx.$$

Now observe that  $\partial_s^2 w = -Hw$  and  $\langle \chi, Hw(s) \rangle_{\mathcal{H}} = 0$  for  $s \in [0, 2]$  (the latter fact following from  $\chi = 1$  near  $[-R, R]$  and  $\operatorname{supp} w(s) \subseteq (-R, R)$  for  $s \in [0, 2]$ ). Thus

$$\chi W(t)f = \frac{1}{2} \begin{pmatrix} \langle \chi, w_1 \rangle_{\mathcal{H}} \chi \\ 0 \end{pmatrix} + e^{-\varepsilon t} \chi W_1(t)f.$$

It now suffices to show that

$$\|\chi W_1(t)f\| \leq C e^{\varepsilon t/2} \|f\|.$$

To prove this, we first use Plancherel's theorem, along with the fact that by (5.1), the operator norm  $\|U(t)\|_{H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \oplus L^2(\mathbb{R})}$  is uniformly bounded for all  $t \in \mathbb{R}$ , as well as the fact that by Corollary 4.1, for any  $\varepsilon > 0$  small enough, the operator norm  $\|\chi(G + \lambda - i\varepsilon)^{-1}\chi\|_{H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \oplus L^2(\mathbb{R})}$  is uniformly bounded for all  $\lambda \in \mathbb{R}$ , to obtain

$$\begin{aligned} \int \|\chi W_1(t)f\|^2 dt &= C \int \|\chi(G + \lambda - i\varepsilon)^{-1}(\varphi' U f)^\sim(\lambda - i\varepsilon)\|^2 d\lambda \\ &\leq C \int \|(\varphi' U f)^\sim(\lambda - i\varepsilon)\|^2 d\lambda \\ &= C \int e^{2\varepsilon t} \|\varphi'(t)U(t)f\|^2 dt \leq C\|f\|^2. \end{aligned} \tag{5.2}$$

Next, compute

$$(\partial_t - iG)\chi W_1(t)f = -i[G, \chi]W_1(t)f + \varepsilon\chi W_1(t)f - i\chi \int e^{-it\lambda}(i\varphi' U f)^\sim(\lambda - i\varepsilon) d\lambda =: \widetilde{W}_1(t)f.$$

Integrating both sides of  $\partial_s(U(t-s)\chi W_1(s)f) = U(t-s)\widetilde{W}_1(s)f$  from  $s = 0$  to  $s = t$  gives

$$\chi W_1(t)f = U(t)\chi W_1(0)f + U(t) \int_0^t U(-s)\widetilde{W}_1(s)f ds.$$

Thus

$$\|\chi W_1(t)f\| \leq C(\|f\| + \int_0^t \|\widetilde{W}_1(s)f\| ds) \leq C(\|f\| + t^{1/2} \left( \int_0^t \|\widetilde{W}_1(s)f\|^2 ds \right)^{1/2}).$$

Now check that, since  $\|[G, \chi]W_1(t)f\| \leq C\|W_1(t)f\|$ , calculating as in (5.2), we obtain  $\int \|\widetilde{W}_1(s)f\|^2 ds \leq C\|f\|^2$ , and hence

$$\|\chi W_1(t)f\| \leq C(1 + t^{1/2})\|f\|$$

as desired. □

## 6. ACKNOWLEDGMENTS

K.D. gratefully acknowledges support under NSF grant DMS 1708511. J.S. gratefully acknowledges support from four sources: ARC grant DP180100589, NSF grant DMS 1440140 while in residence at the Mathematical Sciences Research institute in Berkeley, CA, NSF grant DMS 2204322, and a University of Dayton Catholic Intellectual Tradition grant. Thanks also to the anonymous referee for helpful comments and corrections.

## APPENDIX A. CHARACTERIZATION OF $\mathcal{D}(H^{1/2})$

In this Appendix we show

**Lemma A.1** ([Re22b]). *It holds that  $\mathcal{D}(H^{1/2}) = H^1(\mathbb{R})$ , and that  $u \mapsto \|u\|_{H^1}$ ,  $u \mapsto (\|u\|_{\mathcal{H}}^2 + \|H^{1/2}u\|_{\mathcal{H}}^2)^{1/2}$  are equivalent norms.*

*Proof.* First, recall the well-known fact that  $\mathcal{D}(H^{1/2})$  equals the form domain associated to  $H$ , namely,  $\mathcal{D}(H^{1/2})$  is the completion of  $\mathcal{D}(H)$  with respect to the norm  $\|u\|_{+1}^2 := \langle Hu, u \rangle_{L^2} + \langle u, u \rangle_{L^2}$ . On  $\mathcal{D}(H)$ , it's clear that there exist  $C, c > 0$  so that  $c\|\cdot\|_{H^1}^2 \leq \|\cdot\|_{+1}^2 \leq C\|\cdot\|_{H^1}^2$ .

If  $u \in \mathcal{D}(H^{1/2})$ , then there exists a  $\|\cdot\|_{+1}$ -Cauchy sequence  $u_j \in \mathcal{D}(H)$  converging to  $u$  in  $\mathcal{H}$  (or, equivalently, converging to  $u \in L^2(\mathbb{R})$ ). Because  $\|\cdot\|_{+1}$  and  $\|\cdot\|_{H^1}$  are equivalent on  $\mathcal{D}(H)$ , we

get that the  $u_j$  are also  $\|\cdot\|_{H^1(\mathbb{R})}$ -Cauchy. By completeness of  $H^1(\mathbb{R})$ , we conclude  $u \in H^1(\mathbb{R})$ . We also have

$$\|H^{1/2}u\|_{\mathcal{H}}^2 = \lim_{j \rightarrow \infty} \|H^{1/2}u_j\|_{\mathcal{H}}^2 = \lim_{j \rightarrow \infty} \langle Hu_j, u_j \rangle_{\mathcal{H}} \leq C \lim_{j \rightarrow \infty} \|u_j\|_{H^1}^2 = C\|u\|_{H^1}^2,$$

where the first equals sign follows since  $H^{1/2}$  is a closed operator.

To show  $H^1(\mathbb{R}) \subseteq \mathcal{D}(H^{1/2})$ , first suppose  $u \in H^1(\mathbb{R})$  has compact support. Approximate  $\alpha u'$  in  $L^2(\mathbb{R})$  by  $\tilde{v}_j \in C_0^\infty(\mathbb{R})$  which have support in a fixed compact set. Choose  $\varphi_0 \in C_0^\infty(\mathbb{R})$  with  $\int \varphi_0/\alpha = 1$ , and put

$$v_j := \tilde{v}_j - \left( \int \tilde{v}_j/\alpha \right) \varphi_0.$$

Then  $\int v_j/\alpha = 0$  and the  $v_j/\alpha \rightarrow u'$  in  $L^2(\mathbb{R})$  since  $\int u' = 0$ . We clearly have  $u_j := \int_{-\infty}^x v_j/\alpha \in \mathcal{D}(H)$ . Moreover, because  $\int_{-\infty}^x v_j/\alpha \rightarrow \int_{-\infty}^x u' = u(x)$  locally uniformly in  $x$ , it follows that  $u_j \rightarrow u$  in  $H^1(\mathbb{R})$ , and that the  $u_j$  are  $\|\cdot\|_{+1}$ -Cauchy. Hence  $u \in \mathcal{D}(H^{1/2})$ .

For general  $u \in H^1(\mathbb{R})$ , choose a sequence  $\tilde{u}_j$  of compactly supported functions with  $\|\tilde{u}_j - u\|_{H^1} \leq 2^{-j-1}$ . For each  $j$ , use the construction of the previous paragraph to find  $u_j \in \mathcal{D}(H)$  with  $\|u_j - \tilde{u}_j\|_{H^1} \leq 2^{-j-1}$ . Then the  $u_j \rightarrow u$  in  $H^1(\mathbb{R})$  and

$$\|u_j - u_k\|_{+1}^2 \leq C\|u_j - u_k\|_{H^1} \rightarrow 0 \quad \text{as } j, k \rightarrow \infty.$$

Thus  $u \in \mathcal{D}(H^{1/2})$  and

$$c\|u\|_{H^1}^2 = c \lim_{j \rightarrow \infty} \|u_j\|_{H^1}^2 \leq \lim_{j \rightarrow \infty} \|u_j\|_{+1}^2 = \lim_{j \rightarrow \infty} (\|H^{1/2}u_j\|_{\mathcal{H}}^2 + \|u_j\|_{\mathcal{H}}^2) = \|H^{1/2}u\|_{\mathcal{H}}^2 + \|u\|_{\mathcal{H}}^2.$$

□

## APPENDIX B. ELEMENTARY PROPERTIES OF BV FUNCTIONS

This appendix collects some facts about functions of bounded variation which can be found in [VoHu85] and [AFP00]. The main results are the integration by parts formula (2.5), the product rule (2.6), and the chain rules (2.7) and (2.8). The books [VoHu85] and [AFP00] are mostly concerned with higher dimensional problems, so we present proofs for the much simpler one dimensional case here.

We continue to use the notation (2.1) and (2.3) from Section 2. For  $\psi \in L^1(\mathbb{R})$  compactly supported and satisfying  $\int \psi = 1$ , and for  $\varepsilon > 0$ , let

$$f_\varepsilon(x) = \int f(x - \varepsilon y) \psi(y) dy = \varepsilon^{-1} \int f(y) \psi(\varepsilon^{-1}(x - y)) dy. \quad (\text{B.1})$$

Then, accordingly as  $\psi$  is supported in  $[0, \infty)$  or supported in  $(-\infty, 0]$  or even, we have

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon = f^L \text{ or } f^R \text{ or } f^A, \quad \text{pointwise on } \mathbb{R}. \quad (\text{B.2})$$

Indeed, use the dominated convergence theorem in the first two cases and average them to get the third case.

*Proof of Proposition 2.1.* The integration by parts formula (2.5) follows as in the proof of [Fo07, Theorem 3.36]. Indeed, let  $\Omega = \{(x, y) \in \mathbb{R}^2 : a < x \leq y \leq b\}$ . Since  $\varphi$  is continuous and  $\varphi'$  is piecewise continuous, it holds that  $d\varphi = \varphi' dx$ . Using Fubini's theorem, we evaluate the product measure  $df \times d\varphi$  two different ways,

$$\begin{aligned} \int_{(a,b] \times (a,b]} \mathbf{1}_\Omega(x, y) df(x) \times d\varphi(y) &= \int_{(a,b]} \int_{(a,y]} df(x) d\varphi(y) \\ &= \int_{(a,b]} (f^R(y) - f^R(a)) \varphi'(y) dy = \int_{(a,b]} f(y) \varphi'(y) dy, \end{aligned}$$

where we used that  $f^R = f$  Lebesgue almost everywhere, and that the boundary terms vanish since  $\varphi(a) = \varphi(b) = 0$ . Similarly,

$$\int_{(a,b) \times (a,b)} \mathbf{1}_\Omega(x,y) d_f(x) \times d\varphi(y) = \int_{(a,b)} \int_{[x,b]} d\varphi(y) df(x) = - \int_{(a,b)} \varphi(x) df(x).$$

□

*Proof of Proposition 2.2.* Let  $\psi \in C_0^\infty(\mathbb{R})$  be an even function satisfying  $\int \psi = 1$ . For any  $\varepsilon > 0$ , define  $f_\varepsilon$  by (B.1), and for any,  $\eta > 0$  define  $g_\eta$  similarly. Then

$$(f_\varepsilon g_\eta)' = f_\varepsilon (g_\eta)' + g_\eta (f_\varepsilon)'. \quad (\text{B.3})$$

We now show that taking  $\eta \rightarrow 0^+$  and then  $\varepsilon \rightarrow 0^+$  in (B.3) gives (2.6). Let  $\varphi \in C_0^\infty(\mathbb{R})$ . First, by integration by parts,

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\eta \rightarrow 0^+} \int \varphi (f_\varepsilon g_\eta)' dx = - \lim_{\varepsilon \rightarrow 0^+} \lim_{\eta \rightarrow 0^+} \int \varphi' f_\varepsilon g_\eta dx.$$

Then, we observe that  $\int \varphi' f_\varepsilon g_\eta dx \rightarrow \int \varphi' f_\varepsilon g dx$  by the dominated convergence theorem. Indeed,  $g_\eta \rightarrow g^A \stackrel{\text{a.e.}}{=} g$  by (B.2), and  $|\varphi' f_\varepsilon g_\eta|$  is uniformly bounded for  $\varepsilon$  fixed and  $\eta$  small. Similarly,  $\int \varphi' f_\varepsilon g dx \rightarrow \int \varphi' f g dx$ . Finally,  $-\int \varphi' f g dx = \int \varphi d(fg)$  by (2.5).

Next

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \lim_{\eta \rightarrow 0^+} \int \varphi f_\varepsilon g_\eta' dx &= - \lim_{\varepsilon \rightarrow 0^+} \lim_{\eta \rightarrow 0^+} \int (\varphi f_\varepsilon)' g_\eta dx \\ &= - \lim_{\varepsilon \rightarrow 0^+} \int (\varphi f_\varepsilon)' g dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \int \varphi f_\varepsilon dg \\ &= \int \varphi f^A dg. \end{aligned}$$

For the first equal sign, we integrate by parts; for the second, we use the dominated convergence theorem, as in the previous paragraph. The third equal sign follows from (2.5), and the fourth from another application of the dominated convergence theorem (and (B.2)).

Continuing, by (B.1), (2.5) and Fubini's theorem,

$$\begin{aligned} \int \varphi g^A (f_\varepsilon)' dx &= \int \varphi(x) g^A(x) \varepsilon^{-2} \left[ \int \psi'(\varepsilon^{-1}(x-y)) f(y) dy \right] dx \\ &= \int \varphi(x) g^A(x) \varepsilon^{-1} \left[ \int \psi(\varepsilon^{-1}(x-y)) df(y) \right] dx \\ &= \varepsilon^{-1} \int \left[ \int \varphi(x) g^A(x) \psi(\varepsilon^{-1}(y-x)) dx \right] df(y) = \int (\varphi g^A)_\varepsilon df, \end{aligned} \quad (\text{B.4})$$

where for the third equal sign we used that  $\psi$  is even. Since  $\varphi$  and  $\psi$  have compact support, the integrals against  $df$  make sense, and the application of Fubini's theorem is justified (even though  $df$  may be finite only after it is restricted to a bounded Borel set). Finally,

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\eta \rightarrow 0^+} \int \varphi g_\eta (f_\varepsilon)' dx = \lim_{\varepsilon \rightarrow 0^+} \int \varphi g^A (f_\varepsilon)' dx = \lim_{\varepsilon \rightarrow 0^+} \int (\varphi g^A)_\varepsilon df = \int \varphi g^A df,$$

by the dominated convergence theorem, (B.2), and (B.4).

□

*Proof of Proposition 2.3.* Using the decomposition (2.2), we see that  $e^f = e^{f_{r,+}}e^{-f_{r,-}}$  has locally bounded variation, as it is a product of functions of locally bounded variation.

Let  $\varphi, \psi \in C_0^\infty(\mathbb{R})$ , with  $\psi$  even and satisfying  $\int \psi = 1$ . To show (2.7):

$$\begin{aligned}
\int \varphi d(e^f) &= - \int e^f \varphi' dx \\
&= - \int \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{f^n}{n!} \varphi' dx \\
&= - \lim_{N \rightarrow \infty} \sum_{n=0}^N \int \frac{(f^n)}{n!} \varphi' dx \\
&= \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N \int \frac{\varphi}{n!} df^n - \sum_{n=0}^N \int d\left(\varphi \frac{f^n}{n!}\right) \right) \\
&= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int \frac{\varphi}{(n-1)!} f^{n-1} df \\
&= \int \varphi e^f df.
\end{aligned}$$

The first equal sign follows from (2.5). The third and sixth equal signs use the dominated convergence theorem; the fourth follows by (2.6), and the fifth by (2.4) and the Remark after (2.6).

For (2.8), we first note that, because  $g$  has locally bounded variation, so does  $e^g$ . We compute,

$$\begin{aligned}
\int \varphi d(e^g) &= - \int e^g \varphi' dx \\
&= - \int_{-\infty}^{x_1} e^{r_0} \varphi' dx - \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} e^{r_j} \varphi' dx - \int_{x_N}^{\infty} e^{r_N} \varphi' dx \\
&= \sum_{j=1}^N (e^{r_j} - e^{r_{j-1}}) \varphi(x_j).
\end{aligned}$$

□

## REFERENCES

- [AFP00] L. Ambrosio, N. Fusco, and D. Pallara. Functions of Bounded Variation and Free Discontinuity Problems. Clarendon Press, Oxford (2000) 14
- [AGPP22] A. Arnold, S. Geever, I. Perugia, and D. Ponomarev. On the exponential time-decay for the one-dimensional wave equation with variable coefficients. *Comm. Pure and Appl. Math.* 21(10) (2022), 3389–3405
- [BIZ16] C. Beli, L. Ignat, and E. Zuazua. Dispersion for 1-D Schrödinger and wave equations with BV coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33(6) (2016), 1473–1495
- [Bu98] N. Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.* 180(1) (1998), 1–29
- [Bu03] N. Burq. Global Strichartz estimates for nontrapping geometries: about an article by H. Smith and C. Sogge. *Comm. Partial Differential Equations* 28(9-10) (2003) 1675–1683
- [CaVo02] F. Cardoso and G. Vodev. Uniform Estimates of the Resolvent of the Laplace-Beltrami Operator on Infinite Volume Riemannian Manifolds. II. *Ann. Henri Poincaré* 4(3) (2002), 673–691
- [ChIk20] R. Charao and R. Ikehata. A note on decay rates of the local energy for wave equations with Lipschitz wave speeds. *J. Math. Anal. Appl.* 483(2) (2020), 1–14
- [Da14] K. Datchev. Quantitative limiting absorption principle in the semiclassical limit. *Geom. Func. Anal.* 24(3) (2014), 740–747



- [DaSh20] K. Datchev and J. Shapiro. Semiclassical Estimates for Scattering on the Real Line. *Comm. Math. Phys.* 376(3) (2020), 2301–2308 4, 5
- [DyZw19] S. Dyatlov and M. Zworski. Mathematical Theory of Scattering Resonances. Graduate Studies in Mathematics 200. American Mathematical Society, Providence, RI (2019) 2, 3, 9, 10, 11
- [Fo07] G. Folland. Real analysis: modern techniques and their applications, 2nd ed. Wiley, NY (2007) 3, 14
- [GaSh22] J. Galkowski and J. Shapiro. Semiclassical resolvent bounds for long range Lipschitz potentials. *Int. Math. Res. Not. IMRN* 2022(18) (2022), 14134–14150 4
- [HiZw17] P. Hintz and M. Zworski. Wave decay for star-shaped obstacles in  $\mathbb{R}^3$ : papers of Morawetz and Ralston revisited. *Math Proc. R. Ir. Acad.* 117A(2) (2017), 47–62 2
- [KlVo19] F. Klopp and M. Vogel. Semiclassical resolvent estimates for bounded potentials. *Pure Appl. Anal.* 1(1) (2019), 1–25 4
- [LMP62] P. D. Lax, C. S. Morawetz, and R. S. Phillips. The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. *Bull Amer. Math. Soc.* 68 (1962), 593–595. 2
- [LaPh89] P. D. Lax and R. S. Phillips. Scattering Theory. Revised Edition. Academic Press, Inc., San Diego, 1989. 2
- [Mo61] C. S. Morawetz. The decay of solutions of the exterior initial-boundary value problem for the wave equation. *Comm. Pure Appl. Math.* 14 (1961), 561–568. 2
- [Pi22] I. Pinelis. Chain rule for  $e^f$ , where  $f$  has bounded variation. Math Overflow, <https://mathoverflow.net/q/425346> 4
- [Re22a] C. Remling. Chain rule for  $e^f$ , where  $f$  has bounded variation. Math Overflow, <https://mathoverflow.net/q/425346> 4
- [Re22b] C. Remling. Is the Sobolev space  $H^1(\mathbb{R})$  contained in the domain of  $(-\partial_x \alpha(x) \partial_x)^{1/2}$ ? Math Overflow, <https://mathoverflow.net/q/432325> 1, 13
- [Sh18] J. Shapiro. Local energy decay for Lipschitz wavespeeds. *Comm. Partial Differential Equations* 43 (5) (2018), 839–858 2, 10, 11
- [SjZw91] J. Sjöstrand and M. Zworski. Complex Scaling and the Distribution of Scattering Poles. *J. Amer. Math. Soc.* 4(4) (1991), 729–769 2, 9
- [Va89] B. R. Vainberg. *Asymptotic methods in equations of mathematical physics*. Gordon and Breach Science Publishers. New York, 1989. Translated from the Russian by E. Primrose. 2
- [Vo99] G. Vodev. On the uniform decay of the local energy. *Serdica Math. J.* 25(3) (1999), 191–206 3, 11
- [Vo14] G. Vodev. Semi-classical resolvent estimates and regions free of resonances. *Math. Nach.* 287(7) (2014), 825–835 10, 11
- [VoHu85] A. I. Vol’pert and S. I. Hudjaev. Analysis in classes of discontinuous functions and equations of mathematical physics. Martinus Nijhoff Publishers, Dordrecht (1985) 14
- [Ze05] A. Zettl, Sturm–Liouville Theory. Mathematical Surveys and Monographs 121. American Mathematical Society, Providence, RI (2005) 5