

Derivation Of \mathcal{PT} -symmetric Sine-Gordon model and it's relevance to non-equilibrium

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The Parity-Time \mathcal{PT} -symmetric non-Hermitian Sine-Gordon(nhSG) model derived from the nonequilibrium spin-boson model. We have derived the Keldysh rotation for spin operators, from which the SG model can be derived. We perform renormalization group calculations in the Keldysh fields and compare the fixed points and the flow of the effective couplings of nonequilibrium and non-Hermitian models. Also, we explicitly find the self-energies and compare the two methods to understand the regimes where \mathcal{PT} symmetry preserved regime, and nonequilibrium regimes persist. RG flow of couplings of nonequilibrium model and non-hermitian model both capture the standard Berezinskii-Kosterlitz-Thouless(BKT) physics in a strong coupling regime.

I. INTRODUCTION

The wide applications of the SG model[1] ranging from gravity to superconducting junctions attract attention even in recent times. Nonlinear regimes of the SG model are quite interesting as it hosts soliton states[2]. Recent RG studies on the non-hermitian extension of the Sine Gordon model raise many curiosities in the context of quantum criticality. A generic nonequilibrium problem has various regimes, namely gain-loss balancing, chaotic regimes, and the non-Hermiticity with certain symmetries are essential to explore as they capture gain-loss balancing regimes. SG model shows the unique modes[3] in the noisy regime. These are also treated by Open field theory[4]. Quantization of soliton[5] discuss the two vector solutions. A Ketaeve chain[6] with interaction studied using bosonization, Jordan-Wigner Transformation, and cumulant expansion showed standard BKT physics. The perturbative RG schemes have captured the robust coupling regime in SG field expansion[7] Flow equation renormalization(FeQRG) is carried out on the SG model[8]. Functional renormalization(FRG) and numerical renormalization(NRG) is carried out on the SG model[9, 10] to show a dissipative transition in Josephson junctions.

On soliton resolution conjecture a non linear analysis[11–14] by various approaches having sub-critical mass, massless situations in discrete nonlinear Schrodinger equations. Lie group structure RG flow[15] is discussed through asymptotic freedom on the chiral SG model. The condition is derived and a bifurcation point is attained from the RG solution. Phase transitions at a finite temperature, a diagrammatic approach[16] show restoration of the symmetry above a critical temperature. A generalized universality in massive SG due to the interplay between the UV and IR scaling laws is discussed with its global behavior within RG flow[17]. SG model on lattice gives a universality class $\mathcal{N} = 4$ super-Yang-Mills theory and massive Thirring model[18]. Berezinskii-Kosterlitz-Thouless(BKT) transition in proximity coupled Superconducting Arrays[19] in 2D. The BKT was formulated to describe the superfluidity in Helium-4 thin films; later, it was applied to superconducting films like Josephson junctions. There is BKT physics in the SG

model, so it is widely applied and compared in the 2D XY models.

Asymptotic safety in SG[20] There are nontrivial fixed points in the IR limit, which showed them an irrelevant coupling leads to the same phase structure. Nonconservative kinks shown at the continuum limit with unusual divergence and conventional Kosterlitz-Thouless point consist of normal behavior as that of the Coleman point, which can be anomalous[21]. Wegner-Houghton RG method [22] shows the expansion of the effective coupling beta functions to leading order we get BKT physics. Dual \mathcal{PT} -symmetric quantum field theories[23] have discussed about the mass term breaking the symmetry and its relevance to SG and Thirring models. Boundary Sine Gordon(BSG)[24–26] shows the computing of the current without having the Bethe solution, the self-duality in the impurity problems, and its connection to Seiberg and Witten supersymmetric non-Abelian gauge theory. Out of equilibrium transport and interacting resonant model and BSG relevance[27] shown in 1D interacting resonant level Anderson model. A coherent to incoherent transition is observed in the superconducting qubit and driven Spin-Boson model[28].

A spin-boson model in nonequilibrium is considered, and Keldysh rotations are performed in fermion-bosons. The spin-boson requires spin rotation in Keldysh contours; hence we derived those by Jordan-Wigner transformations for fermion to spin representation. This unitary rotation naturally yields a \mathcal{PT} -symmetric SG model on which we perform the RG. We point out subtle differences in the selfenergies of standard SG and non-Hermitian SG in the strong interaction regime. We explore the universality in the SG model through RG and where they obtain the spiral invariants for Coulomb gas [2] these invariant also obtained in the RG calculations of non-hermitian oscillators[29–31] where various RG calculations yield complex valued ODE which exhibit various limit-cycles[32] are obtained as a solution of RG invariants. The oscillators limit of the self-dual sinh-Gordon model in the article[33] shown to exhibit soliton mass having Coleman bound in analytic continuation and has renormalization analogous to \mathcal{PT} -strength in the non-hermitian oscillators[32]. These complex RG nonlinear ODEs for couplings are multivalued and have Kardar-

Parisi-Zhang(KPZ) criticality.

II. MODEL AND FORMALISM

We start with a generalized spin-boson model as Tomonaga-model[34, 35] of kondo problem for one dimensional system, In order to derive and show the connection to \mathcal{PT} -symmetric Sine-Gordon model within non equilibrium formalism, The spin boson model reads as the following,

$$\begin{aligned} \mathcal{S} = \int d\tau \left(\frac{J_{\perp}}{4\pi a} \sigma_x + \hbar v_f \sum_k a_k^{\dagger} a_k \right. \\ \left. + \frac{1}{2} \left(\frac{J_{\parallel}}{2} - \frac{1}{\rho} \right) \sigma_z \sum_{k < \frac{k_f}{2}} \frac{|k|^{\frac{1}{2}}}{\pi L} (a_k + a_{-k}^{\dagger}) \right) \end{aligned} \quad (1)$$

above model 1 also has connection to kondo model in 1D which shown earlier by various works[36, 37], we will derive the following keldysh rotations for spin and bosons,

$$\begin{aligned} \begin{pmatrix} \sigma_{y+} \\ \sigma_{x-} \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \sigma_{y1} \\ \sigma_{y2} \end{pmatrix} \\ \begin{pmatrix} \sigma_{x+} \\ \sigma_{y-} \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \sigma_{x1} \\ \sigma_{x2} \end{pmatrix} \\ \begin{pmatrix} \sigma_{z+} \\ \sigma_{z-} \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{z1} \\ \sigma_{z2} \end{pmatrix} \\ \begin{pmatrix} a_{+} \\ a_{-} \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \end{aligned} \quad (2)$$

The above transformations are found by spin to fermion transformation which is done by preserving the $\{\sigma^+, \sigma^-\} = \{f_0^{\dagger}, f_0\}$, using the generalized Jordan-Wigner Transformations[38, 39] for local spin to local fermion then using the Keldysh Rotations[40] for fermions and bosons We derive the following Keldysh action using above transformations, Note that the effective action can be written in-terms of the single time due to the Unitary rotations.

$$\begin{aligned} \mathcal{S} = \int_0^{\infty} d\tau \frac{1}{\sqrt{2}} J_{\perp} (\sigma_{\beta}^+ + \sigma_{\beta'}^-) + \sum_{k\beta=\pm} \hbar v_f a_{k\beta}^{\dagger} a_{k\beta} \\ + \sum_{\beta=\pm} \tilde{J}_{\parallel} \beta \sigma_{\beta}^z \sum_{\beta k < \frac{k_f}{2}} \frac{|k|^{\frac{1}{2}}}{\pi L} (a_{k\beta} + \beta' a_{-k\beta'}^{\dagger}) \end{aligned} \quad (3)$$

where $\tilde{J}_{\parallel} = \frac{1}{2} \left(\frac{J_{\parallel}}{2} - \frac{1}{\rho} \right)$

We can do a rotation of the action as in the mapping of spin-boson to a kondo model, It is always possible to define a unitary transformation as the following in the two contours, But it is very difficult to keep track of the mixed spin-boson terms. Hence we rotate them in unitary fashion and write the whole action in single time which

does drastic simplification and all selfenergy calculations are tractable.

$$\mathcal{U} = e \left[\tilde{J}_{\parallel} \sum_{\beta k} \left(\frac{\pi}{|k|L} \right)^{\frac{1}{2}} \sigma_{\beta}^z (a_{k\beta} + \beta' a_{-k\beta'}^{\dagger}) \right] \quad (4)$$

This operator in equation 4 reduce to a simple as $\mathcal{U} = \exp \left(\tilde{J}_{\parallel} \sum_{\beta k} \left(\frac{\pi}{|k|L} \right)^{\frac{1}{2}} \sigma_1^z (a_{k1} + a_{-k1}^{\dagger}) \right)$ after the Keldish rotation defined for spin and boson in equation 2. We can derive the effective action as $\mathcal{S}_{eff} = \mathcal{U} \mathcal{S} \mathcal{U}^{\dagger}$ which is Baker-Campbell-Hausdorff(BCH) expansion with the commutation algebra for the contour bosons $[a_{\alpha}^{\dagger}, a_{\alpha'}] = \delta_{\alpha, \alpha'}$ and for contour spins as $[\sigma_{\alpha}^{\pm}, \sigma_{\alpha'}^z] = \sigma_{\alpha}^{\pm} \delta_{\alpha \alpha'}$, commutation algebra between the spin and boson is derived from Holstien-Primakoff transformation, Transformed action in lowest order does not yield the cross terms which can be written in terms of the spin and boson contours as the following,

$$\begin{aligned} \mathcal{S}_{eff} = \int d\tau \sum_{k\alpha} \hbar v_f a_{k\alpha}^{\dagger} a_{k\alpha} \\ + \frac{J_{\perp}}{4\pi a} (\sigma_1^- + \sigma_2^+ \sigma_1^z) \cosh(\hat{\delta}_1) \\ + \frac{J_{\perp}}{4\pi a} (\sigma_1^- - \sigma_2^+ \sigma_1^z) \sinh(\hat{\delta}_1) \\ + \tilde{\delta}_2 e^{a_{k\alpha} a_{k\alpha'} + a_{k\alpha}^{\dagger} a_{k\alpha'}^{\dagger} - a_{k\alpha}^{\dagger} a_{k\alpha'} - a_{k\alpha} a_{k\alpha'}^{\dagger}} \end{aligned} \quad (5)$$

$$\text{where } \hat{\delta}_{\alpha} = \left(\tilde{J}_{\parallel} \sum_{k\alpha} (a_{k\alpha} + (-1)^{\alpha-1} a_{-k\alpha}^{\dagger}) \right)$$

$$\text{where spinor is } a_k^{\dagger} = \begin{pmatrix} a_{k1}^{\dagger} & a_{k2}^{\dagger} \end{pmatrix}$$

We can define following field operators to write the above action in terms of the scalar fields,

$$\begin{aligned} \Pi_{\alpha}(x) &= \frac{1}{\sqrt{L}} \sum_k \left(\sqrt{\frac{|k|}{2}} \right) (a_{k\alpha} e^{ikx} + a_{k\alpha}^{\dagger} e^{-ikx}) \\ \Phi_{\alpha}(x) &= \frac{1}{\sqrt{L}} \sum_k \left(\sqrt{\frac{|k|}{2}} \right) (a_{k\alpha} e^{ikx} - a_{k\alpha}^{\dagger} e^{-ikx}) \end{aligned} \quad (6)$$

The EP transformation leads spin to boson and commutation algebra between them in the each contour will follow as,

$$\begin{aligned} \sigma_{\alpha}^z &= \left(\frac{1}{2} - a_{\alpha}^{\dagger} a_{\alpha} \right), \sigma_{\alpha}^+ = (1 - a_{\alpha}^{\dagger} a_{\alpha}) a_{\alpha}, \sigma_{\alpha}^- = a_{\alpha}^{\dagger} (1 - a_{\alpha}^{\dagger} a_{\alpha}) \\ [\sigma_{\alpha}^z, a_{\alpha'}^{\dagger}] &= a_{\alpha}^{\dagger} \delta_{\alpha \alpha'}, [\sigma_{\alpha}^z, a_{\alpha'}] = a_{\alpha} \delta_{\alpha \alpha'} \\ [\sigma_{\alpha}^+, a_{\alpha'}^{\dagger}] &= (1 - 2a_{\alpha}^{\dagger} a_{\alpha}) \delta_{\alpha \alpha'}, [\sigma_{\alpha}^+, a_{\alpha'}] = -a_{\alpha}^{\dagger} a_{\alpha}^{\dagger} \delta_{\alpha \alpha'} \\ [\sigma_{\alpha}^-, a_{\alpha'}] &= (1 - 2a_{\alpha}^{\dagger} a_{\alpha}) \delta_{\alpha \alpha'}, [\sigma_{\alpha}^-, a_{\alpha'}^{\dagger}] = -a_{\alpha} a_{\alpha} \delta_{\alpha \alpha'} \\ [\sigma_{\alpha}^+, a_{\alpha'}^{\dagger}] &= 0, [\sigma_{\alpha}^-, a_{\alpha'}^{\dagger}] = 0 \end{aligned} \quad (7)$$

In the new fields ($\tilde{\delta}_1, \tilde{\delta}_2$ each of them scale in different fashion) by defining the inverse transform we can write

the effective action as the following,

$$\begin{aligned}
\mathcal{S}_{eff} = & \int d\tau dx \sum_{\alpha=1,2} \hbar v_f ((\Phi_\alpha \nabla^2 \Phi_\alpha) \\
& + \frac{J_\perp}{4\pi a} (\sigma_1^- + \sigma_2^+ \sigma_1^z) \cos(\hat{\phi}_1) \\
& + \frac{J_\perp}{4\pi a} (\sigma_1^- - \sigma_2^+ \sigma_1^z) \sin(\hat{\phi}_1) \\
& + \hbar v_f \tilde{\phi}_2 e^{\tilde{\phi}_1 \tilde{\phi}_2} \\
\text{where } \hat{\phi}_1 = & \left(\left(\frac{J_\parallel}{4\pi \hbar v_f} - 1 \right) \Phi_1 \right)
\end{aligned} \tag{8}$$

The scalar field action in the (ϕ_1, ϕ_2) can be written as following after the Spin-Boson transformation and Boson to scalar field,

$$\begin{aligned}
\mathcal{S}_{eff} = & \int d\tau dx \sum_{\alpha=1,2} \hbar v_f ((\Phi_\alpha \nabla^2 \Phi_\alpha) \\
& + \frac{J_\perp}{4\pi a} (1 + \phi_2) \cos(\hat{\phi}_1) \\
& + \frac{J_\perp}{4\pi a} (1 - \phi_2) \sin(\hat{\phi}_1) \\
& + \hbar v_f \tilde{\phi}_2 e^{\tilde{\phi}_1 \tilde{\phi}_2} + \dots \\
\text{where } \hat{\phi}_1 = & \left(\left(\frac{J_\parallel}{4\pi \hbar v_f} - 1 \right) \Phi_1 \right)
\end{aligned} \tag{9}$$

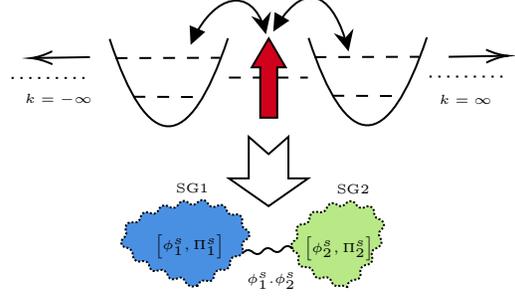
Kinetic term can also be written as the $\Phi^T (\sigma_x \nabla^2) \Phi$ where Φ is spinor of keldysh fields as $\Phi^T = (\Phi_1 \ \Phi_2)$ Using the canonical relations from inverse Keldysh rotations only for spins we get the following Hamiltonian by a Legendre transformation,

$$\begin{aligned}
\mathcal{H}_{eff} = & \int dx \hbar v_f \sum_{\alpha} (\Pi_\alpha^2 + (\nabla \Phi_\alpha)^2) + \\
& \frac{J_\perp}{4\pi a} (\sigma_1^x) \cos(\hat{\phi}_1) + \frac{i J_\perp}{4\pi a} (\sigma_1^y) \sin(\hat{\phi}_1) + \dots
\end{aligned} \tag{10}$$

The above equation 10 show the model is Non-Hermitian and \mathcal{PT} -symmetric at lowest order. We derived this model using a combination of unitary transformation. The omission of higher order terms are usually done in practice particularly in these methods. We use model in equation 8 to rest of our calculations which does not appear as a non-Hermitian at a glance but later it indeed host NH physics. Scaling the operators as $\tilde{\phi}_1 = \Phi_{new} = \left(\frac{J_\parallel}{4\pi \hbar v_f} - 1 \right) \Phi_1$ in terms of the scalar fields we can write the above model as the following,

$$\begin{aligned}
\mathcal{H}_{eff} = & \int dx \left(\frac{\hbar v_f}{\left(\frac{J_\parallel}{4\pi \hbar v_f} - 1 \right)^2} \right) (\tilde{\Pi}_1^2 + (\nabla \tilde{\Phi}_1)^2) \\
& \hbar v_f^2 (\Pi_2^2 + (\nabla \tilde{\Phi}_2)^2) \\
& + \frac{J_\perp}{4\pi a} (\sigma^x) \cos(\tilde{\Phi}_1) + \frac{i J_\perp}{4\pi a} (\sigma^y) \sin(\tilde{\Phi}_1)
\end{aligned} \tag{11}$$

In a broad sense the effective model we have got is a \mathcal{PT} -symmetric SG in the ϕ_1 field but a kinetic term in the ϕ_2 field which is interesting because the two contours mixed interaction term is now diagonal with a reservoir, Interaction terms can be viewed as the cavity-reservoir type open quantum system and can be viewed as schematic below,



III. KELDYSH AND \mathcal{PT} SELF ENERGIES

We expand in terms of the Green function for the SG model and show the relation between the Keldysh and \mathcal{PT} -self-energies. This is matrix laplace equation for the fields ie., $\mathcal{G} = \begin{pmatrix} \langle \phi_1(x), \phi_1(x') \rangle & \langle \phi_1(x), \phi_2(x') \rangle \\ \langle \phi_2(x), \phi_1(x') \rangle & \langle \phi_2(x), \phi_2(x') \rangle \end{pmatrix}$, after the unitary we can write the single time integral, so we need to get the corrections for the both scalar fields as well as the cross terms.

$$\frac{1}{2} \nabla_x^2 \mathcal{G}(x, x') - \int dx'' \Sigma(x, x'') \mathcal{G}(x'', x') = \frac{1}{(2\pi)^2} \delta(x - x'). \tag{12}$$

The selfenergy takes the form as $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ we can show the $\Sigma_{22} = 0$ lowest order in this case. Doing Fourier Transform we get the above in momentum space as the following,

$$\begin{aligned}
\mathcal{G}(q, q') = & \mathcal{G}_0(q) \delta(q + q') \\
& + \mathcal{G}_0(q) \int \frac{dq''}{(2\pi)^2} \mathbb{b}^s(q, q'') \mathcal{G}(-q'', q') \\
\text{where } \Sigma(q, q') = & -\mathbb{b}^s(q, q'), \\
\Sigma(x, x') = & -\mathbb{b}^s(x, x')
\end{aligned} \tag{13}$$

For non-equilibrium case we calculate this quantity above,

$$\begin{aligned} \mathbb{b}^s(x, x') &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & 0 \end{pmatrix} \delta(x - x') \\ &= \mathcal{M} l_I \delta(x - x') \\ \mathcal{M} &= \begin{pmatrix} \tilde{J}_{\parallel}^2 + \tilde{J}_{\parallel} \tilde{J}_{\perp} & J_{\parallel}(\nu_f + \tilde{J}_{\perp})^2 - 2\tilde{J}_{\parallel} \nu_f \\ J_{\parallel}(\nu_f + \tilde{J}_{\perp})^2 - 2\tilde{J}_{\parallel} \nu_f & 0 \end{pmatrix} \\ \mathbb{b}^s(q, q') &= \int dx dx' \mathbb{b}^s(x, x') e^{iqx+iq'x'} \\ \implies \mathbb{b}(q, -q) &= \mathcal{M} l_I(x) \end{aligned} \quad (14)$$

From above we can show the self energy host exceptional points in second order perturbation theory (couplings are third order due to the scaled operators but formalism worked out for only second order) the line of exceptional points correspond to two separatrix $J_{\parallel} = a\nu_f^{\frac{1}{2}} - b\nu_f$ and $\tilde{J}_{\parallel} = -J_{\perp} + c$ the constants (a, b, c) originates from the renormalized \tilde{J}_{\parallel} itself. This can be seen from the diagonalization of \mathcal{M} . The effective action can be written as the following,

$$\begin{aligned} \delta S_{eff}[\phi_1^s, \phi_2^s] &= \int dx dx' \left[\frac{1}{2} \mathcal{G}_0(x, x') \mathbb{b}^s(x, x') \right. \\ &\quad \left. - \frac{1}{4} \mathfrak{a}^s(x)^T \mathcal{G}_0(x, x') \mathfrak{a}^s(x') \right] \end{aligned} \quad (15)$$

Lets write the above action as the two components to scale them separately

$$\delta S_{eff}[\phi_1^s, \phi_2^s] = F_1[\phi_1^s, \phi_2^s] + F_2[\phi_1^s, \phi_2^s] \quad (16)$$

Action expansion Shows the following for the coefficients in the effective action in equation 15 which is also used in computing the vertices for selfenergy up to second order in equation 14.

$$\begin{aligned} \mathfrak{a}_{\alpha}^s(x) &= \tilde{J}_{\parallel} \frac{\delta l_I[\tilde{\phi}_1, \tilde{\phi}_2]}{\delta \tilde{\phi}_{\alpha}(x)} \\ \mathbb{b}^s(x, x') &= \frac{1}{2} \tilde{J}_{\parallel}^2 \frac{\delta^2 l_I[\tilde{\phi}_1, \tilde{\phi}_2]}{\delta \tilde{\phi}_{\alpha}(x) \delta \tilde{\phi}_{\alpha'}(x')} \Big|_{\tilde{\phi}_{\alpha}^s, \tilde{\phi}_{\alpha'}^s} \\ l_I[\tilde{\phi}_1, \tilde{\phi}_2] &= \frac{J_{\perp}}{4\pi a} (\sigma_1^- + \sigma_2^+ \sigma_1^z) \cos(\hat{\phi}_1) \\ &\quad + \frac{J_{\perp}}{4\pi a} (\sigma_1^- - \sigma_2^+ \sigma_1^z) \sin(\hat{\phi}_1) \\ &\quad + \hbar \nu_f \tilde{\phi}_2 e^{\tilde{\phi}_1 \tilde{\phi}_2} \end{aligned} \quad (17)$$

The scaling of $F_1[\phi_1^s, \phi_2^s]$ is as follows,

$$\begin{aligned} F_1[\phi_1^s, \phi_2^s] &= \alpha \frac{J_{\perp} \ln(s)}{4\pi a} \int dx \left(\cos(\phi_1^s) \right. \\ &\quad \left. + \sin(\phi_1^s) + \nu_f^2 J_{\perp} \phi_2^s e^{\nu_f J_{\perp} \phi_1^s \phi_2^s} + \dots \right) \end{aligned} \quad (18)$$

The $F_2[\phi_1^s, \phi_2^s]$ has to be evaluated carefully since we get anomalous corrections due to non hermitian terms,

$$a^s(x)_1 = \tilde{J}_{\perp} \left(\sin(\phi_1^s) + \cos(\phi_1^s) \right) + (\phi_1, \phi_2) \text{terms} \quad (19)$$

We will see now how the $F_2[\phi_1^s, \phi_2^s]$ can be worked out in terms of the $a(x, t)$,

$$\begin{aligned} F_2[\phi_1^s, \phi_2^s] &= \int dx dx' dt dt' \mathcal{G}_0(x - x', t - t') \left(\mathfrak{a}^s(x, t)^2 \right. \\ &\quad + \mathfrak{a}^s(x, t)^T \partial_x \mathfrak{a}^s(x, t) (x - x') \\ &\quad + \mathfrak{a}^s(x, t)^T \partial_t \mathfrak{a}^s(x, t) (t' - t) \\ &\quad \left. + \frac{1}{2} \mathfrak{a}^s(x, t)^T \partial_x^2 \mathfrak{a}^s(x, t) (x' - x)^2 \right) \end{aligned} \quad (20)$$

The correlation function can be written as the following,

$$f_{\partial_q^n, \partial_{\omega}^m} = \frac{i^{m+n}}{\nu} \int_{shell} \frac{dq d\omega}{4\pi^2} \partial_q^n \partial_{\omega}^m \mathcal{G}_0(q, \omega) \delta(|q| - \frac{\Lambda}{s}) \quad (21)$$

The required derivatives for the operators $a^s(x)$ in non equilibrium case using the equation 19 which reads as following,

$$\begin{aligned} (a^s(x))_1^2 &= \tilde{J}_{\perp}^2 + (\phi_1, \phi_2) \text{terms} \\ \dot{a}^s(x)_1 &= \tilde{J}_{\perp} \left(\cos(\phi_1^s) - \sin(\phi_1^s) \right) \dot{\phi}_1^s \\ \partial_t^2 a^s(x)_1 &= \tilde{J}_{\perp} \left(-\sin(\phi_1^s) - \cos(\phi_1^s) \right) \partial_t^2 \phi_1^s \\ \partial_x a^s(x)_1 &= \tilde{J}_{\perp} \left(\cos(\phi_1^s) - \sin(\phi_1^s) \right) \partial_x \phi_1^s \\ \partial_x^2 a^s(x)_1 &= \tilde{J}_{\perp} \left(-\sin(\phi_1^s) - \cos(\phi_1^s) \right) \partial_x^2 \phi_1^s \end{aligned} \quad (22)$$

where in above $\tilde{J}_\perp = \alpha \frac{J_\perp(\ln(s))}{4\pi a}$, second contour involves derivatives with the interaction,

$$\begin{aligned}
(a^s(x))_2^2 &= \tilde{J}_\parallel \nu_f^2 \tilde{\phi}_2^2 e^{2\tilde{\phi}_1 \tilde{\phi}_2} \\
\dot{a}^s(x)_2 &= \tilde{J}_\parallel \nu_f^2 e^{\tilde{\phi}_1 \tilde{\phi}_2} (\dot{\phi}_1^s \tilde{\phi}_2 + \tilde{\phi}_1 \dot{\phi}_2^s) \\
\partial_t^2 a^s(x)_2 &= \tilde{J}_\parallel \nu_f^2 e^{\tilde{\phi}_1 \tilde{\phi}_2} (\tilde{\phi}_2 \partial_t^2 \phi_1^s \\
&\quad + \dot{\phi}_1^s \dot{\phi}_2^s + \tilde{\phi}_1 \partial_t^2 \phi_2^s) \\
\partial_x a^s(x)_2 &= \tilde{J}_\parallel \nu_f^2 \tilde{\phi}_2 e^{\tilde{\phi}_1 \tilde{\phi}_2} (\phi_2^s \partial_x \phi_1^s + \phi_1 \partial_x \phi_2^s) \\
\partial_x^2 a^s(x)_2 &= \tilde{J}_\parallel \nu_f^2 \tilde{\phi}_2 e^{\tilde{\phi}_1 \tilde{\phi}_2} (\tilde{\phi}_2 \partial_x^2 \phi_1^s \\
&\quad + \partial_x \tilde{\phi}_1 \partial_x \tilde{\phi}_2 + \tilde{\phi}_1 \partial_x^2 \phi_2^s)
\end{aligned} \tag{23}$$

Substituting in equation 20 action which is a matrix in (1, 2) contours can be worked out with substitution of equations 22 and 23. This will also give us the scaling equations.

IV. FLOW OF THE ACTION

We can derive the flow equations from the Keldysh fields from the effective action in equation 5 and this will allow us to compare the perturbative RG schemes worked out on NhSG. This is a similar RG procedure as that of the Flow equation RG (FeRG) [8] by choosing the interaction term as the generator $\eta = l_I [a_1^\dagger, a_2^\dagger, a_1, a_2]$ by normal ordering with respect to first contour. Either of the results do not alter since we have bosonic algebra through out.

$$\partial_l \mathcal{Z} = \begin{pmatrix} \frac{\delta^2 \mathcal{Z}}{\delta a_1^\dagger \delta a_1} & \frac{\delta^2 \mathcal{Z}}{\delta a_1^\dagger \delta a_2} \\ \frac{\delta^2 \mathcal{Z}}{\delta a_2^\dagger \delta a_1} & \frac{\delta^2 \mathcal{Z}}{\delta a_2^\dagger \delta a_2} \end{pmatrix} \tag{24}$$

The partition function is written as $\mathcal{Z} = \int \mathcal{D}[a_\alpha^\dagger a_\alpha] e^{-S_{eff}}$. We get the following RG equations,

$$\begin{aligned}
\frac{dJ_\parallel}{d \log l} &= \frac{1}{\left(\frac{J_\parallel}{4\pi \hbar \tilde{\nu}_f} - 1\right)} - J_\perp \left(\frac{J_\parallel}{4\pi \hbar \tilde{\nu}_f} - 1\right) \\
\frac{d\tilde{\nu}_f}{d \log l} &= \frac{\tilde{\nu}_f^2}{\left(\frac{J_\parallel}{4\pi \hbar \tilde{\nu}_f} - 1\right)} \\
\frac{dJ_\perp}{d \log l} &= \frac{\tilde{\nu}_f^2}{\left(\frac{J_\parallel}{4\pi \hbar \tilde{\nu}_f} - 1\right)} + \frac{\tilde{\nu}_f J_\perp}{\left(\frac{J_\parallel}{4\pi \hbar \tilde{\nu}_f} - 1\right)}
\end{aligned} \tag{25}$$

The above beta functions contain the chemical potential in the $\tilde{\nu}_f = \nu_f \pm \mu$. We have done detailed analysis with the non hermitian Sine-Gordon model RG analysis and show the dissipative regimes of the beta functions. The comparison with the original SG model flow from various RG methods Flow Equation RG [8], Field theoretic RG [41] and the RG analysis presented [42, 43] in the context of \mathcal{PT} symmetric non-Hermitian SG model.

V. DISCUSSIONS

A derivation from the spin-boson model in nonequilibrium to the sine-Gordon model gives many insights which connect the exceptional points of non-hermitian problems, which are usually associated with non-Hermitian degeneracy. Selfenergy matrix showing separatrix at the points of vanishing fluctuations in the contours, which compares reasonably well in the RG flow diagrams. At the exceptional point of \mathcal{PT} -symmetric SG model and nonequilibrium SG capture the same physics in a strong coupling regime, both obey the same scaling relations. The renormalization perspective shows that \mathcal{PT} symmetry capture essential physics of nonequilibrium at exceptional points of the model, beyond which we have transition. This may not always have analogs in nonequilibrium formalism. We have the dominant non equilibrium effects at a weak coupling regime, So chemical potential renormalize strongly. **Future Directions:** The two-particle response functions of such systems need a quantitative study—the formalism with appropriate perturbation in non-hermitian shows where the formalism stands regarding observables. Single particle quantities can be easily computed with the selfenergy calculated. However, the transient dynamics of each problem may differ from the ambiguous initial state choice to reach a steady state; hence, the time scales would be different and needs a separate study.

VI. APPENDIX

A. Analytic solution of the flow equations

We separate the solutions of the ν_f and J_\perp RG equations and get following,

$$\int \frac{1+t}{t^2(t^2+t-1)} dt = \int \frac{1}{J_\perp} dJ_\perp, \text{ where } t = \frac{J_\perp}{\nu_f} \tag{26}$$

The complete solution for given boundary conditions can be found either numerically or analytically. Partial fractions of the left integral will make it integrable.

$$\begin{aligned}
\log(J_\perp) &= -\frac{\nu_f}{J_\perp} + 2 \log\left(\frac{J_\perp}{\nu_f}\right) \\
&\quad - \frac{1}{5} (5 - 2\sqrt{5}) \log\left(2\frac{J_\perp}{\nu_f} + \sqrt{5} + 1\right) \\
&\quad - \frac{1}{5} (2\sqrt{5} + 5) \log\left(-2\frac{J_\perp}{\nu_f} + \sqrt{5} - 1\right)
\end{aligned} \tag{27}$$

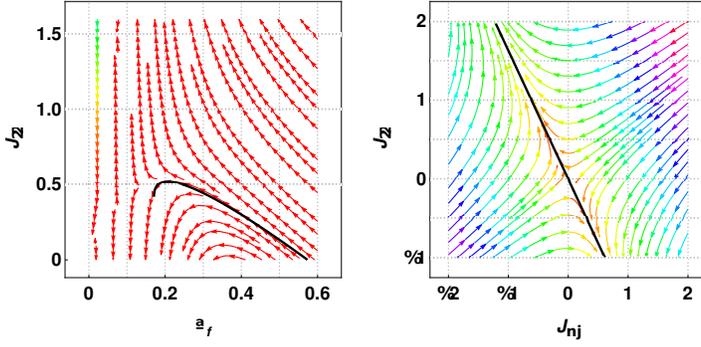


FIG. 1: SG coupling flow showing BKT universality in the $J_{\parallel} - J_{\perp}$ plane where as the $J_{\parallel} - \nu_f$ show steady state of the nonequilibrium regime. The strong coupling to weak coupling regimes are separated by the separatrix (we got from selfenergy matrix) which are denoted by black lines.

$J_{\parallel} - \nu_f$ **solution:** Now we solve for the $\frac{dJ_{\parallel}}{d\nu_f}$ from 25. This is done for simplicity along the separatrix $\tilde{J}_{\parallel} = -\tilde{J}_{\perp}$

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{2\tilde{J}_{\parallel} - 1}{\sqrt{3}}\right) + \frac{1}{3} \ln(\tilde{J}_{\parallel} + 1) - \frac{1}{6} \ln(\tilde{J}_{\parallel}^2 - \tilde{J}_{\parallel} + 1) = -\frac{1}{\nu_f} + c \quad (28)$$

This above curve also has the qualitative feature we got from selfenergy but we could not rescale to plot on same scale in flow diagram 1.

B. Benchmark with the Non-Hermitian SG RG flow of the Ashida's article

Using a perturbative expansion of the action the RG equations derived[43] upto third order particularly we solve the eq 3 of the article and show connection to our flow equations.

$$\begin{aligned} \frac{dK}{dl} &= -(g_r^2 - g_i^2)K^2 \\ \frac{dg_r}{dl} &= (2 - K)g_r + 5g_r^3 - 5g_i^2g_r \\ \frac{dg_i}{dl} &= (2 - K)g_i - 5g_i^3 + 5g_r^2g_i \end{aligned} \quad (29)$$

The above show that $\frac{g_i}{g_r}$ is invariant it denoted as \mathbb{l} in the full solution. we can rewrite the flow equations as just two coupled equations as the following,

$$\begin{aligned} \frac{dK}{dl} &= -g_r^2(1 - \mathbb{l}^2)K^2 \\ \frac{dg_r}{dl} &= (2 - K)g_r + 5g_r^3(1 - \mathbb{l}^2) \\ \implies \frac{dK}{dg_r} &= -\frac{g_r K^2(1 - \mathbb{l}^2)}{2 - K + 5g_r^2(1 - \mathbb{l}^2)} \end{aligned} \quad (30)$$

We find a substitution $a = 5g_r^2(1 - \mathbb{l}^2)$ which solve the non-linear coupled equations as the following,

$$\frac{d \ln(K)}{da} = -\left(1 + \frac{a}{1 - \frac{a}{K}}\right) \quad (31)$$

Now with a substitution of $\frac{a}{K} = t$ solves with partial fractions and we state the result here,

$$\begin{aligned} -\frac{1}{A-1} \log\left(\frac{a}{K}\right) + \frac{3-2A}{(A-1)(2-A)} \log\left((2-A)\frac{a}{K}\right) \\ = \frac{1}{2} \log(a), \quad \text{where } A = 10(1 - \mathbb{l}^2) \end{aligned} \quad (32)$$

the above equation 32 is full analytic solution of the RG equations of Article[43] at exceptional points we have comparison for $J_{\parallel} - \nu_f$ plane and $g_i - g_{real}$ plane flow.

VII. \mathcal{PT} -SELFENERGY

We show the non-hermitian selfenergy from the similar calculation we presented for scalar field theory by expanding in terms of green functions and performing the momentum and frequency integral we get the following,

$$\begin{aligned} \Sigma_{\mathcal{PT}} = & -\frac{f_{q,\omega}}{\sqrt{5(\mathbb{l}^2 - 1)}} e^{-\frac{5\mathbb{l}^2}{g}} \left(5\mathbb{l}^2 c_1 e^{10/g} \right. \\ & - 10\mathbb{l}^2 e^{10/g} \text{Ei}\left(\frac{10(\mathbb{l}^2 - 1)}{g}\right) \\ & + 10e^{10/g} \text{Ei}\left(\frac{10(\mathbb{l}^2 - 1)}{g}\right) \\ & \left. + 2e^{\frac{10\mathbb{l}^2}{g}} - 5c_1 e^{10/g} \right)^{\frac{1}{2}} \end{aligned} \quad (33)$$

The self energy from the non-hermitian model is non-linear but it has a similar root structure as that of the eigenvalues of Keldysh selfenergy matrix and more importantly we have a separatrix solution similar upto a constant. Equating the quantity in the root to zero we have,

$$\begin{aligned} 5\mathbb{l}^2 c_1 e^{10/g} - 10\mathbb{l}^2 e^{10/g} \text{Ei}\left(\frac{10(\mathbb{l}^2 - 1)}{g}\right) \\ + 10e^{10/g} \text{Ei}\left(\frac{10(\mathbb{l}^2 - 1)}{g}\right) + 2e^{\frac{10\mathbb{l}^2}{g}} - 5c_1 e^{10/g} = 0 \end{aligned} \quad (34)$$

In the above 34 $g = |g| = \sqrt{g_r^2 + g_i^2}$, it is to find the sepretrix line for $g_i - g_r$ plane. This compares reasonably with expansion $\frac{\mathbb{l}^2 - 1}{g}$ is small.

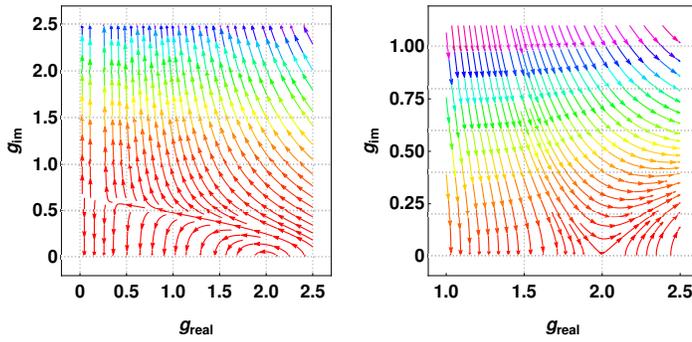


FIG. 2: Left panel is for the $l < 1$ that is symmetry preserved regime which compares well with the left panel of the 1 and right panel for $l > 1$ show the system flow to weak coupling regime which is in contrary to conventional strong coupling regime in standard SG.

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