

Spontaneous entanglement leakage of two static entangled Unruh-DeWitt detectors

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Two entangled two-level Unruh-DeWitt detectors, which are in rest, spontaneously lose entanglement when at least any one of them is not isolated from the environment quantum fields. For eternal interaction between the detectors and environment, the spontaneous emission from the detectors' excited states and vacuum fluctuations of field influence this negative effect. Consequently, it suggests that two entangled qubits become less communicated during their free-fall towards the black hole horizon.

Introduction.— Quantum entanglement, known as a resource of the quantum information, is key to many quantum technologies such as quantum cryptography [1, 2], quantum teleportation [3–6], quantum computation [7], *etc.* Such phenomena drew lot of attention over the past few decades both in non-relativistic (NR) and relativistic regimes. In NR quantum systems, lots of studies have been carried out to understand it [8–10], including its realisation in the lab [11–15]. Moreover this phenomena became very important to understand the quantum nature of gravity [16, 17], the black hole information paradox [18–20], black hole thermodynamics [21, 22], *etc.*

Study of quantum entanglement in the relativistic framework can provide a broader perspective towards the reality. Interestingly the existence of entanglement in vacuum state of a quantum field is capable of harvest entanglement between a pair of two-level detectors (known as Unruh-DeWitt (UD) detectors), even if they are causally disconnected [23–28]. This process of swapping field entanglement to detectors is sensitive to the type of motion of UD detectors [29–33], the switching function of the interactions [34], the nature of the background fields [35–37], presence of black holes [38–42] and other curved spacetimes [42–44], which have been very active areas for last few years. Moreover, the quantum entanglement phenomenon appears to be frame dependent – the measure of entanglement changes as one describes with respect to other reference frames [29, 45–50]. In a bipartite system the measure of entanglement is fruitfully quantified by *Negativity* as well as *Concurrence* [51–53].

In this letter we intend to address a fundamentally important question – whether entanglement between systems remain intact when they are not isolated from the environment? In our environment the background fields always contain vacuum fluctuation energy and as the entanglement in vacuum state of the background fields swaps to a pair of UD detectors (see e.g. [27, 30–44]), it is natural to investigate whether the swapping of vacuum entanglement has any influence in entangled systems. To get the answer to this we consider a relativistic model where two initially entangled UD detectors are individually interacting with the background fields. For simplicity the fields are chosen to be real scalar ones and the

interaction is monopole type. In order to avoid the effect of motion we consider both the detectors to be static eternally with respect to the lab frame in Minkowski spacetime. Also to avoid complexity of calculation and perform an analytical analysis, we choose eternal interaction between the field and detectors. The investigation, like earlier various analysis, is done till second order in perturbation series.

We observe that even if the detectors are eternally static with respect to the lab frame, they lose entanglement communication while interacting with the environment (here the background scalar field). This feature is a bit unexpected as previous results are in favour of entanglement harvesting due to entanglement swapping (like mentioned in [27, 30–44]). Therefore, if two entangled qubit are left open in the environment, they will lose entanglement. We find that such leakage of entanglement within this simple model is caused by collective effects of spontaneous emission of the individual detector and vacuum fluctuation of quantum field. Moreover, we argue that the leakage is unavoidable even for other type of switching function related to interaction. In the latter situation other effects (like excitation of detectors, *etc.*) may also contribute. Interestingly, if any one of the detectors is kept shielded from the environment (while the other one is open to environment quantum field) then also the composite system suffers from entanglement degradation. But in the latter situation the same will be less than the situation where both the detectors are switched on.

Let us now proceed towards the calculation to justify our claim.

The UD detector model.— Consider a pair of UD detectors, A and B , with energy gap $\Delta E_j = E_{e_j} - E_{g_j}$ ($j = A, B$). The detectors are initially entangled and the initial quantum state is taken to be

$$|D\rangle = \alpha |g_A g_B\rangle + \gamma |e_A e_B\rangle, \quad (1)$$

with α and γ are chosen to be real and satisfy $\alpha^2 + \gamma^2 = 1$. Here $|g_j\rangle$ and $|e_j\rangle$ are the ground and excited states of j^{th} detector, respectively and $|g_A g_B\rangle \equiv |g_A\rangle \otimes |g_B\rangle$, so on. The detectors are at rest in $(3 + 1)$ -dimensional Minkowski spacetime and hence their trajectories are de-

noted as

$$\begin{aligned} t_A &= \tau_A, \mathbf{x}_A = \mathbf{0}; \\ t_B &= \tau_B, \mathbf{x}_B = \mathbf{d}, \end{aligned} \quad (2)$$

where \mathbf{d} is a constant vector (measures the distance between the detectors) and τ_j is proper time of j^{th} detector. The action for the interaction between detector and background quantum field $\phi(x)$ is taken to be monopole type [54]:

$$S_{int} = \sum_{j=A,B} C_j \int d\tau_j \chi_j m_j(\tau_j) \phi(x_j(\tau_j)), \quad (3)$$

where C_j is the coupling constant of the interaction and $m_j(\tau_j) = e^{iH_j\tau_j}(|g_j\rangle\langle e_j| + |e_j\rangle\langle g_j|)e^{-iH_j\tau_j}$, is the monopole operator of the j^{th} detector, whose free Hamiltonian is given by H_j . Here χ_j is the switching function which determines the duration of interaction. The initial state of the composite system (field and detectors system) is considered to be $|in\rangle = |0_M\rangle \otimes |D\rangle$, where $|0_M\rangle$ denotes the vacuum state of field in Minkowski spacetime.

The initial density matrix of the detectors' system (by tracing out the field degrees of freedom) is given by

$$\rho_{AB}(t_{in}) = \begin{pmatrix} \alpha^2 & 0 & 0 & \alpha\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma\alpha & 0 & 0 & \gamma^2 \end{pmatrix}. \quad (4)$$

The later time this density matrix is determined as $\rho_{AB}(t) = \text{Tr}_\phi(\text{T}e^{iS_{int}}|in\rangle\langle in|\text{T}e^{-iS_{int}})$, where T denotes the time ordering. Till the second order in perturbation series this turns out to be [50]

$$\rho_{AB}(t) = \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & b_2 & 0 \\ 0 & c_1 & c_2 & 0 \\ d_1 & 0 & 0 & d_2 \end{pmatrix}, \quad (5)$$

with $t > t_{in}$. The explicit forms the matrix elements are presented in [50] (see also Supplemental Material). Consider the detectors are identical and so $\Delta E_A = \Delta E_B \equiv \Delta E$. Also for simplicity assume the coupling constants C_A, C_B are same; i.e. $C_A = C_B \equiv C$. For the present model with $\chi_j = 1$ then they reduce to

$$\begin{aligned} a_1 &= \gamma\gamma(1 - C^2 P_A'' - C^2 P_B''), \\ a_2 &= \gamma\alpha(1 - C^2 M_A - C^2 M_B), \\ b_1 &= \gamma\gamma C^2 P_B'', \\ b_2 &= \gamma\gamma C^2 X_{AB}^*, \\ c_1 &= \gamma\gamma C^2 X_{AB}, \\ c_2 &= \gamma\gamma C^2 P_A'', \\ d_1 &= \alpha\gamma(1 - C^2 M_A^* - C^2 M_B^*), \\ d_2 &= \alpha\alpha; \end{aligned} \quad (6)$$

where

$$\begin{aligned} P_j''(\Delta E) &= \int \int d\tau_j d\tau_j' e^{-i\Delta E(\tau_j - \tau_j')} G_W(x_j', x_j) \\ &= P_j(-\Delta E); \\ M_j(\Delta E) &= \int \int d\tau_j d\tau_j' e^{i\Delta E(\tau_j - \tau_j')} \theta(\tau_j - \tau_j') \\ &\quad \times (G_W(x_j', x_j) + G_W(x_j, x_j')); \\ X_{AB}(\Delta E) &= \int \int d\tau_A d\tau_B' e^{i\Delta E(\tau_B' - \tau_A)} G_W(x_B', x_A). \end{aligned} \quad (7)$$

In the above $G_W(x_j', x_j)$ is the positive frequency Wightman function. As $P_j(\Delta E)$ denotes the transition probability to excited state of j^{th} detector (see e.g. [54]), the first term in (7) corresponds to the de-excitation of it. Since the detectors are static, therefore P_j'' does not have any contribution due to the relative motion of the detectors. Therefore it is completely given by the *spontaneous emission probability* [55]. On the other hand, $G_W(x_j', x_j) + G_W(x_j, x_j')$ in the second term can be realised as $\langle 0_M | \{\phi(x_j'), \phi(x_j)\} | 0_M \rangle$. Therefore M_j is determined from the anti-commutator of the scalar field and as the expectation value of anti-commutator depends on the field state under consideration (contrary to the commutator of field, whose expectation value is independent of state), M_j arises purely due to the *vacuum fluctuation* of field.

Before going to evaluate the integration, let us mention about the choice of χ_j . In detector transition-related studies, one can start with an adiabatic interaction switching function $\chi_j(\tau_j) = e^{-s|\tau_j|}$ to build up a model (e.g see [54, 56]). This helps to suppress spurious transient effects. In that case the interaction is kept switched on for a duration $\tau_j \sim s^{-1}$ with the restriction $s \ll \Delta E$. Moreover to fulfil the adiabatic condition, the interaction is switched on and off in an infinitely slow process. Then if the interaction starts at $t_{in} = \tau_j^{in} = -\tau_0$ and ended at $t = \tau_j^f = \tau_0$, one must take the limits $\tau_0 \rightarrow \infty$ and $s \rightarrow 0$. In that case the limits of integrations are from $-\infty$ to ∞ and χ_j can be chosen to unity. Using this spirit we have chosen $\chi_j = 1$ in the present analysis so that the interaction starts at the asymptotic infinite-past (same has also been done in [54]). For that, the detectors and the field are to be in a product state at the initial time, i.e., $t_{in} = -\infty$.

Evaluation of the integrations yield

$$\begin{aligned} P_j'' &= \frac{\delta(0)}{2c^3} \sqrt{\Delta E^2 - m^2 c^4} \equiv P''; \\ \text{Re}(M_j) &= \frac{\delta(0)}{4c^3} \sqrt{\Delta E^2 - m^2 c^4} \equiv M; \\ X_{AB} &= \frac{\delta(0)}{2c^3} \sqrt{\Delta E^2 - m^2 c^4} \frac{\sin\left(\frac{d}{c} \sqrt{\Delta E^2 - m^2 c^4}\right)}{\left(\frac{d}{c} \sqrt{\Delta E^2 - m^2 c^4}\right)}, \end{aligned} \quad (8)$$

where ‘‘Re’’ denotes the real part only. In the above c is the velocity of light in the free space. Since both of

the detectors are static with respect to the lab frame on Minkowski spacetime, we have $P'_A = P''_B$ and $M_A = M_B$. The Dirac-delta functions in these expressions are arising due to infinite time integrations (i.e. $\chi_j(\tau_j)$ has been chosen to be unity). Then to give a meaning one can interpret the quantities divided by the delta functions as a rate of these quantities by considering

$$\delta(0) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T/2}^{T/2} du, \quad (9)$$

Later we will address this point elaborately.

Quantifying entanglement.– There are two well established quantities, negativity and concurrence, which fruitfully measure the entanglement between two qubits. Here we will find them corresponding to the state described by the density matrix (5). Any difference in them compare to those for (4) will signify the change in the entanglement between our two UD detectors.

Negativity is defined as absolute value of the sum of negative eigenvalues of the partially transposed density matrix, derived from the PPT criterion [51]. The eigenvalues of the partially transposed matrix corresponding to (5) have been calculated in [50]. The analysis showed that out of four eigenvalues, the negative eigenvalue can be either λ_1 or λ_2 , obtained as

$$\lambda_{1,2} \approx \frac{C^2}{2} \left(\sum_j (\alpha^2 P_j + \gamma^2 P'_j) + \alpha\gamma \text{Re}(P'_{AB} + \bar{P}'_{AB}) \right) \pm [\alpha\gamma - C^2 \text{Re}(\alpha^2 \zeta_{AB} + \gamma^2 Y_{AB} + \alpha\gamma(M_A + M_B))]. \quad (10)$$

The explicit forms of the different quantities are provided in Supplemental Material. For the present setup, the above reduces to

$$\lambda_{1,2} \approx \frac{C^2}{2} \sum_j \gamma^2 P'_j \pm [\alpha\gamma - C^2 \alpha\gamma \text{Re}(M_A + M_B)]. \quad (11)$$

(A short discussion on this is elaborated in Supplemental Material as well). Note that only P'_j and $\text{Re}(M_j)$ are contributing and they are real and non-vanishing only when $\Delta E > mc^2$. Under this condition we have $(P'_j, \text{Re}(M_j)) > 0$. Therefore when both α and γ have same sign (e.g. Bell state of the form (1) with $\alpha = \gamma = 1/\sqrt{2}$) then λ_2 is negative, while λ_1 is negative for opposite signs of α and γ (e.g. Bell state of the form (1) with $\alpha = -\gamma = 1/\sqrt{2}$). In both these situations the negativity is given by

$$\mathcal{N} = \max\{0, |\alpha\gamma| - C^2(\gamma^2 P'' + 2|\alpha\gamma|M)\}. \quad (12)$$

Note that since P'' and M are non-vanishing, even though the detectors are static, we will have less negativity. Therefore if an entangled pair of two-level detectors are kept at rest in the environment, there will be entanglement leakage and that phenomenon is driven by two

effects – spontaneous emission of individual detector and the vacuum fluctuation of the background field.

Another independent measure of entanglement is concurrence [53]. The importance of this quantity is due to its connection with the entanglement of formation. The concurrence is defined as

$$\mathcal{C}(\rho) = \max\{0, \lambda'_1 - \lambda'_2 - \lambda'_3 - \lambda'_4\}, \quad (13)$$

where the λ' 's are the square-root of eigenvalues of the matrix $\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)$ and λ'_1 is the largest of them. For our density matrix (5), λ' 's are calculated in Supplemental Material. For our model the concurrence is obtained as

$$\mathcal{C}(\rho) = \max\left\{0, 2|\alpha\gamma| - C^2\left((|\alpha\gamma| + 2\gamma^2)P'' + 2|\alpha\gamma|M\right)\right\}. \quad (14)$$

It decreases from the initial value. So again it is confirmed that there will be leakage of entanglement between the two UD detectors when they are not isolated from the environment and such is due to two phenomenon – spontaneous emission of individual detector and vacuum fluctuation of quantum field.

Note that for this model both P'' and M contain Dirac-delta function $\delta(0)$ and therefore are divergent. This is due to consideration of interaction for infinite time and such issue always arises naturally for the choice of switching function as unity. The same has also appeared in the original calculation for transition probability of an accelerated detector. In this situation, making an analogy with the Fermi's golden rule, the transition probability per unit time (known as detector's response function) is considered to be the relevant physical quantity (see for example section 3.3 of [54]). In the same spirit to tackle the present situation we define the change in negativity or concurrence per unit time as follows. Using the fact (9) one defines the change in negativity and concurrence per unit time as

$$\delta\dot{\mathcal{N}} = (\text{finite quantity}) \times \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-T/2}^{T/2} du, \quad (15)$$

and

$$\delta\dot{\mathcal{C}} = (\text{finite quantity}) \times \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-T/2}^{T/2} du, \quad (16)$$

respectively, where the “finite quantity” in $\delta\dot{\mathcal{N}}$ is determined from $C^2(\gamma^2 P'' + 2|\alpha\gamma|M)$ by removing the common factor $\delta(0)$ in it and so on. In this case to make the perturbative calculation viable we satisfy the condition – initial negativity and concurrence per unit interaction time is greater than their rate of change. Then non-vanishing positive value of $\delta\dot{\mathcal{N}}$ or $\delta\dot{\mathcal{C}}$ can be regarded as the signature of degradation of initial entanglement (the same has been proposed earlier in [50] as well). Introduction of this idea of measuring the entanglement for

the present model then shows the unavoidable leakage of initial entanglement which here depends on the energy gap ΔE of the detectors and mass of the scalar field m under the condition $\Delta E > mc^2$. Moreover, the entanglement leakage will decrease with increasing mass (m) of the scalar field till $mc^2 \sim \Delta E$. Also note that such is independent of the intra-distance ($|\mathbf{d}| = d$) between the detectors and hence situation remains same even if they sit together. Moreover no change will occur when $\Delta E = mc^2$. Interestingly, for massless field one finds degradation for all values of ΔE . The latter discussion seems to indicate that when two entangled detectors are illuminated by photon, that will lead to decrease of quantum communication between them. Additionally it may also be noted that if only one detector (say, A) is switched on while other one (say, B) is shielded from environment, then also entanglement degradation will happen. But in this case P_B'' and M_B will not contribute and hence degradation will be half of the earlier one.

Discussion and implications.— Within this simple model we observed that two static entangled UD detectors loses communication when they are open to environment. This has been confirmed through negativity as well as concurrence of the two detectors system. Such phenomenon is driven by the spontaneous emissions of the individual detectors and the vacuum fluctuation of the background quantum fields. In this regard it may be mentioned that in literature it is already pointed out that in open quantum systems, the environment causes decoherence for the quantum systems (e.g see [57, 58]). Therefore, it is natural to expect that this may cause entanglement leakage (e.g. see [59]), which is observed here as well. On the other hand in the detector context, it is well known that coupling with the background scalar field favours the generation of entanglement between two initially non-entangled detectors due to entanglement swapping from the field vacuum (see e.g. [27, 30–44]). Interestingly here we observe the negative effect of vacuum fluctuation which was observed to be providing entanglement harvesting between two un-entangled UD detector. This degradation can be decreased by shielding one of the detectors from the environment. Although the calculation has been done for eternal interaction between the detector and fields, but other types of switching function should not change the nature of the result. In the latter situation other terms in (10) will contribute, but in any case one can always find a negative eigenvalue and so negativity will decrease. For example, Gaussian type switching function yields non-vanishing value for P_j and in that case the leakage will be driven by transition of detectors from ground state to excited state as well. Hence the phenomenon of entanglement leakage is quite unavoidable and therefore two entangled systems will suffer a spontaneous drainage of communication due to their surroundings.

Finally, we mention that the present model can have

significant impact to understand more about black hole spacetimes. A Minkowski observer is equivalent to a freely-falling observer in black hole spacetime. Therefore the present result indicates that two initially entangled qubits' communication gets fade during their free-fall towards the horizon. Since the quantum nature of a black hole (particularly the black hole information paradox problem) is now being investigated in the light of quantum information, we feel that the present observation can be important in this field of study. This is not more than a suggestive one.

Note added.— After completion of the draft, a very recent work [60] in a similar motivation came to our notice. Our work is based on perturbative approach and therefore can be generalised to any type of switching function. While the previous one has been analysed non-perturbatively and therefore restricted to a specific switching function (Dirac-delta type). Therefore the results were very specific to the model used there. Moreover in the present discussion the actual causes from entanglement leakage have been illuminated.

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- [1] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991), URL <https://link.aps.org/doi/10.1103/PhysRevLett.67.661>.
 - [2] J. Yin, Y.-H. Li, S.-K. Liao, M. Yang, Y. Cao, L. Zhang, J.-G. Ren, W.-Q. Cai, W.-Y. Liu, S.-L. Li, et al., Nature **582**, 501 (2020), URL <https://doi.org/10.1038/s41586-020-2401-y>.
 - [3] M. Hotta, Phys. Rev. D **78**, 045006 (2008), arXiv:0803.2272.
 - [4] M. Hotta, Journal of the Physical Society of Japan **78**, 034001 (2009).
 - [5] M. Frey, K. Funo, and M. Hotta, Phys. Rev. E **90**, 012127 (2014).
 - [6] J. Matson, Nature (2012), URL <https://doi.org/10.1038/nature.2012.11163>.
 - [7] G. J. Mooney, C. D. Hill, and L. C. L. Hollenberg, Scientific Reports **9**, 13465 (2019), URL <https://doi.org/10.1038/s41598-019-49805-7>.
 - [8] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935), URL <https://link.aps.org/doi/10.1103/PhysRev.47.777>.
 - [9] J. S. Bell, Physics Physique Fizika **1**, 195 (1964), URL <https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195>.
 - [10] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969), URL <https://link.aps.org/doi/10.1103/PhysRevLett.23.880>.
 - [11] S. J. Freedman and J. F. Clauser,

- Phys. Rev. Lett. **28**, 938 (1972), URL <https://link.aps.org/doi/10.1103/PhysRevLett.28.938>.
- [12] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982), URL <https://link.aps.org/doi/10.1103/PhysRevLett.49.1804>.
- [13] J.-W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **80**, 3891 (1998), URL <https://link.aps.org/doi/10.1103/PhysRevLett.80.3891>.
- [14] B. Hensen et al., Nature **526**, 682 (2015), arXiv:1508.05949.
- [15] J. Handsteiner, A. S. Friedman, D. Rauch, J. Gallicchio, B. Liu, H. Hosp, J. Kofler, D. Bricher, M. Fink, C. Leung, et al., Phys. Rev. Lett. **118**, 060401 (2017), URL <https://link.aps.org/doi/10.1103/PhysRevLett.118.060401>.
- [16] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. A. Geraci, P. F. Barker, M. S. Kim, and G. Milburn, Phys. Rev. Lett. **119**, 240401 (2017), URL <https://link.aps.org/doi/10.1103/PhysRevLett.119.240401>.
- [17] C. Marletto and V. Vedral, Phys. Rev. Lett. **119**, 240402 (2017), URL <https://link.aps.org/doi/10.1103/PhysRevLett.119.240402>.
- [18] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).
- [19] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, JHEP **02**, 062 (2013), 1207.3123.
- [20] D. Marolf, Reports on Progress in Physics **80**, 092001 (2017), URL <https://doi.org/10.1088/1361-6633/aa77cc>.
- [21] R. Brustein, M. B. Einhorn, and A. Yarom, JHEP **01**, 098 (2006), hep-th/0508217.
- [22] S. N. Solodukhin, Living Rev. Rel. **14**, 8 (2011), 1104.3712.
- [23] A. Valentini, Physics Letters A **153**, 321 (1991), ISSN 0375-9601, URL <https://www.sciencedirect.com/science/article/pii/037596019190052>.
- [24] B. Reznik, Found. Phys. **33**, 167 (2003), quant-ph/0212044.
- [25] S. J. Summers and R. Werner, Physics Letters A **110**, 257 (1985).
- [26] S. J. Summers and R. Werner, Journal of Mathematical Physics **28**, 2440 (1987).
- [27] B. Reznik, A. Retzker, and J. Silman, Phys. Rev. A **71**, 042104 (2005), arXiv:quant-ph/0310058.
- [28] R. A. Thomas, M. Parniak, C. Østfeldt, C. B. Møller, C. Bærentsen, Y. Tsaturyan, A. Schliesser, J. Appel, E. Zeuthen, and E. S. Polzik, Nature Physics **17**, 228 (2021), URL <https://doi.org/10.1038/s41567-020-1031-5>.
- [29] I. Fuentes-Schuller and R. B. Mann, Phys. Rev. Lett. **95**, 120404 (2005), arXiv:quant-ph/0410172.
- [30] E. Martín-Martínez, A. R. H. Smith, and D. R. Terno, Phys. Rev. D **93**, 044001 (2016), arXiv:1507.02688.
- [31] G. Salton, R. B. Mann, and N. C. Menicucci, New J. Phys. **17**, 035001 (2015), arXiv:1408.1395.
- [32] J.-I. Koga, G. Kimura, and K. Maeda, Phys. Rev. A **97**, 062338 (2018), arXiv:1804.01183.
- [33] J.-i. Koga, K. Maeda, and G. Kimura, Phys. Rev. D **100**, 065013 (2019), arXiv:1906.02843.
- [34] A. Pozas-Kerstjens and E. Martín-Martínez, Phys. Rev. D **92**, 064042 (2015), arXiv:1506.03081.
- [35] S. Kukita and Y. Nambu, Entropy **19**, 449 (2017), arXiv:1708.01359.
- [36] D. Barman, S. Barman, and B. R. Majhi, JHEP **07**, 124 (2021), arXiv:2104.11269.
- [37] T. R. Perche, C. Lima, and E. Martín-Martínez, Phys. Rev. D **105**, 065016 (2022), 2111.12779.
- [38] L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. Smith, and J. Zhang, Class. Quant. Grav. **35**, 21LT02 (2018), arXiv:1712.10018.
- [39] E. Tjoa and R. B. Mann, JHEP **08**, 155 (2020), arXiv:2007.02955.
- [40] W. Cong, C. Qian, M. R. R. Good, and R. B. Mann, JHEP **10**, 067 (2020), arXiv:2006.01720.
- [41] K. Gallock-Yoshimura, E. Tjoa, and R. B. Mann, Phys. Rev. D **104**, 025001 (2021), 2102.09573.
- [42] S. Barman, D. Barman, and B. R. Majhi, JHEP **09**, 106 (2022), 2112.01308.
- [43] G. L. Ver Steeg and N. C. Menicucci, Phys. Rev. D **79**, 044027 (2009), arXiv:0711.3066.
- [44] L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith, and J. Zhang, JHEP **05**, 178 (2019), 1809.06862.
- [45] M. Czachor, Phys. Rev. A **55**, 72 (1997), URL <https://link.aps.org/doi/10.1103/PhysRevA.55.72>.
- [46] A. Peres, P. F. Scudo, and D. R. Terno, Phys. Rev. Lett. **88**, 230402 (2002), URL <https://link.aps.org/doi/10.1103/PhysRevLett.88.230402>.
- [47] M. Czachor, Phys. Rev. Lett. **94**, 078901 (2005), URL <https://link.aps.org/doi/10.1103/PhysRevLett.94.078901>.
- [48] R. M. Gingrich and C. Adami, Phys. Rev. Lett. **89**, 270402 (2002), URL <https://link.aps.org/doi/10.1103/PhysRevLett.89.270402>.
- [49] P. M. Alsing and G. J. Milburn, Phys. Rev. Lett. **91**, 180404 (2003), URL <https://link.aps.org/doi/10.1103/PhysRevLett.91.180404>.
- [50] P. Chowdhury and B. R. Majhi, JHEP **05**, 025 (2022), arXiv:2110.11260.
- [51] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996), arXiv:quant-ph/9604005.
- [52] W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997), arXiv:quant-ph/9703041.
- [53] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998), arXiv:quant-ph/9709029.
- [54] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1984).
- [55] E. T. Akhmedov and D. Singleton, Pisma Zh. Eksp. Teor. Fiz. **86**, 702 (2007), arXiv:0705.2525.
- [56] S. Takagi, Progress of Theoretical Physics Supplement **88**, 1 (1986), <http://ptps.oxfordjournals.org/content/88/1.full.pdf+html>.
- [57] W. H. Zurek, Physics Today **44**, 36 (1991).
- [58] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003), URL <https://link.aps.org/doi/10.1103/RevModPhys.75.715>.
- [59] F. Mintert, A. R. R. Carvalho, M. Kuś, and A. Buchleitner, Physics Reports **415**, 207 (2005), quant-ph/0505162.
- [60] K. Gallock-Yoshimura and R. B. Mann, Phys. Rev. D **104**, 125017 (2021), 2109.07495.

Supplemental Material

Spontaneous entanglement leakage of two static entangled Unruh-DeWitt detectors

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SECTION I: ELEMENTS OF THE TIME EVOLVED DENSITY MATRIX EQ. (5)

The explicit forms of the elements of the later time density matrix (5), upto second order perturbation expansion, are already calculated in [S1]. Therefore, instead of giving a detailed derivation, here we just provide the general expressions of them. These are

$$\begin{aligned}
a_1 &= \gamma\gamma^* - \gamma\gamma^*C_A^2P_A'' - \gamma\gamma^*C_B^2P_B'' - \alpha^*\gamma C_A C_B \zeta_{AB}^* - \alpha\gamma^* C_A C_B \zeta_{AB}; \\
a_2 &= \gamma\alpha^* - \alpha\alpha^*C_A C_B \zeta_{AB} - \gamma\gamma^*C_A C_B Y_{AB}^* - \alpha^*\gamma C_A^2 M_A - \alpha\gamma^* C_B^2 M_B; \\
b_1 &= \alpha\alpha^*C_A^2 P_A + \gamma\gamma^*C_B^2 P_B'' + \alpha^*\gamma C_A C_B P_{AB}' + \alpha\gamma^* C_A C_B P_{AB}^*; \\
b_2 &= \alpha\gamma^*C_A^2 \bar{P}_A' + \alpha^*\alpha C_A C_B P_{AB} + \gamma\gamma^*C_A C_B X_{AB}^* + \gamma\alpha^*C_B^2 \bar{P}_B; \\
c_1 &= \alpha^*\gamma C_A^2 \bar{P}_A + \alpha\gamma^*C_B^2 \bar{P}_B' + \gamma\gamma^*C_A C_B X_{AB} + \alpha\alpha^*C_A C_B P_{AB}^*; \\
c_2 &= \gamma^*\gamma C_A^2 P_A'' + \alpha\alpha^*C_B^2 P_B + \gamma\alpha^*C_A C_B \bar{P}_{AB}' + \alpha\gamma^*C_A C_B \bar{P}_{AB}^*; \\
d_1 &= \alpha\gamma^* - \gamma^*\alpha C_A^2 M_A^* - \alpha\gamma^*C_B^2 M_B^* - \gamma\gamma^*C_A C_B Y_{AB} - \alpha\alpha^*C_A C_B \zeta_{AB}^*; \\
d_2 &= \alpha\alpha^* - \alpha\alpha^*C_A^2 P_A - \alpha\alpha^*C_B^2 P_B - C_A C_B \alpha^*\gamma Y_{AB} - C_A C_B \alpha\gamma^* Y_{AB}^*,
\end{aligned} \tag{A1}$$

where

$$\begin{aligned}
P_j(\Delta E) &= \int \int d\tau_j d\tau_j' \chi_j(\tau_j) \chi_j(\tau_j') e^{i\Delta E(\tau_j - \tau_j')} G_W(x_j', x_j); \\
P_j''(\Delta E) &= \int \int d\tau_j d\tau_j' \chi_j(\tau_j) \chi_j(\tau_j') e^{-i\Delta E(\tau_j - \tau_j')} G_W(x_j', x_j); \\
\bar{P}_j(\Delta E) &= \int \int d\tau_j d\tau_j' \chi_j(\tau_j) \chi_j(\tau_j') e^{-i\Delta E(\tau_j + \tau_j')} G_W(x_j', x_j); \\
\bar{P}_j'(\Delta E) &= \int \int d\tau_j d\tau_j' \chi_j(\tau_j) \chi_j(\tau_j') e^{i\Delta E(\tau_j + \tau_j')} G_W(x_j', x_j); \\
M_j(\Delta E) &= \int \int d\tau_j d\tau_j' \chi_j(\tau_j) \chi_j(\tau_j') e^{i\Delta E(\tau_j - \tau_j')} \theta(\tau_j - \tau_j') (G_W(x_j', x_j) + G_W(x_j, x_j')); \\
P_{AB}^*(\Delta E) &= \int \int d\tau_A \tau_B' \chi_A(\tau_A) \chi_B(\tau_B') e^{i\Delta E(\tau_B' - \tau_A)} G_W(x_A, x_B'); \\
P_{AB}'(\Delta E) &= \int \int d\tau_A \tau_B' \chi_A(\tau_A) \chi_B(\tau_B') e^{-i\Delta E(\tau_A + \tau_B')} G_W(x_A, x_B'); \\
\bar{P}_{AB}'(\Delta E) &= \int \int d\tau_A \tau_B' \chi_A(\tau_A) \chi_B(\tau_B') e^{-i\Delta E(\tau_A + \tau_B')} G_W(x_B', x_A); \\
X_{AB}(\Delta E) &= \int \int d\tau_A d\tau_B' \chi_A(\tau_A) \chi_B(\tau_B') e^{i\Delta E(\tau_B' - \tau_A)} G_W(x_B', x_A); \\
Y_{AB}(\Delta E) &= \int \int d\tau_A d\tau_B' \chi_A(\tau_A) \chi_B(\tau_B') e^{-i\Delta E(\tau_B' + \tau_A)} \{iG_F(x_A, x_B')\}; \\
\xi_{AB}(\Delta E) &= \int \int d\tau_A d\tau_B' \chi_A(\tau_A) \chi_B(\tau_B') e^{i\Delta E(\tau_B' + \tau_A)} \{iG_F(x_A, x_B')\}.
\end{aligned} \tag{A2}$$

In the above $G_W(x, x') = \langle 0_M | \phi(x)\phi(x') | 0_M \rangle$ is the positive frequency Wightman function. In Minkowski spacetime this is given by

$$G_W(x, x') = \frac{1}{16\pi^3} \int \frac{d^3p}{E_p} e^{i\vec{p}\cdot(\vec{x}-\vec{x}') - iE_p(t-t')}. \tag{A3}$$

Whereas $iG_F(x, x')$ is the Feynman propagator and it is related to $G_W(x, x')$ by the relation $iG_F(x_A, x'_B) = \theta(t - t')G_W(x, x') + \theta(t' - t)G_W(x', x)$.

For our choice of trajectories (see (2)), $G_W(x, x')$ is time translational invariant with respect to detector's proper time. Using the coordinate transforms $T = \tau - \tau'$ and $\sigma = \tau - \tau'$ we can evaluate all the integrals in (A2). The results are as follows (using infinite switching $\chi_j(\tau_j) = 1$):

$$\begin{aligned}
P_j(\Delta E) &= \int \int d\tau_j d\tau'_j e^{i\Delta E(\tau_j - \tau'_j)} G_W(x'_j, x_j) \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{i\Delta E\sigma} e^{iE_{\mathbf{p}}\sigma} \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \int \frac{dT}{2} 2\pi\delta(\Delta E + E_{\mathbf{p}}) = 0 ;
\end{aligned} \tag{A4}$$

$$\begin{aligned}
P'_j(\Delta E) &= \int \int d\tau_j d\tau'_j e^{i\Delta E(\tau_j - \tau'_j)} G_W(x_j, x'_j) \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{i\Delta E\sigma} e^{-iE_{\mathbf{p}}\sigma} \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \frac{1}{2} 2\pi\delta(0) 2\pi\delta(\Delta E - E_{\mathbf{p}}) \\
&= \frac{\delta(0)}{8\pi} \int \frac{d^3p}{E_{\mathbf{p}}} \delta(\Delta E - E_{\mathbf{p}}) \\
&= \frac{\delta(0)}{8\pi} \int_0^\infty \frac{4\pi p^2 dp}{\sqrt{p^2 c^2 + m^2 c^4}} \delta(\Delta E - \sqrt{p^2 c^2 + m^2 c^4}) \\
&= \frac{\delta(0)}{2} \int_0^\infty \frac{p^2 dp}{\sqrt{p^2 c^2 + m^2 c^4}} \frac{\delta(p - \frac{1}{c}\sqrt{\Delta E^2 - m^2 c^4})}{\frac{c\sqrt{\Delta E^2 - m^2 c^4}}{\Delta E}} \\
&= \frac{\delta(0)}{2} \frac{(\frac{1}{c}\sqrt{\Delta E^2 - m^2 c^4})^2}{\Delta E} \frac{1}{\frac{c\sqrt{\Delta E^2 - m^2 c^4}}{\Delta E}} \\
&= \frac{\delta(0)}{2c^3} \sqrt{\Delta E^2 - m^2 c^4} ;
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\overline{P}_j(\Delta E) &= \int \int d\tau_j d\tau'_j e^{-i\Delta E(\tau_j + \tau'_j)} G_W(x'_j, x_j) \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{-i\Delta E T} e^{iE_{\mathbf{p}}\sigma} \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \frac{1}{2} 2\pi\delta(\Delta E) 2\pi\delta(E_{\mathbf{p}}) = 0 ;
\end{aligned} \tag{A6}$$

$$\overline{P}'_j(\Delta E) = \overline{P}_j(-\Delta E) = 0 ; \tag{A7}$$

$$\begin{aligned}
P_{AB}^*(\Delta E) &= \int \int d\tau_A \tau'_B e^{i\Delta E(\tau'_B - \tau_A)} G_W(x_A, x'_B) \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{-i\Delta E\sigma} e^{-i\vec{p} \cdot \vec{d} - iE_{\mathbf{p}}\sigma} \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} e^{-i\vec{p} \cdot \vec{d}} \frac{1}{2} 2\pi\delta(0) 2\pi\delta(\Delta E + E_{\mathbf{p}}) = 0 ;
\end{aligned}$$

$$\begin{aligned}
P'_{AB}(\Delta E) &= \int \int d\tau_A \tau'_B e^{-i\Delta E(\tau_A + \tau'_B)} G_W(x_A, x'_B) \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{-i\Delta E T} e^{-i\vec{p} \cdot \vec{d} - iE_{\mathbf{p}}\sigma} \\
&= \frac{1}{16\pi^3} \int \frac{d^3p}{E_{\mathbf{p}}} e^{-i\vec{p} \cdot \vec{d}} \frac{1}{2} 2\pi\delta(\Delta E) 2\pi\delta(E_{\mathbf{p}}) = 0 ;
\end{aligned} \tag{A8}$$

$$\begin{aligned}
M_j(\Delta E) &= \int \int d\tau_j d\tau'_j e^{i\Delta E(\tau_j - \tau'_j)} \theta(\tau_j - \tau'_j) (G_W(x'_j, x_j) + G_W(x_j, x'_j)) \\
&= \frac{1}{16\pi^3} \int \frac{d^3 p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{i\Delta E \sigma} \theta(\sigma) (e^{iE_{\mathbf{p}} \sigma} + e^{-iE_{\mathbf{p}} \sigma}) \\
&= \frac{1}{16\pi^3} \int \frac{4\pi p^2 dp}{E_{\mathbf{p}}} \frac{1}{2} \int_{-\infty}^{\infty} dT \int_0^{\infty} d\sigma (e^{i(\Delta E + E_{\mathbf{p}} + i\epsilon)\sigma} + e^{i(\Delta E - E_{\mathbf{p}} + i\epsilon)\sigma}) \\
&= \frac{\delta(0)}{4\pi} \int \frac{p^2 dp}{E_{\mathbf{p}}} \left(PV \left(\frac{i}{\Delta E + E_{\mathbf{p}}} \right) + \pi \delta(\Delta E + E_{\mathbf{p}}) + PV \left(\frac{i}{\Delta E - E_{\mathbf{p}}} \right) + \pi \delta(\Delta E - E_{\mathbf{p}}) \right) \\
&= \frac{\delta(0)}{4} \int \frac{p^2 dp}{E_{\mathbf{p}}} \delta(\Delta E - E_{\mathbf{p}}) + i \frac{\delta(0)}{4\pi} PV \int_0^{\infty} \frac{p^2 dp}{E_{\mathbf{p}}} \left(\frac{2\Delta E}{\Delta E^2 - E_{\mathbf{p}}^2} \right) \\
&= \frac{\delta(0)}{4c^3} \sqrt{\Delta E^2 - m^2 c^4} + i \frac{\delta(0)}{4\pi} PV \int_0^{\infty} \frac{p^2 dp}{E_{\mathbf{p}}} \left(\frac{2\Delta E}{\Delta E^2 - E_{\mathbf{p}}^2} \right) ; \tag{A9}
\end{aligned}$$

where $PV(1/x)$ represents the principal value of $1/x$.

$$\overline{P'}_{AB}(\Delta E) = (P'_{AB}(-\Delta E))^* = 0 ; \tag{A10}$$

$$\begin{aligned}
X_{AB}(\Delta E) &= \int \int d\tau_A d\tau'_B e^{i\Delta E(\tau'_B - \tau_A)} G_W(x'_B, x_A) \\
&= \frac{1}{16\pi^3} \int \frac{d^3 p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{-i\Delta E \sigma} e^{i\vec{p} \cdot \vec{d} + iE_{\mathbf{p}} \sigma} \\
&= \frac{1}{16\pi^3} \int \frac{d^3 p}{E_{\mathbf{p}}} e^{i\vec{p} \cdot \vec{d}} \frac{1}{2} 2\pi \delta(0) 2\pi \delta(\Delta E - E_{\mathbf{p}}) \\
&= \frac{\delta(0)}{8\pi} \int \frac{d^3 p}{E_{\mathbf{p}}} e^{i\vec{p} \cdot \vec{d}} \delta(\Delta E - E_{\mathbf{p}}) \\
&= \frac{\delta(0)}{8\pi} \int \frac{d^3 p}{E_{\mathbf{p}}} e^{i\vec{p} \cdot \vec{d}} \delta(\Delta E - E_{\mathbf{p}}) \\
&= \frac{\delta(0)}{8\pi} \int_0^{\pi} \int_0^{\infty} \frac{2\pi \sin \theta d\theta p^2 dp}{E_{\mathbf{p}}} e^{ipd \cos \theta} \delta(\Delta E - E_{\mathbf{p}}) \\
&= \frac{\delta(0)}{4} \int_0^{\infty} \frac{p^2 dp}{\sqrt{p^2 c^2 + m^2 c^4}} \frac{2 \sin(pd)}{pd} \delta(\Delta E - \sqrt{p^2 c^2 + m^2 c^4}) \\
&= \frac{\delta(0)}{2c^3} \sqrt{\Delta E^2 - m^2 c^4} \frac{\sin \left(\frac{d}{c} \sqrt{\Delta E^2 - m^2 c^4} \right)}{\left(\frac{d}{c} \sqrt{\Delta E^2 - m^2 c^4} \right)} ; \tag{A11}
\end{aligned}$$

$$\begin{aligned}
Y_{AB}(\Delta E) &= \int \int d\tau_A d\tau'_B e^{-i\Delta E(\tau'_B + \tau_A)} \{iG_F(x_A, x'_B)\} \\
&= \int \int d\tau_A d\tau'_B e^{-i\Delta E(\tau'_B + \tau_A)} \{ \theta(\tau'_B - \tau_A) G_W(x'_B, x_A) + \theta(\tau_A - \tau'_B) G_W(x_A, x'_B) \} \\
&= \int \int d\tau_A d\tau'_B e^{-i\Delta E(\tau'_B + \tau_A)} \left\{ \frac{1}{16\pi^2} \int \frac{d^3 p}{E_{\mathbf{p}}} \right. \\
&\quad \times [\theta(\tau'_B - \tau_A) e^{i\vec{p} \cdot \vec{d} - iE_{\mathbf{p}}(\tau'_B - \tau_A)} + \theta(\tau_A - \tau'_B) e^{-i\vec{p} \cdot \vec{d} - iE_{\mathbf{p}}(\tau_A - \tau'_B)}] \} \\
&= \frac{1}{16\pi^2} \int \frac{d^3 p}{E_{\mathbf{p}}} \int \int \frac{dT d\sigma}{2} e^{-i\Delta E T} [\theta(-\sigma) e^{i\vec{p} \cdot \vec{d} + iE_{\mathbf{p}} \sigma} + \theta(\sigma) e^{-i\vec{p} \cdot \vec{d} - iE_{\mathbf{p}} \sigma}] \\
&= \frac{1}{16\pi^2} \int \frac{d^3 p}{E_{\mathbf{p}}} \int \frac{d\sigma}{2} [\theta(-\sigma) e^{i\vec{p} \cdot \vec{d} + iE_{\mathbf{p}} \sigma} + \theta(\sigma) e^{-i\vec{p} \cdot \vec{d} - iE_{\mathbf{p}} \sigma}] 2\pi \delta(\Delta E) = 0 ; \tag{A12}
\end{aligned}$$

$$\xi_{AB}(\Delta E) = Y_{AB}(-\Delta E) = 0 . \tag{A13}$$

SECTION II: NEGATIVITY

Partial transpose of the later time density matrix (5) is obtained as

$$\rho_{AB}^{T_B} = \begin{pmatrix} a_1 & 0 & 0 & b_2 \\ 0 & b_1 & a_2 & 0 \\ 0 & d_1 & c_2 & 0 \\ c_1 & 0 & 0 & d_2 \end{pmatrix}. \quad (\text{A14})$$

The eigenvalues of $\rho_{AB}^{T_B}$ for our static setup are

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2} \left\{ b_1 + c_2 \pm \sqrt{(b_1 - c_2)^2 + 4a_2d_1} \right\} \\ &= \frac{1}{2} \left(\gamma^2(C_A^2 P_A'' + C_B^2 P_B'') \pm \sqrt{\gamma^2(C_A^2 P_A'' - C_B^2 P_B'')^2 + 4\alpha^2\gamma^2|1 - C_A^2 M_A - C_B^2 M_B|^2} \right) \\ &= \frac{1}{2} \left(\gamma^2(C_A^2 P_A'' + C_B^2 P_B'') \pm \left(\gamma^2(C_A^2 P_A'' - C_B^2 P_B'')^2 + 4\alpha^2\gamma^2(1 - C_A^2 M_A - C_B^2 M_B - C_A^2 M_A^* - C_B^2 M_B^* \right. \right. \\ &\quad \left. \left. + C_A^4 M_A M_A^* + C_B^4 M_B M_B^* + C_A^2 C_B^2 M_A M_B^* + C_A^2 C_B^2 M_B M_A^*) \right)^{1/2} \right); \quad (\text{A15}) \\ &= \frac{1}{2} \left(\gamma^2(C_A^2 P_A'' + C_B^2 P_B'') \pm 2|\alpha\gamma| \left(1 - C_A^2 M_A - C_B^2 M_B - C_A^2 M_A^* - C_B^2 M_B^* + \mathcal{O}(C^4) \right)^{1/2} \right) \\ &\simeq \frac{1}{2} \left(\gamma^2(C_A^2 P_A'' + C_B^2 P_B'') \pm 2|\alpha\gamma| \left(1 - \frac{1}{2}(C_A^2 M_A - C_B^2 M_B - C_A^2 M_A^* - C_B^2 M_B^*) \right) \right); \\ &= \frac{1}{2} \left(\gamma^2(C_A^2 P_A'' + C_B^2 P_B'') \pm 2|\alpha\gamma| \left(1 - C_A^2 \text{Re}\{M_A\} - C_B^2 \text{Re}\{M_B\} \right) \right); \end{aligned}$$

and

$$\begin{aligned} \lambda_{3,4} &= \frac{1}{2} \left\{ a_1 + d_2 \pm \sqrt{(a_1 - d_2)^2 + 4b_2c_1} \right\} \\ &= \frac{1}{2} \left(\gamma^2(1 - C_A^2 P_A'' - C_B^2 P_B'') + \alpha^2 \right. \\ &\quad \left. \pm \sqrt{(\gamma^2(1 - C_A^2 P_A'' - C_B^2 P_B'') - \alpha^2)^2 + 4\gamma^4 C_A^2 C_B^2 |X_{AB}|^2} \right) \\ &= \frac{1}{2} \left(\gamma^2(1 - C_A^2 P_A'' - C_B^2 P_B'') + \alpha^2 \pm \sqrt{(\gamma^2 - \alpha^2)^2 - 2\gamma^2(\gamma^2 - \alpha^2)(C_A^2 P_A'' + C_B^2 P_B'') + \gamma^4 \mathcal{O}(C^4)} \right) \quad (\text{A16}) \\ &= \frac{1}{2} \left(\gamma^2(1 - C_A^2 P_A'' - C_B^2 P_B'') + \alpha^2 \pm (\gamma^2 - \alpha^2) \sqrt{1 - \frac{1}{(\gamma^2 - \alpha^2)} (2\gamma^2(C_A^2 P_A'' + C_B^2 P_B'') + \gamma^4 \mathcal{O}(C^4))} \right) \\ &\simeq \frac{1}{2} \left(\gamma^2(1 - C_A^2 P_A'' - C_B^2 P_B'') + \alpha^2 \pm ((\gamma^2 - \alpha^2) - \gamma^2(C_A^2 P_A'' + C_B^2 P_B'')) \right) \\ &= \gamma^2(1 - C_A^2 P_A'' - C_B^2 P_B''), \alpha^2. \end{aligned}$$

Note that $\lambda_{3,4}$ can not be negative. While depending on the relative sign between α and γ , any one among $\lambda_{1,2}$ is negative. Therefore the later ones are considered in (11).

SECTION III: CONCURRENCE

Calculate first the following quantity:

$$\begin{aligned}
& \rho_{AB}(t)(\sigma_y \otimes \sigma_y)\rho_{AB}^*(t)(\sigma_y \otimes \sigma_y) \\
&= \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & b_2 & 0 \\ 0 & c_1 & c_2 & 0 \\ d_1 & 0 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & a_2^* \\ 0 & b_1 & b_2^* & 0 \\ 0 & c_1^* & c_2 & 0 \\ d_1^* & 0 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -a_2 & 0 & 0 & -a_1 \\ 0 & b_2 & b_1 & 0 \\ 0 & c_2 & c_1 & 0 \\ -d_2 & 0 & 0 & -d_1 \end{pmatrix} \begin{pmatrix} -a_2^* & 0 & 0 & -a_1 \\ 0 & b_2^* & b_1 & 0 \\ 0 & c_2 & c_1^* & 0 \\ -d_2 & 0 & 0 & -d_1^* \end{pmatrix} \\
&= \begin{pmatrix} |a_2|^2 + a_1 d_2 & 0 & 0 & a_1 a_2 + a_1 d_1^* \\ 0 & |b_2|^2 + b_1 c_2 & b_1 b_2 + b_1 c_1^* & 0 \\ 0 & b_2^* c_2 + c_1 c_2 & |c_1|^2 + b_1 c_2 & 0 \\ a_2^* d_2 + d_1 d_2 & 0 & 0 & |d_1|^2 + a_1 d_2 \end{pmatrix}.
\end{aligned} \tag{A17}$$

Then the square-root of the eigenvalues of the above matrix (till second order in C) are:

$$\begin{aligned}
\lambda'_{1,2} &= \sqrt{\frac{(|a_2|^2 + |d_1|^2 + 2a_1 d_2) \pm \sqrt{(|a_2|^2 - |d_1|^2)^2 + 4a_1 d_2(|d_1|^2 + |a_2|^2 + 2\text{Re}\{a_2 d_1\})}}{2}} \\
&= \frac{1}{\sqrt{2}} \left(\gamma^2 \alpha^2 (1 - C_A^2 M_A - C_B^2 M_B)^2 + \gamma^2 \alpha^2 (1 - C_A^2 M_A^* - C_B^2 M_B^*)^2 + 2\gamma^2 \alpha^2 (1 - C_A^2 P_A'' - C_B^2 P_B'') \right. \\
&\quad \left. \pm 2\alpha\gamma \sqrt{1 - C_A^2 P_A'' - C_B^2 P_B''} \right. \\
&\quad \left. \times \sqrt{(\gamma^2 \alpha^2 (1 - C_A^2 M_A - C_B^2 M_B)^2 + \gamma^2 \alpha^2 (1 - C_A^2 M_A^* - C_B^2 M_B^*)^2 + 2\alpha^2 \gamma^2 |1 - C_A^2 M_A - C_B^2 M_B|^2)} \right)^{1/2} \\
&= \frac{|\gamma\alpha|}{\sqrt{2}} \left((4 - 2C_A^2 M_A - 2C_B^2 M_B - 2C_A^2 M_A^* - 2C_B^2 M_B^* - 2C_A^2 P_A'' - 2C_B^2 P_B'' + \mathcal{O}_1(C^4)) \right. \\
&\quad \left. \pm 2\sqrt{1 - C_A^2 P_A'' - C_B^2 P_B''} \sqrt{4 - 8C_A^2 \text{Re}\{M_A\} - 8C_B^2 \text{Re}\{M_B\} + \mathcal{O}_2(C^4)} \right)^{1/2} \\
&\approx \frac{2|\gamma\alpha|}{\sqrt{2}} \left(\left(1 - C_A^2 \text{Re}\{M_A\} - C_B^2 \text{Re}\{M_B\} - \frac{C_A^2 P_A'' + C_B^2 P_B''}{2} \right) \right. \\
&\quad \left. \pm \left(1 - \frac{C_A^2 P_A'' + C_B^2 P_B''}{2} \right) (1 - C_A^2 \text{Re}\{M_A\} - C_B^2 \text{Re}\{M_B\}) \right)^{1/2} \\
&\approx 2|\gamma\alpha| \left(1 - \frac{C_A^2 \text{Re}\{M_A\} + C_B^2 \text{Re}\{M_B\}}{2} - \frac{C_A^2 P_A'' + C_B^2 P_B''}{4} \right), 0.
\end{aligned} \tag{A18}$$

and

$$\begin{aligned}
\lambda'_{3,4} &= \sqrt{\frac{(|b_2|^2 + |c_1|^2 + 2b_1 c_2) \pm \sqrt{(|b_2|^2 - |c_1|^2)^2 + 4b_1 c_2(|b_2|^2 + |c_1|^2 + 2\text{Re}\{b_2 c_1\})}}{2}} \\
&= \frac{1}{\sqrt{2}} \left(2\gamma^4 C_A^2 C_B^2 |X_{AB}|^2 + 2\gamma^4 C_A^2 P_A'' C_B^2 P_B'' \pm 2\gamma^2 \sqrt{C_A^2 P_A'' C_B^2 P_B'' (2\gamma^4 C_A^2 C_B^2 |X_{AB}|^2 + 2\gamma^4 C_A^2 C_B^2 |X_{AB}|^2)} \right)^{1/2} \\
&= C_A C_B \gamma^2 \left(|X_{AB}|^2 + P_A'' P_B'' \pm 2\sqrt{P_A'' P_B''} |X_{AB}| \right)^{1/2} \\
&= \begin{cases} C_A C_B \gamma^2 (|X_{AB}| \pm \sqrt{P_A'' P_B''}) & \text{if } |X_{AB}| \geq \sqrt{P_A'' P_B''} \\ C_A C_B \gamma^2 (\sqrt{P_A'' P_B''} \pm |X_{AB}|) & \text{if } |X_{AB}| \leq \sqrt{P_A'' P_B''} \end{cases}
\end{aligned} \tag{A19}$$

From (8), we always have $|X_{AB}| \leq \sqrt{P''_A P''_B}$ as $\frac{\sin x}{x} \leq 1$. Therefore from above two sets we will consider the second set for λ'_3 and λ'_4 . Using this then (13) will lead to (14).

[S1] P. Chowdhury and B. R. Majhi, JHEP **05**, 025 (2022) doi:10.1007/JHEP05(2022)025 [arXiv:2110.11260 [hep-th]].