

LOCAL WELL-POSEDNESS OF UNSTEADY POTENTIAL FLOWS NEAR A SPACE CORNER OF RIGHT ANGLE

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ABSTRACT. In this paper we are concerned with the local well-posedness of the unsteady potential flows near a space corner of right angle, which could be formulated as an initial-boundary value problem of a hyperbolic equation of second order in a cornered-space domain. The corner singularity is the key difficulty in establishing the local well-posedness of the problem. Moreover, the boundary conditions on both edges of the corner angle are of Neumann-type and fail to satisfy the linear stability condition, which makes it more difficult to establish *a priori* estimates on the boundary terms in the analysis. In this paper, extension methods will be updated to deal with the corner singularity, and, based on a key observation that the boundary operators are co-normal, new techniques will be developed to control the boundary terms.

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1. INTRODUCTION

In this paper we are concerned with the local existence of the unsteady inviscid compressible flows near a corner of right angle. Physically, compressible flows near corners can be observed easily and continuously. However, within the best extent of our knowledge, the mathematical analysis for such phenomena is far away from satisfied. In this paper, we are going to study this problem and trying to establish a mathematical theory on the local well-posedness of the unsteady potential flow near a 2-D corner of right angle.

In this paper, the inviscid compressible flow is assumed to be isentropic and irrotational such that its motion can be governed by the following 2-D unsteady potential flow equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla \Phi) = 0 & \text{(Conservation of mass)} \\ \partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + \iota(\rho) = B_0 & \text{(Bernoulli's law)} \end{cases} \quad (1.1)$$

where $\nabla := (\partial_{x_1}, \partial_{x_2})^\top$ is the gradient operator with respect to the spatial variable $\mathbf{x} := (x_1, x_2)$ and t is the time variable. Moreover, ρ is the density, and Φ is the velocity potential, *i.e.*, the gradient $\nabla \Phi$ is the fluid velocity. The fluid is assumed to be a polytropic gas such that the enthalpy $\iota(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1}$, where $\gamma > 1$ is the adiabatic exponent. Finally, $c = \sqrt{\rho^{\gamma-1}}$ is the sonic speed, and B_0 is the Bernoulli's constant.

The Bernoulli's law, the second equation of (1.1), implies that the density ρ can be expressed as a function with respect to $(\partial_t \Phi, \nabla \Phi)$:

$$\rho = \varrho(\partial_t \Phi, \nabla \Phi; \gamma, B_0) := (1 + (\gamma - 1)(B_0 - \partial_t \Phi - \frac{1}{2} |\nabla \Phi|^2))^{\frac{1}{\gamma-1}}. \quad (1.2)$$

Substituting (1.2) into the first equation of (1.1), we deduce that the velocity potential function Φ satisfies the following equation of second order:

$$\partial_{tt} \Phi + 2 \sum_{i=1}^2 \partial_{x_i} \Phi \partial_{tx_i} \Phi - \sum_{i,j=1}^2 (\delta_{ij} c^2 - \partial_{x_i} \Phi \partial_{x_j} \Phi) \partial_{x_i x_j} \Phi = 0, \quad (1.3)$$

where δ_{ij} is the Kronecker delta. Let $a_{00} := 1$ and

$$a_{ij} = a_{ji} := -c^2 \delta_{ij} + \partial_{x_i} \Phi \partial_{x_j} \Phi, \quad \text{for } i, j \geq 1, \quad (1.4)$$

$$a_{0j} = a_{j0} := \partial_{x_j} \Phi, \quad \text{for } j = 1, 2. \quad (1.5)$$

Then equation (1.3) can be denoted by

$$\sum_{i,j=0}^2 a_{ij} \partial_{x_i x_j} \Phi = 0. \quad (1.6)$$

It is well-known that, as $\rho \neq 0$, namely, vacuum does not appear in the flow, the equation (1.3) (or (1.6)) is of hyperbolic type.

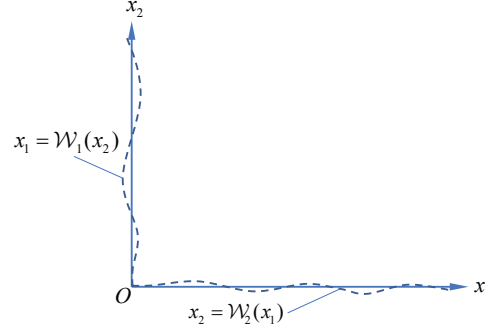


FIGURE 1. A cornered-space domain with slightly curved boundaries.

Let Γ_{w_1} and Γ_{w_2} be two curves defined as

$$\Gamma_{w_1} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = \mathcal{W}_1(x_2), x_2 > 0\},$$

$$\Gamma_{w_2} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0, x_2 = \mathcal{W}_2(x_1)\}.$$

The following two assumptions will be imposed on these two curves:

(A1) $\Gamma_{w_1} \cap \Gamma_{w_2} = \{(0, 0)\}.$

(A2) $\mathcal{W}_i'(0) = \mathcal{W}_i''(0) = \mathcal{W}_i'''(0) = 0$ for $i = 1, 2$, here and after the superscripts $'$, $''$ and $'''$ stand for the derivative of corresponding variable of the first, second and the third order.

Obviously, Γ_{w_1} and Γ_{w_2} are perpendicular to each other at the origin $(0, 0)$. Denote the cornered-space domain (see Figure 1) bounded by Γ_{w_1} and Γ_{w_2} by $\mathcal{D}$, i.e.,

$$\mathcal{D} := \{\mathbf{x} \in \mathbb{R}^2 : x_1 > \mathcal{W}_1(x_2) \text{ and } x_2 > \mathcal{W}_2(x_1)\}.$$

The fluid is confined in $\mathcal{D}$, and the velocity potential function Φ satisfies the following slip boundary conditions on the boundaries Γ_{w_1} and Γ_{w_2} :

$$\partial_{x_1} \Phi - \mathcal{W}_1'(x_2) \partial_{x_2} \Phi = 0, \quad \text{on } \Gamma_{w_1}, \quad (1.7)$$

$$\partial_{x_2} \Phi - \mathcal{W}_2'(x_1) \partial_{x_1} \Phi = 0, \quad \text{on } \Gamma_{w_2}. \quad (1.8)$$

When $t = 0$, the state of the fluid is given such that the velocity potential function Φ is equipped with the initial conditions:

$$\Phi(0, \mathbf{x}) = \Phi_0(\mathbf{x}) \quad \text{and} \quad \partial_t \Phi(0, \mathbf{x}) = \Phi_1(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D}. \quad (1.9)$$

Thus, to show the local existence of the potential flow in the cornered-space domain $\mathcal{D}$, one needs to prove the local well-posedness of the initial-boundary value problem (1.6), (1.7), (1.8), and (1.9). Since the equation (1.6) is of hyperbolic type

which enjoys the property of finite speed of propagation, one may further assume that Γ_{w_1} and Γ_{w_2} are small perturbation of straight lines, and the initial conditions (1.9) describe small perturbation of the state of the flow near the corner $(0, 0)$.

Since Φ satisfies both (1.7) and (1.8) at the corner point $(0, 0)$, one immediately obtains that $\nabla\Phi(t, 0, 0) = (0, 0)$ and the flow is static. Let the density ρ_0 at the corner point $(0, 0)$ be

$$\rho_0 := \varrho(0, 0, 0; \gamma, B_0) = (1 + (\gamma - 1)B_0)^{\frac{1}{\gamma-1}}.$$

Moreover, let

$$\begin{aligned}\bar{\Gamma}_{w_1} &:= \{(x_1, x_2) \in \mathbb{R}^2; x_1 = \bar{\mathcal{W}}_1(x_2) \equiv 0, x_2 > 0\}, \\ \bar{\Gamma}_{w_2} &:= \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0, x_2 = \bar{\mathcal{W}}_2(x_1) \equiv 0\},\end{aligned}$$

and the domain be bounded by them (see Figure 2)

$$\bar{\mathcal{D}} := \{\mathbf{x} \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\}.$$

It is obvious that

$$(\bar{\Phi}(t, \mathbf{x}), \bar{\rho}(t, \mathbf{x})) := (0, \rho_0)$$

is a steady solution to the unsteady potential flow equations (1.1). It can be further verified that $\bar{\Phi}(t, \mathbf{x})$ is a steady solution to equation (1.3) in $\bar{\mathcal{D}}$, satisfying the slip boundary conditions on $\bar{\Gamma}_{w_1}$ and $\bar{\Gamma}_{w_2}$.

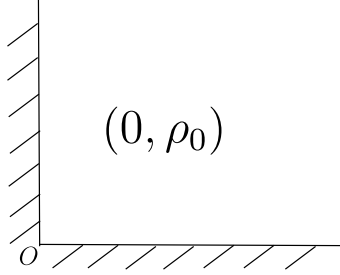


FIGURE 2. A steady solution to (1.1)

Therefore, the initial-boundary value problem (1.6), (1.7), (1.8), and (1.9) for generic smooth initial and boundary data can be reduced to the stability problem for the steady solution $\bar{\Phi}(t, \mathbf{x})$ under small perturbation of the boundary and the initial data.

Problem 1: Suppose Γ_{w_1} and Γ_{w_2} and the initial data $(\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}))$ satisfy the following conditions:

- The boundaries Γ_{w_1} and Γ_{w_2} satisfy assumptions **(A1)** and **(A2)**. Moreover, Γ_{w_1} and Γ_{w_2} are small perturbations of the straight boundaries $\bar{\Gamma}_{w_1}$ and $\bar{\Gamma}_{w_2}$,

respectively such that $\mathcal{W}_1(x_2)$ and $\mathcal{W}_2(x_1)$, as well as their derivatives, are close to zero.

- The initial data are small perturbations of $\overline{\Phi}(t, \mathbf{x})$, i.e., $(\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}))$ is close to $(0, 0)$.

Does there exist a unique local classical solution $\Phi(t, \mathbf{x})$ to the equation (1.3) in the cornered-space domain $\mathcal{D}$ with initial-boundary conditions (1.7)-(1.9), which is still close to $\overline{\Phi}(t, \mathbf{x})$?

This paper is devoted to investigating **Problem 1** and will give a positive answer by proving Theorem 2.1, which is the main theorem of this paper. The key difficulty is the corner singularity on the boundary and one needs to analyse the behaviour of the solution near the corner point. To the best of our knowledge, up to now, a general theory on well-posedness of initial-boundary value problems of hyperbolic systems on non-smooth domains is not available. Nevertheless, there are progresses toward this issue. In [32, 33], Osher gives ill-posed examples of initial-boundary value problems of hyperbolic equations on a non-smooth domain, which shows the complexity of such problems. There are also well-posed results on domains with corners. In particular, as the corner angle is sufficiently small, in [20, 21] Godin derives local well-posedness of smooth solutions for two dimensional Euler system in bounded domains with finite corner points. As the corner is a right angle, under certain symmetry assumptions, Gazzola-Secchi obtains the well-posedness of Euler equations in rectangular cylinders in [17] and Yuan establishes the stability of normal shocks in 2-D flat nozzles for two dimensional unsteady Euler system in [36]. Both in [17] and [36], the symmetry assumptions play an essential role such that extension techniques can be employed to reduced the problem near the corner into an initial-boundary value problem on a domain with smooth boundaries. By developing new techniques based on the extension method, it is established in [18] by Fang-Huang-Xiang-Xiao and in [19] by Fang-Xiang-Xiao the local dynamic stability of steady normal shock solutions for unsteady potential flows under small perturbation of the physical boundary without the symmetry assumptions in [17] and [36].

It turns out that the ideas and techniques developed in [18, 19] help to deal with the difficulties brought by the corner singularity in **Problem 1**. While new difficulties arise because the boundary conditions on both edges of the corner angle do not satisfy the linear stability conditions, which hold on the shock front, one of the edges of the corner angle, in [18, 19]. This fact makes it difficult to establish *a priori* estimates on the boundary trace of the highest order derivatives of the solution and the techniques in [18, 19] and [31] do not work. New techniques will be developed

in this paper to control the boundary terms, based on a key observation that the boundary operators are co-normal (see (ii) in lemma 2.1).

It is well-known that in general unsteady flows governed by quasilinear hyperbolic systems of conservation laws will formulate singularities in finite time and nonlinear waves such as shocks, rarefaction waves, contact discontinuities, etc. may occur. Thanks to continuous efforts of many mathematicians, there have been systematic theory for one space dimensional cases, see [6, 16, 34] and the references cited therein. For multidimensional problems, important progresses have also been made in the past decades. Dynamical stability of the elementary nonlinear waves have been established in, for instance, [1, 2, 13–15, 29–31] by employing the well-established mathematical theory for initial-boundary value problems of hyperbolic systems on smooth domains. There are also progresses on self-similar solutions for important physical phenomena such as shock reflections, supersonic flows onto a wedge, interaction between the elementary nonlinear waves, etc. See, for instance, [3, 4, 7–12, 24, 26, 27, 37]. See also [17–21, 28, 35, 36] for the studies on multidimensional problems on non-smooth domains.

The remainder of this paper is organized as follows. In section 2, the initial boundary value problem in the cornered-space domain $\mathcal{D}$ is reformulated to a new one with straight boundaries by introducing coordinate transformations. Under the transformations, certain coefficients vanish on the boundary of the space domain such that extension technique can be employed. Moreover, the boundary operators remain co-normal. Section 3 is devoted to the well-posedness of the linearized problem. The extension techniques will be employed to establish the existence of the solution in H^2 spaces, similar as in [18, 19]. In order to carry out the nonlinear iteration, H^2 regularity is not sufficient and the solution of the linearized problem should enjoy higher order regularity. However, the regularity of the coefficients of the equation in the extended domain is not sufficient to establish higher order *a priori* estimates, so they have to be established directly in the cornered-space domain. In section 4, based on the observation that the boundary operators are co-normal, new techniques will be developed to establish the higher order estimates, in particular, the estimates of the highest order derivatives of the solution on the boundary. In section 5, a classical iteration scheme will be carried out which converges to the local solution of the reformulated nonlinear initial-boundary value problem in the cornered-space domain.

2. COORDINATE TRANSFORMATION AND MAIN RESULT

In this section, we will introduce two coordinate transformations $\mathbb{T}_1$ and $\mathbb{T}_2$, under which the boundaries Γ_{w_1} and Γ_{w_2} are straightened and the extension technique can be used to solve the initial boundary value problem in new coordinate system. First we introduce the following transformation to straighten Γ_{w_2}

$$\mathbb{T}_1 : \begin{cases} y_0 = t, \\ y_1 = -x_1 - \int_0^{x_1} \mathcal{W}_2'(\tau) d\tau - (x_2 - \mathcal{W}_2(x_1))\mathcal{W}_2'(x_1), \\ y_2 = x_2 - \mathcal{W}_2(x_1). \end{cases} \quad (2.1)$$

Define $\tilde{\Phi}(y_0, y_1, y_2) := \Phi(t, x_1, x_2)$. Then one has

$$\partial_t \Phi(t, x_1, x_2) = \partial_{y_0} \tilde{\Phi}(y_0, y_1, y_2), \quad (2.2)$$

$$\begin{aligned} \partial_{x_1} \Phi(t, x_1, x_2) &= (-1 - (x_2 - \mathcal{W}_2(x_1))\mathcal{W}_2''(x_1))\partial_{y_1} \tilde{\Phi}(y_0, y_1, y_2) \\ &\quad - \mathcal{W}_2'(x_1)\partial_{y_2} \tilde{\Phi}(y_0, y_1, y_2), \end{aligned} \quad (2.3)$$

$$\partial_{x_2} \Phi(t, x_1, x_2) = -\mathcal{W}_2'(x_1)\partial_{y_1} \tilde{\Phi}(y_0, y_1, y_2) + \partial_{y_2} \tilde{\Phi}(y_0, y_1, y_2). \quad (2.4)$$

Let $(y_0, \mathbf{y}) := (y_0, y_1, y_2)$ be the time-spatial variables in new coordinate system, then one has

$$\mathbf{J} := \frac{\partial(y_0, \mathbf{y})}{\partial(t, \mathbf{x})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 - (x_2 - \mathcal{W}_2(x_1))\mathcal{W}_2''(x_1) & -\mathcal{W}_2'(x_1) \\ 0 & -\mathcal{W}_2'(x_1) & 1 \end{pmatrix}.$$

When $\mathcal{W}_2(x_1)$ and its derivatives are sufficiently small, $\mathbb{T}_1$ is invertible. We assume

$$\mathbb{T}_1^{-1} : \begin{cases} t = y_0, \\ x_1 = u(y_1, y_2), \\ x_2 = y_2 + \mathcal{W}_2(u(y_1, y_2)), \end{cases} \quad (2.5)$$

where $u(y_1, y_2)$ is the expression of x_1 in $(y_0, \mathbf{y})$ -coordinate determined by $\mathbb{T}_1$. From now on, for the shortness, we omit the dependence of variables of $\mathcal{W}_1(x_2)$ and $\mathcal{W}_2(x_1)$ and always keep in mind that $\mathcal{W}_1$ is a function of x_2 and $\mathcal{W}_2$ is a function of x_1 . By direct calculation, one has

$$\begin{aligned} D^2\Phi &= \mathbf{J}^\top D_{\mathbf{y}}^2 \tilde{\Phi} \mathbf{J} - \{((x_2 - \mathcal{W}_2)\mathcal{W}_2''' - \mathcal{W}_2'\mathcal{W}_2'')\tilde{\Phi}_{y_1} + \mathcal{W}_2''\tilde{\Phi}_{y_2}\}\mathbf{e}_1^\top \mathbf{e}_1 \\ &\quad - \mathcal{W}_2''\tilde{\Phi}_{y_1}\mathbf{e}_1^\top \mathbf{e}_2 - \mathcal{W}_2''\tilde{\Phi}_{y_1}\mathbf{e}_2^\top \mathbf{e}_1. \end{aligned}$$

where $\mathbf{e}_1 = (0, 1, 0) \in \mathbb{R}^3$ and $\mathbf{e}_2 = (0, 0, 1) \in \mathbb{R}^3$. Let $\mathbf{A} := (a_{ij})_{3 \times 3}$, then one has

$$\begin{aligned}
\sum_{i,j=0}^2 a_{ij} \partial_{x_i x_j} \Phi &= \text{Trace}(\mathbf{A}^\top \mathbf{D}^2 \Phi) \\
&= \text{Trace}(\mathbf{J}^\top \mathbf{A} \mathbf{J} \mathbf{D}_{\mathbf{y}}^2 \tilde{\Phi}) - a_{11} \{((x_2 - \mathcal{W}_2) \mathcal{W}_2''' - \mathcal{W}_2' \mathcal{W}_2'') \partial_{y_1} \tilde{\Phi} + \mathcal{W}_2'' \partial_{y_1} \tilde{\Phi}\} \\
&\quad - 2a_{12} \mathcal{W}_2'' \partial_{y_1} \tilde{\Phi} \\
&= \sum_{i,j=0}^2 \tilde{a}_{ij} \partial_{y_i y_j} \tilde{\Phi} - (a_{11}((x_2 - \mathcal{W}_2) \mathcal{W}_2''' - \mathcal{W}_2' \mathcal{W}_2'') + 2a_{12} \mathcal{W}_2'') \partial_{y_1} \tilde{\Phi} \\
&\quad - a_{11} \mathcal{W}_2'' \partial_{y_2} \tilde{\Phi},
\end{aligned}$$

where $\tilde{a}_{ij}$ is the (i, j) -th entry of $\mathbf{J}^\top \mathbf{A} \mathbf{J}$ such that

$$\tilde{a}_{ij} = \sum_{k,\ell=0}^2 \mathbf{J}_{ki} a_{k\ell} \mathbf{J}_{\ell j}.$$

By direct calculation, we have

$$\tilde{a}_{00} = 1, \tag{2.6}$$

$$\tilde{a}_{02} = \tilde{a}_{20} = -\mathcal{W}_2' \Phi_{x_1} + \Phi_{x_2}, \tag{2.7}$$

$$\tilde{a}_{22} = a_{11} \mathcal{W}_2'^2 - 2a_{21} \mathcal{W}_2' + a_{22}, \tag{2.8}$$

$$\tilde{a}_{01} = \tilde{a}_{10} = -a_{01}(1 + (x_2 - \mathcal{W}_2) \mathcal{W}_2'') - a_{02} \mathcal{W}_2', \tag{2.9}$$

$$\begin{aligned}
\tilde{a}_{12} = \tilde{a}_{21} &= a_{11}(1 + (x_2 - \mathcal{W}_2) \mathcal{W}_2'') \mathcal{W}_2' - a_{12}(1 + (x_2 - \mathcal{W}_2) \mathcal{W}_2'') \\
&\quad - a_{22} \mathcal{W}_2' + a_{21} \mathcal{W}_2'^2,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\tilde{a}_{11} &= a_{11}(1 + (x_2 - \mathcal{W}_2) \mathcal{W}_2'')^2 + a_{21}(-\mathcal{W}_2')(-1 - (x_2 - \mathcal{W}_2) \mathcal{W}_2'') \\
&\quad - \mathcal{W}_2'(-a_{12}(1 + (x_2 - \mathcal{W}_2) \mathcal{W}_2'') - \mathcal{W}_2' a_{22}).
\end{aligned} \tag{2.11}$$

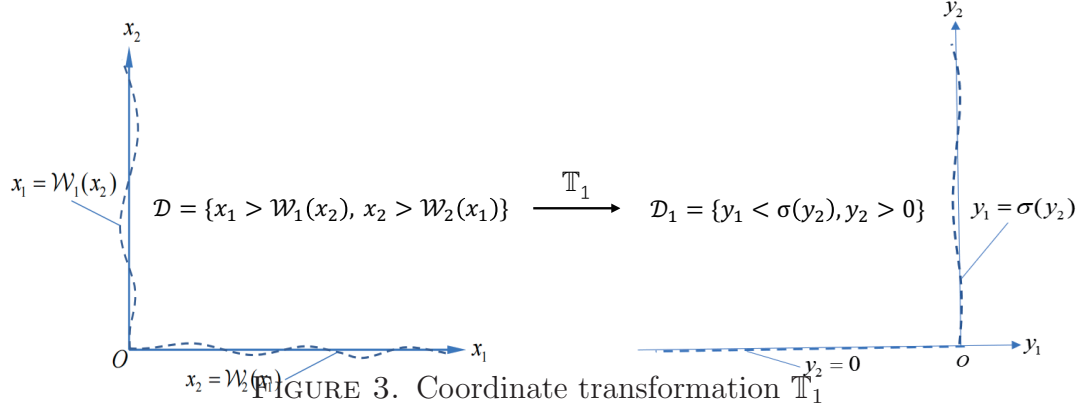
It is clear that boundary Γ_{w_2} becomes $\{y_2 = 0\}$ in the $(y_0, \mathbf{y})$ -coordinate. Let

$$F(y_1, y_2) := x_1(y_1, y_2) - \mathcal{W}_1(x_2(y_1, y_2)).$$

Then one has $F(0, 0) = 0$, and in $(y_0, \mathbf{y})$ -coordinate, the boundary Γ_{w_1} can be expressed as $F(y_1, y_2) = 0$. By direct calculation, one has

$$\frac{\partial F}{\partial y_1} \neq 0,$$

provided that the perturbations $\mathcal{W}_1$ and $\mathcal{W}_2$ are sufficiently small. Then by the implicit function theorem, $F(y_1, y_2) = 0$ determines a function $y_1 = \sigma(y_2)$ with $\sigma(0) = 0$. Hence the boundary Γ_{w_1} in the $\mathbf{y}$ -coordinate can be expressed as $y_1 = \sigma(y_2)$. Therefore, under the transformation $\mathbb{T}_1$, the space domain $\mathcal{D}$ is converted



to

$$\mathcal{D}_1 = \{(y_1, y_2) \in \mathbb{R}^2; y_1 < \sigma(y_2), y_2 > 0\},$$

with boundary $\partial\mathcal{D}_1 = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$, where

$$\tilde{\Gamma}_1 := \{(y_1, y_2) \in \mathbb{R}^2; y_1 = \sigma(y_2), y_2 > 0\},$$

$$\tilde{\Gamma}_2 := \{(y_1, y_2) \in \mathbb{R}^2; y_1 < 0, y_2 = 0\}.$$

By straightforward calculation, we find that the slip boundary conditions become

$$(-1 - y_2 \mathcal{W}_2'' + \mathcal{W}_1' \mathcal{W}_2') \partial_{y_1} \tilde{\Phi} - (\mathcal{W}_1' + \mathcal{W}_2') \partial_{y_2} \tilde{\Phi} = 0, \text{ on } \tilde{\Gamma}_1, \quad (2.12)$$

$$\partial_{y_2} \tilde{\Phi} = 0, \text{ on } \tilde{\Gamma}_2. \quad (2.13)$$

The initial conditions in the $(y_0, \mathbf{y})$ -coordinates are

$$\tilde{\Phi}(0, y_1, y_2) = \Phi_0(u, y_2 + \mathcal{W}_2(u)), \quad (2.14)$$

$$\partial_{y_0} \tilde{\Phi}(0, y_1, y_2) = \Phi_1(u, y_2 + \mathcal{W}_2(u)). \quad (2.15)$$

Next, we straighten Γ_1 by introducing the following coordinate transformation:

$$\mathbb{T}_2 : \begin{cases} z_0 = y_0, \\ z_1 = -y_1 + \sigma(y_2), \\ z_2 = y_2. \end{cases} \quad (2.16)$$

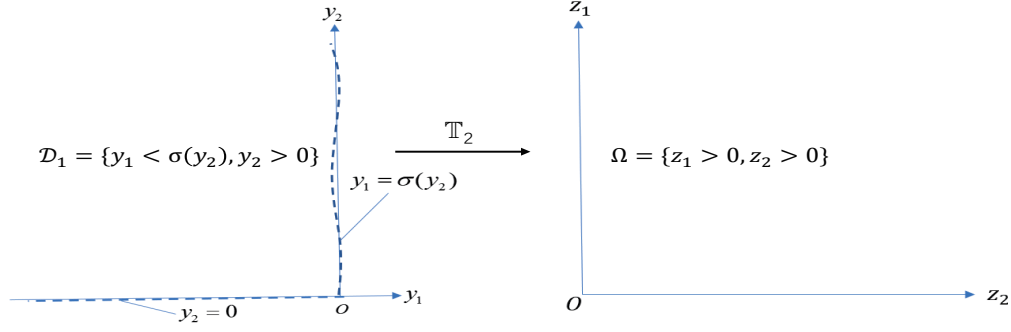
Then one has

$$x_1 = u(-z_1 + \sigma(z_2), z_2),$$

$$x_2 = \mathcal{W}_2(u(-z_1 + \sigma(z_2), z_2)) + z_2,$$

where $u = u(y_1, y_2)$ is the expression of x_1 in $(y_0, \mathbf{y})$ -coordinate determined by $\mathbb{T}_1$, as in (2.5). Under transformation $\mathbb{T}_2$, the space domain $\mathcal{D}_1$ is mapped to

$$\Omega := \{(z_1, z_2) \in \mathbb{R}^2; z_1 > 0, z_2 > 0\},$$

FIGURE 4. Coordinate transformation $\mathbb{T}_2$.

with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 := \{(z_1, z_2) \in \mathbb{R}^2; z_1 = 0, z_2 > 0\},$$

$$\Gamma_2 := \{(z_1, z_2) \in \mathbb{R}^2; z_1 > 0, z_2 = 0\}.$$

Define $\hat{\Phi}(z_0, z_1, z_2) := \tilde{\Phi}(y_0, y_1, y_2)$. Then $\hat{\Phi}(z_0, z_1, z_2)$ satisfies

$$\sum_{i,j=0}^2 \alpha_{ij} \partial_{z_i z_j} \hat{\Phi} + \sum_{i=0}^2 \alpha_i \partial_{z_i} \hat{\Phi} = 0,$$

where

$$\alpha_{00} = \tilde{a}_{00} = 1, \quad (2.17)$$

$$\alpha_{11} = \tilde{a}_{11} - \sigma'(y_2)\tilde{a}_{12} - \sigma'(y_2)\tilde{a}_{21} + (\sigma')^2(y_2)\tilde{a}_{22}, \quad (2.18)$$

$$\alpha_{22} = \tilde{a}_{22}, \quad (2.19)$$

$$\alpha_{02} = \alpha_{20} = \tilde{a}_{02}, \quad (2.20)$$

$$\alpha_{21} = \alpha_{12} = \tilde{a}_{22}\sigma'(y_2) - \tilde{a}_{21}, \quad (2.21)$$

$$\alpha_{01} = \alpha_{10} = \tilde{a}_{02}\sigma'(y_2) - \tilde{a}_{10}, \quad (2.22)$$

and

$$\alpha_0 = 0, \quad (2.23)$$

$$\alpha_1 = a_{11}((x_2 - \mathcal{W}_2)\mathcal{W}_2''' - \mathcal{W}_2'\mathcal{W}_2'') + 2a_{12}\mathcal{W}_2'', \quad (2.24)$$

$$\alpha_2 = -a_{11}\mathcal{W}_2''. \quad (2.25)$$

Boundary conditions (2.12) and (2.13) become

$$\bar{b}_1(z_1, z_2)\partial_{z_1}\hat{\Phi} + \bar{b}_2(z_1, z_2)\partial_{z_2}\hat{\Phi} = 0, \text{ on } \Gamma_1, \quad (2.26)$$

$$\partial_{z_2}\hat{\Phi} = 0, \text{ on } \Gamma_2, \quad (2.27)$$

where

$$\bar{b}_1(z_1, z_2) = p(z_1, z_2) + (\mathcal{W}_2'(x_1) + \mathcal{W}_1'(x_2))\sigma'(z_2) + \mathcal{W}_1'(x_2)\mathcal{W}_2'(x_1), \quad (2.28)$$

$$\bar{b}_2(z_1, z_2) = \mathcal{W}_1'(x_2(z_1, z_2)) + \mathcal{W}_2'(x_1(z_1, z_2)), \quad (2.29)$$

with

$$p(z_1, z_2) = -1 - z_2 \mathcal{W}_2''(x_1(z_1, z_2)).$$

Initial conditions (1.9) become

$$\hat{\Phi}(0, z_1, z_2) = \hat{\Phi}_0(z_1, z_2), \quad (2.30)$$

$$\partial_{z_0} \hat{\Phi}(0, z_1, z_2) = \hat{\Phi}_1(z_1, z_2), \quad (2.31)$$

where

$$\hat{\Phi}_0(z_1, z_2) := \tilde{\Phi}_0(u(z_1 + \sigma(z_2)), \mathcal{W}_2(u(z_1 + \sigma(z_2)))),$$

$$\hat{\Phi}_1(z_1, z_2) := \tilde{\Phi}_1(u(z_1 + \sigma(z_2)), \mathcal{W}_2(u(z_1 + \sigma(z_2)))).$$

Let $\Gamma_0 := \{0\} \times \Omega$ and $\Omega_T := (0, T) \times \Omega$, where T is any positive real number. We summarize the mathematical problem as follows:

$$\begin{cases} \alpha_{ij} \partial_{z_i z_j} \hat{\Phi} + \alpha_i \partial_{z_i} \hat{\Phi} = 0, & \text{in } \Omega_T, \\ G(\partial_{z_1} \hat{\Phi}, \partial_{z_2} \hat{\Phi}; \mathcal{W}_1', \mathcal{W}_2', \mathcal{W}_2'', \sigma') = 0, & \text{on } \Gamma_1, \\ \partial_{z_2} \hat{\Phi} = 0, & \text{on } \Gamma_2, \\ \hat{\Phi}(0, z_1, z_2) = \hat{\Phi}_0(z_1, z_2), \quad \partial_{z_0} \hat{\Phi}(0, z_1, z_2) = \hat{\Phi}_1(z_1, z_2), & \text{on } \Gamma_0, \end{cases} \quad (2.32)$$

where

$$G(\partial_{z_1} \hat{\Phi}, \partial_{z_2} \hat{\Phi}; \mathcal{W}_1', \mathcal{W}_2', \mathcal{W}_2'', \sigma') := \bar{b}_1(z_1, z_2) \partial_{z_1} \hat{\Phi} + \bar{b}_2(z_1, z_2) \partial_{z_2} \hat{\Phi}.$$

The initial data $\hat{\Phi}_0$ and $\hat{\Phi}_1$ are defined in (2.30) and (2.31), respectively and at the background state, one has

$$(\alpha_{ij})_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\rho_0^{\gamma-1} & 0 \\ 0 & 0 & -\rho_0^{\gamma-1} \end{pmatrix}. \quad (2.33)$$

Lemma 2.1. *The coefficients of the initial boundary value problem (2.32) have the following properties:*

(i) $\alpha_{02} = \alpha_{20}$ and $\alpha_{12} = \alpha_{21}$ vanish on Γ_2 , namely, for $i = 0, 1$,

$$\alpha_{i2}(z_0, z_1, 0) = \alpha_{2i}(z_0, z_1, 0) = 0.$$

(ii) The boundary operator of (2.32) is “co-normal”, i.e., the vector $(\alpha_{10}, \alpha_{11}, \alpha_{12}) \in \mathbb{R}^3$ is parallel to $(0, \bar{b}_1, \bar{b}_2) \in \mathbb{R}^3$ on Γ_1 , where $\bar{b}_1$ and $\bar{b}_2$ are the coefficients of the boundary conditions on Γ_1 in (2.32).

(iii) It holds that

$$\bar{b}_2(0, 0) = \partial_{z_2} \bar{b}_2(0, 0) = \partial_{z_2 z_2} \bar{b}_2(0, 0). \quad (2.34)$$

Proof. (i): By the formula of $\tilde{a}_{02}$ in (2.7) and the slip boundary condition (1.8), one has $\tilde{a}_{02} = \tilde{a}_{20} = 0$ on Γ_2 . Then (2.20) implies $\alpha_{02} = \alpha_{20} = 0$ on Γ_2 . By (2.10) and the slip boundary condition (1.8), one has

$$\tilde{a}_{12}|_{z_2=0} = \partial_{x_1} \Phi(\mathcal{W}'_2 \partial_{x_1} \Phi - \partial_{x_2} \Phi) + \mathcal{W}'_2 \partial_{x_2} \Phi(\mathcal{W}'_2 \partial_{x_1} \Phi - \partial_{x_2} \Phi) = 0. \quad (2.35)$$

Differentiating with respect to y_2 on both sides of $x_1(\sigma(y_2), y_2) = \mathcal{W}_1(x_2(\sigma(y_2), y_2))$, one deduces

$$\sigma'(y_2) = \frac{p(x_1, x_2) \mathcal{W}'_1(x_2(\sigma(y_2), y_2)) - \mathcal{W}'_2(x_1(\sigma(y_2), y_2))}{1 - \mathcal{W}'_1 \mathcal{W}'_2},$$

where

$$p(x_1, x_2) := -1 - (x_2 - \mathcal{W}_2) \mathcal{W}_2''.$$

By assumption (A2) and the facts that $\sigma(0) = 0$, $\mathbb{T}_1$ maps the origin in $\mathbf{x}$ -coordinate to the origin in $\mathbf{y}$ -coordinate, and $\mathbb{T}_1$ is invertible, one has

$$\sigma'(0) = \frac{-\mathcal{W}'_1(x_2(\sigma(0), 0)) - \mathcal{W}'_2(x_1(\sigma(0), 0))}{1 - \mathcal{W}'_1(x_2(\sigma(0), 0)) \mathcal{W}'_2(x_1(\sigma(0), 0))} = 0.$$

Combining (2.21), (2.35), and the fact that $\sigma'(0) = 0$, one deduces that $\alpha_{12} = \alpha_{21} = 0$ on Γ_2 (equivalently on $\tilde{\Gamma}_2$). By assumption (A2), it is obvious that $\bar{b}_2 = 0$ on Γ_2 .

(ii) : By the formulas of $\tilde{a}_{10}$, $\tilde{a}_{20}$, and $\sigma'(y_2)$, we obtain

$$\begin{aligned} -\tilde{a}_{10} + \sigma' \tilde{a}_{20} &= -p \partial_{x_1} \Phi + \partial_{x_2} \Phi \mathcal{W}'_2 - \sigma' \mathcal{W}'_2 \partial_{x_1} \Phi + \sigma' \partial_{x_2} \Phi \\ &= -(p + \sigma' \mathcal{W}'_2) \partial_{x_1} \Phi + (\mathcal{W}'_2 + \sigma') \partial_{x_2} \Phi. \end{aligned} \quad (2.36)$$

But it is easy to verify that

$$\frac{\mathcal{W}'_2 + \sigma'}{-p - \sigma' \mathcal{W}'_2} = -\frac{\mathcal{W}'_1}{1}.$$

By the slip boundary condition on Γ_{w_1} , one has $-\tilde{a}_{10} + \sigma' \tilde{a}_{20} = 0$ on Γ_1 , which implies $\alpha_{01} = \alpha_{10} = 0$ on Γ_1 by (2.22). By the formulas of α_{11} and $\alpha_{12} = \alpha_{21}$ in (2.18) and (2.21), one can deduce that

$$\frac{\alpha_{12}}{\alpha_{11}} \Big|_{z_1=0} = \frac{\mathcal{W}'_1 + \mathcal{W}'_2}{p + \mathcal{W}'_1 \mathcal{W}'_2 + \sigma'(\mathcal{W}'_1 + \mathcal{W}'_2)} = \frac{\bar{b}_2}{\bar{b}_1}. \quad (2.37)$$

In fact, we have

$$\begin{aligned}
& (\tilde{a}_{11} - \sigma' \tilde{a}_{12})|_{y_1 = \sigma(y_2)} \\
&= c^2(-p^2 - \mathcal{W}_2'^2 - p\sigma' \mathcal{W}_2' - \sigma' \mathcal{W}_2') + (\partial_{x_1} \Phi)^2(p^2 + p\sigma' \mathcal{W}_2') \\
&\quad + (\partial_{x_2} \Phi)^2(\mathcal{W}_2'^2 + \sigma' \mathcal{W}_2') + \partial_{x_1} \Phi \partial_{x_2} \Phi(-2p \mathcal{W}_2' - p\sigma' - \sigma' \mathcal{W}_2'^2) \\
&= c^2(-p^2 - \mathcal{W}_2'^2 - p\sigma' \mathcal{W}_2' - \sigma' \mathcal{W}_2') \\
&\quad + \partial_{x_1} \Phi(p^2 + p\sigma' \mathcal{W}_2')(\partial_{x_1} \Phi - \mathcal{W}_1' \partial_{x_2} \Phi) \\
&\quad - \frac{(\mathcal{W}_2'^2 + \sigma' \mathcal{W}_2')}{\mathcal{W}_1'} \partial_{x_2} \Phi(\partial_{x_1} \Phi - \mathcal{W}_1' \partial_{x_2} \Phi) \\
&= c^2(-p^2 - \mathcal{W}_2'^2 - p\sigma' \mathcal{W}_2' - \sigma' \mathcal{W}_2'), \tag{2.38}
\end{aligned}$$

where in the last equality, we have used the slip boundary condition (1.7). By direct calculation, one also has

$$(-\tilde{a}_{12} + \sigma' \tilde{a}_{22})|_{y_1 = \sigma(y_2)} = -c^2(p \mathcal{W}_2' + \mathcal{W}_2' + \sigma' \mathcal{W}_2'^2 + \sigma').$$

Thus one has

$$\begin{aligned}
\frac{\tilde{a}_{11} - \sigma \tilde{a}_{12}}{\tilde{a}_{22} \sigma' - \tilde{a}_{12}} \Big|_{z_1=0} &= \frac{p^2 + \mathcal{W}_2'^2 + p\sigma' \mathcal{W}_2' + \sigma' \mathcal{W}_2'}{p \mathcal{W}_2' + \mathcal{W}_2' + \sigma' \mathcal{W}_2'^2 + \sigma'} \\
&= \frac{p + \mathcal{W}_1' \mathcal{W}_2'}{\mathcal{W}_1' + \mathcal{W}_2'}.
\end{aligned}$$

Then by the formula of α_{11} in (2.18) and the formula of α_{12} in (2.21), we obtain

$$\begin{aligned}
\frac{\alpha_{11}}{\alpha_{12}} \Big|_{z_1=0} &= \frac{\tilde{a}_{11} - \sigma \tilde{a}_{12}}{\tilde{a}_{22} \sigma' - \tilde{a}_{12}} \Big|_{z_1=0} + \sigma' \\
&= \frac{p + \mathcal{W}_1' \mathcal{W}_2'}{\mathcal{W}_1' + \mathcal{W}_2'} + \sigma' = \frac{\bar{b}_1}{\bar{b}_2}.
\end{aligned}$$

Hence one has

$$\frac{\alpha_{12}}{\alpha_{11}} \Big|_{z_1=0} = \frac{\bar{b}_2}{\bar{b}_1}.$$

(iii): It is clear that both $\mathbb{T}_1$ and $\mathbb{T}_2$ are invertible and $\mathbb{T}_1 \circ \mathbb{T}_2$ maps the origin in the $\mathbf{x}$ -coordinate to the origin in the $\mathbf{z}$ -coordinate, where $\mathbf{z} := (z_1, z_2)$ is the spatial variable in new coordinate under transformation $\mathbb{T}_2$. So by (2.29) and the assumption (A2) on the perturbations in section 1, it is easy to see that (2.34) holds. This completes the proof of this lemma. $\square$

In $\mathbf{z}$ -coordinate, **Problem 1** can be reformulated as the following problem:

Problem 2: Does there exist a unique local classical solution to problem (2.32), when $(\mathcal{W}_1, \mathcal{W}_2)$ are small perturbations of $(\overline{\mathcal{W}}_1, \overline{\mathcal{W}}_2) = (0, 0)$ and the initial data $(\hat{\Phi}_0(z_1, z_2), \hat{\Phi}_1(z_1, z_2))$ are small perturbations of $(0, 0)$?

Since coordinate transformations $\mathbb{T}_1$ and $\mathbb{T}_2$ are invertible when the perturbations are small, **Problem 1** and **Problem 2** are equivalent. The main theorem of the paper is as follows, which gives a positive answer to **Problem 2**.

Theorem 2.1. *Suppose the perturbed solid walls satisfy the three assumptions (A1) and (A2) in section 1. Then there exists a constant $\epsilon > 0$ such that if*

$$\|\hat{\Phi}_0\|_{H^4(\Omega)} + \|\hat{\Phi}_1\|_{H^3(\Omega)} + \|(\mathcal{W}_1, \mathcal{W}_2)\|_{W^{6,\infty}(\mathbb{R}^+)} \leq \epsilon, \quad (2.39)$$

and $\hat{\Phi}_0$ and $\hat{\Phi}_1$ satisfy the compatibility conditions up to order 2 and are compactly supported in some neighbours of the origin, then there exist two constants $\eta_0 \geq 1$ and $T_0 > 0$ such that the nonlinear problem (2.32) admits a unique smooth solution $\hat{\Phi} \in H^4(\Omega_{T_0})$, satisfying

$$\|e^{-\eta z_0} \hat{\Phi}\|_{H^4(\Omega_T)} \leq C\epsilon \quad (2.40)$$

for $\eta \geq \eta_0$ and $T \leq T_0$, where $C = C(\rho_0, \gamma, \eta_0, T_0)$ is a positive constant.

Remark 2.1. *The compatibility conditions mentioned in Theorem 2.1 come from the requirement that the initial-boundary data of problem (2.32) should be consistent. More precisely, by initial conditions in (2.32) and the first equation of (2.32), we know that at $z_0 = 0$,*

$$D^\beta \hat{\Phi} = D^\beta \hat{\Phi}_0, \quad \partial_{z_0} D^\beta \hat{\Phi} = D^\beta \hat{\Phi}_1$$

and

$$\partial_{z_0}^2 D^\beta \hat{\Phi} = -D^\beta \left(\frac{1}{\alpha_{00}} \left(\sum_{i=0}^2 \alpha_i \partial_{z_i} \hat{\Phi} + \sum_{(i,j) \neq (0,0)}^2 \alpha_{ij} \partial_{z_i z_j} \hat{\Phi} \right) \right),$$

where $D^\beta = \partial_{z_1}^{\beta_1} \partial_{z_2}^{\beta_2}$ is the spatial derivatives and $\beta = (\beta_1, \beta_2)$ is the multi-index corresponds to spatial derivative. Then by the induction on k (i.e., assume we have already known the expression of $\partial_{z_0}^{m+1} D^\beta \hat{\Phi}$ at $z_0 = 0$ for all $m \leq k$.) and by taking derivative $D^\beta \partial_{z_0}^k$ on equation (2.32)₁, we will have the expression of $\partial_{z_0}^{k+2} D^\beta \hat{\Phi}$ at $z_0 = 0$. We omit the details for the shortness. Then we have the expression of $D^\lambda \hat{\Phi}$ at $z_0 = 0$ for all multi-index $\lambda = (\lambda_0, \lambda_1, \lambda_2)$. Let

$$\Phi_\lambda := D^\lambda \hat{\Phi}|_{z_0=0}. \quad (2.41)$$

On the other hand, since we have two boundary conditions in (2.32), for any $(k_0, k_1, k_2) \in \mathbb{N}^3$, we have

$$\begin{aligned} D^{(k_0, 0, k_2)} \left(\sum_{i=1}^2 \bar{b}_i \partial_{z_i} \hat{\Phi} \right) &= 0 \quad \text{on } \Gamma_1, \\ D^{(k_0, k_1, 0)} \partial_{z_2} \hat{\Phi} &= 0 \quad \text{on } \Gamma_2. \end{aligned}$$

Let $z_0 = 0$ and plug (2.41) into the two identities above for all integers $k_0 + k_1 \leq \mathfrak{s}$ and $k_0 + k_2 \leq \mathfrak{s}$. Then we can obtain the identities that the initial and boundary data must satisfy for all integers $k_0 + k_1 \leq \mathfrak{s}$ and $k_0 + k_2 \leq \mathfrak{s}$. These identities are called the compatibility conditions up to order $\mathfrak{s}$. In this paper, the compatibility conditions up to order two are required, i.e., $\mathfrak{s} = 2$.

3. THE LINEARIZED PROBLEM (I): EXISTENCE OF THE SOLUTION IN $H^2(\Omega_T)$

In order to establish the local well-posedness of the nonlinear initial-boundary value problem, the classical iteration scheme will be carried out. Then the unique existence of the solutions to the linearized problem (3.1) as well as the *a priori* estimates of the solution are needed to establish the convergence of the iteration. However, there is no general theory which could be employed, because of the presence of a corner singularity on the boundary of the space domain. Therefore, we have to establish a well-posedness theorem on the unique existence and *a priori* estimates of the solution for the linearized problem, which will be done in this and the next section. In this section, the extension techniques will be employed to establish the existence of the solution in $H^2(\Omega_T)$. Similar as in [18, 19], the regularity of the coefficients in the extended domain is not sufficient to establish higher order estimates of the solution. Therefore, one have to establish the estimates for the higher order derivatives of the solution directly in the cornered space domain, which will be done in the next section.

3.1. The main theorem on the linearized problem. Let us consider following initial-boundary value problem:

$$\begin{cases} \mathcal{L}\varphi = f, & \text{in } \Omega_T, \\ \mathcal{B}\varphi = 0, & \text{on } \Gamma_1, \\ \partial_{z_2}\varphi = 0, & \text{on } \Gamma_2, \\ \varphi(0, z_1, z_2) = \varphi_0, & \text{on } \Gamma_0, \\ \partial_{z_0}\varphi(0, z_1, z_2) = \varphi_1(z_1, z_2), & \text{on } \Gamma_0, \end{cases} \quad (3.1)$$

where

$$\mathcal{L} := \sum_{i,j=0}^2 r_{ij}(z_0, z_1, z_2) \partial_{ij} \quad \text{and} \quad \mathcal{B} := b_1(z_1, z_2) \partial_1 + b_2(z_1, z_2) \partial_2,$$

$\partial_i := \partial_{z_i}$, and $\partial_{ij} = \partial_{z_i z_j}$. The coefficients of $\mathcal{L}$ and $\mathcal{B}$ satisfy

- (i) $\mathcal{L}$ is a hyperbolic differential operator of second order. $r_{ij} = r_{ij}(z_0, z_1, z_2)$ are smooth functions and $r_{00} \equiv 1$. b_i are smooth functions with respect to z_1 and z_2 but do not depend on z_0 .
- (ii) There exist an integer $s_0 \geq 3$, and constants $\delta > 0$ and $\bar{r}_{ij}$ ($0 \leq i, j \leq 2$), such that

$$\sup_{0 \leq z_0 \leq T} \|D^\alpha(r_{ij} - \bar{r}_{ij})\|_{L^2(\Omega)} < \delta \text{ for all } |\alpha| \leq s_0,$$

where $\bar{r}_{00} = 1$, $\bar{r}_{11} = \bar{r}_{22} < 0$, and $\bar{r}_{ij} = 0$ for $i \neq j$. We also require that

$$|\partial_{z_2}^{\ell+1} b_1| + |\partial_{z_2}^\ell b_2| \leq C\delta \quad \text{for } \ell = 0, 1, 2.$$

- (iii) $r_{12} = r_{21} = r_{20} = r_{02} = 0$ on the boundary Γ_2 and $\partial_{z_2}^k b_2(0, 0) = 0$ for $k = 0, 1, 2$.
- (iv) The following identities hold on Γ_1 :

$$\frac{r_{12}}{r_{11}} = \frac{b_2}{b_1} \quad \text{and} \quad r_{01} = r_{10} = 0.$$

Remark 3.1. *The compatibility conditions up to order $\mathfrak{s}$ of problem (3.1) can be defined in the same way as done in remark 2.1.*

Then we have the following proposition.

Proposition 3.1. *There exists $\delta_* > 0$ such that if assumptions (i) – (iv) holds for $\delta \leq \delta_*$, and if φ_0 and φ_1 satisfy the compatibility conditions up to order 2, problem (3.1) admits a smooth solution $\varphi \in H^4(\Omega_T)$ and there exists a constant η_0 , such that for any $T > 0$ and $\eta \geq \eta_0$, it holds that*

$$\begin{aligned} & \sum_{|\alpha| \leq 4} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 3} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} f\|_{H^3(\Omega_T)}^2 \\ & \quad + \|f|_{t=0}\|_{H^2(\Omega)}^2 + \|\varphi_0\|_{H^4(\Omega)}^2 + \|\varphi_1\|_{H^3(\Omega)}^2. \end{aligned} \tag{3.2}$$

Without loss of generality, we assume $(\varphi_0, \varphi_1) = (0, 0)$ in the following two sections below. Otherwise, one can reduce problem (3.1) into a problem with homogeneous initial data by introducing auxiliary functions.

The proof of Proposition 3.1 will be separated into two parts. One is the unique existence of the solution in $H^2(\Omega_T)$, which will be established in this section. The other is the *a priori* estimates of higher order derivatives, which will be done in the next section.

To establish the unique existence of the solution, we are motivated to apply the extension techniques by observing the boundary condition on Γ_2 as well as the

properties of the coefficients enjoyed, such that the well-established theory for initial-boundary value problems of hyperbolic equations can be employed. We are going to show the following lemma in this section, which establishes the well-posedness of problem (3.1) in $H^2(\Omega_T)$.

Lemma 3.1. *There exists $\delta_1 > 0$ such that if assumptions (i) – (iv) hold for $\delta \leq \delta_1$, problem (3.1) admits a solution $\varphi \in H^2(\Omega_T)$ and there exists a constant η_1 , such that for any $T > 0$ and $\eta \geq \eta_1$,*

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 1} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} f\|_{H^1(\Omega_T)}^2 + \|f|_{t=0}\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.3)$$

As mentioned previously, the H^2 solvability of the linearized problem is not sufficient to yield smooth solutions of the nonlinear problem by carrying out nonlinear iteration. Therefore, we have to deduce higher order *a priori* estimate of the solution derived in lemma 3.1. However, one can easily check that the regularity of the coefficients of the equation in the extended domain is not sufficient to establish the needed higher order estimates. Hence, we shall go back to the cornered-space domain to establish the higher order estimates, and the following lemma will be proved in the next section.

Lemma 3.2. *There exists $\delta_2 > 0$ such that if assumptions (i) – (iv) hold for $\delta \leq \delta_2$, then there exists a constant $\eta_2 > 1$ such that for any $T > 0$ and $\eta \geq \eta_2$, the $H^2(\Omega_T)$ solution of problem (3.1) satisfies*

$$\begin{aligned} & \sum_{|\alpha| \leq 4} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 3} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} f\|_{H^3(\Omega_T)}^2 + \|f|_{t=0}\|_{H^2(\Omega)}^2. \end{aligned} \quad (3.4)$$

Proof of Proposition 3.1.

Combining lemma 3.1 and lemma 3.2, one can easily prove Proposition 3.1. In fact, by lemma 3.1 and lemma 3.2, it is easy to see that when $\delta < \delta_* := \min(\delta_1, \delta_2)$, problem (3.1) admits a smooth solution in $H^4(\Omega_T)$ and it satisfies the estimate given in proposition 3.1 for $\eta \geq \eta_0 := \max(\eta_1, \eta_2)$, where (δ_1, η_1) and (δ_2, η_2) are the constants obtained in lemma 3.1 and lemma 3.2, respectively.

Hence, in order to prove Proposition 3.1, it suffices to show that lemma 3.1 and lemma 3.2 hold. In this section, we will give a proof to lemma 3.1 and the proof of lemma 3.2 is postponed to section 4.

3.2. The proof of Lemma 3.1. It is difficult to solve problem (3.1) in the cornered-space domain Ω directly, so we introduce an extended problem first. Precisely, one extends r_{20} , r_{02} , r_{12} , r_{21} , and b_2 oddly with respect to $\{z_2 = 0\}$. Taking r_{02} for example, we define

$$Er_{02} := \begin{cases} r_{02}(z_0, z_1, z_2), & \text{when } z_2 > 0, \\ -r_{02}(z_0, z_1, -z_2), & \text{when } z_2 < 0. \end{cases} \quad (3.5)$$

Other coefficients and the right hand side term f is extended evenly with respect to $\{z_2 = 0\}$. Thanks to assumptions (ii) and (iii), all the extended coefficients are still in $W^{1,\infty}(\tilde{\Omega}_T)$, where $\tilde{\Omega}_T := [0, T] \times \mathbb{R}_+ \times \mathbb{R}$. For the notational simplicity, we omit the “ E ” in all extended functions, the extended vertical boundary is still denoted by Γ_1 and the initial space domain is still denoted by Γ_0 . We try to obtain a solution to (3.1) by solving the following initial boundary value problem:

$$\begin{cases} \mathcal{L}\varphi = f, & \text{in } \tilde{\Omega}_T, \\ \mathcal{B}\varphi = 0, & \text{on } \Gamma_1, \\ \varphi(0, z_1, z_2) = 0, & \text{on } \Gamma_0, \\ \partial_{z_0}\varphi(0, z_1, z_2) = 0, & \text{on } \Gamma_0. \end{cases} \quad (3.6)$$

Proof. The proof of lemma 3.1 is divided into three steps. In the first two steps, we establish the energy estimate of the solutions to (3.6) up to the second order. Then in the third step, we investigate a regularized problem associated to (3.6) by mollifying its coefficients via the convolution with respect to z_2 (since the extended coefficients are non-smooth only in the z_2 direction) with the one dimensional classical Friedrichs mollifier ρ_ϵ and derives its uniform-in- ϵ estimate up to the second order. Then by applying the result in [22] (or [23]) to the regularized problem, one derives a unique solution φ^ϵ to the regularized problem for each $\epsilon > 0$. Owing to the uniform-in- ϵ second order estimate of φ^ϵ , one deduces an $H^2(\Omega_T)$ -solution to the extended problem (3.6) by taking the limit $\epsilon \rightarrow 0^+$ in the regularized problem and it still satisfies the second order estimate. Then by the properties of the extended coefficients and the uniqueness of the solution (the uniqueness is guaranteed by the second order energy inequality), one can show that the unique solution to the extended linear problem (3.6) is actually a solution to the linear problem (3.1).

Step 1. In this step, we will deduce the first order energy estimate of the solution to problem (3.6). Multiplying $2e^{-2\eta z_0}\partial_{z_0}\varphi$ on both sides of equation in (3.6), we have

$$\begin{aligned} 2e^{-2\eta z_0}\mathcal{L}\varphi\partial_{z_0}\varphi &= \partial_i(e^{-2\eta z_0}r_{ij}\partial_j\varphi\partial_0\varphi) + \partial_j(e^{-2\eta z_0}r_{ij}\partial_i\varphi\partial_0\varphi) + e^{-2\eta z_0}P(D\varphi) \\ &\quad - \partial_0(e^{-2\eta z_0}r_{ij}\partial_i\varphi\partial_j\varphi) + 2\eta e^{-2\eta z_0}(2r_{0j}\partial_j\varphi\partial_0\varphi - r_{ij}\partial_i\varphi\partial_j\varphi), \end{aligned}$$

where $P(D\varphi)$ is a quadratic polynomial with respect to $D\varphi$. Clearly we have $P(D\varphi) \leq C|D\varphi|^2$. Integrating the identity over $\tilde{\Omega}_T$ with $z := (z_0, z_1, z_2)$, we obtain

$$\begin{aligned} \int_{\tilde{\Omega}_T} 2e^{-2\eta z_0} \mathcal{L}\varphi Q_\varphi dz &= \left[\int_{\tilde{\Omega}} e^{-2\eta z_0} H_0 dz_1 dz_2 \right]_{z_0=0}^{z_0=T} - \int_0^T \int_{\mathbb{R}} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \\ &\quad + \int_{\tilde{\Omega}_T} e^{-2\eta z_0} P(D\varphi) dz + 2\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} H_0 dz, \end{aligned} \quad (3.7)$$

where for $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3$, we define

$$H_0(\boldsymbol{\xi}) := 2 \sum_{i,k=0}^2 r_{i0} \xi_i Q_k \xi_k - Q_0 \sum_{i,j=0}^2 r_{ij} \xi_i \xi_j, \quad (3.8)$$

$$H_1(\boldsymbol{\xi}) := 2 \sum_{i,k=0}^2 r_{i1} \xi_i Q_k \xi_k - Q_1 \sum_{i,j=0}^2 r_{ij} \xi_i \xi_j. \quad (3.9)$$

At the background state, one has

$$\begin{aligned} H_0(D\varphi) &= 2\partial_0\varphi \sum_{k=0}^2 Q_k \partial_k\varphi - Q_0(r_{11}|\partial_1\varphi|^2 + r_{22}|\partial_2\varphi|^2 + |\partial_0\varphi|^2) \\ &= (D\varphi)\mathbf{M}(D\varphi)^\top, \end{aligned} \quad (3.10)$$

where

$$\mathbf{M} = \begin{pmatrix} Q_0 & Q_1 & Q_2 \\ Q_1 & -r_{11}Q_0 & 0 \\ Q_2 & 0 & -r_{22}Q_0 \end{pmatrix}.$$

Select (Q_0, Q_1, Q_2) properly such that

$$\begin{cases} Q_0 > 0, \\ -Q_0^2 r_{11} - Q_1^2 > 0, \\ r_{22} Q_1^2 Q_0 + Q_0^3 r_{11} r_{22} + r_{11} Q_0 Q_2^2 > 0, \end{cases} \quad (3.11)$$

i.e., such that $\mathbf{M}$ is positive definite. In view of assumption (i), we just need to let

$$\begin{cases} Q_0 > 0, \\ Q_0^2 > \frac{Q_1^2}{-r_{11}}, \\ Q_0^2 > \frac{-Q_1^2 r_{22} - Q_2^2 r_{11}}{r_{11} r_{22}}. \end{cases} \quad (3.12)$$

It is easy to see from (3.12) that Q_1 and Q_2 can be arbitrary. Then $H_0 \geq C_1|D\varphi|^2$ for some positive constant C_1 . On the other hand, it is easy to see $H_0 \leq C_2|D\varphi|^2$

due to assumption (i). By assumption (iv), on the vertical boundary Γ_1 , we have

$$\begin{aligned}
H_1 &= 2 \sum_{k=0}^2 Q_k \partial_k \varphi (r_{11} \partial_1 \varphi + r_{12} \partial_2 \varphi) - Q_1 \sum_{i,j=0}^2 r_{ij} \partial_i \varphi \partial_j \varphi \\
&= 2 \sum_{k=0}^2 Q_k \partial_k \varphi \left(\frac{r_{11}}{b_1} \mathcal{B} \varphi \right) - Q_1 \sum_{i,j=0}^2 r_{ij} \partial_i \varphi \partial_j \varphi \\
&= 0.
\end{aligned} \tag{3.13}$$

In the last equality above, we used the condition $\mathcal{B} \varphi = 0$ on Γ_1 and set $Q_1 = 0$. Hence we have

$$\begin{aligned}
&\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |D\varphi|^2 dz + \int_{\tilde{\Omega}} e^{-2\eta T} |D\varphi|^2 dz_1 dz_2 \\
&\leq C \int_{\tilde{\Omega}_T} e^{-2\eta z_0} ((q\eta + 1) |D\varphi|^2 + \frac{1}{q\eta} |\mathcal{L}\varphi|^2) dz.
\end{aligned} \tag{3.14}$$

Let $q = 1/(2C)$ and $\eta \geq 4$, then we obtain

$$\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |D\varphi|^2 dz + \int_{\tilde{\Omega}} e^{-2\eta T} |D\varphi|^2 dz_1 dz_2 \leq C \frac{1}{\eta} \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\mathcal{L}\varphi|^2 dz. \tag{3.15}$$

Step 2. In this step, we will establish the second order estimate of the solutions based on the first order estimate derived in step 1. Clearly, $\partial_{z_0} \varphi$ satisfies

$$\begin{cases} \mathcal{L}(\partial_{z_0} \varphi) = -[\partial_{z_0}, \mathcal{L}] \varphi + \partial_{z_0} f, & \text{in } \tilde{\Omega}_T, \\ \mathcal{B}(\partial_{z_0} \varphi) = 0, & \text{on } \Gamma_1, \\ \partial_{z_0} \varphi(z_0, z_1, z_2) = 0, & \text{on } \Gamma_0, \\ \partial_{z_0}^2 \varphi(z_0, z_1, z_2) = F|_{z_0=0}, & \text{on } \Gamma_0. \end{cases} \tag{3.16}$$

where

$$F = f - \sum_{(i,j) \neq (0,0)} r_{ij} \partial_{ij} \varphi - \sum_{i=0}^2 r_i \partial_i \varphi = f.$$

It is easy to see that $\|F|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2 = \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2$. By the same argument as done in the first step, we deduce that $\partial_{z_0} \varphi$ satisfies

$$\begin{aligned}
&\eta \|e^{-\eta z_0} D \partial_{z_0} \varphi\|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \|D \partial_{z_0} \varphi(T, \cdot)\|_{L^2(\tilde{\Omega})}^2 \\
&\leq C \left(\frac{1}{\eta} \|\mathcal{L}(\partial_{z_0} \varphi)\|_{L^2(\tilde{\Omega}_T)}^2 + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2 \right).
\end{aligned} \tag{3.17}$$

Then we proceed to estimate $\partial_{z_2}\varphi$. It is clear that $\partial_{z_2}\varphi$ satisfies

$$\begin{cases} \mathcal{L}(\partial_{z_2}\varphi) = -[\partial_{z_2}, \mathcal{L}]\varphi + \partial_{z_2}f, & \text{in } \tilde{\Omega}_T, \\ \mathcal{B}(\partial_{z_2}\varphi) = -[\partial_{z_2}, \mathcal{B}]\varphi, & \text{on } \Gamma_1, \\ \partial_{z_2}\varphi = 0, & \text{on } \Gamma_0, \\ \partial_{z_0}(\partial_{z_2}\varphi) = 0 & , \text{on } \Gamma_0. \end{cases} \quad (3.18)$$

Multiplying $2e^{-2\eta z_0}\partial_{z_0 z_2}\varphi$ on both sides of (3.18)₁ and integrating by parts over Ω_T , we have

$$\begin{aligned} \int_{\tilde{\Omega}_T}^{-2\eta z_0} \mathcal{L}(\partial_{z_2}\varphi) \partial_{z_0} \partial_{z_2} \varphi dz &= -2 \int_0^T \int_{\mathbb{R}} e^{-2\eta z_0} \partial_{z_0 z_2} \varphi \frac{r_{11}}{b_1} \mathcal{B}(\partial_{z_2}\varphi) dz_2 dz_0 \Big|_{z_1=0} \\ &\quad + 2\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} H_0 dz + \left[\int_{\tilde{\Omega}} e^{-2\eta t} H_0 dz_1 dz_2 \right]_{z_0=0}^{z_0=T} \\ &\quad + \int_{\tilde{\Omega}_T} e^{-2\eta z_0} P_1(D\partial_{z_2}\varphi) dz, \end{aligned} \quad (3.19)$$

where $H_0 = |\partial_{z_0}\partial_{z_2}\varphi|^2 - r_{11}|\partial_{z_1}\partial_{z_2}\varphi|^2 - r_{22}|\partial_{z_2}\partial_{z_2}\varphi|^2$ and P_1 is a new quadratic polynomial with respect to $D\partial_{z_2}\varphi$. Since the coefficients are in $W^{1,\infty}(\tilde{\Omega}_T)$, it is easy to see that $|P_1(D\partial_{z_2}\varphi)| \leq C|D\partial_{z_2}\varphi|$.

In order to complete the estimate, we need to deal with the boundary term carefully. Firstly, with the help of the boundary condition $\mathcal{B}\varphi = 0$ on Γ_1 , one has $\mathcal{B}\partial_{z_2}\varphi = -(\partial_{z_2}b_1\partial_{z_1} + \partial_{z_2}b_2\partial_{z_2})\varphi$. Then for $i = 1, 2$, by assumptions (i), (iii), and (iv), and the Gauss theorem

$$\begin{aligned} &2 \int_0^T \int_{\mathbb{R}} e^{-2\eta z_0} \left(\frac{r_{11}}{b_1} \partial_{z_2} b_i \partial_{z_i} \varphi \cdot \partial_{z_0 z_2} \varphi \right) \Big|_{z_1=0} dz_2 dz_0 \\ &= -2 \int_{\tilde{\Omega}_T} \partial_{z_1} \left(e^{-2\eta z_0} \frac{r_{11}}{b_1} \partial_{z_2} b_i \partial_{z_i} \varphi \partial_{z_0 z_2} \varphi \right) dz \\ &= -2 \int_{\tilde{\Omega}_T} e^{-2\eta z_0} \left(\partial_{z_1} \left(\frac{r_{11}}{b_1} \partial_{z_2} b_i \right) \partial_{z_i} \varphi \partial_{z_0 z_2} \varphi + \frac{r_{11}}{b_1} \partial_{z_2} b_i (\partial_{z_1 z_i} \varphi \partial_{z_0 z_2} \varphi + \partial_{z_i} \varphi \partial_{z_0 z_1 z_2} \varphi) \right) \\ &\lesssim \int_0^T e^{-2\eta z_0} (\|\partial_{z_0} \varphi\|_{H^1(\tilde{\Omega})}^2 + \|\varphi\|_{H^2(\tilde{\Omega})}^2) dz_0 + \int_{\tilde{\Omega}_T} e^{-2\eta z_0} \frac{r_{11}}{b_1} \partial_{z_2} b_i \partial_{z_i} \varphi \partial_{z_0 z_1 z_2} \varphi dz \\ &\leq C \int_0^T e^{-2\eta z_0} (\|\partial_{z_0} \varphi\|_{H^1(\tilde{\Omega})}^2 + \|\varphi\|_{H^2(\tilde{\Omega})}^2) dz_0 + \int_{\tilde{\Omega}_T} \partial_{z_0} \left(e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2} b_i \partial_{z_i} \varphi \partial_{z_1 z_2} \varphi \right) dz \\ &\quad + 2\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2} b_i \partial_{z_i} \varphi \partial_{z_1 z_2} \varphi dz - \int_{\tilde{\Omega}_T} e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2} b_i \partial_{z_0 z_i} \varphi \partial_{z_1 z_2} \varphi dz \\ &\quad - \int_{\tilde{\Omega}_T} e^{-2\eta z_0} \partial_{z_0} \left(\frac{r_{22}}{b_1} \partial_{z_2} b_i \right) \partial_{z_i} \varphi \partial_{z_1 z_2} \varphi dz \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^T e^{-2\eta z_0} (\|\partial_{z_0}\varphi\|_{H^1(\tilde{\Omega})}^2 + \|\varphi\|_{H^2(\tilde{\Omega})}^2) dz_0 \\
&\quad + \delta(\eta \int_0^T e^{-2\eta z_0} \|\varphi\|_{H^2(\tilde{\Omega})}^2 dz_0 + e^{-2\eta T} \|\varphi|_{z_0=T}\|_{H^2(\tilde{\Omega})}^2). \tag{3.20}
\end{aligned}$$

It follows from assumptions (i) and (iv) that $H_0 \geq C|D\partial_{z_2}\varphi|^2$. Then in view of (3.19) and (3.20), by the Cauchy inequality, we deduce that

$$\begin{aligned}
&\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |D\partial_{z_2}\varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |D\partial_{z_2}\varphi|_{z_0=T}|^2 dz_1 dz_2 \\
&\leq C \left(\int_{\tilde{\Omega}_T} e^{-2\eta z_0} ((\epsilon\eta + 1)|D\partial_{z_2}\varphi|^2 + \frac{1}{\epsilon\eta} |\mathcal{L}\partial_{z_2}\varphi|^2) dz + \|\varphi_0\|_{H^2(\tilde{\Omega})}^2 + \|\varphi_1\|_{H^1(\tilde{\Omega})}^2 \right. \\
&\quad + \int_0^T e^{-2\eta z_0} (\|\partial_{z_0}\varphi\|_{H^1(\tilde{\Omega})}^2 + \|\varphi\|_{H^2(\tilde{\Omega})}^2) dz_0 \\
&\quad \left. + \delta(\eta \int_0^T e^{-2\eta z_0} \|\varphi\|_{H^2(\tilde{\Omega})}^2 dz_0 + e^{-2\eta T} \|\varphi|_{z_0=T}\|_{H^2(\tilde{\Omega})}^2) \right). \tag{3.21}
\end{aligned}$$

Finally, it follows from the second order equation (3.6)₁ that

$$\partial_{z_1 z_1} \varphi = \frac{1}{r_{11}} \left(\mathcal{L}\varphi - \sum_{(i,j) \neq (1,1)} r_{ij} \partial_{ij} \varphi \right).$$

Therefore, by (3.15), (3.17) and (3.21),

$$\begin{aligned}
&\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\partial_{z_1 z_1} \varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |\partial_{z_1 z_1} \varphi|_{z_0=T}|^2 dz_1 dz_2 \\
&\lesssim \sum_{i=0,2} \left(\eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |D\partial_{z_i} \varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |D\partial_{z_i} \varphi|_{z_0=T}|^2 dz_1 dz_2 \right) \\
&\quad + \eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |D\varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |D\varphi|_{z_0=T}|^2 dz_1 dz_2 \\
&\quad + \eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\mathcal{L}\varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |\mathcal{L}\varphi|_{z_0=T}|^2 dz_1 dz_2 \\
&\lesssim \left(\frac{1}{\eta} \|e^{-\eta z_0} \mathcal{L}\varphi\|_{L^2(\tilde{\Omega}_T)}^2 + \frac{1}{\eta} \|e^{-\eta z_0} \mathcal{L}(\partial_{z_0}\varphi)\|_{L^2(\tilde{\Omega}_T)}^2 + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2 \right) \\
&\quad + \int_{\tilde{\Omega}_T} e^{-2\eta z_0} ((q\eta + 1)|D\partial_{z_2}\varphi|^2 + \frac{1}{q\eta} |\mathcal{L}\partial_{z_2}\varphi|^2) dz \\
&\quad + \int_0^T e^{-2\eta z_0} (\|\partial_{z_0}\varphi\|_{H^1(\tilde{\Omega})}^2 + \|\varphi\|_{H^2(\tilde{\Omega})}^2) dz_0 + \delta\eta \int_0^T e^{-2\eta z_0} \|\varphi\|_{H^2(\tilde{\Omega})}^2 dz_0 \\
&\quad + \delta e^{-2\eta T} \|\varphi|_{z_0=T}\|_{H^2(\tilde{\Omega})}^2 + \eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\mathcal{L}\varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |\mathcal{L}\varphi|_{z_0=T}|^2 dz_1 dz_2. \tag{3.22}
\end{aligned}$$

In order to control the last two terms in (3.22), by conducting integration by parts with respect to z_0 to the integral $\int_{\tilde{\Omega}_T} e^{-2\eta z_0} |v|^2 dz$, we introduce the following inequality

$$\begin{aligned} & \eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |v|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |v|_{z_0=T}|^2 dz_1 dz_2 \\ & \leq \frac{1}{\eta} \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\partial_{z_0} v|^2 dz + \|v|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (3.23)$$

Therefore,

$$\begin{aligned} & \eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\mathcal{L}\varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |\mathcal{L}\varphi|_{z_0=T}|^2 dz_1 dz_2 \\ & \leq \frac{1}{\eta} \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\partial_{z_0}(\mathcal{L}\varphi)|^2 dz + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2 \\ & \lesssim \frac{1}{\eta} \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\mathcal{L}(\partial_{z_0}\varphi)|^2 dz + \frac{1}{\eta} \sum_{|\alpha| \leq 2} \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\tilde{\Omega}_T)}^2 + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (3.24)$$

Then (3.22) and (3.24) imply

$$\begin{aligned} & \eta \int_{\tilde{\Omega}_T} e^{-2\eta z_0} |\partial_{z_1 z_1} \varphi|^2 dz + e^{-2\eta T} \int_{\tilde{\Omega}} |\partial_{z_1 z_1} \varphi|_{z_0=T}|^2 dz_1 dz_2 \\ & \lesssim \frac{1}{\eta} \int_{\tilde{\Omega}_T} e^{-2\eta z_0} \sum_{|\alpha| \leq 1} |\mathcal{L}(D^\alpha \varphi)|^2 dz + \int_{\tilde{\Omega}_T} e^{-2\eta z_0} ((q\eta + 1) |D\partial_{z_2} \varphi|^2 + \frac{1}{q\eta} |\mathcal{L}\partial_{z_2} \varphi|^2) dz \\ & \quad + \int_0^T e^{-2\eta z_0} (C\delta \|\varphi\|_{H^2(\tilde{\Omega})}^2 + \|\partial_{z_0} \varphi\|_{H^1(\tilde{\Omega})}^2) dz_0 + \frac{1}{\eta} \sum_{|\alpha| \leq 2} \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\tilde{\Omega}_T)}^2 \\ & \quad + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (3.25)$$

Since $D\partial_{z_0} \varphi$, $D\partial_{z_2} \varphi$ and $\partial_{z_1 z_1} \varphi$ cover all second order derivatives, by adding (3.15), (3.21) and (3.25) together, letting $q > 0$, $\delta > 0$ and $\frac{1}{\eta}$ be properly small, we deduce

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi|_{z_0=T}\|_{L^2(\tilde{\Omega})}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 1} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\tilde{\Omega}_T)}^2 + \|\mathcal{L}\varphi|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (3.26)$$

Since the coefficients of $\mathcal{L}$ are bounded, by (3.26), it holds for η large enough that

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi|_{z_0=T}\|_{L^2(\tilde{\Omega})}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 1} \|e^{-\eta z_0} D^\alpha f\|_{L^2(\tilde{\Omega}_T)}^2 + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (3.27)$$

Step 3. In this step, we will apply Theorem 1 in [22] (or Theorem 1 in [23]) to derive the existence of an $H^2(\tilde{\Omega}_T)$ -solution to the extended problem, then by the property of the extended coefficients, one shows that the solution is indeed a solution to problem (3.1). In order to apply [22, Theorem 1] (or [23, Theorem 1]), we consider a regularized problem associated to (3.6). For $\epsilon > 0$, let ρ_ϵ be the one dimensional Friedrichs mollifier, i.e., $\rho_\epsilon(s) = \frac{1}{\epsilon} \eta\left(\frac{s}{\epsilon}\right)$, where

$$\eta(s) = \begin{cases} K \cdot \exp(-\frac{1}{1-|s|^2}), & |s| < 1, \\ 0, & |s| \geq 1, \end{cases} \quad (3.28)$$

such that $\int_{\mathbb{R}} \eta(s) ds = 1$. Define $\mathcal{L}^\epsilon$ and $\mathcal{B}^\epsilon$ as

$$\mathcal{L}^\epsilon := \sum_{i,j=0}^2 \tilde{r}_{ij}^\epsilon \partial_{ij}, \quad \mathcal{B}^\epsilon := \sum_{i=1}^2 \tilde{b}_i^\epsilon \partial_i, \quad (3.29)$$

where

$$\tilde{r}_{ij}^\epsilon(z_0, z_1, z_2) = \left(\frac{r_{ij}}{r_{11}}\right)^\epsilon(z_0, z_1, z_2) := \int_{\mathbb{R}} \left(\frac{r_{ij}}{r_{11}}\right)(z_0, z_1, s) \rho_\epsilon(z_2 - s) ds$$

and

$$\tilde{b}_i^\epsilon(z_1, z_2) = \left(\frac{b_i}{b_1}\right)^\epsilon(z_1, z_2) := \int_{\mathbb{R}} \left(\frac{b_i}{b_1}\right)(z_1, s) \rho_\epsilon(z_2 - s) ds.$$

Before going on, we present following lemma, which gives the properties of the coefficients of the regularized problem.

Lemma 3.3. *Under assumptions (i)-(iv), we have:*

- (1). $\tilde{r}_{12}^\epsilon, \tilde{r}_{21}^\epsilon, \tilde{r}_{02}^\epsilon, \tilde{r}_{20}^\epsilon$ and $\tilde{b}_2^\epsilon(0, z_2)$ are odd functions with respect to z_2 , and the other coefficients are even functions with respect to z_2 .
- (2). There exists a positive constant C , independent on ϵ , such that

$$\left\| \tilde{r}_{ij}^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} \leq C\delta + \left| \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right|, \quad \left\| D\tilde{r}_{ij}^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} \leq C\delta,$$

and

$$\left\| \tilde{b}_i^\epsilon \right\|_{W^{1,\infty}(\tilde{\Omega})} \leq C\delta, \quad \left\| \partial_{z_2}^2 \tilde{b}_2^\epsilon(0, \cdot) \right\|_{L^\infty} \leq C\delta.$$

- (3). $\tilde{b}_2^\epsilon(0, 0) = 0$ and $\partial_{z_2} \tilde{b}_2^\epsilon(0, 0) = 0$.
- (4). $\mathcal{B}^\epsilon$ is co-normal to $\mathcal{L}^\epsilon$, i.e.,

$$\left(\frac{r_{10}}{r_{11}}\right)^\epsilon|_{z_1=0} = 0 \quad \text{and} \quad \left(\frac{r_{12}}{r_{11}}\right)^\epsilon|_{z_1=0} = \left(\frac{b_2}{b_1}\right)^\epsilon|_{z_1=0}.$$

With this lemma, it is not difficult to show lemma 3.1 and the proof of this lemma is delayed to the end of this subsection.

Now we consider

$$\begin{cases} \mathcal{L}^\epsilon \varphi^\epsilon = f^\epsilon, & \text{in } \tilde{\Omega}_T, \\ \mathcal{B}^\epsilon \varphi^\epsilon = 0, & \text{on } \Gamma_1, \\ \varphi^\epsilon(0, z_1, z_2) = 0, & \text{on } \Gamma_0, \\ \partial_{z_0} \varphi^\epsilon(0, z_1, z_2) = 0, & \text{on } \Gamma_0, \end{cases} \quad (3.30)$$

where

$$f^\epsilon(z_0, z_1, z_2) = \left(\frac{f}{r_{11}} \right)^\epsilon(z_0, z_1, z_2) := \int_{\mathbb{R}} \left(\frac{f}{r_{11}} \right)(z_0, z_1, s) \rho_\epsilon(z_2 - s) ds.$$

Armed with lemma 3.3, one can immediately obtain the uniform-in- ϵ second order estimate of φ^ϵ by repeating the process in the first two steps. In fact, one has

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \eta \|e^{-\eta z_0} D^\alpha \varphi^\epsilon\|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi^\epsilon|_{z_0=T}\|_{L^2(\tilde{\Omega})}^2 \\ & \leq C \frac{1}{\eta} \sum_{|\alpha| \leq 1} \|e^{-\eta z_0} D^\alpha f^\epsilon\|_{L^2(\tilde{\Omega}_T)}^2 + \|f^\epsilon|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (3.31)$$

Clearly one has

$$\begin{aligned} & \left\| \left(\frac{\mathcal{F}}{r_{11}} \right)^\epsilon(z_0, \cdot) \right\|_{L^2(\tilde{\Omega})}^2 \\ & = \int_{\tilde{\Omega}} \left| \left(\frac{\mathcal{F}}{r_{11}} \right)^\epsilon(z_0, z_1, z_2) \right|^2 dz_1 dz_2 \\ & = \int_{\tilde{\Omega}} \left| \int_{\mathbb{R}} \frac{\mathcal{F}}{r_{11}}(z_0, z_1, s) \rho_\epsilon(z_2 - s) ds \right|^2 dz_1 dz_2 \\ & \leq \int_{\tilde{\Omega}} \left(\int_{\mathbb{R}} \left| \frac{\mathcal{F}}{r_{11}} \right|^2(z_0, z_1, s) \rho_\epsilon(z_2 - s) ds \right) \left(\int_{\mathbb{R}} \rho_\epsilon(z_2 - s) ds \right) dz_1 dz_2 \\ & \leq \left\| \frac{1}{r_{11}} \right\|_{L^\infty(\tilde{\Omega}_T)}^2 \cdot \int_{\tilde{\Omega}} \left[\left(\int_{\mathbb{R}} \rho_\epsilon(z_2 - s) dz_2 \right) |\mathcal{F}|^2(z_0, z_1, s) dz_1 \right] ds \\ & \leq C \int_{\tilde{\Omega}} |\mathcal{F}|^2(z_0, z_1, s) dz_1 ds = C \|\mathcal{F}(z_0, \cdot)\|_{L^2(\tilde{\Omega})}^2, \end{aligned} \quad (3.32)$$

where the constant C depends on ρ_0 and γ , but not on ϵ . Similarly one has

$$\left\| D \left(\frac{\mathcal{F}}{r_{11}} \right)^\epsilon(z_0, \cdot) \right\|_{L^2(\tilde{\Omega})} \leq C \left(\|\mathcal{F}(z_0, \cdot)\|_{L^2(\tilde{\Omega})}^2 + \|D\mathcal{F}(z_0, \cdot)\|_{L^2(\tilde{\Omega})}^2 \right). \quad (3.33)$$

Combining the above estimates and lemma 3.3, we deduce that

$$\sum_{|\alpha| \leq 2} \eta \|e^{-\eta z_0} D^\alpha \varphi^\epsilon\|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi^\epsilon|_{z_0=T}\|_{L^2(\tilde{\Omega})}^2$$

$$\leq C \frac{1}{\eta} \sum_{|\alpha| \leq 1} \|e^{-\eta z_0} D^\alpha f\|_{L^2(\tilde{\Omega}_T)}^2 + \|f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \quad (3.34)$$

By [22, Theorem 1] (or [23, Theorem 1]), lemma 3.3, and inequality (3.34), one concludes that for each $\epsilon > 0$, there exists a solution φ^ϵ to problem (3.30) satisfying the uniform estimate (3.34). Hence there exists a subsequence $\{\varphi^{\epsilon_j}\}_{\epsilon_j > 0}$ converging to a function φ weakly in $H^2(\tilde{\Omega}_T)$. By lemma 3.3 and the uniform estimate (3.34), one can pass the limit $\epsilon \rightarrow 0^+$ in problem (3.30), which implies φ solves problem (3.6) in the weak sense and it also satisfies estimate (3.34). By our extension, it is not difficult to see that $\varphi(z_0, z_1, -z_2)$ is also a solution to problem (3.6). It follows from the uniqueness of the extended problem (3.6) that

$$\varphi(z_0, z_1, z_2) = \varphi(z_0, z_1, -z_2)$$

for all $(z_0, z_1, z_2) \in \tilde{\Omega}_T$. Differentiating on both sides of the above identity then letting $z_2 = 0$, one deduces that $\partial_{z_2} \varphi(z_0, z_1, 0) = 0$. This reveals that φ is indeed a solution to problem (3.1). $\square$

Proof of lemma 3.3.

Proof. (1) They are true due to the constructions of $\tilde{r}_{ij}^\epsilon$'s and $\tilde{b}_i^\epsilon$ and the property of the mollifier. For example, for $\tilde{r}_{12}^\epsilon$, one has

$$\begin{aligned} \tilde{r}_{12}^\epsilon(z_0, z_1, -z_2) &= \int_{\mathbb{R}} \frac{r_{12}}{r_{11}}(z_0, z_1, -z_2 - \tau) \rho_\epsilon(\tau) d\tau \\ &= - \int_{\mathbb{R}} \frac{r_{12}}{r_{11}}(z_0, z_1, z_2 + \tau) \rho_\epsilon(-\tau) d\tau \\ &= - \int_{\mathbb{R}} \frac{r_{12}}{r_{11}}(z_0, z_1, s) \rho_\epsilon(z_2 - s) ds \\ &= -\tilde{r}_{12}^\epsilon(z_0, z_1, z_2), \end{aligned} \quad (3.35)$$

where in the second equality, we have used the oddness of r_{12} , and the evenness of r_{11} and ρ_ϵ , and in the last equality, the changing of variable is used. The properties of the other coefficients can be derived by similar arguments. Since the argument is similar and standard, we omit the details here.

(2) By assumption (ii), properties (1) from the modified coefficients, and the properties of the mollifier, one has

$$\begin{aligned} \|\tilde{r}_{ij}^\epsilon\|_{L^\infty(\tilde{\Omega}_T)} &= \left\| \left(\frac{r_{ij}}{r_{11}} \right)^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} \leq \left\| \left(\frac{r_{ij}}{r_{11}} \right)^\epsilon - \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right\|_{L^\infty(\tilde{\Omega}_T)} + \left| \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right| \\ &= \left\| \left(\frac{r_{ij}}{r_{11}} - \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right)^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} + \left| \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right|. \end{aligned}$$

It is clear that

$$\begin{aligned}
\left\| \left(\frac{r_{ij}}{r_{11}} - \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right)^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} &\leq \left\| \frac{r_{ij}}{r_{11}} - \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right\|_{L^\infty(\Omega_T)} \\
&= \left\| \frac{(r_{ij} - \bar{r}_{ij})\bar{r}_{11} - \bar{r}_{ij}(r_{11} - \bar{r}_{11})}{r_{11}\bar{r}_{11}} \right\|_{L^\infty(\Omega_T)} \\
&\leq \left\| \frac{(r_{ij} - \bar{r}_{ij})}{r_{11}} \right\|_{L^\infty(\Omega_T)} + \left| \frac{\bar{r}_{ij}}{\bar{r}_{11}} \right| \cdot \left\| \frac{(r_{11} - \bar{r}_{11})}{r_{11}} \right\|_{L^\infty(\Omega_T)} \\
&\leq C\delta.
\end{aligned} \tag{3.36}$$

We also have

$$\left\| D\tilde{r}_{ij}^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} = \left\| \left(D\frac{r_{ij}}{r_{11}} \right)^\epsilon \right\|_{L^\infty(\tilde{\Omega}_T)} = \left\| \left(D\frac{r_{ij}}{r_{11}} \right)^\epsilon \right\|_{L^\infty(\Omega_T)} \leq \left\| D\left(\frac{r_{ij}}{r_{11}} \right) \right\|_{L^\infty(\Omega_T)}.$$

It is easy to see that

$$\begin{aligned}
\left\| D\left(\frac{r_{ij}}{r_{11}} \right) \right\|_{L^\infty(\Omega_T)} &= \left\| \frac{1}{r_{11}}Dr_{ij} - \frac{r_{ij}}{r_{11}^2}Dr_{11} \right\|_{L^\infty(\Omega_T)} \\
&\leq \left\| \frac{1}{r_{11}} \right\|_{L^\infty(\Omega_T)} \cdot \|r_{ij} - \bar{r}_{ij}\|_{W^{1,\infty}(\Omega_T)} \\
&\quad + \left\| \frac{r_{ij}}{r_{11}^2} \right\|_{L^\infty(\Omega_T)} \cdot \|r_{11} - \bar{r}_{11}\|_{W^{1,\infty}(\Omega_T)} \\
&\leq C\delta.
\end{aligned} \tag{3.37}$$

Similarly, one can deduce that

$$\left\| \left(\frac{b_i}{b_1} \right)^\epsilon \right\|_{W^{1,\infty}(\tilde{\Omega}_T)} \leq C\delta.$$

In fact, since $b_2(0,0) = \partial_{z_2}^2 b_2(0,0) = 0$, the twice differentiable function of $b_2(0, z_2)$ is bounded at $z_2 = 0$. Hence $|\partial_{z_2} b_2^\epsilon(0, z_2)| \leq C\delta$ for all z_2 .

We remark that the positive constant C in the above inequalities only depends on ρ_0 and γ , but is independent on ϵ .

(3) By the definition of $\tilde{b}_2^\epsilon$, the oddness of $\frac{b_2}{b_1}$ and the evenness of ρ_ϵ , one has

$$\begin{aligned}
\tilde{b}_2^\epsilon(0,0) &= \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, \tau) \rho_\epsilon(-\tau) d\tau \\
&= - \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, -\tau) \rho_\epsilon(\tau) d\tau
\end{aligned}$$

$$\begin{aligned}
&= - \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, z_2 - \tau) \rho_\epsilon(\tau) d\tau \Big|_{z_2=0} \\
&= -\tilde{b}_2^\epsilon(0, 0),
\end{aligned} \tag{3.38}$$

which implies $\tilde{b}_2^\epsilon(0, 0) = 0$. Then

$$\begin{aligned}
\partial_{z_2} \tilde{b}_2^\epsilon(0, 0) &= \frac{1}{\epsilon} \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, \tau) \frac{\partial \eta}{\partial z_2} \left(\frac{z_2 - \tau}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= -\frac{1}{\epsilon} \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, \tau) \frac{\partial \eta}{\partial z_2} \left(\frac{\tau - z_2}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= \frac{1}{\epsilon} \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, \tau) \frac{\partial \eta}{\partial \tau} \left(\frac{\tau - z_2}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= -\frac{1}{\epsilon} \int_{\mathbb{R}} \frac{\partial}{\partial \tau} \left(\frac{b_2}{b_1} \right) (0, \tau) \eta \left(\frac{\tau - z_2}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= -\frac{1}{\epsilon} \int_{\mathbb{R}} \frac{\partial}{\partial \tau} \left(\frac{b_2}{b_1} \right) (0, \tau) \eta \left(\frac{z_2 - \tau}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= \frac{1}{\epsilon} \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, \tau) \frac{\partial \eta}{\partial \tau} \left(\frac{z_2 - \tau}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= -\frac{1}{\epsilon} \int_{\mathbb{R}} \left(\frac{b_2}{b_1} \right) (0, \tau) \frac{\partial \eta}{\partial z_2} \left(\frac{z_2 - \tau}{\epsilon} \right) d\tau \Big|_{z_2=0} \\
&= -\partial_{z_2} \tilde{b}_2^\epsilon(0, 0).
\end{aligned} \tag{3.39}$$

Hence we deduce that $\partial_{z_2} \tilde{b}_2^\epsilon(0, 0) = 0$.

(4) This is an easy consequence of the following fact by lemma 2.1:

$$\left(\frac{r_{10}}{r_{11}} \right) \Big|_{z_1=0} = 0 \quad \text{and} \quad \left(\frac{r_{12}}{r_{11}} \right) \Big|_{z_1=0} = \left(\frac{b_2}{b_1} \right) \Big|_{z_1=0}.$$

□

4. THE LINEARIZED PROBLEM (II): HIGHER ORDER ESTIMATES

It is clear that the $H^2(\Omega_T)$ -solution of the linearized problem obtained in section 3 is not sufficient to yield a smooth solution to the nonlinear problem by nonlinear iteration method. Hence, in order to deduce the existence of smooth solutions of the nonlinear problem, one has to establish higher order the *a priori* estimate of the solution of the linearized problem derived in section 3. However, since the regularity of the coefficients of the equation in the extended domain is not sufficient to establish higher order *a priori* estimates. One has to establish higher order estimate in the cornered-space domain directly. But due to the violation of the linear stability conditions of the boundary operator and the presence of the corner singularity, it is difficult to derive higher order *a priori* estimate of the solutions, in

particular, the estimate of the boundary terms. Based on the observation that the boundary operators are co-normal and the vanishing properties of the coefficients on the boundaries (see lemma 2.1 for details), one expresses the boundary terms in terms of some commutators, which reduces the order of the derivatives contained in the boundary terms. Then by the Gauss theorem and the trace theorem, the estimates of boundary terms of the highest order derivatives of the solution can be established.

In this section, we give a proof to lemma 3.2 by establishing the estimates of the third and fourth order. Due to the corner singularity, the third and the fourth order estimates cannot be derived by same manner, which is similar to the situation in [18, 19]. Hence they will be deduced separately in the next two subsections.

4.1. Third order estimate of the solution. In this subsection, we establish the third order estimate of the solution obtained in lemma 3.1. Since the boundary of the space domain is not smooth, it is difficult for us to find multipliers such that the boundary terms on both sides of the corner point have good sign, which is different from the initial boundary value problems on smooth domains. By the properties of the boundary operator $\mathcal{B}$ and the coefficients of the equation (see lemma 2.1), we can find suitable multipliers such that the boundary terms can be expressed as some commutators, so that the order of the derivative of the solution is reduced. Then the boundary terms can be estimated by control the commutators, which can be done by using the Gauss theorem and integrating by parts with respect to the time derivative. The third order estimate is summarized as the following lemma:

Lemma 4.1. *There exists $\delta_3 > 0$ such that if assumptions (i) – (iv) hold for $\delta \leq \delta_3$, then there exists a constant $\eta_3 > 1$ such that for any $T > 0$ and $\eta \geq \eta_3$, the $H^2(\Omega_T)$ solution of problem (3.1) satisfies*

$$\begin{aligned} & \sum_{|\alpha|=3} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 2} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} f\|_{H^2(\Omega_T)}^2 + \|f|_{t=0}\|_{H^1(\Omega)}^2. \end{aligned} \quad (4.1)$$

Proof. Since ∂_{z_0} is tangential to both boundaries $\{z_1 = 0\}$ and $\{z_2 = 0\}$, one can apply (3.26) to $\partial_{z_0} \varphi$ to obtain that

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \eta \|e^{-\eta z_0} D^\alpha \partial_{z_0} \varphi\|_{L^2(\tilde{\Omega}_T)}^2 + e^{-2\eta T} \|D^\alpha \partial_{z_0} \varphi|_{z_0=T}\|_{L^2(\tilde{\Omega})}^2 \\ & \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 2} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \partial_{z_0} \varphi)\|_{L^2(\tilde{\Omega}_T)}^2 + \|\mathcal{L}(\partial_{z_0} \varphi)|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2. \end{aligned} \quad (4.2)$$

Next, we will consider the first order estimate of $\partial_{z_2}^2 \varphi$. It is clear that $\partial_{z_2}^2 \varphi$ satisfies

$$\begin{cases} \mathcal{L}(\partial_{z_2 z_2} \varphi) = -[\partial_{z_2 z_2}, \mathcal{L}]\varphi + \partial_{z_2 z_2} f, & \text{in } \tilde{\Omega}_T, \\ \mathcal{B}(\partial_{z_2 z_2} \varphi) = -[\partial_{z_2 z_2}, \mathcal{B}]\varphi, & \text{on } \Gamma_1, \\ \partial_{z_2}^2 \varphi = 0, & \text{on } \Gamma_0 \\ \partial_{z_0}(\partial_{z_2}^2 \varphi) = 0, & \text{on } \Gamma_0. \end{cases} \quad (4.3)$$

Via the equation and $\partial_{z_2} \varphi|_{z_2=0} = 0$, we can further derive the boundary condition for $\partial_{z_2 z_2} \varphi$ on $\{z_2 = 0\}$ as

$$\partial_{z_2}(\partial_{z_2 z_2} \varphi) = \frac{1}{r_{22}}(\mathcal{L} - r_{00}\partial_{z_0}^2 - 2r_{02}\partial_{z_0}\partial_{z_2} - 2r_{12}\partial_{z_1}\partial_{z_2} - r_{11}\partial_{z_1}^2)\partial_{z_2} \varphi.$$

As required in assumption (iii), we have $r_{12} = r_{02}$ on $\{z_2 = 0\}$. Moreover, $(\partial_{z_0}^2 \partial_{z_2} \varphi, \partial_{z_0 z_1 z_2} \varphi, \partial_{z_1}^2 \partial_{z_2} \varphi, \partial_{z_0 z_2} \varphi, \partial_{z_1 z_2} \varphi)$ vanishes on $\{z_2 = 0\}$, since $\partial_{z_2} \varphi|_{z_2=0} = 0$. As a result, we obtain

$$\partial_{z_2}^3 \varphi = \frac{1}{r_{22}} \mathcal{L}(\partial_{z_2} \varphi) \text{ on } \{z_2 = 0\}. \quad (4.4)$$

Multiplying $2e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^2 \varphi$ on both sides of (4.3), and integrating by parts over Ω_T , one has

$$\begin{aligned} 2 \int_{\Omega_T} e^{-2\eta z_0} \mathcal{L}(\partial_{z_2}^2 \varphi) \partial_{z_0} \partial_{z_2}^2 \varphi dz &= \left[\int_{\Omega} e^{-2\eta z_0} H_0 dz_1 dz_2 \right]_{z_0=0}^{z_0=T} - \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \\ &\quad + \int_{\Omega_T} e^{-2\eta z_0} P_2(D\varphi) dz + 2\eta \int_{\Omega_T} e^{-2\eta z_0} H_0 dz \\ &\quad - 2 \int_0^T r_{22} e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_2}^3 \varphi|_{z_2=0} dz_1 dz_0 \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} H_0 &= |\partial_{z_0} \partial_{z_2}^2 \varphi|^2 - r_{11} |\partial_{z_1} \partial_{z_2}^2 \varphi|^2 - r_{22} |\partial_{z_2}^3 \varphi|^2, \\ H_1 &= 2((r_{11} \partial_{z_1} + r_{12} \partial_{z_2}) \partial_{z_2}^2 \varphi) \partial_{z_0} \partial_{z_2}^2 \varphi, \end{aligned} \quad (4.6)$$

and $P_2(D\partial_{z_2}^2 \varphi)$ is a quadratic polynomial in $D\partial_{z_2}^2 \varphi$ with bounded coefficients. By assumptions (i) and (ii), it is easy to see that $H_0 \geq C|D\partial_{z_2}^2 \varphi|^2$ for some positive constant C . Hence we deduce that

$$\begin{aligned} &\eta \int_{\Omega_T} e^{-2\eta z_0} |D\partial_{z_2}^2 \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D\partial_{z_2}^2 \varphi|_{z_0=T}|^2 dz_1 dz_2 \\ &\lesssim \int_{\Omega_T} e^{-2\eta z_0} (q\eta + 1) |D\partial_{z_2}^2 \varphi|^2 + \frac{1}{q\eta} |\mathcal{L}(\partial_{z_2}^2 \varphi)|^2 dz \\ &\quad + \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 + \int_0^T r_{22} e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_2}^3 \varphi|_{z_2=0} dz_1 dz_0. \end{aligned} \quad (4.7)$$

Next, we are forced to control the boundary terms on the right hand-side of the above inequality. Employing assumption (iv) and the boundary condition that $\mathcal{B}\varphi = 0$ on $\{z_1 = 0\}$, one arrives at

$$\begin{aligned}
-H_1|_{z_1=0} &= -2r_{11}((\partial_{z_1} + \frac{r_{12}}{r_{11}}\partial_{z_2})\partial_{z_2}^2\varphi)\partial_{z_0}\partial_{z_2}^2\varphi \\
&= -\frac{2r_{11}}{b_1}(b_1\partial_{z_1}\partial_{z_2}^2\varphi + b_2\partial_{z_2}\partial_{z_2}^2\varphi)\partial_{z_0}\partial_{z_2}^2\varphi \\
&= \frac{2r_{11}}{b_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_0}\partial_{z_2}^2\varphi \\
&= \frac{2r_{11}}{b_1}\partial_{z_0}\partial_{z_2}^2\varphi(2(\partial_{z_2}b_1)\partial_{z_1z_2}\varphi + 2(\partial_{z_2}b_2)\partial_{z_2}^2\varphi \\
&\quad + (\partial_{z_2}^2b_1)\partial_{z_1}\varphi + (\partial_{z_2}^2b_2)\partial_{z_2}\varphi).
\end{aligned} \tag{4.8}$$

Therefore, by the Gauss theorem we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \\
&= - \int_{\Omega_T} \partial_{z_1}(e^{-2\eta z_0} \frac{2r_{22}}{b_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_0}\partial_{z_2}^2\varphi) dz \\
&= - \int_{\Omega_T} e^{-2\eta z_0} (\partial_{z_1} \left(\frac{2r_{22}}{b_1} \right) ([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_0}\partial_{z_2}^2\varphi) + \frac{r_{22}}{b_1} \partial_{z_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_0}\partial_{z_2}^2\varphi dz \\
&\quad - \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_0z_1}\partial_{z_2}^2\varphi dz \\
&\leq C\delta \int_0^T e^{-2\eta z_0} \|\partial_{z_0}\varphi(z_0, \cdot)\|_{H^2(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0 \\
&\quad - \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_0z_1}\partial_{z_2}^2\varphi dz \\
&\leq C\delta \int_0^T e^{-2\eta z_0} (\|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\partial_{z_0}\varphi(z_0, \cdot)\|_{H^2(\Omega)}^2) dz_0 + \mathcal{K},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K} &= - \int_{\Omega_T} \partial_{z_0}(e^{-2\eta z_0} \frac{r_{22}}{b_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_1}\partial_{z_2}^2\varphi) dz \\
&\quad - 2\eta \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_1}\partial_{z_2}^2\varphi + \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0}(\frac{r_{22}}{b_1})([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_1}\partial_{z_2}^2\varphi dz \\
&\quad + \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1}\partial_{z_0}([\partial_{z_2}^2, \mathcal{B}]\varphi)\partial_{z_1}\partial_{z_2}^2\varphi dz.
\end{aligned} \tag{4.9}$$

By assumptions (i), (ii), and (iv) and (4.8), we have

$$|\mathcal{K}| \lesssim \delta((\eta + 1) \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0 + e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^3(\Omega)}^2)$$

$$+ \delta \int_0^T e^{-2\eta z_0} (\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^2(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2) dz_0. \quad (4.10)$$

Hence we obtain

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \right| \\ & \lesssim \delta((\eta + 1) \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0 + e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^3(\Omega)}^2) \\ & \quad + \delta \int_0^T e^{-2\eta z_0} \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^2(\Omega)}^2 dz_0. \end{aligned} \quad (4.11)$$

Now we turn to estimate the last term in (4.5), which is the boundary term on $\{z_2 = 0\}$. With the aid of (4.4) and the Gauss theorem, one has

$$\begin{aligned} & -2 \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} r_{22} (\partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_2}^3 \varphi)|_{z_2=0} dz_0 dz_1 \\ & = -2 \int_0^T e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^2 \varphi \mathcal{L}(\partial_{z_2} \varphi) dz_0 dz_1 \\ & = 2 \int_{\Omega_T} \partial_{z_2} (e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^2 \varphi \mathcal{L}(\partial_{z_2} \varphi)) dz dz_0 \\ & = 2 \int_{\Omega_T} e^{-2\eta z_0} (\partial_{z_0} \partial_{z_2}^3 \varphi \mathcal{L}(\partial_{z_2} \varphi) + \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_2} \mathcal{L}(\partial_{z_2} \varphi)) dz dz_0 \\ & := R_1 + R_2. \end{aligned} \quad (4.12)$$

Via integration by parts with respect to z_0 , we have

$$\begin{aligned} |R_1| & = \left| 2 \int_{\Omega_T} \partial_{z_0} (e^{-2\eta z_0} \partial_{z_2}^3 \varphi \mathcal{L}(\partial_{z_2} \varphi)) + e^{-2\eta z_0} (2\eta \partial_{z_2}^3 \varphi \mathcal{L}(\partial_{z_2} \varphi) - \partial_{z_2}^3 \varphi \partial_{z_0} \mathcal{L}(\partial_{z_2} \varphi)) dz \right| \\ & \leq 2e^{-2\eta T} \int_{\Omega} (|\partial_{z_2}^3 \varphi \mathcal{L}(\partial_{z_2} \varphi)|)_{z_0=T} dz_1 dz_2 + 2 \int_{\Omega} (|\partial_{z_2}^3 \varphi \mathcal{L}(\partial_{z_2} \varphi)|)_{z_0=0} dz_1 dz_2 \\ & \quad + 4\eta \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_2}^3 \varphi \mathcal{L}(\partial_{z_2} \varphi)| dz + 2 \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_2}^3 \varphi \partial_{z_0} \mathcal{L}(\partial_{z_2} \varphi)| dz \\ & \lesssim e^{-2\eta T} \int_{\Omega} (q |\partial_{z_2}^3 \varphi|_{z_0=T}^2 + \frac{1}{q} |\mathcal{L}(\partial_{z_2} \varphi)|_{z_0=T}^2) dz_1 dz_2 + \|\varphi_0\|_{H^3(\Omega)}^2 + \|\varphi_1\|_{H^2(\Omega)}^2 \\ & \quad + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2 + 2\eta \int_{\Omega_T} e^{-2\eta z_0} (q |\partial_{z_2}^3 \varphi|^2 + \frac{1}{q} |\mathcal{L}(\partial_{z_2} \varphi)|^2) dz \\ & \quad + \int_{\Omega_T} e^{-2\eta z_0} (q\eta |\partial_{z_2}^3 \varphi|^2 + \frac{1}{q\eta} |\partial_{z_0} \mathcal{L}(\partial_{z_2} \varphi)|^2) dz \end{aligned} \quad (4.13)$$

By (3.23) and the fact that the coefficients of $\mathcal{L}$ are $W^{1,\infty}(\Omega_T)$ functions, we obtain

$$\eta \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(\partial_{z_2} \varphi)|^2 dz + e^{-2\eta T} \int_{\Omega} |\mathcal{L}(\partial_{z_2} \varphi)|_{z_0=T}^2 dz_1 dz_2$$

$$\begin{aligned}
&\leq \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_0} \mathcal{L}(\partial_{z_2} \varphi)|^2 dz + \|\mathcal{L}(\partial_{z_2} \varphi)|_{z_0=0}\|_{L^2(\Omega)}^2 \\
&\lesssim \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2 + |\mathcal{L}(\partial_{z_0 z_2} \varphi)|^2 \right) dz + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2. \quad (4.14)
\end{aligned}$$

Combining above two inequalities, we are led to

$$\begin{aligned}
|R_1| &\lesssim q(\eta \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_2}^3 \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |\partial_{z_2}^3 \varphi|_{z_0=T}|^2 dz_1 dz_2) \\
&\quad + \frac{1}{q} \left(\frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2 + |\mathcal{L}(\partial_{z_0 z_2} \varphi)|^2 \right) dz + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2 \right) \\
&\quad + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2. \quad (4.15)
\end{aligned}$$

Now we proceed to the estimate of R_2 . In fact, one has

$$\begin{aligned}
|R_2| &\leq 2 \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_2} \mathcal{L}(\partial_{z_2} \varphi)| dz \\
&\leq \int_{\Omega_T} e^{-2\eta z_0} (q\eta |\partial_{z_0} \partial_{z_2}^2 \varphi|^2 + \frac{1}{q\eta} |\partial_{z_2} \mathcal{L}(\partial_{z_2} \varphi)|^2) dz \\
&\leq \int_{\Omega_T} e^{-2\eta z_0} (q\eta |\partial_{z_0} \partial_{z_2}^2 \varphi|^2 + \frac{1}{q\eta} (|\mathcal{L}(\partial_{z_2}^2 \varphi)|^2 + \sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2)) dz. \quad (4.16)
\end{aligned}$$

Then inequality (4.16) together with (4.7), (4.11), (4.12), and (4.15) yields the following estimate

$$\begin{aligned}
&\eta \int_{\Omega_T} e^{-2\eta z_0} |D \partial_{z_2}^2 \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D \partial_{z_2}^2 \varphi|_{z_0=T}|^2 dz_1 dz_2 \\
&\lesssim \int_{\Omega_T} e^{-2\eta z_0} (q\eta + 1) |D \partial_{z_2}^2 \varphi|^2 + \frac{1}{q\eta} |\mathcal{L}(\partial_{z_2}^2 \varphi)|^2 dz \\
&\quad + \delta((\eta + 1) \int_{\Omega_T} e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0 + e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^3(\Omega)}^2) \\
&\quad + \delta \int_{\Omega_T} e^{-2\eta z_0} \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^2(\Omega)}^2 \\
&\quad + q(\eta \int_{\Omega_T} e^{-2\eta z_0} (|\partial_{z_2}^3 \varphi|^2 + |\partial_{z_0} \partial_{z_2}^2 \varphi|^2) dz + e^{-2\eta T} \int_{\Omega} |\partial_{z_2}^3 \varphi|_{z_0=T}|^2 dz_1 dz_2) \\
&\quad + \frac{1}{q} \left(\frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2 + |\mathcal{L}(\partial_{z_0 z_2} \varphi)|^2 + |\mathcal{L}(\partial_{z_2}^2 \varphi)|^2 \right) dz + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2 \right) \\
&\quad + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2. \quad (4.17)
\end{aligned}$$

Note that

$$\partial_{z_1}^2 \partial_{z_2} \varphi = \frac{1}{r_{11}} (\mathcal{L}(\partial_{z_2} \varphi) - \sum_{(i,j) \neq (1,1)} r_{ij} \partial_{ij} \partial_{z_2} \varphi), \quad (4.18)$$

$$\partial_{z_1}^3 \varphi = \frac{1}{r_{11}} (\mathcal{L}(\partial_{z_1} \varphi) - \sum_{(i,j) \neq (1,1)} r_{ij} \partial_{ij} \partial_{z_1} \varphi). \quad (4.19)$$

So if we select q and δ properly small and then η appropriately large, the third order derivatives on the right hand-side of the inequality above can be absorbed by the left hand-side terms. Therefore,

$$\begin{aligned} & \eta \int_{\Omega_T} e^{-2\eta z_0} (|\partial_{z_1}^2 \partial_{z_2} \varphi|^2 + |\partial_{z_1}^3 \varphi|^2) dz + e^{-2\eta T} \int_{\Omega} (|\partial_{z_1}^2 \partial_{z_2} \varphi|^2 + |\partial_{z_1}^3 \varphi|^2) dz_1 dz_2 \\ & \lesssim \eta \int_{\Omega_T} e^{-2\eta z_0} |D \partial_{z_2}^2 \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D \partial_{z_2}^2 \varphi|^2 dz_1 dz_2 \\ & \quad + \sum_{|\alpha| \leq 2} (\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \partial_{z_0} \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D^\alpha \partial_{z_0} \varphi|^2 dz_1 dz_2) \\ & \quad + \sum_{|\alpha| \leq 2} (\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D^\alpha \varphi|^2 dz_1 dz_2) \\ & \quad + \sum_{i=1}^2 (\eta \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(\partial_{z_i} \varphi)|^2 dz + e^{-2\eta T} \int_{\Omega} |\mathcal{L}(\partial_{z_i} \varphi)|^2 dz_1 dz_2). \end{aligned} \quad (4.20)$$

Then it follows from (3.23) that

$$\begin{aligned} & \sum_{i=1}^2 (\eta \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(\partial_{z_i} \varphi)|^2 dz + e^{-2\eta T} \int_{\Omega} |\mathcal{L}(\partial_{z_i} \varphi)|^2 dz_1 dz_2) \\ & \leq \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2 + \sum_{i=1}^2 |\mathcal{L}(\partial_{z_i z_0} \varphi)|^2 \right) dz + \sum_{i=1}^2 \|\partial_{z_i} f|_{z_0=0}\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.21)$$

Substituting (4.2), (4.17) and (4.21) into (4.20), one has

$$\begin{aligned} & \eta \int_{\Omega_T} e^{-2\eta z_0} (|\partial_{z_1}^2 \partial_{z_2} \varphi|^2 + |\partial_{z_1}^3 \varphi|^2) dz + e^{-2\eta T} \int_{\Omega} (|\partial_{z_1}^2 \partial_{z_2} \varphi|^2 + |\partial_{z_1}^3 \varphi|^2) dz_1 dz_2 \\ & \lesssim \int_{\Omega_T} e^{-2\eta z_0} (q\eta + 1) |D \partial_{z_2}^2 \varphi|^2 + \frac{1}{q\eta} |\mathcal{L}(\partial_{z_2}^2 \varphi)|^2 dz \\ & \quad + \delta ((\eta + 1) \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0 + e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^3(\Omega)}^2) \\ & \quad + \delta \int_0^T e^{-2\eta z_0} \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^2(\Omega)}^2 dz_0 \\ & \quad + q(\eta \int_{\Omega_T} e^{-2\eta z_0} (|\partial_{z_2}^3 \varphi|^2 + |\partial_{z_0} \partial_{z_2}^2 \varphi|^2) dz + e^{-2\eta T} \int_{\Omega} |\partial_{z_2}^3 \varphi|_{z_0=T}^2 dz_1 dz_2) \\ & \quad + \frac{1}{q} \left(\frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2 + |\mathcal{L}(\partial_{z_0 z_2} \varphi)|^2 + |\mathcal{L}(\partial_{z_2}^2 \varphi)|^2 \right) dz + \|\partial_{z_2} f|_{z_0=0}\|_{L^2(\Omega)}^2 \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\eta} \sum_{|\alpha| \leq 2} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\tilde{\Omega}_T)}^2 + \sum_{|\alpha| \leq 1} \|D^\alpha f|_{z_0=0}\|_{L^2(\tilde{\Omega})}^2 \\
& + \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} |D^\alpha \varphi|^2 \right) dz. \tag{4.22}
\end{aligned}$$

To this end, we are ready to conclude the third order estimate. Adding (3.27), (4.2), (4.17) and (4.22) together, letting q and δ be properly small and η be appropriately large, we deduce that

$$\begin{aligned}
& \sum_{|\alpha| \leq 3} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D^\alpha \varphi|_{z_0=T}^2 dz_1 dz_2 \right) \\
& \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 2} \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz + \sum_{|\alpha| \leq 1} \|D^\alpha f|_{z_0=0}\|_{L^2(\Omega)}^2. \tag{4.23}
\end{aligned}$$

□

4.2. Fourth order estimate of the solution. In this subsection, we establish the fourth order estimate of the solution obtained in lemma 3.1. For the fourth order estimate, since the normal derivative of the solution contained in the boundary term is one order higher than the one contained in the boundary term of third order estimate, the representations of the boundary terms as the commutators are insufficient to derive the desired estimate. Hence more careful analysis and computation is needed to establish the fourth order estimate. We summarize the fourth order estimate as the following lemma:

Lemma 4.2. *There exists $\delta_4 > 0$ such that if assumptions (i) – (iv) hold for $\delta \leq \delta_4$, then there exists a constant $\eta_4 > 1$ such that for any $T > 0$ and $\eta \geq \eta_4$, the $H^2(\Omega_T)$ solution of problem (3.1) satisfies*

$$\begin{aligned}
& \sum_{|\alpha|=4} \eta \|e^{-\eta z_0} D^\alpha \varphi\|_{L^2(\Omega_T)}^2 + e^{-2\eta T} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\
& \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 3} \|e^{-\eta z_0} \mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} f\|_{H^3(\Omega_T)}^2 + \|f|_{t=0}\|_{H^2(\Omega)}^2. \tag{4.24}
\end{aligned}$$

Proof. Since ∂_{z_0} is tangential to both boundaries, applying the third order estimation (4.23) to $\partial_{z_0} \varphi$, one has

$$\begin{aligned}
& \sum_{|\alpha| \leq 3} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \partial_{z_0} \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D^\alpha \partial_{z_0} \varphi|_{z_0=T}^2 dz_1 dz_2 \right) \\
& \lesssim \frac{1}{\eta} \sum_{|\alpha| \leq 2} \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \partial_{z_0} \varphi)|^2 dz + \sum_{|\alpha| \leq 2} \|D^\alpha f|_{z_0=0}\|_{L^2(\Omega)}^2. \tag{4.25}
\end{aligned}$$

Next, we will firstly derive the first order estimate of $\partial_{z_2}^3 \varphi$, *i.e.*, the estimate of $D\partial_{z_2}^3 \varphi$. Then by the equation, we are able to control the other fourth order derivatives. It is easy to verify that $\partial_{z_2}^3 \varphi$ satisfies

$$\begin{cases} \mathcal{L}(\partial_{z_2}^3 \varphi) = -[\partial_{z_2}^3, \mathcal{L}]\varphi + \partial_{z_2}^3 f, & \text{in } \Omega_T, \\ \mathcal{B}(\partial_{z_2}^3 \varphi) = -[\partial_{z_2}^3, \mathcal{B}]\varphi, & \text{on } \Gamma_1, \\ \partial_{z_2}^3 \varphi = 0, & \text{on } \Gamma_0, \\ \partial_{z_0}(\partial_{z_2}^3 \varphi) = 0, & \text{on } \Gamma_0. \end{cases} \quad (4.26)$$

Multiplying $2e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^3 \varphi$ on both sides of (4.26)₁, and integrating by parts over Ω_T , one has

$$\begin{aligned} & 2 \int_{\Omega_T} e^{-2\eta z_0} \mathcal{L}(\partial_{z_2}^3 \varphi) \partial_{z_0} \partial_{z_2}^3 \varphi dz \\ &= \left[\int_{\Omega} e^{-2\eta z_0} H_0 dz_1 dz_2 \right]_{z_0=0}^{z_0=T} + 2\eta \int_{\Omega_T} e^{-2\eta z_0} H_0 dz - \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \\ & \quad - 2 \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} (r_{22} \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_2}^4 \varphi)|_{z_2=0} dz_1 dz_0 + \int_{\Omega_T} e^{-2\eta z_0} P_3(D\varphi) dz \end{aligned} \quad (4.27)$$

where

$$H_0 = |\partial_{z_0} \partial_{z_2}^3 \varphi|^2 - r_{11} |\partial_{z_1} \partial_{z_2}^3 \varphi|^2 - r_{22} |\partial_{z_2}^4 \varphi|^2, \quad (4.28)$$

$$H_1 = 2(r_{11} \partial_{z_1} \partial_{z_2}^3 \varphi + r_{12} \partial_{z_2}^4 \varphi) \partial_{z_0} \partial_{z_2}^3 \varphi, \quad (4.29)$$

and $P_3(D\partial_{z_2}^3 \varphi)$ is a quadratic polynomial in $D\partial_{z_2}^3 \varphi$ with bounded coefficients. Assumptions (i) and (ii) imply that

$$H_0 \geq C|D\partial_{z_2}^3 \varphi|^2 \quad \text{and} \quad |P_3(D\varphi)| \leq C|D\partial_{z_2}^3 \varphi|^2. \quad (4.30)$$

Hence by the Cauchy inequality, one has

$$\begin{aligned} & \eta \int_{\Omega_T} e^{-2\eta z_0} |D\partial_{z_2}^3 \varphi|^2 dz + e^{-2\eta T} \int_{\Omega} |D\partial_{z_2}^3 \varphi|_{z_0=T}^2 dz_1 dz_2 \\ & \lesssim \int_{\Omega_T} e^{-2\eta z_0} ((q\eta + 1)|D\partial_{z_2}^3 \varphi|^2 + \frac{1}{q\eta} |\mathcal{L}(\partial_{z_2}^3 \varphi)|^2) dz + \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \\ & \quad + 2 \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} r_{22} \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_2}^4 \varphi|_{z_2=0} dz_1 dz_0. \end{aligned} \quad (4.31)$$

By assumption (iii), we obtain

$$\begin{aligned} H_1|_{z_1=0} &= 2r_{11}(\partial_{z_1} \partial_{z_2}^3 \varphi + \frac{r_{12}}{r_{11}} \partial_{z_2}^4 \varphi) \partial_{z_0} \partial_{z_2}^3 \varphi \\ &= 2\frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi \{b_1 \partial_{z_1} + b_2 \partial_{z_2}\} \partial_{z_2}^3 \varphi \end{aligned}$$

$$\begin{aligned}
&= 2 \frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi \mathcal{B} \partial_{z_2}^3 \varphi \\
&= -2 \frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi ([\partial_{z_2}^3, \mathcal{B}] \varphi) \\
&= -2 \frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi \sum_{k=0}^2 (\partial_{z_2}^{k+1} b_1 \partial_{z_1} \partial_{z_2}^{2-k} \varphi + \partial_{z_2}^{k+1} b_2 \partial_{z_2}^{3-k} \varphi), \tag{4.32}
\end{aligned}$$

where $[\partial_{z_2}^3, \mathcal{B}]$ is the commutator and we have used the boundary condition that $\mathcal{B}\varphi|_{z_1=0} = 0$ in the forth equality. For $k = 0, 1, 2$, by the Gauss theorem, we deduce that

$$\begin{aligned}
&\left| \int_0^T e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi (\partial_{z_2}^{k+1} b_1) \partial_{z_1} \partial_{z_2}^{2-k} \varphi|_{z_1=0} dz_2 dz_0 \right| \\
&= \left| \int_{\Omega_T} \partial_{z_1} (e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi (\partial_{z_2}^{k+1} b_1) \partial_{z_1} \partial_{z_2}^{2-k} \varphi) dz \right| \\
&\leq \delta \int_{\Omega_T} e^{-2\eta z_0} (\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2) dz_0 \\
&\quad + \left| \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2}^{k+1} b_1 \partial_{z_0 z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^{2-k} \varphi dz \right| \\
&= \delta \int_{\Omega_T} e^{-2\eta z_0} (\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2) dz_0 + \mathcal{A}. \tag{4.33}
\end{aligned}$$

By integrating by parts with respect to z_0 , one has for $k = 0, 1, 2$ that

$$\begin{aligned}
\mathcal{A} &\leq \left| \int_{\Omega_T} \partial_{z_0} (e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2}^{k+1} b_1 \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^{2-k} \varphi) dz \right| \\
&\quad + 2\eta \left| \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2}^{k+1} b_1 \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^{2-k} \varphi dz \right| \\
&\quad + \left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0} (\frac{r_{22}}{b_1} \partial_{z_2}^{k+1} b_1) \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^{2-k} \varphi dz \right| \\
&\quad + \left| \int_{\Omega_T} e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_2}^{k+1} b_1 \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_0 z_1} \partial_{z_2}^{2-k} \varphi dz \right| \\
&\lesssim \delta e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \delta(\eta + 1) \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 dz_0 \\
&\quad + \delta \int_0^T e^{-2\eta z_0} \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0, \tag{4.34}
\end{aligned}$$

where in the last inequality, we have used assumption (iv). By the same argument, we can also obtain

$$\begin{aligned}
& \left| \int_0^T e^{-2\eta z_0} \frac{r_{22}}{b_1} \partial_{z_0} \partial_{z_2}^3 \varphi (\partial_{z_2}^{k+1} b_2) \partial_{z_1} \partial_{z_2}^{3-k} \varphi|_{z_1=0} dz_2 dz_0 \right| \\
& \lesssim \delta e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \delta(\eta + 1) \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 dz_0 \\
& \quad + \delta \int_0^T e^{-2\eta z_0} \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0.
\end{aligned} \tag{4.35}$$

By estimates (4.32)-(4.35), we deduce

$$\begin{aligned}
& \left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} H_1|_{z_1=0} dz_2 dz_0 \right| \\
& \lesssim \delta \left(\eta \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 dz_0 + e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 \right) \\
& \quad + \delta \int_0^T e^{-2\eta z_0} (\|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2) dz_0.
\end{aligned} \tag{4.36}$$

We still need to control the boundary term on boundary $\{z_2 = 0\}$ in (4.31). By the equation and the properties of the coefficients, we know

$$r_{22} \partial_{z_2}^4 \varphi|_{z_2=0} = \{\mathcal{L} - r_{00} \partial_{z_0}^2 - 2r_{01} \partial_{z_0 z_1} - r_{11} \partial_{z_1}^2\} \partial_{z_2}^2 \varphi, \tag{4.37}$$

$$\partial_{z_i} \partial_{z_2}^3 \varphi|_{z_2=0} = \frac{1}{r_{22}} \mathcal{L}(\partial_{z_i z_2} \varphi) \quad (i = 0, 1). \tag{4.38}$$

$$\partial_{z_2}^3 \varphi|_{z_2=0} = \frac{1}{r_{22}} \mathcal{L}(\partial_{z_2} \varphi). \tag{4.39}$$

Therefore,

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} r_{22} \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_2}^4 \varphi|_{z_2=0} dz_1 dz_0 \\
& = \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^3 \varphi (\{\mathcal{L} - r_{00} \partial_{z_0}^2 - 2r_{01} \partial_{z_0 z_1} - r_{11} \partial_{z_1}^2 - \sum_{i=0}^2 r_i \partial_{z_i}\} \partial_{z_2}^2 \varphi) dz_1 dz_0 \\
& := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5.
\end{aligned} \tag{4.40}$$

For $\mathcal{I}_1$, by the Gauss theorem, one has

$$\begin{aligned}
|\mathcal{I}_1| & = \left| \int_{\Omega_T} \partial_{z_2} (e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^3 \varphi \mathcal{L} \partial_{z_2}^2 \varphi) dz \right| \\
& \leq \left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_2} (\mathcal{L} \partial_{z_2}^2 \varphi) dz \right| + \left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^4 \varphi \mathcal{L} (\partial_{z_2}^2 \varphi) dz \right|
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\partial_{z_2} \mathcal{L}(\partial_{z_2}^2 \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 + q\eta \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 \right) dz_0 \\
&\quad + \left| \int_{\Omega_T} \partial_{z_0} (e^{-2\eta z_0} \partial_{z_2}^4 \varphi \mathcal{L}(\partial_{z_2}^2 \varphi)) + e^{-2\eta z_0} (2\eta \partial_{z_2}^4 \varphi \mathcal{L}(\partial_{z_2}^2 \varphi) - \partial_{z_2}^4 \varphi \partial_{z_0} \mathcal{L}(\partial_{z_2}^2 \varphi)) dz \right| \\
&\lesssim \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\partial_{z_2} \mathcal{L}(\partial_{z_2}^2 \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 + q\eta \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 \right) dz_0 \\
&\quad + e^{-2\eta T} \left(q \|\partial_{z_2}^4 \varphi(T, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q} \|\mathcal{L}(\partial_{z_2}^2 \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
&\quad + \eta \int_0^T e^{-2\eta z_0} q \|\partial_{z_2}^4 \varphi(z_0)\|_{L^2(\Omega)}^2 + \frac{1}{q} \|\mathcal{L} \partial_{z_2}^2 \varphi(z_0)\|_{L^2(\Omega)}^2 dz_0 + \|D^2 f|_{t=0}\|_{L^2(\Omega)}^2 \\
&\quad + \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\partial_{z_0} \mathcal{L}(\partial_{z_2}^2 \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 + q\eta \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 \right) dz_0. \tag{4.41}
\end{aligned}$$

By the boundedness of the coefficients of $\mathcal{L}$, we know that

$$\begin{aligned}
&\int_0^T e^{-2\eta z_0} (\|\partial_{z_2} \mathcal{L}(\partial_{z_2}^2 \varphi)\|_{L^2(\Omega)}^2 + \|\partial_{z_0} \mathcal{L}(\partial_{z_2}^2 \varphi)\|_{L^2(\Omega)}^2) dz_0 \\
&\lesssim \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0. \tag{4.42}
\end{aligned}$$

With the help of (3.23), one gets

$$\begin{aligned}
&\eta \int_0^T e^{-2\eta z_0} \|\mathcal{L} \partial_{z_2}^2 \varphi\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|\mathcal{L}(\partial_{z_2}^2 \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \\
&\leq \frac{1}{\eta} \int_0^T e^{-2\eta z_0} \|\partial_{z_0} \mathcal{L}(\partial_{z_2}^2 \varphi)\|_{L^2(\Omega)}^2 dz_0 + \|\mathcal{L}(\partial_{z_2}^2 \varphi)|_{z_0=0}\|_{L^2(\Omega)}^2 \\
&\lesssim \frac{1}{\eta} \int_0^T e^{-2\eta z_0} (\|\mathcal{L}(\partial_{z_0} \partial_{z_2}^2 \varphi)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 4} \|D^\alpha \varphi\|_{L^2(\Omega)}^2) dz_0 \\
&\quad + \|\partial_{z_2}^2 f|_{z_0=0}\|_{L^2(\Omega)}^2. \tag{4.43}
\end{aligned}$$

Combining (4.41), (4.42) and (4.43), we deduce that

$$\begin{aligned}
|\mathcal{I}_1| &\lesssim q \left(\eta \int_{\Omega_T} e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
&\quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \\
&\quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0
\end{aligned}$$

$$+ \frac{1}{q} \|\partial_{z_2}^2 f|_{z_0=0}\|_{L^2(\Omega)}^2. \quad (4.44)$$

Since both $\mathcal{I}_2$ and $\mathcal{I}_3$ contain the time derivative ∂_{z_0} , by the argument similar to the one used for $\mathcal{I}_1$, we obtain

$$\begin{aligned} & |\mathcal{I}_2| + |\mathcal{I}_3| \\ & \lesssim q \left(\eta \int_{\Omega_T} e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot, \cdot)\|_{L^2(\Omega)}^2 \right) \\ & + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0)\|_{L^2(\Omega)}^2 \right) dz_0 \\ & + \frac{1}{q\eta} \int_{\Omega_T} e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 4} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 \right) dz_0 \\ & + \frac{1}{q} \|\partial_{z_2}^2 f|_{z_0=0}\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.45)$$

To estimate $\mathcal{I}_4$, we can not directly use Gauss theorem and then integration by part with respect to z_0 , since $\partial_{z_1}^2 \partial_{z_2}^2 \varphi$ does not contain the time derivative ∂_{z_0} . Instead, we integrate with respect to z_1 , since ∂_{z_1} is also tangential to Γ_2 . Actually, one has

$$\begin{aligned} |\mathcal{I}_4| & \leq \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} \partial_{z_1} \left(e^{-2\eta z_0} \frac{r_{11}}{r_{22}} \mathcal{L}(\partial_{z_0 z_2} \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz_1 dz_0 \right|}_{\mathcal{I}_{4,1}} \\ & + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \frac{r_{11}}{r_{22}} \partial_{z_1} (\mathcal{L} \partial_{z_0 z_2} \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{I}_{4,2}} \\ & + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \partial_{z_1} \left(\frac{r_{11}}{r_{22}} \right) \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{I}_{4,3}}. \end{aligned} \quad (4.46)$$

By Cauchy inequality and the trace theorem, it is easy to see that

$$\begin{aligned} \mathcal{I}_{4,1} & = \left| \int_0^T e^{-2\eta z_0} \left(\frac{r_{11}}{r_{22}} \right) \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi(z_0, 0, 0) dz_0 \right| \\ & = \left| \int_0^T e^{-2\eta z_0} \left(\frac{r_{11}}{b_1 r_{22}^2} \right) (\partial_{z_0} \partial_{z_2}^3 \varphi) \mathcal{B}(\partial_{z_2}^2 \varphi)(z_0, 0, 0) dz_0 \right|. \end{aligned} \quad (4.47)$$

where in the second equality, we have used $b_2(0, 0) = 0$. It is clear that

$$\mathcal{B}(\partial_{z_2}^2 \varphi) = 2(\partial_{z_2} b_1 \partial_{z_1 z_2} \varphi + \partial_{z_2} b_2 \partial_{z_2 z_2} \varphi) + \partial_{z_2 z_2} b_1 \partial_{z_1} \varphi + \partial_{z_2 z_2} b_2 \partial_{z_2} \varphi. \quad (4.48)$$

From the boundary conditions

$$\mathcal{B}\varphi|_{z_1=0} = 0, \quad \partial_{z_2} \varphi|_{z_2=0} = 0, \quad \text{and} \quad b_2(0, 0) = 0,$$

one can see that

$$\partial_{z_1 z_2} \varphi(z_0, 0, 0) = 0 \quad \text{and} \quad \partial_{z_1} \varphi(z_0, 0, 0) = 0. \quad (4.49)$$

Combining (4.48) and (4.49), one has

$$\mathcal{B}(\partial_{z_2}^2 \varphi)(z_0, 0, 0) = 2\partial_{z_2} b_2(0, 0)\partial_{z_2 z_2} \varphi(z_0, 0, 0).$$

By assumption (iii), we conclude that $\mathcal{B}(\partial_{z_2}^2 \varphi)(z_0, 0, 0) = 0$. Hence we have $\mathcal{I}_{4,1} = 0$.

Noticing that $\partial_{z_2} \varphi$ and its tangential derivatives vanish on $\{z_2 = 0\}$, we have

$$\begin{aligned} \mathcal{L}(\partial_{z_0 z_2} \varphi)|_{z_2=0} &= \left\{ -[\partial_{z_0}, \mathcal{L}]\partial_{z_2} \varphi + \partial_{z_0}(\mathcal{L}(\partial_{z_2} \varphi)) \right\} \Big|_{z_2=0} \\ &= -((2(\partial_{z_0} r_{02})\partial_{z_0} \partial_{z_2} + 2(\partial_{z_0} r_{12})\partial_{z_1 z_2} + (\partial_{z_0} r_{22})\partial_{z_2}^2))\partial_{z_2} \varphi|_{z_2=0} \\ &\quad - \partial_{z_0} r_2 \partial_{z_2 z_2} \varphi|_{z_2=0} + \partial_{z_0}(\mathcal{L}(\partial_{z_2} \varphi))|_{z_2=0}. \end{aligned} \quad (4.50)$$

So by integrating by part with respect to z_0 and recalling (4.38) and (4.39), we have

$$\begin{aligned} \mathcal{I}_{4,2} &\leq \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_{02})\partial_{z_0} \partial_{z_2}^2 \varphi + (\partial_{z_0} r_{02})\partial_{z_0 z_1} \partial_{z_2}^2 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_1} \\ &\quad + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_{12})\partial_{z_1} \partial_{z_2}^2 \varphi + (\partial_{z_0} r_{12})\partial_{z_1}^2 \partial_{z_2}^2 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_2} \\ &\quad + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_{22})\partial_{z_2}^3 \varphi + (\partial_{z_0} r_{22})\partial_{z_1} \partial_{z_2}^3 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_3} \\ &\quad + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_2)\partial_{z_2}^2 \varphi + (\partial_{z_0} r_2)\partial_{z_1} \partial_{z_2}^2 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_4} \\ &\quad + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} \mathcal{L}(\partial_{z_2} \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_5}. \end{aligned} \quad (4.51)$$

We estimate $\mathcal{J}_i$ term by term. By the Gauss theorem, it is clear that

$$\begin{aligned}
\mathcal{J}_1 &= \left| \int_{\Omega_T} \partial_{z_2} \left(2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_{02}) \partial_{z_0} \partial_{z_2}^2 \varphi + (\partial_{z_0} r_{02}) \partial_{z_0 z_1} \partial_{z_2}^2 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right| \\
&\leq \underbrace{\left| \int_{\Omega_T} \partial_{z_2} \left(2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{02}) \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right|}_{\mathcal{J}_{1,1}} \\
&\quad + \underbrace{\left| \int_{\Omega_T} \partial_{z_2} \left(2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0} r_{02}) \partial_{z_0 z_1} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right|}_{\mathcal{J}_{1,2}}. \tag{4.52}
\end{aligned}$$

By simple calculation we have

$$\begin{aligned}
\mathcal{J}_{1,1} &\leq \left| \int_{\Omega_T} 2e^{-2\eta z_0} \partial_{z_2} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_0 z_1} r_{02}) \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right| \\
&\quad + \left| \int_{\Omega_T} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1 z_2} r_{02}) \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right| \\
&\quad + \left| \int_{\Omega_T} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{02}) \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right| \\
&\quad + \left| \int_{\Omega_T} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{02}) \partial_{z_0} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^3 \varphi dz dz_0 \right|.
\end{aligned}$$

By the Hölder's inequality, one has

$$\begin{aligned}
\mathcal{J}_{1,1} &\lesssim \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{02}(z_0, \cdot)\|_{L^2} \|\partial_{z_0} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4} dz_0 \\
&\quad + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1 z_2} r_{02}(z_0, \cdot)\|_{L^2} \|\partial_{z_0} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4} dz_0 \\
&\quad + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{02}(z_0, \cdot)\|_{L^4} \|\partial_{z_0} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4} dz_0 \\
&\quad + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{02}(z_0, \cdot)\|_{L^2} \|\partial_{z_0} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4} \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2} dz_0 \\
&\lesssim \delta \int_0^T e^{-2\eta z_0} \left(\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 \right) dz_0, \tag{4.53}
\end{aligned}$$

where in the last inequality, we have used the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and assumption (ii). Similarly, one deduces that

$$\mathcal{J}_{1,2} \lesssim \delta \int_0^T e^{-2\eta z_0} \left(\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 \right) dz_0$$

$$+ \underbrace{\left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0} r_{02} \partial_{z_0 z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|}_{\mathcal{J}_{1,2}^0}. \quad (4.54)$$

Integrating by part with respect to z_0 , one has

$$\begin{aligned} \mathcal{J}_{1,2}^0 &\lesssim \left| \int_{\Omega_T} \partial_{z_0} \left(e^{-2\eta z_0} \partial_{z_0} r_{02} \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right| \\ &\quad + \eta \left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0} r_{02} \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right| \\ &\quad + \left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0} r_{02} \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_0 z_1} \partial_{z_2}^2 \varphi dz dz_0 \right| \\ &\quad + \left| \int_{\Omega_T} e^{-2\eta z_0} \partial_{z_0}^2 r_{02} \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|. \end{aligned}$$

By the Cauchy inequality, one deduces

$$\begin{aligned} \mathcal{J}_{1,2}^0 &\lesssim e^{-2\eta T} \left(q \|\partial_{z_1} \partial_{z_2}^3 \varphi(T, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q} \|\partial_{z_1} \partial_{z_2}^2 \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right) + \|\varphi_0\|_{H^4(\Omega)}^2 \\ &\quad + \eta \left(\int_0^T e^{-2\eta z_0} \left(q \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \right) \\ &\quad + \int_0^T e^{-2\eta z_0} \left(\|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + \|\partial_{z_0 z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \\ &\quad + \delta \int_0^T \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0. \end{aligned} \quad (4.55)$$

Combining (4.52)-(4.55), we obtain

$$\begin{aligned} \mathcal{J}_1 &\lesssim (\delta + 1) \int_0^T e^{-2\eta z_0} \left(\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 \right) dz_0 \\ &\quad + (q\eta + 1) \int_0^T \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2}^2 dz_0 + qe^{-2\eta T} \|\partial_{z_1} \partial_{z_2}^3 \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{q} \left(\eta \int_0^T \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2}^2 dz_0 + e^{-2\eta T} \|\partial_{z_1} \partial_{z_2}^2 \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (4.56)$$

For $\mathcal{J}_2$, we have

$$\mathcal{J}_2 = \left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} \left((\partial_{z_0 z_1} r_{12}) \partial_{z_1} \partial_{z_2}^2 \varphi + (\partial_{z_0} r_{12}) \partial_{z_1}^2 \partial_{z_2}^2 \varphi \right) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|$$

$$\begin{aligned}
& \lesssim \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{12}) \partial_{z_1} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_{2,1}} \\
& + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0} r_{12}) \partial_{z_1}^2 \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_{2,2}}. \tag{4.57}
\end{aligned}$$

By the Gauss theorem and assumption (ii), it is not difficult to deduce that

$$\begin{aligned}
\mathcal{J}_{2,1} &= \left| \int_{\Omega_T} \partial_{z_2} \left(2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{12}) \partial_{z_1} \partial_{z_2}^2 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right| \\
&\lesssim \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{12}(z_0, \cdot)\|_{L^4(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4(\Omega)}^2 dz_0 \\
&+ \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1 z_2} r_{12}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4(\Omega)}^2 dz_0 \\
&+ \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{12}(z_0, \cdot)\|_{L^4(\Omega)} \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4(\Omega)} dz_0 \\
&\lesssim \delta \int_0^T e^{-2\eta z_0} \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{H^1(\Omega)}^2 dz_0, \tag{4.58}
\end{aligned}$$

where the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is used in the last inequality. For $\mathcal{J}_{2,2}$, one has

$$\begin{aligned}
\mathcal{J}_{2,2} &= \left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0} r_{12}) \partial_{z_1} \left(|\partial_{z_1} \partial_{z_2}^2 \varphi|^2 \right) dz_1 dz_0 \right| \\
&\leq \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} \partial_{z_1} \left(e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0} r_{12}) |\partial_{z_1} \partial_{z_2}^2 \varphi|^2 \right) dz_1 dz_0 \right|}_{\mathcal{J}_{2,2}^1} \\
&+ \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \partial_{z_1} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_0} r_{12}) |\partial_{z_1} \partial_{z_2}^2 \varphi|^2 dz_1 dz_0 \right|}_{\mathcal{J}_{2,2}^2} \\
&+ \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{12}) |\partial_{z_1} \partial_{z_2}^2 \varphi|^2 dz_1 dz_0 \right|}_{\mathcal{J}_{2,2}^3}. \tag{4.59}
\end{aligned}$$

We claim $\mathcal{J}_{2,2}^1 = 0$. Indeed, applying $\partial_{z_2}^2$ to the boundary condition $\mathcal{B}\varphi(z_0, 0, z_2) = 0$ and letting $z_2 = 0$, one has

$$b_1 \partial_{z_1} \partial_{z_2}^2 \varphi(z_0, 0, 0) = (\mathcal{B} \partial_{z_2}^2 \varphi)(z_0, 0, 0) - b_2 \partial_{z_2}^3 \varphi(z_0, 0, 0). \tag{4.60}$$

By (4.48), (4.49), and assumption (iii), it is clear that $(\mathcal{B}\partial_{z_2}^2\varphi)(z_0, 0, 0) = 0$, $b_2(0, 0) = 0$, and $b_1(0, 0) = -1 \neq 0$. This together with (4.60) imply that $\partial_{z_1}\partial_{z_2}^2\varphi(z_0, 0, 0) = 0$, hence our claim holds. For $\mathcal{J}_{2,2}^2$, by the trace theorem and assumptions (i) and (ii), one obtains

$$\mathcal{J}_{2,2}^2 \lesssim \int_0^T e^{-2\eta z_0} \|\partial_{z_1}\partial_{z_2}^2\varphi(z_0, \cdot)\|_{H^1(\Omega)}^2 dz_0. \quad (4.61)$$

By the Gauss theorem, the Sobolev embedding theorem, and assumption (ii), we deduce that

$$\begin{aligned} \mathcal{J}_{2,2}^3 &= \int_{\Omega_T} \partial_{z_2} \left(e^{-2\eta z_0} \frac{r_{11}}{r_{22}} \partial_{z_0 z_1} r_{12} |\partial_{z_1}\partial_{z_2}^2\varphi|^2 \right) dz dz_0 \\ &\lesssim \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{12}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1}\partial_{z_2}^2\varphi(z_0, \cdot)\|_{L^4(\Omega)}^2 dz_0 \\ &\quad + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1 z_2} r_{12}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1}\partial_{z_2}^2\varphi(z_0, \cdot)\|_{L^4(\Omega)}^2 dz_0 \\ &\quad + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{12}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1}\partial_{z_2}^2\varphi(z_0, \cdot)\|_{L^4(\Omega)} \|\partial_{z_1}\partial_{z_2}^3\varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0 \\ &\lesssim \delta \int_0^T e^{-2\eta z_0} \|\partial_{z_1}\partial_{z_2}^2(z_0, \cdot)\|_{H^1(\Omega)}^2 dz_0. \end{aligned} \quad (4.62)$$

Hence we conclude that

$$\mathcal{J}_2 \lesssim (\delta + 1) \int_0^T e^{-2\eta z_0} \|\partial_{z_1}\partial_{z_2}^2\varphi(z_0, \cdot)\|_{H^1(\Omega)}^2 dz_0. \quad (4.63)$$

For $\mathcal{J}_3$, one has

$$\begin{aligned} \mathcal{J}_3 &= \left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_{22}) \partial_{z_2}^3 \varphi + (\partial_{z_0} r_{22}) \partial_{z_1} \partial_{z_2}^3 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right| \\ &\leq \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} r_{22}) \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_{3,1}} \\ &\quad + \underbrace{\left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} (\partial_{z_0} r_{22}) \partial_{z_1} \partial_{z_2}^3 \varphi \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right|}_{\mathcal{J}_{3,2}}. \end{aligned} \quad (4.64)$$

By the Gauss theorem and the Hölder inequality, it is not difficult to see that

$$\begin{aligned} \mathcal{J}_{3,1} &\lesssim \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{22}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^4(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4(\Omega)} dz_0 \\ &\quad + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1 z_2} r_{22}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^4(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4(\Omega)} dz_0 \end{aligned}$$

$$\begin{aligned}
& + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{22}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_2}^4 \varphi(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^4(\Omega)} dz_0 \\
& + \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} r_{22}(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^4(\Omega)} \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0 \\
& \lesssim \delta \int_0^T e^{-2\eta z_0} \left(\|\partial_{z_2}^3 \varphi(z_0, \cdot)\|_{H^1(\Omega)}^2 + \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{H^1(\Omega)}^2 \right) dz_0. \tag{4.65}
\end{aligned}$$

For $\mathcal{J}_{3,2}$, the Gauss theorem cannot be used directly, since it contains fourth order derivative $\partial_{z_1} \partial_{z_2}^3 \varphi$ on boundary $\{z_2 = 0\}$. Therefore, we use (4.38) to replace $\partial_{z_1} \partial_{z_2}^3 \varphi$ in $\mathcal{J}_{3,2}$ and then apply the trace theorem and the Cauchy inequality to derive the estimate. In fact, one has

$$\begin{aligned}
\mathcal{J}_{3,2} & = \left| \int_0^T \int_{\mathbb{R}_+} 2e^{-2\eta z_0} \frac{\partial_{z_0} r_{22}}{r_{22}} \mathcal{L}(\partial_{z_1 z_2} \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right| \\
& \lesssim \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_2} r_{22}(z_0, \cdot)\|_{L^4(\Omega)} \|\mathcal{L}(\partial_{z_1 z_2} \varphi)(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0 \\
& + \int_0^T e^{-2\eta z_0} \|\mathcal{L}(\partial_{z_1 z_2} \varphi)(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0 \\
& + \int_0^T e^{-2\eta z_0} \|\partial_{z_2} \mathcal{L}(\partial_{z_1 z_2} \varphi)(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0 \\
& + \int_0^T e^{-2\eta z_0} \|\mathcal{L}(\partial_{z_1 z_2} \varphi)(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^3 \varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0.
\end{aligned}$$

By the Cauchy inequality and assumption (ii), we have

$$\begin{aligned}
\mathcal{J}_{3,2} & \lesssim (\delta + 1) \int_0^T e^{-2\eta z_0} \left(\|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)} + \|\partial_{z_0}^2 \varphi(z_0, \cdot)\|_{H^2(\Omega)} \right) \|\varphi(z_0, \cdot)\|_{H^4(\Omega)} dz_0 \\
& + \int_0^T e^{-2\eta z_0} \left(q\eta \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q\eta} \|\partial_{z_2} \mathcal{L}(\partial_{z_1 z_2} \varphi)(z_0, \cdot)\|^2 \right) dz_0 \\
& \lesssim \left(\delta + 1 + q\eta + \frac{1}{q\eta} \right) \sum_{|\alpha| \leq 4} \int_0^T e^{-2\eta z_0} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 \\
& + \frac{1}{q\eta} \sum_{|\alpha| \leq 3} \int_0^T e^{-2\eta z_0} \|\mathcal{L}(D^\alpha \varphi)(z_0, \cdot)\|^2 dz_0. \tag{4.66}
\end{aligned}$$

Combining (4.64)-(4.66), one deduces that

$$\begin{aligned}
\mathcal{J}_3 & \lesssim \left(\delta + 1 + q\eta + \frac{1}{q\eta} \right) \sum_{|\alpha| \leq 4} \int_0^T e^{-2\eta z_0} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 \\
& + \frac{1}{q\eta} \sum_{|\alpha| \leq 3} \int_0^T e^{-2\eta z_0} \|\mathcal{L}(D^\alpha \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0. \tag{4.67}
\end{aligned}$$

Noticing that $\mathcal{J}_4$ contains no derivatives higher than third order, it can be estimated easily by the Gauss theorem, the trace theorem, and assumption (ii). In fact, we have

$$\begin{aligned} \mathcal{J}_4 &= \left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} ((\partial_{z_0 z_1} r_2) \partial_{z_2}^2 \varphi + (\partial_{z_0} r_2) \partial_{z_1} \partial_{z_2}^2 \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right| \\ &\lesssim (\delta + 1) \sum_{|\alpha| \leq 4} \int_0^T e^{-2\eta z_0} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0. \end{aligned} \quad (4.68)$$

For $\mathcal{J}_5$, by the Gauss theorem, one has

$$\begin{aligned} \mathcal{J}_5 &= \left| \int_0^T \int_{\mathbb{R}^+} 2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi dz_1 dz_0 \right| \\ &= \left| \int_{\Omega_T} \partial_{z_2} \left(2e^{-2\eta z_0} \frac{r_{11}}{r_{22}} (\partial_{z_0 z_1} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right| \\ &= \underbrace{\left| \int_{\Omega_T} 2e^{-2\eta z_0} \partial_{z_2} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_0 z_1} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|}_{\mathcal{J}_{5,1}} \\ &\quad + \underbrace{\left| \int_{\Omega_T} 2e^{-2\eta z_0} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_0 z_1 z_2} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|}_{\mathcal{J}_{5,2}} \\ &\quad + \underbrace{\left| \int_{\Omega_T} 2e^{-2\eta z_0} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_0 z_1} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^3 \varphi dz dz_0 \right|}_{\mathcal{J}_{5,3}}. \end{aligned} \quad (4.69)$$

By the Hölder inequality, one has

$$\begin{aligned} \mathcal{J}_{5,1} &\lesssim \int_0^T e^{-2\eta z_0} \|(\partial_{z_0 z_1} \mathcal{L}(\partial_{z_2} \varphi))(z_0, \cdot)\|_{L^2(\Omega)} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)} dz_0 \\ &\lesssim (\delta + 1 + q\eta) \int_0^T e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 \\ &\quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \|\mathcal{L}(\partial_{z_0 z_1 z_2} \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0. \end{aligned} \quad (4.70)$$

where we have used assumption (ii) and the following fact:

$$\partial_{z_k z_\ell} (\mathcal{L} \partial_{z_2} \varphi) = (\partial_{z_k} \mathcal{L}) \partial_{z_\ell z_2} \varphi + (\partial_{z_\ell} \mathcal{L}) \partial_{z_k z_2} \varphi + (\partial_{z_k z_\ell} \mathcal{L}) \partial_{z_2} \varphi + \mathcal{L}(\partial_{z_k z_\ell z_2} \varphi), \quad (4.71)$$

where

$$\partial_{z_k} \mathcal{L} := \sum_{i,j=0}^2 \partial_{z_k} r_{ij} \partial_{z_i z_j}, \quad (4.72)$$

$$\partial_{z_k z_\ell} \mathcal{L} := \sum_{i,j=0}^2 \partial_{z_k z_\ell} r_{ij} \partial_{z_i z_j}. \quad (4.73)$$

Integrating by parts with respect to z_0 , we have

$$\begin{aligned} \mathcal{J}_{5,2} \leq & \underbrace{\left| \int_{\Omega_T} \partial_{z_0} \left(2e^{-2\eta z_0} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_1 z_2} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi \right) dz dz_0 \right|}_{\mathcal{J}_{5,2}^1} \\ & + 2\eta \underbrace{\left| \int_{\Omega_T} 2e^{-2\eta z_0} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_1 z_2} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|}_{\mathcal{J}_{5,2}^2} \\ & + \underbrace{\left| \int_{\Omega_T} 2e^{-2\eta z_0} \partial_{z_0} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_1 z_2} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|}_{\mathcal{J}_{5,2}^3} \\ & + \underbrace{\left| \int_{\Omega_T} 2e^{-2\eta z_0} \left(\frac{r_{11}}{r_{22}} \right) (\partial_{z_1 z_2} \mathcal{L}(\partial_{z_2} \varphi)) \partial_{z_0 z_1} \partial_{z_2}^2 \varphi dz dz_0 \right|}_{\mathcal{J}_{5,2}^4}. \end{aligned} \quad (4.74)$$

It is clear that

$$\begin{aligned} \mathcal{J}_{5,2}^1 & \lesssim e^{-2\eta T} \int_{\Omega} \left| \partial_{z_1 z_2} \mathcal{L}(\partial_{z_2} \varphi) \partial_{z_1} \partial_{z_2}^2 \varphi \right|_{z_0=T} dz + \|f|_{z_0=0}\|_{H^3(\Omega)}^2 \\ & \lesssim e^{-2\eta T} \left((q + \delta) \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q} \|\partial_{z_1} \partial_{z_2}^2(T, \cdot)\|^2 \right) \\ & \quad + e^{-2\eta T} \left(q\eta^2 \|\partial_{z_1} \partial_{z_2}^2 \varphi(T, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q\eta^2} \|\mathcal{L}(\partial_{z_1} \partial_{z_2}^2 \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\ & \quad + \|f|_{z_0=0}\|_{H^3(\Omega)}^2 \\ & \lesssim e^{-2\eta T} \left((q + \delta) \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q\eta^2} \|\mathcal{L}(\partial_{z_1} \partial_{z_2}^2 \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\ & \quad + \left(q\eta + \frac{1}{q\eta} \right) \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 \\ & \quad + \|f|_{z_0=0}\|_{H^3(\Omega)}^2. \end{aligned} \quad (4.75)$$

In the above estimates, the Cauchy inequality and inequality (3.23) are employed, and δ comes from the L^4 -norm (which can be bounded by the H^1 -norm) of $D^2 r_{ij}$. By a similar argument, we can also deduce that

$$\begin{aligned}
\mathcal{J}_{5,2}^2 &\lesssim \eta \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta^2} \|\mathcal{L}(\partial_{z_1} \partial_{z_2}^2 \varphi)(z_0, \cdot)\|^2 + q\eta^2 \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \\
&\quad + \eta \int_0^T \left(\frac{1}{q} \|\partial_{z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 + (q + \delta) \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \\
&\lesssim \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\mathcal{L}(\partial_{z_1} \partial_{z_2}^2 \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 + \eta(q + \delta) \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \\
&\quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0. \tag{4.76}
\end{aligned}$$

By the Cauchy inequality, it is not difficult to see that

$$\begin{aligned}
&\mathcal{J}_{5,2}^3 + \mathcal{J}_{5,2}^4 \\
&\lesssim \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\mathcal{L}(\partial_{z_1} \partial_{z_2}^2 \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 + q\eta \|\partial_{z_0 z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 \right) dz_0 \\
&\quad + \delta \int_0^T e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0. \tag{4.77}
\end{aligned}$$

Combining (4.74)-(4.77), we are able to conclude that

$$\begin{aligned}
\mathcal{J}_5 &\lesssim \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 \\
&\quad + \left(\delta + (q + \delta)\eta + \frac{1}{q\eta} + 1 \right) \int_0^T e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 \\
&\quad + e^{-2\eta T} \left((q + \delta) \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{q\eta^2} \|\mathcal{L}(\partial_{z_1} \partial_{z_2}^2 \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
&\quad + \left(q\eta + \frac{1}{q\eta} \right) \int_0^T e^{-2\eta z_0} \|\partial_{z_0 z_1} \partial_{z_2}^2 \varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + \|f|_{z_0=0}\|_{H^3(\Omega)}^2. \tag{4.78}
\end{aligned}$$

Collecting the estimates of $\mathcal{J}_1, \dots, \mathcal{J}_5$, we obtain the estimate of $\mathcal{I}_{4,2}$:

$$\begin{aligned}
\mathcal{I}_{4,2} &\lesssim \frac{1}{q\eta} \sum_{|\alpha| \leq 3} \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + \left(\frac{1}{q\eta} + q\eta \right) \sum_{|\alpha| \leq 4} \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz dz_0 \\
&\quad + \|f|_{z_0=0}\|_{H^3(\Omega)}^2 + (1 + q\eta^2) \|\varphi_0\|_{H^4(\Omega)}^2 + \|\varphi_1\|_{H^3(\Omega)}^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q\eta^2} e^{-2\eta T} \left(\|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha)\varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
& + qe^{-2\eta T} \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2.
\end{aligned} \tag{4.79}$$

For $\mathcal{I}_{4,3}$, via (4.38), the trace theorem, and the Cauchy inequality, one has

$$\begin{aligned}
\mathcal{I}_{4,3} & \lesssim \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\mathcal{L}(\partial_{z_0 z_2} \varphi)\|_{H^1(\Omega)}^2 + q\eta \|\partial_{z_1} \partial_{z_2}^2 \varphi\|_{H^1(\Omega)}^2 \right) dz_0 \\
& \lesssim \frac{1}{q\eta} \sum_{|\alpha| \leq 3} \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + \left(\frac{1}{q\eta} + q\eta \right) \sum_{|\alpha| \leq 4} \|e^{-2\eta z_0} D^\alpha \varphi\|_{L^2(\Omega_T)}^2.
\end{aligned} \tag{4.80}$$

Now the sum of the estimate of $\mathcal{I}_{4,1}$, $\mathcal{I}_{4,2}$, and $\mathcal{I}_{4,3}$ yields

$$\begin{aligned}
|\mathcal{I}_4| & \lesssim \frac{1}{q\eta} \sum_{|\alpha| \leq 3} \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 \\
& + \left(\frac{1}{q\eta} + (q + \delta)\eta + \delta + 1 \right) \sum_{|\alpha| \leq 4} \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz dz_0 \\
& + e^{-2\eta T} \left((q + \delta) \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \frac{1}{q\eta^2} \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
& + \|f|_{z_0=0}\|_{H^3(\Omega)}^2.
\end{aligned} \tag{4.81}$$

For $\mathcal{I}_5$, by (4.38) and the trace theorem

$$\begin{aligned}
|\mathcal{I}_5| & = \left| - \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \partial_{z_0} \partial_{z_2}^3 \varphi \sum_{i=0}^2 r_i \partial_{z_i} \partial_{z_2}^2 \varphi dz_1 dz_0 \right| \\
& = \left| \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} \frac{1}{r_{22}} \mathcal{L}(\partial_{z_0 z_2} \varphi) \sum_{i=0}^2 r_i \partial_{z_i} \partial_{z_2}^2 \varphi dz_1 dz_0 \right| \\
& \lesssim \int_0^T e^{-2\eta z_0} \left(\frac{1}{q\eta} \|\mathcal{L} \partial_{z_0 z_2} \varphi(z_0, \cdot)\|_{H^1(\Omega)}^2 + q\eta \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 \right) dz_0 \\
& + \int_0^T e^{-2\eta z_0} \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 dz_0 \\
& \lesssim \int_0^T e^{-2\eta z_0} \frac{1}{q\eta} \left(\sum_{|\alpha| \leq 4} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega)}^2 \right) dz_0 \\
& + (q\eta + 1) \sum_{|\alpha| \leq 4} \int_0^T e^{-2\eta z_0} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 dz_0,
\end{aligned} \tag{4.82}$$

where in the first inequality, we have used the Cauchy inequality. Gathering the estimates of $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_5$, we finally have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^+} e^{-2\eta z_0} r_{22} \partial_{z_0} \partial_{z_2}^3 \varphi \partial_{z_2}^4 \varphi|_{z_2=0} dz_1 dz_0 \\
& \lesssim q \left(\eta \int_0^T e^{-2\eta z_0} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
& \quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0)\|_{L^2(\Omega)}^2 \right) dz_0 \\
& \quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 4} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 \right) dz_0 \\
& \quad + \frac{1}{q\eta^2} e^{-2\eta T} \left(\|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
& \quad + qe^{-2\eta T} \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \frac{1}{q} \|\partial_{z_2}^2 f|_{z_0=0}\|_{L^2(\Omega)}^2. \tag{4.83}
\end{aligned}$$

By (4.31), (4.36), and (4.83), we obtain the estimate of $D\partial_{z_2}^3 \varphi$, i.e.,

$$\begin{aligned}
& \eta \int_{\Omega_T} e^{-2\eta z_0} |D\partial_{z_2}^3 \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |D\partial_{z_2}^3 \varphi|_{z_0=T}|^2 dz_1 dz_2 \\
& \lesssim \delta \left(\eta \int_0^T e^{-2\eta z_0} \|\varphi(z_0, \cdot)\|_{H^4(\Omega)}^2 dz_0 + e^{-2\eta T} \|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 \right) \\
& \quad + \delta \int_0^T e^{-2\eta z_0} (\|\varphi(z_0, \cdot)\|_{H^3(\Omega)}^2 + \|\partial_{z_0} \varphi(z_0, \cdot)\|_{H^3(\Omega)}^2) dz_0 \\
& \quad + (q\eta + 1) \sum_{|\alpha| \leq 4} \int_0^T e^{-2\eta z_0} \|D^\alpha \varphi\|_{L^2(\Omega)}^2 dz_0 + qe^{-2\eta T} \sum_{|\alpha| \leq 4} \|D^\alpha \varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\
& \quad + \frac{1}{q\eta} \int_0^T e^{-2\eta z_0} \left(\sum_{|\alpha| \leq 4} \|D^\alpha \varphi(z_0)\|_{L^2(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(z_0)\|_{L^2(\Omega)}^2 \right) dz_0 \\
& \quad + \frac{1}{q\eta^2} e^{-2\eta T} \left(\|\varphi(T, \cdot)\|_{H^4(\Omega)}^2 + \sum_{|\alpha| \leq 3} \|\mathcal{L}(D^\alpha \varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\
& \quad + \frac{1}{q} \|\partial_{z_2}^2 f|_{z_0=0}\|_{L^2(\Omega)}^2. \tag{4.84}
\end{aligned}$$

By applying the equation,

$$|\partial_{z_1}^2 \partial_{z_2}^2 \varphi| \lesssim |\mathcal{L}(\partial_{z_2}^2 \varphi)| + |D \partial_{z_2}^3 \varphi| + \sum_{|\alpha| \leq 3} |D^\alpha \partial_{z_0} \varphi|, \quad (4.85)$$

$$\begin{aligned} |\partial_{z_1}^3 \partial_{z_2} \varphi| &\lesssim |\mathcal{L}(\partial_{z_1 z_2} \varphi)| + |\partial_{z_1}^2 \partial_{z_2}^2 \varphi| + |\partial_{z_1} \partial_{z_2}^3 \varphi| + \sum_{|\alpha| \leq 3} |D^\alpha \partial_{z_0} \varphi| \\ &\lesssim |\mathcal{L}(\partial_{z_1 z_2} \varphi)| + |\mathcal{L}(\partial_{z_2}^2 \varphi)| + |D \partial_{z_2}^3 \varphi| + \sum_{|\alpha| \leq 3} |D^\alpha \partial_{z_0} \varphi|, \end{aligned} \quad (4.86)$$

$$|\partial_{z_1}^4 \varphi| \lesssim |\mathcal{L}(\partial_{z_1 z_2} \varphi)| + |\mathcal{L}(\partial_{z_2}^2 \varphi)| + |\mathcal{L}(\partial_{z_1}^2 \varphi)| + |D \partial_{z_2}^3 \varphi| + \sum_{|\alpha| \leq 3} |D^\alpha \partial_{z_0} \varphi|. \quad (4.87)$$

It is clear that $\partial_{z_1}^2 \partial_{z_2}^2 \varphi$ can be bounded by the estimated terms, namely, the derivatives of $\partial_{z_0} \varphi$ and $\partial_{z_2} \varphi$, and the commutator. Then $\partial_{z_1}^3 \partial_{z_2} \varphi$ and $\partial_{z_1}^4 \varphi$ can be also bounded by the controlled terms. In fact, one has

$$\begin{aligned} &\eta \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_1}^2 \partial_{z_2}^2 \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |\partial_{z_1}^2 \partial_{z_2}^2 \varphi|_{z_0=T}|^2 dz_1 dz_2 \\ &\quad + \eta \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_1}^3 \partial_{z_2} \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |\partial_{z_1}^3 \partial_{z_2} \varphi|_{z_0=T}|^2 dz_1 dz_2 \\ &\quad + \eta \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_1}^4 \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |\partial_{z_1}^4 \varphi|_{z_0=T}|^2 dz_1 dz_2 \\ &\lesssim \sum_{|\alpha| \leq 2} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |\mathcal{L}(D^\alpha \varphi)|_{z_0=T}|^2 dz_1 dz_2 \right) \\ &\quad + \eta \int_{\Omega_T} e^{-2\eta z_0} |D \partial_{z_2}^3 \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |D \partial_{z_2}^3 \varphi|_{z_0=T}|^2 dz_1 dz_2 \\ &\quad + \sum_{|\alpha| \leq 3} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \partial_{z_0} \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |D^\alpha \partial_{z_0} \varphi|_{z_0=T}|^2 dz_1 dz_2 \right). \end{aligned} \quad (4.88)$$

By (3.23), we have

$$\begin{aligned} &\sum_{|\alpha| \leq 2} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |\mathcal{L}(D^\alpha \varphi)|_{z_0=T}|^2 dz_1 dz_2 \right) \\ &\lesssim \sum_{|\alpha| \leq 2} \left(\frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} |\partial_{z_0} \mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + \|\mathcal{L}(D^\alpha \varphi)|_{z_0=0}\|_{L^2(\Omega)}^2 \right) \\ &\lesssim \sum_{|\alpha| \leq 3} \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + \sum_{|\alpha| \leq 4} \frac{1}{\eta} \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz dz_0. \end{aligned} \quad (4.89)$$

It is clear that the left hand sides of (4.25), (4.84), and (4.88) cover all the fourth order derivatives of φ . Hence by adding up (4.25), (4.84), and (4.88), we obtain a estimate of all the fourth order derivatives, but still with $D^\alpha \partial_{z_0} \varphi$, $D \partial_{z_2}^3 \varphi$ and

$\mathcal{L}(D^\alpha \varphi)$ ($|\alpha| \leq 3$) (which have already been estimated) on the right hand side of the estimate. The resultant inequality is too long, so we omit to write it down. Then we substitute the estimate of $D^\alpha \partial_{z_0} \varphi$ ($|\alpha| \leq 3$) in (4.25), the estimate of $D \partial_{z_2}^3 \varphi$ in (4.84), and the estimate in (4.89) into the right hand side of the resultant estimate. Next, one firstly chooses q and δ properly small, then chooses η appropriately large, one deduces that

$$\begin{aligned} & \sum_{|\alpha| \leq 4} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |D^\alpha \varphi|_{z_0=T}^2 dz_1 dz_2 \right) \\ & \lesssim \frac{1}{\eta^2} \sum_{|\alpha| \leq 3} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |\mathcal{L}(D^\alpha \varphi)|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |\mathcal{L}(D^\alpha \varphi)|_{z_0=T}^2 dz_1 dz_2 \right) \\ & \quad + \|f|_{z_0=0}\|_{H^3(\Omega)}^2. \end{aligned} \quad (4.90)$$

□

Combining lemma 4.1 and lemma 4.2, it is easy to see that lemma 3.2 holds.

5. THE NONLINEAR PROBLEM, PROOF OF THEOREM 2.1

In this section, based on the well-posedness of linear problem (3.1) in Proposition 3.1, we will establish the existence of the non-linear problem by constructing an iteration scheme. The iteration scheme admits an approximate sequence of the solutions. Then by showing the sequence is bounded in the higher order norm and contracted in the lower order norm, one shows that the sequence converges to the desired solution. Hence theorem 2.1 is proved. First from Proposition 3.1, we have the following theorem:

Theorem 5.1. *Under assumptions (i) – (iv), there exists a smooth solution to (3.1) and there exists a constant $\bar{\eta} > 0$, such that for $\eta \geq \bar{\eta}$ and any $T > 0$,*

$$\begin{aligned} & \sum_{|\alpha| \leq 4} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha \varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |D^\alpha \varphi|_{z_0=T}^2 dz_1 dz_2 \right) \\ & \lesssim \frac{1}{\eta^2} \sum_{|\beta| \leq 4} \left(\eta \int_0^T e^{-2\eta z_0} \|D^\beta f(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|D^\beta f(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\ & \quad + \|f|_{z_0=0}\|_{H^3(\Omega)}^2. \end{aligned} \quad (5.1)$$

Proof. Note that

$$\mathcal{L}(D^\alpha \varphi) = -[D^\alpha, \mathcal{L}]\varphi + D^\alpha \mathcal{L}\varphi = -[D^\alpha, \mathcal{L}]\varphi + D^\alpha f. \quad (5.2)$$

For the commutator $[D^\alpha, \mathcal{L}]\varphi$, it is clear that

$$[D^\alpha, \mathcal{L}]\varphi = \sum_{i,j=0}^2 (D^\alpha(r_{ij}\partial_{ij}\varphi) - r_{ij}D^\alpha\partial_{ij}\varphi). \quad (5.3)$$

By the Sobolev embedding theorem and assumption (ii), one has

$$\begin{aligned} \|[D^\alpha, \mathcal{L}]\varphi\|_{L^2(\Omega)}^2 &\lesssim \|\partial_{ij}\varphi\|_{L^\infty(\Omega)}^2 \cdot \left(\sum_{|\beta|\leq 4} \|D^\beta\Phi\|_{L^2(\Omega)}^2 \right) + \|D\partial_{ij}\varphi \cdot D^3\Phi\|_{L^2(\Omega)}^2 \\ &\quad + \|D^2\partial_{ij}\varphi\|_{L^2(\Omega)}^2 \cdot \|D^2\Phi\|_{L^\infty(\Omega)}^2 \\ &\lesssim \delta(\|\varphi\|_{H^4(\Omega)}^2 + \|\partial_{z_0}\varphi\|_{H^3(\Omega)}^2 + \|\partial_{z_0}^2\varphi\|_{H^2(\Omega)}^2) + \|D\partial_{ij}\varphi\|_{L^4(\Omega)}^2 \cdot \|D^3\Phi\|_{L^4(\Omega)}^2 \\ &\lesssim \delta(\|\varphi\|_{H^4(\Omega)}^2 + \|\partial_{z_0}\varphi\|_{H^3(\Omega)}^2 + \|\partial_{z_0}^2\varphi\|_{H^2(\Omega)}^2) + \|D\partial_{ij}\varphi\|_{H^1(\Omega)}^2 \cdot \|D^3\Phi\|_{H^1(\Omega)}^2 \\ &\lesssim \delta(\|\varphi\|_{H^4(\Omega)}^2 + \|\partial_{z_0}\varphi\|_{H^3(\Omega)}^2 + \|\partial_{z_0}^2\varphi\|_{H^2(\Omega)}^2 + \|\partial_{z_0}^3\varphi\|_{H^1(\Omega)}^2) \\ &\lesssim \delta \sum_{|\beta|\leq 4} \|D^\beta\varphi\|_{L^2(\Omega)}^2, \end{aligned} \quad (5.4)$$

where we have used the Hölder's inequality in the second inequality. Hence,

$$\begin{aligned} &\eta \int_0^T e^{-2\eta z_0} \|[D^\alpha, \mathcal{L}]\varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|[D^\alpha, \mathcal{L}]\varphi(T, \cdot)\|_{L^2(\Omega)}^2 \\ &\leq \delta \sum_{|\beta|\leq 4} \left(\eta \int_0^T e^{-2\eta z_0} \|D^\beta\varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|D^\beta\varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (5.5)$$

This together with (5.2) gives

$$\begin{aligned} &\eta \int_0^T e^{-2\eta z_0} \|\mathcal{L}(D^\alpha\varphi)(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|\mathcal{L}(D^\alpha\varphi)(T, \cdot)\|_{L^2(\Omega)}^2 \\ &\lesssim \delta \sum_{|\beta|\leq 4} \left(\eta \int_0^T e^{-2\eta z_0} \|D^\beta\varphi(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|D^\beta\varphi(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\ &\quad + \sum_{|\beta|\leq 3} \left(\eta \int_0^T e^{-2\eta z_0} \|D^\beta f(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|D^\beta f(T, \cdot)\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (5.6)$$

Therefore for properly small δ and appropriately large η , we deduce that

$$\begin{aligned} &\sum_{|\alpha|\leq 4} \left(\eta \int_{\Omega_T} e^{-2\eta z_0} |D^\alpha\varphi|^2 dz dz_0 + e^{-2\eta T} \int_{\Omega} |D^\alpha\varphi|_{z_0=T}^2 dz_1 dz_2 \right) \\ &\lesssim \frac{1}{\eta^2} \sum_{|\beta|\leq 3} \left(\eta \int_0^T e^{-2\eta z_0} \|D^\beta f(z_0, \cdot)\|_{L^2(\Omega)}^2 dz_0 + e^{-2\eta T} \|D^\beta f(T, \cdot)\|_{L^2(\Omega)}^2 \right) \\ &\quad + \|f|_{z_0=0}\|_{H^3(\Omega)}^2. \end{aligned} \quad (5.7)$$

□

Lemma 5.1. *For any smooth function v , we have*

$$\|e^{-\eta z_0} v\|_{H^s(\Omega_T)}^2 \leq \sum_{|\alpha| \leq s} \|e^{-\eta z_0} D^\alpha v\|_{L^2(\Omega_T)}^2 \quad (5.8)$$

provided that $\partial_{z_0}^j v|_{z_0=0} = 0$ for $j = 1, \dots, s-1$.

Proof. For $T > 0$, let

$$A(T) = \int_{\Omega_T} e^{-2\eta z_0} v^2 dz. \quad (5.9)$$

Then one has

$$\begin{aligned} A(T) &= -\frac{1}{2\eta} \int_{\Omega_T} (e^{-2\eta z_0})_{z_0} v^2 dz \\ &= -\frac{1}{2\eta} \int_{\Omega_T} (e^{-2\eta z_0} v^2)_{z_0} - e^{-2\eta z_0} 2v v_{z_0} dz \\ &\leq -\frac{1}{2\eta} e^{-2\eta T} \int_{\Omega} v^2 dz_1 dz_2 + \frac{1}{2\eta} \int_{\Omega_T} e^{-2\eta z_0} (\eta v^2 + \frac{1}{\eta} v_{z_0}^2) dz \\ &\leq \frac{1}{2} A(T) + \frac{1}{2\eta^2} \int_{\Omega_T} e^{-2\eta z_0} v_{z_0}^2 dz. \end{aligned} \quad (5.10)$$

Hence we have

$$A(T) \leq \frac{1}{\eta^2} \int_{\Omega_T} e^{-2\eta z_0} v_{z_0}^2 dz. \quad (5.11)$$

Now we show (5.8) by the induction on s . For $s = 1$,

$$\begin{aligned} \|e^{-\eta z_0} v\|_{H^1(\Omega_T)}^2 &= \eta^2 \|e^{-\eta z_0} v\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} Dv\|_{L^2(\Omega_T)}^2 \\ &\leq \|e^{-\eta z_0} v_{z_0}\|_{L^2(\Omega_T)}^2 + \|e^{-\eta z_0} Dv\|_{L^2(\Omega_T)}^2 \\ &\leq \sum_{|\alpha| \leq 1} \|e^{-\eta z_0} D^\alpha v\|_{L^2(\Omega_T)}^2. \end{aligned} \quad (5.12)$$

Now for $k \in \mathbb{N}$, assume

$$\|e^{-\eta z_0} v\|_{H^k(\Omega_T)}^2 \leq \sum_{|\alpha| \leq k} \|e^{-\eta z_0} D^\alpha v\|_{L^2(\Omega_T)}^2. \quad (5.13)$$

We are going to show

$$\|e^{-\eta z_0} v\|_{H^{k+1}(\Omega_T)}^2 \leq \sum_{|\alpha| \leq k+1} \|e^{-\eta z_0} D^\alpha v\|_{L^2(\Omega_T)}^2. \quad (5.14)$$

Repeating the process for estimate (5.11) above m times where $|v|^2$ in $A(T)$ is replaced by $|D^n v|^2$, we have

$$\int_0^T e^{-2\eta t} \|D^n v\|_{L^2(\Omega)}^2 dt \leq \eta^{-2m} \int_0^T e^{-2\eta t} \|D^{m+n} v\|_{L^2(\Omega)}^2 dt, \quad (5.15)$$

provided that $\partial_t^l v|_{t=0} = 0$, $l = 0, 1, 2, \dots, m+n-1$. Note that

$$\|e^{-\eta z_0} v\|_{H^{k+1}(\Omega_T)}^2 = \|e^{-\eta z_0} v\|_{H^k(\Omega_T)}^2 + \sum_{|\alpha|=k+1} \|D^\alpha(e^{-\eta t} v)\|_{L^2(\Omega_T)}^2 \quad (5.16)$$

and

$$\begin{aligned} \sum_{|\alpha|=k+1} \|D^\alpha(e^{-\eta t} v)\|_{L^2(\Omega_T)}^2 &= \sum_{l_1+l_2=k+1} \|(-\eta)^{l_1} e^{-\eta t} D^{l_2} v\|_{L^2(\Omega_T)}^2 \\ &= \sum_{l_1+l_2=k+1} (\eta)^{2l_1} \|e^{-\eta t} D^{l_2} v\|_{L^2(\Omega_T)}^2. \end{aligned} \quad (5.17)$$

So by (5.15), we have

$$\begin{aligned} \sum_{l_1+l_2=k+1} (\eta)^{2l_1} \|e^{-\eta t} D^{l_2} v\|_{L^2(\Omega_T)}^2 &\leq \sum_{l_1+l_2=k+1} \|e^{-\eta t} D^{l_1+l_2} v\|_{L^2(\Omega_T)}^2 \\ &= \sum_{|\alpha|=k+1} \|e^{-\eta t} D^\alpha v\|_{L^2(\Omega_T)}^2. \end{aligned} \quad (5.18)$$

From (5.13), (5.16)–(5.18), we obtain (5.14). Therefore, we derive the estimate (5.8) for any $s \in \mathbb{N}$ by the induction method. $\square$

Let $\psi(z_0, z_1, z_2) = \sum_{k=0}^3 \frac{\varphi_k}{k!} z_0^k$, where $\varphi_k = \partial_{z_0}^k \hat{\Phi}|_{z_0=0}$, *i.e.*, $\varphi_0 = \hat{\Phi}_0$ and $\varphi_1 = \hat{\Phi}_1$ by the initial conditions in (2.32), and φ_k for $k = 2$ or 3 is defined by equation (2.32)₁ and the initial conditions. Then one defines a approximation sequence in the following manner.

Let $\tilde{\varphi}_0 = 0$ and suppose $\tilde{\varphi}_m$ is given. Then $\tilde{\varphi}_{m+1}$ is defined as the solution to the following initial boundary value problem:

$$\begin{cases} \mathcal{L}(\tilde{\varphi}_m + \psi)\tilde{\varphi}_{m+1} = F_m, & \text{in } \Omega_T, \\ \mathcal{B}\tilde{\varphi}_{m+1} = 0, & \text{on } \Gamma_1, \\ \partial_{z_2}\tilde{\varphi}_{m+1} = 0, & \text{on } \Gamma_2, \\ \tilde{\varphi}_{m+1} = 0, \quad \partial_{z_0}\tilde{\varphi}_{m+1} = 0, & \text{on } \Gamma_0, \end{cases} \quad (5.19)$$

where

$$\mathcal{L}(\Psi')\Psi'' := \sum_{i,j=0}^2 \alpha_{ij}(D\Psi')\partial_{ij}\Psi'', \quad (5.20)$$

and

$$F_m = -\mathcal{L}(\tilde{\varphi}_m + \psi)\psi - \sum_{i=0}^2 \alpha_i(\tilde{\varphi}_m + \psi)\partial_{z_i}\tilde{\varphi}_m. \quad (5.21)$$

By the compatibility conditions, we have $\mathcal{B}\psi = 0$ on Γ_1 and $\partial_{z_2}\psi = 0$ on Γ_2 . Via Theorem 5.1, the sequence $\{\tilde{\varphi}_m\}_{m=0}^\infty$ is well-defined. Now, we will show $\tilde{\varphi}_m$ converges to some function $\tilde{\varphi}$, and then $\tilde{\varphi} + \psi$ is a solution to the non-linear problem (2.32).

Proposition 5.1 (Boundness in the higher order norm). *There exist three constants $\delta_0 > 0$, $\eta_0 \geq 1$, and $T_0 > 0$ such that for $\eta \geq \eta_0$ and $0 < T \leq T_0$, and for all $n \geq 0$, it holds that*

$$\|e^{-\eta z_0}\tilde{\varphi}_n\|_{H^4(\Omega_T)} + e^{-2\eta T} \sum_{k=0}^4 \sup_{0 \leq z_0 \leq T} \|\partial_{z_0}^k \tilde{\varphi}_n(z_0, \cdot)\|_{H^{4-k}(\Omega)} \leq \delta_0. \quad (5.22)$$

Proof. We will prove it by the induction. It is easy to see (5.22) is true when $n = 0$, since ϵ can be selected sufficiently small. Assume (5.22) holds for $n = m \geq 0$. We will show (5.22) holds for $n = m + 1$. By (5.19) and Proposition 3.1, one has

$$\begin{aligned} & \|e^{-\eta z_0}\tilde{\varphi}_m\|_{H^4(\Omega_T)}^2 + e^{-2\eta T} \sum_{k=0}^4 \sup_{0 \leq z_0 \leq T} \|\partial_{z_0}^k \tilde{\varphi}_{m+1}(z_0, \cdot)\|_{H^{4-k}(\Omega)} \\ & \leq \frac{C}{\eta^2} \left(\|e^{-\eta z_0}F_m\|_{H^3(\Omega_T)}^2 + e^{-2\eta T} \sum_{k=0}^3 \|\partial_{z_0}^k F_m(T, \cdot)\|_{H^{3-k}(\Omega)}^2 \right). \end{aligned} \quad (5.23)$$

By (5.21) and (5.22) holds for $n = m$, one has

$$\begin{aligned} \|e^{-\eta z_0}F_m\|_{H^3(\Omega_T)}^2 & \leq C' \|e^{-\eta z_0}\psi\|_{H^5(\Omega_T)}^2 e^{2\eta T} (\delta_0^2 + \|e^{-\eta z_0}\psi\|_{H^5(\Omega_T)}^2) \\ & \quad + C' e^{2\eta T} \delta_0^2 (\delta_0^2 + \|e^{-\eta z_0}\psi\|_{H^5(\Omega_T)}^2). \end{aligned} \quad (5.24)$$

Similarly for $k = 0, 1, 2, 3$, we have

$$\|\partial_{z_0}^k F_m(T, \cdot)\|_{H^{3-k}(\Omega)}^2 \leq C' \epsilon^2 e^{2\eta T} (C\delta_0^2 + \epsilon^2) + C' e^{2\eta T} \delta_0^2 (C\delta_0^2 + \epsilon^2). \quad (5.25)$$

Select $\eta_0 \geq 1$ such that $CC'\eta_0^{-2} \leq \frac{1}{8}$ and let T_0 be small such that $e^{2\eta_0 T_0} \leq 2$. Then for $\delta_0 \leq \sqrt{\frac{1}{C}}$, one sets $0 < \epsilon \leq \delta_0$ in Theorem 2.1 small such that $\|e^{-\eta_0 z_0}\psi(z_0, \cdot)\|_{H^5(\Omega_{T_0})}^2 \leq \delta_0^2$. So it follows from (5.23)-(5.25) that,

$$\|e^{-\eta z_0}\tilde{\varphi}_m\|_{H^4(\Omega_T)}^2 + e^{-2\eta T} \sum_{k=0}^4 \sup_{0 \leq z_0 \leq T} \|\partial_{z_0}^k \tilde{\varphi}_{m+1}(z_0, \cdot)\|_{H^{4-k}(\Omega)}^2 \leq \delta_0^2. \quad (5.26)$$

□

Let $v_m = \tilde{\varphi}_{m+1} - \tilde{\varphi}_m$ for $m \geq 0$, then v_m satisfies the following initial boundary value problem:

$$\begin{cases} \mathcal{L}_1(\tilde{\varphi}_m + \psi)v_m = G_m, & \text{in } \Omega_T, \\ \mathcal{B}v_m = 0, & \text{on } \{z_1 = 0\}, \\ \partial_{z_2}v_m = 0, & \text{on } \{z_2 = 0\}, \\ v_m = 0, \quad \partial_{z_0}v_m = 0, & \text{on } \{z_0 = 0\}, \end{cases} \quad (5.27)$$

where

$$\begin{aligned} G_m = & -[\mathcal{L}(\tilde{\varphi}_m + \psi) - \mathcal{L}(\tilde{\varphi}_{m-1} + \psi)]\psi \\ & - [\alpha_2(\tilde{\varphi}_m + \psi) - \alpha_2(\tilde{\varphi}_{m-1} + \psi)]\partial_{z_2}v_{m-1} \\ & - [\mathcal{L}_1(\tilde{\varphi}_m + \psi) - \mathcal{L}_1(\tilde{\varphi}_{m-1} + \psi)]\tilde{\varphi}_m. \end{aligned} \quad (5.28)$$

Proposition 5.2 (Contraction in the lower order norm). *Under the same assumptions in Theorem 2.1, there exist two constants $\eta_* \geq \eta_0$ and $T_* \leq T_0$ such that, for some $\sigma \in (0, 1)$ for all $m \geq 1$ and for $\eta \geq \eta_*$ and $T \leq T_*$, it holds that:*

$$\begin{aligned} & \|e^{-\eta z_0}v_m\|_{H^1(\Omega_T)} + e^{-2\eta T} \sum_{k=0}^1 \sup_{0 \leq z_0 \leq T} \|\partial_{z_0}^k v_m(z_0, \cdot)\|_{H^{1-k}(\Omega)} \\ & \leq \sigma \left(\|e^{-\eta z_0}v_{m-1}\|_{H^1(\Omega_T)} + e^{-2\eta T} \sum_{k=0}^1 \sup_{0 \leq z_0 \leq T} \|\partial_{z_0}^k v_{m-1}(z_0, \cdot)\|_{H^{1-k}(\Omega)} \right). \end{aligned} \quad (5.29)$$

Proof. Note that

$$\begin{aligned} & [\mathcal{L}(\tilde{\varphi}_m + \psi) - \mathcal{L}(\tilde{\varphi}_{m-1} + \psi)]\tilde{\varphi}_m \\ & = \sum_{i,j=0}^2 [\alpha_{ij}(\tilde{\varphi}_m + \psi) - \alpha_{ij}(\tilde{\varphi}_{m-1} + \psi)]\partial_{ij}\tilde{\varphi}_m \\ & = \sum_{i,j,k=0}^3 \left(\partial_k v_{m-1} \int_0^1 \frac{\partial \alpha_{ij}}{\partial(\partial_k \tilde{\varphi})}(\tilde{\varphi}_{m-1} + \psi + \theta v_{m-1}) d\theta \right) \partial_{ij}\tilde{\varphi}_m. \end{aligned}$$

So for $\eta \geq \eta_0$ and $T \leq T_0$, we have

$$\|e^{-\eta z_0}[\mathcal{L}(\tilde{\varphi}_m + \psi) - \mathcal{L}(\tilde{\varphi}_{m-1} + \psi)]\tilde{\varphi}_m\|_{L^2(\Omega_T)} \leq C(\delta_0)\|e^{-\eta z_0}v_{m-1}\|_{H^1(\Omega_T)}. \quad (5.30)$$

By a similar argument, we also derive that

$$\|[\alpha_i(\tilde{\varphi}_m + \psi) - \alpha_i(\tilde{\varphi}_{m-1} + \psi)]\partial_{z_2}v_{m-1}\|_{L^2(\Omega_T)} \leq C(\delta_0)\|e^{-\eta z_0}v_{m-1}\|_{H^1(\Omega_T)} \quad (5.31)$$

and

$$\|e^{-\eta z_0}[\mathcal{L}(\tilde{\varphi}_m + \psi) - \mathcal{L}(\tilde{\varphi}_{m-1} + \psi)]\psi\|_{L^2(\Omega_T)} \leq C(\delta_0)\|e^{-\eta z_0}v_{m-1}\|_{H^1(\Omega_T)}. \quad (5.32)$$

By (5.30)-(5.32) and the first order energy estimate (3.15), one has

$$\begin{aligned} \|e^{-\eta z_0} v_m\|_{H^1(\Omega_T)} + e^{-2\eta T} \sum_{k=0}^1 \sup_{0 \leq z_0 \leq T} \|\partial_{z_0}^k v_m(z_0, \cdot)\|_{H^{1-k}(\Omega)} \\ \leq C(\delta_0) \eta^{-1} \|e^{-\eta z_0} v_{m-1}\|_{H^1(\Omega_T)}. \end{aligned} \quad (5.33)$$

Choose $\eta_* \geq \eta_0$ such that $\sigma := C(\delta_0) \eta_*^{-1} < 1$. Note that δ_0 does not depend on the weight η . Then select $T_* \leq T_0$ such that $e^{2\eta_* T_*} \leq 2$, we obtain (5.29) for $\eta \geq \eta_*$ and $T \leq T_*$. This completes the proof of this proposition. $\square$

Now, we are ready to conclude this paper by showing the main theorem.

Proof of Theorem 2.1. Proposition 5.2 implies that $\{\tilde{\varphi}_m\}_m^\infty$ is a Cauchy sequence. Hence there exists a function $\tilde{\varphi} \in H^1(\Omega_T)$ such that for $\eta \geq \eta_*$ and $T \leq T_*$,

$$\lim_{m \rightarrow \infty} \left(\|e^{-\eta z_0} (\tilde{\varphi}_m - \tilde{\varphi})\|_{H^1(\Omega_T)} + e^{-2\eta T} \sum_{k=0}^1 \|\partial_{z_0}^k (\tilde{\varphi}_m - \tilde{\varphi})\|_{L^\infty(0, T; H^{1-k}(\Omega))} \right) = 0.$$

Moreover, Proposition 5.1 implies that $\tilde{\varphi} \in H^4(\Omega_T)$. Hence by passing the limit $m \rightarrow \infty$ in the approximation problem (5.19), one deduces that $\tilde{\varphi} + \psi$ is a smooth solution to the non-linear problem (2.1). Moreover, by (5.22) and the assumptions in Theorem 2.1, one has $\|e^{-\eta z_0} (\tilde{\varphi} + \psi)\|_{H^4(\Omega_T)} \leq C\delta_0$, for $\eta \geq \eta_*$ and $T \leq T_*$. This completes the proof of Theorem 2.1. $\square$

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REFERENCES

- [1] S. Alinhac, Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels.(French) [Existence of rarefaction waves for multidimensional hyperbolic quasilinear systems] *Comm. Partial Differential Equations* 14:173–230, 1989.
- [2] S. Alinhac, Unicité d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. (French) [Uniqueness of rarefaction waves for multidimensional hyperbolic quasilinear systems] *Indiana Univ. Math. J.*, 38:345–363, 1989.

- [3] M. Bae, G.-Q. Chen, and M. Feldman, Regularity of solutions to regular shock reflection for potential flow, *Invent. Math.* 175:505–543, 2009.
- [4] M. Bae, G.-Q. Chen, and M. Feldman, Prandtl-Meyer reflection configurations, transonic shocks, and free boundary problems, to appear in *Memoirs of the AMS*, *Preprint at arXiv:1901.05916*, 2019.
- [5] S. Benzoni-Gavage and D. Serre, *Multidimensional hyperbolic partial differential equations. First-order systems and applications*, *Oxford Mathematical Monographs*. The Clarendon Press, Oxford University Press, Oxford, 2007.
- [6] A. Bressan, *Hyperbolic systems of conservation laws: The one dimensional Cauchy problem*, Oxford University Press: Oxford, 2000.
- [7] G.-Q. Chen and M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, *J. Amer. Math. Soc.*, 16:461–494, 2003.
- [8] G.-Q. Chen and M. Feldman, Global solutions of shock reflection by large-angle wedges for potential flow, *Ann. of Math.*, 171:1067–1182, 2010.
- [9] G.-Q. Chen and M. Feldman, *Mathematics of Shock Reflection-Diffraction and Von Neumann’s Conjecture*, *Annals of Mathematics Studies*, Princeton, 2018.
- [10] G.-Q. Chen, M. Feldman, J. Hu, and W. Xiang, Loss of regularity of solutions of shock diffraction problem by a convex cornered wedge to the potential flow equation, *SIAM J. Math. Anal.*, 52:1096–1114, 2020.
- [11] G.-Q. Chen, M. Feldman, and W. Xiang, Convexity of transonic shocks in self-similar coordinates, *Arch. Ration. Mech. Anal.*, 238:47–124, 2020.
- [12] S.-X. Chen, Mach configuration in pseudo-stationary compressible flow, *J. Amer. Math. Soc.*, 21:63–100, 2008.
- [13] J.-F. Coulombel and P. Secchi, The stability of compressible vortex sheets in two space dimensions, *Indiana Univ. Math. J.* 53:941–1012, 2004.
- [14] J.-F. Coulombel and P. Secchi, Non-linear compressible vortex sheets in two space dimensions, *Annales scientifiques de l’école Normale Supérieure, Série 4*, 41:85–139, 2008.
- [15] J.-F. Coulombel and P. Secchi, Uniqueness of 2-D compressible vortex sheets, *Commun. Pure Appl. Anal.*, 8:1439–1450, 2009.
- [16] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Springer-Verlag: Berlin, 2010.
- [17] F. Gazzola and P. Secchi, Inflow-outflow problems for Euler equations in a rectangular cylinder, *Non-linear Differ. Equ. Appl.*, 8:195–217, 2001.
- [18] B.-X. Fang, F.-M. Huang, W. Xiang and F. Xiao, Persistence of the steady planar normal shock structure in 3-D unsteady potential flows, *Preprint at arXiv:2108.09207*, 2021.
- [19] B.-X. Fang, W. Xiang, and F. Xiao, Persistence of the steady normal shock structure for the unsteady potential flow, *SIAM J. Math. Anal.*, 52:6033–6104, 2020.
- [20] P. Godin, On the breakdown of 2D compressible Eulerian flows in bounded impermeable regions with corners, *J. Math. Pures Appl.* 137:178–212, 2020.
- [21] P. Godin, The 2D compressible Euler equations in bounded impermeable domains with corners, *Mem. Amer. Math. Soc.* 269, v+72 pp, 2021.
- [22] M. Ikawa, Mixed problems for hyperbolic equations of second order, *J. Math. Soc. Japan*, 20:580–608, 1968.

- [23] M. Ikawa, On the mixed problem for hyperbolic equations of second order with the neumann boundary condition, *Osaka J. Math.*, 7:203–223, 1970.
- [24] V. Elling and T.-P. Liu, Supersonic flow onto a solid wedge, *Comm. Pure. Appl. Math.*, 61:1347–1448, 2008.
- [25] K.O. Friedrichs and R. Courant, *Supersonic flow and shock waves*, Springer-Verlag, New York, 1948.
- [26] J.-Q. Li and Y.-X. Zheng, Interaction of rarefaction waves of the two-dimensional self-similar Euler equations, *Arch. Ration. Mech. Anal.*, 193:623–657, 2009.
- [27] J.-Q. Li and Y.-X. Zheng, Interaction of four rarefaction waves in the bi-symmetric class of the two-dimensional Euler equations, *Comm. Math. Phys.*, 296:303–321, 2010.
- [28] J. Li, I. Witt, and H.-C. Yin, Global multidimensional shock waves of 2-dimensional and 3-dimensional unsteady potential flow equations, *SIAM J. Math. Anal.*, 50:933–1009, 2018.
- [29] A. Majda, The stability of multidimensional shock fronts, *Mem. Amer. Math. Soc.* 41, 1983.
- [30] A. Majda, The existence of multidimensional shock fronts, *Mem. Amer. Math. Soc.*, 43, 1983.
- [31] A. Majda and E. Thomann, Multi-dimensional shock fronts for second order wave equations. *Comm. Partial Differential Equations*, 12:777–828, 1987.
- [32] S. Osher, An ill posed problem for a hyperbolic equation near a corner, *Bull. Amer. Math. Soc.*, 79:1043–1044, 1973.
- [33] S. Osher, An ill-posed problem for a strictly hyperbolic equation in two unknowns near a corner, *Bull. Amer. Math. Soc.*, 80:705–708, 1974.
- [34] J. Smoller, *Shock Waves and Reaction–Diffusion Equations*, Springer, New York, 1994.
- [35] G. Xu and H.-C. Yin, On global smooth solutions of 3-D compressible Euler equations with vanishing density in infinitely expanding balls, *Discrete Contin. Dyn. Syst.*, 40:2213–2265, 2020.
- [36] H.-R. Yuan, Persistence of shocks in ducts, *Nonlinear Anal.*, 75:3874–3894, 2012.
- [37] Y.-X. Zheng, *Systems of Conservation Laws: Two-Dimensional Riemann Problems*, Birkhäuser, Boston, 2001.

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