ON INFINITE SERIES OF BESSEL FUNCTIONS OF THE FIRST KIND: $\sum_\nu J_{N\nu+p}(x), \sum_\nu(-1)^\nu J_{N\nu+p}(x)$

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Abstract. Infinite series of Bessel function of the first kind, $\sum_\nu^{\pm\infty} J_{N\nu+p}(x)$, $\sum_\nu^{\pm\infty}(-1)^\nu J_{N\nu+p}(x)$, are summed in closed form. These expressions are evaluated by engineering a Dirac comb that selects specific sequences within the Bessel series.

Introduction

Infinite series of Bessel functions of the first kind in the form $\sum_{2\nu} J_{2\nu}(x)$ and $\sum_{3\nu} J_{3\nu}(x)$ arise in many natural systems. They are of particular interest in condensed matter physics when crystals spontaneously break symmetry [1] due to correlated electron effects such as superconductivity, charge density waves, colossal magnetoresistance, and quantum spin liquids. Mathematically, these series can appear when sinusoids exist inside complex exponentials—as described by the Jacobi-Anger relation [2, 3]. Early treatises on Bessel functions by Neumann and Watson [4, 5, 6] provide analytic solutions to the alternating series $\sum_\nu(-1)^\nu J_{2\nu}(x)$ and $\sum_\nu(-1)^\nu J_{2\nu+1}(x)$ which are commonly tabulated [7, 8]. However an analytic expressions for $\sum_\nu J_{3\nu}(x)$, $\sum_\nu J_{3\nu\pm1}(x)$ or more general series $\sum_\nu J_{N\nu+p}(x)$ and $\sum_\nu(-1)^\nu J_{N\nu+p}(x)$ are not readily available. We show closed form expressions to infinite series of Bessel functions of the first kind exist. The expression is evaluated by engineering a Dirac comb that selects specific sequences within the Bessel series.

To illustrate, we find a closed form expression to the series:

$$\sum_{\nu=-\infty}^{\infty} J_{3\nu+p}(x) = \frac{1}{3}\left[1 + 2\cos\left(\frac{x\sqrt{3}}{2} - \frac{2\pi p}{3}\right)\right]$$

$$\nu, p \in \mathbb{Z};\ x \in \mathbb{C}$$

More generally, we find an expression to all series in the class:

$$\sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(x) = \frac{1}{N}\sum_{q=0}^{N-1} e^{ix\sin(2\pi q/N)}e^{-i2\pi pq/N}$$

$$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{N\nu+p}(x) = \frac{1}{N}\sum_{q=0}^{N-1} e^{ix\sin((2q+1)\pi/N)}e^{-i(2q+1)\pi p/N}$$

$$\nu, p, q \in \mathbb{Z};\ N \in \mathbb{Z}^+;\ x \in \mathbb{C}$$

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From these theorem's, we tabulate a family of closed analytic forms to infinite series of Bessel functions of the first kind.

1. Evaluation of Series: $\sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(x)$

Theorem 1. *Infinite series of Bessel functions of the first kind of the form* $\sum_{\nu}^{\pm\infty} J_{N\nu+p}(x)$ *where* $\nu, p, q \in \mathbb{Z},\ N \in \mathbb{Z}^+$ *and* $x \in \mathbb{C}$ *have following closed expression:*

$$\sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(x) = \frac{1}{N}\sum_{q=0}^{N-1} e^{-i2\pi pq/N} e^{ix\sin(2\pi q/N)} \tag{1.1}$$

Proof. Consider the following series of evenly spaced delta functions (i.e. a Dirac comb) on an infinite series of Bessel functions:

$$f(k) = \sum_{h=-\infty}^{\infty}\sum_{p=0}^{N-1} \delta(k-(h+\frac{p}{N})a) \sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(kA) \tag{1.2}$$

where $\nu, h, p \in \mathbb{Z}$; $N \in \mathbb{Z}^+$; $k, a \in \mathbb{R}$; $A \in \mathbb{C}$.

Three summations are re-grouped into two:

$$f(k) = \sum_{h=-\infty}^{\infty}\sum_{\alpha=-\infty}^{\infty} \delta(k-(h+\frac{\alpha}{N})a) J_\alpha(kA) \tag{1.3}$$

The Dirac comb can be represented as a Fourier series:

$$\sum_{h=-\infty}^{\infty} \delta(k-ha) = \frac{1}{a}\sum_{m=-\infty}^{\infty} e^{-i2\pi km/a} \tag{1.4}$$

With (1.4), $f(k)$ becomes:

$$f(k) = \frac{1}{a}\sum_{m=-\infty}^{\infty} e^{-i2\pi km/a} \sum_{\alpha=-\infty}^{\infty} J_\alpha(kA) e^{i\alpha 2\pi m/N} \tag{1.5}$$

The summation over α appears in the Jacobi-Anger relation [2, 3, 6]:

$$e^{ix\sin(\theta)} = \sum_{\alpha=-\infty}^{\infty} J_\alpha(x) e^{i\alpha\theta} \tag{1.6}$$

Using the Jacobi-Anger relation:

$$f(k) = \frac{1}{a}\sum_{m=-\infty}^{\infty} e^{-i2\pi km/a} e^{ikA\sin(2\pi m/N)} \tag{1.7}$$

We split the summation over m into $m = ..., Nn, Nn+1, ..., Nn+(N-1), ...$:

$$f(k) = \frac{1}{a}\sum_{n=-\infty}^{\infty}\Big[... + e^{-i2\pi knN/a} + e^{-i2\pi knN/a}e^{-i2\pi k/a}e^{ikA\sin(2\pi/N)} + ... + e^{-i2\pi knN/a}e^{-i2\pi k(N-1)/a}e^{ikA\sin(2\pi(N-1)/N)} + ...\Big]$$

$$= \frac{1}{a}\sum_{n=-\infty}^{\infty} e^{-i2\pi knN/a}\sum_{q=1}^{N-1} e^{-i2\pi kq/a}e^{ikA\sin(2\pi q/N)} \tag{1.8}$$

Using (1.4) again:

$$f(k) = \frac{1}{N}\sum_{l=-\infty}^{\infty}\delta(k-\frac{la}{N})\sum_{q=0}^{1-N} e^{ikA\sin(2\pi q/N)}e^{-i2\pi kq/a} \tag{1.9}$$

Substitute $k = \frac{la}{N}; l \in \mathbb{Z}$, as f is non-zero only where a delta function exists.

$$f(k) = \frac{1}{N}\sum_{l=-\infty}^{\infty}\delta(k-\frac{la}{N})\sum_{q=0}^{N-1} e^{ikA\sin(2\pi q/N)}e^{-i2\pi ql/N} \tag{1.10}$$

We split the summation into $l = ..., Nh, Nh+1, ..., (N+1)h-1...$ then regroup, similar with (1.8):

$$f(k) = \frac{1}{N}\sum_{h=-\infty}^{\infty}\sum_{p=0}^{N-1}\delta(k-(h+\frac{p}{N})a)\sum_{q=0}^{N-1} e^{ikA\sin(2\pi q/N)}e^{-i2\pi pq/N} \tag{1.11}$$

Initial expression (1.2) must equal (1.11):

$$\sum_{h=-\infty}^{\infty}\sum_{p=0}^{N-1}\delta(k-(h+\frac{p}{N})a)\sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(kA) = \frac{1}{N}\sum_{h=-\infty}^{\infty}\sum_{p=0}^{N-1}\delta(k-(h+\frac{p}{N})a)\sum_{q=0}^{N-1} e^{ikA\sin(2\pi q/N)}e^{-i2\pi pq/N} \tag{1.12}$$

This relation suggests only values on the Dirac comb are equivalent. However, the variable a can take on any real value thus the expression holds for all values of kA. The equivalent Dirac lattices on each side can be disregarded.

$$\therefore \quad \sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(x) = \frac{1}{N}\sum_{q=0}^{N-1} e^{ix\sin(2\pi q/N)}e^{-i2\pi pq/N} \tag{1.13}$$

$\square$

Corollary 1. *From Theorem 1 we comprise a table of closed form expressions:*

N	p	$\sum_{\nu=-\infty}^{\infty} J_{N\nu+p}(x)$	$\frac{1}{N}\sum_{q=0}^{N-1} e^{ix\sin(2\pi q/N)} e^{-i2\pi pq/N}$
1	0	$\sum_{\nu=-\infty}^{\infty} J_{\nu}(x)$	1 [6, 7, 9]
2	0	$\sum_{\nu=-\infty}^{\infty} J_{2\nu}(x)$	1 [6, 7]
	1	$\sum_{\nu=-\infty}^{\infty} J_{2\nu+1}(x)$	0 [6, 7]
3	0	$\sum_{\nu=-\infty}^{\infty} J_{3\nu}$	$\frac{1}{3}\left[1+2\cos(\frac{x\sqrt{3}}{2})\right]$
	1	$\sum_{\nu=-\infty}^{\infty} J_{3\nu+1}$	$\frac{1}{3}\left[1+2\cos(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3})\right]$
	2	$\sum_{\nu=-\infty}^{\infty} J_{3\nu+2}$	$\frac{1}{3}\left[1+2\cos(\frac{x\sqrt{3}}{2}-\frac{4\pi}{3})\right]$
4	0	$\sum_{\nu=-\infty}^{\infty} J_{4\nu}$	$\cos^2(\frac{x}{2})$
	1	$\sum_{\nu=-\infty}^{\infty} J_{4\nu+1}$	$\frac{1}{2}\sin(x)$
	2	$\sum_{\nu=-\infty}^{\infty} J_{4\nu+2}$	$\sin^2(\frac{x}{2})$
	3	$\sum_{\nu=-\infty}^{\infty} J_{4\nu+3}$	$-\frac{1}{2}\sin(x)$
5	0	$\sum_{\nu=-\infty}^{\infty} J_{5\nu}$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{2\pi}{5}))+2\cos(x\sin(\frac{4\pi}{5}))\right]$
	1	$\sum_{\nu=-\infty}^{\infty} J_{5\nu+1}$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{2\pi}{5})-\frac{2\pi}{5})+2\cos(x\sin(\frac{4\pi}{5})-\frac{4\pi}{5})\right]$
	2	$\sum_{\nu=-\infty}^{\infty} J_{5\nu+2}$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{2\pi}{5})-\frac{4\pi}{5})+2\cos(x\sin(\frac{4\pi}{5})-\frac{8\pi}{5})\right]$
	3	$\sum_{\nu=-\infty}^{\infty} J_{5\nu+3}$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{2\pi}{5})-\frac{6\pi}{5})+2\cos(x\sin(\frac{4\pi}{5})-\frac{12\pi}{5})\right]$
	4	$\sum_{\nu=-\infty}^{\infty} J_{5\nu+4}$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{2\pi}{5})-\frac{8\pi}{5})+2\cos(x\sin(\frac{4\pi}{5})-\frac{16\pi}{5})\right]$
6	0	$\sum_{\nu=-\infty}^{\infty} J_{6\nu}$	$\frac{1}{3}\left[1+2\cos(\frac{x\sqrt{3}}{2})\right]$
	1	$\sum_{\nu=-\infty}^{\infty} J_{6\nu+1}$	$\frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2})$
	2	$\sum_{\nu=-\infty}^{\infty} J_{6\nu+2}$	$\frac{1}{3}\left[1-\cos(\frac{x\sqrt{3}}{2})\right]$
	3	$\sum_{\nu=-\infty}^{\infty} J_{6\nu+3}$	0
	4	$\sum_{\nu=-\infty}^{\infty} J_{6\nu+4}$	$\frac{1}{3}\left[1-\cos(\frac{x\sqrt{3}}{2})\right]$
	5	$\sum_{\nu=-\infty}^{\infty} J_{6\nu+5}$	$-\frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2})$

2. Evaluation of Alternating Series: $\sum_{\nu=-\infty}^{\infty}(-1)^{\nu}J_{N\nu+p}(x)$

Theorem 2. *Infinite series of Bessel functions of the first kind of the form* $\sum_{\nu}(-1)^{\nu}J_{N\nu+p}(x)$ *where* $\nu, p, q \in \mathbb{Z},\ N \in \mathbb{Z}^{+}$ *and* $x \in \mathbb{C}$ *have following closed expression:*

$$\sum_{\nu=-\infty}^{\infty}(-1)^{\nu}J_{N\nu+p}(x) = \frac{1}{N}\sum_{q=0}^{N-1} e^{ix\sin\left(\frac{(2q+1)\pi}{N}\right)}e^{-i\frac{(2q+1)\pi p}{N}} \tag{2.1}$$

Proof. Consider the following Dirac comb on an infinite series of Bessel functions:

$$g(k) = \sum_{h=-\infty}^{\infty}\sum_{p=0}^{N-1}\delta(k-(h+\frac{p}{N})a)\sum_{\nu=-\infty}^{\infty}(-1)^{\nu}J_{N\nu+p}(kA) \tag{2.2}$$

where $\nu, h, p \in \mathbb{Z}$; $N \in \mathbb{Z}^{+}$; $k, a \in \mathbb{R}$; $A \in \mathbb{C}$.

Three summations are re-grouped into two:

$$\begin{aligned} g(k) &= \sum_{h=-\infty}^{\infty}\sum_{\alpha=-\infty}^{\infty}\delta(k-(h+\frac{\alpha}{N})a)(-1)^{h}J_{\alpha}(kA) \\ &= \sum_{h=-\infty}^{\infty}\sum_{\alpha=-\infty}^{\infty}\delta(k-(h+\frac{\alpha}{N})a)e^{-i(k-\frac{\alpha}{N}a)\frac{\pi}{a}}J_{\alpha}(kA) \\ &= \sum_{h=-\infty}^{\infty}\sum_{\alpha=-\infty}^{\infty}\delta(k-(h+\frac{\alpha}{N})a)e^{-i\pi k/a}e^{i\pi\alpha/N}J_{\alpha}(kA) \end{aligned} \tag{2.3}$$

The Dirac comb can be represented as a Fourier series (1.4):

$$\begin{aligned} g(k) &= \frac{1}{a}\sum_{m=-\infty}^{\infty}\sum_{\alpha=-\infty}^{\infty} e^{-i2\pi(k-\frac{\alpha}{N}a)m/a}e^{-i\pi k/a}e^{i\pi\alpha/N}J_{\alpha}(kA) \\ &= \frac{1}{a}\sum_{m=-\infty}^{\infty} e^{-i\pi k(2m+1)/a}\sum_{\alpha=-\infty}^{\infty} e^{i\alpha\pi(2m+1)/N}J_{\alpha}(kA) \end{aligned} \tag{2.4}$$

Using the Jacobi-Anger relation (1.6):

$$g(k) = \frac{1}{a}\sum_{m=-\infty}^{\infty} e^{-i\pi k(2m+1)/a}e^{ikA\sin(\pi(2m+1)/N)} \tag{2.5}$$

We split the summation over m into $m = ..., Nn, Nn+1, ..., Nn+(N-1), ...$:

$$\begin{aligned} g(k) = \frac{1}{N}\sum_{n=-\infty}^{\infty}\Big[& ... + e^{-i2\pi knN/a}e^{-i\pi k/a}e^{ikA\sin\left(\frac{1\pi}{N}\right)} \\ & + e^{-i2\pi knN/a}e^{-i3\pi k/a}e^{ikA\sin\left(\frac{3\pi}{N}\right)} + ... \\ & + e^{-i2\pi knN/a}e^{-i(2N-1)\pi k/a}e^{ikA\sin\left(\frac{(2N-1)\pi}{N}\right)} + ...\Big] \\ = \frac{1}{a}\sum_{n=-\infty}^{\infty} & e^{-i2\pi knN/a}\sum_{q=0}^{N-1} e^{-i(2q+1)\pi k/a}e^{ikA\sin\left(\frac{(2q+1)\pi}{N}\right)} \end{aligned} \tag{2.6}$$

Using (1.4) again:

$$g(k) = \frac{1}{N} \sum_{l=-\infty}^{\infty} \delta(k - \frac{a}{N}l) \sum_{q=0}^{N-1} e^{ikA \sin(\frac{(2q+1)\pi}{N})} e^{-i(2q+1)\pi k/a} \tag{2.7}$$

Substitute $k = l\frac{a}{N}$, as f is non-zero where a delta function exists.

$$g(k) = \frac{1}{N} \sum_{l=-\infty}^{\infty} \delta(k - \frac{a}{N}l) \sum_{q=0}^{N-1} e^{ikA \sin(\frac{(2q+1)\pi}{N})} e^{-i\frac{(2q+1)\pi l}{N}} \tag{2.8}$$

We split the summation into $l = ..., Nh, Nh+1, ..., Nh+(N-1), ...$ then regroup:

$$g(k) = \frac{1}{N} \sum_{h=-\infty}^{\infty} \sum_{p=0}^{N-1} \delta(k - (h + \frac{p}{N})a)(-1)^h \sum_{q=0}^{N-1} e^{ikA \sin(\frac{(2q+1)\pi}{N})} e^{-i\frac{(2q+1)\pi p}{N}} \tag{2.9}$$

Initial expression (2.2) must equal (2.9):

$$\begin{aligned} &\sum_{h=-\infty}^{\infty} \sum_{p=0}^{N-1} \delta(k - (h + \frac{p}{N})a) \sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{N\nu+p}(kA) = \\ &\frac{1}{N} \sum_{h=-\infty}^{\infty} \sum_{p=0}^{N-1} \delta(k - (h + \frac{p}{N})a)(-1)^h \sum_{q=0}^{N-1} e^{ikA \sin(\frac{(2q+1)\pi}{N})} e^{-i\frac{(2q+1)\pi p}{N}} \end{aligned} \tag{2.10}$$

The variable a can take on any real value thus the expression holds for all values of kA. The equivalent Dirac lattices on each side can be disregarded.

$$\therefore \quad \sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{N\nu+p}(x) = \frac{1}{N} \sum_{q=0}^{N-1} e^{ix \sin(\frac{(2q+1)\pi}{N})} e^{-i\frac{(2q+1)\pi p}{N}} \tag{2.11}$$

$\square$

Corollary 2. *From Theorem 2 we comprise a table of closed form expressions:*

N	p	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{N\nu+p}(x)$	$\frac{1}{N}\sum_{q=0}^{N-1} e^{ix\sin(\frac{(2q+1)\pi}{N})}e^{-i\frac{(2q+1)\pi p}{N}}$	
1	0	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{\nu}(x)$	1	[6, 7]
2	0	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{2\nu}(x)$	$\cos(x)$	[6]
	1	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{2\nu+1}(x)$	$\sin(x)$	[6]
3	0	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{3\nu}(x)$	$\frac{1}{3}\left[1+2\cos(\frac{x\sqrt{3}}{2})\right]$	
	1	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{3\nu+1}(x)$	$-\frac{1}{3}\left[1-2\cos(\frac{x\sqrt{3}}{2}-\frac{\pi}{3})\right]$	
	2	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{3\nu+2}(x)$	$\frac{1}{3}\left[1+2\cos(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3})\right]$	
4	0	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{4\nu}(x)$	$\cos(\frac{x}{\sqrt{2}})$	
	1	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{4\nu+1}(x)$	$\frac{1}{\sqrt{2}}\sin(\frac{x}{\sqrt{2}})$	
	2	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{4\nu+2}(x)$	0	
	3	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{4\nu+3}(x)$	$\frac{1}{\sqrt{2}}\sin(\frac{x}{\sqrt{2}})$	
5	0	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{5\nu}(x)$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{\pi}{5}))+2\cos(x\sin(\frac{3\pi}{5}))\right]$	
	1	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{5\nu+1}(x)$	$-\frac{1}{5}\left[1+2\cos(x\sin(\frac{\pi}{5})+\frac{4\pi}{5})+2\cos(x\sin(\frac{3\pi}{5})+\frac{2\pi}{5})\right]$	
	2	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{5\nu+2}(x)$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{\pi}{5})+\frac{8\pi}{5})+2\cos(x\sin(\frac{3\pi}{5})+\frac{4\pi}{5})\right]$	
	3	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{5\nu+3}(x)$	$-\frac{1}{5}\left[1+2\cos(x\sin(\frac{\pi}{5})+\frac{12\pi}{5})+2\cos(x\sin(\frac{3\pi}{5})+\frac{6\pi}{5})\right]$	
	4	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{5\nu+4}(x)$	$\frac{1}{5}\left[1+2\cos(x\sin(\frac{\pi}{5})+\frac{16\pi}{5})+2\cos(x\sin(\frac{3\pi}{5})+\frac{8\pi}{5})\right]$	
6	0	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{6\nu}(x)$	$\frac{1}{3}\left[\cos(x)+2\cos(\frac{x}{2})\right]$	
	1	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{6\nu+1}(x)$	$\frac{1}{3}\left[\sin(x)+\sin(\frac{x}{2})\right]$	
	2	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{6\nu+2}(x)$	$-\frac{1}{3}\left[\cos(x)-\cos(\frac{x}{2})\right]$	
	3	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{6\nu+3}(x)$	$-\frac{1}{3}\left[\sin(x)-2\sin(\frac{x}{2})\right]$	
	4	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{6\nu+4}(x)$	$\frac{1}{3}\left[\cos(x)-\cos(\frac{x}{2})\right]$	
	5	$\sum_{\nu=-\infty}^{\infty} (-1)^\nu J_{6\nu+5}(x)$	$\frac{1}{3}\left[\sin(x)+\sin(\frac{x}{2})\right]$	

In Corollary 1, 2 we have noted references to previous reported expressions for $N = 1, 2$ that are closely related. For $N = 1$ the expression is given using the generating function $e^{1/2(t-1/t)x} = \sum_{\nu=-\infty}^{\infty} J_\nu(x) t^\nu$ when $t = 1$. For $N = 2$, the expressions can be re-written such that $\sum_{\nu=-\infty}^{\infty}(-1)^\nu J_{2\nu}(x) = J_0(x) + 2\sum_{\nu=1}^{\infty}(-1)^\nu J_{2\nu}(x) = \cos x$ and $\sum_{\nu=-\infty}^{\infty}(-1)^\nu J_{2\nu+1}(x) = \sum_{\nu=0}^{\infty}(-1)^\nu J_{2\nu+1}(x) = \sin x$ using the identity $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

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