Local quantum field logic

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Abstract

Algebraic quantum field theory, or AQFT for short, is a rigorous analysis of the structure of relativistic quantum mechanics. It is formulated in terms of a net of operator algebras indexed by regions of a Lorentzian manifold. In several cases the mentioned net is represented by a family of von Neumann algebras, concretely, type III factors. Local quantum field logic arises as a logical system that captures the propositional structure encoded in the algebras of the net. In this framework, this work contributes to the solution of a family of open problems, emerged since the 30s, about the characterization of those logical systems which can be identified with the lattice of projectors arising from the Murray-von Neumann classification of factors. More precisely, based on physical requirements formally described in AQFT, an equational theory able to characterize the type III condition in a factor is provided. This equational system motivates the study of a variety of algebras having an underlying orthomodular lattice structure. A Hilbert style calculus, algebraizable in the mentioned variety, is also introduced and a corresponding completeness theorem is established.

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Introduction

Quantum field theory (QFT) is a set of tools that combines three areas of the modern physic: quantum theory, field theory and relativity. This theory underlies elementary particle physics and supplies essentials tools to other branches of the theoretical physics, such as condensed matter physics, statistical mechanics, astrophysics etc. Although quantum field theories have been developed and used for more than 70 years, a generally accepted rigorous description of the structure of these theories has not yet been established. With the aim to establish a consistent mathematical framework for the treatment of QFT, several axiomatic frameworks were formulated since the mid-fifties. One of these is known as *Algebraic quantum field theory* (AQFT). Its origin lies in a seminal work of Haag and Kastler dating back to the early 1960s [11]. On this picture, a collection of observables is assigned to each open region of the Lorentzian spacetime. These observables correspond to physical quantities that can be measured by experiments causally confined to that region. The collection of observables comes equipped with an intrinsic operator algebra structure. AQFT exists in two versions: the original Haag-Kastler formalism [11] based on C^*-algebras and the Haag-Araki formalism [1, 12] which uses von Neumann algebras. Here we adopt the second model mentioned above. Formally, the Haag-Araki model is based on a family $\{\mathcal{N}(\mathcal{O})\}_{\mathcal{O}}$, called *net of local observable algebras over the spacetime*, where $\mathcal{O}$ is an open bounded region of a Lorentzian manifold and $\mathcal{N}(\mathcal{O})$ is a type III factor in the Murray-von Neumann classification of factors [22, 27, 28, 29, 36]. Each algebra $\mathcal{N}(\mathcal{O})$ mathematically represents the set of physical properties in the region $\mathcal{O}$ of the spacetime. In this framework the aim of this work is to develop a logical system describing the propositional structure arising from the algebras of the net.

As is widely known, the elementary propositional structure associated to each algebra $\mathcal{N}(\mathcal{O})$ is encoded in the orthomodular lattice $\mathcal{P}(\mathcal{N}(\mathcal{O}))$ defined by its projectors, each one of them, representing a true/false assertion related to a physical property in the region $\mathcal{O}$. However, the only orthomodularity condition on $\mathcal{P}(\mathcal{N}(\mathcal{O}))$ is not enough to distinguish the most important formal requirement of the algebras of the net, namely, the type III factor condition in the Murray-von Neumann classification. *Local quantum field logic*, or LQF-logic for short, is an expansion of the orthomodular logic that captures the type III factor condition of the algebras of the net. More precisely, this logical system is based on a necessary and sufficient condition formulated by a set of equations on an expanded language of the variety of orthomodular lattices that, when imposed on the projector lattice of a von Neumann factor, implies that this is a type III factor. In this perspective, this work attempts to contribute to the solution of a family of open questions emerged since the 30's about whether it might be possible to establish lattice theoretical conditions in order to characterize each factor of the Murray-von Neumann classification [5, 15, 16].

The paper is structured as follows. Section 1 contains generalities on universal algebra and lattice theory. Some technical results about operations expanding the orthomodular structure are also given. In Section 2 an outline about operator algebras is provided. Moreover, useful facts linking elements of the Murray-von Neumann dimension theory and von Neumann lattices are established. Section 3 provides a detailed motivation of the LQF-logic. In order to do this the physical framework underlying AQFT is briefly described. In Section 4 the action of the partial isometries on the lattice of projectors of a type III factor is studied. This allows us to transplant the Murray-von Neumann equivalence of type III factors in the language of the lattice of projectors. Furthermore, partial isometries define a natural expansion of the language of the orthomdular lattices in which, a set of equations characterizing the type III factor, is formulated. Based on this equational theory, in Section 5, a variety of algebras, called LQF-algebras, is introduced and studied. Via LQF-algebras, alternative proofs of the crucial characteristics regarding to the non atomicity and non modularity of the type III factors are given. Section 6 is devoted to the study of the congruences and filters in LQF-algebras. Finaly, in Section 7, a Hilbert style calculus for LFQ-logic is introduced and a completeness theorem for this calculus is also established.

1 Basic Notions

We first recall from [7] some notion of universal algebra that plays an important role along this article. Let τ be a type of algebras. We denote by $Term_\tau(X)$ the *absolutely free algebra* of type τ built from the denumerable set of variables $X = \{x_1, x_2, ...\}$. Each element of $Term_\tau(X)$ is referred to as a *τ-term*. We denote by $Comp(t)$ the complexity of the term t. An *equation of type τ* is an expression of the form $t = s$ where $t, s \in Term_\tau(X)$. For $t \in Term_\tau(X)$ we usually write $t(x_1, \ldots, x_n)$ to indicate that the variables occurring in t are among $x_1, \ldots, x_n$. A *variety* is a class of algebras of the same type defined by a set of equations. Let $\mathcal{A}$ be a variety of algebras of type τ. In this case, $Term_\tau(X)$ is also denoted by $Term_{\mathcal{A}}(X)$ and each element of $Term_\tau(X)$ is indistinctly referred to as a *τ-term*, *$\mathcal{A}$-term* or simply *term* when there is no confusion. A *$\mathcal{A}$-homomorphism* is a τ-operation preserving map between two algebras of $\mathcal{A}$. If $A \in \mathcal{A}$ then we denote by 1_A the identity $\mathcal{A}$-homomorphism on A. If $\mathcal{B}$ be a subclass of $\mathcal{A}$ then we denote by $\mathcal{V}(\mathcal{B})$ the subvariety of $\mathcal{A}$ generated by the class $\mathcal{B}$, i.e. $\mathcal{V}(\mathcal{B})$ is the smallest subvariety of $\mathcal{A}$ containing $\mathcal{B}$. Let $A \in \mathcal{A}$. Each term $t(x_1, \ldots, x_n)$ in $Term_\tau(X)$ canonically defines an n-ary operation on A denoted by t^A. If $a_1, \ldots, a_n \in A$ then we denote by $t^A(a_1, \ldots, a_n)$ the result of the application of the term operation t^A to the elements $a_1, \ldots, a_n$. A *valuation* in the algebra A is a function of the form $v : X \to A$. By induction on $Comp(t)$ any valuation v in A can be uniquely extended to an $\mathcal{A}$-homomorphism $v : Term_\tau(X) \to A$, that is, $v(t(t_1, \ldots, t_n)) = t^A(v(t_1), \ldots, v(t_n))$ for $t_1, \ldots, t_n \in Term_\tau(X)$. Thus, valuations are identified with $\mathcal{A}$-homomorphisms from the absolutely free algebra. If $t, s \in Term_\tau(X)$, $A \models t = s$ means that for each valuation v in A, $v(t) = v(s)$ and $\mathcal{A} \models t = s$ means that for each $A \in \mathcal{A}$, $A \models t = s$.

For each algebra $A \in \mathcal{A}$, we denote by $Con(A)$ the congruence lattice of A, the diagonal congruence is denoted by Δ_A and the largest congruence A^2 is denoted by ∇_A. A congruence θ is called *factor congruence* iff there is a congruence θ^* on A such that, $\theta \wedge \theta^* = \Delta_A$, $\theta \vee \theta^* = \nabla_A$ and θ permutes with θ^*. In this case the pair (θ, θ^*) is called a *pair of factor congruences* on A and we can prove that $A \cong A/\theta \times A/\theta^*$. The algebra A is *directly indecomposable* iff A is not isomorphic to a product of two non trivial algebras or, equivalently, if Δ_A, ∇_A are the only factor congruences in A. If $\mathcal{A}$ is a variety then we denote by $\mathcal{DI}(\mathcal{A})$ the class of directly indecomposable algebras of $\mathcal{A}$. An algebra A has the *congruence extension property* (CEP) iff for each subalgebra B and $\theta \in Con(B)$ there is a $\phi \in Con(A)$ such that $\theta = \phi \cap A^2$. A variety $\mathcal{A}$ satisfies CEP iff every algebra in $\mathcal{V}$ has the CEP.

The variety $\mathcal{A}$ is said to be *congruence distributive* iff for each $A \in \mathcal{A}$, $Con(A)$ is a distributive lattice. If for each $A \in \mathcal{A}$ the congruences of $Con(A)$ are permutable then we said that $\mathcal{A}$ is a *congruence permutable* variety. The variety $\mathcal{A}$ is *arithmetical* iff it is both congruence distributive and congruence permutable variety.

Let A be an algebra. We say that A is *subdirect product* of a family of $(A_i)_{i\in I}$ of algebras if there exists an embedding $f : A \to \prod_{i\in I} A_i$ such that $\pi_i f : A \to A_i$ is a surjective homomorphism for each $i \in I$ where π_i is the ith-projection onto A_i. The algebra A is *subdirectly irreducible* iff it is trivial or there is a minimum congruence in $Con(A) - \Delta_A$. We denote by $\mathcal{SI}(\mathcal{A})$ the class of subdirectly irreducible algebras of the variety $\mathcal{A}$. It is clear that a subdirectly irreducible algebra is directly indecomposable. Then, for each variety $\mathcal{A}$, we have that

$$\mathcal{SI}(\mathcal{A}) \subseteq \mathcal{DI}(\mathcal{A}). \tag{1}$$

An important result by Birkhoff is the following subdirect representation theorem.

Theorem 1.1 [7, Theorem 8.6] *Let $\mathcal{A}$ be a variety. Then every algebra $A \in \mathcal{A}$ is a subdirect product of subdirectly irreducible algebras of $\mathcal{A}$.* □

Let us notice that, by the above theorem and Eq.(1), in each variety $\mathcal{A}$ the class $\mathcal{SI}(\mathcal{A})$ and the class $\mathcal{DI}(\mathcal{A})$ rule the valid equations in $\mathcal{A}$. That is, for any pair of terms $t, s \in Term_{\mathcal{A}}(X)$ we have that

$$\mathcal{A} \models t = s \quad \text{iff} \quad \mathcal{DI}(\mathcal{A}) \models t = s \quad \text{iff} \quad \mathcal{SI}(\mathcal{A}) \models t = s. \tag{2}$$

An algebra A is said to be *simple* iff $Con(A) = \{\Delta_A, \nabla_A\}$. The class of simple algebras in the variety $\mathcal{A}$ is denoted by $\mathcal{S}im(\mathcal{A})$. The algebra A is *semisimple* iff A is a subdirect product of simple algebras. A variety $\mathcal{A}$ is semisimple iff each algebra of $\mathcal{A}$ is semisimple. A *discriminator term* for the algebra A is a term $t(x, y, z)$ such that

$$t^A(x, y, z) = \begin{cases} x, & x \neq y, \\ z, & x = y. \end{cases}$$

A variety $\mathcal{A}$ is a *discriminator variety* iff there exists a subclass of algebras $\mathcal{K}$ with a common discriminator term $t(x, y, z)$ such that $\mathcal{A} = \mathcal{V}(\mathcal{K})$.

Theorem 1.2 [7, Theorem 9.4] *Let $\mathcal{K}$ be a class of algebras of type τ and $t(x, y, z)$ be a common discriminator τ-term for the class $\mathcal{K}$. If we consider the generated variety $\mathcal{A} = \mathcal{V}(\mathcal{K})$ then*

1. *$\mathcal{A}$ is an arithmetical semisimple variety.*
2. *$\mathcal{DI}(\mathcal{A}) = \mathcal{SI}(\mathcal{A}) = \mathcal{S}im(\mathcal{A})$.*

□

Now we recall from [19] and [26] some notion about orthomodular lattices. A *lattice with involution* [18] is an algebra $\langle L, \vee, \wedge, \neg\rangle$ such that $\langle L, \vee, \wedge\rangle$ is a lattice and $\neg$ is a unary operation on L that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. An *orthomodular lattice* is an algebra $\langle L, \wedge, \vee, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ that satisfies the following conditions:

1. $\langle L, \wedge, \vee, \neg, 0, 1\rangle$ is a bounded lattice with involution,
2. $x \wedge \neg x = 0$,
3. $x \vee (\neg x \wedge (x \vee y)) = x \vee y$. (*orthomodular law*)

We denote by $\mathcal{OML}$ the variety of orthomodular lattices. Upon defining the $\mathcal{OML}$-term tRs as

$$tRs \quad = \quad (t \wedge s) \vee (\neg t \wedge \neg s) \tag{3}$$

an important characterization of the equations in $\mathcal{OML}$ is given by:

$$\mathcal{OML} \models t = s \quad iff \quad \mathcal{OML} \models tRs = 1. \tag{4}$$

Therefore we can safely assume that all $\mathcal{OML}$-equations are of the form $t = 1$, where $t \in Term_{\mathcal{OML}}(X)$.

Remark 1.3 It is clear that the equational characterization given in Eq.(4) is satisfied for each variety $\mathcal{A}$ admitting terms of the language that define, on each $A \in \mathcal{A}$, operations $\vee$, $\wedge$, $\neg$, $0, 1$ such that $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is an orthomodular lattice.

Let L be an orthomodular lattice. An element $a \in L$ is an *atom* iff $a \neq 0$ but $x \leq a$ implies, $x = 0$ or $x = a$. Dually, the notion of *coatom* is established. Let us notice that a is an atom iff $\neg a$ is a coatom. Two elements a, b in L are *orthogonal* (noted $a \perp b$) iff $a \leq \neg b$. For each $a \in L$ let us consider the interval $[0, a] = \{x \in L : 0 \leq x \leq a\}$ and the unary operation on $[0, a]$ given by $\neg_a x = \neg x \wedge a$. As one can readly realize, the structure

$$[0, a]_L = \langle [0, a], \wedge, \vee, \neg_a, 0, a\rangle \tag{5}$$

is an orthomodular lattice. Let $a \in L$. Then the mapping $\mu_a : L \to L$ given by

$$\mu_a(x) = a \wedge (\neg a \vee x) \tag{6}$$

is called the *Sasaki projection* onto $[0, a]$. In [19, p. 156] it is proved that

$$x = \mu_a(x) \quad \text{iff} \quad x \leq a. \tag{7}$$

For elements $a, b \in L$ we said that a *commutes* with b, in symbols aCb, iff $a = (a \vee b) \wedge (a \vee \neg b)$. It is not very hard to see that aCb iff bCa. We also note that aCb, $\neg aCb$, $aC\neg b$, $\neg aC\neg b$ are equivalent conditions in an orthomodular lattices.

Boolean algebras are orthomodular lattices satisfying the *distributive law* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We denote by $\mathbf{2}$ the Boolean algebra of two elements. Let L be an orthomodular lattice. Given a, b, c in L, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, (a,b,c)D^* hold for all permutations of a, b, c. An element z of L is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(L)$ the set of all central elements of L and it is called the *center* of L.

Proposition 1.4 *Let L be an orthomodular lattice. Then we have:*

1. *$Z(L)$ is a Boolean sublattice of L* [26, Theorem 4.15].
2. *$z \in Z(L)$ iff for each $a \in L$, $a = (a \wedge z) \vee (a \wedge \neg z)$* [26, Lemma 29.9].

□

Let L be an orthomodular lattice and $a \in L$. One can define the *central cover* of a, as

$$e(a) = \bigwedge \{z \in Z(L) : a \leq z\} \tag{8}$$

if such infimum exists. By straightforward calculation we can see that

$$e_d(a) = \neg e(\neg a) = \bigvee \{z \in Z(L) : z \leq a\}. \tag{9}$$

if such supremum exists. This element is called the *dual central cover* of a. For example, if L is a complete orthomodular then $e(a)$ and $e_d(a)$ exist for each $a \in L$. Moreover, they are central elements of L [26, Lemma 29.16].

Proposition 1.5 *Let L be an orthomodular lattice and $z \in Z(L)$. Then:*

1. *The binary relation θ_z on L defined by $a\theta_z b$ iff $a \wedge z = b \wedge z$ is a congruence of L.*
2. *The function $f_z : L/\theta_z \to [0, z]_L$ such that $f_z([a]_z) = a \wedge z$ is a $\mathcal{OML}$-isomorphism.*
3. *L is $\mathcal{OML}$-isomorphic to $L/\theta_z \times L/\theta_{\neg z}$ i.e., $(\theta_z, \theta_{\neg z})$ is a pair of factor congruences on L.*
4. *The map $z \to \theta_z$ is a lattice isomorphism between $Z(L)$ and the Boolean subalgebra of $Con_{\mathcal{OML}}(L)$ of factor congruences.*
5. *L is directly indecomposable iff $Z(L) = \{0, 1\}$.*

□

For a proof of Proposition 1.5 we refer to [6, §4]. Let us notice that the above proposition say that in an orthomodular lattice there exists a one to one correspondence between its central elements and the factor congruences. In other words, the presence of non trivial central elements in an orthomodular lattice determines each possible decomposition of the lattice in a direct product of two orthomodular lattices.

Proposition 1.6 *Let L be a directly indecomposable orthomodular lattice. Then the operation*

$$w_0(x) = \begin{cases} 0, & x = 0, \\ 1, & \textit{otherwise}. \end{cases}$$

is the unique operation on L that satisfies the following conditions:

$$w_0(0) = 0, \tag{10}$$

$$X \leq w_0(X), \tag{11}$$

$$Y = (Y \wedge w_0(X)) \vee (Y \wedge w_0(X)^{\perp}). \tag{12}$$

Proof: We first note that w_0 satisfy Eq.(10), Eq.(11) and Eq.(12). Let v be an operation on L satisfying the mentioned equations. Combining Eq.(12), Proposition 1.4-2 and Proposition 1.5-5 we can see that $Imag(v) \subseteq Z(\mathcal{P}(\mathcal{N})) = \{0, 1_{\mathcal{H}}\}$. Then, by Eq.(11), $v(x) = 1$ iff $x \neq 0$. Hence $v = w_0$.

□

Let L be an orthomodular lattice. For elements $a, b \in L$ we say that a is a *complement* of b iff $a \vee b = 1$ and $a \wedge b = 0$.

Proposition 1.7 [19, Proposition 6]. *Let L be an orthomodular lattice and $a \in L$. Then the complements of a are precisely the elements of the image of the following operation*

$$L \ni x \mapsto c_a(x) = \big(x \wedge \neg(x \wedge a)\big) \vee \neg(x \vee a). \tag{13}$$

Let L be an orthomodular lattice and $a, b \in L$. We say that *a and b are perspective* and we write $a \sim_p b$ iff they have a common complement, i.e. if there exists $c \in L$ such that $a \vee c = 1 = b \vee c$ and $a \wedge c = 0 = b \wedge c$.

Proposition 1.8 *Let L be an orthomodular lattice and $a, b \in L$. Then the following statement are equivalent:*

1. *There exists $x \in L$ such that $a \vee x = b \vee x$ and $a \wedge x = b \wedge x$.*
2. *$a \sim_p b$.*

Proof: If we assume that $a \vee x = b \vee x$ and $a \wedge x = b \wedge x$ then, by Proposition 1.7, $c_a(x) = (x \wedge \neg(x \wedge a)) \vee \neg(x \vee a) = (x \wedge \neg(x \wedge b)) \vee \neg(x \vee b) = c_b(x)$ is a common complement of a and b. The other direction is immediate.

□

Proposition 1.9 *Let L be an orthomodular lattice and $a, b \in L$ such that $a \sim_p b$. Then,*

1. *For each $z \in Z(L)$, $a \wedge z \sim_p b \wedge z$.*
2. *If $z \in Z(L)$ and $z \leq a$ then $z \leq b$.*

3. $e_d(a) = e_d(b)$ *whenever such elements exist in* L.

Proof: Let us assume that $a \sim_p b$ in L.

1) Let $c \in L$ be a common complement of a and b and $z \in C(L)$. Note that $(a \wedge z) \wedge c = 0 = a \wedge (c \wedge z)$ and $(a \wedge z) \vee c = (a \vee c) \wedge (z \vee c) = (z \vee c) = (b \vee c) \wedge (z \vee c)$. Then, by Proposition 1.8, $a \wedge z \sim_p b \wedge z$.

2) By item 1 we have that $z \sim_p b \wedge z$ because $z = a \wedge z$. Since z is a central element then, necessarily, $\neg z$ is the common complement of z and $b \wedge z$. Thus, $1 = \neg z \vee (b \wedge z) = \neg z \vee b$ and $z = z \wedge 1 = z \wedge (\neg z \vee b) = z \wedge b$. Hence $z \leq b$.

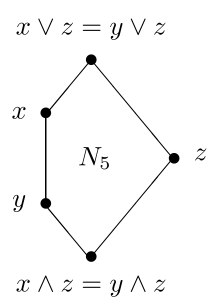
3) By item 1 we have that $e_d(a) = a \wedge e_d(a) \sim_p b \wedge e_d(a)$. Since $e_d(a) \in Z(L)$ then $\neg e_d(a)$ is the common complement of $e_d(a)$ and $b \wedge e_d(a)$. Thus, $b \wedge e_d(a) \in Z(L)$ and $b \wedge e_d(a) = e_d(b \wedge e_d(a)) = e_d(b) \wedge e_d(a)$. In this way, $1 = \neg e_d(a) \vee \big(e_d(b) \wedge e_d(a)\big) = \neg e_d(a) \vee e_d(b)$ proving that $e_d(a) \leq e_d(b)$. Similarly we can prove that $e_d(b) \leq e_d(a)$. Hence, $e_d(a) = e_d(b)$.

□

Let L be an orthomodular lattice and $a, b \in L$. We say that (a, b) is a *modular pair*, in symbols $(a, b)M$, iff for every $x \leq b$, $(x \vee a) \wedge b = x \vee (a \wedge b)$. By substituting x with $x \wedge b$, $(a, b)M$ is equivalent to the following equation:

$$(x \wedge b) \vee (a \wedge b) = ((x \wedge b) \vee a) \wedge b. \tag{14}$$

The lattice L is called *modular* iff $(a, b)M$ holds for all elements a and b of L. In other words, L is modular iff it satisfies Eq.(14). We can show that if $a \perp b$ then $(a, b)M$ holds. This property motivated Kaplansky to introduce the name "orthomodular" as an abbreviation for "orthogonal pairs are modular". In order to characterize the modularity we introduce the following lattice known as N_5.



Proposition 1.10 *Let* L *be an orthomodular lattice. Then the following statements are equivalent:*

1. L *is non modular.*

2. *There exists* $x, y \in L$ *such that* $x < y$ *and* $x \sim_p y$.

3. N_5 *is a sublattice* L.

Proof: $1 \iff 2$) Immediate. $1 \iff 3$) See [10, §2, Theorem 2].

□

The technical results provided by the following two propositions turn out to be useful in the rest of the paper.

Proposition 1.11 *Let L be an orthomodular lattice, $a \in L$ such that $a \neq 0$ and let us consider a function $w : [0,a]_L \rightarrow L$. Then the following statement are equivalent:*

1. $w(\mu_a(\neg x)) = \neg w(x \wedge a)$.

2. *For each $x \in [0,a]$, $w(\neg_a x) = \neg w(x)$ i.e. w preserves relatives complements related to a.*

Proof: 1 $\Longrightarrow$ 2) Let us suppose that $x \leq a$. Since $\neg a \leq \neg x$ then we have that $\neg w(x) = \neg w(x \wedge a) = w(\mu_a(\neg x)) = w(a \wedge (\neg a \vee \neg x)) = w(a \wedge \neg x) = w(\neg_a x)$.

2 $\Longrightarrow$ 1) Let $x \in L$. Since $x \wedge a \in [0,a]$ then $\neg w(x \wedge a) = w(\neg_a(x \wedge a)) = w(a \wedge \neg(x \wedge a)) = w(a \wedge (\neg a \vee \neg x)) = w(\mu_a(\neg x))$.

□

Proposition 1.12 *Let L be an orthomodular lattice and $a \in L$ such that $a \neq 0$. Suppose that there exist two unary order preserving operations w_a and w_a^* on L such that $w_a w_a^* = id_L$ and $w_a^* w_a = \mu_a$. Then,*

1. $w_a^* : L \rightarrow [0,a]_L$ *is an order isomorphism.*

2. *The restriction $w_a \restriction_{[0,a]_L} \rightarrow L$ is an order isomorphism.*

3. $w_a^*(a) = a$ *iff* $a = 1$.

Proof: 1) We first prove that $Imag(w_a^*) = [0,a]$. Let $x \in L$. Then $w_a^*(x) = w_a^*\big(w_a w_a^*(x)\big) = \big(w_a^* w_a\big) w_a^*(x) = \mu_a(w_a^*(x)) \in [0,a]$. Let $y \in [0,a]_L$. By Eq.(7) and by hypothesis we have that $y = \mu_a(y) = w_a^* w_a(y)$ and then, $w_a(y)$ is a preimage of y by w_a^*. Thus, $Imag(w_a^*) = [0,a]$. Let $x, y \in L$ and let us assume that $w_a^*(x) = w_a^*(y)$. Then, by hypothesis, $x = w_a w_a^*(x) = w_a w_a^*(y) = y$. Therefore, w_a^* is injective. Since w_a^* is an order preserving map, by the above conditions, $w_a^* : L \rightarrow [0,a]_L$ is an order isomorphism.

2) If $x \neq y$ in $[0,a]_L$ then, by hypothesis, $w_a^* w_a(x) = \mu_a(x) = x \neq y = \mu_a(y) = w_a^* w_a(y)$. Thus, by item 1, $w_a(x) \neq w_a(y)$. Therefore, the restriction $w_a \restriction_{[0,a]_L}$ is an injective map. Let $y \in L$. If $x = w_a^*(y)$ then $w_a(x) = w_a w_a^*(y) = y$. Thus w_a is a surjective map. Since w_a is an order preserving map, by the above conditions, the restriction $w_a \restriction_{[0,a]_L} \rightarrow L$ is an order isomorphism.

3) Let us suppose that $w_a^*(a) = a$. Then $a = w_a w_a^*(a) = w_a(a)$ and, by item 2, $a = 1$. For the converse, if we suppose that $a = 1$ then, by item 1, $w_1^*(1) = 1$.

□

2 Murray-von Neumann dimension theory and von Neumann lattices

In this section we study some general properties about the lattice of projectors of a von Neumann algebra arising from the Murray-von Neumann classification of factors. In order to do this we first summarize some basic notions about the Murray-von Neumann dimension theory.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Two elements x, y are said to be *orthogonal* iff $\langle x, y \rangle = 0$ which can be written as $x \perp y$. For a set $X \subseteq \mathcal{H}$ the *orthogonal complement* of X is given by

$$X^{\perp} = \{y \in \mathcal{H} : \forall x \in X, \;\; \langle x, y \rangle = 0\}.$$

Two subspaces X, Y are *orthogonal*, in symbols $X \perp Y$, iff $X \subseteq Y^{\perp}$ (equivalently $X^{\perp} \subseteq Y$). If X, Y are subspaces of $\mathcal{H}$ then the *sum of* X *and* Y is given by the subspace $X + Y = \{x + y : x \in X, y \in Y\}$. In particular, $\mathcal{H}$ is said to be the *direct sum* of X and Y iff $\mathcal{H} = X + Y$ and $X \cap Y = \{0\}$. In this case we write $\mathcal{H} = X \oplus Y$. Let us remark that $\mathcal{H} = X \oplus Y$ iff any $a \in \mathcal{H}$ has a unique decomposition $a = a_X + a_Y$ where $a_X \in X$ and $a_Y \in Y$. The following proposition provides well known results on Hilbert spaces (see [13, §12]).

Proposition 2.1 *Let $\mathcal{H}$ be a Hilbert space and X, Y subspaces of $\mathcal{H}$. Then,*

1. *$X^{\perp}$ is a closed subspace of $\mathcal{H}$.*
2. *If X is closed then $\mathcal{H} = X \oplus X^{\perp}$.*
3. *If X and Y are closed and $X \perp Y$ then $X + Y$ is the closed subspace of $\mathcal{H}$ generated by $X \cup Y$.*

$\square$

An indexed subset $\{e_i\}_i$ of $\mathcal{H}$ is said to be *orthonormal* iff, $\|e_i\| = 1$ for all i and $e_i \perp e_j$ for all $i \neq j$. The space $\mathcal{H}$ is *separable* iff it has a countable basis, i.e., there exists a denumerable orthonormal set $\{e_i\}_{i=1}^{\infty}$ such that for each $x \in \mathcal{H}$, $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$.

From here on, we confine ourselves to separable Hilbert spaces.

A linear operator $L : \mathcal{H} \to \mathcal{H}$, or operator for short, is said to be *bounded* iff $\sup_{\|x\| \leq 1} \{\|L(x)\|\} < \infty$. If $(e_i)_{i \in I}$ is a basis of $\mathcal{H}$ where $I \subseteq \mathbb{N}$ then the *trace* of L is defined as $tr(L) = \sum_{i \in I} \langle L(e_i), e_i \rangle$. We can prove that the trace of L is independent of the choice of basis. The *kernel* of L is defined as the set $Ker(L) = \{x \in \mathcal{H} : L(x) = 0\}$. Let us remark that $Ker(L)$ and the image of L, denoted by $Imag(L)$, are subspaces of $\mathcal{H}$. The set of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. The identity operator on $\mathcal{H}$ is denoted by $1_{\mathcal{H}}$ and immediately follows that it is a bounded operator. For each operator $L \in \mathcal{B}(\mathcal{H})$ we denote by L^* the *adjoint of* L i.e., the unique operator in $\mathcal{B}(\mathcal{H})$ such that $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$ for each $x, y \in \mathcal{H}$.

The relationship between $Imag(L)$ and $Ker(L^*)$ is given by the following proposition.

Proposition 2.2 *Let $L \in \mathcal{B}(\mathcal{H})$. Then,*

1. $Ker(L) = Imag^{\perp}(L^*)$.
2. $Ker^{\perp}(L) = \overline{Imag(L^*)}$ *i.e. the closure of* $Imag(L^*)$.

$\square$

Remark 2.3 Combining Proposition 2.1 and Proposition 2.2 we see that for each $L \in \mathcal{B}(\mathcal{H})$, $Ker(L)$ and $Ker^{\perp}(L)$ are closed subspaces of $\mathcal{H}$.

The set $\mathcal{B}(\mathcal{H})$ can be endowed with a linear space structure over $\mathbb{C}$ by considering the sum operation $(L+M)(x) = L(x) + M(x)$ and a product by complex scalars defined by $(\lambda L)(x) = \lambda L(x)$. In $\mathcal{B}(\mathcal{H})$ we can also define the product operation $L \cdot M$ given by the composition $(L \cdot M)(x) = L(M(x))$. For the sake of simplicity the product operation $L \cdot M$ is written LM.

An operator P on $\mathcal{H}$ is an *orthogonal projection* or simply a *projector* iff $P = P^* = P^2$. Note that if P is a projector on $\mathcal{H}$, then $Imag(P)$ is a closed subspace of $\mathcal{H}$. Reciprocally, if E is a closed subspace of $\mathcal{H}$ then there exists a projector P_E such that $Imag(P_E) = E$ and $Ker(P_E) = E^{\perp}$. In this way, for each closed subspace E of $\mathcal{H}$ we denote by P_E the projector onto E. Thus, the concepts of closed subspace and projector are interchangeable and, by the usual abuse of language, the identification

$$E \cong P_E \tag{15}$$

will be used along this article.

We denote by $\mathcal{P}(\mathcal{H})$ the set of all closed projectors on $\mathcal{H}$. We also remark that $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. It is well known that the structure

$$\langle \mathcal{P}(\mathcal{H}), \vee, \wedge, ^{\perp}, 0, 1_{\mathcal{H}} \rangle \tag{16}$$

where $P_X \vee P_Y = P_{X \vee Y}$ being $X \vee Y$ the smallest closed subspace of $\mathcal{H}$ containing X and Y, $P_X \wedge P_Y = P_{X \cap Y}$, $0 \cong \{0\}$ and $1_{\mathcal{H}} \cong \mathcal{H}$, is a complete orthomodular lattice. These kind of structures are called *Hilbert lattices*.

Proposition 2.4 [13, §28] *Let $\mathcal{H}$ be a Hilbert space and P_X, P_Y two projectors. Then, $P_X + P_Y$ is a projector iff $X \perp Y$. If this condition is satisfied then $P_{X \vee Y} = P_X + P_Y$*

$\square$

In [32] Sasaki noticed that for two closed subspaces E, X of $\mathcal{H}$ the orthogonal projection of X to the subspace E i.e., $P_E(X)$, can be expressed without the use of the inner product, more precisely, by using only the lattice operations and orthocomplements of a Hilbert lattice. Formally, taking into account the Sasaki projection introduced in Eq.(6), we have that

$$P_E(X) = \mu_E(X) = E \wedge (E^{\perp} \vee X). \tag{17}$$

Thus, in a Hilbert lattice the Sasaki projection μ_E exactly encodes the action of the orthogonal projector P_E that projects onto the subspace E.

Let W be an operator in $\mathcal{B}(\mathcal{H})$. Then W is said to be *unitary* iff $W^*W = WW^* = 1_{\mathcal{H}}$. The operator W is an *isometry* iff $\|W(x)\| = \|x\|$ for every $x \in \mathcal{H}$. Let us notice that unitary operators are isometries.

Proposition 2.5 *Let $W \in \mathcal{B}(\mathcal{H})$ be an isometry and X be a closed subspace of $\mathcal{H}$. Then $W(X)$ is a closed subspace of $\mathcal{H}$.*

Proof: We shall prove that $W(X)$ contains all its accumulation points. Let $\big(W(x_n)\big)_{n\in\mathbb{N}}$ be a sequence in $W(X)$ which is convergent to $y \in \mathcal{H}$. Since $\big(W(x_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence and W preserves distances we have that $\|x_n - x_m\| = \|W(x_n) - W(x_m)\| \to 0$ whenever $n, m \to \infty$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and x_n converges to $x \in \mathcal{H}$ because $\mathcal{H}$ is complete. Note that $x \in X$ because X is closed. Then, by the continuity of W, we have that $W(x) = W(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} W(x_n) = y$. Hence $y \in W(X)$ and $W(X)$ is a closed subspace of $\mathcal{H}$.

$\square$

An operator W in $\mathcal{B}(\mathcal{H})$ is a *partial isometry* iff the restriction $W\restriction_{Ker^{\perp}(W)}$ is an isometry. There are many familiar examples of partial isometries: every isometry is one and every projection is one also. Partial isometries have been intensively studied since the early stages of the theory of operator on Hilbert spaces. In particular, they form the cornerstone of the dimension theory of von Neumann algebras. One of the earliest results in operator theory is the following characterization of partial isometries.

Theorem 2.6 [30, §2.2.8] *Let $W \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent,*

1. *W is a partial isometry,*
2. *$WW^* = P_{_{Imag(W)}}$,*
3. *$W^*W = P_{_{Ker(W)^{\perp}}}$,*
4. *$WW^*W = W$,*
5. *$W^*WW^* = W^*$,*
6. *W^* is a partial isometry.*

$\square$

Let $\mathcal{H}$ be a Hilbert space. A $*$-*subalgebra* of $\mathcal{B}(\mathcal{H})$ is a linear subspace $\mathcal{N}$ of $\mathcal{B}(\mathcal{H})$ closed by the operations $\cdot$ and *. In particular $\mathcal{N}$ is said to be *unital* iff the identity operator $1_{\mathcal{H}} \in \mathcal{N}$. Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$. The *commutant of* $\mathcal{N}$ is the set

$$\mathcal{N}' = \{X \in \mathcal{B}(\mathcal{H}) : \forall T \in \mathcal{N}, \;\; XT = TX\}.$$

In particular $\mathcal{N}'' = (\mathcal{N}')'$ is called the *bicommutant of* $\mathcal{N}$.

Definition 2.7 Let $\mathcal{H}$ be a Hilbert space. A *von Neumann algebra* is a unital $*$*-subalgebra* $\mathcal{N}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{N}'' = \mathcal{N}$.

Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. We denote by $\mathcal{P}(\mathcal{N})$ the set of all projectors in $\mathcal{N}$ i.e. $\mathcal{P}(\mathcal{N}) = \mathcal{N} \cap \mathcal{P}(\mathcal{H})$. An important fact is that $\mathcal{P}(\mathcal{N})$ generates $\mathcal{N}$ in the sense that $\mathcal{P}(\mathcal{N})'' = \mathcal{N}$. If $A \in \mathcal{N}$ then we define the *left annihilator* of A as the set

$$Ann_L(A) = \{S \in \mathcal{N} : SA = 0\}. \tag{18}$$

Proposition 2.8 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $A \in \mathcal{N}$. Then there exists a unique $A' \in \mathcal{P}(\mathcal{N})$ such that*

$$Ann_L(A) = \mathcal{N}A' = \{NA' : N \in \mathcal{N}\}.$$

Proof: See [26, Definition 37.3 and Remark 37.15]. □

Proposition 2.9 [14, §D, p.72] *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then, $\mathcal{P}(\mathcal{N})$ is complete orthomodular lattice with respect to the lattice operations inherited from $\mathcal{P}(\mathcal{H})$.* □

If $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra then the orthomodular structure related to $\mathcal{P}(\mathcal{N})$ is called *von Neumann lattice*. By the above proposition we can also identify the projectors of $\mathcal{P}(\mathcal{N})$ with the respective closed subspaces of $\mathcal{H}$.

A crucial relation between projectors in operator theory is the notion of unitary equivalence. Two projectors P and Q in a von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ are said to be *unitary equivalent* iff there exists an unitary operator $W \in \mathcal{N}$ such that $Q = WPW^*$.

Theorem 2.10 [9, Theorem 1]. *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $P, Q \in \mathcal{P}(\mathcal{N})$. Then P and Q are unitary equivalent iff $P \sim_p Q$ in the lattice $\mathcal{P}(\mathcal{N})$.*

□

The above theorem allows us to see the notion of perspectivity as a lattice order representation for the notion of unitary equivalence between projectors of a von Neumann algebra.

Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. The *center of* $\mathcal{N}$ is defined as the set $\mathcal{Z}(\mathcal{N}) = \mathcal{N} \cap \mathcal{N}'$. Note that $\mathcal{Z}(\mathcal{N})$ is a commutative von Neumann sub algebra of $\mathcal{N}$. The algebra $\mathcal{N}$ is called *factor* iff its center is trivial, that is, $\mathcal{Z}(\mathcal{N}) = \{\lambda I : \lambda \in \mathbb{C}\}$. An example of a factor is $\mathcal{B}(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$.

The notion of factor is closely related to the directly decomposability of the lattice $\mathcal{P}(\mathcal{N})$. Indeed: Let us notice that each von Neumann algebra $\mathcal{N}$ has an underlying ring structure, more precisely, it is a Baer*-ring [26, Remark 37.15]. Then, $\mathcal{N}$ is a factor iff the central idempotent elements of the underlying ring structure of $\mathcal{N}$ are $\{0, 1_{\mathcal{H}}\}$ which is equivalent to $Z(\mathcal{P}(\mathcal{N})) = \mathbf{2}$. Thus, by Proposition 1.5-5, we can establish the following result.

Proposition 2.11 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then the following statements are equivalent:*

1. *$\mathcal{N}$ is a factor.*
2. *$\mathcal{P}(\mathcal{N})$ is a directly indecomposable lattice i.e. $Z(\mathcal{P}(\mathcal{N})) = \mathbf{2}$.*

$\square$

Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Two projectors $P, Q \in \mathcal{P}(\mathcal{N})$ are said to be *Murray-von Neumann equivalent* iff there exists a partial isometry $W \in \mathcal{N}$ such that $WW^* = P$ and $W^*W = Q$. If P and Q are Murray-von Neumann equivalent then we write $P \sim Q$.

Proposition 2.12 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $P, Q \in \mathcal{P}(\mathcal{N})$. If $P \sim_p Q$ then $P \sim Q$.*

Proof: Let us suppose that $P \sim_p Q$. By Proposition 2.10 there exists a unitary operator $U \in \mathcal{N}$ such that $Q = UPU^*$. If we define $W = UP$ then $WW^*W = UP(UP)^*UP = UPP^*U^*UP = UPP(U^*U)P = UPP1_{\mathcal{H}}P = UP = W$. Thus, by Proposition 2.6-4, W is a partial isometry. We also note that $W^* = (UP)^* = PU^* = (U^*QU)U^* = U^*Q$. Hence $WW^* = Q$, $W^*W = P$ and $P \sim Q$.

$\square$

The relation $\sim$ defines an equivalence on $\mathcal{P}(\mathcal{N})$ and the description of the quotient $\mathcal{P}(\mathcal{N})/_\sim$ is known as the *dimension theory* for $\mathcal{N}$. We write $P \preceq Q$, and say that P is *Murray-von Neumann sub-equivalent* to Q if there exists a partial isometry $W \in \mathcal{N}$ such that $WW^* = P$ and $W^*W \leq Q$. The projector P is said to be *finite* iff, whenever $P \sim Q$ and $Q \preceq P$ then $P = Q$. Otherwise P is said to be *infinite*.

If $\mathcal{N} = \mathcal{B}(\mathcal{H})$ then it is immediate to see that two projections are equivalent iff their image have the same dimension. Thus, the ordering in $\mathcal{P}(\mathcal{H})/_\sim$ is isomorphic to $\{0, 1, \ldots, n\}$ if $dim(\mathcal{H}) = n$ and isomorphic to the ordinal number $\mathbb{N} \cup \{\infty\}$ if $\mathcal{H}$ is a infinite separable dimensional Hilbert space. Hence, the idea of equivalence of projectors via partial isometries represents an abstract notion of dimension for an arbitrary factor an the first result confirming this is the following theorem whose proof can be found in [27].

Proposition 2.13 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a factor. Then $\langle \mathcal{P}(\mathcal{N})/_\sim, \preceq \rangle$ is a totally ordered set.* $\square$

Let us notice that the order $\preceq$ is defined through the ring structure of a factor. Thus, two factors can not be isomorphic if the orderings of the corresponding quotient $/_\sim$ are different.

Proposition 2.14 [9, Lemma 4] *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a factor and $P, Q, R \in \mathcal{P}(\mathcal{N})$ such that $Q \sim P \leq Q$ and $P \preceq R \leq Q^\perp$. Then P and Q are perspective in $[0, Q \vee R]_{\mathcal{P}(\mathcal{N})}$.* $\square$

The Murray-von Neumann classification of factors consists in determining the order type of $\mathcal{P}(\mathcal{N})/_\sim$ for a factor $\mathcal{N}$. In order to do this, the following notion of dimension function introduced in [27, Definition 8.2.1] plays a crucial role.

Definition 2.15 Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A function of the form $D : \mathcal{P}(\mathcal{N}) \to [0, \infty]$ is called *dimension function* if it satisfies the following requirements:

1. $D(P) = 0$ iff $P = 0$,
2. $P \sim Q$ iff $D(P) = D(Q)$,
3. if $P \perp Q$ then $D(P + Q) = D(P) + D(Q)$.

Theorem 2.16 [27, Theorem VII] *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a factor. Then there exists a dimension function $D : \mathcal{P}(\mathcal{N}) \to [0, \infty]$ uniquely determined up to positive constant multiple. Further, $Imag(D)$ falls into exactly one of four possible cases, depending on which of the following sets is the range of some scaling of D.*

Type $\mathbf{I}_n$: $Imag(D) = \{0, 1 \ldots n\}$.

Type $\mathbf{I}_\infty$: $Imag(D) = \{0, 1 \ldots \infty\}$.

Type $\mathbf{II}_1$: $Imag(D) = [0, 1]$.

Type $\mathbf{II}_\infty$: $Imag(D) = [0, \infty]$.

Type $\mathbf{III}$: $Imag(D) = \{0, \infty\}$.

□

The above theorem implies that if $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is a factor then $\langle \mathcal{P}(\mathcal{N})/_\sim, \preceq \rangle$ is order isomorphic to the image of the uniquely determined dimension function $D : \mathcal{P}(\mathcal{N}) \to [0, \infty]$. In this way all von Neumann factor were found to belong to the classes type **I** or type **II** or type **III**. The following proposition, sometime referred as Borchers condition [4, 8], provides a crucial characterization of type III factors.

Proposition 2.17 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a factor such that $\mathcal{N} \neq \mathcal{Z}(\mathcal{N})$ and $D : \mathcal{P}(\mathcal{N}) \to [0, \infty]$ be the dimension function. Then the following statement are equivalent,*

1. *$\mathcal{N}$ is a type III factor.*
2. *For all $P_1, P_2 \in \mathcal{P}(\mathcal{N}) - \{0\}$, $P_1 \sim P_2$.*
3. *For each $P \in \mathcal{P}(\mathcal{N}) - \{0\}$, $P \sim 1_\mathcal{H}$.* (*Borchers condition*)

□

Proposition 2.18 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a factor and $P_X, P_Y \in \mathcal{P}(\mathcal{N}) - \{0, 1_{\mathcal{H}}\}$ such that $X \subseteq Y$. Then*

1. $D(P_Y) = D(P_X) + D(P_{X^\perp \cap Y})$.
2. *If* $P_X \sim_p P_Y$ *and* $X \neq Y$ *then* $D(P_X) = D(P_Y) = D(1_{\mathcal{H}}) = \infty$.

Proof: 1) By the orthomodular law $Y = X \vee (X^\perp \cap Y)$ and $X \perp (X^\perp \cap Y)$. Thus, by Proposition 2.4, $P_Y = P_{X \vee (X^\perp \cap Y)} = P_X + P_{X^\perp \wedge Y}$. Hence, by condition 3 in Definition 2.15, we have that $D(P_Y) = D(P_X) + D(P_{X^\perp \cap Y})$.

2) By Proposition 2.12 we have that $D(P_X) = D(P_Y)$ because $X \sim_p Y$. Let us suppose that $D(P_Y) < \infty$. Thus, by item 1, we have that

$$D(P_Y) = D(P_X) + D(P_{X^\perp \wedge Y}) = D(P_Y) + D(P_{X^\perp \wedge Y})$$

and consequently, $D(P_{X^\perp \wedge Y}) = 0$. Then, by Condition 1 in Definition 2.15, $X^\perp \cap Y = X^\perp \wedge Y = \{0\}$. Let us notice that $X^\perp \cap Y = \neg_Y X$ is the complement of X in the interval lattice $[0, Y]_{\mathcal{P}(\mathcal{N})} \approx [0, P_Y]_{\mathcal{P}(\mathcal{N})}$. Since $X \vee \neg_Y X = Y$ and $X < Y$ then $X^\perp \cap Y = \neg_Y X \neq 0$ which is a contradiction. Hence, $D(P_Y) = D(P_X) = \infty$ and, by item 1, $D(1_{\mathcal{H}}) = D(P_Y + P_{Y^\perp}) = D(P_Y) + D(P_{Y^\perp}) = \infty + D(P_{Y^\perp}) = \infty$.

$\square$

3 Motivating LQF-logic: Logics from Murray-von Neumann dimension theory and related open questions

In order to motivate our logical system we first need to introduce a brief description of the AQFT whose formalism is supported by two theories: the algebraic approach to quantum mechanics and general relativity.

On one side, the standard operator algebra formulation of quantum mechanics, or *algebraic quantum mechanics* for short, starts with the concept of separable complex Hilbert space. This formalism was described in a kind of postulates by J. von Neumann in his celebrated 1932 book [34]. In this approach we can understand quantum mechanics as a theory whose primitive concepts are *quantum system*, *states*, *observables* and *measurement* subject to the following basic interpretations:

- To every quantum system, we associate a complex separable Hilbert space $\mathcal{H}$ called the *state space of the system.*

Intuitively speaking, a physical system consists of a region of spacetime and all the entities contained within it.

- The *state of the system* is completely specified by a density operator ρ on $\mathcal{H}$, that is, a self-adjoint ($\rho = \rho^*$), positive ($\langle -, \rho(-)\rangle \geq 0$) and unit-trace ($Tr(\rho) = 1$) operator.

The state of a quantum system encodes all of the physical information that we know about the system. Traditional quantum mechanics distinguishes between pure states and mixed states. If our knowledge of the state of the system is complete we say that the system is in a *pure state*. This special case is represented by a density operator degenerates to a projector on a closed 1-dimensional subspace of $\mathcal{H}$. Thus, a pure state is equally well characterized by a unit vector which defines this 1-dimensional subspace. Otherwise, if our knowledge of the state is incomplete but the statistical ensemble associated to the system is known, we say that the system is in a *mixed state*.

- The *observables* of a quantum system are represented by self-adjoint operators on the space $\mathcal{H}$.

Intuitively, an observable represents a physical property $\mathcal{A}$ as energy, position, momentum, etc. If $\mathcal{A}$ is represented by the self-adjoint operator A then the eigenvalues of A are real numbers representing all the possible values of the physical property $\mathcal{A}$.

There are other postulates, involving measurement process and time evolution of a quantum system, that we do not mention because they do not play any role in the logical systems treated here.

Let us notice that a projector is a self-adjoint operator having only the two eigenvalues 0 and 1. Along these lines, von Neumann proposed a correspondence between projectors and logical propositions in his 1932 book [34]. We refer to such propositions as *quantum propositions* or *q-propositions* for short. In order to clarify this concept, let us consider the spectral decomposition of a self-adjoint operator A representing an observable $\mathcal{A}$ i.e.

$$A = \sum_i a_i P_{a_i}, \qquad \sum_i P_{a_i} = 1_{\mathcal{H}} \tag{19}$$

where $(a_i)_i$ are the eigenvalues of A and $(P_{a_i})_i$ is the family of projectors over the respective eigenspaces. In this way, each projector P_{a_i} represents the proposition "*the value of the observable $\mathcal{A}$ is a_i*". Thus, by the spectral decomposition, each observable can be decomposed into elementary true/false propositions. We can also analyze under which conditions these kind of propositions are true or false. Indeed: Let us consider an arbitrary state of the system represented by a density operator ρ. In this case we only know that, in the state ρ, the measurement of $\mathcal{A}$ will yield one of the values a_i, but we don't know which one. Then, for each projector P_{a_i} of the spectral decomposition, mentioned in Eq.(19), the number in the real interval $[0, 1]$ given by the trace $Tr(\rho P_{a_i})$ represents the probability that the proposition "*the value of the observable $\mathcal{A}$ is a_i*" is true in the state ρ. In the particular case where $Tr(\rho P_{a_i}) = 1$, for example when $\rho = P_{a_i}$, we can regard the proposition "*the value of the observable $\mathcal{A}$ is a_i*" as true in the state ρ. But, if $Tr(\rho P_{a_i}) = 0$, as is the case of $\rho = P_{a_i}^{\perp}$, a measurement of $\mathcal{A}$ will never provide result a_i. In this case we can consider the proposition "*the value of the* *observable $\mathcal{A}$ is a_i*" as a false proposition in the state ρ. Therefore, projectors correspond to true/false propositions but they are not organized in a Boolean structure. For example, the distributivity condition between projectors fails. In this way, the successive work in logic related to quantum systems, developed by Birkhoff and von Neumann in their seminal article in 1936 [3], substituted Boolean algebras with the lattice of closed subspaces of a Hilbert space, or Hilbert lattice, for encoding the structure of q-propositions. This structure was successively named *quantum logic*.

Soon after the publication of von Neumann book in 1932, his interest in ergodic theory, group representations and quantum mechanics contributed significantly to von Neumann realization that a theory of operator algebras was the next important stage in the development of quantum mechanics [33]. Operator algebras can trace its origin to the appearing of the four "*Rings of Operators*" papers of F. J. Murray and J. von Neumann, where the first one [27] appeared in 1936. Those operator algebras, originally known as rings of operators or W^*-algebras, were later renamed von Neumann algebras by J. Dixmier and J. Dieudonné. That was when the orthomodular lattice theory grew out of the theory of von Neumann algebras as a general framework to include all projection lattices of these algebras. More precisely, the Murray-von Neumann classification of factors suggests that the projector lattices of each factor can be quite different from one on another giving rise to a wide family of lattices whose common characteristic is the orthomodularity. Indeed:

Type I_n and type I_∞ factors always correspond to the whole algebra of bounded operators on a separable Hilbert space. This is the case usually considered for describing quantum systems with a state space having finite or infinite dimension, respectively. The standard model of type I factor is the algebra $\mathcal{B}(\mathcal{H})$ where $\mathcal{P}(\mathcal{B}(\mathcal{H})) = \mathcal{P}(\mathcal{H})$. Thus, the logic associated to this factor is the usual Birkhoff and von Neumann quantum logic. If $Dim(\mathcal{H}) = n$ then $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ is an atomic modular orthomodular lattice. Differently, if $Dim(\mathcal{H}) = \infty$ then $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ loses the modularity [2, §10.3.8].

Type II_1 and type II_∞ factors play an important role in non-relativistic statistical quantum mechanics. The projector lattice of a II_1 factor is a modular orthomodular lattice and it has no atoms. While, the projector lattice of a type II_∞ factor is non modular orthomodular lattice and it has no atoms too.

Type III factors was the most mysterious case, however, they became relevant in relativistic quantum field theories. At the beginning, von Neumann regarded the type III factors as a kind of pathological class of operator algebras. Indeed, it took four years after the discovery of the classification of factors, in 1936, to construct the first example of type III factor [36]. Moreover, it was only in the mid of the sixties that the existence of a continuous number of non isomorphic type III factors was proven [31]. The projector lattice of a type III factor is a non modular orthomodular lattice and it has no atoms. In the next section Proposition 4.9 provides an alternative proof of these lattice order properties about type III factors.

The other theory which AQFT is based on is the theory of relativity. In general relativity, the space of events is represented by a *Lorentzian manifold*, i.e. a pair $\langle \mathcal{M}, g\rangle$ where $\mathcal{M}$ is a smooth $(n+1)$-dimensional manifold and g is a *Lorentzian tensor metric*, that is, g associates to each point $p \in \mathcal{M}$ the following inner product $g_p(-,-)$ on the tangent space $T_p\mathcal{M}$: chosen a basis $\{e_0 \dots e_n\}$ in $T_p\mathcal{M}$ and $x, y \in T_p\mathcal{M}$, $g_p(x,y) = -x^0y^0 + \sum_{i=1}^n x^iy^i$ where $(x^0 \dots x^n)$ and $(y^0 \dots y^n)$ are the components of x, y with respect to the chosen basis. For each $x \in T_p\mathcal{M}$ the tensor metric defines the following classification:

$$x \text{ is } \begin{cases} \textit{spacelike} & \text{iff } g_p(x,x) > 0 \\ \textit{null} & \text{iff } g_p(x,x) = 0 \\ \textit{timelike} & \text{iff } g_p(x,x) < 0. \end{cases} \tag{20}$$

The *causal cone at* $p \in \mathcal{M}$ is defined as $C_p = \{x \in T_p\mathcal{M} : g_p(x,x) \leq 0\}$. A *causal curve* is a smooth curve $\gamma : I \to \mathcal{M}$ such that for each $s \in I$, $\dot\gamma(s) \in C_{\gamma(s)}$. An important principle of general relativity states that observers can move only on causal curves. Let $\mathcal{O}$ be an open bounded region in $\mathcal{M}$ and $x \in \mathcal{M}$. We said that x lies in the causal cone of $\mathcal{O}$, denoted by $C(\mathcal{O})$, iff there exists $p \in \mathcal{O}$ and a causal curve $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = x$. The *causal complement* of the open bounded region $\mathcal{O}$ is defined as $\mathcal{O}^c = \mathcal{M} \smallsetminus \overline{C(\mathcal{O})}$. We say that an open bounded region $\mathcal{O}_1$ is *spacelike separated from* the open bounded region $\mathcal{O}_2$ iff $\mathcal{O}_1 \subseteq \mathcal{O}_2^c$. In general relativity events happening at spacelike separated regions cannot influence each other. A *space time* or *timelike oriented* Lorentzian manifold is a 3-tuple $\langle \mathcal{M}, g, \mathbf{t}\rangle$ such that $\langle \mathcal{M}, g\rangle$ is a Lorentzian manifold and $\mathbf{t}$ is a smooth vectorial field on $\mathcal{M}$ i.e., $p \mapsto \mathbf{t}(p) \in T_p\mathcal{M}$, such that $g_p(\mathbf{t}(p), \mathbf{t}(p)) < 0$. The vector field $\mathbf{t}$ determines a time orientation on $\mathcal{M}$. More precisely, if $\gamma : I \to M$ is a causal curve then we say that γ is *future directed* (resp. *past directed*) provided for each $s \in I$, $g_{\gamma(s)}(\mathbf{t}(\gamma(s)), \dot\gamma(s)) > 0$ (resp. $g_{\gamma(s)}(\mathbf{t}(\gamma(s)), \dot\gamma(s)) < 0$).

These geometrical notions on relativity and the basic concepts of the algebraic formulation of quantum mechanics introduced above allow us to briefly describe the Haag-Araki formalism of AQFT and a related logical systems.

Let $\mathcal{M}$ be a spacetime represented by a Lorentzian manifold, $\mathfrak{R}(\mathcal{M})$ be the set of open bounded region and $\mathcal{H}$ be a separable Hilbert space. The basic object of the Haag-Araki model for AQFT is a net of von Neumann algebras, called *net of observables over the spacetime* $\mathcal{M}$ defined as

$$\mathfrak{R}(\mathcal{M}) \ni \mathcal{O} \mapsto \mathcal{N}(\mathcal{O})$$

where $\mathcal{N}(\mathcal{O}) \subseteq \mathcal{B}(\mathcal{H})$ is a Type *III* factor and the following basic conditions are satisfied

$$\text{If } \mathcal{O}_1 \subseteq \mathcal{O}_2 \text{ then } \mathcal{N}(\mathcal{O}_1) \subseteq \mathcal{N}(\mathcal{O}_2). \tag{\textit{Isotony}}$$

$$\text{If } \mathcal{O}_1 \subseteq \mathcal{O}_2^c \text{ then } \mathcal{N}(\mathcal{O}_1) \subseteq \mathcal{N}(\mathcal{O}_2)'. \tag{\textit{Locality}}$$

Let us remark that there are other conditions imposed on the net of observables [12, 22, 17] that we do not mention because they do not play any role in the argument treated here.

Each factor $\mathcal{N}(\mathcal{O})$ is called *local observable algebra of the net* and it represents the observables in the region $\mathcal{O}$ of the spacetime. In this perspective, projectors in $\mathcal{P}\big(\mathcal{N}(\mathcal{O})\big)$ represent q-propositions related to $\mathcal{O}$. A *state of the net* is a density operator of the von Neumann algebra generated by $\bigcup_{\mathcal{O}\in\mathfrak{R}(\mathcal{M})}\mathcal{N}(\mathcal{O}) \subseteq \mathcal{B}(\mathcal{H})$.

The role of type III factors in the net of observables turns out to be a consequence of general physical requirements compatible with the relativity. For example, one of these is the fact that measurements of observables happening in space-like separated regions are not allowed to influence each other. The next proposition formally expresses the mentioned physical requirement.

Proposition 3.1 *Let $\mathfrak{R}(\mathcal{M}) \ni \mathcal{O} \mapsto \mathcal{N}(\mathcal{O})$ be a net of observables in a Hilbert space $\mathcal{H}$, ρ be a state of the net and P be a projector of the local algebra $\mathcal{N}(\mathcal{O})$. Then there exists a state $\rho_{_P}$ of the net such that*

1. *$tr(\rho_{_P} P) = 1$.*
2. *If $\mathcal{O} \subseteq \mathcal{O}_1^c$ and A is a selfadjoint operator in $\mathcal{N}(\mathcal{O}_1)$ then $tr(\rho_{_P} A) = tr(\rho A)$.*

Proof: Let ρ be a state of the net and P be a projector in the local algebra $\mathcal{N}(\mathcal{O})$. Since $\mathcal{N}(\mathcal{O})$ is a type III factor then, by Proposition 2.17-3, there exists a partial isometry $W \in \mathcal{N}(\mathcal{O})$ such that $WW^* = 1_{\mathcal{H}}$ and $W^*W = P$. Let us define $\rho_{_P} = W^*\rho W$. We first show that $\rho_{_P}$ is a density operator. Indeed: By a straightforward calculations we can see that $\rho_{_p}^* = \rho_{_p}$. In order to prove that ρ_p is positive, let us consider $x \in \mathcal{H}$ and, by Proposition 2.1, the orthogonal decomposition $x = x_{_W} + x_{_{W^\perp}}$ where $x_{_W} \in Ker(W)$ and $x_{_{W^\perp}} \in Ker(W)^\perp$. Since $W^*\rho W(x_{_{W^\perp}}) \in ker(W)^\perp$ and W is an isometry over $ker(W)^\perp$ then we have that $\langle x, \rho_{_P}(x)\rangle = \langle x, W^*\rho W(x)\rangle = \langle x_{_{W^\perp}}, W^*\rho W(x_{_{W^\perp}})\rangle + 0 = \langle W(x_{_{W^\perp}}), WW^*\rho W(x_{_{W^\perp}})\rangle = \langle W(x_{_{W^\perp}}), \rho W(x_{_{W^\perp}})\rangle$. Thus we also have that $\langle x, \rho_{_P}(x)\rangle = \langle W(x_{_{W^\perp}}), \rho W(x_{_{W^\perp}})\rangle \geq 0$ since ρ is a positive operator. Note that $\rho_{_P}$ is a unit-trace operator because $tr(\rho_{_P}) = tr(W^*\rho W) = tr(WW^*\rho) = tr(\rho) = 1$. Hence, $\rho_{_p}$ is a density operator.

Now we prove that $tr(\rho_{_P} P) = 1$. Indeed:

$$\begin{aligned} tr(\rho_{_P} P) &= tr(W^*\rho W P) = tr(PW^*\rho W) = tr(W^*WW^*\rho W) = \\ &= tr(W^*\rho W) = tr(\rho_{_P}) = 1. \end{aligned}$$

Let us suppose that $\mathcal{O} \subseteq \mathcal{O}_1^c$ and A is a selfadjoint operator in $\mathcal{N}(\mathcal{O}_1)$. Note that $WA = AW$ since, by the postulate of locality in the net of observables, A belongs to the commutant of $\mathcal{N}(\mathcal{O})$. Thus, $Tr(\rho_{_P} A) = Tr(W^*\rho W A) = Tr(W^*\rho A W) = Tr(WW^*\rho A) = Tr(\rho A)$. Hence our claim.

$\square$

As we have seen in the above proof, the type III factor condition imposed to each algebra $\mathcal{N}(\mathcal{O})$ implies that every projector $P \in \mathcal{N}(\mathcal{O})$ can be written as $W^*W = P$ where $W \in \mathcal{N}(\mathcal{O})$ is a partial isometry. Consequently, if ρ is a state of the net then the transformation $\rho \mapsto \rho_{_P} = W^*\rho W$ change the state ρ into the eigenstate $\rho_{_P}$ of P by the local operation W without disturbing the causal complement of the region we are dealing with. This is precisely the motivation of the type III factor in order to show that the relativity is not violated. For more details regarding the role of type III factors in quantum field theory we refer to [38].

Proposition 3.1 also provides an interesting insight into the local character of the logic of q-propositions related to each region $\mathcal{O}$ of the Lorentzian spacetime. Indeed: Let us consider a q-proposition, represented by the projector $P \in \mathcal{N}(\mathcal{O})$. Since every state ρ of the net can be changed into an eigenstate ρ_P of P via a local operation, the q-proposition P can be evaluated as true in the state ρ_P i.e., $tr(\rho_P P) = 1$ without affecting the truth values of the q-propositions of the spacelike separated regions of $\mathcal{O}$. In other words, the type III condition of each algebra of the net defines a local logical system such that in spacelike separated regions of the spacetime the respective notions of truth are independent each other. In this way, the goal of the LQF-logic is to capture the type III factor condition of the local observable algebras.

In the aim to develop a such logical system, unavoidably, we deal with a family of open questions related to the problem of characterizing those logical systems which can be identified with the lattice of projections of factors of the Murray-von Neumann classification. Indeed: Since the beginning of the Murray-von Neumann classification, in the 30's, it has been noticed that each factor of the classification is associated to a particular kind of von Neumann lattice. The subject aroused great interest among the logical community giving rise to the study of logical structures emerging from the factors of the classification. At first, von Neumann introduced the concept of continuous geometry in the mid of the thirties as a lattice theoretical generalization of the projective geometry. More precisely, continuous geometries define a subclass of modular lattices. These basic results were published in [37] after his death from his Princeton lecture notes during 1935-1937. In this work it was shown that in a directly indecomposable continuous geometry the quotient by the perspectivity relation $\sim_p$ univocally defines a notion of dimension function onto the real interval $[0, 1]$.

Remark 3.2 It is interesting to note that in a factor $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ the relations $\sim$ and $\sim_p$ coincide whenever $1_{\mathcal{H}}$ is a finite element [9, 20]. Moreover, the condition of the finiteness of $1_{\mathcal{H}}$ it can be equivalently formulated in equational terms through the equation of modularity (see Eq.(14)). Thus, the modularity imposed to the lattice of projectors of the factor $\mathcal{N}$ forcing it to be either a type I_n factor or a type II_1 factor (see [2, §10.3.8] for more detail). In this way, the notion of continuous geometries provides a natural common framework for type I_n and II_1 factors.

From the remark above, naturally arises the question whether it might be possible to establish lattice theoretical conditions in order to characterize each factor of the Murray-von Neumann classification [21, 24]. In this way and in separated works Loomis [23] and Maeda [25], assuming the existence of a particular equivalence relation on an orthomodular lattice, derived several properties related to the Murray-von Neumann dimension theory in a purely algebraic way.

This kind of structure is known as *dimension lattices*. Although the dimension lattices theory provides a general theoretical framework for von Neumann lattices, from a logical algebraic point of view, to establish an equational theory able to capture the Murray-von Neumann dimension equivalence remains an undone tasks of orthomodular lattice theory. The great difficult in order to deal with this problem is that, in general, we can not transplant the Murray-von Neumann equivalence to a lattice since it involves elements of the theory of operators on Hilbert spaces. However, as we will see in the next section, this is possible to do for the special case of type III factors. Therefore, and in a similar way as is mentioned in Remark 3.2, LQF-logic will be based on an equational system that, when imposed on the projector lattice of a factor it determines univocally the type III factor condition. This result will be proved in Theorem 4.10. In this way, the logical system studied here, that describes the propositional structure of the local observable algebras in the Haag-Araki model of AQFT, is closely related to the question about whether it might be possible to establish lattice theoretical conditions in order to characterize the factors of the Murray-von Neumann classification.

4 An equational characterization for the type III factor

In this section we establish a set of equations able to capture, in a purely algebraic way, the dimension function and the Murray-von Neumann equivalence of a type III factor. More precisely, the mentioned equational system, imposed to a von Neumann lattice, univocally determines the type III factor. In order to do this, we first need to establish a representation of the dimension function into the projector lattice of a type III factor.

Definition 4.1 Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a type III factor. Then we define the *internal dimension function* as the unary operation $w_0 : \mathcal{P}(\mathcal{N}) \to \{0, 1_{\mathcal{H}}\}$ such that

$$w_0(X) = \begin{cases} 0, & X = 0, \\ 1_{\mathcal{H}}, & otherwise. \end{cases}$$

The following proposition can be easily proved.

Proposition 4.2 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a type III factor, $D : \mathcal{P}(\mathcal{N}) \to \{0, \infty\}$ be the dimension function and w_0 be the internal dimension function. Then*

1. $w_0(X) = 0$ *iff* $D(X) = 0$,
2. $X \sim Y$ *iff* $w_0(X) = w_0(Y)$,
3. $w_0(X \vee Y) = w_0(X) \vee w_0(Y)$,
4. $w_0(w_0(X)) = w_0(X)$,

5. $w_0(X \wedge w_0(Y)) = w_0(X) \wedge w_0(Y)$,

$\square$

Remark 4.3 Let us notice that items 1 and 2 in the above proposition show that w_0 exactly describes the dimension function and the Murray-von Neumann equivalence within the ordered structure of the projectors of a type III factor. Furthermore, if we consider the category whose objects are the von Neumann lattices of the type III factors expanded by the internal dimension function and whose arrows are internal dimension function preserving $\mathcal{OML}$-homomorphisms, then it is equivalent to the category of von Neumann lattices of type III factors whose arrows are the following commutative triangles

$$\begin{tikzcd} \mathcal{P}(\mathcal{N}_1) \arrow[rr, "f"] \arrow[dr, "D_1"'] \arrow[rr, phantom, shift right=4ex, "\equiv"] & & \mathcal{P}(\mathcal{N}_2) \arrow[dl, "D_2"] \\ & \{0, \infty\} & \end{tikzcd}$$

In other words, the preservation of the internal dimension function is equivalent to the preservation of the Murray-von Neumann equivalence through $\mathcal{OML}$-homomorphisms between von Neumann lattices of type III factors.

Proposition 4.4 *Let $W \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Then the following statements are equivalent:*

1. $WW^* = 1_{\mathcal{H}}$.
2. *The restriction $W \restriction_{Ker^{\perp}(W)}$ defines an isometry onto $\mathcal{H}$.*
3. *W^* defines a surjective isometry of the form $W^* : \mathcal{H} \rightarrow Ker^{\perp}(W)$.*

Proof: 1 $\Longrightarrow$ 2) Since W is a partial isometry we only need to prove that $W \restriction_{Ker^{\perp}(W)}$ is a surjective map. Let $y \in \mathcal{H}$. Then, by hypothesis, $y = WW^*(y) = W(x)$ where $x = W^*(y)$. By Proposition 2.1 and Remark 2.3 x can be decomposed as $x = x_{_{Ker(W)}} + x_{_{ker^{\perp}(W)}}$ where $x_{_{Ker(W)}} \in Ker(W)$ and $x_{_{ker^{\perp}(W)}} \in Ker^{\perp}(W)$. Then, $y = W(x) = W(x_{_{Ker(W)}} + x_{_{ker^{\perp}(W)}}) = W(x_{_{Ker(W)}}) + W(x_{_{ker^{\perp}(W)}}) = 0 + W(x_{_{ker^{\perp}(W)}}) = W(x_{_{ker^{\perp}(W)}})$. Thus, $W \restriction_{Ker^{\perp}(W)}$ is a surjective map.

2 $\Longrightarrow$ 3) By hypothesis we have that $Imag(W) = \mathcal{H}$. Then, by Proposition 2.2-1, $Ker(W^*) = Imag^{\perp}(W^{**}) = Imag^{\perp}(W) = \mathcal{H}^{\perp} = \{0\}$ i.e., W^* is injective. Therefore, W^* is an isometry on $\mathcal{H}$ and, by Proposition 2.5, $Imag(W^*)$ is a closed subspace of $\mathcal{H}$. Then, by Proposition 2.2-2, $Imag(W^*) = \overline{Imag(W^*)} = Ker^{\perp}(W)$. In this way W^* defines a surjective isometry of the form $W^* : \mathcal{H} \rightarrow Ker^{\perp}(W)$.

3 $\Longrightarrow$ 1) By hypothesis $Ker(W^*) = \{0\}$. Thus, by Proposition 2.2-1, $Imag^{\perp}(W) = Imag^{\perp}(W^{**}) = Ker(W^*) = \{0\}$ and then $Imag(W) = \mathcal{H}$. Hence, by Theorem 2.6-2, $WW^* = P_{_{Imag(W)}} = P_{\mathcal{H}} = 1_{\mathcal{H}}$.

$\square$

Proposition 4.5 *Let $W \in \mathcal{B}(\mathcal{H})$ be a partial isometry such that $WW^* = 1_{\mathcal{H}}$. Then for each $P_X \in \mathcal{P}(\mathcal{H})$*

1. *$W(X)$ and $W^*(X)$ are closed subspaces of $\mathcal{H}$.*
2. *W^*P_X is a partial isometry.*
3. *$P_{W^*(X)} = W^*P_XW$.*

Proof: 1) We first prove that $W(X)$ is a closed subset. Since $Ker(W)$ is a closed subspace of $\mathcal{H}$, by Proposition 2.1-2, $\mathcal{H} = Ker(W) \oplus Ker^{\perp}(W)$. Then each $x \in X$ can be written as $x = x_{Ker(W)} + x_{Ker^{\perp}(W)}$ where $x_{Ker(W)} \in Ker(W)$ and $x_{Ker^{\perp}(W)} \in Ker^{\perp}(W)$. Let us notice that

$$X_{Ker^{\perp}(W)} = \{x_{Ker^{\perp}(W)} \in \mathcal{H} : x \in X\} = P_{Ker^{\perp}(W)}(X)$$

is a closed subspace of $Ker^{\perp}(W)$ and $W(X) = W(X_{Ker^{\perp}(W)})$. Consequently, by Proposition 4.4-2, $W(X)$ is a closed subspace of $\mathcal{H}$. By Proposition 4.4-3 and Proposition 2.5 immediately follows that $W^*(X)$ is also a closed subspace of $\mathcal{H}$.

2) By Proposition 4.4-3, W^* is an isometry onto $Ker^{\perp}(W)$ and therefore $Ker(W^*) = \{0\}$. Thus, $Ker(W^*P_X) = Ker(P_X) = X^{\perp}$. Consequently, $W^*P_X \upharpoonright_{Ker^{\perp}(W^*P_X)} = W^*P_X \upharpoonright_X = W^* \upharpoonright_X$ is an isometry too. In this way W^*P_X is a partial isometry.

3) By item 2 W^*P_X is a partial isometry. Then, by Proposition 2.6-2, $P_{W^*(X)} = P_{Imag(W^*P_X)} = W^*P_X(W^*P_X)^* = W^*P_XP_XW = W^*P_XW$.

□

Proposition 4.6 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $W \in \mathcal{N}$ be a partial isometry such that $WW^* = 1_{\mathcal{H}}$. Then for each $P_X \in \mathcal{P}(\mathcal{N})$*

$$P_{W^*(X)} \in \mathcal{N} \ \text{ and } \ P_{W(X)} \in \mathcal{N}.$$

Proof: By Proposition 4.5-3 it is immediate to see that $P_{W^*(X)} \in \mathcal{N}$. In order to prove that $P_{W(X)} \in \mathcal{N}$, let us consider the left annihilator of the operator WP_X in the algebra $\mathcal{N}$, that is,

$$Ann_L(WP_X) = \{S \in \mathcal{N} : S(WP_X) = 0\}.$$

By Proposition 2.8 there exists unique projector $(WP_X)' \in \mathcal{P}(\mathcal{N})$ such that $Ann_L(WP_X) = \mathcal{N}(WP_X)'$. We shall prove that $(WP_X)' = P_{W(X)^{\perp}}$. Indeed:

Let $NP_{W(X)^{\perp}} \in \mathcal{N}P_{W(X)^{\perp}}$ and $x \in \mathcal{H}$. Then $(NP_{W(X)^{\perp}})(WP_X)(x) = (NP_{W(X)^{\perp}})(y)$ where $y = (WP_X)(x) \in W(X)$. Thus $P_{W(X)^{\perp}}(y) = 0$ and $(NP_{W(X)^{\perp}})(WP_X)(x) = 0$. It proves that $\mathcal{N}P_{W(X)^{\perp}} \subseteq Ann_R(WP_X)$. For the other inclusion let us consider $S \in Ann_R(WP_X)$. In this way $S \upharpoonright_{W(X)} = 0$. By Proposition 4.5-1, $W(X)$ is a closed subspace of $\mathcal{H}$ and then, for each $x \in \mathcal{H}$ we can decompose x as $x = x_{W(X)} + x_{W(X)^{\perp}}$ where $x_{W(X)} \in W(X)$ and $x_{W(X)^{\perp}} \in$

$W^{\perp}(X)$. Thus, $S(x) = S(x_{W(X)}) + S(x_{W(X)^{\perp}}) = 0 + S(x_{W(X)^{\perp}}) = SP_{W(X)^{\perp}}(x)$ and then $S = SP_{W(X)^{\perp}} \in \mathcal{N}P_{W(X)^{\perp}}$. It proves that $Ann_R(WP_X) \subseteq \mathcal{N}P_{W(X)^{\perp}}$. Hence, $P_{W(X)^{\perp}} = (WP_X)' \in \mathcal{P}(\mathcal{N})$ and, by Proposition 2.4, $P_{W(X)} = 1_{\mathcal{H}} - P_{W(X)^{\perp}} \in \mathcal{P}(\mathcal{N})$.

□

Let us notice that Proposition 4.6 suggests that each partial isometry W in a von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ satisfying $WW^* = 1_{\mathcal{H}}$ defines, in a natural way, two operations on the lattice $\mathcal{P}(\mathcal{N})$ given by

$$\mathcal{P}(\mathcal{N}) \ni P_X \longmapsto P_{W(X)} \quad \text{and} \quad \mathcal{P}(\mathcal{N}) \ni P_X \longmapsto P_{W^*(X)}.$$

For the sake of simplicity and in order to study these operations will be more convenient to work with closed subspaces rather than projectors. Then, by considering the usual identification $P_X \cong X$ for elements of $\mathcal{P}(\mathcal{N})$ the following definition formally introduce the above operations.

Definition 4.7 Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $W \in \mathcal{N}$ be a partial isometry such that $WW^* = 1_{\mathcal{H}}$. By denoting $Z = Ker^{\perp}(W)$ we define the pair of *Borchers operations associated* to W as the operations $\{w_Z, w_Z^*\}$ on $\mathcal{P}(\mathcal{N})$ given by

$$\mathcal{P}(\mathcal{N}) \ni P_X \cong X \longmapsto w_Z(X) = W(X) \cong P_{W(X)}. \tag{21}$$

$$\mathcal{P}(\mathcal{N}) \ni P_X \cong X \mapsto w_Z^*(X) = W^*(X) \cong P_{W^*(X)} \tag{22}$$

Proposition 4.8 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $W \in \mathcal{N}$ be a partial isometry such that $WW^* = 1_{\mathcal{H}}$ and $Z = Ker^{\perp}(W)$. Then, the Borchers operations w_Z and w_Z^* associated to W satisfy the following:*

1. *$w_Z(w_Z^*(X)) = X$ and $w_Z^*(w_Z(X)) = \mu_Z(X)$.*
2. *w_Z defines an order preserving operation on $\mathcal{P}(\mathcal{N})$ such that the restriction $w_Z \upharpoonright_{[0,Z]_{\mathcal{P}(\mathcal{N})}}$ is a $\mathcal{OML}$-isomorphism onto $\mathcal{P}(\mathcal{N})$.*
3. *w_Z^* is a $\mathcal{OML}$-isomorphism of the form $w_Z^* : \mathcal{P}(\mathcal{N}) \to [0, Z]_{\mathcal{P}(\mathcal{N})}$.*

Proof: 1) They follow by the condition $WW^* = 1_{\mathcal{H}}$ and by Proposition 2.6-3 respectively.

2) We first note that w_Z is an order preserving operation on $\mathcal{P}(\mathcal{N})$. Then, by Proposition 1.12-2, the restriction $w_Z \upharpoonright_{[0,Z]_{\mathcal{P}(\mathcal{N})}}$ is an order isomorphism onto $\mathcal{P}(\mathcal{N})$. We have to prove that w_Z preserves orthogonal complements i.e., for each $X \in [0, Z]_{\mathcal{P}(\mathcal{N})}$, $w_Z(\neg_Z X) = w_Z(X)^{\perp}$. Indeed: Let $y \in w_Z(\neg_Z X)$. Then there exists $x \in \neg_Z X = X^{\perp} \cap Z$ such that $y = W(x)$. Let $y_1 \in w_Z(X)$. Then there exists $x_1 \in X$ such that $y_1 = W(x_1)$. Let us notice that $\langle x, x_1 \rangle = 0$ and $x, x_1 \in Z = Ker^{\perp}(W)$. Thus, by Proposition 4.4-2, $\langle y, y_1 \rangle = \langle W(x), W(x_1) \rangle = \langle x, x_1 \rangle = 0$. It proves that $y \perp w_Z(X)$ and consequently $y \in w_Z(X)^{\perp}$. Therefore, $w_Z(\neg_Z X) \subseteq w_Z(X)^{\perp}$. For the converse, let $y \in w_Z(X)^{\perp}$. Then, for each $z \in X$, we have that $\langle y, W(z) \rangle = 0$. By Proposition 4.4-2 there exists $x \in Z$

such that $y = W(x)$. Let us notice that $x \perp X$ because $\langle x, z\rangle = \langle W(x), W(z)\rangle = \langle y, W(z)\rangle = 0$ for each $z \in X$. Thus, $x \in X^\perp \cap Z$ and $y = W(x) \in w_Z(X^\perp \cap Z) = w_Z(\neg_z X)$. It proves that $w_Z(X)^\perp \subseteq w_Z(\neg_z X)$. Hence, the restriction $w_Z \restriction_{[0,Z]_{\mathcal{P}(\mathcal{N})}}$ preserves orthogonal complements and then, $w_Z \restriction_{[0,Z]_{\mathcal{P}(\mathcal{N})}}$ is a $\mathcal{OML}$-isomorphism onto $\mathcal{P}(\mathcal{N})$.

3) Let us notice that w_z^* defines an order preserving operation on $\mathcal{P}(\mathcal{N})$. Then, by Proposition 1.12-1, w_Z^* is an order isomorphisms onto $[0, Z]_{\mathcal{P}(\mathcal{N})}$. We have to prove that w_Z^* preserves orthogonal complements i.e., for each $X \in \mathcal{P}(\mathcal{N})$, $w_Z^*(X^\perp) = \neg_z w_Z^*(X)$. Indeed: We first note that $w_Z\big(w_Z^*(X^\perp)\big) = X^\perp$ because of item 1. We also note that $w_Z^*(X) \in [0, Z]_{\mathcal{P}(\mathcal{N})}$. Then, by item 2, $w_Z\big(\neg_z w_Z^*(X)\big) = \big(w_Z w_Z^*(X)\big)^\perp = X^\perp$. Thus, $w_Z\big(w_Z^*(X^\perp)\big) = w_Z\big(\neg_z w_Z^*(X)\big)$ and , since the restriction $w_Z \restriction_{[0,Z]_{\mathcal{P}(\mathcal{N})}}$ is bijective, we have that $w_Z^*(X^\perp) = \neg_z w_Z^*(X)$. Hence w_Z^* preserves orthogonal complements and it is a $\mathcal{OML}$-isomorphism from $\mathcal{P}(\mathcal{N})$ onto $[0, Z]_{\mathcal{P}(\mathcal{N})}$.

□

As we will see below, Proposition 4.8 provides a useful result in order to establish an alternative proof of the well known properties of non-modularity and non-atomicity of the lattice of projectors of a type III factor.

Proposition 4.9 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a type III factor. Then*

1. *If $0 < Y < X < 1_\mathcal{H}$ then $Y \sim_p X$.*
2. *$\mathcal{P}(\mathcal{N})$ is a non modular lattice.*
3. *$\mathcal{P}(\mathcal{N})$ has no atoms.*

Proof: 1) By Proposition 2.17-2, we have that $X \sim Y < X$. Moreover, by Proposition 4.8, there exists a partial isometry defining an order isomorphism $w_{X^\perp}^* : \mathcal{P}(\mathcal{N}) \to [0, X^\perp]_{\mathcal{P}(\mathcal{N})}$. Therefore, $w_{X^\perp}^*(X^\perp) \leq X^\perp$ and, by Proposition 2.17-2 again, we also have that $Y \preceq w_{X^\perp}^*(X^\perp)$. In this way $Y \preceq w_{X^\perp}^*(X^\perp) \leq X^\perp$. Then, by Proposition 2.14, Y and X are perspective in $[0, X \vee w_{X^\perp}^*(X^\perp)]_{\mathcal{P}(\mathcal{N})}$. Thus, by Proposition 1.8, $Y \sim_p X$.

2) Immediately follows from item 1.

3) Let $Z \in \mathcal{P}(\mathcal{N}) - \{0, 1_\mathcal{H}\}$. By Proposition 2.17 there exists a partial isometry W such that $WW^* = 1_\mathcal{H}$ and $W^*W = P_{Z^\perp}$. Thus, by Proposition 4.8-3, we have that $0 < w_Z^*(Z) < Z$ and, consequently, Z is not an atom. Hence $\mathcal{P}(\mathcal{N})$ has no atoms.

□

The following theorem provides a purely lattice order characterization of type III factors.

Theorem 4.10 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be factor. Then the following statements are equivalent:*

1. *$\mathcal{N}$ is a a non trivial type III factor.*

2. *$\mathcal{P}(\mathcal{N})$ admits two binary operations $w(-,-)$ and $w^*(-,-)$ such that, upon defining $w_z(X) = w(Z,X)$ and $w^*_z(X) = w^*(Z,X)$, the following conditions are satisfied:*

   III*1.* $w_0(0) = 0$,

   III*2.* $X \leq w_0(X)$,

   III*3.* $Y = (Y \wedge w_0(X)) \vee (Y \wedge w_0(X)^\perp)$,

   III*4.* $w_z(X \wedge Y) \leq w_z(X)$,

   III*5.* $w_0(Z) \wedge w^*_z(X \wedge Y) \leq w^*_z(X)$,

   III*6.* $w^*_0(Z) \wedge Z = w^*_0(Z) \wedge w^*_z(Z)$
   $$= \Big(w_0(Z^\perp)^\perp \wedge w^*_0(Z)\Big) \vee \Big(w_0(Z^\perp) \wedge w_0\big(w^*_0(Z) \wedge Z\big)\Big),$$

   III*7.* $w^*_0(Z) \vee Z = w^*_0(Z) \vee w^*_z(Z) = w_0\big(w^*_0(Z) \vee Z\big)$,

   III*8.* $w_0(Z) \leq w_z\big(w^*_z(X)\big) R X$,

   III*9.* $w_0(Z) \leq w^*_z\big(w_z(X)\big) R \mu_z(X)$,

   III*10.* $w_0\big(w^*_0(1)\big) = w_0\big(w^*_0(1)^\perp\big)$.

   In this case w_0 is the internal dimension function on $\mathcal{P}(\mathcal{N})$ and w^*_0 satisfies the following conditions

$$w^*_0(0) = 0. \tag{23}$$

$$0 < w^*_0(1_\mathcal{H}) < 1_\mathcal{H}. \tag{24}$$

$$w^*_0(Z) \textit{ is a common complement of } \{Z, w^*_z(Z)\} \textit{ for } Z \neq 0, 1_\mathcal{H}. \tag{25}$$

Proof: $1 \Longrightarrow 2$) Let us suppose that $\mathcal{N}$ is a non trivial type III factor. Let w_0 be the internal dimension function on $\mathcal{P}(\mathcal{N})$. Then, by Proposition 1.6, conditions III*1*, III*2* and III*3* are satisfied. By the Borchers condition, mentioned in Proposition 2.17, for each $Z \in \mathcal{P}(\mathcal{N}) - \{0\}$ there exists a partial isometry W_Z such that $W_Z W^*_Z = 1_\mathcal{H}$ and $W^*_Z W_Z = P_z$, thereby defining a pair of Borchers operations $\{w_Z, w^*_Z\}$ on $\mathcal{P}(\mathcal{N})$ associated to W_Z. By Proposition 4.8, w_Z and w^*_Z are order preserving operations such that

$$w_Z(w^*_Z(X)) = X, \quad w^*_Z(w_Z(X)) = \mu_z(X). \tag{26}$$

In this way, by Proposition 4.8-3, for each $Z \in \mathcal{P}(\mathcal{N}) - \{0, 1_\mathcal{H}\}$ we have that $0 < w^*_z(Z) < Z$ and, by Proposition 4.9-1, $w^*_z(Z) \sim_p Z$. This allow us to define an unary operation $w^*_0(-)$ on $\mathcal{P}(\mathcal{N})$ such that

$w^*_0(0) = 0$,

$0 < w^*_0(1_\mathcal{H}) < 1_\mathcal{H}$,

$w^*_0(Z)$ is a common complement of $\{Z, w^*_z(Z)\}$ for $Z \neq 0, 1_\mathcal{H}$.

In this way w_0^* satisfies Eq.(23), Eq.(24) and Eq.(25). Moreover, $w_0\big(w_0^*(1)\big) = w_0\big(w_0^*(1)^\perp\big)$ because $w_0^*(1) \neq 0, 1_{\mathcal{H}}$. Thus, condition III*10* is also satisfied.

III*4*. Let us notice that w_0 is an order preserving map. Then, $w_0(X \wedge Y) \leq w_0(X)$. If $Z \neq 0$ then the partial isometry W_Z preserves the inclusion of closed subspaces. Thus, $w_{_Z}(X \wedge Y) \leq w_{_Z}(X)$.

III*5*. If $Z = 0$ then $w_0(0) \wedge w_0^*(X \wedge Y) = 0 \leq w_0^*(X)$. Let us suppose that $Z \neq 0$. Then W_Z^* preserves the inclusion of closed subspaces because it is also a partial isometry. Thus, $w_0(Z) \wedge w_{_Z}^*(X \wedge Y) = 1_{\mathcal{H}} \wedge w_{_Z}^*(X \wedge Y) \leq w_{_Z}^*(X)$.

III*6*. By straightforward calculation we can see that condition III*6* is satisfied for $Z \in \{0, 1_{\mathcal{H}}\}$. Let us suppose that $0 < Z < 1_{\mathcal{H}}$. By definition of w_0^*, $w_0^*(Z)$ is a common complement of $\{Z, w_{_Z}^*(Z)\}$ and then, $w_0^*(Z) \wedge Z = w_0^*(Z) \wedge w_{_Z}^*(Z) = 0$. Let also notice that $0 < Z^\perp < 1_{\mathcal{H}}$. Then, $\Big(w_0(Z^\perp)^\perp \wedge w_0^*(Z)\Big) \vee \Big(w_0(Z^\perp) \wedge w_0\big(w_0^*(Z) \wedge Z\big)\Big) = \big(0 \wedge w_0^*(Z)\big) \vee \big(1 \wedge w_0\big(w_0^*(Z) \wedge Z\big)\big) = 0 \vee (1 \wedge w_0(0)) = 0$. Thus III*6* is also satisfied in this case.

III*7*. If $Z \in \{0, 1_{\mathcal{H}}\}$ then, by straightforward calculation, III*7* is satisfied. The case $0 < Z < 1_{\mathcal{H}}$ follows along the same lines as the proof of condition III*6*.

III*8*, III*9*. If $Z = 0$ then III*8* and III*9* are trivially satisfied. If $Z \neq 0$ then, by Eq.(26), $w_{_Z}(w_{_Z}^*(X))RX = 1$ and $w_{_Z}^*(w_{_Z}(X))R\mu_{_Z}(X) = 1$. Thus III*8* and III*9* are also satisfied in this case.

Finally, if we define the binary operations $w(Z, X) = w_Z(X)$ and $w^*(Z, X) = w_Z^*(X)$ then our claim is established.

$2 \Longrightarrow 1$) Let us suppose that $\mathcal{P}(\mathcal{N})$ admits two binary operations $w(-,-)$ and $w^*(-,-)$ such that, upon defining $w_{_Z}(X) = w(Z, X)$ and $w_{_Z}^*(X) = w^*(Z, X)$, conditions III*1* ... III*10* are satisfied. Since $\mathcal{N}$ is a factor, combining conditions III*1*, III*2*, III*3* and Proposition 1.6, $w_0(0) = 0$ and $w_0(X) = 1_{\mathcal{H}}$ whenever $X \neq 0$. By condition III*10* we have that $0 < w_0^*(1_{\mathcal{H}}) < 1_{\mathcal{H}}$. Thus, Eq.(24) is satisfied and $\mathcal{N}$ is a non trivial factor. By condition III*6* we have that $0 = w_0^*(0) \wedge 0 = w_0^*(0) \wedge w_0^*(0) = w_0^*(0)$. Therefore, Eq.(23) is satisfied. Let us suppose that $0 < Z < 1_{\mathcal{H}}$. Then $w_0(Z) = w_0(Z^\perp) = 1$ and, by condition III*6*, $w_0^*(Z) \wedge Z = w_0^*(Z) \wedge w_{_Z}^*(Z) = w_0\big(w_0^*(Z) \wedge Z\big) \in \{0, 1_{\mathcal{H}}\}$. If $w_0^*(Z) \wedge Z = 1_{\mathcal{H}}$ then $Z = 1_{\mathcal{H}}$ which is a contradiction. Then,

$$w_0^*(Z) \wedge Z = w_0^*(Z) \wedge w_{_Z}^*(Z) = 0. \tag{27}$$

By condition III*7*, $w_0^*(Z) \vee Z = w_0^*(Z) \vee w_{_Z}^*(Z) = w_0\big(w_0^*(Z) \vee Z\big) \in \{0, 1_{\mathcal{H}}\}$. If $w_0^*(Z) \vee Z = 0$ then $Z = 0$ which is a contradiction. Then,

$$w_0^*(Z) \vee Z = w_0^*(Z) \vee w_{_Z}^*(Z) = 1. \tag{28}$$

Thus, by Eq.(27) and Eq.(28), $w_0^*(Z)$ is a common complement of $\{Z, w_{_Z}^*(Z)\}$ whenever $Z \neq 0, 1_{\mathcal{H}}$. Consequently, Eq.(25) is also satisfied.

By Theorem 2.16, there exists a dimension function $D : \mathcal{P}(\mathcal{N}) \to [0, \infty]$ uniquely determined up to positive constant multiple. Then we shall prove that for each $Z \in \mathcal{P}(\mathcal{N}) - \{0, 1_{\mathcal{H}}\}$, $D(Z) = \infty$. Indeed:

Since $Z \neq 0$ we have that $w_0(Z) = 1$. Then, by conditions III*8* and III*9*, $w_{_Z}(w_{_Z}^*(X))RX = 1$ and $w_{_Z}^*(w_{_Z}(X))R\mu_Z(X) = 1$ respectively. Therefore, by

Eq.(4), we have that

$$w_Z(w_Z^*(X)) = X \ \ and \ \ w_Z^*(w_Z(X)) = \mu_Z(X). \tag{29}$$

We also note that, by condition III*4* and III*5*, w_Z and w_Z^* are order preserving maps in $\mathcal{P}(\mathcal{N})$. Thus, by Eq.(29) and Proposition 1.12, we have that

$$0 < w_Z^*(Z) < Z < 1. \tag{30}$$

We also note that, by Eq.(27) and Eq.(28), $w_Z^*(Z) \sim_p Z$. Then, by Eq.(30) and by Proposition 2.18-2, we have that $D(Z) = \infty$. It proves that $\mathcal{N}$ is a type III factor and w_0 is the internal dimension function.

□

Let us notice that conditions III*1* ... III*10* can be rephrased as an equational system because $\mathcal{P}(\mathcal{N})$ is a lattice.

Proposition 4.11 *Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a type III factor and let us consider two binary operations $w(-,-)$ and $w^*(-,-)$ on $\mathcal{P}(\mathcal{N})$ satisfying the conditions III1 ... III10 of the Theorem 4.10. Then,*

1. $w_X(y \wedge w_0(Z)) = w_{X \wedge w_0(Z)}(Y \wedge w_0(Z)) = w_X(Y) \wedge w_0(Z)$.
2. $w_X^*(Y \wedge w_0(Z)) = w_{X \wedge w_0(Z)}^*(Y \wedge w_0(Z)) = w_X^*(Y) \wedge w_0(Z)$.

Proof: 1) Let us suppose that $Z = 0$. Then, $w_X(Y \wedge w_0(0)) = w_X(0) = 0$, $w_{X \wedge w_0(0)}(Y \wedge w_0(0)) = w_0(Y \wedge 0) = 0$ and $w_X(Y) \wedge w_0(0) = w_X(Y) \wedge 0 = 0$. Thus, the equations are satisfied in this case. Let us suppose that $Z \neq 0$. Then $w_X(y \wedge w_0(Z)) = w_X(Y \wedge 1) = w_X(Y)$, $w_{X \wedge w_0(Z)}(Y \wedge w_0(Z)) = w_{X \wedge 1}(Y \wedge 1) = w_X(Y)$ and $w_X(Y) \wedge w_0(Z) = w_X(Y) \wedge 1 = w_X(Y)$. Thus the equations are also satisfied in this case.

2) Let us suppose that $Z = 0$. Then, by Eq.(23) and Proposition 1.12-1, $w_X^*(Y \wedge w_0(0)) = w_X^*(0) = 0$, $w_{X \wedge w_0(0)}^*(Y \wedge w_0(0)) = w_0^*(0) = 0$ and $w_X^*(Y) \wedge w_0(0) = 0$. Thus, the equations are satisfied in this case. Let us suppose that $Z \neq 0$. Then, $w_X^*(Y \wedge w_0(Z)) = w_X^*(Y \wedge 1) = w_X^*(Y)$, $w_{X \wedge w_0(Z)}^*(Y \wedge w_0(Z)) = w_{X \wedge 1}^*(Y \wedge 1) = w_X^*(Y)$ and $w_X^*(Y) \wedge w_0(Z) = w_X^*(Y) \wedge 1 = w_X^*(Y)$. Thus the equations are also satisfied in this case.

□

5 The algebraic model of LQF-logic

Let us consider a net of observables $\mathfrak{R}(\mathcal{M}) \ni \mathcal{O} \mapsto \mathcal{N}(\mathcal{O})$ over the spacetime $\mathcal{M}$. As we have already mentioned in Section 3, each local observable algebra $\mathcal{N}(\mathcal{O})$ defines a local propositional system encoded in the von Neumann lattice $\mathcal{P}(\mathcal{N}(\mathcal{O}))$. Furthermore, as shown in Theorem 4.10, a set of equation formulated in an expanded language of $\mathcal{P}(\mathcal{N}(\mathcal{O}))$ is able to capture the fundamental requirement of the net of observables, namely, the type III factor condition of each algebra of the net. Thus, we are led to the following extension of the orthomodular structure that defines the algebraic model for the LQF-logic.

Definition 5.1 A *LQF-algebra* is an algebra $\langle A, \wedge, \vee, w, w^*, \neg, 0, 1\rangle$ of type $\langle 2,2,2,2,1,0,0\rangle$ such that, upon defining $w_z(x) = w(z,x)$ and $w^*_z(x) = w^*(z,x)$, the following conditions are satisfied:

LQF0. $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ is an orthomodular lattice,

LQF1. $w_0(0) = 0$,

LQF2. $x \leq w_0(x)$,

LQF3. $y = (y \wedge w_0(x)) \vee (y \wedge \neg w_0(x))$,

LQF4. $w_z(x \wedge y) \leq w_z(y)$,

LQF5. $w_0(z) \wedge w^*_z(x \wedge y) \leq w^*_z(x)$,

LQF6. $w^*_0(z) \wedge z = w^*_0(z) \wedge w^*_z(z)$
$$= \Big(\neg w_0(\neg z) \wedge w^*_0(z)\Big) \vee \Big(w_0(\neg z) \wedge w_0\big(w^*_0(z) \wedge z\big)\Big),$$

LQF7. $w^*_0(z) \vee z = w^*_0(z) \vee w^*_z(z) = w_0\big(w^*_0(z) \vee z\big)$,

LQF8. $w_0(z) \leq w_z\big(w^*_z(x)\big) R x$,

LQF9. $w_0(z) \leq w^*_z\big(w_z(x)\big) R \mu_z(x)$,

LQF10. $w_0\big(w^*_0(1)\big) = w_0\big(\neg w^*_0(1)\big)$,

LQF11. $w_x(y \wedge w_0(z)) = w_{x \wedge w_0(z)}(y \wedge w_0(z)) = w_x(y) \wedge w_0(z)$,

LQF12. $w^*_x(y \wedge w_0(z)) = w^*_{x \wedge w_0(z)}(y \wedge w_0(z)) = w^*_x(y) \wedge w_0(z)$.

In the same way as in Theorem 4.10 conditions LQF0 ... LQF12 can be rephrased as an equational system. Thus, the class of LQF-algebras defines a variety of algebras that we denote by $\mathcal{LQF}$. By a LQF-homomorphism we mean a $\langle \wedge, \vee, w, w^*, \neg, 0, 1\rangle$-preserving function between LQF-algebras.

Remark 5.2 Let us notice that LQF8 and LQF9 allow us to represent the Borchers condition, that characterize the type III factor condition, in a purely algebraic way.

Example 5.3 Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a non trivial type III factor. From Theorem 4.10 and Proposition 4.11, it immediately follows that the lattice of projector $\mathcal{P}(\mathcal{N})$ defines a LQF-algebra.

Proposition 5.4 *Let A be a LQF-algebra. Then*

1. $w_0(1) = 1$.
2. $w^*_0(0) = 0$.
3. $w_0(x) \in Z(A)$.

4. *If $x \leq y$ then $w_z(x) \leq w_z(y)$.*

5. *If $z < 1$ and $w_0(z) = 1$ then $0 < w_z^*(z) < z$.*

Proof: 1) It follows from LQF2. 2) If $z = 0$ then, by LQF1 and LQF12, we have that $w_0^*(0) = w_0^*(0 \wedge w_0(0)) = w_0^*(0) \wedge w_0(0) = w_0^*(0) \wedge 0 = 0$. 3) It follows from LQF3 and Proposition 1.4-2. 4) If $x \leq y$ then, by LQF4, $w_z(x) = w_z(x \wedge y) \leq w_z(y)$. 5) Let us suppose that $z < 1$ and $w_0(z) = 1$. By LQF5 w_z^* is an order preserving map and, by LQF8 and LQF9, $w_z w_z^* = id_A$ and $w_z^* w_z = \mu_z$ respectively. Thus, by Proposition 1.12-1, $w_z^* : A \to [0, z]$ is an order isomorphism. Consequently, $0 < w_z^*(z) < w_z^*(1) = z$ because $0 < z < 1$. 6) It follows from LQF7 and item 3.

$\square$

Proposition 5.5 *Let A be a LQF-algebra. Then, for each $z \in A$, the relation $\theta_{w_0(z)}$ on A given by*

$$(a, b) \in \theta_{w_0(z)} \quad \textit{iff} \quad a \wedge w_0(z) = b \wedge w_0(z)$$

is a factor congruence and $A/_{\theta_{w_0(z)}} \simeq_{LQF} [0, w_0(z)]_A$.

Proof: By Proposition 5.4-3 we have that $w_0(z) \in Z(A)$. Then $\theta_{w_0(z)}$ is a factor $\mathcal{OML}$-congruence and, by Proposition 1.5, $A/_{\theta_{w_0(z)}} \simeq_{OML} [0, w_0(z)]_A$. We have to prove that $w(-,-)$ and $w^*(-,-)$ are compatible operations with respect to $\theta_{w_0(z)}$. Indeed: Let us suppose that

$$a \wedge w_0(z) = b \wedge w_0(z).$$

In order to prove the compatibility of w let us notice that, by Axiom LFQ11, for each $x \in A$ we have that

$$\begin{aligned} w(x, a) \wedge w_0(z) &= w_x(a) \wedge w_0(z) = w_x(a \wedge w_0(z)) = w_x(b \wedge w_0(z)) \\ &= w_x(b) \wedge w_0(z) = w(x, b) \wedge w_0(z) \end{aligned}$$

and

$$\begin{aligned} w(a, x) \wedge w_0(z) &= w_a(x) \wedge w_0(z) = w_{a \wedge w_0(z)}(x \wedge w_0(z)) \\ &= w_{b \wedge w_0(z)}(x \wedge w_0(z)) = w_b(x) \wedge w_0(z) \\ &= w(b, x) \wedge w_0(z). \end{aligned}$$

The above equations then give immediately the compatibility of w with respect to $\theta_{w_0(z)}$.

For the compatibility of w^* let us notice that, by Axiom LFQ12, for each $x \in A$ we have that

$$\begin{aligned} w^*(x, a) \wedge w_0(z) &= w_x^*(a) \wedge w_0(z) = w_x^*(a \wedge w_0(z)) = w_x^*(b \wedge w_0(z)) \\ &= w_x^*(b) \wedge w_0(z) = w^*(x, b) \wedge w_0(z) \end{aligned}$$

and

$$\begin{aligned} w^*(a, x) \wedge w_0(z) &= w_a^*(x) \wedge w_0(z) = w_{a \wedge w_0(z)}^*(x \wedge w_0(z)) \\ &= w_{b \wedge w_0(z)}^*(x \wedge w_0(z)) = w_b^*(x) \wedge w_0(z) \\ &= w^*(b, x) \wedge w_0(z). \end{aligned}$$

The above equations then give immediately the compatibility of w^* with respect to $\theta_{w_0(z)}$.

Thus, the orthomodular lattice $[0, w_0(z)]_A$ endowed with the operations

$$[0, w_0(z)]_A \ni (x, y) \mapsto w(x, y)_{/\theta_{w_0(z)}} = w(x, y) \wedge w_0(z)$$

$$[0, w_0(z)]_A \ni (x, y) \mapsto w^*(x, y)_{/\theta_{w_0(z)}} = w^*(x, y) \wedge w_0(z)$$

is a LQF-algebra and $A/_{\theta_{w_0(z)}} \simeq_{LQF} [0, w_0(z)]_A$.

$\square$

Proposition 5.6 *Let A be a LQF-algebra. Then the following conditions are equivalent:*

1. *A is directly indecomposable algebra.*
2. *$w_0(z) = 1$ iff $z \neq 0$.*
3. *For each $z \in A - \{0, 1\}$, $w_0^*(z)$ is a common complement of $\{w_z^*(z), z\}$.*
4. *$Z(A) = \{0, 1\}$.*

Proof: $1 \Longrightarrow 2$) If A is directly indecomposable algebra then, by Proposition 5.5, $w_0(z) \in \{0, 1\}$. Thus, by Axiom LFQ1 and LQF2, we have that $w_0(z) = 1$ iff $x \neq 0$.

$2 \Longrightarrow 3$) Let $z \in A - \{0, 1\}$. Then $0 < \neg z < 1$ and, by hypothesis, $w_0(z) = w_0(\neg z) = 1$. Therefore, by LFQ6, we have that

$$\begin{aligned} w_0^*(z) \wedge z = w_0^*(z) \wedge w_z^*(z) &= \Big(\neg 1 \wedge w_0^*(z)\Big) \vee \Big(1 \wedge w_0\big(w_0^*(z) \wedge z\big)\Big) \\ &= w_0(w_0^*(z) \wedge z) \in \{0, 1\}. \end{aligned}$$

In this way $w_0^*(z) \wedge z = 0$; otherwise $z = 1$, which is a contradiction. Thus,

$$w_0^*(z) \wedge z = w_0^*(z) \wedge w_z^*(z) = 0. \tag{31}$$

By hypothesis and LFQ7 we also have that $w_0^*(z) \vee z = w_0^*(z) \vee w_z^*(z) = w_0\big(w_0^*(z) \vee z\big) \in \{0, 1\}$. In this way $w_0^*(z) \vee z = 1$; otherwise $z = 0$, which is a contradiction. Thus,

$$w_0^*(z) \vee z = w_0^*(z) \vee w_z^*(z) = 1. \tag{32}$$

Hence, by Eq.(31) and Eq.(32), $w_0^*(z)$ is a common complement of $\{z, w_z^*(z)\}$.

$3 \Longrightarrow 4$) Let us suppose that there exists $z \in Z(A)$ such that $0 < z < 1$. By hypothesis we have that $w_0^*(z)$ is a common complement of $\{z, w_z^*(z)\}$. Let us notice that $\neg z$ is the unique complement of z because $z \in Z(A)$. Therefore $w_0^*(z) = \neg z$ and

$$z = w_z^*(z). \tag{33}$$

Next, let us observe that $w_0(z) = 1$ then, by LFQ8 and LFQ9, $w_z\big(w_z^*(x)\big)Rx = 1$ i.e., $w_z\big(w_z^*(x)\big) = x$ and $w_z^*\big(w_z(x)\big)R\mu_z(x) = 1$ i.e., $w_z^*\big(w_z(x)\big) = \mu_z(x)$

respectively. Thus, by Proposition 1.12-1, $w_z^*(z) < z$ contradicting Eq.(33). Hence, $Z(A) = \{0, 1\}$.

$4 \Longrightarrow 1$) If $Z(A) = \{0, 1\}$ then A is directly indecomposable as orthomodular lattice. Then A is directly indecomposable as LQF-algebra.

$\square$

Proposition 5.7 *Let A be a LQF-algebra. Then:*

1. *A has no atoms.*
2. *$Card(A) \geq \aleph_0$.*
3. *A is a non modular lattice.*

Proof: 1) We first prove that $1 \in A$ is not an atom. Let us notice that $w_0^*(1) \notin \{0, 1\}$; otherwise, by Proposition 5.6-2, $w_0(w_0^*(1)) \neq w_0(\neg w_0^*(1))$, that contradicts LQF10. Thus, $0 < w_0^*(1) < 1$ and 1 cannot be an atom. In other words $A \neq \{0, 1\}$. Let $a \in A - \{0, 1\}$.

First suppose that A is a directly indecomposable LQF-algebra. By Proposition 5.6-2, $w_0(a) = 1$. Then, by Proposition 5.4-5, we have that $0 < w_a^*(a) < a$, so a is not an atom.

For the general case, suppose that a is an atom. Let us consider a subdirect representation $f : A \hookrightarrow \prod_{i \in I} A_i$ in the variety $\mathcal{LQF}$. Since $a > 0$ and f is injective then $f(a) = (a_i)_{i \in I} > 0$. Thus, there exists $j \in I$ such that $a_j = \pi_j f(a) > 0$. Since A_j is directly indecomposable algebra, a_j is not an atom, so there exists $x_j \in A_j$ such that

$$0 < x_j < a_j. \tag{34}$$

Since $\pi_j f$ is a surjective LQF-homomorphism then there exists $x \in A - \{0\}$ such that $\pi_j f(x) = x_j$. Note that $x \not< a$ since we assumed that a is an atom. Then we have to consider two possible cases:

If $a \leq x$ then $a_j = \pi_j f(a) \leq \pi_j f(x) = x_j$ that contradicts Eq.(34). Otherwise, if a is not comparable to x then, $a \wedge x = 0$ since a is an atom. Thus, $0 = \pi_j f(a \wedge x) = \pi_j f(a) \wedge \pi_j f(x) = a_j \wedge x_j = x_j$ that also contradicts Eq.(34). Hence, A has no atoms.

2) It immediately follows from the above item.

3) First let us suppose that A is a directly indecomposable LQF-algebra. By item 1 there exists $z \in A$ such that $0 < z < 1$ and then, by Proposition 5.6-2, $w_0(z) = 1$. Therefore, by Proposition 5.4-5, we have that $0 < w_z^*(z) < z$. Furthermore, by Proposition 5.6-3, $w_0^*(z)$ is a common complement of $\{w_z^*(z), z\}$ and, consequently, $w_z^*(z) \sim_p z$. Hence, by Proposition 1.10-2, A is a non modular lattice.

For the general case let us suppose that A is a modular lattice. Let us consider a subdirect representation $f : A \hookrightarrow \prod_{i \in I} A_i$ in the variety $\mathcal{LQF}$. For $i \in I$ let us consider $z_i \in A_i$ such that $0 < z_i < 1$. Thus $0 < w_{z_i}^*(z) < z_i$

where $w_0^*(z_i)$ is a common complement of $\{w_{z_i}^*(z), z_i\}$ because A_i is a directly indecomposable LQF-algebra. From this it follows that

$$(w_{z_i}^*(z_i) \wedge z_i) \vee (w_0^*(z_i) \wedge z_i) \neq \Big((w_{z_i}^*(z_i) \wedge z_i) \vee w_0^*(z_i)\Big) \wedge z_i. \qquad (35)$$

By the subjectivity of $\pi_i f$ there exists $z \in A$ such that $\pi_i f(z) = z_i$. Moreover, $(w_z^*(z) \wedge z) \vee (w_0^*(z) \wedge z) = \big((w_z^*(z) \wedge z) \vee w_0^*(z)\big) \wedge z$ since we assumed that A is a modular lattice. Thus,

$$\begin{aligned}(w_{z_i}^*(z_i) \wedge z_i) \vee (w_0^*(z_i) \wedge z_i) &= \pi_i f\big((w_z^*(z) \wedge z) \vee (w_0^*(z) \wedge z)\big) \\ &= \pi_i f\Big(\big((w_z^*(z) \wedge z) \vee w_0^*(z)\big) \wedge z\Big) \\ &= \Big((w_{z_i}^*(z_i) \wedge z_i) \vee w_0^*(z_i)\Big) \wedge z_i\end{aligned}$$

that contradicts Eq.(35). Hence, A is a non modular lattice.

□

Remark 5.8 Combining Theorem 5.6 and Proposition 5.7 we can observe that the class of LQF-algebras successfully captures the most remarkable properties of the projector lattices of the type III factors mentioned in Section 3.

Proposition 5.9 *Let A be a LQF-algebra. Then:*

1. *For each $z \in Z(A)$, $w_0(z) = z$*
2. *For each $a \in A$, $w_0(a) = e(a)$ and $\neg w_0(\neg a) = e_d(a)$.*

Proof: 1) Let A be a LQF-algebra and, by Theorem 1.1, let us consider a subdirect representation $f : A \hookrightarrow \prod_{i \in I} A_i$ in the variety $\mathcal{LQF}$. Let $z \in Z(A)$. We first claim that, for each $i \in I$, $\pi_i f(z) \in Z(A_i)$. Indeed: Let $a_i \in A_i$. Since $\pi_i f$ is surjective then there exists $a \in A$ such that $\pi_i f(a) = a_i$. By Proposition 1.4-2 we have that $a = (a \wedge z) \vee (a \wedge \neg z)$ and therefore, $a_i = \pi_i(a) = \big(\pi_i(a) \wedge \pi_i(z)\big) \vee \big(\pi_i(a) \wedge \pi_i(\neg z)\big) = (a_i \wedge \pi_i(z)) \vee (a_i \wedge \neg \pi_i(z))$. Thus, again by Proposition 1.4-2, $\pi_i f(z) \in Z(A_i)$ as claimed. Since A_i is a directly indecomposable LQF-algebras, so $Z(A_i) = \mathbf{2}$, then $f(z) = (z_i)_{i \in I}$ where $z_i \in \mathbf{2}$. Thus, $f(w_0(z)) = w_0(f(z)) = (w_0(z_i))_{i \in I} = (z_i)_{i \in I} = f(z)$. Hence, $z = w_0(z)$ because f is an injective map.

2) Let $a \in A$ and let us consider the set $Z^{\uparrow}(a) = \{z \in Z(A) : a \leq z\}$. By LFQ2 and Proposition 5.4-3 we have that $w_0(a) \in Z^{\uparrow}(a)$. Let $z \in Z^{\uparrow}(a)$. Then, by Proposition 5.4-4 and item 1, $w_0(a) \leq w_0(z) = z$. It proves that $w_0(a) = \bigwedge Z^{\uparrow}(a) = e(a)$. Finally, by Eq.(9), $\neg w_0(\neg a) = e_d(a)$.

□

Proposition 5.10 *Let A be a LQF-algebra. Then,*

1. *$e_d(e_d(x)) = e_d(x)$.*
2. *If $x \leq y$ then $e_d(x) \leq e_d(y)$.*

3. $e_d(x \wedge y) = e_d(x) \wedge e_d(y)$.

4. $e_d(x_1 R x_2) \wedge e_d(y_1 R y_2) \leq w(x_1, y_1) R w(x_2, y_2)$.

5. $e_d(x_1 R x_2) \wedge e_d(y_1 R y_2) \leq w^*(x_1, y_1) R w^*(x_2, y_2)$.

Proof: 1) It immediately follows from Proposition 5.9.

2) If $x \leq y$ then $e_d(x) \leq y$. Since $e_d(x) \in Z(A)$ then, $e_d(x) \leq \bigvee\{z \in Z(A) : z \leq y\} = e_d(y)$.

3) We first note that $e_d(x) \wedge e_d(y) \in Z(A)$ and $e_d(x) \wedge e_d(y) \leq x \wedge y$. Let $z \in Z(A)$ such that $z \leq x \wedge y \leq x, y$. Then, by item 2 and Proposition 5.9-1, $z = e_d(z) \leq e_d(x), e_d(y)$, so $z \leq e_d(x) \wedge e_d(y)$. Hence, $e_d(x) \wedge e_d(y) = \bigvee\{z \in Z(A) : z \leq x \wedge y\} = e_d(x \wedge y)$.

4,5) Let us remark that the inequalities of these items can be equivalently formulated by equations. Then, by Eq.(2), it is enough to study these inequalities in the class $\mathcal{DI}(\mathcal{LQF})$. Let A be a directly indecomposable LQF-algebra. By Proposition 5.4-3 and Proposition 5.6-4 we have that $e_d(x_1 R x_2) \wedge e_d(y_1 R y_2) \in Z(A) = \{0, 1\}$. Let us assume that $e_d(x_1 R x_2) \wedge e_d(y_1 R y_2) = 1$. Then, by Eq.(4), $x_1 = x_2$ and $y_1 = y_2$ and so $w(x_1, y_1) = w(x_2, y_2)$, $w^*(x_1, y_1) = w^*(x_2, y_2)$. Again by Eq.(4), we have that $w(x_1, y_1) R w(x_2, y_2) = 1$ and $w^*(x_1, y_1) R w^*(x_2, y_2) = 1$. Hence our claim.

□

Proposition 5.11 *Let A be a directly indecomposable LQF-algebra. Then the term*

$$t(x, y, z) = \Big(x \wedge \neg e_d(xRy)\Big) \vee \Big(z \wedge e_d(xRy)\Big)$$

is a discriminator term for A and $\mathcal{LQF}$ is a discriminator variety.

Proof: Let A be a directly indecomposable LQF-algebra. We first note that, by Proposition 5.6, $Z(A) = \{0, 1\}$. Moreover, by Proposition 5.9, $e_d(x) = 0$ for each $x \neq 1$. Let $x, y, z \in A$. Suppose that $x = y$. Therefore, by Eq.(4), $xRy = 1$ and then, $e_d(xRy) = 1$. Thus $t(x, y, z) = z$. Now let us suppose that $x \neq y$. Therefore, by Eq.(4), $xRy < 1$ and then $e_d(xRy) = 0$. Thus $t(x, y, z) = x$. Hence, $t(x, y, z)$ is a discriminator term in A. Since $\mathcal{DI}(\mathcal{LQF})$ generates $\mathcal{LQF}$ it is a discriminator variety.

□

Combining Proposition 5.11 and Proposition 1.2 we can establish the following result.

Proposition 5.12 *$\mathcal{LQF}$ is an arithmetical semisimple variety. Therefore,*

$$\mathcal{DI}(\mathcal{LQF}) = \mathcal{SI}(\mathcal{LQF}) = \mathcal{S}im(\mathcal{LQF}).$$

□

6 Filters and congruences in LQF-algebras

In this section we develop the filter theory for LQF-algebras. In order to do this, we first recall some basic results about filter theory for orthomodular lattices and Boolean algebras. Let L be an orthomodular lattice. An *increasing set* in L is a subset F of L such that if $a \in F$ and $a \leq x$ then $x \in F$. An *OML-filter* (also called *perspective filter* [19]) in L is a subset $F \subseteq L$ satisfying the following conditions:

1. F is an increasing set.
2. If $a, b \in F$ then $a \wedge b \in F$.
3. If $a \in F$ and $a \sim_p b$ then $b \in F$.

We denote by $Filt_{_{OML}}(L)$ complete lattice of OML-filters in L. A OML-filter in L is called *proper* iff $F \neq L$, or equivalently, iff $0 \notin F$. A *maximal OML-filter* is a proper OML-filter maximal with respect to inclusion. In the particular case in which L is a Boolean algebra the notion of filter, referred as *BA-filter*, is characterized by the first two conditions above. If L is a Boolean algebra then we denote by $Filt_{_{BA}}(L)$ the complete lattice of BA-filter of L. It is well known that a BA-filter F in the Boolean algebra L is maximal iff, for each $x \in L$, $x \in F$ or $\neg x \in F$.

Let L be an orthomodular lattice. If we denote by $Con_{_{OML}}(L)$ the congruence lattice of L then the map

$$Con_{_{OML}}(L) \ni \theta \mapsto F_\theta = \{x \in L : (x, 1) \in \theta\} \tag{36}$$

is a lattice order isomorphism from $Con_{_{OML}}(L)$ onto $Filt_{_{OML}}(L)$ (see [19, §2 Theorem 6]) whose inverse is given by

$$Filt_{_{OML}}(L) \ni F \mapsto \theta_F = \{(x, y) \in L^2 : xRy \in F\}. \tag{37}$$

Definition 6.1 Let A be a LQF-algebra. A *LQF-filter* in A is an OML-filter of A which is closed under ϵ_d.

Example 6.2 Let $f : A \to B$ be a LQF-homomorphism. Clearly $Ker(f) = \{x \in A : f(x) = 1\}$ is a OML-filter which is closed under e_d. Hence, $Ker(f)$ is a LQF-filter.

We denote by $Filt_{_{LQF}}(A)$ the set of all LQF-filters in the LQF-algebra A. Recall that $Filt_{_{LQF}}(A)$ is a closure system, hence it is also a complete lattice under the set inclusion. The notion of *maximal LQF-filter* is defined in an analogous way to the maximal OML-filters. By $Con_{_{LQF}}(A)$, we denote the congruence lattice of A.

Proposition 6.3 *Let A be a LQF-algebra. Then the maps $\theta \mapsto F_\theta$ and $F \mapsto \theta_F$, defined in Eq.(36) and Eq.(37) respectively, are mutually inverse lattice order isomorphisms between $Con_{LQF}(A)$ and $Filt_{_{LQF}}(A)$.*

Proof: Let us suppose that $\theta \in Con_{LQF}(A)$. Since θ_F is a OML-congruence, $F_\theta = \{x \in L : (x,1) \in \theta_F\}$ is OML-filter. Let $x \in F_\theta$, that is, $(x,1) \in \theta$. Since θ is compatible with ϵ_d then $(\epsilon_d(x),1) = (\epsilon_d(x), \epsilon_d(1)) \in \theta$. Thus, $\epsilon_d(x) \in F_\theta$ and $F_\theta \in Filt_{_{LQF}}(A)$.

For the converse, let us suppose that $F \in Filt_{_{LQF}}(A)$. We first note that θ_F is a OML-congruence. Thus, we only need to look at the compatibility of θ_F with respect to the binary operations w and w^*. Indeed: Let $(x_1,x_2) \in \theta_F$ and $(y_1,y_2) \in \theta_F$, that is, $x_1Rx_2 \in F$ and $y_1Ry_2 \in F$ respectively. Note that $\epsilon_d(x_1Rx_2) \in F$ and $\epsilon_d(y_1Ry_2) \in F$ because F is closed by ϵ_d. Therefore, $e_d(x_1Rx_2) \wedge e_d(y_1Ry_2) \in F$ because F is also closed under the infimum. Since F is an increasing set, by Proposition 5.10-4 and 5, we have that

$$e_d(x_1Rx_2) \wedge e_d(y_1Ry_2) \leq w(x_1,y_1)Rw(x_2,y_2) \in F,$$

$$e_d(x_1Rx_2) \wedge e_d(y_1Ry_2) \leq w^*(x_1,y_1)Rw^*(x_2,y_2) \in F.$$

It proves that $(w(x_1,y_1), w(x_2,y_2)) \in \theta_F$ and $(w^*(x_1,y_1), w^*(x_2,y_2)) \in \theta_F$, so θ_F is compatible with w and w^*. Therefore, $\theta_F \in Con_{LQF}(A)$.

Summarizing, by the above results, we have that

$$F \in Filt_{_{LQF}}(A) \ \text{ iff } \ \theta_F \in Con_{_{LQF}}(A). \tag{38}$$

Since the maps $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse lattice-isomorphisms between $Con_{_{OML}}(A)$ and $Filt_{_{OML}}(A)$ with respect to the OML-reduct $\langle A, \vee, \wedge, \neg, 0, 1\rangle$, by Eq.(38), we have that $Filt_{_{LQF}}(A)$ and $Con_{_{LQF}}(A)$ are lattice-order isomorphic.

$\square$

Let A be a LQF-algebra and $M \subseteq A$. Since $Filt_{_{LQF}}(A)$ is a complete lattice we define the *LQF-filter generated by M* as

$$F_{_{LQF}}(M) = \bigcap \{F \in Filt_{_{LQF}}(A) : M \subseteq F\}$$

that is, the smallest LQF-filter containing M. In particular, if $M = \{a\}$ then $F_{_{LQF}}(a)$ is called the *principal filtyer associated to a*.

Proposition 6.4 *Let A be LQF-algebra, $M \subseteq A$ and $a \in A$. Then:*

1. $F_{_{LQF}}(M) = \{x \in L : \exists x_1 \ldots x_n \in M$ *such that* $e_d(x_1 \wedge \ldots \wedge x_n) \leq x\}$.
2. $F_{_{LQF}}(a) = [e_d(a), 1]$.
3. $F_{_{LQF}}(M)$ *is a proper filter iff each finite subset* $\{x_1 \ldots x_n\} \subseteq M$ *satisfies* $e_d(x_1 \wedge \ldots \wedge x_n) > 0$.

Proof: We first prove that $M_0 = \{x \in L : \exists x_1 \ldots x_n \in M$ such that $e_d(x_1 \wedge \ldots \wedge x_n) \leq x\}$ is a LQF-filter. Let us notice that M_0 is an increasing set and, by Proposition 5.10-1 it is closed under e_d. Then it remains to prove that M_0 is closed under $\wedge$ and closed under perspectivity. Indeed:

To see that M_0 is closed under $\wedge$, let $x, y \in M_0$. Then $e_d(x_1 \wedge \ldots \wedge x_n) \leq x$ and $e_d(y_1 \wedge \ldots \wedge y_m) \leq y$ where $x_1, \ldots, x_n, y_1, \ldots y_n \in M$. Thus, by Proposition 5.10-3, $e_d(x_1 \wedge \ldots \wedge x_n \wedge y_1, \ldots y_n) = e_d(x_1 \wedge \ldots \wedge x_n) \wedge e_d(y_1 \wedge \ldots \wedge y_m) \leq x \wedge y$, so $x \wedge y \in M_0$.

To show that M_0 is closed under perspectivity, let $x \in M_0$ i.e., there exists $\{x_1 \ldots x_n\} \subseteq M$ such that $e_d(x_1 \wedge \ldots \wedge x_n) \leq x$, and $y \in A$ such that $x \sim_p y$. Then, by Proposition 1.9-2, $e_d(x_1 \wedge \ldots \wedge x_n) \leq y$ because $e_d(x_1 \wedge \ldots \wedge x_n) \in Z(A)$. Thus $y \in M_0$, so M_0 is closed under perspectivity.

Therefore, M_0 is a LQF-filter. Let us notice that $M \subseteq M_0$ and, for each $F \in Filt_{LQF}(A)$ containing M, we have that $M_0 \subseteq F$. Thus, M_0 is the smallest LQF-filter containing M i.e., $M_0 = F_{LQF}(M)$. This proves item 1. Item 2 and item 3 are handled similarly.

$\square$

Proposition 6.5 *Let A be LQF-algebra and F be a LQF-filter. Then:*

1. *$F \cap Z(A) \in Filt_{BA}(Z(A))$.*
2. *$F = F_{LQF}(F \cap Z(A))$.*
3. *The map $Filt_{BA}(Z(A)) \ni G \mapsto F_{LQF}(G)$ defines a lattice order isomorphism between $Filt_{BA}(Z(A))$ and $Filt_{LQF}(A)$.*

Proof: 1) Since F is a LQF-filter it follows that, $F \cap Z(A)$ is closed under $\wedge$ and it is also an increasing set in $Z(A)$. Thus $F \cap Z(A) \in Filt_{BA}(Z(A))$.

2) Let us notice that $F_{LQF}(F \cap Z(A)) \subseteq F$ because $F \cap Z(A) \subseteq F$. To prove equality of these two sets, let $x \in F$. Then, $e_d(x) \in F \cap Z(A)$ and $e_d(x) \leq x$. Thus, $x \in F_{LQF}(F \cap Z(A))$ and $F \subseteq F_{LQF}(F \cap Z(A))$. Hence, $F = F_{LQF}(F \cap Z(A))$.

3) We first note that $G \mapsto F_{LQF}(G)$ is an order inclusion preserving map and, by the above items, it is also surjective. Then we prove that the map $G \mapsto F_{LQF}(G)$ is injective. Let $G_1, G_2 \in Filt_{BA}(Z(A))$ such that $G_1 \neq G_2$. Then, there exists $z \in Z(A)$ such that $z \in G_1$ and $z \notin G_2$. Let us suppose that $F_{LQF}(G_1) = F_{LQF}(G_2)$. Then $z \in F_{LQF}(G_1)$ and $z \in F_{LQF}(G_2)$. By Proposition 6.4-1 there exists $z_1 \ldots z_n \in G_2 \subseteq Z(A)$ such that $e_d(z_1 \wedge \ldots \wedge z_n) \leq z$. Note that $z_1 \wedge \ldots \wedge z_n \in G_2$ because G_2 is closed under $\wedge$ and, by Proposition 5.10-3, $z_1 \wedge \ldots \wedge z_n = e_d(z_1) \wedge \ldots \wedge e_d(z_n) = e_d(z_1 \wedge \ldots \wedge z_n) \leq z$. Since G_2 is an increasing set then $z \in G_2$ which is a contradiction. Thus, $F_{LQF}(G_1) \neq F_{LQF}(G_2)$ and the map is injective. Hence, the map $Filt_{BA}(Z(A)) \ni G \mapsto F_{LQF}(G)$ defines a lattice order isomorphism between $Filt_{BA}(Z(A))$ and $Filt_{LQF}(A)$.

$\square$

The above proposition shows that the filter theory of a LQF-algebra is completely determined by the BA-filter theory of its center.

Proposition 6.6 *Let A be LQF-algebra and F be a LQF-filter. Then the following statement are equivalent*

1. *F is maximal.*

2. $F \cap Z(A)$ *is a maximal* BA*-filter in* $Z(A)$.

3. *For each* $x \in A$, $x \in F$ *or* $\neg e_d(x) \in F$.

Proof: $1 \Longrightarrow 2$) Let us assume that F is maximal. Let $z \in Z(A)$ such that $z \notin F \cap Z(A)$, i.e. $z \notin F$. Then, $A = F_{_{LQF}}(F \cup \{z\})$ because F is maximal. Therefore, $0 \in F_{_{LQF}}(F \cup \{z\})$ and, by Proposition 6.4-1, there exists $x_1 \dots x_n \in F$ such that $e_d(x_1) \wedge \dots \wedge e_d(x_n) \wedge z \leq 0$ where $e_d(x_1) \dots e_d(x_n) \in F \cap Z(A)$. In this way, $e_d(x_1) \wedge \dots \wedge e_d(x_n) \leq \neg z$ and $\neg z \in F$ because F is an increasing set. Hence, $\neg z \in F \cap Z(A)$ and $F \cap Z(A)$ is a maximal BA-filter in $Z(A)$.

$2 \Longrightarrow 3$) Let $x \in A$ such that $x \notin F$. Then, $e_d(x) \notin F \cap Z(A)$ and, consequently, $\neg e_d(x) \in F \cap Z(A)$ because $F \cap Z(A)$ is a maximal BA-filter in $Z(A)$. Hence our claim.

$3 \Longrightarrow 1$) Let us suppose that F is not maximal. Then, there exists $x \in A$ such that $x \notin F$ and $F_{_{LQF}}(F \cup \{x\})$ is a proper LQF-filter. Thus, $e_d(x) \in F_{_{LQF}}(F \cup \{x\})$ and, by hypothesis, $\neg e_d(x) \in F \subseteq F_{_{LQF}}(F \cup \{x\})$. Consequently, $F_{_{LQF}}(F \cup \{x\})$ is not proper which is a contradiction. Hence, F is maximal.

$\square$

Proposition 6.7 *$\mathcal{LQF}$ satisfies CEP.*

Proof: Let A be a LQF-algebra and let B be a sub LQF-algebra of A. For each $F \in Filt_{_{LQF}}(B)$, let $F^A_{_{LQF}}(F)$ be the LQF-filter of A generated by F. Clearly $F \subseteq B \cap F^A_{_{LQF}}(F)$. To prove equality of these two sets, let $x \in B \cap F^A_{_{LQF}}(F)$. By Proposition 6.4-1 there exist $x_1, \cdots, x_n \in F$ such that $e_d(x_1 \wedge \cdots \wedge x_n) \leq x$. Since $x \in B$ and F is an LQF-filter of B, hence upward closed, it follows that $x \in F$, so $B \cap F^A_{_{LQF}}(F) \subseteq F$. Thus, $\mathcal{LQF}$ satisfies CEP.

$\square$

7 A Hilbert style calculus for $\mathcal{LQF}$

In this section we give a Hilbert-style presentation for LQF-logic and we prove strong completeness with respect to the variety $\mathcal{LQF}$.

Let X be a denumerable set of variable. The language of the calculus is given by the absolutely free algebra $Term_{_{\mathcal{LQF}}}(X)$. In this case, valuations are homomorphisms of the form $v : Term_{_{\mathcal{LQF}}}(X) \to A$ where $A \in \mathcal{LQF}$. A term $t \in Term_{_{\mathcal{LQF}}}(X)$ is said to be a *tautology* iff for each valuation v, $v(t) = 1$. In this framework we regard LQF-terms as propositions and valid equations of the form $t = 1$ as tautologies. Each subset $T \subseteq Term_{_{\mathcal{LQF}}}(X)$ is referred as a *theory*. If v is a valuation then $v(T) = 1$ means that $v(t) = 1$ for each $t \in T$. Let $t \in Term_{_{\mathcal{LQF}}}(X)$ and T be a theory. We use $T \models_{_{\mathcal{LQF}}} t$, read t is a *semantic consequence* of T, in the case in which when $v(T) = 1$ then $v(t) = 1$ for each valuation v. In order to establish a Hilbert style calculus for $\mathcal{LQF}$ let us again consider the following notation

$$\begin{array}{rll} w_t(s) & \text{for} & w(t,s), \\ w_t^*(s) & \text{for} & w^*(t,s), \\ tRs & \text{for} & (t\wedge s)\vee(\neg t\wedge\neg s), \\ e_d(t) & \text{for} & \neg w_0(\neg t). \end{array}$$

Definition 7.1 The calculus $\langle Term_{\mathcal{LQF}}(X), \vdash\rangle$ is given by the following axioms:

A0. $(t\vee\neg t)R1$ and $(t\wedge\neg t)R0$,

A1. tRt,

A2. $\neg(tRs)\vee(\neg(sRr)\vee(tRr))$,

A3. $\neg(tRs)\vee(\neg tR\neg s)$,

A4. $\neg(tRs)\vee((t\wedge r)R(s\wedge r))$,

A5. $(t\wedge s)R(s\wedge t)$,

A6. $(t\wedge(s\wedge r))R((t\wedge s)\wedge r)$,

A7. $(t\wedge(t\vee s))Rt$,

A8. $(\neg t\wedge t)R((\neg t\wedge t)\wedge s)$,

A9. $tR\neg\neg t$,

A10. $\neg(t\vee s)R(\neg t\wedge\neg s)$,

A11. $(t\vee(\neg t\wedge(t\vee s))R(t\vee s)$,

A12. $(tRs)R(sRt)$,

A13. $\neg(tRs)\vee(\neg t\vee s)$,

A14. $w_0(0)R0$,

A15. $xR(x\wedge w_0(x))$,

A16. $yR\big((y\wedge w_0(x))\vee(y\wedge\neg w_0(x))\big)$,

A17. $\big(w_z(x\wedge y)\vee w_z(y)\big)Rw_z(y)$,

A18. $\Big(\big(w_0(z)\wedge w_z^*(x\wedge y)\big)\vee w_z^*(y)\Big)Rw_z^*(y)$,

A19. $\big(w_0^*(z)\wedge z\big)R\big(w_0^*(z)\wedge w_z^*(z)\big)$,

A20. $\big(w_0^*(z)\wedge z\big)R\Big(\neg w_0(\neg z)\wedge w_0^*(z)\Big)\vee\Big(w_0(\neg z)\wedge w_0\big(w_0^*(z)\wedge z\big)\Big)$, A21. $\big(w_0^*(z) \vee z\big) R \big(w_0^*(z) \vee w_z^*(z)\big)$,

A22. $\big(w_0^*(z) \vee z\big) R w_0 \big(w_0^*(z) \vee z\big)$,

A23. $w_0(z) R \Big(w_0(z) \wedge \big(w_z(w_z^*(x)) R x\big)\Big)$,

A24. $w_0(z) R \Big(w_0(z) \wedge \big(w_z^*\big(w_z(x)\big) R \mu_z(x)\big)\Big)$,

A25. $w_0\big(w_0^*(1)\big) R w_0\big(\neg w_0^*(1)\big)$,

A26. $w_x(y \wedge w_0(z)) R w_{x \wedge w_0(z)}(y \wedge w_0(z))$,

A27. $w_x(y \wedge w_0(z)) R \big(w_x(y) \wedge w_0(z)\big)$,

A28. $w_x^*(y \wedge w_0(z)) R w_{x \wedge w_0(z)}^*(y \wedge w_0(z))$,

A29. $w_x^*(y \wedge w_0(z)) R \big(w_x^*(y) \wedge w_0(z)\big)$,

A30. $\neg e_d(tRs) \vee \big(w_r(t) R\, w_r(s)\big)$,

A31. $\neg e_d(tRs) \vee \big(w_t(r) R\, w_s(r)\big)$,

A32. $\neg e_d(tRs) \vee \big(w_r^*(t) R\, w_r^*(s)\big)$,

A33. $\neg e_d(tRs) \vee \big(w_t^*(r) R\, w_s^*(r)\big)$.

and the following inference rules:

$$\frac{t, \neg t \vee s}{s}, \qquad \textit{disjunctive syllogism (DS)}$$

$$\frac{t}{e_d(t)}. \qquad \textit{necessitation (N)}$$

Let T be a theory in $Term_{\mathcal{LQF}}(X)$. A *proof* from T is a sequence $t_1, ..., t_n$ in $Term_{\mathcal{LQF}}(X)$ such that each member is either an axiom or a member of T or follows from some preceding member of the sequence using *DS* or *N*. If $t \in Term_{\mathcal{LQF}}(X)$, $T \vdash t$ means that t is provable from T, that is, t is the last element of a proof from T. If $T = \emptyset$ then we use the notation $\vdash t$ and in this case we will say that t is a theorem of the calculus $\langle Term_{\mathcal{LQF}}(X), \vdash\rangle$. The theory T is *inconsistent* if and only if $T \vdash t$ for each $t \in Term_{\mathcal{LQF}}(X)$; otherwise it is *consistent*.

Proposition 7.2 *1. Axioms of $\langle Term_{\mathcal{LQF}}(X), \vdash\rangle$ are tautologies.*

2. Let $t, s \in Term_{\mathcal{LQF}}(X)$ and v be a valuation such that $v(t) = v(\neg t \vee s) = 1$. Then $v(s) = 1$ i.e., DS preserves 1-valuations.

3. *Let $t \in Term_{\mathcal{LQF}}(X)$ and v be a valuation such that $v(t) = 1$. Then $v(e_d(t)) = 1$ i.e., N preserves 1-valuations.*

Proof: 1) For axioms A0 ... A13 we refer to [19, §15]. Since axioms A14 ... A29 are rephrased forms of the axioms of LQF-algebras, by Eq.(4), they are tautologies. Lastly, by Proposition 5.10-4 and 5, A30 ... A33 are also tautologies,

2,3) If $v(t) = v(\neg t \vee s) = 1$ then $v(s) = 0 \vee v(s) = \neg v(t) \vee v(s) = v(\neg t \vee s) = 1$. Thus, DS preserves 1-valuations. The preservation of 1-valuations across the inference rule N is immediate.

$\square$

An immediate consequence of the last proposition is the following.

Theorem 7.3 [Soundness] *Let T be a theory in $Term_{\mathcal{LQF}}(X)$ and $t \in Term_{\mathcal{LQF}}(X)$. Then:*

$$T \vdash t \Longrightarrow T \models_{\mathcal{LQF}} t.$$

$\square$

Proposition 7.4 *Let T be a theory in $Term_{\mathcal{LQF}}(X)$ and $t, s, r \in Term_{\mathcal{LQF}}(X)$. Then:*

1. $\vdash t \vee \neg t$,
2. $\vdash 1$,
3. $T \vdash tRs \Longrightarrow T \vdash sRt$,
4. $T \vdash tRs$ *and* $T \vdash sRr \implies T \vdash tRr$,
5. $T \vdash tRs \implies T \vdash \neg tR\neg s$,
6. $T \vdash tRs$ *and* $T \vdash t \wedge r \implies T \vdash s \wedge r$,
7. $T \vdash tRs$ *and* $T \vdash t \vee s \implies T \vdash s \vee r$,
8. $T \vdash tRs \implies T \vdash w_r(t)R\, w_r(s)$,
9. $T \vdash tRs \implies T \vdash w_t(r)R\, w_s(r)$,
10. $T \vdash tRs \implies T \vdash w^*_r(t)R\, w^*_r(s)$,
11. $T \vdash tRs \implies T \vdash w^*_t(r)R\, w^*_s(r)$.

Proof: 1) It follows from A1 and A13.
2)

(1) $\vdash t \vee \neg t$ — by item 1

(2) $\vdash (t \vee \neg t)R1$ — by A0

(3) $\vdash \neg((t \vee \neg t)R1) \vee (\neg(t \vee \neg t) \vee 1)$ — by A13

(4) $\vdash \neg(t \vee \neg t) \vee 1$ by DS 2,3

(5) $\vdash 1$ by DS 4,1

3)

(1) $T \vdash tRs$

(2) $T \vdash (tRs)R(sRt)$ by A12

(3) $T \vdash \neg((tRs)R(sRt)) \vee (\neg(tRs) \vee (sRt))$ by A13

(4) $T \vdash (\neg(tRs) \vee (sRt))$ by DS 2,2

(5) $T \vdash sRt$ by DS 1,4

4) It easily follows from A2 and two application of the DS.
5) It follows from A3.
6)

(1) $T \vdash tRs$

(2) $T \vdash t \wedge r$

(3) $T \vdash \neg(tRs) \vee ((t \wedge r)R(s \wedge r))$ by A4

(4) $T \vdash (t \wedge r)R(s \wedge r)$ by DS 1,2

(5) $\vdash \neg((t \wedge r)R(s \wedge r)) \vee (\neg(t \wedge r) \vee (s \wedge r))$ by A13

(6) $T \vdash s \wedge r$ by DS 5,4,2

7) It follows by item 4, A9 and A10.
8)

(1) $T \vdash tRs$

(2) $T \vdash e_d(tRs)$ by N 1

(3) $\vdash \neg e_d(tRs) \vee \big(w_r(t)R\ w_r(s)\big)$ by A30

(4) $T \vdash w_r(t)R\ w_r(s)$ by DS 2,3

9,10,11) These items can be proved in an exact way as the item 8 by taking into account axioms A31, A32, A32 respectively.

$\square$

Proposition 7.5 *Let T be a theory in $Term_{\mathcal{LQF}}(X)$ and let us consider the binary relation in $Term_{\mathcal{LQF}}(X)$ given by*

$$t \equiv_T s \quad \textit{iff} \quad T \vdash tRs.$$

Then $\equiv_T$ is an equivalence in $Term_{\mathcal{LQF}}(X)$. Moreover if we define the following operations on $\mathcal{L}_T(X) = Term_{\mathcal{LQF}}(X)/_{\equiv_T}$

$$
\begin{aligned}
[t]_T \wedge [s]_T &= [t \wedge s]_T, & \neg[t]_T &= [\neg t]_T,\\
[t]_T \vee [s]_T &= [t \vee s]_T, & 0 &= [0]_T,\\
w([t]_T, [s]_T) &= [w(t,s)]_T, & 1 &= [1]_T,\\
w^*([t]_T, [s]_T) &= [w^*(t,s)]_T. & &
\end{aligned}
$$

Then we have

1. $\langle \mathcal{L}_T(X), \wedge, \vee, w, w^*, \neg, 0, 1\rangle$ *is a LQF-algebra.*
2. $T \vdash t$ *if and only if* $[t]_T = 1$.

Proof: By Axiom A1 and Proposition 7.4-3 and 4, $\equiv_T$ is an equivalence in $Term_{\mathcal{LQF}}(X)$. We first prove that the operations $\langle \wedge, \vee, w, w^*, \neg, 0, 1\rangle$ are well defined on $\mathcal{L}_T(X)$ i.e, they are compatible operations with respect to $\equiv_T$. Let us suppose that

$$T \vdash t_1 R t_2 \quad \text{and} \quad T \vdash s_1 R s_2.$$

We prove that $T \vdash (t_1 \wedge t_2) R (s_1 \wedge s_2)$.

(1) $T \vdash t_1 R t_2$	hypothesis
(2) $\vdash \neg(t_1 R t_2) \vee ((t_1 \wedge s_1) R (t_2 \wedge s_1))$	A4
(3) $T \vdash (t_1 \wedge s_1) R (t_2 \wedge s_1)$	by DS 1,2
(4) $T \vdash s_1 R s_2$	hypothesis
(5) $\vdash \neg(s_1 R s_2) \vee ((s_1 \wedge t_2) R (s_2 \wedge t_2))$	A4
(6) $T \vdash (s_1 \wedge t_2) R (s_2 \wedge t_2)$	by DS 4,5
(7) $T \vdash (t_1 \wedge s_1) R (t_2 \wedge s_2)$	by 3,6, A5, Proposition 7.4-4

It proves that $\wedge$ is well defined on $\mathcal{L}_T(X)$. By Proposition 7.4-5 we can easily show that $\neg$ is well defined on $\mathcal{L}_T(X)$. Next, combining A10 with the previous results it is easily seen that $T \vdash (t_1 \vee t_2) R (s_1 \vee s_2)$. Thus $\vee$ is also well defined on $\mathcal{L}_T(X)$.

We prove that $T \vdash w(t_1, s_1) R w(t_2, s_2)$.

(1) $T \vdash s_1 R s_2$	hypothesis
(2) $T \vdash w(t_1, s_1) R\, w(t_1, s_2)$	Proposition 7.4-8
(3) $T \vdash t_1 R t_2$	hypothesis
(4) $T \vdash w(t_1, s_2) R\, w(t_2, s_2)$	Proposition 7.4-9

(5) $T \vdash w(t_1, s_1)Rw(t_2, s_2)$ by 3,5 and Proposition 7.4-4

It proves that w is well defined on $\mathcal{L}_T(X)$. Analogously, by using Proposition 7.4-10 and 11, we can also prove that w^* is well defined on $\mathcal{L}_T(X)$.

1) By straightforward calculation it can be seen that the reduct $\langle \mathcal{L}_T(X), \wedge, \vee, \neg, 0, 1\rangle$ is an orthomodular lattice (for more details we refer to [19, §15, 1. Proposition]). Since Axioms A14 ... A33 are the axioms of LQF-algebras rephrased in terms of the secondary connective tRs, we have that $\langle \mathcal{L}_T(X), \wedge, \vee, w, w^*, \neg, 0, 1\rangle$ is a LQF-algebra.

2) Let us notice that $[tR1]_T = [t]_T R[1]_T = [t]_T$. Thus, $T \vdash t$ if and only if $T \vdash tR1$ iff $[t]_T = 1$.

□

Theorem 7.6 [Strong Completeness] *Let $t \in Term_{\mathcal{LQF}}(X)$ and T be a theory in $Term_{\mathcal{LQF}}(X)$. Then,*

$$T \models_{\mathcal{LQF}} t \Longrightarrow T \vdash t.$$

Proof: If T is inconsistent, this result is trivial. Let us assume that T is consistent and that $T \models_{\mathcal{LQF}} t$. Let us suppose that T does not prove t. If we consider the valuation $v : Term_{\mathcal{LQF}}(X) \rightarrow \mathcal{L}_T(X)$ such that $v(s) = [s]_T$ then, by Proposition 7.5-2, $[t]_T \neq 1$ which is a contradiction. Hence $T \vdash t$.

□

Corollary 7.7 (Compactness) *Let $t \in Term_{\mathcal{LQF}}(X)$ and T be a theory in $Term_{\mathcal{LQF}}(X)$. Then, $T \models_{\mathcal{LQF}} t$ iff there exists a finite subset $T_0 \subseteq T$ such that $T_0 \models_{\mathcal{LQF}} t$.*

Proof: Let us suppose that $T \models_{\mathcal{LQF}} t$. Then, by Theorem 7.6, $T \vdash t$ and we can suppose that $t_1, \cdots t_m, t$ is a proof of t from T. If we consider the finite set $T_0 = T \cap \{t_1, \cdots t_n\}$ then $T_0 \vdash t$ and, by Theorem 7.3, we have that $T_0 \models_{\mathcal{LQF}} t$.

□

We can also establish a kind of deduction theorem for $\langle Term_{\mathcal{LQF}}(X), \vdash\rangle$.

Theorem 7.8 *Let $s, t \in Term_{\mathcal{LQF}}(X)$ and T be a theory in $Term_{\mathcal{LQF}}(X)$. Then we have that:*

$$T \cup \{s\} \vdash t \quad iff \quad T \vdash \neg e_d(s) \vee t.$$

Proof: Let us suppose $T \cup \{s\} \vdash t$. Then, by Corollary 7.7, there exists $t_1 \ldots t_n \in T$ such that $(t_1 \wedge \ldots \wedge t_n) \wedge s \models_{\mathcal{LQF}} t$. Let $r = t_1 \wedge \ldots \wedge t_n$. Then $r \wedge s \models_{\mathcal{LQF}} t$ implies that $(r \wedge s) \vee \neg e_d(s) \models_{\mathcal{LQF}} \neg e_d(s) \vee t$. Therefore, $r \vee \neg e_d(s) \models_{\mathcal{LQF}} \neg e_d(s) \vee t$ because for each valuation v, $v(e_d(s))$ is a central element and $v(s \vee \neg e_d(s)) = 1$. Since $r \models_{\mathcal{LQF}} r \vee \neg e_d(s)$ then $r \models_{\mathcal{LQF}} \neg e_d(s) \vee t$. Thus, $T \models_{\mathcal{LQF}} \neg e_d(s) \vee t$ and, by Theorem 7.6, $T \vdash \neg e_d(s) \vee t$.

On the other hand, let us suppose that $T \vdash \neg e_d(s) \vee t$. Then, there exist $t_1 \ldots t_n \in T$ such that, by defining $r = t_1 \wedge \ldots \wedge t_n$, $r \models_{\mathcal{LQF}} \neg e_d(s) \vee t$. Therefore, we also have that $r \wedge e_d(s) \models_{\mathcal{LQF}} e_d(s) \wedge (\neg e_d(s) \vee t)$ and, consequently, $r \wedge$

$e_d(s) \models_{\mathcal{LQF}} e_d(s) \wedge t$ because for each valuation v, $v(e_d(s) \wedge (\neg e_d(s) \vee t)) = v(e_d(s) \wedge t)$. Since $e_d(s) \wedge t \models_{\mathcal{LQF}} t$ we have that

$$r \wedge e_d(s) \models_{\mathcal{LQF}} t. \tag{39}$$

Let v be a valuation such that $v(T) = 1$ and $v(s) = 1$. Then, $v(r) = 1$ and $v(e_d(s)) = e_d(v(s)) = e_d(1) = 1$. Thus, by Eq.(39), $v(t) = 1$ proving that $T \cup \{s\} \models_{\mathcal{LQF}} t$. Hence, by Theorem 7.6, $T \cup \{s\} \vdash t$.

$\square$

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