

A reverse duality for the ASEP with open boundaries

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Abstract

We prove a duality between the asymmetric simple exclusion process (ASEP) with non-conservative open boundary conditions and an asymmetric exclusion process with particle-dependent hopping rates and conservative reflecting boundaries. This is a reverse duality in the sense that the duality function relates the measures of the dual processes rather than expectations. Specifically, for a certain parameter manifold of the boundary parameters of the open ASEP this duality expresses the time evolution of a family of shock product measures with N microscopic shocks in terms of the time evolution of N particles in the dual process. The reverse duality also elucidates some so far poorly understood properties of the stationary matrix product measures of the open ASEP given by finite-dimensional matrices.

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1 Introduction

Duality expresses time-dependent expectations for one Markov process in terms of time-dependent expectations of another Markov process and thus is an important tool for the study of stochastic interacting particle systems [50, 51, 40], particularly if the dual process has some simple properties that are “hidden” in the original process. The key object is the duality function D : Processes $\boldsymbol{\eta}(t)$ with state space Ω and $\boldsymbol{\xi}(t)$ with state space Ξ respectively are said to be dual w.r.t. a bounded measurable function $D : \Omega \times \Xi \rightarrow \mathbb{R}$ if and only if for all $\boldsymbol{\eta} \in \Omega, \boldsymbol{\xi} \in \Xi$ and $t \geq 0$ one has $\mathbf{E}_{\boldsymbol{\eta}} D(\boldsymbol{\eta}(t), \boldsymbol{\xi}) = \mathbf{E}_{\boldsymbol{\xi}} D(\boldsymbol{\eta}, \boldsymbol{\xi}(t))$. The subscript denotes expectation with the initial states $\boldsymbol{\eta} = \boldsymbol{\eta}(0)$ and $\boldsymbol{\xi} = \boldsymbol{\xi}(0)$ resp. Considering the duality function $D(\boldsymbol{\eta}, \boldsymbol{\xi})$ to represent a family of functions $f^{\boldsymbol{\xi}}(\boldsymbol{\eta}) = D(\boldsymbol{\eta}, \boldsymbol{\xi})$ indexed by $\boldsymbol{\xi}$ then duality expresses for all members of this family the expectation of this function at time t in terms of the expectation at time t of the family of functions $g^{\boldsymbol{\eta}}(\boldsymbol{\xi}) = D(\boldsymbol{\eta}, \boldsymbol{\xi})$ indexed by $\boldsymbol{\eta}$ of the dual process.

In the last decade a very considerable body of work on applications of duality to interacting particle systems defined on the one-dimensional integer lattice has emerged see e.g. [10, 45, 52, 11, 18, 16, 46, 47] and references therein specifically for very recent work on duality for processes closely related to the ASEP which is studied in the present paper. To push this field further we introduce the concept of reverse duality which expresses the time evolution of probability measures of one process in terms of the dual process rather than relating time-dependent expectations. Reverse duality can, however, be rephrased as a conventional duality for the time-reversed processes and thus extends the framework of conventional duality. We use it to derive a duality that has so far remained elusive, viz., for the asymmetric simple exclusion process (ASEP) on a one-dimensional integer lattice of L sites with two non-conservative open boundaries [49, 44, 25, 63]. On a certain manifold of boundary rates for injection and absorption the open ASEP is shown to be reverse-dual to an ASEP with particle-dependent jump rates on the same lattice but with two conservative reflecting boundaries. Thus we go beyond a recent proof of self-duality for one open and one reflective boundary condition [46, 47] and an earlier failure to construct a non-trivial duality for two open boundaries [57].

The reverse duality for the open ASEP constructed here allows for expressing the time evolution of a certain family of Bernoulli shock measures with N shocks in terms of the evolution of the probability measure of the dual ASEP with N particles and thus reduces the complexity of the time evolution of the system with 2^L states to $\binom{L}{N}$ states. Specifically, the time evolution of a single shock in the many-particle ASEP turns into the simple problem of a single biased random walker. The evolution of initial measures with more than one shock can be studied through the complete integrability [37, 27] which provides an explicit system of Bethe equations for the spectrum of the generator [55, 56, 22, 23, 70] and for which results on the eigenfunctions are available [21]. Along the way one also obtains insight into some of the properties of the stationary matrix product measures of the open ASEP introduced in [25] which on the parameter manifold studied here is given by finite-

dimensional matrices [54].

The paper is organized as follows. In Sec. 2 we introduce reverse duality in general terms, and define the ASEP with open boundaries, shock measures for the ASEP (which give rise to the duality function), and the dual shock ASEP with particle-dependent hopping rates (which turns out to be the reverse dual). The results, in particular the reverse duality, are stated in precise form as theorems in Sec. 3, with comments on the significance of these results. The theorems are formally proved in Sec. 4. In Appendix A we show how the generator of the open ASEP in the parametrization of the present work is related to the integrable Heisenberg quantum spin chain with non-hermitian boundary fields of [55, 56] and in Appendix B we discuss some properties of the boundary parameters.

2 Definitions

2.1 General definitions and conventions

We consider the finite integer lattice $\Lambda_L = [L_-, L_+] \cap \mathbb{Z}$ with $L = L_+ - L_- + 1 \geq 2$ sites. The Kronecker- δ is defined by

$$\delta_{x,y} := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else} \end{cases} \quad (2.1)$$

for x, y from any set.

For a finite or countably infinite vector space column vectors with components ϕ_i are denoted by the quantum mechanical ket symbol $|\phi\rangle$ while row vectors denoted by the bra symbol $\langle\phi|$ have complex conjugate components $\bar{\phi}_i$. The standard scalar product is written in the Dirac convention

$$\langle\phi|\psi\rangle = \sum_i \bar{\phi}_i \psi_i \quad (2.2)$$

Notice the complex conjugation of the left argument.

Definition 2.1 (Kronecker product) *Let A and B be two finite-dimensional matrices with $m_A \geq 1$ ($m_B \geq 1$) rows and $n_A \geq 1$ ($n_B \geq 1$) columns with matrix elements A_{ij} and B_{kl} respectively. The Kronecker product $A \otimes B$ is a $m_A m_B \times n_A n_B$ -matrix C with matrix elements*

$$C_{(i-1)m_B+k, (j-1)n_B+l} = A_{ij} B_{kl} \quad (2.3)$$

with $i \in \{1, \dots, m_A\}, j \in \{1, \dots, n_A\}, k \in \{1, \dots, m_B\}, l \in \{1, \dots, n_B\}$. The n -fold Kronecker product of a matrix A with itself is denoted by $A^{\otimes n}$ and by convention $A^{\otimes 0} = 1$ and $A^{\otimes 1} = A$.

Definition 2.2 (Local and bond operators) *(i) With the two-dimensional unit matrix denoted by $\mathbb{1}$ and an arbitrary two-dimensional matrix \mathbf{a} the Kronecker product*

$$\mathbf{a}_k := \mathbb{1}^{\otimes(k-L_-)} \otimes \mathbf{a} \otimes \mathbb{1}^{\otimes(L_+-k)}, \quad k \in \Lambda_L \quad (2.4)$$

is called a local operator acting on site $k \in \Lambda_L$.

(ii) For any four-dimensional matrix \mathbf{b} , bond operators $\mathbf{b}_{k,k+1}$ with $L_- \leq k \leq L_+ - 1$ are defined by the Kronecker product

$$\mathbf{b}_{k,k+1} = \mathbb{1}^{\otimes(k-L_-)} \otimes \mathbf{b} \otimes \mathbb{1}^{\otimes(L_+-1-k)}. \quad (2.5)$$

2.2 Duality and reverse duality

For many applications it is convenient to express the generator of a Markov process $\boldsymbol{\eta}(t)$ with countable state space in terms of the intensity matrix W whose off-diagonal matrix elements $W_{\boldsymbol{\eta},\boldsymbol{\eta}'}$ are the transition rates from configuration $\boldsymbol{\eta}$ to $\boldsymbol{\eta}'$ and with diagonal elements given by the negative sum $W_{\boldsymbol{\eta}\boldsymbol{\eta}} = -\sum_{\boldsymbol{\eta}'} W_{\boldsymbol{\eta},\boldsymbol{\eta}'}$ of outgoing transition rates from state $\boldsymbol{\eta}$. In terms of intensity matrices W for the process $\boldsymbol{\eta}(t)$ and Q for the process $\boldsymbol{\xi}(t)$, and with the duality matrix D with the matrix elements $D_{\boldsymbol{\eta}\boldsymbol{\xi}} = D(\boldsymbol{\eta}, \boldsymbol{\xi})$ duality is expressed in matrix form as the relation [72]

$$WD = DQ^T. \quad (2.6)$$

This notion of duality has the following natural generalization.

Definition 2.3 Let $\boldsymbol{\eta}(t)$ be a Markov process with intensity matrix W . A Markov process $\boldsymbol{\xi}(t)$ with intensity matrix Q satisfying

$$RW = Q^T R \quad (2.7)$$

with a duality matrix R with matrix elements $R_{\boldsymbol{\xi}\boldsymbol{\eta}}$ is called reverse dual to $\boldsymbol{\eta}(t)$ w.r.t. the duality function $R(\boldsymbol{\xi}, \boldsymbol{\eta}) = R_{\boldsymbol{\xi}\boldsymbol{\eta}}$.

Obviously, reverse duality is a mutual relation between two processes as can be seen by transposing both sides of the defining equation (2.7) which then says that $\boldsymbol{\eta}(t)$ is a reverse dual to $\boldsymbol{\xi}(t)$ with duality matrix R^T . To give some motivation for the notion of reverse duality assume that the processes $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ have invertible invariant measures π_W and π_Q resp., i.e., $\pi_W \neq 0$ for all $\boldsymbol{\eta}$ and $\pi_Q \neq 0$ for all $\boldsymbol{\xi}$. This implies that the diagonal matrices $\hat{\pi}_W^{-1}$ with diagonal entries $\pi_W^{-1}(\boldsymbol{\eta})$ and $\hat{\pi}_Q^{-1}$ with diagonal entries $\pi_Q^{-1}(\boldsymbol{\xi})$ exist. By definition, the intensity matrices for the reversed processes are then given by $W_{rev} = \hat{\pi}_W^{-1} W^T \hat{\pi}_W$ and $Q_{rev} = \hat{\pi}_Q^{-1} Q^T \hat{\pi}_Q$, resp., so that $W^T = \hat{\pi}_W W_{rev} \hat{\pi}_W^{-1}$ and $Q = \hat{\pi}_Q^{-1} Q_{rev}^T \hat{\pi}_Q$. After transposition and with the duality matrix $D = \hat{\pi}_W^{-1} R^T \hat{\pi}_Q^{-1}$ the defining relation (2.7) for reverse duality turns into the conventional duality relation $W_{rev} D = D Q_{rev}^T$ for the reversed processes. In other words, for reversible processes reverse duality implies a conventional duality for the reversed processes and vice versa.

While conventional duality relates expectation values of the dual processes, reverse duality can be used to relate the measures for all times $t \geq 0$. To see this, we denote by $\mu(t)$ the probability of the configuration $\boldsymbol{\eta}(t)$ for some unspecified initial measure $\mu(0)$ and for the process $\boldsymbol{\xi}(t)$ we denote by $P(\boldsymbol{\xi}, t | \boldsymbol{\xi}', 0)$ the transition probability at time t from a configuration $\boldsymbol{\xi}'$ to a configuration $\boldsymbol{\xi}$.

Theorem 2.4 For a countable set Ξ let \mathcal{F}_Ξ be a family of probability measures for the process $\boldsymbol{\eta}(t)$ with countable state space Ω and let the process $\boldsymbol{\xi}(t)$ with state space Ξ be reverse dual to $\boldsymbol{\eta}(t)$ with duality function $R(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mu_{\boldsymbol{\eta}}^{\boldsymbol{\xi}} \in \mathcal{F}_\Xi$. Then at time $t \geq 0$ the measure $\mu^{\boldsymbol{\xi}}(t)$ is given by the convex combination

$$\mu^{\boldsymbol{\xi}}(t) = \sum_{\boldsymbol{\xi}'} P(\boldsymbol{\xi}, t | \boldsymbol{\xi}', 0) \mu^{\boldsymbol{\xi}'}(0) \quad (2.8)$$

of initial measures $\mu_0^{\boldsymbol{\xi}'} \in \mathcal{F}_\Xi$.

Proof: The time evolution property (2.8) of the measure follows directly from the semigroup property of the Markov processes $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ and Definition 2.3 of reverse duality. \square

Thus if the duality function $R(\boldsymbol{\xi}, \boldsymbol{\eta})$ represents a family of probability measures $\mu_{\boldsymbol{\eta}}^{\boldsymbol{\xi}}$ indexed by $\boldsymbol{\xi}$ then reverse duality expresses the Markovian time evolution of all members of this family of measures in terms of the probability measure of the reverse dual process starting from configuration $\boldsymbol{\xi}$.¹ In contrast, considering the duality function of conventional duality to be member of a measurable family of functions $f^{\boldsymbol{\xi}}(\boldsymbol{\eta}) := D(\boldsymbol{\eta}, \boldsymbol{\xi})$ of the states $\boldsymbol{\eta}$ and denoting the expectation of this function at time $t \geq 0$ w.r.t. some unspecified initial measure by $\mathbf{E}f^{\boldsymbol{\xi}}(t)$, conventional duality reads

$$\mathbf{E}f^{\boldsymbol{\xi}}(t) = \sum_{\boldsymbol{\xi}'} P(\boldsymbol{\xi}, t | \boldsymbol{\xi}', 0) \mathbf{E}f^{\boldsymbol{\xi}'}(0). \quad (2.9)$$

This highlights the difference between reverse duality and conventional duality in terms of the role of the transition probability of the (reverse) dual process.

For reversible processes $\boldsymbol{\xi}(t)$ with reversible measure π Theorem 2.4 can be stated in a slightly modified form.

Theorem 2.5 For a countable set Ξ let \mathcal{F}_Ξ be a family of probability measures for the process $\boldsymbol{\eta}(t)$ with countable state space Ω and let the process $\boldsymbol{\xi}(t)$ with state space Ξ be (a) reversible with reversible measure π and (b) reverse dual to $\boldsymbol{\eta}(t)$ with duality function $R(\boldsymbol{\xi}, \boldsymbol{\eta}) = \pi(\boldsymbol{\xi}) \mu_{\boldsymbol{\eta}}^{\boldsymbol{\xi}} \in \mathcal{F}_\Xi$. Then at time $t \geq 0$ the measure $\mu^{\boldsymbol{\xi}}(t)$ is given by the convex combination

$$\mu^{\boldsymbol{\xi}}(t) = \sum_{\boldsymbol{\xi}'} P(\boldsymbol{\xi}', t | \boldsymbol{\xi}, 0) \mu^{\boldsymbol{\xi}'}(0) \quad (2.10)$$

of initial measures $\mu^{\boldsymbol{\xi}'}(0) \in \mathcal{F}_\Xi$.

Proof: With $S := \hat{\pi}^{-1}R$ reverse duality as defined in (2.7) implies for $Q = Q^{rev}$ the intertwining relation

$$SW = QS. \quad (2.11)$$

The time evolution property (2.10) then follows from the semigroup property of the Markov processes $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ and reversibility of $\boldsymbol{\xi}(t)$. \square

¹Duality between measures is discussed in a different setting also in [19].

Remark 2.6 *The difference to Theorem 2.4 appears in the time-reversal of the transition probability of the reverse dual process and in the relationship between the duality function $R(\boldsymbol{\xi}, \boldsymbol{\eta})$ and the measures $\mu^{\boldsymbol{\xi}}$.*

Remark 2.7 *For self-dual processes with $W = Q$ the intertwining relation (2.11) expresses a symmetry of the generator [64, 35, 15] and for invertible S it constitutes a similarity transformation between processes with intensity matrices W, Q [36].*

2.3 Intensity matrices and quantum Hamiltonians

We recall that by construction the sum of all matrix elements in each row of the intensity matrices add up to zero. It is well-known that the transposed intensity matrix often takes the form of a Hamiltonian operator $H = -W^T$ for quantum mechanical spin chains [1, 53, 68] and concepts from quantum mechanics can be imported to solve probabilistic problems even though in general H is not a Hermitian matrix. To illustrate the point in the context of duality we mention that an invariant measure is a ground state vector of the quantum system and non-trivial self-dualities arise from non-Abelian symmetries of the generator [65, 66, 35, 15]. An general relation between duality and spectral properties of the generator - which play a crucial role in quantum mechanics and can be computed exactly for integrable systems - was recently worked out in [58]. Complete integrability can also be used to directly construct novel dualities [69].

With this in mind we express below duality and reverse duality in terms of the negative transpose H of the intensity matrix and in slight abuse of language we also call H the generator of the process. Time-invariance of a measure π means that the vector $|\pi\rangle$ is a right eigenvector of H with eigenvalue 0. Defining the diagonal matrix $\hat{\pi}$ with the components $\pi(\cdot)$ on the diagonal, the generator of the time-reversed process is given by

$$H_{rev} := \hat{\pi} H^T \hat{\pi}^{-1} \quad (2.12)$$

and has the same invariant measure π . With $H = -W^T$ and $G = -Q^T$ duality as defined in (2.6) then reads

$$D^T H = G^T D^T \quad (2.13)$$

with transposed duality matrix D^T with matrix elements $D_{\boldsymbol{\xi}\boldsymbol{\eta}}^T = D(\boldsymbol{\eta}, \boldsymbol{\xi})$. Reversed duality as defined in (2.7) becomes

$$H R^T = R^T G^T \quad (2.14)$$

and for reversible G this yields the intertwining relation

$$H S^T = S^T G \quad (2.15)$$

with $S^T = R^T \hat{\pi}^{-1}$.

To rephrase Theorem 2.5 we use quantum mechanical bra-ket notation for vectors and the scalar product (2.2). Then a probability measure for the process $\boldsymbol{\eta}(t)$

is a ket vector $|\mu(t)\rangle$ whose components $\mu_{\boldsymbol{\eta}}(t)$ are the probabilities of configuration $\boldsymbol{\eta}(t)$. The Markov semigroup property yields $|\mu(t)\rangle = [\langle \mu(0) | \exp(Wt)]^T = \exp(-Ht)|\mu(0)\rangle$ so that the transition probability $\mu(\boldsymbol{\eta}', t|\boldsymbol{\eta}, 0)$ from a configuration $\boldsymbol{\eta}$ to a configuration $\boldsymbol{\eta}'$ is given by

$$\mu(\boldsymbol{\eta}', t|\boldsymbol{\eta}, 0) = \langle \boldsymbol{\eta} | e^{Wt} | \boldsymbol{\eta}' \rangle = \langle \boldsymbol{\eta}' | e^{-Ht} | \boldsymbol{\eta} \rangle. \quad (2.16)$$

and similarly for the process $\boldsymbol{\xi}(t)$ generated by G . The evolution formula (2.10) of Theorem 2.5 reads in vector form

$$|\mu^{\boldsymbol{\xi}}(t)\rangle = \sum_{\boldsymbol{\xi}'} P(\boldsymbol{\xi}', t|\boldsymbol{\xi}, 0) |\mu^{\boldsymbol{\xi}'}(0)\rangle. \quad (2.17)$$

with the transition probability $P(\boldsymbol{\xi}', t|\boldsymbol{\xi}, 0)$ of the dual process.

Remark 2.8 *Under the conditions of Theorem 2.5 the family \mathcal{F}_{Ξ} of probability vectors belongs to a right-invariant subspace of the generator H with dimension $d_{\Xi} \leq |\mathcal{F}_{\Xi}|$.*

2.4 ASEP with open boundaries

2.4.1 Generator

In the ASEP each lattice site $k \in \Lambda_L$ can be occupied or empty as expressed by the occupation numbers $\eta_k \in \{0, 1\}$. The particle configuration on the whole lattice λ_L is denoted by $\boldsymbol{\eta} := \{\eta_{L_-}, \eta_{L_-+1}, \dots, \eta_{L_+}\}$. For a given configuration $\boldsymbol{\eta}$ we also define the flipped configuration $\boldsymbol{\eta}^k$ with occupation numbers

$$\eta_l^k = \begin{cases} 1 - \eta_k & \text{if } l = k \\ \eta_l & \text{else.} \end{cases} \quad (2.18)$$

and the swapped configuration $\boldsymbol{\eta}^{kk+1}$ with occupation numbers

$$\eta_l^{kk+1} = \begin{cases} \eta_{k+1} & \text{if } l = k \\ \eta_k & \text{if } l = k + 1 \\ \eta_l & \text{else.} \end{cases} \quad (2.19)$$

Expectations are denoted by $\langle \cdot \rangle$. In particular, $\langle \eta_k \rangle$ is called the local density.

Informally, the Markovian dynamics of the open ASEP can be described as follows. A particle on a bulk site $k \in [L_- + 1, L_+ - 1]$ jumps with rate r (ℓ) independently of the other particles to right (left) neighbouring site $k + 1$ ($k - 1$), provided the target site is empty. Otherwise the jump attempt is rejected. On the left boundary site L_- a particle jumps with rate r to the right (if site $L_- + 1$ is empty) or is annihilated with rate γ . If site L_- is empty then a particle is created with rate α . On the right boundary site L_+ a particle jumps with rate ℓ to the left (if site $L_+ - 1$ is empty) or is annihilated with rate β . If site L_+ is empty then a particle is created with rate δ .

With the local jump rate

$$w_k(\boldsymbol{\eta}) = r\eta_k(1 - \eta_{k+1}) + \ell(1 - \eta_k)\eta_{k+1} \quad (2.20)$$

and the boundary rates

$$b^-(\boldsymbol{\eta}) = \alpha(1 - \eta_{L_-}) + \gamma\eta_{L_-}, \quad b^+(\boldsymbol{\eta}) = \delta(1 - \eta_{L_+}) + \beta\eta_{L_+} \quad (2.21)$$

the generator \mathcal{L} of this process is thus given by

$$\mathcal{L} = \mathcal{L}^{bulk} + \mathcal{L}^- + \mathcal{L}^+ \quad (2.22)$$

with the bulk generator

$$\mathcal{L}^{bulk} f(\boldsymbol{\eta}) = \sum_{k=L_-}^{L_+-1} w_k(\boldsymbol{\eta}) [f(\boldsymbol{\eta}^{kk+1}) - f(\boldsymbol{\eta})] \quad (2.23)$$

and the boundary generators

$$\mathcal{L}^- f(\boldsymbol{\eta}) = b^-(\boldsymbol{\eta}) [f(\boldsymbol{\eta}^{L_-}) - f(\boldsymbol{\eta})], \quad \mathcal{L}^+ f(\boldsymbol{\eta}) = b^+(\boldsymbol{\eta}) [f(\boldsymbol{\eta}^{L_+}) - f(\boldsymbol{\eta})]. \quad (2.24)$$

Throughout this work it is assumed that all rates $r, \ell, \alpha, \beta, \gamma, \delta$ are strictly positive and that $r \neq \ell$.

To write the corresponding intensity matrix we denote the two-dimensional unit matrix by $\mathbf{1}$ and define the matrices

$$w^b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\ell & \ell & 0 \\ 0 & r & -r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.25)$$

$$w^- = \begin{pmatrix} -\alpha & \alpha \\ \gamma & -\gamma \end{pmatrix} \quad (2.26)$$

$$w^+ = \begin{pmatrix} -\delta & \delta \\ \beta & -\beta \end{pmatrix}. \quad (2.27)$$

and the Kronecker products $w_{L_-}^- := w^- \otimes \mathbf{1}^{\otimes L_+ - L_-}$, $w_{L_+}^+ := \mathbf{1}^{\otimes L_+ - L_-} \otimes w^+$, and $w_{k,k+1}^b := \mathbf{1}^{\otimes -L_- + k} \otimes w^b \otimes \mathbf{1}^{\otimes L_+ - 1 - k}$. In terms of these matrices the intensity matrix for the open ASEP is given by

$$W = W_{bulk} + W_{edge} \quad (2.28)$$

with

$$W_{bulk} = \sum_{k=L_-}^{L_+-1} w_{k,k+1}^b, \quad W_{edge} = w_{L_-}^- + w_{L_+}^+. \quad (2.29)$$

In $H := -W^T$ one recognizes a non-hermitian version of the XXZ quantum spin chain with non-diagonal boundary fields [55, 56], see the appendix.

To make contact with the well-known properties of the XXZ chain, the computations below are done using H rather than W . Defining the particle number operator

$$\hat{n} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.30)$$

and the Kronecker product $\hat{n}_k := \mathbf{1}^{\otimes -L_- + k} \otimes \hat{n} \otimes \mathbf{1}^{\otimes L_+ - k}$ one gets

$$H = H_{bulk} + H_{edge} \quad (2.31)$$

with

$$H_{bulk} = \sum_{k=1}^{L-1} \tilde{h}_{k,k+1}, \quad H_{edge} = \tilde{h}_{L_-}^- + \tilde{h}_{L_+}^+ \quad (2.32)$$

where

$$\tilde{h}_{k,k+1} = -(w_{k,k+1}^b)^T + (r - \ell) (\hat{n}_{k+1} - \hat{n}_k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r & -r & 0 \\ 0 & -\ell & \ell & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{k,k+1} \quad (2.33)$$

$$\tilde{h}_{L_-}^- = -(w_{L_-}^-)^T + (r - \ell) \hat{n}_{L_-} = \begin{pmatrix} \alpha & -\gamma \\ -\alpha & \gamma + r - \ell \end{pmatrix}_{L_-}, \quad (2.34)$$

$$\tilde{h}_{L_+}^+ = -(w_{L_+}^+)^T + (r - \ell) \hat{n}_{L_+} = \begin{pmatrix} \delta & -\beta \\ -\delta & \beta - (r - \ell) \end{pmatrix}_{L_+}. \quad (2.35)$$

Notice the addition of the discrete gradient $(r - \ell) (\hat{n}_{k+1} - \hat{n}_k)$ to the local bulk generators which cancel in (2.31) with the corresponding contributions to the boundary generators in (2.34) and (2.35).

2.4.2 Particle current

Particle conservation in the bulk is reflected in the discrete continuity equation

$$\mathcal{L}\eta_k = j_{k-1} - j_k \quad (2.36)$$

where

$$j_k = r\eta_k(1 - \eta_{k+1}) - \ell(1 - \eta_k)\eta_{k+1}, \quad L_- \leq k < L_+ \quad (2.37)$$

is the instantaneous current across bond $(k, k + 1)$. At the boundaries one has

$$\mathcal{L}\eta_{L_-} = j^- - j_{L_-}, \quad \mathcal{L}\eta_{L_+} = j_{L_+ - 1} - j^+ \quad (2.38)$$

where

$$j^- = \alpha(1 - \eta_{L_-}) - \gamma\eta_{L_-}, \quad j^+ = \beta\eta_{L_+} - \delta(1 - \eta_{L_-}) \quad (2.39)$$

are the boundary currents with a source term a sink term each. Stationarity implies constant expectation

$$j := \langle j^- \rangle = \langle j_k \rangle = \langle j^+ \rangle \quad (2.40)$$

in the invariant measure which was first computed for the TASEP ($\gamma = \delta = \ell = 0$) in two different ways, viz., with a matrix product ansatz in [25] and simultaneously by generating function techniques in [63]. The general case was solved subsequently in [61].

2.4.3 Invariant matrix product measures

To discuss some properties of the invariant measure it is convenient to introduce the jump asymmetry q and the time scale w by

$$q := \sqrt{\frac{r}{\ell}}, \quad w := \sqrt{r\ell} \quad (2.41)$$

and to parametrize the boundary rates as

$$\alpha = (r + \omega_-)\rho_-, \quad \gamma = (\ell + \omega_-)(1 - \rho_-) \quad (2.42)$$

$$\beta = (r + \omega_+)(1 - \rho_+), \quad \delta = (\ell + \omega_+)\rho_+ \quad (2.43)$$

where the parameters ω_{\pm} may be interpreted as boundary jump barriers and, as seen below, the parameters ρ_{\pm} play the role of boundary densities. It is convenient to also consider the fugacity

$$z \equiv z(\rho) = \frac{\rho}{1 - \rho} \quad (2.44)$$

as a function of the density ρ . For an indexed density ρ_i with i from any index set we shall use the notation

$$z_i \equiv z(\rho_i) \quad (2.45)$$

as shorthand.

The invariant measure of the open ASEP is conveniently characterized in terms of the functions [61]

$$\kappa_{\pm}(x, y) := \frac{1}{2x}(y - x + r - \ell \pm \sqrt{(y - x + r - \ell)^2 + 4xy}) \quad (2.46)$$

which are the roots of the quadratic equation

$$x\kappa^2 - (y - x + r - \ell)\kappa - y = 0 \quad (2.47)$$

and which are related by

$$\kappa_+(x, y)\kappa_-(x, y) = -\frac{y}{x}. \quad (2.48)$$

As shown in [54] on the parameter manifold \mathcal{B}_N defined by the relation

$$\kappa_+(\alpha, \gamma)\kappa_+(\beta, \delta) = q^{2N} \quad (2.49)$$

the invariant measure for the open ASEP can be expressed in terms of a matrix product measure (MPM) with $(N + 1)$ -dimensional representation matrices of a

quadratic algebra satisfied by the matrices of the matrix product ansatz developed in [25]. On the submanifold $\mathcal{B}_N^M \subset \mathcal{B}_N$ defined by the further constraint

$$\kappa_-(\alpha, \gamma)\kappa_-(\beta, \delta) = q^{-2M}, \quad 1 \leq M \leq N \quad (2.50)$$

the MPM has the special property that it exists only for system sizes $L > N - M + 1$ [54] whereas away from this submanifold it exists for all system sizes [29]. Boundary parameters satisfying (2.49) and (2.50) for $1 \leq N \leq L$ define the setting that the main results proved below apply to.

To get further insight into the significance of the parameter manifold \mathcal{B}_N we note that for $N = 0$ the condition (2.49) reduces to $z_+ = z_-$. As shown in [61], with this constraint on the boundary parameters the invariant measure is a Bernoulli product measure with density $\rho_+ = \rho_-$ and the manifolds (B.3) and (B.4) in Appendix B turn into the stationarity condition (2.40) for the expectation of the current. Following [49] this observation may be understood as follows. In a semi-infinite system with $L_+ = +\infty$ the left boundary rates (2.42) allow for an invariant Bernoulli product measure with stationary density $\rho_- \leq 1/2$ and stationary current given by (B.3) (corresponding to $j = \langle j^- \rangle$) while in a semi-infinite system with $L_- = -\infty$ the right boundary rates (2.43) allow for an invariant Bernoulli product measure with a stationary density $\rho_+ \geq 1/2$ and stationary current given by (B.4) (corresponding to $j = \langle j^+ \rangle$). When $\rho_+ = \rho_-$ or $\rho_+ = 1 - \rho_-$ the two currents $\langle j^- \rangle$ and $\langle j^+ \rangle$ match. Then stationarity of the Bernoulli product measure holds even in the finite lattice Λ_L for any L_\pm and any density $\rho := \rho_+ = \rho_-$.

On the other hand, when $\rho_+ = 1 - \rho_-$ stationary can only be attained with a non-homogeneous density profile which has been argued to be a shock profile [63, 67] analogous to a domain wall in driven non-equilibrium systems [30, 48, 20, 41, 17] but with diffusive fluctuations of the domain wall position [28, 62, 23] proved rigorously for the ASEP on the infinite integer lattice in [32]. Any mismatch of the boundary currents in the range $\rho_- < 1/2$ and $\rho_+ > 1/2$ is then expected to be still realized by a shock profile, but with a shock localized close to the left boundary for $\langle j^+ \rangle < \langle j^- \rangle$ or close to the right boundary for $\langle j^+ \rangle > \langle j^- \rangle$, thus explaining the origin of the phase diagram of the open ASEP in terms of domain wall dynamics [42]. From a mathematical perspective, this reasoning is a conjecture based on microscopic shock stability proved only in the hydrodynamic limit [2]. As will be seen below, the reverse duality constructed below provides a rigorous proof of this conjecture on microscopic scale for the parameter manifold \mathcal{B}_N^1 defined by (2.49) and (2.50).

Finally we note that for system size $L = N$ the invariant measure on the parameter manifold \mathcal{B}_N^1 is a zero-current blocking measure restricted to the finite lattice [12]. This blocking measure, originally defined for the infinite system [50], is a product measure with strictly increasing marginal fugacities $z_k \propto q^{2(k-L_-)}$, $k \in \Lambda_L$, is also an invariant measure of the ASEP with reflecting boundaries [60] where $\alpha = \beta = \gamma = \delta = 0$.

2.5 Shocks in the ASEP

2.5.1 Bernoulli shock measures

In the spirit of [6, 10] we extend the notion of Bernoulli shock measures to the finite lattice. To this end and to allow for more compact notation auxiliary boundary sites are introduced.

Definition 2.9 (*Bernoulli shock measures*) For auxiliary boundary sites $x_0 := L - 1$ and $x_{N+1} := L + 1$ and a nonempty ordered set $\mathbf{x} := \{x_1, \dots, x_N\}$ of N lattice sites the product measure

$$\mu_{\boldsymbol{\eta}}^{\mathbf{x}} = \prod_{k=L_-}^{L_+} p_{\eta_k}^{\mathbf{x}} \quad (2.51)$$

with marginals

$$p_{\eta_k}^{\mathbf{x}} = \begin{cases} (1 - \rho_i^*)(1 - \eta_k) + \rho_i^* \eta_k & k = x_i, \quad 1 \leq i \leq N \\ (1 - \rho_i)(1 - \eta_k) + \rho_i \eta_k & x_i < k < x_{i+1}, \quad 0 \leq i \leq N \end{cases} \quad (2.52)$$

is called a Bernoulli shock measure with N microscopic shocks at positions $x_i \in \Lambda_L$ and left boundary density ρ_0 , right boundary density ρ_N , bulk densities ρ_i for $1 \leq i \leq N - 1$, and shock densities ρ_i^* for $1 \leq i \leq N$.

This definition is motivated by the fact that seen from a shock position the local particle configuration of an infinite system becomes uncorrelated at large distances [31, 24]. The Bernoulli shock measures extend this property to all lattice sites which was found in [26] for specific shock densities whose fugacities satisfy the relation

$$\frac{z_i}{z_{i-1}} = q^2 \quad (2.53)$$

and second-class particles at the shock positions.

The vector representation of a shock measure with N shocks is given by

$$|\mu^{\mathbf{x}}\rangle := \sum_{\boldsymbol{\eta}} \mu_{\boldsymbol{\eta}}^{\mathbf{x}} |\boldsymbol{\eta}\rangle \quad (2.54)$$

and with the column vector $|\rho.\rangle := (1 - \rho., \rho.)^T$ one gets the Kronecker product

$$|\mu^{\mathbf{x}}\rangle = |\rho_0\rangle^{\otimes(x_1-L_-)} \otimes |\rho_1^*\rangle \otimes |\rho_1\rangle^{\otimes(x_2-x_1-1)} \otimes \dots \otimes |\rho_N^*\rangle \otimes |\rho_N\rangle^{\otimes(L_+-x_N)} \quad (2.55)$$

which reflects the factorized form of the shock measure.

2.5.2 Microscopic and macoscopic shocks

For a Bernoulli shock measure the expectation of the particle current (2.37) inside a domain with density ρ_i is given by

$$j_i \equiv j(\rho_i) = (r - \ell)\rho_i(1 - \rho_i). \quad (2.56)$$

As shown in [6] for shock densities $\rho_i^* = 1$ the quantities

$$d_i^\ell = \frac{j_{i-1}}{\rho_i - \rho_{i-1}} \quad (2.57)$$

$$d_i^r = \frac{j_i}{\rho_i - \rho_{i-1}} \quad (2.58)$$

are the shock hopping rates of a microscopically stable shock with bulk densities ρ_{i-1}, ρ_i satisfying (2.53) which we therefore call the microscopic shock stability condition. In [43, 4, 9, 10] the same microscopic shock jump rates were found with a second-class particle [31] at the shock position which is equivalent to $\rho_i^* = 0$. The average shock velocity of shock i between bulk densities ρ_i and ρ_{i+1} is the difference

$$v_i = d_i^r - d_i^\ell \quad (2.59)$$

of shock jump rates and the variance of the shock position is described by the diffusion coefficient

$$D_i = \frac{1}{2}(d_i^r + d_i^\ell) \quad (2.60)$$

which appear also on macroscopic scale in the random motion of a macroscopically stable shock in the ASEP [31, 32]. Notice, however, that because of the convexity of the current-density relation $j \propto \rho(1-\rho)$ only the milder condition Rankine Hugoniot condition [59] is required for the macroscopic stability of a shock which for the ASEP only implies $\rho_{i+1} > \rho_i$ rather than the strict condition (2.53). For consecutive shocks this convexity also yields $v_i > v_{i+1}$ for all i so that after finite microscopic time consecutive shocks become very close and form a bound state for all subsequent times [6]. On macroscopic scale this phenomenon corresponds to a coalescence of shocks [33]. Also other particle-conserving models with random-walking shocks are known [43, 3, 39, 7, 5]

2.5.3 Shock random walks and MPMs for the open ASEP

A shock measure with N stable shocks which all satisfy the microscopic stability criterion (2.53) then has the property

$$\frac{z_N}{z_0} = q^{2N}. \quad (2.61)$$

If the boundary parameters of the open ASEP satisfy (2.49) with $z_- = z_0$ then $z_+ = z_N$ then also (2.61) is satisfied and the $(N+1)$ -dimensional representations of [54] of the stationary matrix product algebra of [25] are convex combinations of homogenous Bernoulli product measures with fugacities z_0 and z_N and Bernoulli shock measures with N shocks satisfying (2.53) and (2.49) [38]. Therefore we refer to (2.49) as boundary shock stability condition.

In fact, it was suggested even earlier in [43] that the shocks whose invariant distribution in the finite system is described by the MPM remain stable during the stochastic time evolution and perform a random walk dynamics. Here we focus on

boundary rates satisfying also the condition (2.61) and use reverse duality to prove this conjecture and make it precise. In particular, we identify the shock densities ρ_i^* which allows us to elucidate special properties of MPMs satisfying (2.50).

2.6 Shock ASEP with particle-dependent hopping rates

Using duality it is proved below that on microscopic scale shocks in the open ASEP on Λ , or more precisely the microscopic shock positions perform a random motion that we introduced in more general form in [10] under the name *shock exclusion process* in the context of self-duality for a multi-species ASEP. Here we go beyond [10] by considering non-conservative open boundaries instead of the more straightforward case of conservative dynamics on \mathbb{Z} . However, we focus on only one species of particles.

For a single species of exclusion particles the shock exclusion process, has particles, labelled by i with $1 \leq i \leq N$ that are located on lattice sites x_i with the single-file condition $x_0 < x_1 < x_2 \cdots < x_N < x_{N+1}$ with the fixed auxiliary boundary coordinates $x_0 := L_- - 1$ and $x_{N+1} = L_+ + 1$. Thus the configuration of all particles is represented by the coordinate vector $\mathbf{x} := (x_1, x_2, \dots, x_N)$ and each particle has its individual jump rate

$$w_i(\mathbf{x}) = w_i^\ell(\mathbf{x}) + w_i^r(\mathbf{x}) \quad (2.62)$$

with left- and right jump rates

$$w_i^\ell(\mathbf{x}) = d_i^\ell(1 - \delta_{x_i, x_{i-1}+1})(1 - \delta_{x_i, x_{i+1}}) \quad (2.63)$$

$$w_i^r(\mathbf{x}) = d_i^r(1 - \delta_{x_i, x_{i+1}-1})(1 - \delta_{x_i, x_{i-1}}) \quad (2.64)$$

given by the shock jump rates (2.57) and (2.58). The terms with the Kroneckersymbols entail the single-file exclusion principle and the boundary terms x_0 and x_{N+1} appearing in the rates $w_1(\mathbf{x})$ and $w_N(\mathbf{x})$ express the particle number conservation at the reflecting boundaries.

The generator of the shock ASEP can be defined more formally by introducing for a given coordinate vector $\mathbf{x} = (x_1, \dots, x_N)$ the locally shifted coordinate vectors $\mathbf{x}^{i\pm}$ with the coordinates

$$x_j^{i\pm} = x_j \pm \delta_{i,j}. \quad (2.65)$$

Definition 2.10 (*Shock exclusion process*) *The N -particle shock exclusion process of particles i , $1 \leq i \leq N$ that are located on lattice sites $x_i \in \Lambda_L$ with the single-file condition $x_0 < x_1 < x_2 \cdots < x_N < x_{N+1}$ and with jump rates (2.63), (2.64) is defined by the generator*

$$\mathcal{M}f(\mathbf{x}) = \sum_{i=1}^N \mathcal{M}_i f(\mathbf{x}) \quad (2.66)$$

with the single-particle hopping generators

$$\mathcal{M}_i f(\mathbf{x}) = w_i^\ell(\mathbf{x})(f(\mathbf{x}^{i-}) - f(\mathbf{x})) + w_i^r(\mathbf{x})(f(\mathbf{x}^{i+}) - f(\mathbf{x})) \quad (2.67)$$

for fixed parameters $\rho_i \in [0, 1]$, $0 \leq i \leq N$.

Remark 2.11 For $N = 1$ the shock exclusion process reduces to a biased random walk on Λ_L with generator

$$\mathcal{M}f(x) = w^\ell(x)(f(x-1) - f(x)) + w^r(x)(f(x+1) - f(x)) \quad (2.68)$$

and with left- and right jump rates

$$w^\ell(x) = d_1^\ell(1 - \delta_{x,L_-})(1 - \delta_{x,L_++1}) \quad (2.69)$$

$$w^r(x) = d_1^r(1 - \delta_{x,L_+})(1 - \delta_{x,L_- -1}) \quad (2.70)$$

for any fixed pair of densities $\rho_0, \rho_1 \in [0, 1]$. The intensity matrix Q is the tridiagonal Toeplitz matrix with matrix elements

$$Q_{xy} = d_1^\ell(1 - \delta_{x,L_-})(\delta_{y,x-1} - \delta_{y,x}) + d_1^r(1 - \delta_{x,L_+})(\delta_{y,x+1} - \delta_{y,x}) \quad (2.71)$$

for $x, y \in \Lambda_L$.

3 Reverse duality for the open ASEP

The main result is the reverse duality of the non-conservative open ASEP and the conservative shock ASEP with reflecting boundaries. The duality function is given by the Bernoulli shock measures of Definition 2.9 with N shocks and a specific choice of shock densities determined by condition 2.50. Since the main ideas in the proof for general N appear already in the simpler case of a single shock, we separate the single-shock case (which also allows for more explicit information on the time evolution of the shock measure) from the general case. Before stating these results we assert reversibility of the shock exclusion process.

Proposition 3.1 *The N -particle shock exclusion process 2.10 on Λ_L with reflecting boundaries is reversible w.r.t. the unnormalized measure*

$$\pi(\mathbf{x}) = \prod_{i=1}^N d_i^{2x_i} \quad (3.1)$$

where

$$d_i := \sqrt{\frac{d_i^r}{d_i^\ell}} \quad (3.2)$$

is the hopping asymmetry of particle i .

Now we are in a position to present the main results.

Theorem 3.2 *Let W be the intensity matrix (2.28) of the open ASEP with boundary rates (2.42) and (2.43) and for parameters ρ_0 and ρ_1 let Q be the intensity matrix*

of a simple biased random walk with generator (2.68). Further, let μ^x be the shock measure (2.51) with left bulk density $\rho_0 = \rho_-$ and shock density

$$\rho_1^* = \frac{\alpha}{\alpha + \gamma}. \quad (3.3)$$

The intensity matrices W and Q satisfy the reverse-duality relation

$$RW = Q^T R \quad (3.4)$$

w.r.t. the duality matrix R with matrix elements $R_{x,\eta} = d_1^{2x} \mu^x(\eta)$ if and only if the following three conditions are satisfied:

- (i) The bulk shock stability condition (2.53) is satisfied for $i = N = 1$,
- (ii) The boundary rates are on the manifold \mathcal{B}_1^1 .

Theorem 3.3 Denote by μ_t^x the distribution at time t of the open ASEP, starting at $t = 0$ from a shock measure $\mu_0^x = \mu^x$ and let Conditions (i) - (iii) of Theorem 3.2 be satisfied. Then, for any $x \in \Lambda_L$

$$\mu^x(t) = \sum_{y=L_-}^{L_+} P(y, t|x, 0) \mu^y(0) \quad (3.5)$$

where

$$P(y, t|x, 0) = \frac{d_1^2 - 1}{d_1^{2L} - 1} d_1^{2(y-L_-)} + \frac{2}{L} \sum_{p=1}^{L-1} d_1^{y-x} \psi_p(x) \psi_p(y) \frac{w}{\epsilon_p} e^{-\epsilon_p t} \quad (3.6)$$

with

$$\psi_p(y) := d_1 \sin\left(\frac{\pi p}{L}(y+1-L_-)\right) - \sin\left(\frac{\pi p}{L}(y-L_-)\right) \quad (3.7)$$

$$\epsilon_p = w \left[d_1 + d_1^{-1} - 2 \cos\left(\frac{\pi p}{L}\right) \right] \quad (3.8)$$

is the transition probability of the biased random walk generated by the intensity matrix Q and starting at time $t = 0$ from x . The limit $\mu_1^* := \lim_{t \rightarrow \infty} \mu^x(t)$ is the unique invariant measure of the open ASEP with boundary parameters as specified by Conditions (ii) and (iii) and is given by the convex combination

$$\mu_1^* = \frac{d_1^2 - 1}{d_1^{2L} - 1} \sum_{y=L_-}^{L_+} d_1^{2(y-L_-)} \mu^y. \quad (3.9)$$

of shock measures μ^y .

Remark 3.4 For asymmetric exclusion processes with open boundaries similar random walk dynamics were found earlier for a single anti-shock with bulk densities related by the inverse stability condition $z_1 = q^{-2} z_0$ under conditioning the process on a special atypical value of the time-integrated particle current [8]. Boundary parameters in that work corresponded to $\omega_+ = \omega_- = 0$. In [39] random walk dynamics of a single shock were proved for a discrete-time totally asymmetric simple exclusion process with deterministic sublattice update and stochastic open boundaries. In this model there is no constraint on the boundary parameters.

Remark 3.5 According to condition (iii) in Theorem 3.2 the invariant measure (3.9) can be expressed by the two-dimensional representation of the stationary matrix product algebra characterized in the Appendices of [54] which exists for any $L \geq 2$ and which is analyzed further in [12]. The homogeneous Bernoulli measure with densities ρ_0 and ρ_1 generically contribute to the two-dimensional MPM [38] but are absent in the convex combination (3.9). It would be interesting to explore whether two-dimensional time-dependent matrices can capture the dynamics of some subspace as has been proved for the SSEP with open boundaries [68] and also for a non-conservative annihilating random walks with pair deposition [65, 69].

Remark 3.6 The spectrum of the generator given by the eigenvalues (3.8) yields a subset of eigenvalues of the generator of the open ASEP and is in agreement with the picture of spectral properties arising from a shock random walk explored in [28, 62, 23].

As pointed out above, Theorem 3.2 is a special case of the following result for duality functions given by Bernoulli shock measures with $N > 1$ shocks.

Theorem 3.7 Let W be the intensity matrix (2.28) of the open ASEP with boundary rates (2.42) and (2.43) and for parameters ρ_0, \dots, ρ_N let Q be the intensity matrix of the N -particle shock exclusion process of Definition 2.10 and reversible measure (3.1). Further, let μ^x be the shock measure (2.51) with left boundary density $\rho_0 = \rho_-$ and shock fugacities

$$z_i^* = \frac{\alpha}{\gamma} q^{2(i-1)} \quad (3.10)$$

for $1 \leq i \leq N$. The intensity matrices W and Q satisfy the reverse-duality relation

$$RW = Q^T R \quad (3.11)$$

w.r.t. the duality matrix R with matrix elements $R_{\mathbf{x}, \boldsymbol{\eta}} = \pi(\mathbf{x}) \mu_{\boldsymbol{\eta}}^x$ if and only if the following three conditions are satisfied:

- (i) The bulk shock stability condition (2.53) is satisfied for all $i \in \{1, \dots, N\}$,
- (ii) The boundary rates are on the manifold \mathcal{B}_N^1 for $1 \leq N \leq L$.

Remark 3.8 The conservative reflective boundaries of the reverse dual are in contrast to the conventional duality for the open symmetric simple exclusion process (SSEP) which is dual to the SSEP with nonconservative absorbing boundaries noticed in [71] and later fully developed in [13, 34], including the generalization to symmetric partial exclusion [64]. An open question is whether a similar conservative reverse duality can be constructed for the conservative asymmetric partial exclusion process studied in [14] when non-conservative open boundaries are added.

Since the shock exclusion process with reflecting boundaries is reversible for any N also Theorem 3.3 has a natural generalization for $N > 1$.

Theorem 3.9 *Let $\mu^{\mathbf{x}}(t)$ denote the distribution at time t of the ASEP of Theorem 3.7, starting from an N -shock measure $\mu^{\mathbf{x}} = \mu^{\mathbf{x}}(0)$. Then*

$$\mu^{\mathbf{x}}(t) = \sum_{\mathbf{y}} P(\mathbf{y}, t | \mathbf{x}, 0) \mu^{\mathbf{y}} \quad (3.12)$$

where $P(\mathbf{y}, t | \mathbf{x}, 0)$ is the transition probability of the shock exclusion process 2.10. The limit $\mu_N^* := \lim_{t \rightarrow \infty} \mu^{\mathbf{x}}(t)$ is the unique invariant measure of the open ASEP with boundary parameters as specified by Conditions (ii) and (iii) and is given by the unnormalized convex combination

$$\tilde{\mu}_N^* = \frac{j_N^{2y_N}}{j_1^{2y_1}} \sum_{\mathbf{y}} \prod_{i=1}^{N-1} j_i^{2(y_i - y_{i+1})} \mu^{\mathbf{y}} \quad (3.13)$$

of shock measures with N shocks.

Remark 3.10 *The spectrum of the generator of the N -particle shock exclusion process can be calculated from Bethe ansatz [55, 56, 22, 23, 70]. However, there is no simple expression in closed form. For results on the eigenfunctions see [21]. Further progress along these lines has been made recently for the SSEP with open boundaries by using duality [34]. It would be most interesting to generalize this approach to the open ASEP to cover the full manifold of boundary parameters.*

4 Proofs

We prove reverse duality and its consequences for the time evolution of shock measures by studying the action of the quantum Hamiltonian on the vector representation of the shock measures which are shown to form an invariant subspace. The coefficients appearing in these computations yield the reverse duality with the shock exclusion process. Sufficient conditions for a similar property were conjectured earlier in [43] for second-class particles at the shock positions and in [38] for a single shock. Here we obtain rigorously the necessary and sufficient conditions on the parameters of the ASEP for the reverse duality for a specific choice of shock densities for $N \geq 1$ shocks..

4.1 Preliminaries

Everywhere below a vector denoted by $|abc\dots\rangle$ is defined to be the Kronecker product $|a\rangle \otimes |b\rangle \otimes |c\rangle \otimes \dots$ of two-dimensional vectors $|a\rangle, |b\rangle, |c\rangle \dots \in \mathbb{R}^2$ with components $(a_0, a_1), (b_0, b_1), (c_0, c_1), \dots$. For vectors with components of the form $a_0 = 1 - \rho, a_1 = \rho$ and $0 \leq \rho \leq 1$ we use greek letters and write $|\rho\rangle$ and call ρ a density. In slight abuse of wording we call for any vector $|a\rangle$ with $a_0 \neq 0$ the ratio

$$z(a) := \frac{a_1}{a_0} \quad (4.1)$$

the *fugacity*. The functions

$$\epsilon_-(\cdot) := \alpha - \gamma z(\cdot), \quad \epsilon_+(\cdot) := \delta - \beta z(\cdot) \quad (4.2)$$

play a role as eigenvalues of the boundary operators h^\pm .

For two vectors $|a\rangle, |b\rangle$ we introduce the determinant

$$\Delta(a, b) := a_0 b_1 - a_1 b_0 \quad (4.3)$$

and recall that linear dependence of $|a\rangle$ and $|b\rangle$ is equivalent to $\Delta(a, b) = 0$. For $z(b) = q^2 z(a)$ one has

$$\Delta(a, b) = (r - \ell) \frac{a_0 b_1}{r} = (r - \ell) \frac{a_1 b_0}{\ell}. \quad (4.4)$$

For linearly independent vectors we also define the functions

$$d(a, b) := (r - \ell) \frac{a_1 a_0}{\Delta(a, b)}, \quad \tilde{d}(a, b) := (r - \ell) \frac{a_1 b_0}{\Delta(a, b)}. \quad (4.5)$$

$$d_\pm(a, b) := \frac{\epsilon_\pm(a) a_0 (a_0 + a_1)}{\Delta(a, b)}, \quad \tilde{d}_\pm(a, b) := \frac{\epsilon_\pm(a) a_0 (b_0 + b_1)}{\Delta(a, b)}. \quad (4.6)$$

which play a role for the transition rates of the dual process. We recall that in all assertions and in all the proofs below it is tacitly assumed that all boundary rates and bulk jump rates of the open ASEP are strictly positive and that $q \neq 1$.

Lemma 4.1 (Eigenvectors) *(i) For any two vectors $|a\rangle, |b\rangle$ the bulk eigenvalue property*

$$h|ab\rangle = 0 \quad (4.7)$$

is satisfied if and only if $\Delta(a, b) = 0$.

(ii) For vectors $|a^\pm\rangle$ with components such that $a_0^\pm a_1^\pm \neq 0$ and $a_0^\pm + a_1^\pm \neq 0$ the two boundary eigenvalue equations

$$h^\pm |a^\pm\rangle = \epsilon_\pm(a^\pm) |a^\pm\rangle \quad (4.8)$$

have a unique solution with strictly positive fugacities given by

$$z(a^-) = \kappa^{-1}(\alpha, \gamma), \quad z(a^+) = \kappa_+(\beta, \delta) \quad (4.9)$$

with $\kappa_+(\cdot, \cdot)$ defined in (2.46). The eigenvalues are

$$\epsilon_-(\cdot) = \alpha - \gamma z(\cdot) = (r - \ell) \frac{z(\cdot)}{1 + z(\cdot)} \quad (4.10)$$

$$\epsilon_+(\cdot) = \delta - \beta z(\cdot) = -(r - \ell) \frac{z(\cdot)}{1 + z(\cdot)}. \quad (4.11)$$

For $a_0^\pm a_1^\pm = 0$ or $a_0^\pm + a_1^\pm = 0$ the eigenvalue equation (4.8) has no solution for strictly positive boundary rates and $r \neq \ell$.

Proof: Consider two vectors $|a\rangle$ with components a_0, a_1 and $|b\rangle$ with components b_0, b_1 and determinant Δ_{ab} defined in (4.3).

(i) For the product vector one gets from the definition (2.33)

$$h|ab\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r & -r & 0 \\ 0 & -\ell & \ell & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix} = \Delta_{ab} \begin{pmatrix} 0 \\ r \\ -\ell \\ 0 \end{pmatrix} \quad (4.12)$$

which proves part (i) of the Lemma.

(ii) To make notation less heavy we drop the superscripts \pm in the vectors $|a^\pm\rangle$ in the following computations. From the definition (2.34) of the boundary matrix one obtains

$$h^-|a\rangle = \begin{pmatrix} \alpha a_0 - \gamma a_1 \\ -\alpha a_0 + \gamma a_1 + (r - \ell)a_1 \end{pmatrix} \quad (4.13)$$

from which one reads off immediately that for strictly positive rates α, γ there exists no eigenvector with $a_0 = 0$ or $a_1 = 0$, i.e., for $a_0 a_1 = 0$. To proceed with the proof of the eigenvalue properties for a vector $|a\rangle$ such that $a_0 a_1 \neq 0$ we define

$$g_\pm(a) := \mp \frac{r - \ell}{\epsilon_-(a)} - \frac{1}{z(a)} \quad (4.14)$$

and note that for any linearly independent vector $|b\rangle$ the equality

$$\begin{aligned} h^-|a\rangle &= \epsilon_-(a) \begin{pmatrix} a_0 \\ g_-(a)a_1 \end{pmatrix} \\ &= \left(\tilde{d}_-(a, b) - \tilde{d}(a, b) \right) |a\rangle - \left(d_-(a, b) - d(a, b) \right) |b\rangle \end{aligned} \quad (4.15)$$

holds without further constraints on the vectors $|a\rangle$ and $|b\rangle$.

If an eigenvector exists (which requires $d_-(a, b) = d(a, b)$ or equivalently $g_-(a) = 1$) then the eigenvalue is of the form $\epsilon_-(a)$ as defined in (4.2) and one deduces that there exists no eigenvector with $a_1 = -a_0$ since then $z = -1$ and therefore $g(a) \neq 1$ for $r \neq \ell$. From $g_-(a) = 1$ one also obtains the second equality in (4.10) and together with the first equality one arrives at the quadratic equation

$$z^2 + \frac{\gamma - \alpha + r - \ell}{\gamma} z - \frac{\alpha}{\gamma} = 0 \quad (4.16)$$

for the fugacity $z \equiv z(a)$. The positive root is given by $z = \kappa^{-1}(\alpha, \gamma)$ which yields (up to an arbitrary multiplicative factor) the components of the eigenvector $|a\rangle$ in terms of the boundary rates as in (4.9).

For the right boundary matrix one gets

$$h^+|a\rangle = \begin{pmatrix} \delta a_0 - \beta a_1 \\ -\delta a_0 + \beta a_1 - (r - \ell)a_1 \end{pmatrix}$$

$$\begin{aligned}
&= \epsilon_+(a) \begin{pmatrix} a_0 \\ g_+(a)a_1 \end{pmatrix} \\
&= \left(\tilde{d}_+(a, b) + \tilde{d}(a, b) \right) |a\rangle - (d_+(a, b) + d(a, b)) |b\rangle. \quad (4.17)
\end{aligned}$$

with a linearly independent vector $|b\rangle$. Eigenvectors with $a_0a_1 = 0$ or $a_0 + a_1 = 0$ cannot exist as can be seen by arguments analogous to the case treated above. For an eigenvector one immediately reads off the first equality in (4.11). Moreover, an eigenvector satisfies $d_+(a, b) = -d(a, b)$ which gives the second equality in (4.11) and together with the first equality in (4.11) the quadratic equation

$$z^2 - \frac{\delta - \beta + r - \ell}{\beta} z - \frac{\delta}{\beta} = 0 \quad (4.18)$$

for the fugacity. The positive root is given by $z = \kappa_+(\beta, \delta)$ which yields (4.9) for the right boundary. \square

Corollary 4.2 *A probability vector $|\pi\rangle$ with factorized boundary marginals such that $|\pi\rangle = |\rho_-\rangle \otimes |\tilde{\pi}\rangle \otimes |\rho_+\rangle$ and with $z_+ = q^{2N}z_-$ is an eigenvector of the boundary operator $h_{L_-}^- + h_{L_+}^+$ if and only if (2.49) holds.*

Lemma 4.3 (Projection) *(i) For vectors $|a^{(i)}\rangle, |\tilde{a}^{(i)}\rangle, i \in \{1, 2\}$, such that $a_0^{(i)}a_1^{(i)} \neq 0$ and $\tilde{a}_0^{(i)}\tilde{a}_1^{(i)} \neq 0$ and a pair of linearly independent vectors $(|b^{(i)}\rangle, |\tilde{b}^{(i)}\rangle)$ such that $|b^{(1)}\rangle$ is linearly independent of $|\tilde{a}^{(1)}\rangle$ and $|b^{(2)}\rangle$ is linearly independent of $|\tilde{a}^{(2)}\rangle$ the two-site jump generator h has the three-dimensional projection properties*

$$(A) \quad h|a^{(1)}b^{(1)}\rangle = u^{(1)}|a^{(1)}b^{(1)}\rangle - v^{(1)}|b^{(1)}\tilde{a}^{(1)}\rangle - w^{(1)}|a^{(1)}\tilde{b}^{(1)}\rangle \quad (4.19)$$

with coefficients

$$u^{(1)} = \ell - \tilde{d}(b^{(1)}, \tilde{b}^{(1)}), \quad v^{(1)} = d(a^{(1)}, \tilde{a}^{(1)}), \quad w^{(1)} = -d(b^{(1)}, \tilde{b}^{(1)}) \quad (4.20)$$

and

$$(B) \quad h|b^{(2)}\tilde{a}^{(2)}\rangle = u^{(2)}|b^{(2)}\tilde{a}^{(2)}\rangle - v^{(2)}|a^{(2)}b^{(2)}\rangle - w^{(2)}|\tilde{b}^{(2)}\tilde{a}^{(2)}\rangle \quad (4.21)$$

with coefficients

$$u^{(2)} = r + \tilde{d}(b^{(2)}, \tilde{b}^{(2)}), \quad v^{(2)} = -d(\tilde{a}^{(2)}, a^{(2)}), \quad w^{(2)} = d(b^{(2)}, \tilde{b}^{(2)}) \quad (4.22)$$

if and only if the condition

$$\frac{z(\tilde{a}^{(i)})}{z(a^{(i)})} = q^2, \quad (4.23)$$

on the vectors $|a^{(i)}\rangle, |\tilde{a}^{(i)}\rangle$ are satisfied.

(ii) For a pair of linearly independent vectors $(|a\rangle, |\tilde{a}\rangle)$ such that $a_0a_1 \neq 0$ and

$\tilde{a}_0\tilde{a}_1 \neq 0$ and vectors $|\tilde{c}\rangle$ linearly independent of $|a\rangle$, $|c\rangle$ linearly independent of $|\tilde{a}\rangle$ the two-site jump generator h has the three-dimensional projection property

$$(C) \quad h|a\tilde{a}\rangle = u|a\tilde{a}\rangle - w|\tilde{c}\tilde{a}\rangle - v|ac\rangle \quad (4.24)$$

with coefficients

$$u = \tilde{d}(a, \tilde{c}) - \tilde{d}(\tilde{a}, c), \quad v = -d(\tilde{a}, c), \quad w = d(a, \tilde{c}) \quad (4.25)$$

if and only if the condition (4.23) is satisfied.

(iii) If $a_0a_1 = 0$ or $\tilde{a}_0\tilde{a}_1 = 0$ the projection equations (4.19), (4.21), and (4.24) have no solution for strictly positive boundary rates and $r \neq \ell$.

Proof: We focus on the projection property (A) and drop the superscript (1) to make notation less heavy. The proof of the projection properties (B) and (C) is analogous. In simplified notation (4.19) reads $h|ab\rangle = u|ab\rangle - v|b\tilde{a}\rangle - w|a\tilde{b}\rangle$ from which using (4.12) one obtains are the four linear equations

$$0 = ua_0b_0 - vb_0\tilde{a}_0 - wa_0\tilde{b}_0 \quad (4.26)$$

$$r(a_0b_1 - a_1b_0) = ua_0b_1 - vb_0\tilde{a}_1 - wa_0\tilde{b}_1 \quad (4.27)$$

$$-\ell(a_0b_1 - a_1b_0) = ua_1b_0 - vb_1\tilde{a}_0 - wa_1\tilde{b}_0 \quad (4.28)$$

$$0 = ua_1b_1 - vb_1\tilde{a}_1 - wa_1\tilde{b}_1 \quad (4.29)$$

for the three coefficients u, v, w . (a) Consider $a_0 = 0$ which implies by linear independence $b_0 \neq 0$. Then the first two equations yield $v = r = 0$ in contradiction to $r > 0$. Hence there exists no solution for $a_0 = 0$.

(b) Consider $a_0 \neq 0$ and $\tilde{a}_0 = 0$. The first and third equation reduce to

$$0 = ub_0 - w\tilde{b}_0 \quad (4.30)$$

$$-\ell(a_0b_1 - a_1b_0) = a_1 \left(ub_0 - w\tilde{b}_0 \right) \quad (4.31)$$

which due to linear independence yields $\ell = 0$, in contradiction to the assumption $\ell > 0$. Hence there exists no solution for $\tilde{a}_0 = 0$. The proof that there exists no solution for $a_1 = 0$ or $\tilde{a}_1 = 0$ is analogous.

(c) To proceed with the generic case we introduce the fugacities

$$\tilde{z} := \frac{\tilde{a}_1}{\tilde{a}_0}, \quad z := \frac{a_1}{a_0} \quad (4.32)$$

and consider next $b_0 = 0$. Using linear independence the four equations yield immediately the unique solution

$$u = r, \quad v = \ell \frac{a_0}{\tilde{a}_0}, \quad w = 0 \quad (4.33)$$

if and only if $\tilde{z} = q^2z$ in agreement with the assertion of the theorem. For $b_0 \neq 0$ it is useful to define

$$\tilde{v} := v \frac{\tilde{a}_0}{a_0}, \quad \tilde{w} := w \frac{\tilde{b}_0}{b_0}. \quad (4.34)$$

The four equations turn into

$$u = \ell + \tilde{w} \quad (4.35)$$

$$\tilde{v} = \ell \quad (4.36)$$

and

$$r(a_0 b_1 - a_1 b_0) = a_0 \left[(\ell + \tilde{w}) b_1 - \tilde{w} \frac{b_0}{\tilde{b}_0} \tilde{b}_1 \right] - \ell \frac{a_0}{\tilde{a}_0} b_0 \tilde{a}_1 \quad (4.37)$$

$$0 = a_1 \left[(\ell + \tilde{w}) b_1 - \tilde{w} \frac{b_0}{\tilde{b}_0} \tilde{b}_1 \right] - \ell \frac{a_0}{\tilde{a}_0} b_1 \tilde{a}_1. \quad (4.38)$$

Inserting the second into the first equation then yields

$$\tilde{w} = \frac{(r - \ell) b_1 \tilde{b}_0}{b_1 \tilde{b}_0 - b_0 \tilde{b}_1} = -\frac{(r - \ell) b_1 \tilde{b}_0}{\Delta_{b\tilde{b}}} \quad (4.39)$$

and this solution exists if and only if $\tilde{z} = q^2 z$ as the second equation shows. Thus with (4.3) one arrives at

$$u = \ell - (r - \ell) \frac{b_1 \tilde{b}_0}{\Delta_{b\tilde{b}}} \quad (4.40)$$

$$v = \ell \frac{a_0}{\tilde{a}_0} = (r - \ell) \frac{a_0 a_1}{\Delta_{a\tilde{a}}} \quad (4.41)$$

$$w = -(r - \ell) \frac{b_0 b_1}{\Delta_{b\tilde{b}}} \quad (4.42)$$

and with the functions $d(\cdot, \cdot)$ and $\tilde{d}(\cdot, \cdot)$ defined in (4.5) one obtains the coefficients (4.20). \square

Corollary 4.4 *For shocks satisfying the shock stability criterion (2.53) the shock jump rates $d_{i,r,\ell}$ satisfy the identities*

$$d(\rho_{i-1}, \rho_i) = \ell \frac{1 - \rho_{i-1}}{1 - \rho_i} = r \frac{\rho_{i-1}}{\rho_i} = \ell(1 - \rho_{i-1}) + r\rho_{i-1} = d_i^\ell \quad (4.43)$$

$$-d(\rho_i, \rho_{i-1}) = \ell \frac{\rho_i}{\rho_{i-1}} = r \frac{1 - \rho_i}{1 - \rho_{i-1}} = r(1 - \rho_i) + \ell\rho_i = d_i^r \quad (4.44)$$

$$\epsilon_+(\rho_i) = d_i^r - r \quad (4.45)$$

$$\epsilon_-(\rho_{i-1}) = d_i^\ell - \ell. \quad (4.46)$$

which also imply $d_i^r d_i^\ell = r\ell$ independently of i .

4.2 Proof of Proposition 3.1

The proof of reversibility of the shock exclusion process) is essentially trivial. According to Definition 2.10 the intensity matrix of the N -particle shock exclusion process is given by the matrix elements

$$Q_{\mathbf{x}\mathbf{x}'} = \sum_{i=1}^N [w_i^\ell(\mathbf{x})(\delta_{\mathbf{x}',\mathbf{x}^{i-}} - \delta_{\mathbf{x}',\mathbf{x}}) + w_i^r(\mathbf{x})(\delta_{\mathbf{x}',\mathbf{x}^{i+}} - \delta_{\mathbf{x}',\mathbf{x}})]. \quad (4.47)$$

Using $\delta_{\mathbf{y}^{i\pm},\mathbf{x}} = \delta_{\mathbf{y},\mathbf{x}^{i\mp}}$ one obtains

$$d_i^2 w_i^\ell(\mathbf{x}') \delta_{\mathbf{x},(\mathbf{x}')^{i-}} = w_i^r(\mathbf{x}) \delta_{\mathbf{x}',\mathbf{x}^{i+}} \quad (4.48)$$

$$d_i^{-2} w_i^r(\mathbf{x}') \delta_{\mathbf{x},(\mathbf{x}')^{i+}} = w_i^\ell(\mathbf{x}) \delta_{\mathbf{x}',\mathbf{x}^{i-}} \quad (4.49)$$

and $\pi(\mathbf{x}^{i\pm})/\pi(\mathbf{x}) = d_i^{\pm 2}$ yields $Q_{\mathbf{y}\mathbf{x}} = Q_{\mathbf{x}\mathbf{y}}\pi(\mathbf{x})/\pi(\mathbf{y})$ which is the definition of reversibility, viz., $Q^T = \hat{\pi}Q\hat{\pi}^{-1}$, in terms of the matrix elements. \square

4.3 Reverse duality involving one shock

The modified duality matrix S with elements $S_{x,\eta} = \langle \eta | \mu^x \rangle = \mu^x(\eta)$ is the matrix with the column vectors $|\mu^x\rangle$ as rows. For one shock the vector representation (2.55) of the shock measure of Theorem 2.4 is given by

$$|\mu^x\rangle = |\rho_0\rangle^{\otimes(x-L_-)} \otimes |\rho^*\rangle \otimes |\rho_1\rangle^{\otimes(L_+-x)}, \quad L_- \leq x \leq L_+ \quad (4.50)$$

with ρ^* given by (3.3).

4.3.1 Proof of Theorem 3.2

One has to find the necessary and sufficient conditions under which the shock measures $|\mu^x\rangle$ form an invariant subspace under the action of H , i.e.,

$$H|\mu^x\rangle = \sum_{y=L_-}^{L_+} G_{yx}|\mu^y\rangle \quad (4.51)$$

with a matrix G and then show that $G = -Q^T$ for the intensity matrix Q of the biased random walk defined in Remark 2.11. To do so, we first employ the projection lemma 4.3 together with the eigenvector properties detailed in Lemma 4.1 and its Corollary 4.2 to prove (4.51) if and only if (2.49), (2.50) and (2.61) hold with $N = m + 1 = 1$.

Shock in the range $L_- < x < L_+$: From the product form of the shock measure, the eigenvalue properties (4.7) and (4.8) in Lemma 4.1 and Corollary 4.2 one concludes

$$H|\mu^x\rangle = (\tilde{h}_{L_-}^- + \tilde{h}_{x-1,x} + \tilde{h}_{x,x+1} + \tilde{h}_{L_+}^+)|\mu^x\rangle$$

$$\begin{aligned}
&= |\rho_0\rangle^{\otimes(x-L-1)} \otimes [(\tilde{h}_{12} + \tilde{h}_{23})|\rho_0\rangle \otimes |\rho_1^*\rangle \otimes |\rho_1\rangle] \otimes |\rho_1\rangle^{\otimes(L+x-1)} \\
&\quad + (\epsilon_-(\rho_0) + \epsilon_+(\rho_1))|\mu^x\rangle.
\end{aligned} \tag{4.52}$$

Taking $a^{(1)} = a^{(2)} = \rho_0$, $\tilde{a}^{(1)} = \tilde{a}^{(2)} = \rho_1$, $b^{(1)} = b^{(2)} = \rho_1^*$ in (4.19) and (4.21) of Lemma 4.3 yields after taking Kronecker products on the right in (4.19) and on the left in (4.21)

$$\begin{aligned}
\tilde{h}_{12}|\rho_0\rho_1^*\rho_1\rangle &= \left(\ell - \tilde{d}(\rho_1^*, \tilde{b}^{(1)})\right)|\rho_0\rho_1^*\rho_1\rangle \\
&\quad - d(\rho_0, \rho_1)|\rho_1^*\rho_1\rangle + d(\rho_1^*, \tilde{b}^{(1)})|\rho_0\tilde{b}^{(1)}\rho_1\rangle \\
\tilde{h}_{23}|\rho_0\rho_1^*\rho_1\rangle &= \left(r + \tilde{d}(\rho_1^*, \tilde{b}^{(2)})\right)|\rho_0\rho_1^*\rho_1\rangle \\
&\quad + d(\rho_1, \rho_0)|\rho_0\rho_0\rho_1^*\rangle - d(\rho_1^*, \tilde{b}^{(2)})|\rho_0\tilde{b}^{(2)}\rho_1\rangle.
\end{aligned} \tag{4.53}$$

Since $\tilde{b}^{(1)}$ and $\tilde{b}^{(2)}$ are arbitrary (except for the immaterial inequality with ρ_1^*) we can without loss of generality take $\tilde{b}^{(1)} = \tilde{b}^{(2)} \neq \rho_1^*$. Thus the projection lemma asserts that for $q > 1$, $\rho_1^* \in [0, 1]$ and any pair of densities $\rho_0, \rho_1 \in (0, 1)$ such that $0 < \rho_0 < \rho_1 < 1$ and $z_1 = q^2 z_0$ the three-site jump generator $\tilde{h}_{12} + \tilde{h}_{23}$ projects the vector $|\rho_0\rho_1^*\rho_1\rangle := |\rho_0\rangle \otimes |\rho_1^*\rangle \otimes |\rho_1\rangle$ on the three-dimensional subspace of spanned by $|\rho_0\rho_1^*\rho_1\rangle$, $|\rho_1^*\rho_1\rangle := |\rho_1^*\rangle \otimes |\rho_1\rangle^{\otimes 2}$, and $|\rho_0\rho_0\rho_1^*\rangle := |\rho_0\rangle^{\otimes 2} \otimes |\rho_1^*\rangle$ as

$$(\tilde{h}_{12} + \tilde{h}_{23})|\rho_0\rho_1^*\rho_1\rangle = (r + \ell)|\rho_0\rho_1^*\rho_1\rangle - v_{01}|\rho_1^*\rho_1\rangle - w_{01}|\rho_0\rho_0\rho_1^*\rangle \tag{4.54}$$

with constants

$$u_{01} = r + \ell, \quad w_{01} = -d(\rho_1, \rho_0), \quad v_{01} = d(\rho_0, \rho_1) \tag{4.55}$$

that do not depend on ρ_1^* . The identities (4.43) - (4.46) then yield for arbitrary shock density ρ_1^* the bulk closure relation

$$\begin{aligned}
H|\mu^x\rangle &= [r + \ell + \epsilon_-(\rho_0) + \epsilon_+(\rho_1)]|\mu^x\rangle - d_1^\ell|\mu^{x-1}\rangle - d_1^r|\mu^{x+1}\rangle \\
&= (d_1^\ell + d_1^r)|\mu^x\rangle - d_1^\ell|\mu^{x-1}\rangle - d_1^r|\mu^{x+1}\rangle
\end{aligned} \tag{4.56}$$

if and only if the conditions (2.49) and (2.61) are satisfied with $N = 1$.

Shock at the left boundary: From the eigenvalue properties in Lemma 4.1 one gets

$$\begin{aligned}
H|\mu^{L-}\rangle &= (h_{L-}^- + \tilde{h}_{L-,L-+1} + h_{L+}^+)|\mu^{L-}\rangle \\
&= [(h_1^- + h)|\rho_1^*\rangle \otimes |\rho_1\rangle] \otimes |\rho_1\rangle^{\otimes L-2} + \epsilon_+(\rho_1)|\mu^{L-}\rangle.
\end{aligned} \tag{4.57}$$

To handle the left boundary term in (4.57) use case B in the projection lemma 4.3 by choosing in (4.21) $a^{(2)} = \rho_0$, $b^{(2)} = \rho_1^*$, and $\tilde{a}^{(2)} = \rho_1$ and by choosing in (4.15) $a = \rho_1^*$ taking the Kronecker product with $|\rho_1\rangle$ on the right. This yields

$$h|\rho_1^*\rho_1\rangle = \left(r + \tilde{d}(\rho_1^*, \tilde{b}^{(2)})\right)|\rho_1^*\rho_1\rangle$$

$$+d(\rho_1, \rho_0)|\rho_0\rho_1^*\rangle - d(\rho_1^*, \tilde{b}^{(2)})|\tilde{b}^{(2)}\rho_1\rangle \quad (4.58)$$

and

$$\tilde{h}_1|\rho_1^*\rho_1\rangle = \left(\tilde{d}_-(\rho_1^*, b) - \tilde{d}(\rho_1^*, b)\right)|\rho_1^*\rho_1\rangle - (d_-(\rho_1^*, b) - d(\rho_1^*, b))|b\rho_1\rangle \quad (4.59)$$

Taking without loss of generality $b = \tilde{b}^{(2)} \neq \rho_1^*$ one finds

$$\begin{aligned} (h + \tilde{h}_1)|\rho_1^*\rho_1\rangle &= \left(r + \tilde{d}_-(\rho_1^*, b)\right)|\rho_1^*\rho_1\rangle \\ &\quad + d(\rho_1, \rho_0)|\rho_0\rho_1^*\rangle - d_-(\rho_1^*, b)|b\rho_1\rangle. \end{aligned} \quad (4.60)$$

Observe next that for ρ_1^* as specified in the Theorem one has $\epsilon_-(\rho_1^*) = 0$ so that $\tilde{d}_-(\rho_1^*, b) = d_-(\rho_1^*, b) = 0$ for any vector $b \neq \rho_1^*$. Therefore

$$(h + \tilde{h}_1)|\rho_1^*\rho_1\rangle = r|\rho_1^*\rho_1\rangle + d(\rho_1, \rho_0)|\rho_0\rho_1^*\rangle. \quad (4.61)$$

Using (4.44) and (4.45) the conclusion from these computations is the left closure relation

$$H|\mu^{L-}\rangle = [r + \epsilon_+(\rho_1)]|\mu^{L-}\rangle + d(\rho_1, \rho_0)|\mu^{L-+1}\rangle = d_1^r(|\mu^{L-}\rangle - |\mu^{L-+1}\rangle) \quad (4.62)$$

if and only if $\rho_1^* = \alpha/(\alpha + \gamma)$ as specified in the Theorem.

Shock at the right boundary: From Lemma 4.1 one gets in a similar fashion

$$\begin{aligned} H|\mu^{L+}\rangle &= (h_{L-}^- + \tilde{h}_{L+-1, L+} + h_{L+}^+)|\mu^{L+}\rangle \\ &= \epsilon_-(\rho_0)|\mu^{L+}\rangle + |\rho_0\rangle^{\otimes(L-2)} \otimes [(h + h_2^+)|\rho_0\rangle \otimes |\rho^*\rangle] \end{aligned} \quad (4.63)$$

To treat (4.63) we use the projection formula (4.19) in Lemma 4.3 with the replacements $a^{(1)} \rightarrow \rho_0$, $b^{(1)} \rightarrow \rho_1^*$, $\tilde{a}^{(1)} \rightarrow \rho_1$ and take $a = \rho_1^*$ in (4.17). After left Kronecker multiplication with $|\rho_0\rangle$ this yields

$$\begin{aligned} h|\rho_0\rho_1^*\rangle &= \left(\ell - \tilde{d}(\rho_1^*, \tilde{b}^{(1)})\right)|\rho_0\rho_1^*\rangle \\ &\quad - d(\rho_0, \rho_1)|\rho_1^*\rho_1\rangle + d(\rho_1^*, \tilde{b}^{(1)})|\rho_0\tilde{b}^{(1)}\rangle \end{aligned} \quad (4.64)$$

and

$$h_2^+|\rho_0\rho_1^*\rangle = \left(\tilde{d}(\rho_1^*, b) + d_+(\rho_1^*, b)\right)|\rho_0\rho_1^*\rangle - (d(\rho_1^*, b) + d_+(\rho_1^*, b))|\rho_0b\rangle \quad (4.65)$$

With $b = \tilde{b}^{(1)} \neq \rho_0$ one thus finds the projection property

$$\begin{aligned} (h + h_2^+)|\rho_0\rho_1^*\rangle &= \left(\ell - \tilde{d}(\rho_1^*, \tilde{b}^{(1)}) + \tilde{d}(\rho_1^*, b) + d_+(\rho_1^*, b)\right)|\rho_0\rho_1^*\rangle \\ &\quad - d(\rho_0, \rho_1)|\rho_1^*\rho_1\rangle + d(\rho_1^*, \tilde{b}^{(1)})|\rho_0\tilde{b}^{(1)}\rangle \\ &\quad - (d(\rho_1^*, b) + d_+(\rho_1^*, b))|\rho_0b\rangle \end{aligned} \quad (4.66)$$

which with (4.44) and (4.46) and by choosing without loss of generality $b = \tilde{b}^{(1)} \neq \rho_1^*$ leads to

$$\begin{aligned} (h + h_2^+) |\rho_0 \rho_1^* \rangle &= (\ell + d_+(\rho_1^*, b)) |\rho_0 \rho_1^* \rangle \\ &\quad - d(\rho_0, \rho_1) |\rho_1^* \rho_1 \rangle - d_+(\rho_1^*, b) |\rho_0 b \rangle \end{aligned} \quad (4.67)$$

Since by assumption 2.50 holds one has $\epsilon_+(\rho_1) = 0$ so that $d_+(\rho_1^*, b) = 0$ for all $b \neq \rho_1^*$. With (4.43) we arrive at the right closure relation

$$H |\mu^{L+} \rangle = d_1^\ell (|\mu^{L+} \rangle - |\mu^{L+1} \rangle). \quad (4.68)$$

Conversely, if 2.50 does not hold then closure is violated.

It is thus proved that the set of vectors $|\mu^x \rangle$, $x \in \Lambda_L$ forms an invariant subspace under the action of H if and only if the conditions (i) - (iii) hold. This allows for writing (4.51) with coefficients

$$G_{yx} = \begin{cases} d_1^r (\delta_{y, L_-} - \delta_{y, L_-+1}) & x = L_- \\ (d_1^\ell + d_1^r) \delta_{x, y} - d_1^\ell \delta_{x, y+1} - d_1^r \delta_{x, y-1} & L_- < x < L_+ \\ d_1^\ell (\delta_{y, L_+} - \delta_{y, L_+-1}) & x = L_+ \end{cases} \quad (4.69)$$

Comparing with the intensity matrix Q of the biased random walk of 2.11 one finds $G_{xy} = Q_{xy}$. Thus with $H = -W^T$ and $S_{x\eta} := \pi^{-1}(x)R(x, \eta) = \mu^x(\eta)$ one gets $SW = QS$ which according to (2.11) in the proof of Theorem 2.5 is equivalent to (3.4) due to reversibility of the shock exclusion process w.r.t. the measure $\pi(x) = d_1^x$ of the biased random walk. \square

4.3.2 Proof of Theorem 3.3

The assertion (3.5) as such, i.e., without specifying the transition probability, is an immediate corollary of the time-reversed duality established in Theorem (3.2). The specific formula (3.6) follows from the fact that the intensity matrix for a biased simple random walk with constant rates is a tridiagonal Toeplitz matrix. For readers not familiar with such matrices we outline how to arrive at (3.6): (a) The transition probability is by definition the matrix element $P(y, t|x, 0) = (\exp Gt)_{x,y}$ of the exponential of the intensity matrix. (b) The non-symmetric intensity matrix can be symmetrized by the ground state transformation $\tilde{G} = \hat{\pi}G\hat{\pi}^{-1}$ where $\hat{\pi}$ is the diagonal matrix with elements $d_1^{x-L_-}$ on the diagonal. (c) Diagonalizing the tridiagonal Toeplitz matrix G (which is textbook) then yields the eigenvalues $\epsilon_0 = 0$ and ϵ_p (3.8) for $1 \leq p \leq L-1$ and the associated eigenvectors of \tilde{G} with components that can be expressed as

$$\Psi_0(y) = \sqrt{\frac{d_1^2 - 1}{d_1^{2L} - 1}} d_1^{y-L_-} \quad (4.70)$$

$$\Psi_p(y) = \sqrt{\frac{2}{L}} \frac{\psi_p(y)}{d_1 - e^{-\frac{i\pi p}{L}}} \quad (4.71)$$

and satisfy the orthogonality relations

$$\sum_{y=-L_-}^{L_+} \Psi_p(y) \Psi_q^*(y) = \delta_{p,q} \quad \forall p, q \in \{0, 1, \dots, L-1\}. \quad (4.72)$$

Orthogonality can be proved by expanding for $d_1 \neq 1$ the denominator in the power series

$$\frac{1}{d_1 - e^{-\frac{i\pi p}{L}}} = \begin{cases} \frac{1}{1 - d_1^{-2L}} \sum_{n=0}^{2L-1} d_1^{-n-1} e^{-\frac{i\pi p n}{L}} & d_1 > 1 \\ -\frac{1}{1 - d_1^{2L}} \sum_{n=0}^{2L-1} d_1^n e^{\frac{i(n+1)\pi p}{L}} & d_1 < 1 \end{cases} \quad (4.73)$$

and using the orthogonality relations

$$\frac{2}{L} \sum_{y=L_-}^{L_+} \sin \frac{\pi p(y+1-L_-)}{L} \sin \frac{\pi q(y+1-L_-)}{L} = \sum_{k \in \mathbb{Z}} (\delta_{p-q, 2kL} - \delta_{p+q, 2kL}) \quad (4.74)$$

$$\frac{2}{L} \sum_{p=1}^{L-1} \sin \frac{\pi p(x+1-L_-)}{L} \sin \frac{\pi p(y+1-L_-)}{L} = \sum_{k \in \mathbb{Z}} (\delta_{x-y, 2kL} - \delta_{x+y, 2kL}) \quad (4.75)$$

and similar addition formulae involving three trigonometric functions. (d) It then follows that $P(y, 0|x, 0) = \delta_{y,x}$ and eigenvector decomposition of the matrix element $(\exp Gt)_{x,y}$ leads to (3.6). Finally, we note that existence of the limit $t \rightarrow \infty$ can be read off the fact that the eigenvalues (3.8) of the intensity matrix are strictly positive for all $p \in \{1, \dots, L-1\}$. Uniqueness of the invariant measure follows from ergodicity of the process. \square

4.4 Reverse duality involving N shocks

The strategy of the proof is analogous to the case of one shock which can be used as it is for shock measures that do not have neighbouring shock positions.

4.4.1 Proof of Theorem 3.7

Isolated shocks If all shocks are isolated, i.e., there is at least one site with density ρ_i between consecutive shocks with densities $\rho_i^* \rho_{i+1}^*$, then according to (A) and (B) of the projection lemma 4.3 the generator H acts on the shock measure by acting on all shocks separately as in the case of a single shock. In other words, for N shocks at positions x_i such that $x_{i+1} - x_i > 1$ for all i one gets

$$\begin{aligned} H|\mu^{\mathbf{x}}\rangle &= \left(h_{L_-}^- + \sum_{i=1}^N (\tilde{h}_{x_{i-1}} + \tilde{h}_{x_i}) + h_{L_+}^+ \right) |\mu^{\mathbf{x}}\rangle \\ &= \sum_{i=1}^N \left[d_i^\ell (1 - \delta_{x_i, L_-}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i-}}\rangle \right) \right] \end{aligned} \quad (4.76)$$

$$+d_i^r (1 - \delta_{x_N, L_+}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i+}}\rangle \right) \Big] \quad (4.77)$$

which implies invariance of the subspace with N shocks with bulk densities satisfying the microscopic stability condition (2.53) and arbitrary shock densities provided that no shocks at nearest neighbour sites appear in the sum over the shock positions.

The new situation for which closure needs to be proved are bunches of neighbouring shocks. We call n shocks on consecutive sites $x_k, x_{k+1}, \dots, x_{k+n-1}$ with shock densities $\rho_{i+1}^*, \rho_{i+2}^*, \dots, \rho_{i+n}^*$ and bulk densities ρ_i on site x_{k-1} and ρ_{i+n} on site x_{k+n} satisfying (2.61) with $N = n$ a shock bunch of size n at position x_k . An isolated shock is called a bunch of size 1. Evidently, for a total of N shocks, the bunch size is in the range $1 \leq n \leq N$, the shock index is in range $0 \leq i \leq N - n$, and the bunch position is in the range $L_- \leq k \leq L_+ - N + 1$. The new technical ingredient to prove closure for bunches is assertion (C) in Lemma 4.3.

Bulk bunches of size n : For fixed n we define the product vectors

$$|\Xi_i\rangle := |\rho_i \rho_{i+1}^* \rho_{i+2}^* \cdots \rho_{i-1+n}^* \rho_{i+n}^* \rho_{i+n}\rangle \quad (4.78)$$

$$|\Omega_i\rangle := \left(\tilde{h}_{12} + \sum_{k=2}^n \tilde{h}_{kk+1} + \tilde{h}_{n+1, n+2} \right) |\Xi_i\rangle \quad (4.79)$$

$$|\Xi_{i+1}^\ell\rangle := |\rho_{i+1}^* \rho_{i+1} \rho_{i+2}^* \cdots \rho_{i-1+n}^* \rho_{i+n}^* \rho_{i+n}\rangle \quad (4.80)$$

$$|\Xi_{i+n}^r\rangle := |\rho_i \rho_{i+1}^* \rho_{i+2}^* \cdots \rho_{i-1+n}^* \rho_{i-1+n} \rho_{i+n}^*\rangle \quad (4.81)$$

$$|\Upsilon_{ij}\rangle := |\rho_i \rho_{i+1}^* \cdots \rho_{j-1}^* c_j \rho_{j+1}^* \cdots \rho_{i-1+n}^* \rho_{i+n}\rangle, \quad i < j \leq i+n \quad (4.82)$$

$$|\tilde{\Upsilon}_{ij}\rangle := |\rho_i \rho_{i+1}^* \cdots \rho_{j-1}^* \tilde{c}_j \rho_{j+1}^* \cdots \rho_{i-1+n}^* \rho_{i+n}\rangle, \quad i < j \leq i-1+n. \quad (4.83)$$

To compute $|\Omega_i\rangle$ we use Lemma 4.3, specifically assertion (A) for the action of \tilde{h}_{12} , assertion (B) for the action of $\tilde{h}_{n+1, n+2}$, and assertion (A) for the action of the remaining local bulk jump operators h_{kk+1} in the summation symbol. Setting $a^{(1)} = \rho_i$, $\tilde{a}^{(1)} = \rho_{i+1}$, $b^{(1)} = \rho_{i+1}^*$, $\tilde{b}^{(1)} = c_{i+1}$ in (A), $a = \rho_k^*$, $\tilde{a} = \rho_{k+1}^*$, $\tilde{c} = \tilde{c}_k$, $c = c_{k+1}$ for $i+1 \leq k \leq i+n-1$ in (C), and $a^{(2)} = \rho_{i+n-1}$, $\tilde{a}^{(2)} = \rho_{i+n}$, $b^{(2)} = \rho_{i+n}^*$, $\tilde{b}^{(2)} = \tilde{c}_{i+n}$ this yields

$$\begin{aligned} |\Omega_i\rangle &= \left(r + \ell + \sum_{k=i+1}^{i+n} \left(\tilde{d}(\rho_k^*, \tilde{c}_k) - \tilde{d}(\rho_k^*, c_k) \right) \right) |\Xi_i\rangle \\ &\quad - d(\rho_i, \rho_{i+1}) |\Xi_{i+1}^\ell\rangle + d(\rho_{i+n}, \rho_{i+n-1}) |\Xi_{i+n}^r\rangle \\ &\quad + \sum_{k=i+1}^{i+n} \left(d(\rho_k^*, c_k) |\Upsilon_{ik}\rangle - d(\rho_k^*, \tilde{c}_k) |\tilde{\Upsilon}_{ik}\rangle \right) \end{aligned} \quad (4.84)$$

Without loss of generality we choose $c_k = \tilde{c}_k \neq \rho_k^*$ so that the sums vanish and using (4.43) and (4.44) one finds

$$\begin{aligned} |\Omega_i\rangle &= (r + \ell) |\Xi_i\rangle \\ &\quad - d_{i+1}^\ell |\Xi_{i+1}^\ell\rangle - d_{i+n}^r |\Xi_{i+n}^r\rangle \end{aligned}$$

$$\begin{aligned}
&= [d_{i+1}^\ell + d_{i+1}^r + (r - \ell)(\rho_i - \rho_{i-1})] |\Xi_i\rangle \\
&\quad - d(\rho_i, \rho_{i+1}) |\Xi_{i+1}^\ell\rangle + d(\rho_{i+n}, \rho_{i+n-1}) |\Xi_{i+n}^r\rangle
\end{aligned} \tag{4.85}$$

if and only if (2.53) which is equivalent to (4.23) holds for all consecutive shock densities and all consecutive bulk densities.

Applying this relation to a measure $|\mu^{\mathbf{x}}\rangle$ with N shocks at bulk positions $L_- < x_1 < \dots < x_N < L_+$ arranged in k bunches one thus finds an invariant subspace under the action of the generator by noting

$$\begin{aligned}
H|\mu^{\mathbf{x}}\rangle &= [\epsilon_-(\rho_0) + \epsilon_+(\rho_N)] |\mu^{\mathbf{x}}\rangle \\
&\quad + \sum_{i=1}^N (1 - \delta_{x_{i-1}, x_i}) \left([d_i^\ell - (r - \ell)\rho_{i-1}] |\mu^{\mathbf{x}}\rangle - d_i^\ell |\mu^{\mathbf{x}^{i-}}\rangle \right) \\
&\quad + \sum_{i=1}^N (1 - \delta_{x_{i+1}, x_i}) \left([d_i^r + (r - \ell)\rho_i] |\mu^{\mathbf{x}}\rangle - d_i^r |\mu^{\mathbf{x}^{i+}}\rangle \right) \\
&= \sum_{i=1}^N d_i^\ell (1 - \delta_{x_{i-1}, x_i}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i-}}\rangle \right) \\
&\quad + \sum_{i=1}^N d_i^r (1 - \delta_{x_{i+1}, x_i}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i+}}\rangle \right) \\
&\quad + (r - \ell) (\delta_{x_1, L_-} \rho_0 - \delta_{L_+, x_N} \rho_N) |\mu^{\mathbf{x}}\rangle
\end{aligned} \tag{4.86}$$

where in the first equality the bunch property and in the second equality we have used cancellation of the terms in the telescopic sum over the diagonal density terms with the boundary eigenvalues. Observing finally that $\delta_{x_1, L_-} = \delta_{L_+, x_N} = 0$ for $L_- < x_1 < \dots < x_N < L_+$ we arrive at

$$\begin{aligned}
H|\mu^{\mathbf{x}}\rangle &= \sum_{i=1}^N d_i^\ell (1 - \delta_{x_{i-1}, x_i}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i-}}\rangle \right) \\
&\quad + \sum_{i=1}^N d_i^r (1 - \delta_{x_{i+1}, x_i}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i+}}\rangle \right)
\end{aligned} \tag{4.87}$$

$$= \sum_{i=1}^N \left[w_i^\ell \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i-}}\rangle \right) + w_i^r \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i+}}\rangle \right) \right] \tag{4.88}$$

for shocks not at the boundary sites. It remains to prove closure when one or two shocks are at the boundary sites L_\pm .

Bunch of shocks at the boundaries: We first consider a bunch of n shocks at the left boundary but no shock at the right boundary, i.e., $x_1 = L_-$ and $x_N < L_+$. For fixed n we define the product vectors

$$|\Xi^-\rangle := |\rho_1^* \rho_2^* \dots \rho_{n-1}^* \rho_n^* \rho_n\rangle \tag{4.89}$$

$$|\Omega^-\rangle := \left(h_1^- + \sum_{k=1}^{n-1} \tilde{h}_{kk+1} + \tilde{h}_{n,n+1}\right) |\Xi_-\rangle \quad (4.90)$$

$$|\Xi_n^-\rangle := |\rho_1^* \rho_2^* \cdots \rho_{n-1}^* \rho_n^*\rangle \quad (4.91)$$

$$|\Upsilon_j^-\rangle := |\rho_1^* \cdots \rho_{j-1}^* c_j \rho_{j+1}^* \cdots \rho_{n-1}^* \rho_n^*\rangle, \quad 1 \leq j \leq n \quad (4.92)$$

$$|\tilde{\Upsilon}_j^-\rangle := |\rho_1^* \cdots \rho_{j-1}^* \tilde{c}_j \rho_{j+1}^* \cdots \rho_{n-1}^* \rho_n^*\rangle, \quad 1 \leq j \leq n \quad (4.93)$$

$$(4.94)$$

To compute $|\Omega^-\rangle$ we use assertion (B) in Lemma 4.3 for the action of $\tilde{h}_{n,n+1}$ and assertion (C) for the action of the remaining local bulk jump operators \tilde{h}_{kk+1} in the summation symbol. With $a = \rho_1^*$ and $b = c_1$ for the boundary terms, $a = \rho_k^*$, $\tilde{a} = \rho_{k+1}^*$, $c = c_{k+1}$, and $\tilde{c} = \tilde{c}_k$ for assertion (C), and $b^{(2)} = \rho_n^*$, $\tilde{a}^{(2)} = \rho_n$, $a^{(2)} = \rho_{n-1}$, $\tilde{b}^{(2)} = \tilde{c}_n$ for assertion (B) this yields

$$\begin{aligned} |\Omega^-\rangle &= \left(\tilde{d}_-(\rho_1^*, c_1) - \tilde{d}(\rho_1^*, c_1)\right) |\Omega^-\rangle \\ &\quad - (d_-(\rho_1^*, c_1) - d(\rho_1^*, c_1)) |\Upsilon_1^-\rangle \\ &\quad + \sum_{k=1}^{n-1} \left(\tilde{d}(\rho_k^*, \tilde{c}_k) - \tilde{d}(\rho_{k+1}^*, c_{k+1})\right) |\Omega^-\rangle \\ &\quad - \sum_{k=1}^{n-1} d(\rho_k^*, \tilde{c}_k) |\tilde{\Upsilon}_k^-\rangle + \sum_{k=1}^{n-1} d(\rho_{k+1}^*, c_{k+1}) |\Upsilon_{k+1}^-\rangle \\ &\quad + \left(r + \tilde{d}(\rho_n^*, \tilde{c}_n)\right) |\Omega^-\rangle \\ &\quad + d(\rho_n, \rho_{n-1}) |\Xi_n^-\rangle - d(\rho_n^*, \tilde{c}_n) |\tilde{\Upsilon}_n^-\rangle \\ &= \left(r + \tilde{d}_-(\rho_1^*, c_1)\right) |\Omega^-\rangle - d_-(\rho_1^*, c_1) |\Upsilon_1^-\rangle + d(\rho_n, \rho_{n-1}) |\Xi_n^-\rangle \\ &\quad + \sum_{k=1}^n \left(d(\rho_k^*, c_k) |\Upsilon_k^-\rangle - d(\rho_k^*, \tilde{c}_k) |\tilde{\Upsilon}_k^-\rangle\right). \end{aligned} \quad (4.95)$$

Now we choose use without loss of generality $\tilde{c}_k = c_k$ and observe that by assumption $\rho_1^* = \alpha/(\alpha + \gamma)$ so that $d_-(\rho_1^*, c_1) = \tilde{d}_-(\rho_1^*, c_1) = 0$ for any choice of c_1 . Therefore

$$\begin{aligned} |\Omega^-\rangle &= r |\Omega^-\rangle + d(\rho_n, \rho_{n-1}) |\Xi_n^-\rangle \\ &= [d_n^r + (r - \ell)\rho_n] |\Omega^-\rangle - d_n^r |\Xi_n^-\rangle \end{aligned} \quad (4.96)$$

where in the second equality (4.43) was used and one gets for consecutive bunches with a total of N shocks

$$\begin{aligned} H|\mu^{\mathbf{x}}\rangle &= \sum_{i=1}^N d_i^\ell (1 - \delta_{x_{i-1}, x_{i-1}}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i-}}\rangle\right) \\ &\quad + \sum_{i=1}^N d_i^r (1 - \delta_{x_{i+1}, x_{i+1}}) \left(|\mu^{\mathbf{x}}\rangle - |\mu^{\mathbf{x}^{i+}}\rangle\right) \end{aligned}$$

$$-(r - \ell)\rho_0 (1 - \delta_{x_1, L_-}) |\mu^{\mathbf{x}}\rangle. \quad (4.97)$$

Since for a shock at the left boundary site one has $1 - \delta_{x_1, L_-} = 0$ one recovers (4.88) also for $x_1 = L_-$. For a bunch of shocks only at the right boundary or two bunches on both boundaries similar computations using (2.50) with $N = m + 1$ also lead to (4.88) which is therefore valid for any set of sites \mathbf{x} .

Hence invariance of the subspace spanned by the vectors $|\mu^{\mathbf{x}}\rangle$ is proved. The closure relation (4.88) can be recast as

$$H|\mu^{\mathbf{x}}\rangle = - \sum_{\mathbf{y}} Q_{\mathbf{xy}} |\mu^{\mathbf{y}}\rangle \quad (4.98)$$

with the matrix elements $Q_{\mathbf{xy}}$ (4.47) of the intensity matrix for N -particle shock exclusion process. With the same arguments as at the end of the proof of Theorem 3.4 one arrives at (3.11). \square

4.4.2 Proof of Theorem 3.9

The time evolution formula (3.12) follows from Theorem 2.5. Existence of the limit (3.13) follows trivially from the finite state space of the ASEP defined on Λ_L (which guarantees existence of an invariant measure) and uniqueness follows from ergodicity which is guaranteed by the strict positivity of the boundary rates. \square

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A The XXZ quantum spin chain with non-diagonal boundary fields

We denote by $\mathbb{1}$ the two-dimensional unit matrix and by $\mathbf{1}$ the unit matrix of dimension 2^L . We also recall the standard definitions the Pauli matrices and some of their linear combinations.

Definition A.1 *The Pauli matrices are the 2×2 matrices*

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.1})$$

and

$$\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y) \quad (\text{A.2})$$

are the particle annihilation and creation operators. The diagonal matrices

$$\hat{n} = \frac{1}{2}(\mathbb{1} - \sigma^z), \quad \hat{v} = \frac{1}{2}(\mathbb{1} + \sigma^z) \quad (\text{A.3})$$

are the particle and vacancy projectors respectively.

With these notations the negative transpose H (2.31) of the intensity matrix M (2.28) is given by

$$\begin{aligned} H = & - \sum_{k=L_-}^{L_+-1} [r(\sigma_k^+ \sigma_{k+1}^- - \hat{n}_k \hat{v}_{k+1}) + \ell(\sigma_k^- \sigma_{k+1}^+ - \hat{v}_k \hat{n}_{k+1})] \\ & - [\alpha(\sigma_{L_-}^- - \hat{v}_{L_-}) + \gamma(\sigma_{L_-}^+ - \hat{n}_{L_-}) + \delta(\sigma_{L_+}^- - \hat{v}_{L_+}) + \beta(\sigma_{L_+}^+ - \hat{n}_{L_+})] \end{aligned} \quad (\text{A.4})$$

A transformation with the diagonal matrix

$$\hat{Q} := \prod_{k=L_-}^{L_+-1} q^{k\hat{n}_k} \quad (\text{A.5})$$

symmetrizes the bulk jump matrices. With the parameters

$$\theta := \ln q, \quad w := \sqrt{r\ell} \quad (\text{A.6})$$

$$E_0 := (L-1) \cosh(\theta) + \frac{\alpha + \beta + \gamma + \delta}{w} \quad (\text{A.7})$$

and with the parametrization

$$\alpha = \frac{w}{2} \sinh \theta \frac{e^{\phi_- - \psi_-}}{\sinh \phi_- \cosh \psi_-} \quad (\text{A.8})$$

$$\gamma = \frac{w}{2} \sinh \theta \frac{e^{\psi_- - \phi_-}}{\sinh \phi_- \cosh \psi_-} \quad (\text{A.9})$$

$$\beta = \frac{w}{2} \sinh \theta \frac{e^{\phi_+ - \psi_+}}{\sinh \phi_+ \cosh \psi_+} \quad (\text{A.10})$$

$$\delta = \frac{w}{2} \sinh \theta \frac{e^{-\phi_+ + \psi_+}}{\sinh \phi_+ \cosh \psi_+} \quad (\text{A.11})$$

$$\theta_- = \psi_- - \phi_- + \theta L_- \quad (\text{A.12})$$

$$\theta_+ = \phi_+ - \psi_+ + \theta L_+ \quad (\text{A.13})$$

one gets the quantum Hamiltonian

$$H^{XXZ} = \hat{Q}^{-1} H \hat{Q} \quad (\text{A.14})$$

$$\begin{aligned}
&= -\frac{w}{2} \left\{ \sum_{k=L_-}^{L_+-1} [\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \cosh(\theta) \sigma_k^z \sigma_{k+1}^z] - E_0 \mathbf{1} \right. \\
&\quad + \frac{\sinh \theta}{\sinh \phi_- \cosh \psi_-} \left(e^{-\theta_-} \sigma_{L_-}^- + e^{\theta_-} \sigma_{L_-}^+ + \sinh \psi_- \cosh \phi_- \sigma_{L_-}^z \right) \\
&\quad \left. + \frac{\sinh \theta}{\sinh \phi_+ \cosh \psi_+} \left(e^{-\theta_+} \sigma_{L_+}^- + e^{\theta_+} \sigma_{L_+}^+ - \sinh \psi_+ \cosh \phi_+ \sigma_{L_+}^z \right) \right\} \text{A.15}
\end{aligned}$$

of the XXZ spin chain with non-diagonal and non-hermitian boundary fields in the form presented in [55] for $w = 1$ and $E_0 = 0$.

In terms of the parameters (A.6) - (A.13) the boundary functions (2.46) are given by

$$\kappa_+(\alpha, \gamma) = e^{\phi_- + \psi_- - \theta L_- + \theta_-} \quad (\text{A.16})$$

$$\kappa_+(\beta, \delta) = e^{\phi_+ + \psi_+ - \theta_+ + \theta L_+} \quad (\text{A.17})$$

so that the condition (2.49) reads

$$\phi_- + \psi_- + \phi_+ + \psi_+ = \theta_+ - \theta_- + (2N - L + 1)\theta. \quad (\text{A.18})$$

This is the integrability condition found in [55]. \square

The special condition (2.50) reads

$$\frac{\alpha\beta}{\gamma\delta} = e^{2(\phi_- + \phi_+ - \psi_- - \psi_+)} = e^{-2\theta(N-M)} \quad (\text{A.19})$$

which can be recast as

$$\theta_+ - \theta_- + \theta(N - M - L + 1) = 0. \quad (\text{A.20})$$

B On the parametrization of the open ASEP

With the parametrization (2.42) and (2.43) one gets

$$\kappa_+(\alpha, \gamma) = z_-^{-1}, \quad \kappa_-(\alpha, \gamma) = -\frac{\ell + \omega_-}{r + \omega_-} \quad (\text{B.1})$$

$$\kappa_+(\beta, \delta) = z_+, \quad \kappa_-(\beta, \delta) = -\frac{\ell + \omega_+}{r + \omega_+}. \quad (\text{B.2})$$

The independence of the function $\kappa_+(\alpha, \gamma)$ of ω_- can be seen by noting that $x/\rho - y/(1-\rho) = r - \ell$ and therefore $y - x + r - \ell = xz_-^{-1} - yz_-$ so that the argument of the square root becomes $(xz_-^{-1} + yz_-)^2$. With the role of ρ and $1 - \rho$ interchanged one then also realizes that $\kappa_+(\beta, \delta)$ is independent of ω_+ . Thus on the parameter manifold given by

$$(r - \ell)\rho_-(1 - \rho_-) = \alpha(1 - \rho_-) - \gamma\rho_- \quad (\text{B.3})$$

one has $\kappa_+(\alpha, \gamma) = z_-^{-1}$ and on the manifold

$$(r - \ell)\rho_+(1 - \rho_+) = \beta\rho_+ - \delta(1 - \rho_+) \quad (\text{B.4})$$

one has $\kappa_+(\beta, \delta) = z_+$.

For $q > 1$, boundary densities $\rho_{\pm} \in (0, 1)$, and boundary parameters in \mathbb{R}^+ one has $\omega_{\pm} > -\ell$ and correspondingly $-1 < \kappa_-(x, y) < 0$. Moreover, $\kappa_+(x, y) \in \mathbb{R}^+$. Then (2.48) implies $\rho_0 < \sigma_1^* < \rho_1$. On the other hand, for $q < 1$ where a downward shock with $\rho_0 > \rho_1$ is macroscopically stable one gets $-\infty < \kappa_-(x, y) < -1$ and therefore $\rho_1 < \sigma_1^* < \rho_0$. Hence either way the shock density is between the boundary densities of the shock.

Alternatively, (2.50) can be written

$$\frac{\alpha\beta}{\gamma\delta} = q^{2(N-M)} \quad (\text{B.5})$$

In terms of the parameters ρ_{\pm} the manifold \mathcal{B}_N defined by (2.49) is given by

$$\frac{z_+}{z_-} = q^{2N}, \quad N \in \mathbb{N}^+ \quad (\text{B.6})$$

and (2.50) defining the submanifold \mathcal{B}_N^M can be written as the constraint

$$\frac{(r + \omega_-)(r + \omega_+)}{(\ell + \omega_-)(\ell + \omega_+)} = q^{2M}, \quad 1 \leq M \leq N \quad (\text{B.7})$$

on the boundary parameters ω_{\pm} .

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