

## NS and equivalence conjectures.

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**Abstract:** *General considerations on the Equivalence conjectures and a review of few mathematical results.*

### I. INTRODUCTION

Reversible equations conjectured to be equivalent for the purpose of modeling stationary states, under large scale forcing, of incompressible Navier-Stokes evolutions have been introduced since the 90’s, [1, 2]. Equivalence has been first conjectured to be asymptotic at vanishing viscosity, then at all viscosity for observables of large scale, [3], or just for observables of scale up to  $\sim$ Kolmogorov’s scale, [4].

There are very few rigorous results, stressed in [3, 4], supporting the conjectures. And some breakthrough results, [5, 6], on the classical NS equation can be directly applied to the reversible equations.

The NS equations, incompressible and in a periodic container  $\Omega = [-\pi, \pi]^d$ ,  $d = 2, 3$ , deal with a velocity field which can be expressed in terms of its Fourier’s coefficients as

$$\mathbf{u}(\mathbf{x}) = \sum_{\substack{\mathbf{0} \neq \mathbf{k} \in \mathbb{Z}^d, \\ c=1, \dots, d-1}} u_{\mathbf{k}}^c \mathbf{e}_{\mathbf{k}}^c e^{-i\mathbf{k} \cdot \mathbf{x}} \quad (1.1)$$

where  $\mathbf{e}_{\mathbf{k}}^c$ ,  $c = 1, \dots, d-1$ , are  $d-1$  unit “elicity” vectors, orthogonal to  $\mathbf{k}$  and with  $\mathbf{e}_{-\mathbf{k}}^c = -\mathbf{e}_{\mathbf{k}}^c$ , and  $u_{-\mathbf{k}}^c = \bar{u}_{\mathbf{k}}^c$  are the complex harmonics of  $\mathbf{u}$ , with  $|u_{\mathbf{0}}^c| \equiv 0$  (fixing the baricenter).

With  $\mathbf{u}$  so represented the well known NS equations are  $\partial_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x}) = 0$  (incompressibility) and:

$$\dot{\mathbf{u}}(\mathbf{x}) = -(\mathbf{u}(\mathbf{x}) \cdot \partial_{\mathbf{x}}) \mathbf{u}(\mathbf{x}) + \nu \Delta \mathbf{u}(\mathbf{x}) - \partial_{\mathbf{x}} p(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \quad (1.2)$$

with  $\nu =$  kinematic viscosity. Or, in terms of the  $u_{\mathbf{k}}^c$ :

$$\dot{u}_{\mathbf{k}}^c = \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} \\ a, b}} T_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}}^{a, b, c} u_{\mathbf{k}_1}^a u_{\mathbf{k}_2}^b - \nu \mathbf{k}^2 u_{\mathbf{k}}^c + f_{\mathbf{k}}^c \quad (1.3)$$

$$T_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}}^{a, b, c} = -(\mathbf{e}_{\mathbf{k}_1}^a \cdot \mathbf{k}_2)(\mathbf{e}_{\mathbf{k}_2}^b \cdot \mathbf{e}_{\mathbf{k}}^c), \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$$

with  $\mathbf{k}^2 \stackrel{def}{=} \sum_{i=1, \dots, d} k_i^2$  and the forcing  $\mathbf{f} \neq \mathbf{0}$  is supposed fixed once and for all and to act only on ‘large scale’:  $f_{\mathbf{k}}^c = 0$  unless  $0 < |\mathbf{k}| = \max_{i=1, \dots, d} |k_i| \leq k_f < \infty$ .

Without restriction, suppose  $\|\mathbf{f}\|^2 = \sum_{\mathbf{k}, c} |f_{\mathbf{k}}^c|^2 = 1$ : so the equation has viscosity as the only free parameter, whose inverse will also be called Reynolds number  $R = \nu^{-1}$ . The equation will be called “INS”, or *irreversible* NS.

### II. EQUIVALENT EQUATION

Equivalent equations can be obtained by replacing the viscous force  $-\nu \mathbf{k}^2 u_{\mathbf{k}}^c$  by  $-\alpha(\mathbf{u}) \mathbf{k}^2 u_{\mathbf{k}}^c$  determining  $\alpha$  so that the equation:

$$\dot{u}_{\mathbf{k}}^c = \sum_{\substack{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} \\ a, b}} T_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}}^{a, b, c} u_{\mathbf{k}_1}^a u_{\mathbf{k}_2}^b - \alpha(\mathbf{u}) \mathbf{k}^2 u_{\mathbf{k}}^c + f_{\mathbf{k}}^c \quad (2.1)$$

will admit a selected observable as an exact constant of motion.

In [2, 3] the selected observable is the “Enstrophy”:

$$\begin{aligned} \mathcal{D}(\mathbf{u}) &= \sum_{c, \mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}^c|^2 = \int \mathcal{D}(\mathbf{u}, x) \frac{dx}{(2\pi)^d} \\ \mathcal{D}(\mathbf{u}, x) &= \frac{1}{2} \sum_{i, j} (\partial_i u(x)_j + \partial_j u(x)_i)^2 \end{aligned} \quad (2.2)$$

Other observables have been considered (*e.g.* in [7] the selected observable is  $E = \sum_{c, \mathbf{k}} |u_{\mathbf{k}}^c|^2$ ). Selecting  $\mathcal{D}$  leads, if  $d \geq 3$ , to:

$$\begin{aligned} \alpha(\mathbf{u}) &= \frac{\sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^* \sum_{a, b, c} \mathbf{k}_3^2 u_{\mathbf{k}_1}^a u_{\mathbf{k}_2}^b \bar{u}_{\mathbf{k}_3}^c T_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3}^{a, b, c}}{\sum_{c, \mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}^c|^2} \\ &+ \frac{\sum_{c, \mathbf{k}} f_{\mathbf{k}}^c \mathbf{k}^2 \bar{u}_{\mathbf{k}}^c}{\sum_{c, \mathbf{k}} \mathbf{k}^4 |u_{\mathbf{k}}^c|^2} \end{aligned} \quad (2.3)$$

where the  $*$  reminds  $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{0}$ . See Eq.(4.2) for a possibly more natural expression of  $\alpha(\mathbf{u})$ .

If  $d = 2$  the multiplier  $\alpha$  would simply be the second term in Eq.(2.3): because the first term would cancel.<sup>1</sup> Furthermore selecting  $E$ , instead of  $D$ , yields  $\alpha = \frac{\sum_{c, \mathbf{k}} f_{\mathbf{k}}^c \bar{u}_{\mathbf{k}}^c}{\sum_{c, \mathbf{k}} \mathbf{k}^2 |u_{\mathbf{k}}^c|^2}$  in any dimension <sup>2</sup>

The Eq.(2.1) will be called RNS, *reversible* NS: because if  $t \rightarrow \mathbf{u}(t)$  is a solution for Eq.(2.1) also  $-\mathbf{u}(-t)$  is a solution. Correspondingly  $\alpha(\mathbf{u})$  will also be named “*reversible viscosity*”.

Here only properties of the RNS equation with  $\alpha$  such that the enstrophy is constant will be considered.

The equivalence conjectures concern the stationary distributions of INS and of RNS and the averages that they assign to the “*local observables*”  $O(\mathbf{u})$ , which are functions  $O(\mathbf{u})$  of the velocity fields which depend on finitely many harmonics  $u_{\mathbf{k}}^c$ , possibly subject to the further condition that the waves  $\mathbf{k}$  are  $|\mathbf{k}| \ll K_\nu$  where  $K_\nu$  is Kolmogorov’s inverse length scale  $K_\nu = (\frac{En}{\nu^2})^{\frac{1}{4}}$ .

<sup>1</sup>By the well known identity which implies the Enstrophy conservation (only) in the 2-dimensional Euler-equation.

<sup>2</sup>By the identity which implies the energy conservation in the Euler equation.

To formulate mathematically precise conjectures introduce the *regularized equations*  $INS^N$  and  $RNS^N$  with ultraviolet cut-off  $N$ :

**Definition:** *The equations  $INS^N$  and  $RNS^N$  are the Eq.(1.3),(2.1) with  $\nu$  and  $\alpha(\mathbf{u})$  respectively as above with the further restriction on the  $u_{\mathbf{k}}^c$  that  $|u_{\mathbf{k}}^c| = 0, |\mathbf{k}| > N$ .*

Therefore both regularized equations are ODE's on a phase space of (real) dimension  $M = (d-1)((2N+1)^d - 1)$  as each component has  $0 < |\mathbf{k}| = \max_i |k_i| \leq N$ .

Hence it makes sense to consider initial data in  $R^M$  randomly selected with a distribution  $\rho(\mathbf{u})d\mathbf{u}$ , with  $\rho$  a continuous density *i.e.* “volume continuous”. Starting evolution  $t \rightarrow S_t(\mathbf{u})$  with an initial datum  $\mathbf{u}$  so chosen, it is assumed:

**Hypothesis:** *Given  $\nu$  or  $D$  there are, for the  $INS^N$  or  $RNS^N$  evolutions, finitely many stationary ergodic probability distributions denoted, for  $i = 1, \dots, \mathcal{N}$ ,  $\mu_{\nu,i}^N$  or, respectively,  $\gamma_{D,i}^N$  which control the statistics of the local observables  $O$ . This means that, on motions starting with initial data chosen with a volume-continuous distribution, the average of such observables is given, with probability 1, by  $\mu_{\nu,i}^N(O)$  or, respectively,  $\gamma_{D,i}^N(O)$  for some  $i$ .*

When the motion is “chaotic”, [8–10], (or, a particular case, satisfies the “Chaotic Hypothesis”, [11, 12]) and has  $\mathcal{N}$  attractors then each  $\mu_i$  is called a “SRB-measure”, [10].

It is expected that in most cases  $\mathcal{N} = 1$ , *greatly simplifying* the hypothesis: which embodies the classical ergodic hypothesis if applied to the chaotic microscopic motions of Hamiltonian systems of many particles.

**Remark:** the above hypothesis is intended to apply also to cases in which the attractors are periodic orbits (typically if viscosity is large). In [3] it is suggested that the reason for its validity at fixed  $\nu$  and large enough  $N$  could be looked in the microscopic motions, from which the NS evolution is derived as a scaling limit without modifications to the microscopic equations. But at fixed cut-off  $N$  the hypothesis can be related, and possibly hold, to the chaoticity of the motions only at small enough  $\nu$ : manifestly a less interesting case.

However the latter key point will not be further discussed here, as attention is devoted to mathematical properties of the regularized equations in the light of the following conjectures and the regularization removal, *i.e.*  $N \rightarrow \infty$ .

### III. EQUIVALENCE CONJECTURES

Given the UV cut-off  $N$  the stationary SRB distributions (think, at first, that there is only one such) form a collection  $\mathcal{E}^N$  of probability distributions on  $M$ , the  $(d-1)((2N+1)^d - 1)$ -dimensional phase space, parameterized by  $\nu$  in the  $INS^N$  case or by  $D$  in the  $RNS^N$  case and possibly other  $\mathcal{N}$  labels distinguishing the SRB

distributions on the  $\mathcal{N}$  attractors (there is no relation between the cut-off  $N$  and the number of degrees of freedom  $\mathcal{N}$ ).

Denote  $\mathcal{E}^N$  and  $\mathcal{G}^N$  the collection of the SRB distributions  $\mu_{\nu}^N, \nu > 0$  for  $INS^N$ , or respectively,  $\gamma_D^N, D > 0$  for  $RNS^N$ : such collections will be called *viscosity ensemble* or *enstrophy ensemble*.

**Conjecture 1:** *(1) if  $\nu$  is small enough the number  $\mathcal{N}$  of attractors for  $INS^N$  with viscosity  $\nu$  and average enstrophy  $\mu_{\nu}^N(\mathcal{D}) = D$  is eventually (as  $N \rightarrow \infty$ ) the same as the number of attractors for  $RNS^N$  with enstrophy  $D$ .<sup>3</sup> (2) If  $O$  is an observable and  $\mu_{\nu}^N(\mathcal{D}) = D$  it is*

$$\lim_{\nu \rightarrow 0} \mu_{\nu}^N(O) = \lim_{D \rightarrow \infty} \gamma_D^N(O) \quad (3.1)$$

The conjecture, [2, 13, 14], can be regarded as a “homogenization” property: at large  $D$  the  $RNS^N$  equation generates chaotic motion (a feature shared quite generally by strongly forced ODE's) and  $\alpha(\mathbf{u})$  fluctuates around a constant value and induces averages of observables equal to those of  $INS^N$  with constant viscosity, as in [15]. A second conjecture is:

**Conjecture 2:** *(1) if  $N$  is large enough the number  $\mathcal{N}$  of attractors for  $INS^N$  with viscosity  $\nu$  and average enstrophy  $\mu_{\nu}^N(\mathcal{D}) = D$  is the same as the number of attractors for  $RNS^N$  with enstrophy  $D$ . (2) If  $O$  is a local observable and  $\mu_{\nu}^N(\mathcal{D}) = D$  it is*

$$\lim_{N \rightarrow \infty} \mu_{\nu}^N(O) = \lim_{N \rightarrow \infty} \gamma_D^N(O) \quad (3.2)$$

Therefore conjecture 1 deals with the limit  $\nu \rightarrow 0$  at fixed UV-cut-off  $N$  and holds for any observable, while conjecture 2 deals with the physically relevant limit  $N \rightarrow \infty$  at fixed  $\nu$  and holds for local (*i.e.* large scale) observables. A much weaker conjecture:

**Conjecture 3:** *(1) same as in conjecture 2. (2) If  $O$  is an observable localized on scale sufficiently small compared to Kolmogorov's scale  $K_{\nu} = (\frac{\nu D}{\nu^3})^{\frac{1}{4}}$  and  $\mu_{\nu}^N(\mathcal{D}) = D$  it is*

$$\lim_{N \rightarrow \infty} \mu_{\nu}^N(O) = \lim_{N \rightarrow \infty} \gamma_D^N(O) \quad (3.3)$$

This restricts the observables  $O$  to depend on the Fourier's components  $u_{\mathbf{k}}^c$  of the velocity with  $|\mathbf{k}| < K_{\nu}$ , provided  $\mu_{\nu}^N(\mathcal{D}) = D$  as in the previous conjectures. Hence an observable  $O$  depending on just one  $\mathbf{k}$  will be a local observable relevant for conjecture 2 but not for conjecture 3 unless  $|\mathbf{k}|$  is “sufficiently smaller” than  $K_{\nu}$ .

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<sup>3</sup>In the relation  $\mu_{\nu}^N(\mathcal{D}) = D$  given  $\nu$  the average  $D$  depends also on  $N$ : the extra label  $N$  on  $D$  will be always omitted in the following to simplify the notation if clear from the context.

Conjecture 3 is introduced in [4] to cover at least the results of the corresponding simulations: the simulations were not developed enough to allow stating that the failure of conjecture 2 on observables of scales over  $\simeq \frac{1}{8}K_\nu$ , as apparently shewed by the simulations, could be firmly confirmed; the point was left for consideration in future work.

**Remark:** Introducing the “viscosity” and “enstrophy” ensembles leads to a strong analogy between equilibrium statistical mechanics (where the finite volume regularizes the dynamics) and stationary properties of NS evolution (where the UV-cut-off regularizes the dynamics): aspects of the analogy have been pointed out, for instance, in [3, 16, 17]. The conjectures (in particular conjecture 2) make the thermodynamic limit (infinite volume) analogous to the  $N \rightarrow \infty$  limit.

#### IV. $RNS^N$ -UNIFORM REGULARITY

It is well known that in dimension 3 an algorithm for the construction of a smooth solution for INS with  $C^\infty$ -smooth initial data and smooth force is an open problem, [18, 19]: the difficulty being to establish an *a priori* bound on the enstrophy, [20].

In the  $RNS^N$  case the enstrophy of a smooth datum  $\mathbf{u}$  is finite and evolves at time  $t$  into  $\mathbf{u}(t) = S_t^N \mathbf{u}$  with the same enstrophy. To investigate the regularity it is natural to study first the size of  $\alpha(u)$  in a velocity field of given enstrophy  $D$ .

**Theorem 1:** *If  $\mathbf{u} \in C^\infty$  has enstrophy  $\mathcal{D}(\mathbf{u}) = D$  then the multiplier  $\alpha(\mathbf{u})$  in  $RNS^N$ , Eq.(2.3) is bounded by*

$$|\alpha(\mathbf{u})| \leq C_1(D^{\frac{1}{2}} + D^{-\frac{1}{2}}) \quad (4.1)$$

with  $C_1$  a universal constant, independent of the UV-cut-off  $N$ . [4].

This kinematic inequality (*i.e.* depending on  $\mathbf{u} \in C^\infty$  and unrelated to the  $RNS^N$ ) is obtained by combining the Hölder and Sobolev’s inequalities, see for instance [4, Appendix A], applied to  $\Lambda(\mathbf{u}) = -\int[(\mathbf{u} \cdot \boldsymbol{\varrho})\mathbf{u}] \cdot \Delta \mathbf{u} \, d\mathbf{x}$  which appears in the expression of the first addend in Eq.(2.3) rewritten in the form:

$$\alpha(\mathbf{u}) = \frac{\Lambda(\mathbf{u}) + \int \mathbf{f} \cdot \Delta \mathbf{u} \, d\mathbf{x}}{\int (\Delta \mathbf{u})^2 \, d\mathbf{x}} \quad (4.2)$$

A remarkable regularity, uniform in time and in the UV-cut-off, holds for solutions of the  $RNS^N$ .

Suppose  $0 < \varepsilon < \alpha(\mathbf{u}(t)) \leq \kappa$ , for some  $\varepsilon, \kappa > 0$ , and that the initial data  $\mathbf{u}(0)$  and the forcing  $\mathbf{f}$  satisfy  $|\mathbf{u}_\mathbf{k}|, |\mathbf{f}_\mathbf{k}| < c_p |\mathbf{k}|^{-p}$  for all  $p > 0$  (recall that we consider only initial data and force with a finite number of modes,  $\leq N$ , for simplicity). Let  $a(t, \tau) = \int_\tau^t \alpha(\mathbf{u}(t')) dt'$ ; then  $\varepsilon(t - \tau) < a(t, \tau) < \kappa(t - \tau)$ .

**Theorem 2:** *If  $\mathbf{u}(t)$  is a solution of  $RNS^N$  with enstrophy  $D$  and  $\alpha(\mathbf{u}(t)) > \varepsilon > 0$  then  $\mathbf{u}(t)$  is  $C^\infty$ -regular*

with  $C^k$  norm  $\|\mathbf{u}(t)\|_{C^k} < c_k(\varepsilon, \|\mathbf{u}(0)\|_{C^\infty})$  where  $c_k$  is independent of  $t$  and of the UV-cut-off  $N$ . [4].

*proof:* Following, for instance, [21] and clarifying the notations, write  $\mathbf{u}_\mathbf{k}(t) = e^{-a(t,0)\mathbf{k}^2} \mathbf{u}_\mathbf{k}(0) + \int_0^t e^{-a(t,\tau)\mathbf{k}^2} (\mathbf{n}_\mathbf{k}(\mathbf{u}(\tau)) + \mathbf{f}_\mathbf{k}) d\tau$ , where  $\mathbf{n}_\mathbf{k}(\mathbf{u}(\tau))$  is the non-linear term of the NSE. Therefore, the sum of the first and last term can be bounded by  $\frac{c_p}{|\mathbf{k}|^p}$ ,  $c_p = \|\Delta^p \mathbf{u}(0)\|_2 + \|\Delta^p \mathbf{f}\|_2$ , while the integral is bounded by

$$\int_0^t e^{-\varepsilon \mathbf{k}^2(t-\tau)} \|\boldsymbol{\varrho} \mathbf{u}(\tau)\|_2 \|\mathbf{u}(\tau)\|_2 \, d\tau \leq \frac{\sqrt{D\mathcal{E}}}{\varepsilon \mathbf{k}^2} \quad (4.3)$$

where  $\mathcal{E}$  is an *a priori* bound on  $\sum_\mathbf{k} |\mathbf{u}_\mathbf{k}|^2$  (*e.g.*  $D$  itself as  $|\mathbf{k}| \geq 1$ ) so that adding the two bounds:  $|\mathbf{u}_\mathbf{k}(t)|^2 < \frac{C_2}{\mathbf{k}^2}$  for a suitable  $C_2$  (*e.g.*  $C_2 = \frac{\sqrt{D\mathcal{E}}}{\varepsilon} + c_2$ ). Therefore, again,  $|\mathbf{u}_\mathbf{k}(t)|$  can be bounded by adding  $\frac{c_p}{|\mathbf{k}|^p}$ , contributed from the initial datum, and a bound on

$$\int_0^t e^{-\varepsilon \mathbf{k}^2(t-\tau)} \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{|\mathbf{k}_1| |\mathbf{u}_{\mathbf{k}_1}| |\mathbf{k}_2|^2 |\mathbf{u}_{\mathbf{k}_2}|}{|\mathbf{k}_1| |\mathbf{k}_2|} \, d\tau \quad (4.4)$$

A bound on the latter integral is obtained via the Schwartz inequality and the remark that  $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$  implies  $|\mathbf{k}_1| |\mathbf{k}_2| \geq \frac{k_0}{2} |\mathbf{k}|$ ,  $k_0 = 1$ , and

$$\begin{aligned} \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{|\mathbf{k}_1| |\mathbf{u}_{\mathbf{k}_1}| |\mathbf{k}_2|^2 |\mathbf{u}_{\mathbf{k}_2}|}{|\mathbf{k}_1| |\mathbf{k}_2|} &\leq 2C_2 \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{|\mathbf{k}_1| |\mathbf{u}_{\mathbf{k}_1}|}{|\mathbf{k}_1| |\mathbf{k}_2|} \\ &\leq 2C_2 \sqrt{D} \left( \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{1}{(|\mathbf{k}_1| |\mathbf{k}_2|)^2} \right)^{\frac{1}{2}} \\ &\leq 2^{1+\frac{1}{8}} C_2 |\mathbf{k}|^{-\frac{1}{8}} \sqrt{D} \left( \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \frac{1}{(|\mathbf{k}_1| |\mathbf{k}_2|)^{2-\frac{1}{4}}} \right)^{\frac{1}{2}} \\ &\leq 2^{1+\frac{1}{8}} C_2 \sqrt{D} |\mathbf{k}|^{-\frac{1}{8}} \left( \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|^{4-\frac{1}{2}}} \right)^{\frac{1}{2}} \end{aligned} \quad (4.5)$$

where  $\mathbf{k}_1$  has been changed to  $\mathbf{n}$  just to make clear that summing over  $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$  allows using the Schwartz inequality. Hence integration over  $t$ , as in Eq.(4.3), yields for suitable  $C_3$ , proportional to  $\sqrt{D}$ :

$$|\mathbf{u}_\mathbf{k}(t)| \leq \frac{C_3}{k^{2+\frac{1}{8}}} \quad (4.6)$$

Thus if  $D$  is finite the bound  $|\mathbf{u}_\mathbf{k}| < \gamma k^{-2}$ , Eq. (4.3), can be improved into  $|\mathbf{u}_\mathbf{k}| < \gamma_1 |\mathbf{k}|^{-2-\frac{1}{8}}$ .

Iterating a *autoregularization* phenomenon sets in and

$$\|\mathbf{u}_\mathbf{k}(t)\|_2 \leq \frac{\gamma_p}{k^{2+\frac{1}{4}p}} \quad \text{for all } p \geq 1 \quad (4.7)$$

so that  $\mathbf{u}(t)$  is a  $C^\infty$ -functions and all its derivatives can be bounded in terms of the enstrophy  $D$ , uniformly in  $N$ . See Sec. 3.3 in [21] for related results on the classic autoregularization.

**Remarks:** (1) the theorem shows that the condition  $\alpha(\mathbf{u}(t)) > \varepsilon$  for some  $\varepsilon > 0$  has an *extremely unlikely possibility*. Besides providing a well defined prescription to construct uniformly smooth approximations of the RNS equations as sequences of solutions to  $RNS^N$ , it would imply that the  $RNS^N$  attractors consist of uniformly  $\mathcal{C}^\infty$  velocities (*i.e.* with  $\mathcal{C}^k$  norms uniformly bounded for each  $k$  and independently of  $N$ ).

(2) the simulations with large  $N$  show that  $\alpha(\mathbf{u}(t))$  is observed, after a transient, not only  $> 0$  but also quite close to 1: in [4] evidence is provided that this might be an illusion: following the evolution of  $\alpha(t)$  on typical  $RNS^N$  solutions it is found that as  $N$  increases the negative values of  $\alpha$  have a rapidly decreasing probability. So the negative values of  $\alpha$  might be not observable within the precision of the simulation and the time available to it.

(3) of course the latter comment also indicates that, if the above conjectures are confirmed, there could be an alternative way of studying the INS equation regularity: rather than looking for solutions in suitable function spaces it would be particularly relevant to study solutions of the  $RNS$  equations trading the search of singularities with the search of extremely unlikely events with  $\alpha < 0$ .

(4) the closeness to 1 of  $\alpha(\mathbf{u}(t))$  in the simulations mentioned in (2) above, raises questions on whether the restriction on the notion of locality in conjecture 3 can be considered as really needed: it arises from simulations in which not only  $\alpha(\mathbf{u}(t))$  stays  $> 0$  but also fluctuates close to 1. If this is not due to the precision of the simulation and the time available to it, it is difficult to believe that the difficulties (which seem unsurmountable at constant viscosity) disappear if viscosity only slightly fluctuates, leaving valid the key *a priori* bounds based on the positivity of the viscosity in, for instance, [20].

## V. INS LYAPUNOV SPECTRUM

The linearization of the  $INS^N$  or  $RNS^N$  flows is the  $M \times M$  matrix,  $M = (d-1)((2N+1)^3 - 1)$ , formally defined as  $J_{c,\mathbf{k};b,\mathbf{h}} = \frac{\partial \dot{u}_c}{\partial u_b}$ . To really define it the  $u_{\mathbf{k}}^c$  can be represented a real  $M$  components vectors  $\{U_s\}_{s=0,\dots,M}$  holding the  $u_{\mathbf{k}}^c$ , if  $d = 3$ , as

(a) for  $c = 0$ :  $U_{2i}$  are real parts of  $u_{\mathbf{k}}^0$  after labeling half of the  $\mathbf{k}$  arbitrarily with  $i \in [0, M/4)$ ; and  $U_{2i+1}$  are the corresponding imaginary parts of  $u_{\mathbf{k}}^0$

(b) for  $c = 1$ :  $U_{2i+M/2}$  and  $U_{2i+1+M/2}$  are labeled, from the  $u_{\mathbf{k}}^1$ , likewise as  $i \in [0, M/4)$ .

Consider first the  $INS^N$  equations.

Then the equation can be written  $\dot{U}_s = N_s(U, U)$  and its Jacobian as  $J_{s;r}(\mathbf{u}) = \frac{\partial \dot{U}_s}{\partial U_r}$ .<sup>4</sup>

<sup>4</sup>A (arbitrary) way to define, in  $d = 3$ , the labels  $i$  is to consider first the  $\mathbf{k} = (k_0, k_1, k_2)$  with  $k_0 > 0, k_1 = 0, k_2 = 0$  assigning them

The matrix  $J$  can also be represented as an operator on the velocity fields acting by multiplication by  $T_{c;d}(\mathbf{x}) = -\partial_d \dot{u}_c(\mathbf{x})$ , *i.e.*  $(T\mathbf{v})(\mathbf{x})_c = \sum_d T_{c;d}(\mathbf{x})v_d(\mathbf{x})$ , to which the viscosity contribution has to be added.

The symmetrized  $J(\mathbf{u})$ , if  $W_{c,d}(x) = -\frac{1}{2}(\partial_d u_c(\mathbf{x}) + \partial_c u_d(\mathbf{x}))$  is therefore:

$$J_{c,d}^s(\mathbf{u}) = \nu \delta_{cd} \Delta + W_{c,d}(x) \quad (5.1)$$

Following [5] introduce  $w(\mathbf{x})$  as the largest eigenvalue of the matrix  $J^s(\mathbf{x})$ . The (negative of the) Schrödinger operator  $H = \nu \Delta + w$  will be considered as an operator on  $L_2^N(\Omega)$  of divergenceless velocity fields. The  $w(\mathbf{x})$  could be bounded by  $(Tr W^2)^{\frac{1}{2}}$  which is bounded above by the enstrophy density  $\frac{1}{2} \sum_{a,b} (\partial_a u_b(\mathbf{x}))^2 = \varepsilon(\mathbf{x})$ ; the bound, [5], can be improved into:

$$w(\mathbf{x}) \leq \frac{(d-1)}{d} \varepsilon(\mathbf{x}) \quad (5.2)$$

as shown in [6] taking advantage that the trace of  $W(\mathbf{x})$  is zero.

Denoting  $\mu^{0,N}(\mathbf{u}(0)) \geq \dots \geq \mu^{M,N}(\mathbf{u}(0))$  the Lyapunov exponents for an ergodic stationary distribution  $\rho$  for the  $IRS^N$  or  $RNS^N$ , the sum of the  $n$  largest exponents, defined for  $\rho$ -almost all initial  $\mathbf{u}(0)$ , is

$$\begin{aligned} & \sum_{i=0}^{n-1} \mu^{i,N}(\mathbf{u}(0)) \\ &= \lim_{t \rightarrow \infty} \frac{1}{2t} \log \|\phi_0(t) \wedge \phi_1(t) \wedge \dots \wedge \phi_{n-1}(t)\|^2 \end{aligned} \quad (5.3)$$

with  $\phi_j(t) = S_t^N(\phi(0))$  for almost all choices of the  $n$  fields  $\phi_j(0)$ 's in the  $M$ -dimensional phase space.

The time derivative of the log in Eq.(5.3) yields the expectation value of  $J(\mathbf{u}(t))$  in the state  $\frac{\phi_0(t) \wedge \dots \wedge \phi_{n-1}(t)}{\|\phi_0(t) \wedge \dots \wedge \phi_{n-1}(t)\|}$  and via the max-min principle leads to an estimate about the Lyapunov exponents in terms of the the eigenvalues  $a^{k,N}(\mathbf{u}(t))$ , in decreasing order, of the operator  $\nu \Delta + w$  on  $L_2(\Omega)$ , [5, p.291]:

**Theorem 3:** For all  $n$ :

$$\sum_{k=0}^{n-1} \mu^{(k)} \leq \sum_{k=0}^{n-1} \langle a^{k,N} \rangle \leq \sum_{k=0}^{n-1} \langle a^{k,\infty} \rangle \quad (5.4)$$

where the  $\langle \cdot \rangle$  denote time average or, equivalently, average with respect to the invariant  $\rho$ .

the labels  $i = k_0 - 1 \in [0, N - 1]$ , then consider the  $\mathbf{k} = (k_0, k_1 > 0, k_2 = 0)$  assigning them  $i = N + (k_1 - 1)(2N + 1) + k_0 + N$  (hence  $0 \leq i < 2N(N + 1) = ((2N + 1)^2 - 1)/2$ ), and finally assign to  $\mathbf{k} = (k_0, k_1, k_2 > 0)$  the label  $i = 2N(N + 1) + (k_2 - 1)(2N + 1)^2 + (k_1 + N)(2N + 1) + k_0 + N$  and  $0 \leq i < ((2N + 1)^3 - 1)/2$ . The total number of labels  $i$  is  $n = ((2N + 1)^3 - 1)/2$ . For each  $\mathbf{k}$  there are two complex components  $u_{\mathbf{k}}^c, c = 0, 1$ : then given  $\mathbf{k}, c$  assign  $U_{2i+n} = Re(u_{\mathbf{k}}^c)$ ,  $U_{2i+1+n} = Im(u_{\mathbf{k}}^c)$ .

This is obtained in [5, Eq.(1.7)] for the INS evolution, and as a consequence of the variational principle the *argument applies as well to  $INS^N$*  (and to  $RNS^N$  if  $\alpha(\mathbf{u}(t))$  is eventually  $> \varepsilon > 0$  for some  $\varepsilon$ ).

A particularly remarkable estimate is derived, [5, 6], as

**Theorem 5:** For  $d = 2, 3$  and  $\gamma \geq 0$

$$\sum_{a^{k,N} \geq 0} (a^{k,N})^\gamma \leq L_{\gamma,d} \nu^{-\frac{d}{2}} \int_{\Omega} \mathcal{D}(\mathbf{u}, x)^{\frac{\gamma}{2} + \frac{d}{4}} dx \quad (5.5)$$

with  $L_{\gamma,d} < \infty$  for  $\gamma > 0$  if  $d = 2, 3$  and  $L_{0,3} < \infty$ . Furthermore the best constant  $L_{\gamma,d}$  is  $\Omega$ -independent.

(a) Whether  $L_{0,2} < \infty$  is an open problem. The restriction  $\gamma \geq 1$  in [5] is improved to  $\gamma \geq 0$  in [6] if  $d = 3$  and  $\gamma > 0$  if  $d = 2$ .

(b) The interest of the case  $\gamma = 0$  is that it estimates the number  $\overline{N}$  of positive Lyapunov exponents as bounded in terms of the viscosity and of  $\eta$ =average energy dissipation per unit time (finite for all  $N$  in  $INS^N, RNS^N$ , and for  $INS$  (*i.e.*  $N = \infty$ ) conjectured to be finite and to have a positive finite limit even as  $\nu \rightarrow 0$ , [22, 23]).

(c) Hence for  $\gamma = 0, d = 3$  the bound on  $\overline{N}$ , implied by Hölder's inequality with  $p = \frac{4}{3}, q = 4$  applied to Eq.(5.5), is (using also convexity of  $x \rightarrow x^{\frac{3}{4}}$ ):

$$\overline{N} \leq L_{0,3} \frac{|\Omega|}{3^{\frac{1}{4}}} \left( \frac{\nu \langle \mathcal{D}(\cdot) \rangle}{\nu^3} \right)^{\frac{3}{4}} = \frac{L_{0,3} |\Omega|}{3^{\frac{1}{4}}} \left( \frac{\eta}{\nu^3} \right)^{\frac{3}{4}} \quad (5.6)$$

and the constant  $L_{0,3}$  can be taken  $\frac{4}{\pi^{2.3} 3^{\frac{3}{2}}}$  as in [6, p.475], and  $|\Omega| = L^3$  if  $L$  is the container side.

Since the Kolmogorov momentum scale  $K_\nu$  is proportional to  $K_\nu = \frac{1}{L} \left( \frac{\eta}{\nu^3} \right)^{\frac{1}{4}}$ , [24], this can be interpreted as saying that the number of degrees of freedom responsible for the chaotic evolution  $\overline{N}$  is of the order of the number of harmonics with momentum below Kolmogorov's momentum scale (*i.e.* with wave length above Kolmogorov's length scale).

The above statements hold for  $INS^N$  independently of  $N$ ; they would also apply, with minor variations, to  $RNS^N$  equations in the (very unlikely) case that  $\alpha(\mathbf{u}_t)$  is eventually  $\geq \varepsilon$  for some  $\varepsilon > 0$ .

## VI. RNS LYAPUNOV SPECTRUM

The Lyapunov exponents (LE) and the average spectrums (AS) of the symmetrized linearization matrix are not averages of local observables, so that the conjectures do not imply a relation between such quantities under the equivalence condition.

Nevertheless asymptotic, as  $N \rightarrow \infty$ , equality to  $\nu$  of the averages of  $\alpha(\mathbf{u})$ , Eq.(4.2), in corresponding  $INS^N$  and  $RNS^N$ , appears to hold in 2D simulations seems (at large  $N$ ): since  $\alpha(\mathbf{u})$  is non local, if regarded as an observable, this in  $RNS^N$  can be shown to be a simple consequence of the conjecture, [3, 25], *but not in  $INS^N$* .

Therefore it is natural to look if there are other non local observables to which the equivalence can be extended: and results in SM provide important examples of non local observables which have equal average values in corresponding distributions of different ensembles.

A few simulations have been performed on the NS problem in the above context. The results confirm the equality of the averaged Lyapunov spectra, [3, Sec.18],[25].

In 2D a further property emerges from the simulations: if the  $M$  average local LE's (defined in the previous section) for  $RNS^N$  are labeled  $\lambda_k$  with  $k \in [0, M)$ , ordered by decreasing size, and if  $\overline{div}$  is the average phase space contraction rate<sup>5</sup> an approximate "pairing rule" appears:

$$\frac{1}{2}(\lambda_k + \lambda_{M-k}) = \frac{1}{M} \overline{div}, \quad k = 0, \dots, \frac{M}{2} - 1 \quad (6.1)$$

The latter relation is well verified in the simulations with small regularization  $N$ , so far performed, for all but a few small  $k$ 's: and a question is whether the discrepancies remain as  $N \rightarrow \infty$ . However (as hinted in [26]) it is expected that  $(\lambda_k + \lambda_{M-k})$  for large  $k$  (hence large  $N$ ) is a concave curve.

A pairing rule, rigorously holds in systems governed by a Hamiltonian of the form  $H = \frac{1}{2} \mathbf{p} \cdot \mathbf{p}$  and subject to a force  $\mathbf{f}(\mathbf{q})$  only locally conservative and to the constraint of maintaining constant kinetic energy  $\frac{1}{2} \mathbf{p}^2$  via a force  $-\alpha(\mathbf{p}, \mathbf{q}) \mathbf{p}$  (hence  $\alpha = \frac{\mathbf{p} \cdot \mathbf{f}(\mathbf{q})}{\mathbf{p} \cdot \mathbf{p}}$ ). Hence it is expected to hold for the fluids described by adding a linear friction ("Ekman friction",  $-\nu \mathbf{u}$ ) to the Euler equations in Lagrangian form (*i.e.* with a second equation describing the individual fluid elements trajectories, thus doubling the number of degrees of freedom and of exponents): in [3, 26] the question is contemplated about a possible connection between the above pairings.

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<sup>5</sup>Contraction rate = trace of the above matrix  $J(\mathbf{u}, x)$ .

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