Non-equilibrium relaxations: ageing and finite-size effects[1]

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Abstract

The long-time behaviour of spin-spin correlators in the slow relaxation of systems undergoing phase-ordering kinetics is studied in geometries of finite size. A phenomenological finite-size scaling ansatz is formulated and tested through the exact solution of the kinetic spherical model, quenched to below the critical temperature, in $2 < d < 4$ dimensions.

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1 Critical relaxations in finite-size systems

Collective phenomena arise in many-body systems with dynamically created long-range interactions and thereby often show new qualitative properties which cannot be obtained in systems with a small number of degrees of freedom. An important class are *critical phenomena*, characterised by *scale-invariance*. We are interested here in time-dependent phenomena with time-dependent or 'dynamical' scaling. As a physical example, we consider many-body spin systems, initially prepared in a disordered state with at most short-ranged correlations and then suddenly quenched to a temperature $0 < T < T_c$ below the critical temperature $T_c > 0$, with at least two physically distinct phases. Such a quenched spin system is then said to undergo *phase-ordering kinetics* [17]. For a spatially infinite geometry, observables such as correlation functions are then expected to be invariant under the time-space dilatation

$$t \mapsto t' = \kappa^z t \quad , \quad \boldsymbol{r} \mapsto \boldsymbol{r}' = \kappa \boldsymbol{r} \tag{1.1}$$

where κ is a constant re-scaling factor and the *dynamical exponent* z serves to distinguish the scaling between time and space. The relaxation of the system after the quench can be measured through connected correlators of the time- and space-dependent spin variables $S_{\boldsymbol{r}}(t)$, namely

$$C(t;\boldsymbol{r}) := \langle S_{\boldsymbol{r}}(t) S_{\boldsymbol{0}}(t)\rangle - \langle S_{\boldsymbol{r}}(t)\rangle \langle S_{\boldsymbol{0}}(t)\rangle = F_C\left(\frac{|\boldsymbol{r}|}{t^{1/z}}\right) \tag{1.2a}$$

$$C(t,s) := \langle S_{\boldsymbol{r}}(t) S_{\boldsymbol{r}}(s)\rangle - \langle S_{\boldsymbol{r}}(t)\rangle \langle S_{\boldsymbol{r}}(s)\rangle = f_C\left(\frac{t}{s}\right) \tag{1.2b}$$

where the quoted scaling forms are meant to hold in the limit of large times and large distances, such that $|\boldsymbol{r}|^z/t$ and t/s are kept fixed. In (1.2b), t is the *observation time* and s is the *waiting time*. Asymptotically, the scaling function $f_C(y)$ in (1.2b) should be algebraic

$$f_C(y) \sim y^{-\lambda/z} \quad , \quad \text{as } y \to \infty \tag{1.3}$$

where $\lambda = \lambda_C$ is the *autocorrelation exponent*. A many-body non-stationary system whose slow relaxation dynamics also breaks time-translation-invariance and is such that the single-time correlator $C(t,\boldsymbol{r})$ and the two-time auto-correlator $C(t,s)$ obey the dynamical scaling (1.2), is said to be *ageing* [71, 32, 49, 73].

For phase-ordering, with a non-conserved order parameter, some general exact results exist for models with short-ranged interactions. First, the dynamical exponent $z = 2$ for a non-conserved order parameter [18].[1] Second, the Yeung-Rao-Desai inequality states that $\lambda \geq d/2$ [76]. Third, for the $2D$ Ising model one has the Fisher-Huse inequality $\lambda \leq \frac{5}{4}$ [39]. Some typical values for z and λ are listed in table 1. They illustrate the sharpness of these exact bounds and permit a comparison between short-ranged and long-ranged interactions. The agreement with the available experiments [61, 5] is very satisfying. For more detailed tables, see [49].

How is the scaling behaviour, encoded in the scaling forms (1.2), modified in a system confined to a domain of finite size, e.g. because it is placed into a box ?

For a phenomenological answer, consider figure 1. For a fully finite hyper-cubic lattice with N^d sites and periodic boundary conditions, the single-time correlator $C(t;\boldsymbol{r})$ is shown in

[1] In this work, we restrict to this *model-A dynamics*.

material/model			z	λ	Refs.
Merck (CCH-501)			1.94(5)	1.246(79)	[61]
nematic TNLC			2.01(1)	1.28(11)	[5]
Ising	$1D$	LR	$1+\sigma$	0.5	[28, 29]
Ising	$2D$	LR	$1+\sigma$	1	[24, 25]
Ising	$2D$	SR	2	1.24(2)	[56]
			2	1.25	[24, 25]
			2	1.3	[62, 63, 74]
Ising	$3D$	SR	2	1.60(2)	[48]
			2	1.6	[62, 63]
Potts-3	$2D$	SR	2	1.19(3)	[56]
			2	1.22(2)	[27]
Potts-8	$2D$	SR	2	1.25(1)	[56]
XY	$3D$	SR	2	1.7(1)	[2]
			2	1.6	[62]
spherical		SR	2	$d/2$	[45]
spherical		LR	σ	$d/2$	[20, 11]

Table 1: Dynamical exponent z and autocorrelation exponent λ, as measured experimentally or found in some spin models. Long-range (LR) behaviour occurs in the Ising model for $\sigma < 1$ and in the spherical model for $\sigma < 2$. The spherical model is considered for dimensions $d > z$.

figure 1a, where $\boldsymbol{r}$ is oriented along one of coordinate axes. If the spatial distances $r = |\boldsymbol{r}|$ are not too large, the shape of the correlator does not depend sensitively on N. Only if $r \lesssim \frac{N}{2}$, does the correlator also receive contributions 'from around the world', such that for $r \approx \frac{N}{2}$ it no longer tends towards zero, but rather saturates at a N-dependent constant $C_{\rm lim}^{(1)}(N) > 0$. Figure 1b shows the two-time autocorrelator $C(ys, s)$. For large s, but y small enough, there is a clear data collapse. However, for larger values of y, C begins to decrease more rapidly than the infinite-size curve (1.2b).[2] As $y \gg 1$, C finally saturates at the limit value $C_\infty^{(2)}(N) > 0$.

Although the single-time correlator does not display strong finite-size effects, this is different for the length scale $L = L(t)$ of the growing clusters, estimated from the second moment

$$L^2(t) = \frac{\sum_{\boldsymbol{r}} |\boldsymbol{r}|^2 C(t;\boldsymbol{r})}{\sum_{\boldsymbol{r}} C(t;\boldsymbol{r})} \tag{1.4}$$

The precise extent of the sums will be specified below. Figure 2 shows that for sufficiently short times, the length $L^2(t) \sim t$ behaves as for the infinite system, but as t grows further, finally there occurs a cross-over towards a finite constant $L_\infty(N)$. We shall see how to explain the findings of figures 1 and 2 in terms of phenomenological finite-size scaling. The resulting predictions will be tested in the exactly solved kinetic spherical model, for dimensions $2 < d < 4$.

The *spherical model* of a ferromagnet [13, 55] has served as an exactly solvable, yet non-trivial, model for the detailed analysis of general concepts of critical phenomena, see [40] for a

[2]Since for lattices large enough that the system is just leaving the effective finite-size regime, the local exponent estimates $\lambda_{\rm eff}(y)$ may slightly over-estimate λ. In certain cases this might lead to claims of violation of exact upper bounds such as the Fisher-Huse inequality.

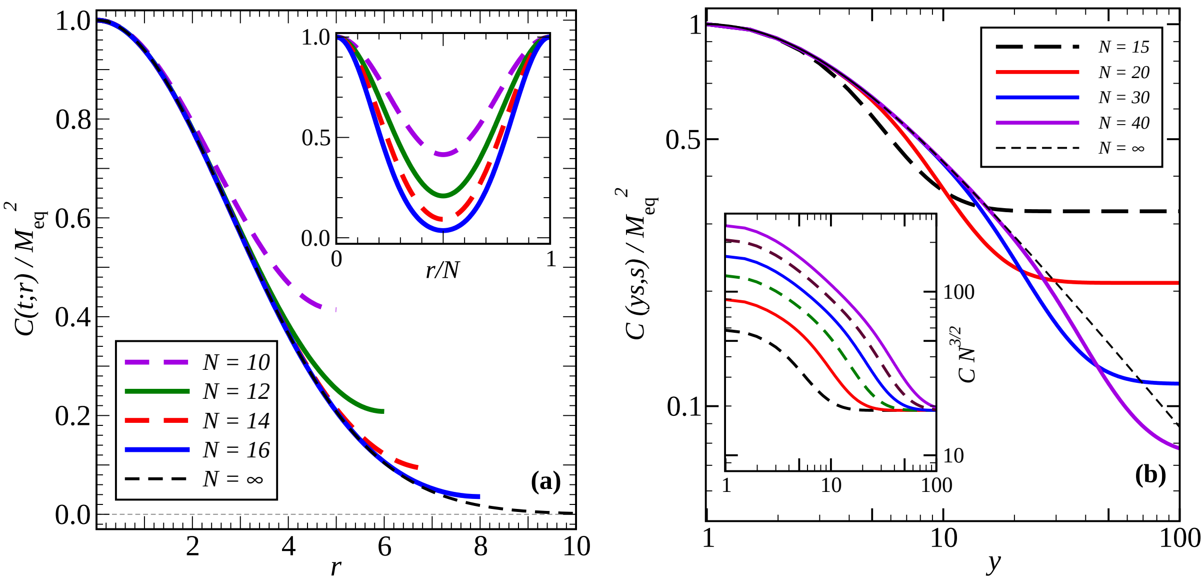

Figure 1: **(a)** Finite-size effects for the single-time correlator $C(t,\boldsymbol{r})$ in the fully finite spherical model at $T < T_c$, with $t = 50$ and $N = [10, 12, 14, 16, \infty]$ from top to bottom. The inset shows the periodicity over the interval $0 \leq r/N \leq 1$.
(b) Finite-size effects for the two-time autocorrelator $C(ys, s)$ in the $3D$ fully finite spherical model at $T < T_c$, for $N = [15, 20, 30, 40]$ from top to bottom (at the right) and s fixed. The thin dashed line gives the infinite-size autocorrelator. The inset shows the data collapse of the re-scaled correlator $CN^{3/2}$ for $y = t/s$ large, with $N = [15, 20, 25, 30, 35, 40]$ from left to right (arbitrary units).

historical perspective. Its non-equilibrium behaviour after a quench has also been thoroughly analysed, see [67, 30, 31, 45, 20, 43, 64, 47, 6, 11, 34, 7, 49]. The related Arcetri model provides a qualitative description of the dynamics in the non-equilibrium growth of interfaces [50, 33]. Finite-size effects at equilibrium have also been analysed at great depth in the spherical model and have been of value to test the theory of finite-size scaling derived from the renormalisation group, see [8, 19, 9, 57, 69, 70, 65, 4, 16, 22] and refs. therein. For dimensions $d > d_c = 4$, that is above the upper critical dimension, the standard finite-size scaling *ansatz* must be considerably modified [14, 65, 51, 41, 42, 46, 12].

Finite-size scaling techniques have been applied in studies of phase-ordering kinetics [63, 74], the ageing of polymer collapse [58, 23, 59, 60] or the dynamics of mitochondrial networks [77]. Explicit studies of finite-size scaling in an ageing system have been carried out in Ising spin glasses [52] and notably on the dimensional cross-over between the $3D$ and $2D$ Edwards-Anderson spin glass [37] as motivated by extremely accurate experiments on CuMn films [78, 79]. In addition, finite-size effects analogous to figures 1 and 2 are clearly visible in the time-evolution of characteristic cluster sizes in long-ranged Ising models quenched to $T < T_c$ [24] or in the auto-correlator [25]. Since the bulk $3D$ spherical model and the bulk ($p = 2$) spherical spherical spin glass are in the same dynamic universality class [31], one might hope that finite-size effects could be similar as well. Not so ! Rather, detailed studies of the ($p = 2$) spherical spin glass [44, 10] show that this equivalence only holds in the spin glass for times $t \ll t_{\rm cross} \sim N^{2/3}$. For time scales $t \gg t_{\rm cross}$, ageing still holds with a new set of universal exponents [44], to be followed by a second cross-over to a regime of exponential decay at extremely large times [10].

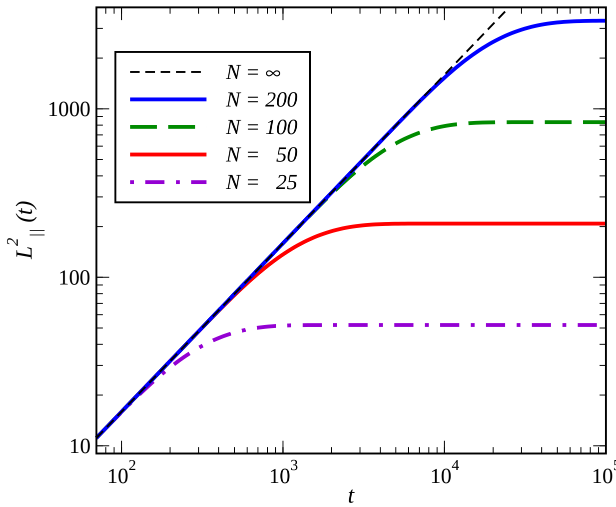

Figure 2: Finite-size effects for the longitudinal characteristic length $L_{\|}^2(t)$, measured along a coordinate axis, in the fully finite spherical model with lattice sizes $N = [25, 50, 100, 200]$ from bottom to top. The thin dashed line indicates the infinite-size behaviour $L(t) \sim t^{1/2}$.

This work is organised as follows. In section 2, we recall the main features of dynamical scaling in ageing phase-ordering kinetics. In section 3, we extend this phenomenological treatment to finite systems, using the hyper-cubic geometry $\overbrace{N \times \cdots \times N}^{d^* \text{ factors}} \times \overbrace{\infty \times \cdots \times \infty}^{d-d^* \text{ factors}}$, where the first $d^* \leq d$ directions are finite and periodic and the other $d - d^*$ directions are infinite. The finite-size forms so obtained will be checked in section 4 using the exact solution of the kinetic spherical model in $2 < d < 4$ dimensions, quenched to $T > T_c$ from a totally disordered state and in section 5 we conclude. Technical details of the exact solution are given in the appendix.

2 Dynamical scaling description

A central ingredient of ageing is dynamical scaling. For the general two-time and spatial bulk correlator, our starting point is (below the upper critical dimension $d < d_c$; for short-ranged interactions usually $d_c^{(\text{short})} = 4$)

$$C\left(\kappa^z t, \kappa^z s; \kappa \boldsymbol{r}\right) = \kappa^{\phi} C\left(t, s; \boldsymbol{r}\right) \tag{2.1}$$

where t, s are the observation and the waiting time, z is the dynamical exponent, ϕ a scaling exponent and $\boldsymbol{r}$ is the spatial distance. Writing (2.1) means that we assume negligible all finite-time and finite-distance corrections to scaling. Choosing $\kappa = s^{-1/z}$, this gives

$$C\left(t, s; \boldsymbol{r}\right) = s^{\phi/z} C\left(\frac{t}{s}, 1; \frac{\boldsymbol{r}}{s^{1/z}}\right) \tag{2.2}$$

In phase-ordering, the single-time correlator at $\boldsymbol{r} = \boldsymbol{0}$ is finite; namely either $C(t; \boldsymbol{0}) = 1$ in Ising-like systems or else $C(t; \boldsymbol{0}) = M_{\text{eq}}^2$ for order parameters with a continuous global symmetry. Setting $s = t$ in (2.2), this leads to $\phi = 0$ and[3] further to $C(t; \boldsymbol{r}) = C\big(1, 1; |\boldsymbol{r}| t^{-/z}\big) =:$

[3]If more generally, one would expect $C(t, s) = s^{-b} f_C(t/s)$, this would lead to the identification $b = -\phi/z$, but for $\phi \neq 0$, this is incompatible with $C(t; \boldsymbol{0})$ being finite and constant for $t \to \infty$.

$F_C\big(|\boldsymbol{r}|t^{-/z}\big)$. On the other hand, setting now $\boldsymbol{r}=\mathbf{0}$, the two-time auto-correlator is $C(t,s) = C\big(t,s;\mathbf{0}\big) = C(t/s,1;\mathbf{0}) =: f_C(t/s)$. These results fully reproduce (1.2).

3 Dynamical finite-size scaling

According to the original definition, *finite-size scaling* [38] is the scaling behaviour in a nearly critical system confined to a *geometry of finite linear extent* N. For finite geometries, the natural generalisation of (2.1) consists, as at equilibrium [8, 72, 9, 65], to consider $1/N$ as a further relevant scaling field.[4] While this hypothesis was originally specified for the order parameter at the critical point [72], we adapt this to the situation at hand and write down the finite-size scaling (FSS) *ansatz* for the full two-time correlator

$$C\left(\kappa^z t, \kappa^z s; \kappa\boldsymbol{r}; \kappa^{-1}\frac{1}{N}\right) = \kappa^{\phi} C\left(t, s; \boldsymbol{r}; \frac{1}{N}\right) \tag{3.1}$$

meant to hold in the hyper-cubic geometry $\overbrace{N\times\cdots\times N}^{d^* \text{ factors}} \times \overbrace{\infty\times\cdots\times\infty}^{d-d^* \text{ factors}}$ where N describes the finite length in the system. For simplicity, we consider a single length of this kind.[5] Of course, for $N\to\infty$, one is back to the bulk scaling form (2.1), and hence (1.2).

Choose the re-scaling factor $\kappa = s^{-1/z}$. For phase-ordering kinetics, recall from section 2 that $\phi = 0$. Then (3.1) can be equivalently expressed as

$$C\left(t, s; \boldsymbol{r}; \frac{1}{N}\right) = C\left(\frac{t}{s}, 1; \frac{\boldsymbol{r}}{s^{1/z}}; \frac{s^{1/z}}{N}\right) \tag{3.2}$$

As above in section 2, we then expect for the correlators (provided spatial rotation-invariance can be assumed)

$$C\big(t; \boldsymbol{r}; N^{-1}\big) = F_C\left(\frac{|\boldsymbol{r}|^z}{t}; \frac{N^z}{t}\right) \quad , \quad C\big(t, s; N^{-1}\big) = f_C\left(\frac{t}{s}; \frac{N^z}{t}\right) \tag{3.3}$$

such that the corresponding scaling functions are now functions of two variables. Finite-size scaling in ageing can be analysed in the asymptotic FSS limit where $t\to\infty$, $s\to\infty$, $|\boldsymbol{r}|\to\infty$ and $N\to\infty$ such that the three scaling variables

$$y = \frac{t}{s} \quad , \quad \boldsymbol{\varrho} = \frac{\boldsymbol{r}}{t^{1/z}} \quad , \quad Z = \frac{N^z}{t} \tag{3.4}$$

are kept fixed. The precise form of the finite-size scaling functions (3.3) will depend on the universality class under study, and on the boundary conditions [9, 65, 16].

As a first consequence, consider the characteristic length $L(t)$ of the clusters. From (1.4) and (3.3), we derive the finite-size scaling form

$$L^2(t; N^{-1}) = \frac{\sum_{\boldsymbol{r}} |\boldsymbol{r}|^2 C(t; \boldsymbol{r}; N^{-1})}{\sum_{\boldsymbol{r}} C(t; \boldsymbol{r}; N^{-1})} \simeq t^{2/z} \frac{\int\!\mathrm{d}\boldsymbol{r}\, \left(|\boldsymbol{r}|t^{-1/z}\right)^2 F_C(|\boldsymbol{r}|^z/t; N^z/t)}{\int\!\mathrm{d}\boldsymbol{r}\, F_C(|\boldsymbol{r}|^z/t; N^z/t)} = t^{2/z} f_L\left(\frac{N^z}{t}\right) \tag{3.5}$$

[4] Very interesting adaptations of this idea have been brought forward in the study of the kinetics of polymer collapse, where N is now the finite number of monomers, but the spatial geometry of the system was not specified [59, 60].

[5] Spatially anisotropic finite-size effects could be taken into account by introducing distinct finite sizes N_j in different spatial directions.

For $Z \gg 1$, the behaviour of an effectively infinite system requires that $f_L(Z) \stackrel{Z\gg1}{\simeq} f_0 = \text{cste}$. and for $Z \ll 1$, the time-independent saturation in figure 2 is captured by $f_L(Z) \stackrel{Z\ll1}{\sim} Z^{2/z}$ such that $L_\infty(N) \sim N$, as would have been expected from dimensional analysis.

Next, we consider the plateau in the two-time auto-correlator $C(ys,s)$ for $y \gg 1$, see figure 1b. Recall that for the infinite system, we expect from (1.2b,1.3) that $C(t,s;\mathbf{0};0) = f_C(t/s) \sim \big(t/s\big)^{-\lambda/z}$. For $N < \infty$, we reformulate (3.2) as follows

$$C\big(t,s;N^{-1}\big) = C\left(t,s;\mathbf{0};\frac{1}{N}\right) = C\left(\frac{t}{s},1;\mathbf{0};\frac{s^{1/z}}{N}\right) = \left(\frac{t}{s}\right)^{-\lambda/z} \mathscr{F}_C\left(\left(\frac{t}{s}\right)^{1/z}, \frac{s^{1/z}}{N}\right) \tag{3.6}$$

Herein, the first argument in the scaling function $\mathscr{F}_C = \mathscr{F}_C\big(y,u\big)$ will be considered large and be kept fixed, $y \gg 1$. In that case, the scaling function will describe the cross-over between (i) the infinite-system behaviour (when $u = s^{1/z}/N \to 0$) $f_C(y) = \mathscr{F}_C(y,0) \sim y^{-\lambda/z}$ which is independent of s and (ii) the fully finite-system behaviour (when $u = s^{1/z}/N \to \infty$) when $C \xrightarrow{y\gg1} C_\infty^{(2)}$ no longer depends on $y = t/s$. The first limit case is taken into account by admitting $\mathscr{F}_C(y,u) \simeq \mathtt{F}\big(yu\big)$ and $\mathtt{F}(0) = \text{cste}$. Then the second limit case leads to

$$C\left(t,s;\mathbf{0};\frac{1}{N}\right) \stackrel{t/s\gg1}{\simeq} \left(\frac{t}{s}\right)^{-\lambda/z} \mathtt{F}\left(\left(\frac{t}{s}\right)^{1/z}\cdot\frac{s^{1/z}}{N}\right) \sim \left(\frac{t}{s}\right)^{-\lambda/z} \left(\left(\frac{t}{s}\right)^{1/z}\cdot\frac{s^{1/z}}{N}\right)^{\omega} \tag{3.7}$$

where in the last step, we assumed a power-law form of $\mathtt{F}(yu) \sim \big(yu\big)^{\omega}$ for $yu \gg 1$. The y-independent plateau $C_\infty^{(2)}$ observed for fully finite systems (see figure 1b for s fixed) is reproduced if we choose $\omega = \lambda$. Hence, for finite systems with $y = t/s \gg 1$

$$C\left(t,s;\mathbf{0};\frac{1}{N}\right) \xrightarrow{t/s\gg1} C_\infty^{(2)} \sim \left(\frac{s^{1/z}}{N}\right)^{\lambda} \tag{3.8}$$

Herein, s is still kept fixed whereas N must be taken large enough such that the system under study is indeed in its finite-size scaling regime (in other word, $Ns^{-1/z}$ must be large enough).

Hence for fully finite systems, quenched to $T < T_c$, the auto-correlator $C(ys,s) = f_C(y) \xrightarrow{y\gg1} C_\infty^{(2)}$, such that the plateau value $C_\infty^{(2)} = C_\infty^{(2)}(s,N)$ should obey the scalings

$$C_\infty^{(2)} \sim N^{-\lambda} \;\text{ with } s \text{ fixed} \quad , \quad C_\infty^{(2)} \sim s^{\lambda/z} \;\text{ with } N \text{ fixed} \tag{3.9}$$

These are the sought scalings for the plateau of the autocorrelator and the main result of this section.

The inset in figure 1b shows the data collapse of $N^{\lambda}C(ys,s)$ to a y-independent constant for y large enough and s fixed, in the $3D$ spherical model, where $\lambda = \frac{3}{2}$. In the next section, (3.9) will be verified analytically from the exact solution of the quenched kinetic spherical model in dimensions $2 < d < 4$.

A simple heuristic argument to establish (3.8) goes as follows. For widely different times $t \gg s \gg \tau_{\text{mic}}$, the asymptotic form of the autocorrelator is expressed through the cluster sizes L as $C(t,s) \sim \big(L(t)/L(s)\big)^{-\lambda}$. If furthermore t is so large that $L(t) \sim N$ while s is small enough such that still $L(s) \sim s^{1/z}$, the scaling (3.8) of the plateau $C_\infty^{(2)}$ follows.

4 The kinetic spherical model

Following standard developments [67, 30, 45, 50], the kinetic spherical model is defined in terms of real spin variables $S_{\boldsymbol{n}} = S_{\boldsymbol{n}}(t) \in \mathbb{R}$ at each lattice site $\boldsymbol{n} \in \Lambda \subset \mathbb{Z}^d$, subject to the *spherical constraint* $\sum_{\boldsymbol{n}\in\Lambda} S_n^2(t) = |\Lambda|$, where $|\Lambda| = \prod_{j=1}^d N_j$ is the number of sites of the lattice $\Lambda \subset \mathbb{Z}^d$. Its dynamics is given by the Langevin equation

$$\partial_t S_{\boldsymbol{n}}(t) = D\Delta_{\boldsymbol{n}} S_{\boldsymbol{n}}(t) - \mathfrak{z}(t) S_{\boldsymbol{n}}(t) + \eta_{\boldsymbol{n}}(t) \tag{4.1}$$

with the spatial laplacian $\Delta_{\boldsymbol{n}}$ and the thermal white noise $\eta_{\boldsymbol{n}} = \eta_{\boldsymbol{n}}(t)$. It has the first two moments

$$\left\langle \eta_{\boldsymbol{n}}(t) \right\rangle = 0 \;\;,\;\; \left\langle \eta_{\boldsymbol{n}}(t)\eta_{\boldsymbol{m}}(t') \right\rangle = 2DT\delta(t-t')\delta_{\boldsymbol{n},\boldsymbol{m}} \tag{4.2}$$

where T is the bath temperature and D a kinetic coefficient. The Lagrange multiplier $\mathfrak{z}(t)$ is fixed from the spherical constraint. The Fourier representation

$$S_{\boldsymbol{n}}(t) = \frac{1}{|\Lambda|} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_d=0}^{N_d-1} \exp\left(2\pi\mathrm{i} \sum_{j=1}^d \frac{k_j}{N_j} n_j \right) \widehat{S}(t,\boldsymbol{k}) \tag{4.3}$$

achieves the formal solution of the model which reads

$$\widehat{S}(t,\boldsymbol{k}) = \widehat{S}(0,\boldsymbol{k}) \frac{\exp\bigl(-2D\omega(\boldsymbol{k})t\bigr)}{\sqrt{g(t)}} + \int_0^t \mathrm{d}\tau\, \widehat{\eta}(\tau,\boldsymbol{k}) \sqrt{\frac{g(\tau)}{g(t)}}\, \exp\bigl(-2D\omega(\boldsymbol{k})(t-\tau)\bigr) \tag{4.4a}$$

with the abbreviations (nearest-neighbour interactions assumed)

$$\omega(\boldsymbol{k}) = \sum_{j=1}^d \left(1 - \cos \frac{2\pi}{N_j} k_j \right) \;\;,\;\; g(t) = \exp\left(2 \int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau) \right) \tag{4.4b}$$

In what follows, we restrict to a totally disordered initial state, such that $\left\langle S_{\boldsymbol{n}}(0) \right\rangle = 0$ and $\left\langle S_{\boldsymbol{n}}(0) S_{\boldsymbol{m}}(0) \right\rangle = \delta_{\boldsymbol{n},\boldsymbol{m}}$. In momentum space, the second moments of initial and thermal noises become

$$\left\langle \widehat{S}(0,\boldsymbol{k})\widehat{S}(0,\boldsymbol{k}') \right\rangle = |\Lambda|\delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{0}} \;\;,\;\; \left\langle \widehat{\eta}(t,\boldsymbol{k})\widehat{\eta}(t',\boldsymbol{k}') \right\rangle = 2DT|\Lambda|\delta(t-t')\delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{0}} \tag{4.4c}$$

Then the spherical constraint can be cast into a Volterra integral equation for $g = g(t)$

$$g(t) = f(t) + 2DT \int_0^t \mathrm{d}\tau\, g(\tau) f(t-\tau) \;\;,\;\; f(t) := \frac{1}{|\Lambda|} \sum_{\boldsymbol{k}} \exp\left(-4D\omega(\boldsymbol{k})t\right) \tag{4.4d}$$

Here and below, we abbreviate $\sum_{\boldsymbol{k}} := \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_d=0}^{N_d-1}$. Eqs. (4.4) specify the exact solution of the kinetic spherical model. We are interested in

(I) the two-time correlation function $\widehat{C}(t,s;\boldsymbol{k})$ in momentum space, defined by

$$\left\langle \widehat{S}(t,\boldsymbol{k})\widehat{S}(s,\boldsymbol{k}') \right\rangle =: |\Lambda|\delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{0}}\, \widehat{C}(t,s;\boldsymbol{k}) \tag{4.5a}$$

$$\widehat{C}(t,s;\boldsymbol{k}) = \frac{e^{-2D\omega(\boldsymbol{k})(t+s)}}{\sqrt{g(t)g(s)}} + 2DT \int_0^{\min(t,s)} \mathrm{d}\tau\, \frac{g(\tau)}{\sqrt{g(t)g(s)}}\, e^{-2D\omega(\boldsymbol{k})(t+s-2\tau)} \tag{4.5b}$$

and especially the two-time autocorrelator

$$C(t,s) := \frac{1}{|\Lambda|} \sum_{\boldsymbol{k}} \widehat{C}(t,s;\boldsymbol{k}) = C(s,t) \tag{4.6}$$

(II) the single-time correlator in momentum space $\widehat{C}(t;\boldsymbol{k}) := \widehat{C}(t,t;\boldsymbol{k})$, obtained from (4.5) by setting $s=t$. The time-space correlator reads

$$C(t;\boldsymbol{n}) = \frac{1}{|\Lambda|}\sum_{\boldsymbol{k}} \exp\left(2\pi\mathrm{i}\sum_{j=1}^{d}\frac{k_j}{N_j}n_j\right)\widehat{C}(t;\boldsymbol{k}) \tag{4.7}$$

The well-known bulk critical temperature [13] ($I_0(u)$ is a modified Bessel function [1])

$$\frac{1}{T_c(d)} = \int_0^\infty \mathrm{d}u\ \left(e^{-2u}I_0(2u)\right)^d \tag{4.8}$$

is finite and positive for $d>2$. Explicitly [21, 15]

$$\frac{1}{T_c(3)} = \frac{\sqrt{3}-1}{192\pi^3}\left(\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right)^2 \approx 0.25273\ldots \tag{4.9}$$

In what follows, we consider a hyper-cubic geometry $\overbrace{N\times\cdots\times N}^{d^*\text{ factors}}\times\overbrace{\infty\times\cdots\times\infty}^{d-d^*\text{ factors}}$, where the first $d^*\le d$ directions are finite and periodic and the other $d-d^*$ directions are infinite. We also restrict to $2<d<4$ and rescale the temporal units such that $8\pi D \stackrel{!}{=} 1$. After a quench from the disordered initial state (4.4c) to a temperature $T<T_c(d)$, we find in the FSS limit (3.4) (see the appendix for the calculations)

(A) the single-time temporal-spatial correlator, namely

$$C(t;\boldsymbol{n}) = M_{\mathrm{eq}}^2\exp\left(-\pi\sum_{j=1}^{d}\frac{n_j^2}{t}\right)\prod_{j=1}^{d^*}\frac{\vartheta_3\left(\mathrm{i}\pi\frac{Nn_j}{t}, e^{-\pi Z}\right)}{\vartheta_3\left(0,e^{-\pi Z}\right)} \tag{4.10a}$$

$$= M_{\mathrm{eq}}^2\exp\left(-\pi\sum_{j=d^*+1}^{d}\frac{n_j^2}{t}\right)\prod_{j=1}^{d^*}\frac{\vartheta_3\left(\pi n_j/N, \exp(-\pi/Z)\right)}{\vartheta_3\left(0,\exp(-\pi/Z)\right)} \tag{4.10b}$$

where $M_{\mathrm{eq}}^2 = 1-T/T_c(d)$ is the squared equilibrium magnetisation and Z was defined in (3.4) with $z=2$. Finally, $\vartheta_3(z,q)=\sum_{p=-\infty}^{\infty}q^{p^2}\cos(2pz)$ is a Jacobi Theta function [1].[6] See figure 1a for illustration. From (4.10a) we identify the finite-size scaling function $F_C=F_C(\boldsymbol{\varrho},Z)$ in (3.3). The shape of this function is temperature-independent. Indeed, an universal shape of F_C is expected, since the temperature T should be irrelevant in phase-ordering kinetics [17].

Eq. (4.10a) gives a factorisation of $C(t,\boldsymbol{n}) = C_{\mathrm{bulk}}(t;\boldsymbol{n})\cdot C_{\mathrm{red}}(t;\boldsymbol{n};N)$ into a size-independent 'bulk' part and a 'reduced' part which contains the finite-size effects. Because of the identity $\vartheta_3(z+\pi,q)=\vartheta_3(z,q)$, it is seen from (4.10b) that the correlator repeats periodically when $n_j\mapsto n_j+N$ is in the finite directions, as illustrated in the inset of figure 1a. For Z large enough[7] the central peak of the correlator around $\boldsymbol{n}=\boldsymbol{0}$ decays as in the bulk with a length scale $L(t)\sim t^{1/2}$ such that the system decomposes into separate and independent clusters of linear size $L(t)$, as expected. The bulk gaussian decay $\sim e^{-\boldsymbol{n}^2/t}$, rather than an exponential

[6] Analogous expressions of the finite-size scaling functions in terms of Jacobi Theta functions are known for the particle density in several $1D$ reaction-diffusion processes for both periodic and open boundary conditions [53, 54] and for the single-time correlator in the periodic $1D$ Glauber-Ising model at temperature $T=0$ [3].

[7] Actually for $Z\gtrsim 25$, which in physical units corresponds to $L(t)\lesssim 5N$.

$\sim e^{-|\boldsymbol{n}|/\sqrt{t}}$, is a peculiar property of the spherical model which distinguishes it from the Ising universality class.

(B) the two-time autocorrelator, for all $T < T_c(d)$, reads

$$C(ys,s) = M_{\rm eq}^2 \left(\frac{2\sqrt{y}}{1+y}\right)^{d/2} \left(\frac{\vartheta_3\big(0,\exp(-\pi\frac{2Z}{1+1/y})\big)^2}{\vartheta_3\big(0,\exp(-\pi Z)\big)\vartheta_3\big(0,\exp(-\pi Zy)\big)}\right)^{d^*/2} \tag{4.11a}$$

$$= M_{\rm eq}^2 \left(\frac{2\sqrt{y}}{1+y}\right)^{(d-d^*)/2} \left(\frac{\vartheta_3\big(0,\exp(-\pi\frac{1+1/y}{2Z})\big)^2}{\vartheta_3\big(0,\exp(-\pi/Z)\big)\vartheta_3\big(0,\exp(-\pi/Zy)\big)}\right)^{d^*/2} \tag{4.11b}$$

as illustrated in figure 1b. We identify from (4.11a) the finite-size scaling function $f_C = f_C(y,Z)$ in (3.3), whose shape is once more temperature-independent. As above for the single-time correlator, (4.11a) displays a natural factorisation into the bulk two-time autocorrelator $C_{\rm bulk}(ys,s) = M_{\rm eq}^2\left(\frac{2\sqrt{y}}{1+y}\right)^{d/2}$ and a 'reduced' factor which alone contains all finite-size effects. Eq. (4.11a) shows that for $Z \gg 1$, finite-size corrections with respect to the bulk behaviour are exponentially small. On the other hand, eq. (4.11b) shows that for $Z \ll 1$, the system behaves effectively as if it had only $d-d^*$ dimensions, up to exponentially small corrections.[8]

Having verified the generic finite-size scaling forms (3.3), we now test the validity of the finite-size scaling predictions (3.9) for the plateau values $C_\infty^{(2)}$. To be specific, we consider a fully finite system, with $d^* = d$. Fix the system size N and the waiting time s and consider the changes in $y = t/s$ by varying the observation time t. Physically, finite-size effects will be felt first by the larger length $L(t) \sim t^{1/2}$. Since $t \gg s$, we expect that $L(t) \gg L(s)$. The limit $y \gg 1$ is realised by taking $t \gg 1$. With the identity $\vartheta_3\big(0,e^{-\pi y}\big) = y^{-1/2}\,\vartheta_3\big(0,e^{-\pi/y}\big)$, we have

$$C(t,s) = M_{\rm eq}^2 \left(\frac{t\big(s/t\big)^{1/2}}{(t+s)/2}\right)^{d/2} \left(\frac{\big(2\frac{N^2}{t+s}\big)^{-1/2}\vartheta_3\big(0,\exp(-\frac{\pi}{2}\frac{t+s}{N^2})\big)}{\sqrt{\big(\frac{N^2}{t}\big)^{-1/2}\vartheta_3\big(0,\exp(-\pi\frac{t}{N^2})\big)\vartheta_3\big(0,\exp(-\pi\frac{N^2}{s})\big)}}\right)^{d^*} \tag{4.12}$$

For N^2/s finite but large enough (such that the plateaux in figure 1b are reached) , the last of the Theta functions in (4.12) is very close to unity. Because of the condition $t/N^2 \gg 1$, the other two Theta-functions in (4.12) are also close to unity. Up to constants, we obtain

$$C(t,s) \stackrel{t\to\infty}{\sim} \left(\frac{s}{t}\right)^{d/4} \left(\left(\frac{t+s}{N^2}\right)^{1/2}\left(\frac{N^2}{t}\right)^{1/4} \text{cste.}\right)^{d^*} \sim \left(\frac{s}{t}\right)^{d/4}\left(\frac{t(1+s/t)}{t^{1/2}}\right)^{d^*/2}\left(N^{-2\frac{1}{2}+2\frac{1}{4}}\right)^{d^*} \tag{4.13}$$

Finally, now admitting a fully finite system such that $d = d^*$, we have (for $2 < d < 4$)

$$C(t,s) \sim \left(\frac{s}{t}\right)^{d/4} t^{d/4} N^{-d/2} = s^{d/4} N^{-d/2} \tag{4.14}$$

which in view of the well-known results $\lambda = d/2$ [45] and $z = 2$ [17] does indeed reproduce of (3.8), or (3.9) if either s or N is kept fixed.

[8] Finite-temperature and finite-time effects merely give a corrective factor $1+{\rm O}(Ts^{1-d/2})$, negligible for large waiting times $s \to \infty$, if $d > 2$.

(C) Characteristic time-dependent length scales $L(t)$ of the ordered clusters can be measured as second moments of the single-time correlator

$$L^2(t) := \frac{\sum_{\boldsymbol{n}} \boldsymbol{n}^2 C(t;\boldsymbol{n})}{\sum_{\boldsymbol{n}} C(t;\boldsymbol{n})} \tag{4.15}$$

Precise expressions follow from (4.10a) once the range of summation of the distances $|\boldsymbol{n}|$ is fixed. For example, if one measures the distances along one of the coordinate axes of one of the infinite directions, one obtains the *'transverse'* length scale $L_{\perp}^2(t) = 4Dt$, as for a fully infinite system [35]. On the other hand, if the distances are measured along the coordinates axes of one of the finite directions, we find a *'longitudinal'* length scale, which reads for sufficiently thick films, and in agreement with (3.5)

$$L_{\parallel}^2(t) = \frac{1}{\pi} t f_L(Z) \ \ , \ \ f_L(Z) = \frac{\pi}{6} Z \left(1 + \frac{12}{\pi^2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2}\, e^{-\pi p^2/Z} \right) \simeq \left\{ \begin{array}{ll} \frac{\pi}{6} Z & ; \text{ if } Z \ll 1 \\ 1 & ; \text{ if } Z \gg 1 \end{array} \right. \tag{4.16}$$

The scaling function f_L is temperature-independent. This describes the cross-over shown in figure 2, such that for $Z = N^2/t$ small enough, we obtain saturation at $L_{\parallel}^2(t) \to L_{\infty}^2 \sim N^2$, but on the other hand one has $L_{\parallel}^2(t) \sim t$ of an effectively infinite system for Z large enough.

5 *Ad conclusio*

We studied finite-size scaling in the ageing relaxation of phase-ordering kinetics after a quench from a disordered initial state into the two-phase coexistence regime with temperature $0 < T < T_c$. The finite-size scaling *ansatz* (3.1) is the natural extension of dynamic finite-size scaling at equilibrium [72]. Phenomenologically, the observations to be gleaned from figure 1 for the single-time and two-correlations and figure 2 for the characteristic length scale are captured by the finite-size scaling forms (3.3). The *form* of the associated scaling functions is temperature-independent, which confirms the expectation that the temperature should be irrelevant in phase-ordering kinetics [17]. From these, the finite-size scaling (3.5) for the length scale $L_{\parallel}(t)$ and especially (3.9) for the plateaux $C_{\infty}^{(2)}$ in the two-time autocorrelator of a fully finite system were derived. We checked that these predictions are fully bourne out in the phase-ordering of the exactly solved kinetic spherical model, for $2 < d < 4$ dimensions.

Clearly, several open questions remain, including:

1. Do the FSS predictions (3.3,3.5,3.9) also hold for other universality classes ? For kinetic Ising models with either short-ranged or long-ranged interactions, detailed tests on all these have been carried out recently and will be reported elsewhere [26].

2. Although the discussion was entirely formulated here in terms of classical dynamics, a finite-size scaling ansatz such as (3.3) should *a priori* also work for relaxations in quantum systems, either closed or open.

3. Our analysis is restricted to below the upper critical dimension $d < d_c$. At equilibrium, it is well-known that dangerous irrelevant variables lead to essential modifications of the finite-size scaling ansatz (3.1,3.3) [14, 65, 51, 41, 42, 46, 12]. Such modifications should also become necessary for the dynamics.

Considerations of this kind might become crucial either for long-range interactions, where d_c is lowered with respect to the value $d_c^{(\mathrm{short})} = 4$ of short-ranged systems or else for d-dimensional *quantum* systems (possibly with long-ranged interactions as well), for which at least the equilibrium quantum phase transitions at $T = 0$ are known to be in the same universality class as the corresponding $(d+\theta)$-dimensional classical universality class at finite temperature, where the anisotropy exponent $\theta \geq 1$ [68].

4. From figure 1b it appears that finite-size effects might create a spurious regime where the autocorrelator $C(ys, s) \sim y^{-\lambda_{\mathrm{eff}}}$ might look algebraic in a certain window; but rather the system already is the transition region between the rapid fall-off after having left the infinite-size behaviour of $C_{\mathrm{bulk}}(ys, s)$ and the turn-around towards the saturation plateau $C_\infty^{(2)}$. Since $\lambda_{\mathrm{eff}} > \lambda$, not recognising this effect carries the risk of systematic over-estimation of the auto-correlation exponent λ, in simulations or in experiments.
5. One may generalise dynamical FSS to critical quenches and to two-time response functions as well. The theory and numerical tests thereof will be presented elsewhere [26].
6. Can one use (3.9) to devise improved methods for the measurement of λ ?

Appendix. Analytical derivations

The exact solution of the kinetic spherical model at $T < T_c(d)$, starting from (4.4), is described.

A.1 Spherical constraint

The Volterra integral equation (4.4d) gives the long-time behaviour of $g(t)$ in a large, but finite system, as follows. The first part retraces the steps used at equilibrium [69, 70], with the notation adapted for dynamics. The second part gives the new ingredients needed for non-equilibrium dynamics.

1. Through a Laplace transform we formally solve (4.4d)

$$\overline{g}(p) = \mathscr{L}(g)(p) := \int_0^\infty \mathrm{d}t\, e^{-pt} g(t) = \frac{\overline{f}(p)}{1 - 2DT\overline{f}(p)} \tag{A.1}$$

Standard Tauberian theorems [36] relate the behaviour of $\overline{g}(p)$ in the $p \to 0$ limit to the asymptotic long-time behaviour of $g(t)$ for $t \to \infty$. One needs the leading terms of $\overline{f}(p)$ as $p \to 0$. Recall the generalised Poisson re-summation formula [75]

$$\sum_{n=a}^{b} f(n) = \sum_{q=-\infty}^{\infty} \int_a^b \mathrm{d}x\, e^{2\pi \mathrm{i} q x} f(x) + \frac{1}{2} f(a) + \frac{1}{2} f(b) \tag{A.2}$$

and use this to deduce the important identity, for $m \in \mathbb{Z}$ and $x \in \mathbb{R}$

$$\sum_{k=0}^{N-1} \exp\left(\frac{2\pi \mathrm{i}}{N} km + x \cos \frac{2\pi k}{N} \right) = N \sum_{q=-\infty}^{\infty} I_{qN+m}(x) \tag{A.3}$$

where $I_n(x)$ is a modified Bessel function [1].

Now, one writes as in [69], using eq. (A.3) with $m = 0$ in the second line d times

$$
\begin{aligned}
2D\overline{f}(p) &= \frac{2D}{|\Lambda|}\sum_{\boldsymbol{k}}\int_0^\infty \mathrm{d}t\, \exp\left[-\left(p + 4D\sum_{j=1}^{d}\left(1-\cos\frac{2\pi}{N_j}k_j\right)\right)t\right] \\
&= 2D\int_0^\infty \mathrm{d}t\, e^{-(p+4Dd)t}\sum_{q_1,\ldots,q_d\in\mathbb{Z}}\prod_{j=1}^{d} I_{N_j q_j}(4Dt) \\
&= \frac{1}{2}\int_0^\infty \mathrm{d}u\, e^{-\frac{1}{2}\phi u}\left(e^{-u}I_0(u)\right)^d + \frac{1}{2}{\sum_{\boldsymbol{q}\in\mathbb{Z}^d}}'\int_0^\infty \mathrm{d}u\, e^{-\frac{1}{2}\phi u}\prod_{j=1}^{d}\left(e^{-u}I_{N_j q_j}(u)\right) \qquad \text{(A.4)}
\end{aligned}
$$

where one sets $\phi := p/2D$. In the last line, the bulk contribution which arises from $\boldsymbol{q} = \boldsymbol{0}$, is separated from the finite-size terms which have $\boldsymbol{q} \neq \boldsymbol{0}$ (indicated by $\sum'$).

In what follows, restrict throughout to dimensions $2 < d < 4$. First, standard techniques [8, 19, 57, 45] give the leading order of the Watson function $W_d(\phi)$ for $\phi \ll 1$, as follows

$$
\begin{aligned}
W_d(\phi) &:= \frac{1}{2}\int_0^\infty \mathrm{d}u\, e^{-\frac{1}{2}\phi u}\left(e^{-u}I_0(u)\right)^d \\
&\simeq W_d(0) - (4\pi)^{-d/2}\left|\Gamma\left(1-\frac{d}{2}\right)\right|\phi^{(d-2)/2}\left(1+\mathrm{o}(\phi)\right) \qquad \text{(A.5)}
\end{aligned}
$$

with an implied analytic continuation in d. Next, the finite-size terms are evaluated in the hyper-cubic geometry, such that the first d^* dimensions are finite ($0 < d^* \leq d$), with periodic boundary conditions (for simplicity, set $N_j = N$ for all $j = 1, \ldots, d^*$). The remaining $d - d^*$ dimensions are assumed to be infinite, formally $N_j = \infty$. With the asymptotic identity [69] $I_\nu(x) = (2\pi x)^{-1/2}e^{x-\nu^2/2x}\big(1 + \mathrm{O}(1/x)\big)$ one has

$$
\begin{aligned}
\frac{1}{2}\int_0^\infty \mathrm{d}u\, e^{-\frac{1}{2}\phi u}\prod_{j=1}^{d}\left(e^{-u}I_{N_j q_j}(u)\right) &\simeq \frac{1}{2}\int_0^\infty \mathrm{d}u\, e^{-\frac{1}{2}\phi u}\big(2\pi u\big)^{-d/2}\prod_{j=1}^{d^*}e^{-(Nq_j)^2/2u} \\
&= (4\pi)^{-d/2}\phi^{d/2-1}\int_0^\infty \mathrm{d}v\, v^{-d/2}\exp\left(-v - \frac{1}{v}\frac{\phi}{4}\sum_{j=1}^{d^*}N^2 q_j^2\right) \\
&= \frac{2}{(4\pi)^{d/2}}\left(\frac{2\psi}{N}\right)^{d-2}\left(\frac{1}{\psi|\boldsymbol{q}|}\right)^{(d-2)/2} K_{(d-2)/2}\big(2\psi|\boldsymbol{q}|\big) \qquad \text{(A.6)}
\end{aligned}
$$

with the *thermo-geometric parameter* $\psi := \frac{1}{2}N\phi^{1/2}$, the short-hand $|\boldsymbol{q}|^2 := \sum_{j=1}^{d^*} q_j^2$, the other modified Bessel function $K_\nu(x)$ [1] and where the identity [69]

$$
\int_0^\infty \mathrm{d}x\, x^{\nu-1}e^{-\beta x-\alpha/x} = 2\left(\frac{\alpha}{\beta}\right)^{\nu/2}K_\nu\big(2\sqrt{\alpha\beta}\,\big) \qquad \text{(A.7)}
$$

was used in the last line. In the infinite directions, only the terms with $q_j = 0$ contribute in (A.6), for $j = d^* + 1, \ldots, d$. The final result of the first part is, for $2 < d < 4$ [69, 70]

$$
2D\overline{f}(p) = W_d(0) - \frac{1}{(4\pi)^{d/2}}\left(\left|\Gamma\left(1-\frac{d}{2}\right)\right| - 2{\sum_{\boldsymbol{q}\in\mathbb{Z}^{d^*}}}'\frac{K_{(d-2)/2}(2\psi|\boldsymbol{q}|)}{(\psi|\boldsymbol{q}|)^{(d-2)/2}}\right)\left(\frac{2\psi}{N}\right)^{d-2} + \ldots \qquad \text{(A.8)}
$$

2. We define the abbreviation

$$H_\alpha(\psi) := \frac{1}{(4\pi)^{d/2}} \left(|\Gamma(-\alpha)| - 2{\sum_{\boldsymbol{q}^*}}' \frac{K_\alpha(2\psi|\boldsymbol{q}|)}{(\psi|\boldsymbol{q}|)^\alpha} \right) \tag{A.9}$$

where $\sum_{\boldsymbol{q}^*} = \sum_{\boldsymbol{q}\in\mathbb{Z}^{d^*}}$ is only extended over the finite directions. In the spherical model, the equilibrium magnetisation $M_{\rm eq}^2 = 1 - T/T_c$, where the critical temperature $1/T_c = W_d(0)$ [13, 8, 69, 67, 45]. For quenches to $T < T_c$ one has $M_{\rm eq}^2 > 0$. Then, using (A.1) and (A.8)

$$\begin{aligned}
\overline{g}(p) &\simeq \frac{1}{2D} \frac{W_d(0) - H_{(d-2)/2}(\psi)\left(\frac{2\psi}{N}\right)^{d-2} + \ldots}{1 - TW_d(0) + TH_{(d-2)/2}(\psi)\left(\frac{2\psi}{N}\right)^{d-2} + \ldots} \\
&\simeq \frac{1}{2DT_c} \frac{1}{M_{\rm eq}^2} - \frac{1}{2D}\frac{1}{M_{\rm eq}^4} H_{(d-2)/2}(\psi) \left(\frac{2\psi}{N}\right)^{d-2} + \ldots \\
&= \frac{1}{2DT_c}\frac{1}{M_{\rm eq}^2} - \frac{1}{2DM_{\rm eq}^4} \frac{|\Gamma(1-d/2)|}{(4\pi)^{d/2}} \left(\frac{p}{2D}\right)^{(d-2)/2} \\
&\quad + \frac{2}{2DM_{\rm eq}^4}\frac{1}{(4\pi)^{d/2}} \left(\frac{p}{2D}\right)^{(d-2)/4} {\sum_{\boldsymbol{q}^*}}' \left(\frac{N|\boldsymbol{q}|}{2}\right)^{(2-d)/2} K_{(d-2)/2}\left(\frac{N|\boldsymbol{q}|}{\sqrt{2D}} p^{1/2}\right)
\end{aligned} \tag{A.10}$$

gives the leading terms of $\overline{g}(p)$ for small values of p. The first two of these terms are the bulk contributions, while the remaining ones give the leading finite-size effects.

The leading long-time behaviour of $g(t)$ is then obtained via the identities [66]

$$\mathscr{L}^{-1}\left(p^{\nu/2} K_\nu(2ap^{1/2})\right)(t) = \frac{1}{2}\frac{a^\nu}{t^{\nu+1}}\, e^{-a^2/t} \tag{A.11a}$$

$$\mathscr{L}^{-1}\left(p^{-\nu}\right)(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} \tag{A.11b}$$

and we find, where from now on both d and d^* can be considered as continuous parameters

$$\begin{aligned}
g(t) &= \frac{1}{2DT_c}\frac{1}{M_{\rm eq}^2}\delta(t) + \frac{1}{M_{\rm eq}^4}\frac{1}{(8\pi Dt)^{d/2}} + \frac{1}{M_{\rm eq}^4 (8\pi Dt)^{d/2}} {\sum_{\boldsymbol{q}^*}}' e^{-\pi \frac{N^2}{8\pi Dt}|\boldsymbol{q}|^2} \\
&= \frac{1}{2DT_c}\frac{1}{M_{\rm eq}^2}\,\delta(t) + \frac{(8\pi Dt)^{-d/2}}{M_{\rm eq}^4}\,\vartheta_3\left(0, \exp\left(-\pi\frac{N^2}{8\pi Dt}\right)\right)^{d^*}
\end{aligned} \tag{A.12}$$

with the Jacobi Theta function ϑ_3 [1], which obeys the functional identity

$$\vartheta_3\left(0, e^{-\pi y}\right) = y^{-1/2}\,\vartheta_3\left(0, e^{-\pi/y}\right) \tag{A.13}$$

Figure 3 illustrates the rapid cross-over (essentially in the interval $\frac{1}{2} \lesssim y \lesssim 2$) between the two asymptotic regimes. Therefore, we have the following asymptotic limits, for $2 < d < 4$ and $T < T_c$

$$g(t) = \frac{1}{2DT_c}\frac{1}{M_{\rm eq}^2}\delta(t) + \begin{cases} \frac{(8\pi Dt)^{-d/2}}{M_{\rm eq}^4} & ; \text{ if } N^2/t \gg 1 \quad \text{infinite-size system} \\ \frac{(8\pi Dt)^{-(d-d^*)/2}}{M_{\rm eq}^4} N^{-d^*} & ; \text{ if } N^2/t \ll 1 \quad \text{finite-size system} \end{cases} \tag{A.14}$$

This shows that the long-time behaviour of the spherical constraint in a finite geometry is effectively $(d-d^*)$-dimensional. The singular terms in (A.12,A.14) will become very important for the calculation of the correlators, as we shall see below.

Eq. (A.12) is the main result of this sub-section.

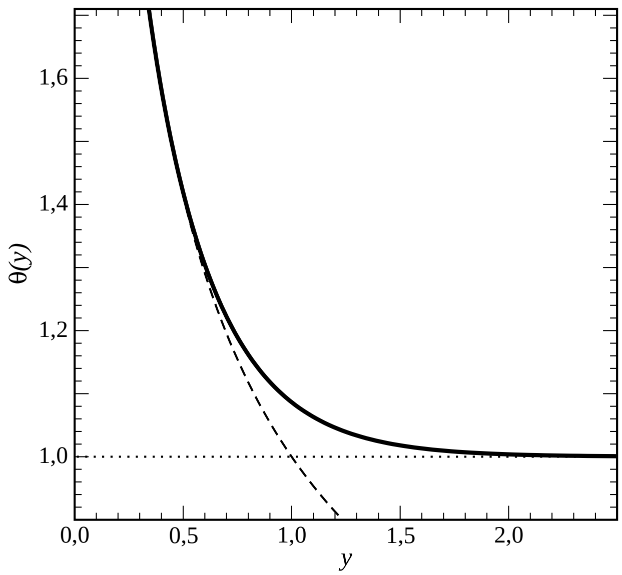

Figure 3: The function $\theta(y) := \vartheta_3\big(0, e^{-\pi y}\big)$ and its cross-over between the regimes where $y \gg 1$ and $\theta(y) \simeq 1$ (dotted line) and $y \ll 1$ and $\theta(y) \simeq y^{-1/2}$ (dashed line).

A.2 Two-time autocorrelator

We decompose in (A.12) $g(t) = g_{\rm sing}(t) + g_{\rm reg}(t)$, where $g_{\rm sing}(t) = \frac{1}{2DT_c}\frac{1}{M_{\rm eq}^2}\delta(t)$. In momentum space, with the convention $t > s$, we have from (4.5), for large times, the decomposition

$$
\begin{aligned}
\widehat{C}(t,s;\boldsymbol{k}) &= \frac{e^{-2D\omega(\boldsymbol{k})(t+s)}}{\sqrt{g_{\rm reg}(t)g_{\rm reg}(s)}}\left\{1 + \frac{2DT}{2DT_c}\frac{1}{M_{\rm eq}^2}\int_0^s \mathrm{d}\tau\, \delta(\tau)e^{2D\omega(\boldsymbol{k})2\tau} + 2DT\int_0^s \mathrm{d}\tau\, g_{\rm reg}(\tau)e^{2D\omega(\boldsymbol{k})2\tau}\right\} \\
&= \frac{1}{M_{\rm eq}^2}\frac{e^{-2D\omega(\boldsymbol{k})(t+s)}}{\sqrt{g_{\rm reg}(t)g_{\rm reg}(s)}} + 2DT\int_0^s \mathrm{d}\tau\, \frac{g_{\rm reg}(\tau)}{\sqrt{g_{\rm reg}(t)g_{\rm reg}(s)}}\, e^{-2D\omega(\boldsymbol{k})(t+s-2\tau)} \qquad \text{(A.15)}
\end{aligned}
$$

for all temperatures $T < T_c$. With (4.6), this gives the two-time autocorrelator $C(t,s) = C^{[1]}(t,s) + C^{[2]}(t,s)$. The first term in (A.15) leads to

$$
\begin{aligned}
C^{[1]}(t,s) &= \frac{|\Lambda|^{-1}M_{\rm eq}^{-2}}{\sqrt{g_{\rm reg}(t)g_{\rm reg}(s)}}\sum_{\boldsymbol{k}}\exp\left[-2D\sum_{j=1}^{d}\left(1-\cos\frac{2\pi}{N_j}k_j\right)(t+s)\right] \\
&= \frac{M_{\rm eq}^{-2}}{\sqrt{g_{\rm reg}(t)g_{\rm reg}(s)}}\prod_{j=1}^{d}\sum_{q_j\in\mathbb{Z}} e^{-2D(t+s)}I_{N_j q_j}(2D(t+s)) \\
&\simeq \frac{M_{\rm eq}^{-2}}{\sqrt{g_{\rm reg}(t)g_{\rm reg}(s)}}\frac{1}{(4\pi D(t+s))^{d/2}}\prod_{j=1}^{d^*}\sum_{q_j\in\mathbb{Z}}\exp\left[-\frac{(Nq_j)^2}{4D(t+s)}\right]\Big(1+\mathrm{O}\big((t+s)^{-1}\big)\Big) \\
&= M_{\rm eq}^2\left(\frac{t^{d/2}s^{d/2}}{\big((t+s)/2\big)^d}\right)^{1/2}\left(\frac{\vartheta_3\big(0,\exp(-\pi\frac{N^2}{4\pi D(t+s)})\big)}{\sqrt{\vartheta_3\big(0,\exp(-\pi\frac{N^2}{8\pi Dt})\big)\,\vartheta_3\big(0,\exp(-\pi\frac{N^2}{8\pi Ds})\big)}}\right)^{d^*} \qquad \text{(A.16)}
\end{aligned}
$$

where in the first line (A.3) with $m = 0$ was used once more. In the second line, we use the asymptotic expansion of the modified Bessel function $I_n(x)$. In the third and forth lines, $g_{\rm reg}(t)$

was inserted with $N_j = N$ for $j = 1, \ldots, d^*$ from (A.12) and the sums in the same line were expressed in terms of the Jacobi Theta function ϑ_3. Both d and d^* can be taken as continuous variables.

The second term in (A.15) can be expressed as a convolution

$$C^{[2]}(ys, s) = \frac{2DT}{\sqrt{g_{\text{reg}}(ys)g_{\text{reg}}(s)}} \mathscr{L}^{-1}\left(\overline{g_{\text{reg}}}(p)\left(\overline{\left[e^{-4D((y+1)s/2)}I_0(4D(y+1)s/2)\right]^d}\right)(p)\right)(s) \tag{A.17}$$

For $s \to \infty$, a Tauberian theorem relates the leading behaviour to the one of the Laplace transform at $p \to 0$ [36]. In turn, the behaviour of the two factors should be dominated by the long-time behaviour of the original functions. Therefore, one expects the leading contribution to be of the order (g_0 is the amplitude of $g_{\text{reg}}(\tau)$)

$$\begin{aligned} C^{[2]}(ys, s) &\simeq \frac{2DT}{\sqrt{g_{\text{reg}}(ys)g_{\text{reg}}(s)}} \int_0^s \mathrm{d}\tau\, g_0 \tau^{-d/2}\left(8\pi D\frac{y+1}{2}(t+s-2\tau)\right)^{-d/2} \\ &\simeq 2DT(ys^2)^{d/4}s^{1-d}\int_0^1 \mathrm{d}v\, v^{-d/2}\left(4\pi D(y+1)(y+1-2v)\right)^{-d/2} \\ &= \mathrm{O}\big(Ts^{1-d/2}\big) \end{aligned} \tag{A.18}$$

up to an s-independent amplitude. For $d > 2$, $C^{[2]}(ys, s)$ is negligible in the scaling limit where $s \to \infty$. Hence for all temperatures $T < T_c$, the leading term of the autocorrelator is $C(t, s) = C^{[1]}(t, s)$.

Finally, introducing the scaling variables Z and y in (A.15), and with the scaling $8\pi D \stackrel{!}{=} 1$, we arrive at (4.11a). With (A.13), the equivalent form (4.11b) is obtained.

A.3 Single-time correlator

We re-use the decomposition $g(t) = g_{\text{sing}}(t) + g_{\text{reg}}(t)$ from above. In momentum space, we decompose $\widehat{C}(t;\boldsymbol{k}) = \widehat{C}^{[1]}(t;\boldsymbol{k}) + \widehat{C}^{[2]}(t;\boldsymbol{k})$ and have for all $T < T_c$

$$\begin{aligned} \widehat{C}(t;\boldsymbol{k}) &= \frac{e^{-4D\omega(\boldsymbol{k})t}}{g_{\text{reg}}(t)} + \frac{2DT}{g_{\text{reg}}(t)}\int_0^t \mathrm{d}\tau\left(\frac{1}{2DT_c}\frac{1}{M_{\text{eq}}^2}\delta(\tau) + g_{\text{reg}}(\tau)\right)e^{-4D\omega(\boldsymbol{k})(t-\tau)} \\ &= \frac{e^{-4D\omega(\boldsymbol{k})t}}{M_{\text{eq}}^2\, g_{\text{reg}}(t)} + 2DT\int_0^t \mathrm{d}\tau\, \frac{g_{\text{reg}}(\tau)}{g_{\text{reg}}(t)}\, e^{-4D\omega(\boldsymbol{k})(t-\tau)} \end{aligned} \tag{A.19}$$

Herein, the first term is analysed as follows

$$\begin{aligned} C^{[1]}(t;\boldsymbol{n}) &= \frac{|\Lambda|^{-1}}{M_{\text{eq}}^2\, g_{\text{reg}}(t)}\sum_{\boldsymbol{k}}\exp\left[\sum_{j=1}^d \frac{2\pi\mathrm{i}}{N_j}k_j n_j - 4D\left(1-\cos\frac{2\pi}{N_j}k_j\right)t\right] \\ &= \frac{e^{-4Ddt}}{M_{\text{eq}}^2\, g_{\text{reg}}(t)}\sum_{\boldsymbol{q}\in\mathbb{Z}^d}\prod_{j=1}^d I_{N_j q_j + n_j}(4Dt) \\ &\simeq \frac{M_{\text{eq}}^2}{\vartheta_3\big(0, e^{-\pi N^2/(8\pi Dt)}\big)^{d^*}}\prod_{j=1}^d\sum_{q_j\in\mathbb{Z}} e^{-(q_jN_j+n_j)^2/(8Dt)} \end{aligned} \tag{A.20}$$

where first the full identity (A.3) is used d times, then the asymptotic form of the modified Bessel function $I_n(x)$ is used for $x \gg 1$ and finally, in the chosen finite-size geometry, the asymptotic form (A.12) is inserted. The product over the sums in the last line of (A.20) is evaluated as follows: (i) in the $d-d^*$ infinite directions where formally $N_j=\infty$, only the terms with $q_j=0$ contribute and lead to a factor $\exp\bigl[-\frac{1}{8Dt}\sum_{j=d^*+1}^{d} n_j^2\bigr]$. (ii) the d^* finite directions with $N_j=N$ produce d^* factors, each of the form

$$\sum_{q_j\in\mathbb{Z}} \exp\left[-\frac{(q_jN+n_j)^2}{8Dt}\right] = e^{-n_j^2/(8Dt)} \sum_{q_j\in\mathbb{Z}} \exp\left[-\frac{Nn_j}{4Dt}q_j - \frac{N^2}{8Dt}q_j^2\right] \tag{A.21}$$

With the identity

$$e^{-n_j^2/(8Dt)}\,\vartheta_3\left(\mathrm{i}\pi\frac{Nn_j}{8\pi Dt}, e^{-\pi N^2/(8\pi Dt)}\right) = \frac{\sqrt{8\pi Dt}}{N}\vartheta_3\left(\pi\frac{n_j}{N}, e^{-\pi (N^2/(8\pi Dt))^{-1}}\right) \tag{A.22}$$

we finally obtain (and used again (A.13))

$$\begin{aligned} C^{[1]}(t;\boldsymbol{n}) &= M_{\rm eq}^2 \exp\left(-\pi\sum_{j=1}^{d}\frac{n_j^2}{8\pi Dt}\right)\prod_{j=1}^{d^*}\frac{\vartheta_3\bigl(\mathrm{i}\pi\frac{Nn_j}{8\pi Dt}, e^{-\pi N^2/(8\pi Dt)}\bigr)}{\vartheta_3\bigl(0, e^{-\pi N^2/(8\pi Dt)}\bigr)} \\ &= M_{\rm eq}^2 \exp\left(-\pi\sum_{j=d^*+1}^{d}\frac{n_j^2}{8\pi Dt}\right)\prod_{j=1}^{d^*}\frac{\vartheta_3\bigl(\pi\frac{n_j}{N}, e^{-\pi (N^2/(8\pi Dt))^{-1}}\bigr)}{\vartheta_3\bigl(0, e^{-\pi (N^2/(8\pi Dt))^{-1}}\bigr)} \end{aligned} \tag{A.23}$$

The second term can be re-written as follows

$$C^{[2]}(t;\boldsymbol{n}) = 2DT\sum_{\boldsymbol{q}}\int_0^t \mathrm{d}\tau\,\frac{g_{\rm reg}(\tau)}{g_{\rm reg}(t)}\prod_{j=1}^{d} e^{-4D(t-\tau)} I_{q_jN_j+n_j}(4D(t-\tau)) \tag{A.24}$$

and takes the form of a convolution. For large times $t\to\infty$, we estimate this asymptotically by appealing to Tauberian theorems [36]. Then the leading term should become

$$\begin{aligned} C^{[2]}(t;\boldsymbol{n}) &\simeq \frac{2DT}{(8\pi D)^{d/2}}\int_0^t \mathrm{d}\tau\, t^{-d/2}\left(1-\frac{\tau}{t}\right)^{-d/2}\exp\left[-\pi\sum_{j=1}^{d}\frac{n_j^2}{8\pi D(t-\tau)}\right] \\ &\quad\times\prod_{j=1}^{d^*}\frac{\vartheta_3\bigl(\mathrm{i}\pi\frac{Nn_j}{8\pi D(t-\tau)}, e^{-\pi N^2/(8\pi D(t-\tau))}\bigr)\,\vartheta_3\bigl(0, e^{-\pi N^2/(8\pi D\tau)}\bigr)}{\vartheta_3\bigl(0, e^{-\pi N^2/(8\pi Dt)}\bigr)} \\ &\sim \mathrm{O}(Tt^{1-d/2}) \end{aligned} \tag{A.25}$$

which becomes negligible in the long-time limit $t\to\infty$ for $d>2$.

Therefore, in the long-time limit $t\to\infty$, $C(t;\boldsymbol{n}) = C^{[1]}(t;\boldsymbol{n})$. Introducing the scaling variables (3.5) into (A.23), and re-using (A.13,A.22) and scaling $8\pi D \stackrel{!}{=} 1$, we arrive at eqs. (4.10).

A.4 Characteristic length

The characteristic lengths $L(t)$ are defined from (4.15), with the single-time correlator given by (A.23). If the distances are calculated along the coordinates axes in one of the d^* finite directions, i.e. $\boldsymbol{n} = (n, 0, \ldots, 0)$, we find a *longitudinal length* $L_{\|}$. If n is measured along one of the infinite directions, we find a *transverse length* $L_{\perp}(t)$.

The most simple example of a transverse length arises if the distances are measured along one of the coordinate axes in one of the infinite directions (i.e. $\boldsymbol{n} = (0, 0, \ldots, n)$ with $d^* \leq d-1$)

$$L_{\perp}^2(t) \;=\; \frac{\sum_{n=-\infty}^{\infty} n^2 \exp\left[-\pi \frac{n^2}{8\pi D t}\right]}{\sum_{n=-\infty}^{\infty} \exp\left[-\pi \frac{n^2}{8\pi D t}\right]} \;\simeq\; 8\pi D t \, \frac{\int_{-\infty}^{\infty} \mathrm{d}n \, n^2 \, e^{-\pi n^2}}{\int_{-\infty}^{\infty} \mathrm{d}n \, e^{-\pi n^2}} \;=\; 4D\, t \tag{A.26}$$

which is identical to the known result for the bulk system [35].

A longitudinal length is found when $\boldsymbol{n} = (n, 0, \ldots, 0)$ with $d^* \geq 1$ is measured along one of the coordinate axes in a finite direction. If $N = 2M$ is even, we have

$$L_{\|}^2(t) = \frac{\sum_{n=-M+1}^{M} n^2 \, \vartheta_3\big(\pi \frac{n}{2M}, e^{-\pi/Z}\big)}{\sum_{n=-M+1}^{M} \vartheta_3\big(\pi \frac{n}{2M}, e^{-\pi/Z}\big)} \tag{A.27}$$

Using the definition of the Jacobi Theta function ϑ_3, we have

$$\begin{aligned}
\sum_{n=-M+1}^{M} \vartheta_3\big(\pi \frac{n}{2M}, e^{-\pi/Z}\big) &= \sum_{p\in\mathbb{Z}} \sum_{n=-M+1}^{M} \exp\left[-\pi\mathrm{i}\frac{n}{M}p - \frac{\pi p^2}{Z}\right] \\
&= \; 2M + \sum_{p\neq 0} e^{-\pi p^2/Z} \left(1 + e^{-\pi \mathrm{i} p} + \sum_{n=1}^{M-1} e^{-\pi\mathrm{i}(n/M)p} + \sum_{n=1}^{M-1} e^{\pi\mathrm{i}(n/M)p}\right) \;=\; 2M
\end{aligned} \tag{A.28}$$

and

$$\begin{aligned}
\sum_{n=-M+1}^{M} n^2 \, \vartheta_3\big(\pi \frac{n}{2M}, e^{-\pi/Z}\big) \;&=\; \sum_{p\in\mathbb{Z}} \sum_{n=-M+1}^{M} n^2 \, \exp\left[-\pi\mathrm{i}\frac{n}{M}p - \frac{\pi p^2}{Z}\right] \\
&= \; \sum_{n=-M+1}^{M} n^2 + \sum_{p\neq 0} e^{-\pi p^2/Z} \left(0 + M^2 e^{-\pi\mathrm{i}p} + \sum_{n=1}^{M-1} n^2\, e^{-\pi\mathrm{i}(n/M)p} + \sum_{n=1}^{M-1} n^2 \, e^{\pi\mathrm{i}(n/M)p}\right) \\
&\simeq \; \frac{2}{3}M^3 + M^2 + \sum_{p\neq 0} e^{-\pi p^2/Z} \left(M^2(-1)^p + \frac{4(-1)^p}{\pi^2 p^2} M^3 + (-1)^p M^2\right) + \mathrm{O}(M) \\
&\simeq \; \frac{2}{3}M^3 + \frac{8M^3}{\pi^2} \sum_{p=1}^{\infty} e^{-\pi p^2/Z} \frac{(-1)^p}{p^2} + \mathrm{O}\big(M^2\big)
\end{aligned} \tag{A.29}$$

where in the third line, an asymptotic expansion for M large was made. Inserting (A.28,A.29) into (A.27) and fixing $8\pi D = 1$ gives (4.16). The same leading result also holds if $N = 2M + 1$ is odd.

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