

Non reciprocal odd viscosity: Coarse graining the kinetic energy and exceptional instability

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Odd viscosity couples stress to strain rate in a dissipationless way. It has been studied in plasmas under magnetic fields, superfluid He³, quantum Hall fluids, and recently in the context of chiral active matter. In most of these studies odd terms in the viscosity obey Onsager reciprocal relations. Although this is expected in equilibrium systems, it is not obvious that Onsager relations hold in active materials. By directly coarse graining the kinetic energy and using both the Poisson-bracket formalism and a kinetic theory derivation, we find that the appearance of a non-vanishing angular momentum density necessarily breaks Onsager reciprocal relations, which leads to a non-Hermitian dynamical matrix for the total momentum and to the appearance of odd viscosity and other non-dissipative contributions to the viscosity. Furthermore, by accounting for both the angular momentum density and interactions that lead to odd viscosity, we find regions in the parameter space in which 3D odd mechanical waves propagate and regions in which they are mechanically unstable. The lines separating these regions are continuous lines of exceptional points, suggesting of a non-reciprocal phase transition.

Chiral active materials are composed of complex particles that break both parity and time reversal symmetry (TRS) at the microscale. This is generally a result of continuous injection of energy and angular momentum through local torques. Realizations of such fluids are found in a variety of systems across length scales, from nanoscale biomolecular motors [1–3], actomyosin networks [4], and microscale active colloids [5–7], to macroscale-driven chiral grains [8]. An important consequence of the breaking of parity and TRS is the possible appearance of *odd viscosity*. Unlike the regular viscosity that dissipates energy, odd viscosity is ‘reactive’ [9, 10] and can be obtained using a Hamiltonian theory with no need of adding dissipation [11]. Importantly, Onsager reciprocal relations [12–16] predict that such odd viscosity only appears when TRS is broken [17–19], be it due to an external magnetic field as was studied in gases [20], plasmas [21, 22], and in the context of quantum-Hall fluids [23–27], or due to local torques injected at the particle level [10, 13, 28–30].

By construction, the stress only enters the dynamics via $\nabla \cdot \boldsymbol{\sigma}$. As a result dynamics is not modified by the transformation $\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + \nabla \times \mathbf{A}$ [31–33]. Moreover, one can write the stress tensor associated with the center-of-mass (CM) or with the total momentum density, where the latter can always be made symmetric [33]. In passive systems where the angular momentum density relaxes rapidly in comparison to the linear momentum density, there is no difference between the two stress tensors (in the hydrodynamic limit). However, in chiral active systems where the angular momentum density is driven, a significant difference may arise. We argue that the experimentally measured surface forces are those related to the total momentum density $\mathbf{g}$, which account for both the

center-of-mass (CM) momentum density and the angular momentum density.

In all studies we are aware of, odd terms in the viscosity always obey Onsager reciprocal relations, mainly due to the fact that their origin is usually associated with reciprocal inter-particle collisions, even though these may break parity [31]. Even when the possible existence of such non-reciprocal odd terms is assumed phenomenologically from symmetry arguments [11, 16, 28], molecular dynamic simulations accounting for the inter-particles interactions [28] have supported reciprocity. This is quite surprising considering the fact that in active materials reciprocity need not be obeyed and non-reciprocity is quite common [34, 35]. Of course microscopic interactions obey Newton’s third law and are reciprocal, but effective interactions mediated by an ‘active’ nonequilibrium medium are not necessarily so, and they are often responsible for the frequent occurrence of non-reciprocity [36–38]. The observation that active materials do not obey Onsager reciprocity is not new [36, 39–41], but in this work we find a quite remarkable and simple instance of this – the mere existence of non-vanishing angular momentum breaks Onsager reciprocal relations. In passive systems, this has no consequence as angular momentum relaxes rapidly and does not affect the hydrodynamic equations. However, as we show below, in the presence of active torques this is not longer the case, and reciprocity is generically broken.

In Ref. [10] we coarse-grained the total momentum density $\mathbf{g}$ and found that it obeys the Belinfante-Rosenfeld relation [42, 43], which for classical fields means that the total momentum density equals the CM momentum density plus half of the curl of the internal angular momentum density [33]. Then, by writing the

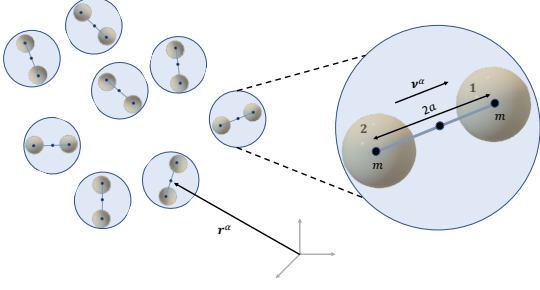


FIG. 1. Illustration of our model system that is composed of many diatomic molecules. Each of these molecules is composed of two equal point masses, m , separated by distance $2a$. The vector $\boldsymbol{\nu}$ points from mass ‘2’ to mass ‘1’, and $\mathbf{r}^\alpha$ points to the CM of the α ’s molecule. Diatomic molecules are the minimal molecules that has internal angular momentum, and therefore this is the minimal system that will exhibit ‘kinetic’ odd viscosity solely due to particles spinning, which could be a result of external field or internal *active* torques. The existence of angular momentum also breaks Onsager reciprocal relations.

kinetic energy in the common form $H = \int d\mathbf{r} \mathbf{g}^2 / (2\rho)$ we found that odd-viscosity, which obeys the Onsager reciprocal relations, emerges from the Poisson-bracket (PB) formalism [44, 45]. The resultant excitation spectrum exhibits 3D odd mechanical waves with no exceptional points, even in non-interacting systems.

In this paper we choose a more direct route and coarse-grain (CG) the microscopic kinetic energy directly. Surprisingly we find that these two CG methods are not equivalent. Indeed, odd viscosity still naturally emerges, but more terms appear in the viscosity including one that couples vorticity and pressure (odd pressure [11]). These new viscosity terms exactly cancel terms in the dynamics of the total momentum and do not allow for propagation of 3D odd mechanical waves. Instead, they require a longitudinal wave to always be accompanied by a transverse wave but not vice versa. Importantly, this means that the dynamics are non-reciprocal and indeed the dynamical matrix is non-Hermitian. In the presence of an additional odd viscosity that can originate in fluctuations or inter-particle interactions [28] we find instability lines of exceptional points [46], which suggests the existence of a non-reciprocal phase transition [35].

In this work we consider a fluid of dumbbells, which are diatomic molecules composed of two point masses, m , separated by distance $2a$ as depicted in Fig. 1. We define the CM position of each dumbbell to be $\mathbf{r}^\alpha$ while $\boldsymbol{\nu}^\alpha$ is a unit vector pointing from mass ‘2’ to mass ‘1’. It is important to note that we only consider a fluid of dumbbells for clearer presentation. The derivation below also applies to general complex rigid particles (index α) that are composed of multiple sub-particles (atoms) with mass $m^{\alpha\mu}$ and momentum $\mathbf{p}^{\alpha\mu}$ located at $\mathbf{r}^{\alpha\mu}$. We show this in the SI [47].

TOTAL MOMENTUM

We start by introducing the concept of total momentum, a density field that captures the momentum of all atoms (not only the molecules CM) in the hydrodynamic limit. As we show below, the total momentum flux (or the total momentum stress tensor) is very different from the one associated with the CM. Clearly, the experimentally measurable force in a rheological experiment is the one related to this total momentum stress.

To model a fluid of dumbbells, we write the momentum of the two point masses in terms of the CM momentum, $\mathbf{p}^\alpha$, and $\boldsymbol{\nu}^\alpha$ as $\mathbf{p}_{1,2}^\alpha = \frac{1}{2}\mathbf{p}^\alpha \pm m a \boldsymbol{\nu}^\alpha$, such that the total momentum density for the diatomic fluid, $\hat{g}_i(\mathbf{r}) = \sum_{\alpha\mu} p_i^{\alpha\mu} \delta(\mathbf{r} - \mathbf{r}^{\alpha\mu})$, can be split into the CM momentum density, $\hat{\mathbf{g}}^c \equiv \sum_{\alpha} \mathbf{p}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$, and a spin-like momentum density, $\hat{\mathbf{g}} = \hat{\mathbf{g}}^c + \hat{\mathbf{g}}^s$, with

$$\hat{g}_i^s(\mathbf{r}) = \frac{1}{2} \nabla \times \hat{\boldsymbol{\ell}} - \frac{1}{2} \nabla \cdot \hat{\mathbf{A}}, \quad (1)$$

where $\hat{\boldsymbol{\ell}}(\mathbf{r}) = \sum_{\alpha} \boldsymbol{\ell}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$ and $\hat{\mathbf{A}}(\mathbf{r}) = I \sum_{\alpha} \dot{\mathbf{Q}}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$. Here $Q_{ij}^\alpha = \nu_i^\alpha \nu_j^\alpha - \delta_{ij}/d$ is the alignment tensor of molecule α (d is the spatial dimension) and $\hat{Q}_{ij}(\mathbf{r}) = \sum_{\alpha} Q_{ij}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$. The angular momentum of each molecule is related to its angular velocity, $\Omega^\alpha \equiv \boldsymbol{\nu}^\alpha \times \dot{\boldsymbol{\nu}}^\alpha$, by $\boldsymbol{\ell}^\alpha = I \Omega^\alpha$. The moment of inertia of each molecule is $I = M a^2$ with $M = 2m$ being the molecule mass. Note that these expressions are written in the long wavelength limit (see SI [47]).

Coarse-graining (1) gives $\mathbf{g}^s(\mathbf{r}) = \frac{1}{2} \nabla \times \boldsymbol{\ell} - \frac{1}{2} \nabla \cdot \mathbf{A}$. Throughout this work we use a coarse-graining method [48] from unpublished notes of P. C. Martin that is described in Materials and Methods. Assuming the system is in its isotropic state, $\mathbf{A} = \mathbf{Q} = 0$, the well-known form of the total momentum density is recovered [10, 33].

$$\mathbf{g}(\mathbf{r}) = \mathbf{g}^c + \frac{1}{2} \nabla \times \boldsymbol{\ell}, \quad (2)$$

where $\mathbf{g}^c = \rho \mathbf{v}^c$ is the CM momentum density, ρ the mass density, $\mathbf{v}^c$ the CM velocity, and $\boldsymbol{\ell}(\mathbf{r}) = \rho(\mathbf{r}) \tilde{I} \boldsymbol{\Omega}(\mathbf{r})$ is the angular-momentum density of spinning particles, where $\tilde{I} = I/M = a^2$ is the moment of inertia per unit mass and $\boldsymbol{\Omega}$ is the rotation-rate vector. The relation of (2) is well known for classical fields [33] and is referred to in the quantum physics literature as the Balinfante-Rosenfeld relation in which case the CM momentum is the canonical momentum (derived from Noether theorem) and the angular momentum is the particle spin [43].

KINETIC ENERGY OF COMPLEX MOLECULES

In this section we use the coarse-graining procedure that is detailed in Materials and Methods to directly CG the kinetic energy of a fluid of dumbbells (derivation for

a general complex molecule is deferred to the SI [47]). The microscopic kinetic Hamiltonian is:

$$H_k = \sum_{\alpha\mu} \frac{(\mathbf{p}^{\alpha\mu})^2}{2m^{\alpha\mu}}. \quad (3)$$

Using the CG described in Materials and Methods, and taking the long wavelength limit, the kinetic energy reads (for complete derivation see SI [47])

$$H_k = \int d\mathbf{r} \left[\frac{(\mathbf{g}^c)^2}{2\rho} + \frac{\ell^2}{2I} + \frac{1}{2} \nabla \cdot (\ell \times \mathbf{v} - \mathbf{v} \cdot \mathbf{A}) \right]. \quad (4)$$

Importantly, the last term in Eq. (4) is a boundary term and can therefore be ignored in the bulk. Then, the Hamiltonian in terms of the CM momentum assumes a well-known form [13, 45, 49] that does not produce odd viscosity. Indeed, below we show that no odd viscosity appears in the CM stress, but it does appear in the coarse-grained total momentum of (2).

Intuitively this can be understood by writing the kinetic Hamiltonian in terms of the total momentum (up to terms $\sim (\nabla \ell)^2$):

$$H_k = \int d\mathbf{r} \left[\frac{\mathbf{g}^2}{2\rho} + \frac{\ell^2}{2I} - \ell \cdot \boldsymbol{\omega} + \frac{1}{2} \nabla \cdot (\ell \times \mathbf{v} - \mathbf{v} \cdot \mathbf{A}) \right] \quad (5)$$

where $\boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{v}$ is the rotation vector. Here the third term is the term that was responsible for the appearance of odd viscosity in Refs. [10, 29]. (The last term remains a boundary term.) This result differs from the CG Hamiltonian we used in Ref. [10] in which the third term is absent. This term is responsible for the appearance of odd viscosity in Refs. [10, 29]. Below we show that odd viscosity emerges from (5) as well, alongside other non-dissipative terms.

Note that although the last term in Eqs. (4-5) can be ignored in the bulk it will affect the boundary conditions and may play a significant role in the study of surface states in fluids of rotating particles [7, 50, 51].

DYNAMICS OF THE TOTAL MOMENTUM

In terms of the CM momentum the kinetic Hamiltonian of (4) is quite standard (disregarding the boundary term) and the complete Hamiltonian is [10, 44]:

$$H = \int d\mathbf{r} \left(\frac{(\mathbf{g}^c)^2}{2\rho} + \frac{\ell^2}{2I} + F[\rho(\mathbf{r})] \right), \quad (6)$$

where $F[\rho] = \int d\mathbf{r} \mathcal{F}(\rho, \nabla \rho)$ is the free energy, which in the isotropic case is only a functional of $\rho(\mathbf{r})$. The latter statement assumes that interactions are only central force, i.e., they only depend on the CM positions of the particles. When friction between particles is considered

this might not be the case and other types of interactions that depend on ℓ may appear [28] – we ignore such interactions for now.

To derive the dynamics of the CG fields we use the PB formalism for fields [10, 44] (see Materials and Methods) and the Hamiltonian of (6) to derive the reactive part of the dynamics for $\mathbf{g}$, ℓ and ρ . Using the fundamental PBs (see Materials and Methods) we find that the reactive part of the dynamics follows

$$\dot{\rho} + \nabla \cdot \mathbf{g} = 0, \quad (7a)$$

$$\dot{g}_i^c + \nabla_j (v_j^c g_i^c) = -\nabla_i P + f_i, \quad (7b)$$

$$\dot{\ell}_i + \nabla_j (v_j^c \ell_i) = \tau_i, \quad (7c)$$

where $\mathbf{f}$ and $\boldsymbol{\tau}$ are, respectively, external force and torque densities that were added to the dynamics of linear and angular momentum. Here $P = \rho \frac{\delta F}{\delta \rho} - \mathcal{F}$ is the thermodynamic pressure and $F = \int d\mathbf{r} \mathcal{F}$. This result is not surprising or interesting, we simply recovered the continuity equation, the reactive parts of the Navier-Stokes equation and the angular momentum density dynamics.

The odd viscosity is revealed upon the realization that the CM momentum density does not account for all of the momentum. Using the definition of the total momentum, (2), and the above Eqs. (7a-7c) we can write the dynamics of the total momentum:

$$\dot{g}_i + \nabla_j (g_i v_j) = \nabla_j \sigma_{ij} + f_i, \quad (8a)$$

$$\sigma_{ij} = -P \delta_{ij} + (\eta_{ijkl}^o + \eta_{ijkl}^e) \nabla_l v_k + \frac{1}{2} \varepsilon_{ijk} \tau_k, \quad (8b)$$

$$\eta_{ijkl}^o = -\frac{\ell_n}{4} (\gamma_{ijkl;n}^o + \varepsilon_{ilk} \delta_{nj} + \varepsilon_{jlk} \delta_{ni} - 2\varepsilon_{lkn} \delta_{ij}), \quad (8c)$$

with the usual [10, 22, 29, 52] odd viscosity tensor

$$\gamma_{ijkl;n}^o = \varepsilon_{iln} \delta_{jk} + \varepsilon_{ikn} \delta_{jl} + \varepsilon_{jkn} \delta_{il} + \varepsilon_{jln} \delta_{ik}. \quad (9)$$

Note that in deriving the above equations we divided the velocity gradient tensor into its symmetric and anti-symmetric parts and dropped a non-hydrodynamic term $\sim \nabla (\nabla \times \ell)^2$. Furthermore, we have added the dissipative viscosity tensor η_{ijkl}^e , which for an isotropic system (as we assume here, disregarding anisotropic terms associated with ℓ) assumes the known form $\eta_{ijkl}^e = \lambda \delta_{ij} \delta_{kl} + \eta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$, where λ and η are constants.

The total momentum stress thus contains the odd viscosity, the so-called odd pressure [11] that couples vorticity (or rotations) to pressure, and two more terms that couple vorticity to shears that involve the direction of ℓ [53], which do not appear in 2D. Importantly, the existence of these non-dissipative terms does not depend on F , hence, these terms will appear even in a non-interacting systems.

Because we find that the two ways of CG do not give the same results (compare the result of this section and [10]), we support our current findings using another,

unrelated derivation, of the dynamics using a kinetic theory that is detailed in Materials and Methods.

EXCITATION SPECTRUM

We continue by calculating the excitation spectrum in the presence of odd viscosity. For simplicity we assume that $\boldsymbol{\tau}$ and $\boldsymbol{\ell}$ are constants, and that $\mathbf{f} = 0$, such that the dynamics for the total momentum obeys

$$\dot{g}_i + \nabla_j (v_j g_i) = -\nabla_i \tilde{P} + \eta \nabla^2 v_i - \frac{1}{2} \varepsilon_{ikn} \ell_n \nabla_k \nabla \cdot \mathbf{v} \quad (10)$$

with $\tilde{P} \equiv P - (\lambda + \eta) \nabla \cdot \mathbf{v}$ being the mechanical pressure (diagonal part of the stress). To find the excitation spectrum we linearize (10) and use the Fourier-transform $[\mathbf{v}, \delta \rho] = \int \frac{d\mathbf{k}}{(2\pi)^3} [\tilde{\mathbf{v}}, \delta \tilde{\rho}] e^{i(\mathbf{k} \cdot \mathbf{r} - st)}$, leading to:

$$\begin{pmatrix} s + i\nu k^2 & 0 & 2i\ell_r k k_\perp & 0 \\ 0 & s + i\nu k^2 & 0 & 0 \\ 0 & 0 & s + i\nu_L k^2 & -kc \\ 0 & 0 & -kc & s \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_L \\ \tilde{h} \end{pmatrix} = 0. \quad (11)$$

Here $\nu \equiv \eta/\rho_0$, $\nu_L \equiv 2\nu + \lambda/\rho_0$, $\ell_r \equiv \ell/(4\rho_0)$, $\boldsymbol{\ell} = \ell \hat{z}$, and $h = \delta\rho/(c\rho_0)$ where $\rho = \rho_0 + \delta\rho$ and c is the sound speed ($c^2 = \partial P/\partial\rho$). We further define $v_{1,2} \equiv \mathbf{v} \cdot \hat{\mathbf{e}}_{1,2}$ and $v_L \equiv \mathbf{v} \cdot \hat{\mathbf{k}}$, where $\hat{\mathbf{k}} = (k_x, k_y, k_z)/k$, $\hat{\mathbf{e}}_1 = (-k_y, k_x, 0)/k_\perp$, and $\hat{\mathbf{e}}_2 = (-k_x k_z, -k_y k_z, k_\perp^2)/(k k_\perp)$ are a set of orthonormal vectors with $k = \sqrt{\mathbf{k}^2}$, $k_\perp = \sqrt{k_x^2 + k_y^2}$, and $k_\parallel = \mathbf{k} \cdot \boldsymbol{\ell}/\ell$.

This matrix is not Hermitian, but its eigenvalues can be easily found using the determinant:

$$(s + i\nu k^2)^2 (s^2 - c^2 k^2 + i s \nu_L k^2) = 0. \quad (12)$$

The dispersion relation is thus the one found for simple fluids [54], where there are two dissipative transverse modes with $s_T = -i\nu k^2$ and two longitudinal decaying waves that obey the usual sound waves relation $s_L = \frac{1}{2} k \left[-i\nu_L k \pm \sqrt{4c^2 - \nu_L^2 k^2} \right]$. The transverse eigenvectors $[\vec{v} = (v_1, v_2, v_L, h)]$ are trivial $[\vec{v}_T = (1, 0, 0, 0); (0, 1, 0, 0)]$ while the longitudinal eigenvectors to lowest order in $g = 2\ell_r k_\perp/c$ and $g_\nu = k\nu_L/c$ are $\vec{v}_L = (\mp ig, 0, 1, \pm 1 + ig_\nu/2)$.

Interestingly, this means that although the presence of angular momentum does not affect the dispersion relation, it creates (a non-reciprocal) coupling between longitudinal and transverse modes such that a longitudinal wave is always accompanied by a transverse wave (but not vice-versa).

The result of this work stands in contrast with Ref. [10] in which it was predicted that the presence of angular momentum itself is sufficient to produce new type of transverse ‘odd’ mechanical waves. It is nevertheless expected [28] that spin-spin interactions (that were neglected in this work so far) will result in odd viscosity such that (8c) will have an additional term $(\eta_n^o/4)\gamma_{ijkl;n}^o$.

Because this ‘interaction odd viscosity’ is a result of spin-spin interaction [28], which together with $\boldsymbol{\ell}$ defines the broken symmetry direction of the system, we expect that $\boldsymbol{\eta}^o \parallel \boldsymbol{\ell}$ [55]. Then, assuming that $\boldsymbol{\tau}$, $\boldsymbol{\ell}$, and $\boldsymbol{\eta}^o$ are constants (and $\mathbf{f} = 0$), the linearized equations in the Fourier space, (11), become

$$\begin{pmatrix} s + i\nu k^2 & -i\nu^o k k_\parallel & 2i k k_\perp (\ell_r + \nu^o) & 0 \\ i\nu^o k k_\parallel & s + i\nu k^2 & 0 & 0 \\ -2i\nu^o k k_\perp & 0 & s + i\nu_L k^2 & -kc \\ 0 & 0 & -kc & s \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_L \\ \tilde{h} \end{pmatrix} = 0 \quad (13)$$

where $\nu^o = |\boldsymbol{\eta}^o|/(4\rho_0)$. This matrix is again non-Hermitian, and when dissipation is negligible, $\nu = \nu_L = 0$, the dimensionless eigenvalue equation is

$$x^4 - x^2 \left[1 + (\tilde{\eta}_\parallel^o)^2 + 4\tilde{\eta}_\perp^o (\tilde{\eta}_\perp^o + \tilde{\ell}_\perp) \right] + (\tilde{\eta}_\parallel^o)^2 = 0, \quad (14)$$

where we define $x = s/(kc)$, $\tilde{\eta}_\parallel^o = \nu^o k_\parallel/c$, $\tilde{\eta}_\perp^o = \nu^o k_\perp/c$, and $\tilde{\ell}_\perp = \ell_r k_\perp/c$. The solutions for this equation can be written as $x^2 = [\chi \pm \sqrt{D}]/2$ where $\chi = 1 + (\tilde{\eta}_\parallel^o)^2 + 4\tilde{\eta}_\perp^o (\tilde{\eta}_\perp^o + \tilde{\ell}_\perp)$ and $D = \chi^2 - 4(\tilde{\eta}_\parallel^o)^2$. If the discriminant $D \geq 0$ then x^2 is real. If also $\chi > 0$ then $x^2 > 0$ and there are four modes of non-decaying waves. When $D \geq 0$ but $\chi < 0$, we have $x^2 < 0$ and x is pure imaginary, such that there are two modes of non-decaying waves and two unstable modes. When the discriminant is negative x^2 becomes a complex number and $x = \Re\{x\} + i\Im\{x\}$ with $\Re\{x\} = \pm \frac{1}{2} \left[\left(\sqrt{|D| + \chi^2} + \chi \right) \right]^{1/2}$ and $\Im\{x\} = \pm \frac{1}{2} \left[\sqrt{|D| + \chi^2} - \chi \right]^{1/2}$, which means that there are two unstable modes with positive imaginary part. Therefore if $D < 0$ or $\chi < 0$ the system is linearly unstable, hence the onset of instability is given by $D < 0$. A special case is when $\tilde{\eta}_\parallel^o = 0$, for which there are only two modes of non-decaying waves (the regular sound waves) and two zero-modes.

The various instability regions are plotted in Fig. 2 in the $(\tilde{\ell}_\perp, \tilde{\eta}_\perp)$ [for (a) and (d)], $(\tilde{\ell}_\perp, \tilde{\eta}_\parallel)$ [for (b) and (e)], and $(\tilde{\eta}_\perp, \tilde{\eta}_\parallel)$ [for (c) and (f)] phase-spaces. A necessary condition for instability is $\tilde{\ell}_\perp \tilde{\eta}_\perp < 0$. In the stable regions there are four propagating modes. When $\tilde{\ell}_\perp = 0$, we recover the results of [10] in which odd waves always propagate and reciprocity is restored.

Remarkably, for any non-vanishing $\tilde{\ell}_\perp$, the lines separating stable and unstable regions are lines of exceptional points [46]. Along these lines, instead of having four propagating modes (two pairs of $\pm$ eigenvalues), each pair of eigenvectors (and eigenvalues) collapse into one, leaving only two propagating modes. Recently, it has been suggested such exceptional lines marks a new type of phase transition denoted as *non-reciprocal phase transition* [35]. However, our system does not precisely fall within the class of theories examined in [35] as here

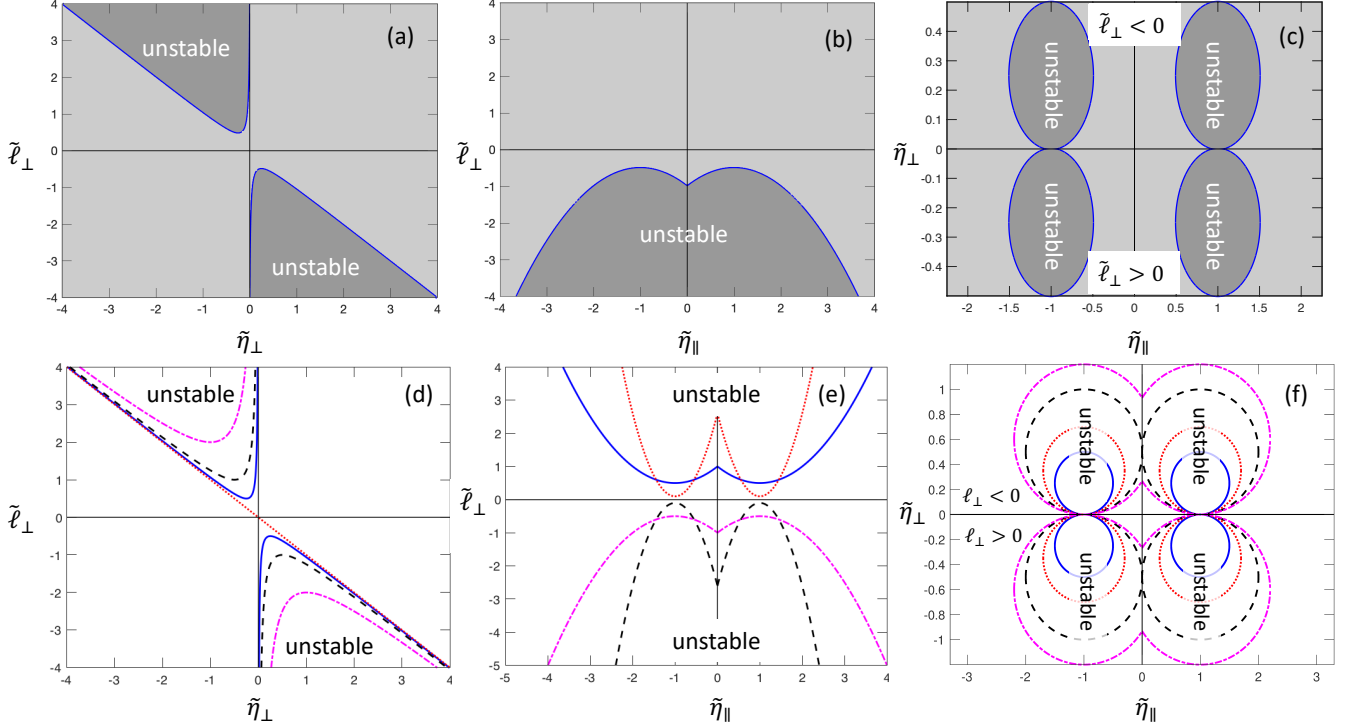


FIG. 2. Regions of instability in the $(\tilde{\ell}_\perp, \tilde{\eta}_\perp)$ [for (a) and (d)], $(\tilde{\ell}_\perp, \tilde{\eta}_\parallel)$ [for (b) and (e)], and $(\tilde{\eta}_\perp, \tilde{\eta}_\parallel)$ [for (c) and (f)] phase-spaces. Instability regions are the dark grey areas in (a), (b), and (c). Lines of exceptional points marks the transition suggesting a non-reciprocal phase transition [35]. A necessary condition for instability is $\tilde{\ell}_\perp \tilde{\eta}_\perp < 0$. Parameters used are as follows. Top row: (a) $\tilde{\eta}_\parallel = -0.5$, (b) $\tilde{\eta}_\perp = 0.5$ and (c) $|\tilde{\ell}_\perp| = 0.5$; Bottom row: The lines (solid blue, dotted red, dashed black, magenta dashe-dotted) correspond to (d) $\tilde{\eta}_\parallel = (-0.5, 1, 2, 3)$, (e) $\tilde{\eta}_\perp = (-0.5, -0.1, 0.1, 0.5)$, and (f) $\tilde{\ell}_\perp = (0.5, 0.7, 1, 1.2)$.

there is no spontaneous symmetry breaking and the exceptional points have non-zero eigenvalues. It is worth noting that in these exceptional points there are two other modes, but they are not of the form of regular normal modes (see SI [47] for details), instead they have the following form: $(\mathbf{W} + (t + c)\mathbf{V})e^{i(\mathbf{k}\cdot\mathbf{r} - st)}$, where $\mathbf{V}$ is an eigenvector of the dynamical matrix and $\mathbf{W}$ is a generalized eigenvector[56] of the dynamical matrix in Eq. 13. These solutions grows slowly at short times but decay exponentially at long times if $\Re s > 0$. They may, however be stabilized in the presence of viscosity, such that the system may be stable along the exceptional lines.

CONCLUSION

The study of chiral active matter has increased interest in odd viscosity dramatically in recent years, from pure theoretical work [10, 11, 29, 50] to molecular dynamic simulations [28] and experimental work [7]. So far, and quite surprisingly, there is no evidence that odd terms in the viscosity breaks Onsager relations. In our recent work [10] we proposed a microscopic model for odd viscosity in active matter, which reproduced an odd vis-

cosity that obeys Onsager relations in the total momentum stress tensor. For that purpose we have used the CG total momentum density, which accounts for both the center-of-mass linear momentum density and the angular momentum density. Importantly, when measuring surface forces in a fluid with spinning particles, these must be determined using the stress tensor of the total momentum density rather than the one related to the CM momentum density.

In this paper we directly CG the kinetic energy which results in somewhat different results. The CM stress tensor does not contain any trace of the angular momentum density and therefore has no odd viscosity. It trivially obeys Onsager reciprocal relations. However, the CM momentum density does affect the dynamics of the angular momentum density. This, by itself is a manifestation of non-reciprocity. Indeed, when writing the dynamics of the total momentum density, which includes both CM linear momentum density and angular momentum density, odd viscosity appears together with an odd pressure and two other terms that couple vorticity to shears involving the direction of ℓ . We, therefore, conclude that the mere existence of a non-vanishing angular momentum density breaks Onsager reciprocal relations and gives rise

to odd viscosity. Unlike our previous work [10], using our direct CG approach we find that there are no new excitations in a 3D fluid (or gas) of non-interacting spinning constituents. Instead, a longitudinal wave will always be accompanied by a transverse wave, but not vice versa, thus breaking Onsager reciprocity. To verify the discrepancy between our two CG approaches we have verified our current results using a kinetic theory.

It is expected that in sufficiently dense chiral fluids, interactions will play an increasing role. Although we did not propose a microscopic model for such interactions, recent work [28, 31] suggests that these exist and give rise to another odd viscosity that obeys Onsager reciprocal relations. Considering such an effect together with the “kinetic odd viscosity” gives rise to a non-Hermitian dynamical matrix. Analysing the excitation spectrum of this matrix in 3D reveals regions in which waves propagate (as we found in [10]) and regions that are linearly unstable, suggesting an emergence of a new inhomogeneous phase. Surprisingly, the plains that separates these regions are densely packed surfaces of exceptional points, which may indicate the existence of a non-reciprocal phase transition [35]. As there is no spontaneous symmetry breaking in our system and also because ℓ breaks both chirality and reciprocity simultaneously, our system does not seem to precisely fit within the framework of [35]. To further study the transition and classify it, one must go beyond the linearisation scheme we used to obtain the excitation spectrum. We intend to investigate this in future work.

In a broader perspective, our results highlight the significance of using the total momentum density in the study of chiral active materials where angular momentum density does not vanish. Without accounting for the stress caused by the spinning of the molecules one cannot resolve the true forces that acts on the system boundaries.

MATERIALS AND METHODS

Coarse-graining according to Paul C. Martin

The PB formalism requires the knowledge of the CG Hamiltonian. Since PB are strictly mechanistic, they do not require any ensemble average or the introduction of temperature. Here, we instead perform a strictly mechanical average. In most of literature smoothing $\hat{\mathcal{O}} = \sum_{\alpha} \mathcal{O}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha})$ is done as follows (see, e.g., [57]):

$$\mathcal{O}(\mathbf{r}) \equiv [\hat{\mathcal{O}}(\mathbf{r})]_c = \int d\mathbf{r}' W(\mathbf{r} - \mathbf{r}') \hat{\mathcal{O}}(\mathbf{r}') = \sum_{\alpha} \mathcal{O}^{\alpha} W(\mathbf{r} - \mathbf{r}^{\alpha}),$$

where W is a smooth function such as a Gaussian or a Heaviside function that obeys $\int d\mathbf{r} W(\mathbf{r}) = 1$.

We, however, are not interested in this homogenization procedure (which we refer to as usual), but rather in

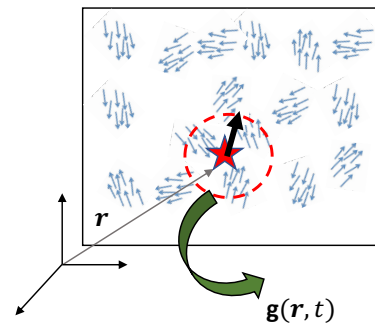


FIG. 3. Illustration of coarse-graining the momentum density at point $\mathbf{r}$ in time t . Small arrows are the momentum of various molecules in the system. The large thick arrow shows the average momentum within the CG volume, which is marked by the red dashed circle.

performing spatial averages within cells of volume ΔV , see Fig. 3. In this process of averaging we disregard any intra-cell information. This intra-cell information, is the cause of (thermal) fluctuations, which can be added to our hydrodynamic equations [58]. We discuss this briefly in the end of the kinetic theory derivation below. Our aim is to approximate, within such CG volume, functions of the form:

$$\hat{\mathcal{O}}(\mathbf{r}) = \sum_{\alpha} \mathbf{A}^{\alpha} \mathbf{B}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}), \quad (15)$$

using the CG fields $\mathbf{A}(\mathbf{r}) = \sum_{\alpha} \mathbf{A}^{\alpha} W(\mathbf{r} - \mathbf{r}^{\alpha})$ and $\mathbf{B}(\mathbf{r}) = \sum_{\alpha} \mathbf{B}^{\alpha} W(\mathbf{r} - \mathbf{r}^{\alpha})$. To do so we follow unpublished notes by Paul C. Martin and replace $\mathbf{B}^{\alpha}$ (or $\mathbf{A}^{\alpha}$) with its average within the CG volume, $\bar{\mathbf{B}}(\mathbf{r}) = \mathbf{B}(\mathbf{r})/n(\mathbf{r})$, where $n(\mathbf{r}) \equiv \sum_{\alpha} W(\mathbf{r} - \mathbf{r}^{\alpha})$. With this we can write,

$$\mathcal{O}(\mathbf{r}) \simeq \bar{\mathbf{B}}(\mathbf{r}) \sum_{\alpha} \mathbf{A}^{\alpha} W(\mathbf{r} - \mathbf{r}^{\alpha}) = \frac{\mathbf{A}(\mathbf{r}) \mathbf{B}(\mathbf{r})}{n(\mathbf{r})}. \quad (16)$$

In the SI [47] we provide simple examples and some extension of this method.

Interestingly, if one chooses cells that are of the size of the particles, such that only one particle can be within each cell, an exact relation for the the microscopic fields $\hat{\mathcal{O}}(\mathbf{r})$ can be obtained [48]:

$$\hat{\mathcal{O}}(\mathbf{r}) = \sum_{\alpha} \mathbf{A}^{\alpha} \mathbf{B}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) \frac{\sum_{\beta} \delta(\mathbf{r} - \mathbf{r}^{\beta})}{\sum_{\gamma} \delta(\mathbf{r} - \mathbf{r}^{\gamma})} \quad (17)$$

$$= \frac{1}{\hat{n}(\mathbf{r})} \sum_{\alpha, \beta} \mathbf{A}^{\alpha} \mathbf{B}^{\beta} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) \delta(\mathbf{r} - \mathbf{r}^{\beta}) \quad (18)$$

$$= \frac{\hat{\mathbf{A}}(\mathbf{r}) \hat{\mathbf{B}}(\mathbf{r})}{\hat{n}(\mathbf{r})}, \quad (19)$$

where $\hat{\mathcal{O}}(\mathbf{r})$ denotes a microscopic field. The second line of (19) is exact because no two particles occupy the same

cell. In fact, using this equation we can write the Hamiltonian of (3) in terms of the microscopic fields and calculate the microscopic PBs thus getting the reactive part of the dynamics in terms of the microscopic fields. This microscopic dynamics assumes exactly the same form as Eq. (7a-7c) (but with the microscopic fields $\hat{\mathcal{O}}_i$), such that odd viscosity is present even in this microscopic formulation. In the microscopic formulation there is, however, no dissipation (in contrast to (8a)) and $F[\rho]$ of (6) can no longer be thought of as free energy and it must be written in terms of the microscopic fields (see, e.g., F in the kinetic theory derivation below).

Poisson-Bracket for fields

In the main text we use the PB formalism for fields [10, 44, 54, 59] to derive the reactive part of the dynamics for $\mathbf{g}$, ℓ and ρ . In this section we provide a brief reminder of this method. We are interested in the dynamics of the CG total momentum and angular momentum densities. The microscopic fields $\hat{\Phi}_\mu(\{\mathbf{q}_i^\alpha\}, \{\boldsymbol{\pi}_i^\alpha\}, t)$ are then coarse-grained to give mesoscopic fields $\Phi_\mu(\mathbf{r}, t) = [\hat{\Phi}_\mu(\{\mathbf{q}_i^\alpha\}, \{\boldsymbol{\pi}_i^\alpha\}, t)]_c$ (the definition of $[\hat{\mathcal{O}}]_c$ can be found in the previous section), and the statistical mechanics of the latter is determined by the CG Hamiltonian $H[\{\Phi_\mu\}]$. The reactive (non-dissipative) part of the dynamics of the CG fields is found by using [10, 44]

$$\left. \frac{\partial \Phi_\mu(\mathbf{r}, t)}{\partial t} \right|_{\text{reactive}} = - \int d\mathbf{r}' \{ \Phi_\mu(\mathbf{r}), \Phi_\nu(\mathbf{r}') \} \frac{\delta H}{\delta \Phi_\nu(\mathbf{r}')} \quad (20)$$

where $\{ \Phi_\mu(\mathbf{r}), \Phi_\nu(\mathbf{r}') \} = [\{ \hat{\Phi}_\mu(\mathbf{r}), \hat{\Phi}_\nu(\mathbf{r}') \}]_c$ and

$$\{ \hat{\Phi}_\mu(\mathbf{r}), \hat{\Phi}_\nu(\mathbf{r}') \} = \sum_{\alpha i} \left[\frac{\partial \hat{\Phi}_\mu(\mathbf{r})}{\partial \pi_i^\alpha} \frac{\partial \hat{\Phi}_\nu(\mathbf{r}')}{\partial q_i^\alpha} - \frac{\partial \hat{\Phi}_\mu(\mathbf{r})}{\partial q_i^\alpha} \frac{\partial \hat{\Phi}_\nu(\mathbf{r}')}{\partial \pi_i^\alpha} \right] \quad (21)$$

However, unlike the microscopic dynamics, PBs do not produce the complete mesoscopic dynamics. The CG procedure neglects many degrees of freedom, which appear as an additional dissipative term in the CG dynamics

$$\left. \frac{\partial \Phi_\mu(\mathbf{r}, t)}{\partial t} \right|_{\text{dissipative}} = - \int d\mathbf{r}' \Gamma_{\mu\nu}(\mathbf{r}, \mathbf{r}') \frac{\delta H}{\delta \Phi_\nu(\mathbf{r}')} \quad (22)$$

The dissipative tensor Γ is symmetric and positive semi-definite. It is generally a function of all fields $\{\Phi_\mu\}$, and following Curie's symmetry principle it must obey the system symmetries. Moreover, Φ_μ is dissipatively coupled to Φ_ν only if they have the same sign under time reversal. This is the signature of dissipation where the flux $\partial \Phi_\mu / \partial t$ has the opposite sign under time reversal from its conjugate force $\delta H / \delta \Phi_\nu$.

For our specific case we use the following fundamental PB (all other PB vanish) and (20) to derive the dynamics

of the CM momentum:

$$\{ \ell_i(\mathbf{r}), \ell_j(\mathbf{r}') \} = - \varepsilon_{ijk} \ell_k(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \quad (23a)$$

$$\{ g_i^c(\mathbf{r}), \ell_j(\mathbf{r}') \} = - \nabla'_i [\ell_j(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}')] \quad (23b)$$

$$\{ \ell_i(\mathbf{r}), g_j^c(\mathbf{r}') \} = \ell_i(\mathbf{r}') \nabla_j \delta(\mathbf{r} - \mathbf{r}') \quad (23c)$$

$$\{ g_i^c(\mathbf{r}), \rho(\mathbf{r}') \} = \rho(\mathbf{r}) \nabla_i \delta(\mathbf{r}' - \mathbf{r}) \quad (23d)$$

Kinetic theory derivation

The kinetic derivation is essentially the same as the Irving-Kirkwood derivation of the equations of hydrodynamics [60], which uses the Liouville equation for the probability density and ensemble average. The only difference between the approaches lies in the type of averaging. In the Irving-Kirkwood approach an ensemble average is used, while in the kinetic theory derivation [57] an average over some small volume (coarse-graining) is used. If the coarse-grained volume contains a large number of particles these two averages coincide.

We start from the microscopic equation for the total momentum density (see (1)) and ignore the $\hat{\mathbf{Q}}$ term that will vanish once the equation is coarse-grained:

$$\hat{\mathbf{g}}(\mathbf{r}) \simeq \sum_\alpha \left[\mathbf{p}^\alpha + \frac{1}{2} \nabla \times \ell^\alpha \right] \delta(\mathbf{r} - \mathbf{r}^\alpha) \quad (24)$$

where as before $\mathbf{p}^\alpha = \sum_\mu \mathbf{p}^{\alpha\mu}$ and $\ell^\alpha = I \boldsymbol{\nu}^\alpha \times \dot{\boldsymbol{\nu}}^\alpha$. Assuming the particles are only interacting via two-body central forces (this is of course not general – we expect interactions involving mutual torques will lead to another odd viscosity term on their own [28]) we have $\dot{\mathbf{p}}^{c,\alpha} = - \nabla_\alpha U(\{\mathbf{r}^\beta\}) + \mathbf{f}^\alpha$ and $\dot{\ell}^\alpha = \mathbf{T}^\alpha$ such that:

$$\partial_t \hat{g}_i = \sum_\alpha \left[f_i^\alpha - \nabla_\alpha U(\{\mathbf{r}^\beta\}) - p_i^\alpha v_j^\alpha \nabla_j \right. \quad (25)$$

$$\left. - \frac{1}{2} \varepsilon_{ikn} \ell_n^\alpha v_j^\alpha \nabla_j \nabla_k + \frac{1}{2} \varepsilon_{ijk} T_k^\alpha \nabla_j \right] \delta(\mathbf{r} - \mathbf{r}^\alpha) \quad (26)$$

Here $U(\{\mathbf{r}^\beta\}) = \frac{1}{2} \sum_{\alpha \neq \beta} \phi(|\mathbf{r}^\alpha - \mathbf{r}^\beta|)$ is a general two-body potential. Coarse-graining using (16) and using (2) we get

$$\dot{g}_i + \nabla_j (g_i v_j) = f_i - \nabla_i P \quad (27)$$

$$+ \nabla_j \left[- \frac{1}{2} \ell_n (\varepsilon_{iln} \delta_{jk} + \varepsilon_{jln} \delta_{ik}) \nabla_l v_k + \frac{1}{2} \varepsilon_{ijk} \tau_k \right] \quad (28)$$

with $\mathbf{f} \equiv \sum_\alpha \mathbf{f}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$ and $\boldsymbol{\tau}(\mathbf{r}) \equiv \sum_\alpha \mathbf{T}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$ being the external force and torque densities, respectively, and $F = \int d\mathbf{r} d\mathbf{r}' [\rho(\mathbf{r}) \phi(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}')]$. The pressure is as in the main text, $P = \rho \frac{\delta F}{\delta \rho} - \mathcal{F}$, and we have dropped a non hydrodynamic term $\sim \nabla (\nabla \times \ell)^2$. This result can be shown to be equivalent to that obtained in the main text from the PB approach for the standard kinetic energy (see SI [47]).

Note that within the kinetic derivation above, the thermal pressure term is absent (the pressure above is a result of particle interactions). In some literature (e.g., [57]) the kinetic part of the stress (which is the thermal pressure) is written in a way similar to that of the Irving-Kirkwood kinetic stress [60], $\sigma_{ij}^K = -\sum_{\alpha} m^{\alpha} \left(\frac{\mathbf{p}^{\alpha}}{m^{\alpha}} - \mathbf{v} \right)_i \left(\frac{\mathbf{p}^{\alpha}}{m^{\alpha}} - \mathbf{v} \right)_j W(\mathbf{r} - \mathbf{r}^{\alpha})$. We use the fluctuating hydrodynamics approach [58, 59] in which any deviation from the average motion appears as thermal noise. In such an approach, σ^K is absent before any ensemble average. It does appear when considering the ensemble average of the dynamics. By writing $\mathbf{v} = \langle \mathbf{v} \rangle + \delta \mathbf{v}$ and $\rho = \langle \rho \rangle + \delta \rho$ (where $\langle \dots \rangle$ denotes average over noise realizations), the streaming term becomes $\nabla_j (g_i v_j) = \nabla_j (\langle \rho \rangle \langle v_i \rangle \langle v_j \rangle + \langle \rho \rangle \langle \delta v_i \delta v_j \rangle)$, where the second term gives the ideal gas pressure as desired. Note also that by adding the ideal gas entropy $-T \int d\mathbf{r} [n \log n - n]$ to F (thus making it a free-energy rather than just internal energy) the dynamics obtained (without the addition of noise) are the same as after the thermal averaging discussed above.

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Supplementary Material

Non reciprocal odd viscosity: Coarse graining the kinetic energy and exceptional instability

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I. TOTAL MOMENTUM AND KINETIC ENERGY OF A FLUID OF DUMBBELLS

In this section we detail the derivation of Eqs. (1) and (4) of the main text. Considering a fluid of dumbbells as described in the main text, the momentum of the two point masses can be written in terms of the CM momentum, $\mathbf{p}^\alpha$, and $\boldsymbol{\nu}^\alpha$ as $\mathbf{p}_{1,2}^\alpha = \frac{1}{2}\mathbf{p}^\alpha \pm m a \boldsymbol{\nu}^\alpha$, such that the total momentum density for the diatomic fluid is

$$\begin{aligned}\hat{g}_i(\mathbf{r}) &= \sum_{\alpha\mu} p_i^{\alpha\mu} \delta(\mathbf{r} - \mathbf{r}^{\alpha\mu}) \\ &= \sum_{\alpha} \left[p_i^{\alpha,1} \delta(\mathbf{r} - \mathbf{r}^\alpha - a\boldsymbol{\nu}^\alpha) + p_i^{\alpha,2} \delta(\mathbf{r} - \mathbf{r}^\alpha + a\boldsymbol{\nu}^\alpha) \right] \\ &\simeq \sum_{\alpha} \left[p_i^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha) - I \dot{\nu}_i^\alpha \nu_j^\alpha \nabla_j \delta(\mathbf{r} - \mathbf{r}^\alpha) \right],\end{aligned}\quad (1)$$

where we have used $\mathbf{r}^{\alpha,1} - \mathbf{r}^{\alpha,2} = 2a\boldsymbol{\nu}^\alpha$, and the moment of inertia of each molecule is $I = Ma^2$ with $M = 2m$ being the molecule mass. As we are interested in the long wavelength limit, $a \ll |\mathbf{r} - \mathbf{r}^\alpha|$, we only keep terms $\mathcal{O}(a^2)$. It is useful to write $\hat{\mathbf{g}} = \hat{\mathbf{g}}^c + \hat{\mathbf{g}}^s$, with $\hat{\mathbf{g}}^c \equiv \sum_{\alpha} \mathbf{p}^\alpha \delta(\mathbf{r} - \mathbf{r}^\alpha)$ and

$$\hat{g}_i^s(\mathbf{r}) = -\frac{I}{2} \left[\varepsilon_{ijk} \varepsilon_{klm} \sum_{\alpha} \dot{\nu}_l^\alpha \nu_m^\alpha \nabla_j \delta(\mathbf{r} - \mathbf{r}^\alpha) + \sum_{\alpha} \dot{Q}_{ij}^\alpha \nabla_j \delta(\mathbf{r} - \mathbf{r}^\alpha) \right]. \quad (2)$$

Here $Q_{ij}^\alpha = \nu_i^\alpha \nu_j^\alpha - \delta_{ij}/d$ is the alignment tensor and d is the number of dimensions. The first term of Eq. (2) includes the angular momentum of each molecule, $\boldsymbol{\ell}^\alpha = I \boldsymbol{\nu}^\alpha \times \dot{\boldsymbol{\nu}}^\alpha$. By using the definition of the angular velocity of each molecule, $\boldsymbol{\Omega}^\alpha \equiv \boldsymbol{\nu}^\alpha \times \dot{\boldsymbol{\nu}}^\alpha$, one gets $\boldsymbol{\ell}^\alpha = I \boldsymbol{\Omega}^\alpha$. This is Eq. (1) of the main text.

We continue by directly coarse-graining the kinetic Hamiltonian:

$$H_k = \sum_{\alpha\mu} \frac{(\mathbf{p}^{\alpha\mu})^2}{2m^{\alpha\mu}}. \quad (3)$$

Defining the kinetic energy density $\mathcal{H}$ such that $H_k = \int d\mathbf{r} \mathcal{H}_k$ we have

$$\begin{aligned}\mathcal{H}_k(\mathbf{r}) &= \frac{1}{2m} \sum_{\alpha} \left[\left(\frac{1}{2}\mathbf{p}^\alpha + m a \dot{\boldsymbol{\nu}}^\alpha \right)^2 \delta(\mathbf{r} - \mathbf{r}^\alpha - a\boldsymbol{\nu}^\alpha) + \left(\frac{1}{2}\mathbf{p}^\alpha - m a \dot{\boldsymbol{\nu}}^\alpha \right)^2 \delta(\mathbf{r} - \mathbf{r}^\alpha + a\boldsymbol{\nu}^\alpha) \right] \\ &\simeq \sum_{\alpha} \left[\frac{\mathbf{p}_\alpha^2}{2M} + \frac{1}{2} I \dot{\boldsymbol{\nu}}_\alpha^2 - \frac{I}{M} (\dot{\boldsymbol{\nu}}^\alpha \cdot \mathbf{p}^\alpha) (\boldsymbol{\nu}^\alpha \cdot \nabla) \right] \delta(\mathbf{r} - \mathbf{r}^\alpha).\end{aligned}\quad (4)$$

We now use the coarse-graining method described in Materials and Methods to write Eq. (4) in terms of the various fields. Let us start with the first term of Eq. (4), which after coarse-graining is written as:

$$\sum_{\alpha} \frac{\mathbf{p}_\alpha^2}{2M} \delta(\mathbf{r} - \mathbf{r}^\alpha) = \frac{(\mathbf{g}^c)^2}{2\rho}, \quad (5)$$

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where $\rho(\mathbf{r}) = Mn(\mathbf{r})$ and $\hat{n}(\mathbf{r}) = \sum_{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha})$. Noting that the relations $\boldsymbol{\Omega}^{\alpha} = \boldsymbol{\nu}^{\alpha} \times \dot{\boldsymbol{\nu}}^{\alpha}$ and $\dot{\boldsymbol{\nu}}^{\alpha} \cdot \boldsymbol{\nu}^{\alpha} = 0$ implies that $(\dot{\boldsymbol{\nu}}^{\alpha})^2 = (\boldsymbol{\Omega}^{\alpha})^2$, coarse-graining the second term of Eq. (4) yields

$$\sum_{\alpha} \frac{1}{2} I \dot{\boldsymbol{\nu}}_{\alpha}^2 \delta(\mathbf{r} - \mathbf{r}^{\alpha}) = \sum_{\alpha} \frac{(\boldsymbol{\ell}^{\alpha})^2}{2I} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) = \frac{\boldsymbol{\ell}^2}{2\tilde{I}\rho}. \quad (6)$$

The third term of Eq. (4) can be written as

$$\begin{aligned} -\nabla_j \sum_{\alpha} \tilde{I} \dot{\nu}_i^{\alpha} \nu_j^{\alpha} p_i^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) &= -\frac{1}{2} \nabla_j \sum_{\alpha} \tilde{I} \left[(\dot{\nu}_i^{\alpha} \nu_j^{\alpha} + \dot{\nu}_j^{\alpha} \nu_i^{\alpha}) + \varepsilon_{ijk} \varepsilon_{kmn} \dot{\nu}_m^{\alpha} \nu_n^{\alpha} \right] p_i^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) \\ &= -\frac{1}{2} \nabla_j \sum_{\alpha} \left(\tilde{I} \dot{Q}_{ij}^{\alpha} - \frac{1}{M} \varepsilon_{ijk} \ell_k^{\alpha} \right) p_i^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}), \end{aligned} \quad (7)$$

which after coarse-graining reads

$$-\nabla_j \sum_{\alpha} \tilde{I} \dot{\nu}_i^{\alpha} \nu_j^{\alpha} p_i^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) = \frac{1}{2} \nabla \cdot (\boldsymbol{\ell} \times \mathbf{v} - \mathbf{v} \cdot \mathbf{A}), \quad (8)$$

such that the kinetic energy is the one of Eq. (4) of the main text:

$$H_k = \int d\mathbf{r} \left[\frac{(\mathbf{g}^c)^2}{2\rho} + \frac{\boldsymbol{\ell}^2}{2I} + \frac{1}{2} \nabla \cdot (\boldsymbol{\ell} \times \mathbf{v} - \mathbf{v} \cdot \mathbf{A}) \right]. \quad (9)$$

II. COARSE-GRAINED KINETIC ENERGY FOR GENERAL COMPLEX MOLECULES

Here we extend the model described in the main text to account for a fluid of general complex (rigid) molecules (rather than dumbbells), each of which composed of multiple sub-particles (atoms) with mass $m^{\alpha\mu}$ and momentum $\mathbf{p}^{\alpha\mu}$ located at $\mathbf{r}^{\alpha\mu}$. As already shown in the supplementary of [1] the CG total momentum for such fluid is $\mathbf{g} = \mathbf{g}^c + \frac{1}{2} \nabla \times \boldsymbol{\ell} + \nabla \cdot \mathbf{A}$, where the definitions of $\boldsymbol{\ell}$ and $\mathbf{A}$ are similar to those in the main text of the current paper. The angular momentum density is $\hat{\boldsymbol{\ell}}(\mathbf{r}) = \sum_{\alpha} \boldsymbol{\ell}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha})$ with

$$\ell_i^{\alpha} = \varepsilon_{ijk} \sum_{\mu} \Delta p_k^{\alpha\mu} \Delta r_j^{\alpha\mu}, \quad (10)$$

where $\Delta \mathbf{r}^{\alpha\mu} \equiv \mathbf{r}^{\alpha\mu} - \mathbf{r}^{\alpha}$ and $\Delta \mathbf{p}^{\alpha\mu} = m^{\alpha\mu} \partial_t \Delta \mathbf{r}^{\alpha\mu}$. The CM momentum $\mathbf{p}^{\alpha} \equiv \sum_{\mu} \mathbf{p}^{\alpha\mu}$ such that $\hat{\mathbf{g}}^c(\mathbf{r}) \equiv \sum_{\alpha} \mathbf{p}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha})$. We further define $\hat{\mathbf{K}}(\mathbf{r}) = \sum_{\alpha} \mathbf{K}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha})$, which is a generalization of the alignment tensor and should be thought of as an order parameter (see supplementary of [1]) with $\mathbf{K}^{\alpha} = \frac{3}{2} (\mathbf{R}^{\alpha} + \frac{1}{2} \mathbf{I}^{\alpha})$ and [2]

$$R_{ij}^{\alpha} = \sum_{\mu} m^{\alpha\mu} \left[\Delta r_i^{\alpha\mu} \Delta r_j^{\alpha\mu} - \frac{1}{3} (\Delta \mathbf{r}^{\alpha\mu})^2 \delta_{ij} \right], \quad (11)$$

$$I_{ij}^{\alpha} = \sum_{\mu} m^{\alpha\mu} \left[(\Delta \mathbf{r}^{\alpha\mu})^2 \delta_{ij} - \Delta r_i^{\alpha\mu} \Delta r_j^{\alpha\mu} \right]. \quad (12)$$

Then, $\hat{\mathbf{A}}(\mathbf{r}) = \frac{1}{2} \sum_{\alpha} \dot{\mathbf{K}}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha})$.

Notably, the fundamental PBs (see Eq. (22) in Materials and Methods) are not modified, hence, the only thing left to find is the form of the Hamiltonian, which we will show to have the same form as in Eq. (4) of the main text such that the dynamics we use in the main text applies also for a fluid composed of general complex molecules. We begin with the Hamiltonian of Eq. (3) where now the kinetic energy density $\mathcal{H}_k$, defined as before via $H_k = \int d\mathbf{r} \mathcal{H}_k$ is:

$$\mathcal{H}_k(\mathbf{r}) = \sum_{\alpha} \frac{1}{2m^{\alpha\mu}} \left(\frac{m^{\alpha\mu}}{M^{\alpha}} \mathbf{p}^{\alpha} + \Delta \mathbf{p}^{\alpha\mu} \right)^2 \delta(\mathbf{r} - \mathbf{r}^{\alpha} - \Delta \mathbf{r}^{\alpha\mu}). \quad (13)$$

Here $\mathbf{p}^{\alpha\mu} = \mathbf{p}^{\alpha} (m^{\alpha\mu}/M^{\alpha}) + \Delta \mathbf{p}^{\alpha\mu}$ and $M^{\alpha} = \sum_{\mu} m^{\alpha\mu}$ is the molecule mass. Assuming the molecules are small compares to typical distances (long wavelength limit), $|\Delta \mathbf{r}^{\alpha\mu}| \ll |\mathbf{r} - \mathbf{r}^{\alpha}|$ we write $\delta(\mathbf{r} - \mathbf{r}^{\alpha} - \Delta \mathbf{r}^{\alpha\mu}) \simeq \delta(\mathbf{r} - \mathbf{r}^{\alpha}) - \Delta \mathbf{r}^{\alpha\mu} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}^{\alpha})$ such that Eq. (13) becomes

$$\mathcal{H}_k(\mathbf{r}) = \sum_{\alpha} \frac{(\mathbf{p}^{\alpha})^2}{2M^{\alpha}} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) + \sum_{\alpha\mu} \left[\frac{(\Delta \mathbf{p}^{\alpha\mu})^2}{2m^{\alpha\mu}} - \frac{\mathbf{p}^{\alpha} \cdot \Delta \mathbf{p}^{\alpha\mu}}{2M^{\alpha}} \Delta \mathbf{r}^{\alpha\mu} \cdot \nabla \right] \delta(\mathbf{r} - \mathbf{r}^{\alpha}), \quad (14)$$

where we have used $\sum_{\mu} m^{\alpha\mu} \Delta \mathbf{r}^{\alpha\mu} = \sum_{\mu} \Delta \mathbf{p}^{\alpha\mu} = 0$ and neglected a term $\sim \Delta \mathbf{p}^{\alpha\mu} \Delta \mathbf{r}^{\alpha\mu}$ that is proportional to the square of the molecule size (we only keep linear order). The second term in Eq. (14) can be written in more familiar terms:

$$\sum_{\mu} \frac{(\Delta \mathbf{p}^{\alpha\mu})^2}{2m^{\alpha\mu}} = \frac{1}{2} \ell_i^{\alpha} (I_{ij}^{\alpha})^{-1} \ell_j^{\alpha}, \quad (15)$$

with $\boldsymbol{\Omega}^{\alpha}$ being the angular velocity of a molecule around its CM which for rigid molecules obeys $\partial_t \Delta \mathbf{r}^{\alpha\mu} = -\Delta \mathbf{r}^{\alpha\mu} \times \boldsymbol{\Omega}^{\alpha}$. The third term of Eq. (14) can also be simplified with the help of Eqs. (10)-(11):

$$\sum_{\mu} \frac{\mathbf{p}^{\alpha} \cdot \Delta \mathbf{p}^{\alpha\mu}}{2M^{\alpha}} \Delta \mathbf{r}^{\alpha\mu} = \frac{\mathbf{p}^{\alpha}}{2M^{\alpha}} \left(\dot{K}_{ij}^{\alpha} - \frac{1}{2} \varepsilon_{ijk} \ell_k^{\alpha} \right). \quad (16)$$

Substituting Eqs. (15) and (16) into (14) we get

$$\mathcal{H}_k(\mathbf{r}) = \sum_{\alpha} \left[\frac{(\mathbf{p}^{\alpha})^2}{2M^{\alpha}} + \frac{1}{2} \ell_i^{\alpha} (I_{ij}^{\alpha})^{-1} \ell_j^{\alpha} + \frac{1}{2M^{\alpha}} \nabla \cdot \left(\ell^{\alpha} \times \mathbf{p}^{\alpha} - \mathbf{p}^{\alpha} \cdot \dot{\mathbf{K}}^{\alpha} \right) \right] \delta(\mathbf{r} - \mathbf{r}^{\alpha}). \quad (17)$$

The last step is to CG using the method described in Materials and Methods, which gives a very similar Hamiltonian to the one used in Eq. (4) of the main text,

$$H_k = \int d\mathbf{r} \left[\frac{(\mathbf{g}^c)^2}{2\rho} + \frac{1}{2} \boldsymbol{\ell} \cdot \mathbf{I}^{-1} \cdot \boldsymbol{\ell} + \frac{1}{2} \nabla \cdot (\boldsymbol{\ell} \times \mathbf{v} - \mathbf{v} \cdot \mathbf{A}) \right], \quad (18)$$

where $\mathbf{A}$ here is a generalization of the one in the main text (for diatomic molecules $\mathbf{K}^{\alpha} = \mathbf{Q}^{\alpha}$).

III. TOTAL MOMENTUM DENSITY DYNAMICS

Deriving the dynamics of the total momentum density (Eq. (8) of the main text) from that of the CM momentum density (Eq. (7) of the main text) requires some non-trivial manipulation that we details in this section.

Starting from Eq. (7) of the main text and using Eq. (2) of the main text we can write the dynamics of the total momentum:

$$\begin{aligned} \dot{g}_i &= \dot{g}_i^c + \frac{1}{2} \varepsilon_{ijk} \nabla_j \dot{\ell}_k \\ &= -\nabla_j (v_j g_i) + [\nabla_j (v_j g_i) - \nabla_j (v_j^c g_i^c)] - \nabla_i P - \frac{1}{2} \varepsilon_{kmn} \nabla_j \nabla_m (v_m^c \ell_n) + \frac{1}{2} \varepsilon_{ijk} \nabla_j \tau_k + f_i. \end{aligned} \quad (19)$$

Converting $\nabla_j (g_i^c v_j^c)$ to $\nabla_j (g_i v_j)$ with the help of Eq. (2) of the main text we arrive at

$$\dot{g}_i + \nabla_j (g_i v_j) = f_i - \nabla_i P + \nabla_j \left[-\frac{1}{2} \ell_n (\varepsilon_{iln} \delta_{jk} + \varepsilon_{jln} \delta_{ik}) \nabla_l v_k + \frac{1}{2} \varepsilon_{ijk} \tau_k \right], \quad (20)$$

where we have dropped a non hydrodynamic term $\sim \nabla (\nabla \times \boldsymbol{\ell})^2$, and we have also took advantage of the identity $\nabla_j \varepsilon_{jln} \nabla_l (\ell_n v_i) = \nabla_j \left[v_i (\nabla \times \boldsymbol{\ell})_j + \varepsilon_{jln} \ell_n \nabla_l v_i \right] = 0$. The final step is to separate the velocity gradient tensor to its symmetric and antisymmetric parts. Then, the symmetric part of first term in the square brackets of Eq. (20) gives the odd viscosity, while its antisymmetric part can be written as

$$[\varepsilon_{iln} \delta_{jk} + \varepsilon_{jln} \delta_{ik} - \varepsilon_{ikn} \delta_{jl} - \varepsilon_{jkn} \delta_{il}] \nabla_l v_k = \varepsilon_{lkm} \varepsilon_{mst} (\varepsilon_{iln} \delta_{jk} + \varepsilon_{jln} \delta_{ik}) \nabla_s v_t. \quad (21)$$

We then use

$$\varepsilon_{lkm} \varepsilon_{mst} \varepsilon_{iln} \delta_{jk} \nabla_s v_t = (\varepsilon_{ilk} \delta_{nj} - \varepsilon_{lkn} \delta_{ij}) \nabla_l v_k, \quad (22)$$

to obtain

$$[\varepsilon_{iln} \delta_{jk} + \varepsilon_{jln} \delta_{ik} - \varepsilon_{ikn} \delta_{jl} - \varepsilon_{jkn} \delta_{il}] \nabla_l v_k = (\varepsilon_{ilk} \delta_{nj} + \varepsilon_{jlk} \delta_{ni} - 2\varepsilon_{lkn} \delta_{ij}) \nabla_l v_k, \quad (23)$$

where we also utilized Eq. (22) with interchanging $i \leftrightarrow j$. This directly gives the nondissipative part of Eq. (8) of the main text.

IV. SIMPLE EXAMPLES FOR OUR COARSE-GRAINING PROCESS

To verify the validity of our coarse-graining approach let us apply it to the kinetic energy density of a monoatomic gas

$$\mathcal{H}_k(\mathbf{r}) = \sum_{\alpha} \frac{(\mathbf{p}^{\alpha})^2}{2m} \delta(\mathbf{r} - \mathbf{r}^{\alpha}). \quad (24)$$

This is done straightforwardly by writing $\mathbf{A}^{\alpha} = \mathbf{p}^{\alpha}$ and $\mathbf{B}^{\alpha} = \mathbf{p}^{\alpha}/(2m)$. Then, using Eq. (17) in Materials and Methods the coarse-grained kinetic energy is

$$\epsilon(\mathbf{r}) = \frac{\mathbf{g}(\mathbf{r})^2}{2\rho(\mathbf{r})}, \quad (25)$$

as expected. Here $\rho(\mathbf{r}) = mn(\mathbf{r})$ and $\mathbf{g}(\mathbf{r}) = \sum_{\alpha} \mathbf{p}^{\alpha} W(\mathbf{r} - \mathbf{r}^{\alpha})$.

Note that we can apply our prescription also for functions incorporating more microscopic variables. For example, if the particles masses are not equal

$$\mathcal{H}_k(\mathbf{r}) = \sum_{\alpha} \frac{(\mathbf{p}^{\alpha})^2}{2m^{\alpha}} \delta(\mathbf{r} - \mathbf{r}^{\alpha}). \quad (26)$$

Let us write in general

$$\hat{\mathcal{O}}(\mathbf{r}) = \sum_{\alpha} \mathbf{A}^{\alpha} (\mathbf{C}^{\alpha})^{-1} \mathbf{B}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}). \quad (27)$$

Then, after using $\bar{\mathbf{B}}(\mathbf{r}) = \mathbf{B}(\mathbf{r})/n(\mathbf{r})$ as we do in Materials and Methods, and replacing $(\mathbf{C}^{\alpha})^{-1}$ with $\overline{(\mathbf{C}^{\alpha})^{-1}}$ within the coarse-graining volume:

$$\overline{(\mathbf{C}^{\alpha})^{-1}} \equiv \frac{\int_{\Delta V} d\mathbf{u} \sum_{\alpha} (\mathbf{C}^{\alpha})^{-1} W(\mathbf{r} - \mathbf{u} - \mathbf{r}^{\alpha})}{\int_{\Delta V} d\mathbf{u} \sum_{\alpha} W(\mathbf{r} - \mathbf{u} - \mathbf{r}^{\alpha})} \simeq [\bar{\mathbf{C}}(\mathbf{r})]^{-1}, \quad (28)$$

we find that

$$\mathcal{O}(\mathbf{r}) \simeq \mathbf{A}(\mathbf{r}) [\mathbf{C}(\mathbf{r})]^{-1} \mathbf{B}(\mathbf{r}). \quad (29)$$

For the example of the kinetic energy above this clearly gives $\mathcal{H}_k \simeq \mathbf{g}^2/(2\rho)$ as desired. Note that in Eq. (28) we have used

$$\frac{1}{\Delta V} \int_{\Delta V} d\mathbf{u} \sum_{\alpha} (\mathbf{C}^{\alpha})^{-1} W(\mathbf{r} - \mathbf{u} - \mathbf{r}^{\alpha}) \simeq n(\mathbf{r}) [\bar{\mathbf{C}}(\mathbf{r})]^{-1}. \quad (30)$$

Next we want to examine coarse-graining of functions that include gradients such as

$$\hat{\mathcal{L}}(\mathbf{r}) = \sum_{\alpha} \mathbf{A}^{\alpha} \mathbf{B}^{\alpha} \nabla \delta(\mathbf{r} - \mathbf{r}^{\alpha}). \quad (31)$$

Clearly the integrated value of $\hat{\mathcal{L}}(\mathbf{r})$ must give a boundary term because:

$$\mathbf{L} \equiv \int d\mathbf{r} \hat{\mathcal{L}}(\mathbf{r}) = \int d\mathbf{r} \nabla \left[\sum_{\alpha} \mathbf{A}^{\alpha} \mathbf{B}^{\alpha} \delta(\mathbf{r} - \mathbf{r}^{\alpha}) \right] = \oint_S d\mathbf{S} \sum_{\alpha} \mathbf{A}^{\alpha} \mathbf{B}^{\alpha} \delta(\mathbf{s} - \mathbf{s}^{\alpha}). \quad (32)$$

Within our coarse-graining scheme we have

$$\hat{\mathcal{L}}(\mathbf{r}) \simeq \nabla \left[\frac{\mathbf{A}(\mathbf{r}) \mathbf{B}(\mathbf{r})}{n(\mathbf{r})} \right], \quad (33)$$

which gives a boundary term as it should.

V. EXCEPTIONAL POINTS

Our analysis showed that the non-Hermitian matrices characterizing the excitations of active systems of rotating objects exhibit an exceptional point at which two eigenvalues coalesce and the system appears to “lose” an eigenvalue. In terms of the differential equation we are solving this means we lost a fundamental solution, which is of course not true. In much of the literature (see e.g., Refs. [3–5]) the discussion on exceptional points (EPs) was focused on the EPs marking a breaking of the (generalized) $\mathcal{PT}$ symmetry and non-reciprocal phase transition [3]. Here follow these recent works that explains EPs through the textbook example of a damped harmonic oscillator (Ref. [6] is the most elaborated, but see also the SI of Ref. [3] and Ref. [4]) and explain where is the “lost” fundamental solution.

A. Exceptional points in a damped harmonic oscillator

Let us start with the equation of a one-dimensional damped harmonic oscillator:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0. \quad (34)$$

Here, ω_0 is the natural frequency of the undamped oscillator and $\gamma = \Gamma/m$, the inverse decay time, is the friction coefficient Γ divided by particle mass m . Following standard procedure, we assume a solution of the form $x(t) \propto e^{-\alpha t}$ which leads to

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad (35)$$

with two solutions for α ,

$$\alpha_{\pm} = \frac{1}{2} \left(\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2} \right), \quad (36)$$

and associated eigenfunctions, $\exp(-\alpha_{\pm}t)$. When $\gamma^2 > 4\omega_0^2$, the over-damped regime, the two modes decay exponentially, and when $\gamma^2 < 4\omega_0^2$, both modes oscillate and decay (underdamped regime). When $\gamma^2 = 4\omega_0^2$, there is only one solution, $\alpha = \gamma/2$, and the system is at the exceptional point. Of course even when $\gamma^2 = 4\omega_0^2$, Eq. (34) must have two solutions, the first of which is the simple exponential one just discussed and the second of which can be found by considering the limit as the EP is approached. To this end, we set

$$\alpha_{\pm} = \frac{1}{2}\gamma \pm \sqrt{\Delta}, \quad (37)$$

where $\Delta = (\gamma^2/4) - \omega_0^2$, and we construct two fundamental solutions using the sum and difference of the two eigenfunctions:

$$\psi_1(t) = \frac{1}{2} \left(e^{-\gamma t/2 + \sqrt{\Delta}t} + e^{-\gamma t/2 - \sqrt{\Delta}t} \right), \quad (38)$$

$$\psi_2(t) = \frac{1}{2\sqrt{\Delta}} \left(e^{-\gamma t/2 + \sqrt{\Delta}t} - e^{-\gamma t/2 - \sqrt{\Delta}t} \right). \quad (39)$$

The first fundamental solution, ψ_1 has a well-defined $\Delta \rightarrow 0$ limit that is $e^{-\gamma t/2}$. The second fundamental solution, ψ_2 , is divided by $\sqrt{\Delta}$ to give a proper limit as $\Delta \rightarrow 0$, which is $te^{-\gamma t/2}$. Thus we have two linearly independent solutions at the EP:

$$\psi_1(t) = e^{-\gamma t/2}, \quad \text{and} \quad \psi_2(t) = te^{-\gamma t/2}. \quad (40)$$

We can then write a general solution to Eq. (34) (at the EP) for some initial condition $x(t=0) = x_0$ as $x(t) = a\psi_1(t) + b\psi_2(t)$, where a and b are determined from the initial condition.

Though linearly independent, these ψ_1 and ψ_2 are not orthogonal because $\int_0^\infty dt \psi_1(t) \psi_2(t) = \gamma^{-2}$. Although not necessary, one can create an orthogonal pair by adding to $\psi_2(t)$ a term proportional to ψ_1 . A trivial exercise establishes that the function $\tilde{\psi}_2 = (1 - \gamma t)e^{-\gamma t/2} = \psi_1 - \gamma\psi_2$ is orthogonal to ψ_1 .

B. Exceptional points using matrix representation

The damped oscillator equation can be couched in terms of a matrix operating on the vector $\mathbf{u}(t) = [x(t), v(t) = \dot{x}(t)]^T$, which may be more familiar to those who study exceptional points. The equation of motion for $\mathbf{u}(t)$ is

$$\dot{\mathbf{u}} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{pmatrix} \mathbf{u} \equiv \mathbf{M} \mathbf{u} . \quad (41)$$

$\mathbf{M}$ has two eigenvectors, $\mathbf{u}^\pm(t) = [1, -\alpha_\pm]^T$, associated with the two eigenvalues $\alpha_\pm$. At the EP, where $\Delta = 0$, the matrix $\mathbf{M}$ is defective (not diagonalizable) and has only one eigenvector $\mathbf{u}_1 = [1, -\gamma/2]^T$. It does not have a set of linearly independent eigenvectors that form a basis.

Nevertheless, in general, for any $n \times n$ matrix one can form a basis using *generalized eigenvectors* generated by Jordan chains. For a generic matrix $\mathbf{M}$ and eigenvalue λ a generalized eigenvector of rank m is defined as

$$[\mathbf{M} - \lambda \mathbf{I}]^m \mathbf{v}_m = 0 \quad ; \quad [\mathbf{M} - \lambda \mathbf{I}]^{m-1} \mathbf{v}_m \neq 0 , \quad (42)$$

where $\mathbf{v}_1$ (a generalized eigenvector of rank 1) is clearly an ordinary eigenvector. Once one knows a generalized eigenvector $\mathbf{v}_m$ the Jordan chain of generalized eigenvectors can be found using $[\mathbf{M} - \lambda \mathbf{I}] \mathbf{v}_{m+1} = \mathbf{v}_m$. The generalized eigenvectors can be used as a generalized modal matrix $\mathbf{P}$ (a matrix in which the columns are the generalized eigenvectors) to find its Jordan normal form [6] using $\mathbf{J} = \mathbf{P}^{-1} \mathbf{M} \mathbf{P}$. The Jordan normal form is a block diagonal matrix, where each block is a Jordan block - it has the eigenvalues on the diagonal, ones on the superdiagonal and zeros elsewhere. This form simplifies matrix functions (similarly to matrix functions on diagonal matrices).

In our example we solve $[\mathbf{M} - \lambda \mathbf{I}] \mathbf{u}_2 = \mathbf{u}_1$ for $\lambda = \gamma/2$ and find the generalized eigenvector $\mathbf{u}_2 = [0, 1]^T$. With the help of the modal matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -\gamma/2 & 1 \end{pmatrix} , \quad (43)$$

we write $\mathbf{M}$ in Jordan normal form

$$\mathbf{J} = \mathbf{P}^{-1} \mathbf{M} \mathbf{P} = \begin{pmatrix} -\gamma/2 & 1 \\ 0 & -\gamma/2 \end{pmatrix} . \quad (44)$$

This matrix is a Jordan block of size two, hence, the EP is of order two (in general a Jordan block of size m is related to an EP of order m [3]).

Notice that the generalized eigenvectors we have found $\mathbf{u}_1$ and $\mathbf{u}_2$ are not orthogonal. It is, however, straightforward to find an orthogonal generalized eigenvector because $\tilde{\mathbf{u}}_2 = \mathbf{u}_2 + c\mathbf{u}_1$ (c being a scalar) is also a generalized eigenvector, and for $c = -\mathbf{u}_1 \cdot \mathbf{u}_2 / \mathbf{u}_1^2$ we get that $\mathbf{u}_1 \cdot \tilde{\mathbf{u}}_2 = 0$. Now that we know the generalized eigenvectors, we can write the formal solution to Eq. (41):

$$\mathbf{u}(t) = \exp(\mathbf{M}t) \mathbf{u}_0 , \quad (45)$$

where $\mathbf{u}_0 = \mathbf{u}(t=0)$, and since $\mathbf{u}_1$ and $\tilde{\mathbf{u}}_2$ span the space we can write $\mathbf{u}_0 = a\mathbf{u}_1 + b\tilde{\mathbf{u}}_2$. Next, we wish to be able to write the general solution to Eq. (41) as in Sec. V A. To do so we calculate

$$\begin{aligned} \exp(\mathbf{M}t) \mathbf{u}_1 &= \sum_{n=0}^{\infty} \frac{\mathbf{M}^n}{n!} t^n \mathbf{u}_1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} t^n \mathbf{u}_1 = e^{\lambda t} \mathbf{u}_1 , \\ \exp(\mathbf{M}t) \mathbf{u}_2 &= \mathbf{u}_2 + \sum_{n=1}^{\infty} \frac{\mathbf{M}^{n-1}}{n!} t^n (\mathbf{u}_1 + \lambda \mathbf{u}_2) = e^{\lambda t} \mathbf{u}_2 + \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} t^n n \mathbf{u}_1 = e^{\lambda t} (\mathbf{u}_2 + t \mathbf{u}_1) , \end{aligned} \quad (46)$$

and find that $\mathbf{u}(t) = a\psi_1(t)\mathbf{u}_1 + b[\psi_1(t)\tilde{\mathbf{u}}_2 + \psi_2(t)\mathbf{u}_1]$, with $\psi_1(t)\mathbf{u}_1$ being an eigenfunction (the usual normal mode) and $\mathbf{w}(t) = \psi_1(t)\tilde{\mathbf{u}}_2 + \psi_2(t)\mathbf{u}_1$ is a generalized eigenfunction. This solution is very similar to what we found in Sec. V A, where the generalized eigenfunctions are composed of ψ_1 and ψ_2 , the two linearly independent solutions.

To conclude, the generalized eigenvectors generate the linearly independent solutions of the differential equations as they should. Notably, without using the generalized eigenvectors one cannot construct a general solution for the differential equation (in our case, Eq. (41)).

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