# CUTOFF AND METASTABILITY FOR THE HOMOGENEOUS BLOCK MEAN-FIELD POTTS MODEL

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ABSTRACT. In this paper, we study the Potts model on a general graph whose vertices are partitioned into $m$ blocks of the same size. The interaction between two spins depends only on whether they belong to the same block or not. We denote the interaction coefficients for the spins at the different blocks and the same block by $a, b$, respectively. We show that $J = (b + (m-1)a)/m$ plays the role of the effective interaction coefficient of the overall system. In [10], for Curie-Weiss-Potts model with $a = 0,\ b = 1$, and $m = 1$, critical phase transition appears at the inverse-temperature $\beta_c$, and mixing transition occurs at $\beta_s$. Generalizing this fact to our model, we explicitly find the critical inverse temperatures $\beta_c/J,\ \beta_s/J$ for the phase transition and mixing, respectively. Enhancing the aggregate path coupling, we prove in a high-temperature regime $\beta < \beta_s/J$, a cutoff occurs at time $[2(1 - 2\beta J/q)]^{-1} n \log n$ with window size $n$. Moreover, metastability deduces the exponential mixing in the low-temperature regime $\beta > \beta_s/J$. These results first show the cutoff of the block Potts model and suggest a novel extension of the aggregate path coupling.

## 1. Introduction

Curie-Weiss models are fundamental models in statistical physics explaining the spin system. Various mathematical analyses on these models occurred in the perspective of probability language [31]. Many mathematicians researched the dynamic phase transitions of these models in [30, 9, 10, 35, 25, 18, 24, 32, 13, 14, 1, 16, 4, 28, 8]. Also, there are common exciting phenomena in these spin systems. The first is a cutoff phenomenon in a high-temperature regime. The cutoff is the asymptotically abrupt convergence of a dynamic system to its equilibrium state at some specified time (the cutoff time) with negligible error time (the cutoff window). It happens when the temperature is sufficiently high, and enough mixing can be achieved at the equilibrium of the state. With varied conditions of statistical models, the cutoff phenomenon was researched in [30, 10, 35, 34, 33]. In contrast, metastability occurs when the temperature is sufficiently low. It is a phenomenon due to the appearance of multiple equilibrium macrostates, which are local minima points of the free energy functional. The states tend to stay near these local extreme points. So its exit time is exponential, or it has slow mixing. Metastability was studied in [30, 12, 10, 35, 20, 3, 7, 2, 14, 5, 23, 6, 22, 19].

Starting with the most straightforward spin system, the Ising model was first understood on the complete graphs in [30]. When the inverse-temperature $\beta$ is given, [30] investigates asymptotic analysis on mixing time as the total number $n$ of vertices in the given graph goes to $\infty$. There is a cutoff at time $O(n \log n)$ with window size $n$ in high-temperature regime $\beta < 1$, polynomial mixing of order $n^{3/2}$ in critical regime $\beta = 1$, and slow mixing in the low-temperature

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regime $\beta > 1$. It shows that critical mixing slowdown happens when inverse-temperature satisfies $\beta = 1$ and thoroughly analyzes mixing times on each temperature region.

With interest in the above phenomena, mathematicians and physicists dream of completely understanding the nature of complex interactions through rigorous analysis of microscopic dynamics. These are why researchers tend to generalize the Ising model to the Potts model and the Potts model on complete graphs to the Potts model on a lattice, such as $\mathbb{Z}^2$. They try to explain the more complex model, which imitates nature rigorously. Nevertheless, an understanding of spin systems on the lattice needs to be improved. In this manner, we think of a spin system on a graph with the general interaction suggesting an excellent approximation of the lattice system.

Hence, the natural attempts began to generalize [30] to the more complex case. The next level is investigating critical temperature and phase transitions, where the complicated interactions between particles are given. Mathematically, the Hamiltonian of the system could be more complex described in the interaction matrix $\mathbf{K}$. [25] proves that dynamic phase transition occurs at the same inverse-temperature 1 as [30] in the viewpoint of spectral norm of the interaction matrix $\mathbf{K}$. Firstly, [21] proves a cutoff phenomenon for the multi-partite Ising model in high temperature, and [35] generalized cutoff to the general block Ising model in high temperature. [35] also proved $O(n^{3/2})$ mixing times in the critical regime and exponential mixing time in the low-temperature regime. Their results demonstrate that it is enough to investigate the interaction matrix's spectral norm to determine the overall system's mixing phase.

Back to the Potts model, in [13], applying Varadhan's lemma, Ellis and Wang showed that for a given number $q > 2$ of spins,

$$\beta_c(q) = \frac{q-1}{q-2}\log(q-1)$$

is a critical temperature of the phase transition. According to [11], the phase means how much spins prefer to be aligned. Physically, when the spin system is given at a high temperature since entropy overrides the energy, spins would be roughly independent. But, as inverse-temperature $\beta$ increases to such a critical point $\beta_c$, spins would be aligned in an ordered fashion because the contribution of energy gets bigger than that of entropy. This separation of phase via inverse-temperature $\beta$ is called a phase transition.

Moreover, [10] identifies the mixing critical inverse-temperature

$$\beta_s(q) = \sup\Big\{\beta > 0 : \big(1 + (q-1)e^{2\beta\frac{1-qx}{q-1}}\big)^{-1} - x \neq 0 \text{ for all } x \in \big(\frac{1}{q}, 1\big)\Big\}.$$

The results are similar to [30]. There is a cutoff at $O(n\log n)$ time in high temperature $\beta < \beta_s$, polynomial mixing of $n^{4/3}$ in critical temperature $\beta = \beta_s$, and exponential mixing in low-temperature $\beta > \beta_s$.

A disagreement between phase transition critical temperature and mixing critical temperature makes the Potts model more difficult than the Ising model. In Potts model, the mixing critical temperature $\beta_s$ is always smaller than phase transition critical temperature $\beta_c$, *i.e.* $\beta_s < \beta_c$. Indeed, the Ising model is a special case of the Potts model satisfying $\beta_s(2) = \beta_c(2) = 1$.

Developing [13], [24] analyzes for a homogeneous interaction matrix

$$\mathbf{K}_{a,b} = \begin{bmatrix} b & a & \cdots & a \\ a & b & \cdots & a \\ & & \ddots & \\ a & a & \cdots & b \end{bmatrix} \tag{1.1}$$

in $\mathbb{R}^{m\times m}$ with inter-block interaction coefficient $a$ and self-interaction interaction coefficient $b$. The phase transition occurs when $\beta(b+(m-1)a)/m$ is the same as $\beta_c$ defined in [10], where $m$ is the number of blocks in a graph. Moreover, [18] proves the central limit theorem for normalized magnetization matrices in the homogeneous block Potts model. Furthermore, this paper will show that the critical mixing transition happens when $\beta(b+(m-1)a)/m = \beta_s$.

Another difficulty in the Potts model is that, unlike the Ising model, the magnetization vector has freedom of degree $q-1>1$. Hence it cannot be determined by the sole spin. For this reason, complicated coupling methods were constructed in [10] to estimate the upper bound of cutoff time in the high-temperature regime. On the other hand, [27] describes mixing time divisions on the Blume-Capel model using an aggregate path coupling. This path coupling reveals the essence of high-temperature mixing implicitly applying negative drift of the magnetization vector. [26] applies this aggregate path coupling to show $O(n\log n)$ mixing time in complete Potts model. Also, in [17], authors extend the path coupling onto the bipartite graphs, obtaining similar mixing time differentiation.

This logic can be naturally generalized to multi-partite graphs. However, multi-partite graphs do not consider the self-interaction between particles. This paper will develop the aggregate path coupling on a general block model with $\mathbf{K}_{a,b}$.

This paper is organized as follows. In Section 2, we firmly explain our model and investigate the critical temperature of phase transition according to the large deviations theory. We will identify the mixing time threshold in this process, differentiating temperature regions. Since the unique equilibrium macrostate is guaranteed in the high-temperature regime, we can construct the aggregate path coupling in Section 3. The aggregate path coupling holds on an arbitrary starting configuration. Also, in Section 4, we further develop diverse contractions, including Frobenius norm contraction, which mainly determines the cutoff time in the high-temperature regime. Finally, we prove the cutoff in the high temperature in Section 5 and show slow mixing in the low temperature in Section 6.

### 1.1. Main Results.

Mixing time differentiation agrees with the results on the complete Potts model [10]. The parameter $J = (b+(m-1)a)/m$ represents the effective interaction of the overall graph. Hence, $\beta J$ takes the same role as the inverse-temperature $\beta$ in the complete graph. We show that the dynamics of the general block Potts model with homogeneity undergo critical slowdown when $\beta < \beta_s/J$.

**Theorem 1.** *Let $q>2,\ m>0$ be integers. If $\beta<\beta_s/J$, then the Glauber dynamics of general $m$-block, $q$-spin Potts model with interaction $\mathbf{K}=\mathbf{K}_{a,b}$ in (1.1) exhibits a cutoff at mixing time*

$$t_{\mathrm{MIX}}(n) = \xi(\beta, J, q)\, n\log n$$

*with cutoff window $n$, where $J = (b + (m-1)a)/m$ and*

$$\xi(\beta, J, q) = [2(1 - 2\beta J/q)]^{-1}. \tag{1.2}$$

In the low-temperature regime, we gain a slow mixing for big enough $n$.

**Theorem 2.** *Let $q > 2$, $m > 0$ be integers. In our model, if $\beta > \beta_s/J$, then there exists constants $C, c > 0$ such that for uniformly $n > 0$ big enough,*

$$t_{\mathrm{MIX}}(n) > Ce^{cn}.$$

## 2. Model and Large Deviations Theory

2.1. **Notations.** In this paper, we will use the following notations;

- For matrix $\mathbf{A} \in \mathbb{R}^{m\times q}$, $\mathbf{A}^\intercal$ denotes a transpose of $\mathbf{A}$, $\mathbf{A}^\complement$ denotes a complement of $\mathbf{A}$, and $\mathrm{Tr}(\mathbf{A})$ denotes the trace of $\mathbf{A}$.
- For matrix $\mathbf{B} \in \mathbb{R}^{m\times q}$, the entry located on the position $(i,j)$ is written as $\mathbf{B}^{ij}$, and for integers $i, j, r, s$,

$$\mathbf{B}^{i*} = [\mathbf{B}^{i1}, \mathbf{B}^{i2}, \cdots, \mathbf{B}^{iq}], \quad \mathbf{B}^{*j} = [\mathbf{B}^{1j}, \mathbf{B}^{2j}, \cdots, \mathbf{B}^{mj}]^\intercal,$$

$$\mathbf{B}^{i[r,s]} = [\mathbf{B}^{ir}, \mathbf{B}^{i(r+1)}, \cdots, \mathbf{B}^{is}], \quad \mathbf{B}^{[r,s]j} = [\mathbf{B}^{rj}, \mathbf{B}^{(r+1)j}, \cdots, \mathbf{B}^{sj}]^\intercal.$$

- We use the matrix norms as; for matrix $\mathbf{C} \in \mathbb{R}^{m\times q}$,

$$\|\mathbf{C}\|_{(1,1)} = \sum_{i,j} |\mathbf{C}^{ij}|, \ \|\mathbf{C}\|_F = \Big[\sum_{i,j} \big(\mathbf{C}^{ij}\big)^2\Big]^{1/2},$$

  and $\|\mathbf{C}\|_2$ is the spectral norm or Hilbert-Schmidt norm.
- We denote $\langle\cdot,\cdot\rangle$ as the standard Euclidean inner product between two vectors, and write $\langle\cdot,\cdot\rangle_F$ for the Frobenius inner product between two matrices.
- We write $\mathbf{1}_{r\times s}, \mathbf{1}_m$ as a $r \times s$, $1 \times m$ dimensional matrices with identical entries 1, respectively, and we denote $\mathbf{I}_m$ as $m$ dimensional identity matrix *i.e.*

$$\mathbf{1}_{r\times s} = (1)_{r\times s}, \ \mathbf{1}_m = (1)_{1\times m}, \ \mathbf{I}_m = \mathrm{diag}(1, 1, \cdots, 1),$$

  and $e_{ij}$ as a matrix with entry 0, but 1 for only $(i,j)$-th coordinate.
- We will fix the notation $m$ for the number of blocks and $q$ for the number of colors (spins) in the system. And we write

$$\mathbf{\Delta} = q^{-1}\mathbf{1}_{m\times q}. \tag{2.1}$$

2.2. **Glauber dynamics for general block Potts model.** We study the dynamics of the Curie-Weiss-Potts model on a graph $G = (V, E)$ having finite $n$-vertices, where $V$ is a vertex set and $E$ is a set of edges in graph $G$. For given $q \geq 3$, a number of colors, we define the state space to be $\Sigma = \{1, 2, \cdots, q\}^V$, and each element $\sigma \in \Sigma$ is called a configuration. In other words, a configuration is an assignment of colors in $\{1, 2, \cdots, q\}$ on the sites in $G$.

Generalized Potts model on a graph $G$ is constructed by the Hamiltonian on configuration space $\Sigma$ as follows;

$$H(\sigma) = -\frac{1}{n} \sum_{v \sim w} \mathbf{K}(v, w) \mathbb{1}_{\{\sigma(v) = \sigma(w)\}},$$

where the equivalence relation $v \sim w$ indicates that $(v, w) \in E$, and the interaction matrix $\mathbf{K} \in \mathbb{R}^{n \times n}$ is given. The corresponding Gibbs measure has the form in inverse-temperature $\beta > 0$ as;

$$\mu_*[\sigma] = Z_\beta^{-1} \exp(-\beta H(\sigma)),$$

where $Z_\beta$ is a partition function in inverse-temperature $\beta$ defined as

$$Z_\beta = \sum_{\sigma \in \Sigma} \exp\big(-\beta H(\sigma)\big).$$

We employ the discrete-time Glauber dynamics on $(\sigma_t)_{t \geq 0}$, starting from $\sigma_0$. At each step, we choose a vertex $v \in V$ uniformly and create a new configuration on the feasible set

$$\Sigma^v = \{\sigma' \in \Sigma : \sigma'(w) = \sigma(w) \quad \forall w \neq v\}$$

according to the Gibbs measure $\mu_*$, which means that the probability of updating $v$ to color $k$ is

$$\mu_*[\sigma(u) = k \mid \sigma(w) = \sigma_t(w), \quad \forall w \neq u].$$

2.3. **Proposed Model.** We assume that the sites are equally partitioned into a finite number of blocks, and their interaction strength depends on the block they belong. And block-wise interaction has homogeneity giving symmetry over blocks.

Formally, suppose that a vertex set $V$ of the overall graph is partitioned into $m$-blocks as;

$$V = V_1 \cup V_2 \cup \cdots \cup V_m$$

with equal proportion $n/m$, where $n$ is a number of overall vertices in $G$. In this paper, the interaction is given by

$$\mathbf{K}(v, w) = (a\mathbf{1}_{m \times m} + (b - a)\mathbf{I}_m)^{ij} \text{ if } v \in V_i, w \in V_j,$$

where $a$ is an inter-block interaction coefficient and $b$ is a self-interaction coefficient with $0 < a < b$. From now on, we tolerate the abuse of notation and re-define $\mathbf{K}$ into $m \times m$ dimensional matrix with off-diagonal entry $a$ and on-diagonal entry $b$ as in (1.1), *i.e.*

$$\mathbf{K}_{a,b} = a\mathbf{1}_{m \times m} + (b - a)\mathbf{I}_m.$$

We set $a < b$ to assure that the interaction matrix $\mathbf{K}$ is positive definite. Furthermore, this interaction form suggests homogeneity along blocks and limits inter-block interaction $a$ to be below self-interaction $b$. Now, we can write the Hamiltonian of given configuration $\sigma$ as;

$$H(\sigma) = -\frac{b}{n} \sum_{v \approx w} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}} - \frac{a}{n} \sum_{v \not\approx w} \mathbb{1}_{\{\sigma(v)=\sigma(w)\}},$$

where $v \approx w$ means that $v, w$ are in the same block. With this proposed model, we can consider the Potts model on multi-partite graphs as the extreme case of our model.

We decompose the given configuration $\sigma$ as;

$$\sigma = (\sigma^1, \sigma^2, \cdots, \sigma^m),$$

where $\sigma^i$ is the natural restriction of $\sigma$ on $\{1, 2, \cdots, q\}^{V_i}$. To analyze the configuration easily, we define magnetization matrix (proportion matrix) $\mathbb{S}(\sigma)$ in $\mathbb{R}^{m \times q}$ by

$$\mathbb{S}(\sigma)^{ij} = \frac{1}{|V_i|} \sum_{v \in V_i} \mathbb{1}_{\{\sigma^i(v)=j\}}$$

for each $i \in [1, m],\ j \in [1, q]$. We can understand $\mathbb{S}$ as the operator $\mathbb{S} : \Sigma \to \mathcal{S}$ with codomain

$$\mathcal{S} = \left\{ \mathbf{z} \in \mathbb{R}^{m \times q} :\ \text{each } \mathbf{z}^{ij} \geq 0, \sum_{j=1}^{q} \mathbf{z}^{ij} = 1\ \forall i \in [1, m] \right\}.$$

To understand proportion of restricted configuration $\sigma^i$, we also define the following simplex

$$\mathcal{P} = \left\{ x \in \mathbb{R}^q :\ \text{each } x_j \geq 0, \sum_{j=1}^{q} x_j = 1 \right\}.$$

We often write $\mathbb{S}$ for $\mathbb{S}(\sigma)$ when the given configuration $\sigma$ is obvious. For stochastic processes $(\sigma_t)_{t \geq 0}$ under Glauber dynamics on our model, we write proportion matrix processes $\mathbb{S}_t = \mathbb{S}(\sigma_t)$.

With this operator, we can re-interpret the Hamiltonian as follow;

$$H(\sigma) = -\frac{n}{m^2} \operatorname{Tr}((\mathbb{S}(\sigma))^\intercal \mathbf{K} \mathbb{S}(\sigma)).$$

Following the Glauber dynamics on our model, when we choose a vertex $v \in V_i$ and update its color to $j$, we say that $\sigma^i$ changes into $\sigma^{i,(v,j)}$. And the probability of this update denoted by $\mathbb{P}[(\sigma^1, \cdots, \sigma^i, \cdots \sigma^m) \to (\sigma^1, \cdots, \sigma^{i,(v,j)}, \cdots, \sigma^m)]$ is equal to

$$\mathbb{P}[(\sigma^1, \cdots, \sigma^i, \cdots \sigma^m) \to (\sigma^1, \cdots, \sigma^{i,(v,j)}, \cdots, \sigma^m)] = g_\beta^{(i,j)}\big(\mathbb{S}(\sigma)\big) + O(n^{-1}),$$

where for $(i, j) \in [1, m] \times [1, q]$ and $\mathbf{z} \in \mathcal{S}$,

$$g_\beta^{(i,j)}(\mathbf{z}) = \frac{\exp(\frac{2\beta}{m}(\mathbf{K}\mathbf{z})^{ij})}{\sum_{k=1}^{q} \exp(\frac{2\beta}{m}(\mathbf{K}\mathbf{z})^{ik})}.$$

Furthermore, we can express the drift of $\mathbb{S}_t^{ij}$ as; for $i \in [1, m],\ j \in [1, q]$,

$$\mathbb{E}[\mathbb{S}_{t+1}^{ij} - \mathbb{S}_t^{ij} \mid \mathcal{F}_t] = \frac{1}{n}\Big( - \mathbb{S}_t^{ij} + g_\beta^{(i,j)}(\mathbb{S}_t)\Big) + O(n^{-2}). \tag{2.2}$$

2.4. **Large Deviations Results.** Following the similar logic from [24], we find the specified equilibrium macrostate for our model and observe first-order phase transition decided by the parameter $b+(m-1)a$.

**Proposition 2.1.** [24, Theorem 3.2.] *Under the stationary measure $\mu_*$, the matrix valued random variable $\mathbb{S}$ obeys Large Deviation Principle with speed $n$ and rate function $I$ given by*

$$I(\nu_1,\nu_2,\cdots,\nu_m)=\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{q}\nu_i(j)\log(q\cdot\nu_i(j)),$$

*where $\nu_i\in\mathcal{P}$ for each $i\in[1,m]$.*

Thus, we can say that for any closed subset $C\subset\mathcal{S}$ and open subset $U\subset\mathcal{S}$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\Big(\mathbb{P}\big[\mathbb{S}\in C\big]\Big)\le-\inf\{I(\mathbf{z}):\mathbf{z}\in C\},\tag{2.3}$$

$$\liminf_{n\to\infty}\frac{1}{n}\log\Big(\mathbb{P}\big[\mathbb{S}\in U\big]\Big)\ge-\inf\{I(\mathbf{z}):\mathbf{z}\in U\}.\tag{2.4}$$

**Proposition 2.2.** [24, Theorem 4.7.] *Denote by $\beta_c=(q-1)\log(q-1)/(q-2)$, the critical inverse temperature in the q-color Potts model on the complete graph, and $J=(b+(m-1)a)/m$. Then our model has a phase transition. More precisely, if $\beta<\beta_c/J$, then the distribution of $\frac{1}{m}\mathbb{S}$ under the Gibbs measure $\mu_*$ concentrates in the unique point $\frac{1}{m}\mathbf{\Delta}$, where $\mathbf{\Delta}$ is defined in (2.1). To describe the "low-temperature behavior", define the function $\psi:[0,1]\to\frac{1}{m}\cdot\mathcal{P}$ :*

$$\psi(t)=\Big(\frac{1+(q-1)t}{mq},\frac{1-t}{mq},\cdots,\frac{1-t}{mq}\Big),$$

*and let $u(\beta J)$ be the largest solution of the equation*

$$u=\frac{1-e^{-\beta Ju}}{1+(q-1)e^{-\beta Ju}}.$$

*Finally let $n^1(\beta J)=\psi(u(\beta J))$ and $n^i(\beta J)$ be $n^1(\beta J)$ with the $i$-th and the first coordinate interchanged, $i=2,\cdots,q$. Let $\nu^i(\beta J)$ be the matrix with all rows identical to $n^i(\beta J)$. Then, if $\beta>\beta_c/J$, then the distribution of $\frac{1}{m}\mathbb{S}$ under the Gibbs measure $\mu_*$ concentrates in a uniform mixture of the Dirac measures in $\nu^1(\beta J),\cdots,\nu^q(\beta J)$. At $\beta=\beta_c/g$, the limit points of $\frac{1}{m}\mathbb{S}$ under the Gibbs measure $\mu_*$ are $\frac{1}{m}\mathbf{\Delta}$ and $\nu^1(\beta J),\cdots,\nu^q(\beta J)$.*

We set $J$, an effective interaction coefficient as

$$J=\frac{b+(m-1)a}{m}.\tag{2.5}$$

We will prove $J$ determines division of mixing regimes. To do so, we first define the critical inverse temperatures defined in Potts model on complete graphs described in [10] as follows;

$$\beta_c=\frac{q-1}{q-2}\log(q-1),$$
$$\beta_s=\sup\Big\{\beta>0:\big(1+(q-1)e^{2\beta\frac{1-qx}{q-1}}\big)^{-1}-x\neq0\text{ for all }x\in(1/q,1)\Big\}.$$

**Proposition 2.3.** *If $\beta < \beta_s/J$, then for any $\mathbf{z} \in \mathcal{S}$ with $(\mathbf{Kz})^{ij} > mJ/q$ for some $(i,j)$, the following statements holds*

$$g_\beta^{(i,j)}(\mathbf{z}) < \frac{(\mathbf{Kz})^{ij}}{mJ}.$$

*Proof.* By Jensen's inequality,

$$\begin{aligned} g_\beta^{(i,j)}(\mathbf{z}) - \frac{(\mathbf{Kz})^{ij}}{mJ} &= \frac{\exp\left(\frac{2\beta}{m}(\mathbf{Kz})^{ij}\right)}{\sum_{k=1}^q \exp\left(\frac{2\beta}{m}(\mathbf{Kz})^{ik}\right)} - \frac{(\mathbf{Kz})^{ij}}{mJ} \\ &< \frac{1}{1+(q-1)\exp\left(\frac{2\beta J}{q-1}\left(1-q\frac{(\mathbf{Kz})^{ij}}{mJ}\right)\right)} - \frac{(\mathbf{Kz})^{ij}}{mJ} \\ &= D_{\beta J}\left(\frac{(\mathbf{Kz})^{ij}}{mJ}\right) \leq 0, \end{aligned}$$

where the function $D_\cdot(\cdot)$ is defined in [10, Proposition 3.1.]. □

## 3. Aggregate Path Coupling

In this section, we will develop aggregate path coupling for our model. For this coupling, the conditions of the starting configurations are almost arbitrary. Hence, it would start contraction for two configurations in $O(n)$ time in the high-temperature regime.

3.1. **Greedy Coupling.** We define greedy coupling of $(\sigma_t, \tau_t)_{t\geq 0}$, traditionally used to gain the optimized upper bound of mixing time in probability theory. Suppose that starting configurations $\sigma = (\sigma^1, \sigma^2, \cdots, \sigma^m), \tau = (\tau^1, \tau^2, \cdots, \tau^m)$ are given. We define the greedy coupling by updating two configurations as follows;

(1) Choose a vertex $v$ uniformly in total $n$-vertices. Let $v \in V_i$. In this case, we denote the following probabilities for each $1 \leq k \leq q$,

$$\begin{aligned} u_{i,k} &= \mathbb{P}\big[(\sigma^1, \cdots, \sigma^i, \cdots, \sigma^m) \to (\sigma_1, \cdots, \sigma^{i,(v,k)}, \cdots, \sigma^m)\big], \\ v_{i,k} &= \mathbb{P}\big[(\tau^1, \cdots, \tau^i, \cdots, \tau^m) \to (\tau^1, \cdots, \tau^{i,(v,k)}, \cdots, \tau^m)\big], \\ Q_{i,k} &= \min\{u_{i,k}, v_{i,k}\}, \text{ and } Q_i = \sum_{k=1}^q Q_{i,k}. \end{aligned}$$

(2) Update both configurations' color at vertex $v$ according to the following joint transition probabilities; for distinct colors $k \neq l$,

$$\mathbb{P}\big[\sigma \to (\sigma^1, \cdots, \sigma^{i,(v,k)}, \cdots, \sigma^m),\ \tau \to (\tau^1, \cdots, \tau^{i,(v,k)}, \cdots, \tau^m)\big] = \frac{1}{n} Q_{i,k},$$
$$\mathbb{P}\big[\sigma \to (\sigma^1, \cdots, \sigma^{i,(v,k)}, \cdots, \sigma_m),\ \tau \to (\tau^1, \cdots, \tau^{i,(v,l)}, \cdots, \tau^m)\big] = \frac{1}{n}\frac{(u_{i,k} - Q_{i,k})(v_{i,l} - Q_{i,l})}{1 - Q_i}.$$

We write $\mathbb{P}_{\sigma,\tau}^{GC}$ for the measure defined by this greedy coupling. Before constructing a contraction by greedy coupling, we define a mean distance between two configurations.

**Definition 3.1.** *For two configurations $\sigma = (\sigma^1, \sigma^2, \cdots, \sigma^m)$, $\tau = (\tau^1, \tau^2, \cdots, \tau^m)$, a metric $d$ on $\Omega$ is defined to be a number of vertices having distinct colors;*

$$d(\sigma, \tau) = \sum_{i=1}^{m} \sum_{v \in V_i} \mathbb{1}_{\{\sigma^i(v) \neq \tau^i(v)\}}.$$

**Lemma 3.2.** *Under Greedy Coupling, the mean coupling distance after one iteration of the coupling process started from $\sigma = (\sigma^1, \sigma^2, \cdots, \sigma^m)$, $\tau = (\tau^1, \tau^2, \cdots, \tau^m)$ with $d = d(\sigma, \tau)$ is bounded as follows; for given $\epsilon$ small enough and $n$ large enough, there exists a constant $c > 0$ such that*

$$\mathbb{E}^{GC}_{\sigma,\tau}\big[d(\sigma_1, \tau_1)\big] \leq \Big[1 - \frac{1}{n}\Big(1 - \frac{\kappa + c\epsilon^2}{d/n}\Big)\Big] d,$$

*where for $i \in [1, m]$,*

$$\kappa = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{q} \Big| \Big\langle (\mathbb{S}(\sigma) - \mathbb{S}(\tau)), \nabla g_\beta^{(i,j)}\big(\mathbb{S}(\sigma)\big) \Big\rangle_F \Big|.$$

*Proof.* Define $\mathcal{I}_i$ to be a set of vertices in the block $V_i$ at which the color of the two configurations $\sigma, \tau$ disagree, and let $\kappa(e^l)$ be the probability that the coupled processes update differently, when the chosen vertex $v \notin \bigcup_{i=1}^m \mathcal{I}_i$ had color $l$.

By construction of greedy coupling, since we update the vertex $v$ to the same color $k$ in both processes with the largest possible probability, the probability of vertex $v$ updating to color $k$ in exactly one of the two processes(but not in the other) should be one of

$$\Big| \mathbb{P}\big[(\sigma^1, \cdots, \sigma^i, \cdots, \sigma^m) \to (\sigma^1, \cdots, \sigma^{i,(v,k)}, \cdots, \sigma^m)\big] - \mathbb{P}\big[(\tau^1, \cdots, \tau^i, \cdots, \tau^m) \to (\tau^1, \cdots, \tau^{i,(v,k)}, \cdots, \tau^m)\big] \Big|$$

for $1 \leq i \leq m$, depending on which block vertex $v$ was selected from. So, if the chosen vertex $v$ had previous color $l$, then

$$\begin{aligned} \kappa(e^l) &= \sum_{i=1}^{m} \frac{1}{m} \sum_{k=1}^{q} \Big| \mathbb{P}\big[(\sigma^1, \cdots, \sigma^i, \cdots, \sigma^m) \to (\sigma^1, \cdots, \sigma^{i,(v,k)}, \cdots, \sigma^m)\big] \\ &\quad - \mathbb{P}\big[(\tau^1, \cdots, \tau^i, \cdots, \tau^m) \to (\tau^1, \cdots, \tau^{i,(v,k)}, \cdots, \tau^m)\big] \Big| \\ &= \sum_{i=1}^{m} \frac{1}{m} \sum_{k=1}^{q} \Big| g_\beta^{(i,j)}(\mathbb{S}(\sigma)) - g_\beta^{(i,j)}(\mathbb{S}(\tau)) \Big| + O(n^{-1}). \end{aligned}$$

Now, by Taylor expansion of $g_\beta^{(i,j)}$, there exists a constant $C' > 0$ such that

$$\Bigg| \kappa(e^l) - \sum_{i=1}^{m} \frac{1}{m} \sum_{j=1}^{q} \Big| \Big\langle \big(\mathbb{S}(\sigma) - \mathbb{S}(\tau)\big), \nabla g_\beta^{(i,j)}\big(\mathbb{S}(\sigma)\big) \Big\rangle \Big| \Bigg| < C'\epsilon^2$$

for all $1 \leq l \leq q$ and $n > 0$ large enough, if $\epsilon < \|\mathbb{S}(\sigma) - \mathbb{S}(\tau)\|_F < 2\epsilon$. Denote for $1 \leq i \leq m$,

$$\kappa_i = \frac{1}{m} \sum_{j=1}^{q} \Big| \Big\langle \big(\mathbb{S}(\sigma) - \mathbb{S}(\tau)\big), \nabla g_\beta^{(i,j)}\big(\mathbb{S}(\sigma)\big) \Big\rangle \Big|.$$

Hence, we can estimate $\kappa(e^l)$ by the sum of $\kappa_i$ for any previous color $l$. Moreover, we can apply the same logic for $\tau$. Therefore, the mean distance between two configurations after one step of greedy coupling has the form for some $c > 0$,

$$\mathbb{E}_{\sigma,\tau}^{GC}\big[d(\sigma_1, \tau_1) - d\big] \leq -\frac{1}{n} d(\sigma_0, \tau_0) + \sum_{i=1}^{m} \kappa_i + c\epsilon^2 \leq -\frac{d(\sigma_0, \tau_0)}{n}\Big[1 - \frac{\kappa + c\epsilon^2}{d/n}\Big].$$

In conclusion, we finally obtain the wanted estimate

$$\mathbb{E}_{\sigma,\tau}^{GC}\big[d(\sigma_1, \tau_1)\big] \leq \left[1 - \frac{1}{n}\Big(1 - \frac{\kappa + c\epsilon^2}{d/n}\Big)\right] d.$$

$\square$

3.2. **Aggregate Path Coupling.** One of the reasons why analysis on the Potts model is more challenging than on the Ising model is the absence of monotonic coupling. However, with this aggregate path coupling, we can imitate monotone coupling from the Ising model by constructing a monotone path on the proportion matrix.

**Definition 3.3.** *For two configurations $\sigma, \tau \in \Sigma$, we say a path $\pi$ linking $\sigma$ and $\tau$ on $\Sigma$ denoted by*

$$\pi : \sigma = x_0,\ x_1,\ \cdots,\ x_r = \tau$$

*is called a monotone path, if*

(1) $d(\sigma, \tau) = \sum_{s=1}^{r} d(x_{s-1}, x_s)$,
(2) *For each $(i,j)$, $\mathbb{S}(x_s)^{ij}$ is monotonic as $s$ increases from $0$ to $r$.*

To construct a monotone path in the configuration space, we define a similar object in proportion space $\mathcal{S}$.

**Definition 3.4.** *Define the aggregate $g$-variation between a pair of two points $\mathbf{x}, \mathbf{z} \in \mathcal{S}$ along a continuous monotone (in each entry) path $\rho$ to be*

$$D_\rho^g(\mathbf{x}, \mathbf{z}) = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{q} \int_\rho \Big| \big\langle \nabla g_\beta^{(i,j)}(\mathbb{S}), d\mathbb{S} \big\rangle \Big|,$$

*and define the corresponding pseudo-distance between a pair of points $\mathbf{x}, \mathbf{z} \in \mathcal{S}$ as;*

$$d_g(\mathbf{x}, \mathbf{z}) = \inf_\rho D_\rho^g(\mathbf{x}, \mathbf{z}),$$

*where the infimum is taken over all continuous monotone paths in $\mathcal{S}$ linking $\mathbf{x}$ and $\mathbf{z}$.*

**Definition 3.5.** *With some $\delta > 0$ small enough, we denote $NG_\delta$, a family of neo-geodesic curves, monotone in each coordinate such that for each $\mathbf{z} \neq \mathbf{\Delta}$ in $\mathcal{S}$, there is exactly one curve $\rho = \rho_{\mathbf{z}}$ in the family $NG_\delta$ connecting $\mathbf{\Delta}$ to $\mathbf{z}$, and satisfying*

$$\frac{D^g_\rho(\mathbf{z}, \mathbf{\Delta})}{\|\mathbf{z} - \mathbf{\Delta}\|_{(1,1)}} \leq \frac{2\beta J}{q} - \delta,$$

*where $J$ is defined on (2.5).*

Now, we prove that in the high-temperature regime, proportion matrix $\mathbb{S}$ coalesces into $\mathbf{\Delta}$, which is the unique equilibrium macrostate.

**Proposition 3.6.** *Assume $\beta < \beta_s/J$. Since $\mathbf{\Delta}$ is the unique equilibrium marcostate, then there exists $\delta \in (0,1)$ such that uniformly for $\mathbf{z} \in \mathcal{S}$,*

$$\frac{d_g(\mathbf{z}, \mathbf{\Delta})}{\|\mathbf{z} - \mathbf{\Delta}\|_{(1,1)}} < 1 - \delta.$$

*Proof.* Let $\rho$ be a line path connecting $\mathbf{\Delta}$ and $\mathbf{z}$ in $\mathcal{S}$ parametrized by $\mathbf{z}(t) = \mathbf{\Delta}(1-t) + \mathbf{z}t$ for $t \in [0,1]$. Then,

$$D^g_\rho(\mathbf{z}, \mathbf{\Delta}) = \frac{1}{m}\sum_{i=1}^m \sum_{j=1}^q \int_\rho \left|\left\langle \nabla g^{(i,j)}_\beta(\mathbf{z}(t)), d\mathbf{z}(t)\right\rangle\right| = \frac{1}{m}\sum_{i=1}^m \sum_{j=1}^q \int_0^1 \left|\frac{d}{dt} g^{(i,j)}_\beta\big(\mathbf{z}(t)\big)\right| dt.$$

For fixed $(i,j)$,

$$\frac{d}{dt} g^{(i,j)}_\beta(\mathbf{z}(t)) = \frac{2}{m} g^{(i,j)}_\beta \left[ (\mathbf{K}(\mathbf{z} - \mathbf{\Delta}))^{ij} - \left\langle \big(\mathbf{K}(\mathbf{z} - \mathbf{\Delta})\big)^{i*}, g^{(i,*)}_\beta \right\rangle \right], \tag{3.1}$$

where $g^{(i,j)}_\beta = g^{(i,j)}_\beta(\mathbf{z}(t))$. Since

$$\frac{d}{dt}\left\langle \big(\mathbf{K}(\mathbf{z} - \mathbf{\Delta})\big)^{i*}, g^{(i,j)}_\beta \right\rangle = \frac{2}{m} \mathbb{V}ar_{g^{(i,*)}_\beta}\left[ \big(\mathbf{K}(\mathbf{z} - \mathbf{\Delta})\big)^{i*} \right] \geq 0$$

and $\left\langle \big(\mathbf{K}(\mathbf{z}-\mathbf{\Delta})\big)^{i*}, g^{(i,j)}_\beta\big(\mathbf{z}(0)\big) \right\rangle = 0$, then $\left\langle (\mathbf{K}(\mathbf{z}-\mathbf{\Delta}))^{i*}, g^{(i,j)}_\beta\big(\mathbf{z}(0)\big) \right\rangle \geq 0$ for any time $t \in [0,1]$.

Now we define a subset of $[1,m] \times [1,q]$ for each $\mathbf{z} \in \mathcal{S}$ by

$$A^{\mathbf{z}} = \big\{(i,j) \in [1,m] \times [1,q] : (\mathbf{K}(\mathbf{z} - \mathbf{\Delta}))^{ij} > 0\big\}.$$

If $(i,j) \notin A^{\mathbf{z}}$, then by (3.1), $g^{(i,j)}_\beta$ is monotonically decreasing and so,

$$\int_0^1 \left|\frac{d}{dt} g^{(i,j)}_\beta\right| dt = -\int_0^1 \frac{d}{dt} g^{(i,j)}_\beta \ dt = g^{(i,j)}_\beta(\mathbf{\Delta}) - g^{(i,j)}_\beta(\mathbf{z}).$$

Else if $(i,j) \in A^{\mathbf{z}}$, then there is at most one critical point $t^{ij}_*$ of $g^{(i,j)}_\beta$ in $(0,1)$. Writing $t^{ij} = \max\{t^{ij}_*, 1\}$,

$$\int_0^1 \left|\frac{d}{dt} g^{(i,j)}_\beta\right| dt = \int_0^{t^{ij}_*} \frac{d}{dt} g^{(i,j)}_\beta \ dt + \int_{t^{ij}_*}^1 -\left(\frac{d}{dt} g^{(i,j)}_\beta\right) dt = 2g^{(i,j)}_\beta(\mathbf{z}(t^{ij})) - g^{(i,j)}_\beta(\mathbf{z}) - g^{(i,j)}_\beta(\mathbf{\Delta}).$$

Therefore, $D^g_\rho$ has the form

$$D^g_\rho(\mathbf{z},\boldsymbol{\Delta}) = \frac{1}{m}\left[\sum_{(i,j)\in A^{\mathbf{z}}}\Big(2g_\beta^{(i,j)}\big(\mathbf{z}(t^{ij})\big) - g_\beta^{(i,j)}(\mathbf{z}) - g_\beta^{(i,j)}(\boldsymbol{\Delta})\Big) + \sum_{(i,j)\notin A^{\mathbf{z}}}\Big(g_\beta^{(i,j)}(\boldsymbol{\Delta}) - g_\beta^{(i,j)}(\mathbf{z})\Big)\right]$$
$$= \frac{1}{m}\sum_{(i,j)\in A^{\mathbf{z}}} 2\big(g_\beta^{(i,j)}(\mathbf{z}(t^{ij})) - g_\beta^{(i,j)}(\boldsymbol{\Delta})\big).$$

For $(i,j)\in A^{\mathbf{z}}$, because $\mathbf{K}(\mathbf{z}(t^{ij}))$ is in the interval $\Big((\mathbf{K}\boldsymbol{\Delta})^{ij}, (\mathbf{K}\mathbf{z})^{ij}\Big)$,

$$\Big(\mathbf{K}\big(\mathbf{z}(t^{ij})\big)\Big)^{ij} > \frac{mJ}{q}.$$

Therefore, using $\beta < \beta_s/J$ with Proposition 2.3,

$$g_\beta^{(i,j)}\big(\mathbf{z}(t^{ij})\big) < \frac{\big(\mathbf{K}\mathbf{z}(t^{ij})\big)^{ij}}{mJ} \le \frac{(\mathbf{K}\mathbf{z})^{ij}}{mJ}.$$

Since $\|\mathbf{K}\|_{(1,1)} = m^2 J$,

$$D^g_\rho(\mathbf{z},\boldsymbol{\Delta}) < \frac{2}{m}\sum_{(i,j)\in A^{\mathbf{z}}}\Big(\frac{(\mathbf{K}\mathbf{z})^{ij}}{mJ} - \frac{1}{q}\Big) = \frac{\|\mathbf{K}(\mathbf{z}-\boldsymbol{\Delta})\|_{(1,1)}}{\|\mathbf{K}\|_{(1,1)}} \le \|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}.$$

Then, we finish the proof since $\rho$ is continuous monotone path in $\mathcal{S}$. □

**Proposition 3.7.** *For given $\epsilon > 0$ small enough, and $\delta$ from Proposition 3.6, if $\beta < \beta_s/J$, then there exists a neo-geodesic family $NG_\delta$ such that for each $\mathbf{z}\in\mathcal{S}$ with $\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)} \ge \epsilon$, the curve $\rho = \rho_{\mathbf{z}}$ in $NG_\delta$ connecting from $\boldsymbol{\Delta}$ to $\mathbf{z}$ satisfies*

$$\frac{\sum_{i=1}^m\sum_{j=1}^q\sum_{s=1}^r\Big|\big\langle \mathbf{z}_s - \mathbf{z}_{s-1}, \nabla g_\beta^{(i,j)}(\mathbf{z}_{s-1})\big\rangle\Big|}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} \le \frac{2\beta J}{q} + c\epsilon$$

*for some $c > 0$ and any sequence of points $\boldsymbol{\Delta} = \mathbf{z}_0, \mathbf{z}_1, \cdots, \mathbf{z}_r = \mathbf{z}$ interpolating $\rho$ with $\epsilon \le \|\mathbf{z}_s - \mathbf{z}_{s-1}\|_{(1,1)} < 2\epsilon$ for $s = 1, 2, \cdots, r$.*

To prove this proposition, we apply the following lemma.

**Lemma 3.8.** *Let $\beta < \beta_s/J$. Then,*

$$\limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{m^{-1}\|g_\beta(\mathbf{z}) - g_\beta(\boldsymbol{\Delta})\|_{(1,1)}}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} \le \frac{2\beta J}{q} < 1.$$

*Proof of Lemma 3.8.* For each $i\in[1,m]$, by Taylor expansion of $g_\beta^{(i,j)}(\mathbf{z})$,

$$\limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{\|g_\beta^{(i,*)}(\mathbf{z}) - g_\beta^{(i,*)}(\boldsymbol{\Delta})\|_1}{\|(\mathbf{K}(\mathbf{z}-\boldsymbol{\Delta}))^{i*}\|_1} = \frac{2\beta}{qm}$$

following from [26, Lemma 10.4]. This statement implies

$$\limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{\|g_\beta(\mathbf{z}) - g_\beta(\boldsymbol{\Delta})\|_{(1,1)}}{\|\mathbf{K}(\mathbf{z}-\boldsymbol{\Delta})\|_{(1,1)}} \leq \frac{2\beta}{qm}.$$

Since the norm $\|\cdot\|_{(1,1)}$ is sub-multiplicative, we finally obtain

$$\limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{m^{-1}\|g_\beta(\mathbf{z}) - g_\beta(\boldsymbol{\Delta})\|_{(1,1)}}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} \leq \frac{2\beta J}{q}.$$

□

Applying Lemma 3.8, we prove the Proposition 3.7.

*Proof of Proposition 3.7.* For $\mathbf{z} \in \mathcal{S}$, define $\rho = \rho_{\mathbf{z}}$ be the straight line linking $\boldsymbol{\Delta}$ and $\mathbf{z}$ as the proof of Proposition 3.6. Then, since $\rho$ is a line path and by Lemma 3.8,

$$\limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{D^g_\rho(\mathbf{z},\boldsymbol{\Delta})}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} = \limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{m^{-1}\|g_\beta(\mathbf{z}) - g_\beta(\boldsymbol{\Delta})\|_{(1,1)}}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} \leq \frac{2\beta J}{q}.$$

Using definition of $d_\rho$,

$$\limsup_{\mathbf{z}\to\boldsymbol{\Delta}} \frac{d_\rho(\mathbf{z},\boldsymbol{\Delta})}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} \leq \frac{2\beta J}{q}.$$

This guarantees that there is $\delta \in (0,1)$ such that

$$\{\rho : \mathbf{z}(t) = \boldsymbol{\Delta}(1-t) + \mathbf{z}t, \mathbf{z} \in \mathbb{S}\} \text{ is } NG_\delta \text{ family of smooth curves.}$$

Furthermore, since the family of line paths $\rho$ is neo-geodesic family $NG_\delta$, the integrals

$$D^g_\rho(\mathbf{z},\boldsymbol{\Delta}) = \frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{q}\int_\rho \left|\left\langle \nabla g_\beta^{(i,j)}(\mathbb{S}), d\mathbb{S}\right\rangle\right|$$

can be approximated by Riemann sums of small enough steps. Then, there exists a constant $C > 0$ such that for $\epsilon > 0$ small enough,

$$\left| m^{-1}\sum_{(i,j)}\sum_{s=1}^{r} \left|\left\langle \mathbf{z}_s - \mathbf{z}_{s-1}, \nabla g_\beta^{(i,j)}(\mathbf{z}_{s-1})\right\rangle\right| - D^g_\rho(\mathbf{z},\boldsymbol{\Delta})\right| < Cr\epsilon^2 \quad \forall \mathbf{z} \in \mathcal{S} \text{ with } \|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)} \geq \epsilon$$

for each sequences of points $\boldsymbol{\Delta} = \mathbf{z}_0, \mathbf{z}_1, \cdots, \mathbf{z}_r = \mathbf{z}$ interpolating $\rho$ in $\mathcal{S}$ such that $\epsilon \leq \|\mathbf{z}_s - \mathbf{z}_{s-1}\|_{(1,1)} < 2\epsilon$. Therefore, for $\epsilon$ small enough

$$\frac{m^{-1}\sum_{(i,j)}\sum_{s=1}^{r} \left|\left\langle \mathbf{z}_s - \mathbf{z}_{s-1}, \nabla g_\beta^{(i,j)}(\mathbf{z}_{s-1})\right\rangle\right|}{\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)}} \leq \frac{2\beta J}{q} + C\epsilon.$$

□

In high-temperature regime $\beta < \beta_s/J$, we apply the same logic in [26, 17] to construct the aggregate path coupling from greedy coupling and the monotone paths by Proposition 3.7. In this aggregate path coupling started from $(\sigma_0, \tau_0)$, we use the probability measure $\mathbb{P}^{APC}_{\sigma_0,\tau_0}$. We now prove the contraction result for the mean coupling distance when one of the coupled processes becomes near the equilibrium $\boldsymbol{\Delta}$;

**Proposition 3.9.** *Assume $\beta < \beta_s/J$. Let $(X,Y)$ be the aggregate path coupling for one step with starting configuration $\sigma = (\sigma^1, \cdots, \sigma^m)$, $\tau = (\tau^1, \cdots, \tau^m)$. Then there exists $\epsilon' > 0$ small enough such that for uniformly large enough $n$, if $\|\mathbb{S}(\sigma) - \mathbf{\Delta}\|_{(1,1)} < \epsilon'$, then*

$$\mathbb{E}^{APC}_{\sigma,\tau}\big[d(X,Y)\big] \leq \Big(p + O\big(n^{-2}\big)\Big) d(\sigma,\tau),$$

*where*

$$p = 1 - \frac{1 - 2\beta J/q}{n}. \tag{3.2}$$

*Proof.* Suppose that $\epsilon > 0$ is sufficiently small. For configurations $\sigma, \tau \in \Sigma$, let $\mathbf{z} = \mathbb{S}(\sigma)$, $\mathbf{w} = \mathbb{S}(\tau)$ be the corresponding proportion matrices in simplex $\mathcal{S}$, and consider a continuous monotone path $\rho$ connecting $\mathbf{\Delta}$ to $\mathbf{w}$, defined as follows;

$$\rho = \{\mathbf{w}(t) = (1-t)\mathbf{\Delta} + t\mathbf{w} : t \in [0,1]\}.$$

By Proposition 3.6, there exists $\delta \in (0,1)$ such that

$$\frac{D^g_\rho(\mathbf{w}, \mathbf{\Delta})}{\|\mathbf{w} - \mathbf{\Delta}\|_{(1,1)}} \leq 1 - \delta.$$

Now let $\epsilon' = \epsilon^2\delta/M$ for $M$ large enough.

**Case I.** Suppose that $\|\mathbf{w} - \mathbf{z}\|_{(1,1)} \geq \epsilon + \epsilon'$, and $\|\mathbf{z} - \mathbf{\Delta}\|_{(1,1)} < \epsilon'$. By Proposition 3.7, there exists a discrete monotone path in $\mathcal{S}$,

$$\mathbf{z} = \mathbf{z}_0, \mathbf{z}_1, \cdots, \mathbf{z}_s = \mathbf{w}$$

approximating the continuous monotone path $\rho$ such that $\epsilon \leq \|\mathbf{z}_s - \mathbf{z}_{s-1}\|_{(1,1)} < 2\epsilon$ $\forall s \in [1,r]$ for which, with some $c > 0$,

$$\frac{m^{-1}\sum_{(i,j)}\sum_{s=1}^r \left|\left\langle \mathbf{z}_s - \mathbf{z}_{s-1}, \nabla g^{(i,j)}_\beta(\mathbf{z}_{s-1})\right\rangle\right|}{\|\mathbf{z} - \mathbf{w}\|_{(1,1)}} < \frac{2\beta J}{q} + c\epsilon.$$

Following the same idea in [26, Lemma 9.1.], one can construct a monotone path in configuration level $\pi :$ $\sigma = x_0, x_1, \cdots, x_r = \tau$ connecting configurations $\sigma, \tau$ such that $\mathbf{z}_s = \mathbb{S}(x_s)$. Then, by Lemma 3.2,

$$\begin{aligned}
\mathbb{E}^{APC}_{\sigma,\tau}\big[d(X,Y)\big] &\leq \sum_{s=1}^r \mathbb{E}^{APC}_{x_{s-1},x_s}\big[d(X_{s-1}, X_s)\big] \\
&\leq \sum_{s=1}^r \left\{ d(x_{s-1}, x_s)\left[1 - \frac{1}{n}\Big(1 - \frac{\kappa_s + c\epsilon^2}{d(x_{s-1}, x_s)}\Big)\right]\right\} \\
&= d(\sigma,\tau)\left[1 - \frac{1}{n}\Big(1 - \frac{\sum_{s=1}^r \kappa_s + c\epsilon^2}{d(\sigma,\tau)/n}\Big)\right] \\
&\leq d(\sigma,\tau)\left[1 - \frac{1}{n}\Big(1 - \frac{m\big(\sum_{s=1}^r \kappa_s + c\epsilon^2\big)}{\|\mathbf{z} - \mathbf{w}\|_{(1,1)}}\Big)\right],
\end{aligned}$$

where $\kappa_s = m^{-1}\sum_{i=1}^m\sum_{j=1}^q \left|\left\langle \mathbf{z}_s - \mathbf{z}_{s-1}, \nabla g_\beta^{(i,j)}(\mathbf{z}_{s-1})\right\rangle\right|$ for $s \in [1,r]$, and the last inequality holds because $\|\mathbb{S}(\sigma)-\mathbb{S}(\tau)\|_{(1,1)} \le \frac{m}{n} d(\sigma,\tau)$. So,

$$\mathbb{E}_{\sigma,\tau}^{APC}\big[d(X,Y)-d(\sigma,\tau)\big] \le -\frac{1}{n}\Big(1-\frac{2\beta J}{q}+c\epsilon^2\Big)d.$$

**Case II.** Suppose that $\|\mathbf{w}-\mathbf{z}\|_{(1,1)} < \epsilon+\epsilon'$, and $\|\mathbf{z}-\boldsymbol{\Delta}\|_{(1,1)} < \epsilon'$. Simliar to the previous path coupling,

$$\begin{aligned}\mathbb{E}_{\sigma,\tau}^{APC}\big[d(X,Y)\big] &\le d(\sigma,\tau)\left[1-\frac{1}{n}\Big(1-\frac{m^{-1}\|g_\beta(\mathbf{z})-g_\beta(\mathbf{w})\|_{(1,1)}+c\epsilon^2}{\|\mathbf{z}-\mathbf{w}\|_{(1,1)}}\Big)\right]+O(n^{-2})\\ &\le d(\sigma,\tau)\Big[1-\frac{1}{n}\Big(1-\frac{2\beta J}{q}\Big)\Big]+O(\epsilon)+O(n^{-2}).\end{aligned}$$

Therefore we obtain that

$$\mathbb{E}_{\sigma,\tau}^{APC}[d(X,Y)-d(\sigma,\tau)] \le -\frac{1}{n}\Big(1-\frac{2\beta J}{q}\Big)d+O(\epsilon)+O(n^{-2}).$$

Combining both cases, we finish the proof. □

## 4. Contractions in the high-temperature regime

In this section, we construct fundamental couplings and contractions to estimate mixing time in the high-temperature regime. We first take hitting time estimates for general supermartingales, which have already been studied in much mathematical literature.

### 4.1. Hitting Time Estimates for General Supermartingales.

The following lemmas are from [10].

**Lemma 4.1.** [10, Lemma 2.1.(2)] *Let $(Z_t)_{t\ge 0}$ be a non-negative supermartingale with respect to filtrations $\{\mathcal{F}_t\}$, and let $N$ be a stopping time for $\{\mathcal{F}_t\}$. Suppose that*

1. *$Z_0 = z_0$,*
2. *there exists $B>0$ such that $|Z_{t+1}-Z_t| \le B$ for all $t \ge 0$,*
3. *there exists a constant $\sigma > 0$ such that, for each $t \ge 0$, the inequality $\mathbb{V}ar\,[Z_{t+1} \mid \mathcal{F}_t] \ge \sigma^2$ holds on the event $\{N>t\}$.*

*If $u > 4B^2/3\sigma^2$, then*

$$\mathbb{P}_{z_0}[N>u] \le \frac{4z_0}{\sigma\sqrt{u}}.$$

**Lemma 4.2.** [10, Lemma 2.3.] *For $x_0 \in \mathbb{R}$, let $(X_t)_{t\ge 0}$ be a discrete-time process, adapted to $(\mathcal{F}_t)_{t\ge 0}$ which satisfies,*

1. *$\exists\delta \ge 0$ s.t. $\mathbb{E}_{x_0}[X_{t+1}-X_t] \ge -\delta$ on $\{X_t \ge 0\}$ for all $t \ge 0$,*
2. *$\exists R>0$ s.t. $|X_{t+1}-X_t| \le R \quad \forall t \ge 0$,*
3. *$X_0 = x_0$,*

*where $\mathbb{P}_{x_0}$ is the underlying probability measure. Let $\tau_x^+ = \inf\{t \geq 0 : X_t \geq x\}$. Then, if $x_0 \leq 0$, then for any $x > 0$ and $t \geq 0$,*

$$\mathbb{P}_{x_0}[\tau_x^+ \leq t] \leq 2\exp\left\{-\frac{(x-R)^2}{8tR^2}\right\}.$$

4.2. **Concentration of magnetization chains.** We achieved the negative drift of the magnetization chain by the aggregate path coupling. The next step is to deduce a sufficiently chunked concentration of magnetization matrices after $O(n)$ time. Indeed, when the starting magnetization is close to $\mathbf{\Delta}$ in the sense of $O(n^{-1/2})$ scale of Frobenius norm, the magnetization chains would remain near $\mathbf{\Delta}$ until $O(n)$ time. To be rigorous, we define a restricted region of configuration space $\Sigma$ having proportion near $\mathbf{\Delta}$. For $\rho > 0$ small, we define

$$\Sigma_F^\rho = \left\{\sigma \in \Sigma : \|\mathbb{S}(\sigma) - \mathbf{\Delta}\|_F < \rho\right\}.$$

**Proposition 4.3.** *Assume $\beta < \beta_s/J$ and $\epsilon > 0$ be given.*
*For all $r > r_0 > 0$, there exists $\gamma > 0$ such that,*

$$\mathbb{P}_{\sigma_0}\left[\exists 0 \leq s \leq \gamma n : \sigma_s \notin \Sigma_F^{r/\sqrt{n}}\right] \leq \epsilon$$

*for all $\sigma_0 \in \Sigma_F^{r_0/\sqrt{n}}$ and uniformly large enough $n$.*

*Proof.* In this proof, we use analysis of function $D_{\beta J}(y)$ for $y \in (1/q, 1]$ where $\beta J < \beta_s$, defined in [10]. Recall that

$$D_{\beta J}(y) = -y + \frac{1}{1 + (q-1)\exp\left(\frac{2\beta J}{q-1}(1-qy)\right)},$$

and it has Taylor expansion of the form

$$D_{\beta J}(y) = -\left(1 - \frac{2\beta J}{q}\right)\left(y - \frac{1}{q}\right) + O\left(\left(y - \frac{1}{q}\right)^2\right)$$

near $y = \frac{1}{q}$. We define the hitting time for $\Sigma_F^{r/\sqrt{n}}$ as $\tau_* = \inf\left\{t : \sigma_t \notin \Sigma_F^{r/\sqrt{n}}\right\}$, and let $\mathbf{X}_t = \mathbf{K}\mathbb{S}_{t\wedge\tau_*}$. Then, by (2.2),

$$\begin{aligned}\mathbb{E}\left[\mathbb{S}_{t+1}^{ij} - \mathbb{S}_t^{ij}\middle|\mathcal{F}_t\right] &= \frac{1}{n}\Big(-\mathbb{S}_t^{ij} + g_\beta^{(i,j)}(\mathbb{S}_t)\Big) + O(n^{-2}) \\ &\leq \frac{1}{n}\left(-\mathbb{S}_t^{ij} + \frac{(\mathbf{K}\mathbb{S}_t)^{ij}}{mJ} + D_{\beta J}\Big(\frac{(\mathbf{K}\mathbb{S}_t)^{ij}}{mJ}\Big)\right) + O(n^{-2}).\end{aligned}$$

To calculate drift of $\mathbf{X}_t^{ij}$ for $i \in [1,m], j \in [1,q]$,

$$\begin{aligned}&\mathbb{E}\left[(\mathbf{K}\mathbb{S}_{t+1})^{ij} - (\mathbf{K}\mathbb{S}_{t+1})^{ij}\middle|\mathcal{F}_t\right] - O(n^{-2}) \\ &= n^{-1}\left(-(\mathbf{K}\mathbb{S}_t)^{ij} + b\frac{(\mathbf{K}\mathbb{S}_t)^{ij}}{mJ} + a\sum_{u\neq i}\frac{(\mathbf{K}\mathbb{S}_t)^{uj}}{mJ} + bD_{\beta J}\Big(\frac{(\mathbf{K}\mathbb{S}_t)^{ij}}{mJ}\Big) + a\sum_{u\neq i}D_{\beta J}\Big(\frac{(\mathbf{K}\mathbb{S}_t)^{uj}}{mJ}\Big)\right).\end{aligned}$$

Because $\big|(\mathbf{K}\mathbb{S}_t)^{ij} - mJ/q\big| \le C \max\Big\{|\mathbb{S}_t^{ij} - 1/q| : i \in [1,m], j \in [1,q]\Big\}$ for some $C > 0$ and $\Big|\mathbf{X}_t^{ij} - mJ/q\Big|^2 = O(n^{-1})$, we obtain that by Taylor expansion for $D_{\beta J}$,

$$\mathbb{E}\big[\mathbf{X}_{t+1}^{ij} - \mathbf{X}_t^{ij}\big|\mathcal{F}_t\big] = \frac{1}{n}\Bigg( - \mathbf{X}_t^{ij} + \frac{2\beta}{mq}(b-a)\mathbf{X}_t^{ij} + \frac{2\beta}{mq} a \sum_{u\in[1,m]} \mathbf{X}_t^{uj} + \Big(1 - \frac{2\beta J}{q}\Big)\cdot\frac{mJ}{q}\Bigg) + O(n^{-2}).$$

Using (1.2), we can calculate

$$\begin{aligned}
\mathbb{E}\big[\mathbf{X}_{t+1}^{ij}\big|\mathcal{F}_t\big] &= \mathbf{X}_t^{ij} + \frac{1}{n}\Bigg( - \mathbf{X}_t^{ij} + \frac{2\beta}{mq}\Big((b-a)\mathbf{X}_t^{ij} + a\sum_u \mathbf{X}_t^{uj}\Big) + \frac{mJ}{2q\xi}\Bigg) + O(n^{-2}),\\
\mathbb{E}\big[(\mathbf{X}_{t+1}^{ij})^2\big|\mathcal{F}_t\big] &= (\mathbf{X}_t^{ij})^2 + \frac{2}{n}\mathbf{X}_t^{ij}\Bigg( - \mathbf{X}_t^{ij} + \frac{2\beta}{mq}\Big((b-a)\mathbf{X}_t^{ij} + a\sum_u \mathbf{X}_t^{uj}\Big) + \frac{mJ}{2q\xi}\Bigg) + O(n^{-2}).
\end{aligned}$$

Regarding $\mathbf{1}_m$ is the eigenvector of the largest eigenvalue $mJ$ of $\mathbf{K}$, we multiply $\mathbf{1}_m$ on

$$\Big[\mathbb{E}\big[(\mathbf{X}_{t+1}^{1j})^2\big|\mathcal{F}_t\big], \mathbb{E}\big[(\mathbf{X}_{t+1}^{2j})^2\big|\mathcal{F}_t\big], \cdots, \mathbb{E}\big[(\mathbf{X}_{t+1}^{mj})^2\big|\mathcal{F}_t\big]\Big].$$

Thus, when $t < \tau_*$, we obtain that

$$\begin{aligned}
\mathbb{E}\Big[\sum_{i=1}^m (\mathbf{X}_{t+1}^{ij})^2\big|\mathcal{F}_t\Big] &= \Big[1 - \frac{2}{n}\Big(1 - \frac{2\beta(b-a)}{mq}\Big)\Big]\sum_{i=1}^m (\mathbf{X}_t^{ij})^2 + \frac{2}{n}\frac{2\beta a}{mq}(\sum_{i=1}^m \mathbf{X}_t^{ij})^2 + \frac{2}{n}\frac{mJ}{2q\xi}\sum_{i=1}^m \mathbf{X}_t^{ij} + O(n^{-2})\\
&\le \Big[1 - \frac{2}{n}\Big(1 - \frac{2\beta(b-a)}{mq}\Big)\Big]\sum_{i=1}^m (\mathbf{X}_t^{ij})^2 + \frac{2}{n}\frac{2\beta a}{mq}\cdot m\sum_{i=1}^m (\mathbf{X}_t^{ij})^2 + \frac{2}{n}\frac{mJ}{2q\xi}\sum_{i=1}^m \mathbf{X}_t^{ij} + O(n^{-2})\\
&= \Big[1 - \frac{2}{n}\Big(1 - \frac{2\beta J}{q}\Big)\Big]\sum_{i=1}^m (\mathbf{X}_t^{ij})^2 + \frac{2}{n}\frac{mJ}{2q\xi}\sum_{i=1}^m \Big(\frac{mJ}{q} + O(n^{-1})\Big) + O(n^{-2})\\
&= \Big(1 - \frac{1}{n\xi}\Big)\sum_{i=1}^m (\mathbf{X}_t^{ij})^2 + \frac{m}{n\xi}\Big(\frac{mJ}{q}\Big)^2 + O(n^{-2}).
\end{aligned}$$

Thus, there exists $C_1 > 0$ such that on $\Big\{\sum_{i=1}^m (\mathbf{X}_t^{ij})^2 > m\cdot(mJ/q)^2\Big\}$, for $t \ge 0$,

$$\mathbb{E}\Big[\sum_{i=1}^m (\mathbf{X}_{t+1}^{ij})^2 - \sum_{i=1}^m (\mathbf{X}_t^{ij})^2\Big|\mathcal{F}_t\Big] < -\frac{C_1}{n}.$$

Besides, we define process $Z_t^j = \sum_{i=1}^m (\mathbf{X}_t^{ij})^2$ where the color $j \in [1,q]$ is given, and calculate as;

$$\begin{aligned}
Z_t^j &= \sum_{i=1}^m \Big[ b\mathbb{S}_{t\wedge\tau_*}^{ij} + a \sum_{u\neq i} \mathbb{S}_{t\wedge\tau_*}^{uj} \Big]^2 \\
&= \sum_{i=1}^m \left[ b\Big(\mathbb{S}_{t\wedge\tau_*}^{ij} - \frac{1}{q}\Big) + a \sum_{u\neq i} \Big(\mathbb{S}_{t\wedge\tau_*}^{uj} - \frac{1}{q}\Big) + \frac{mJ}{q} \right]^2 \\
&= \sum_{i=1}^m \left[ b^2\Big(\mathbb{S}_{t\wedge\tau_*}^{ij} - \frac{1}{q}\Big)^2 + a \sum_{u\neq i} \Big(\mathbb{S}_{t\wedge\tau_*}^{ij} - \frac{1}{q}\Big)^2 + \frac{m^2J^2}{q^2} + O(n^{-1/2}) \right] \\
&= \frac{m^3J^2}{q^2} + \big(b^2 + (m-1)a^2\big) \sum_{i=1}^m \Big(\mathbb{S}_{t\wedge\tau_*}^{ij} - \frac{1}{q}\Big)^2 + O(n^{-1/2}).
\end{aligned}$$

Hence, if $\|\mathbb{S}_t - \mathbf{\Delta}\|_F > r/\sqrt{n}$, then there exists $j \in [1,q]$ such that $\sum_{i=1}^m (\mathbb{S}_t^{ij} - 1/q)^2 \geq r^2/qn$ and

$$Z_t^j \geq \frac{m^3J^2}{q^2} + \frac{c_2}{\sqrt{n}} + \frac{r^2}{nq}$$

for some constant $c_2 > 0$. Since process $(Z_t^j - m^3J^2/q^2)$ satisfies three conditions for Lemma 4.2 with $R = O(n^{-1})$ and $x_0 = c_2/\sqrt{n} + r^2/nq$, then there exists $\gamma > 0$ and some $c_3 > 0$ such that uniformly in $n > 0$ large enough,

$$\mathbb{P}\Big[\exists 0 \leq s \leq \gamma n : \sigma_s \notin \Sigma_F^{r/\sqrt{n}}\Big] \leq 2\exp\left\{-\frac{(x_0 - O(n^{-1}))^2}{8\cdot\gamma n\cdot O(n^{-2})}\right\} \leq 2e^{-c_3/\gamma} < \epsilon.$$

□

4.3. **Synchronized Coupling.** The synchronized coupling is a coupling of two $\rho$-bounded dynamics by $\Sigma_F^\rho$ in which the two chains "synchronize" their steps as much as possible. We define $(\sigma_t)_{t\geq 0}, (\widetilde{\sigma}_t)_{t\geq 0}$ on the same probability space such that starting from $\sigma_0, \widetilde{\sigma}_0$, at time $t+1$:

1. Select block number $l$ uniformly from $[1,m]$.
2. Choose colors $I_{t+1}, \widetilde{I}_{t+1}$ according to the optimal coupling of $\mathbb{S}_t^{l*}$, $\widetilde{\mathbb{S}}_t^{l*}$.
3. Choose colors $J_{t+1}, \widetilde{J}_{t+1}$ according to the optimal coupling of $g_\beta^{(l,*)}(\mathbb{S}_t - \frac{m}{n}e_{lI_{t+1}})$, $g_\beta^{(l,*)}(\widetilde{\mathbb{S}}_t - \frac{m}{n}e_{l\widetilde{I}_{t+1}})$.
4. Change a uniformly chosen vertex in $V_l$ of color $I_{t+1}$ in $\sigma_t$ to have color $J_{t+1}$ in $\sigma_{t+1}$, but only if $\sigma_{t+1} \in \Sigma_F^\rho$. Unless, do not change its color.
5. Change a uniformly chosen vertex in $V_l$ of color $\widetilde{I}_{t+1}$ in $\widetilde{\sigma}_t$ to have color $\widetilde{J}_{t+1}$ in $\widetilde{\sigma}_{t+1}$, but only if $\widetilde{\sigma}_{t+1} \in \Sigma_F^\rho$. Unless, do not change its color.

We shall write $\mathbb{P}_{\sigma_0,\widetilde{\sigma}_0}^{SC,\rho}$ for the underlying measure and omit $\rho$ if it is large enough for the dynamics not to be bounded.

**Proposition 4.4.** *If $\beta < q/(2J)$, under $\rho$-bounded synchronized coupling started from $(\sigma_0, \widetilde{\sigma}_0) \in \Sigma_F^\rho \times \Sigma_F^\rho$, for time $t \geq 0$,*

$$
\begin{bmatrix} \mathbb{E}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0}[\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{1*}\|_1] \\ \mathbb{E}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0}[\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{2*}\|_1] \\ \vdots \\ \mathbb{E}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0}[\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{m*}\|_1] \end{bmatrix} \leq \mathbf{A}^t \begin{bmatrix} \|(\mathbb{S}_0 - \widetilde{\mathbb{S}}_0)^{1*}\|_1 \\ \|(\mathbb{S}_0 - \widetilde{\mathbb{S}}_0)^{2*}\|_1 \\ \vdots \\ \|(\mathbb{S}_0 - \widetilde{\mathbb{S}}_0)^{m*}\|_1 \end{bmatrix}, \tag{4.1}
$$

*where with some matrix $\mathbf{C}$,*

$$
\mathbf{A} = \Big(1 - \frac{1}{n}\Big)\mathbf{I}_m + \frac{2\beta}{nqm}(1 + O(\rho))\mathbf{K} + \frac{\mathbf{C}}{n^2}.
$$

*Proof.* Suppose we select block number $l$ with probability $m^{-1}$. By definition of optimal coupling, $I_{t+1} \neq \widetilde{I}_{t+1}$ implies that $\mathbb{S}_t^{lI_{t+1}} > \widetilde{\mathbb{S}}_t^{lI_{t+1}}$, $\mathbb{S}_t^{l\widetilde{I}_{t+1}} < \widetilde{\mathbb{S}}_t^{l\widetilde{I}_{t+1}}$, and $J_{t+1} \neq \widetilde{J}_{t+1}$ implies that $(\mathbf{K}(\mathbb{S}_t - \frac{m}{n}e_{lI_{t+1}}))^{lJ_{t+1}} > (\mathbf{K}(\widetilde{\mathbb{S}}_t - \frac{m}{n}e_{l\widetilde{I}_{t+1}}))^{lJ_{t+1}}$, $(\mathbf{K}(\mathbb{S}_t - \frac{m}{n}e_{lI_{t+1}}))^{l\widetilde{J}_{t+1}} < (\mathbf{K}(\widetilde{\mathbb{S}}_t - \frac{m}{n}e_{l\widetilde{I}_{t+1}}))^{l\widetilde{J}_{t+1}}$ Then, for each $i \in [1, m]$,

$$
\|(\mathbb{S}_{t+1} - \widetilde{\mathbb{S}}_{t+1})^{i*}\|_1 - \|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{i*}\|_1 = \frac{2m}{n}\mathbb{1}_{\{i=l\}}\big[\mathbb{1}_{\{J_{t+1}\neq\widetilde{J}_{t+1}\}} - \mathbb{1}_{\{I_{t+1}\neq\widetilde{I}_{t+1}\}}\big].
$$

Besides, since $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2}\|\mu - \nu\|_1$ holds for two probability measure $\mu, \nu$, then using Taylor expansion for $g_\beta$,

$$
\begin{aligned}
\mathbb{P}^{SC,\rho}_{\sigma_t,\widetilde{\sigma}_t}\big[J_{t+1} \neq \widetilde{J}_{t+1} \mid l = i\big] &= \|g_\beta^{(i,*)}(\mathbb{S}_t) - g_\beta^{(i,*)}(\widetilde{\mathbb{S}}_t)\|_{\mathrm{TV}} \\
&= \frac{\beta}{qm}\|(\mathbf{K}(\mathbb{S}_t - \widetilde{\mathbb{S}}_t))^{i*}\|_1\Big(1 + O(\|(\mathbf{K}(\mathbb{S}_t - \mathbf{\Delta}))^{i*}\|_1) + O(\|(\mathbf{K}(\widetilde{\mathbb{S}}_t - \mathbf{\Delta}))^{i*}\|_1)\Big) + O(n^{-1}) \\
&\leq \frac{\beta}{qm}(1 + O(\rho))\sum_{v=1}^m \mathbf{K}^{iv}\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{v*}\|_1 + O(n^{-1}).
\end{aligned}
$$

Moreover, $O\big(\|(\mathbf{K}(\mathbb{S}_t - \mathbf{\Delta})^{i*}\|_1\big), O\Big(\|\big(\mathbf{K}(\widetilde{\mathbb{S}}_t - \mathbf{\Delta})\big)^{i*}\|_1\Big) = O(\rho)$ and $\mathbb{P}^{SC,\rho}_{\sigma_t,\widetilde{\sigma}_t}\big[I_{t+1} \neq \widetilde{I}_{t+1} \mid l = i\big] = \frac{1}{2}\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{i*}\|_1$. Taking expectation,

$$
\begin{aligned}
\mathbb{E}^{SC,\rho}_{\sigma_t,\widetilde{\sigma}_t}\big[\|(\mathbb{S}_{t+1} - \widetilde{\mathbb{S}}_{t+1})^{i*}\|_1\big] =& \Big(1 - \frac{1}{n}\Big)\mathbb{E}[\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{i*}\|_1] \\
&+ \frac{2\beta(1 + O(\rho))}{nqm}\sum_{v=1}^m \mathbf{K}^{iv}\mathbb{E}[\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{v*}\|_1] + O(n^{-2}).
\end{aligned}
$$

By induction on time $t$, we complete the proof. $\square$

Since matrix $\mathbf{K}$ is positive definite and positive, Perron-Frobenius theorem reveals that there is the largest eigenvalue $mJ = b + (m-1)a$ of $\mathbf{K}$ and its corresponding left eigenvector $\mathbf{1}_m$ exists. By multiplying $\mathbf{1}_m^\intercal$ left on both sides of (4.1), we obtain the following result.

**Corollary 4.5.** *If $\beta < q/(2J)$, then for the $\rho$-bounded Synchronized Coupling $(\sigma_t, \widetilde{\sigma}_t)$ starting at $(\sigma_0, \widetilde{\sigma}_0) \in \Sigma_F^\rho \times \Sigma_F^\rho$, there exists a constant $C > 0$ such that for big enough $n$,*

$$\mathbb{E}^{SC,\rho}_{\sigma_0,\widetilde{\sigma}_0}\left[\|\mathbb{S}_t - \widetilde{\mathbb{S}}_t\|_{(1,1)}\right] \leq \left(p + \frac{C\rho}{n}\right)^t \|\mathbb{S}_0 - \widetilde{\mathbb{S}}_0\|_{(1,1)},$$

*where $p = 1 - \frac{1-2\beta J/q}{n}$ is same as (3.2).*

4.4. **Uniform Variance Bound.** To prove that the variance of magnetization matrices is uniform $O(n^{-1})$-bounded in the high-temperature regime, we use the following Proposition proven in [18].

**Proposition 4.6.** [18, Theorem 1.] *Consider the high-temperature case i.e. $\beta < \beta_c/J$. Then, for arbitrary $\sigma \in \Sigma$, the normalized magnetization satisfies the following Central Limit Theorem as $n \to \infty$;*

$$\sqrt{\frac{n}{m}}(\mathbb{S}(\sigma) - \mathbf{\Delta}) \to W,$$

*where $W \sim \mathcal{N}(0, \Lambda)$ is an $m \times q$-dimensional centered Gaussian random variable with singular covariance matrix*

$$\Lambda = \frac{m}{2}(\beta\mathbf{K})^{-1}\left(\left(\mathbf{I}_{m\times q} + \frac{2}{m}\beta\mathbf{K}\big(\mathbf{I}_m \otimes (q^{-2}\mathbf{1}_{q\times q} - q^{-1}\mathbf{I}_q)\big)\right)^{-1} - \mathbf{I}_{m\times q}\right).$$

Since this theorem holds for arbitrary configuration $\sigma$ and $\Lambda$ is independent of $n$, then we can obtain the uniform $O(n^{-1})$ variance bound starting from stationary distribution $\mu_*$ immediately.

**Corollary 4.7.** *Assume $\beta < \beta_c/J$, and $\sigma \in \Sigma$ to be arbitrary starting configuration. For uniformly large enough $n$, there exists a constant $C > 0$ such that for big enough $n$,*

$$\mathbb{V}ar_{\mu_*}[\mathbb{S}(\sigma) - \mathbf{\Delta}] \leq \frac{C}{n},$$

*where $\mu_*$ is the stationary Gibbs measure.*

We extend [35, Lemma 4.10.] to the matrix form;

**Lemma 4.8.** [35, Lemma 4.10] *Let $(\mathbf{Z}_t)_{t\geq 0}$ be a Markov chain in a finite state space $\Omega \subset \mathbb{R}^{m\times q}$. Write $\mathbb{P}_{\mathbf{z}_0}$, $\mathbb{E}_{\mathbf{z}_0}$ for its probability measure and expectation respectively, when $\mathbf{Z}_0 = \mathbf{z}_0$. Suppose that there is some $0 < \eta < 1$ such that for all pairs of starting states $(\mathbf{z}_0, \widetilde{\mathbf{z}}_0) \in \Omega \times \Omega$,*

$$\|\mathbb{E}_{\mathbf{z}_0}\mathbf{Z}_t - \mathbb{E}_{\widetilde{\mathbf{z}}_0}\mathbf{Z}_t\|_{(1,1)} \leq \eta^t \|\mathbf{z}_0 - \widetilde{\mathbf{z}}_0\|_{(1,1)}.$$

*Then, variance of $\mathbf{Z}_t$*

$$v_t = \sup_{\mathbf{z}_0\in\Omega} \mathbb{V}ar_{\mathbf{z}_0}(\mathbf{Z}_t) = \sup_{\mathbf{z}_0\in\Omega} \mathbb{E}_{\mathbf{z}_0}\|\mathbf{Z}_t - \mathbb{E}_{\mathbf{z}_0}\mathbf{Z}_t\|_F^2$$

*satisfies that*

$$v_t \leq mqv_1 \min\{t, (1-\eta^2)^{-1}\}.$$

**Proposition 4.9.** *Assume $\beta < \beta_c/J$. Then, there exists $\rho_0$ such that if $0 < \rho \le \rho_0$, under $\rho$-bounded $\Sigma_F^\rho$ dynamics,*

$$\mathbb{V}ar^\rho_{\sigma_0}[\mathbb{S}_t] = O(n^{-1})$$

*uniformly in $\sigma_0 \in \Sigma_F^\rho$ and $t \ge 0$, and there exists $\gamma_0 > 0$ such that*

$$\mathbb{V}ar_{\sigma_0}[\mathbb{S}_t] = O(n^{-1})$$

*uniformly in $\sigma_0 \in \Sigma_F^{\rho_0}$ and $t \le e^{\gamma_0 n}$.*

*Proof.* Since $\beta < q/(2J)$ and $\rho$ is small enough, we can set

$$\eta = p + \frac{C\rho}{n} \le 1 - \frac{1-2\beta J/q}{2n},$$

for large enough $n$ according to Corollary 4.7. So, $\frac{1}{1-\eta^2} = O(n)$ and

$$\mathbb{V}ar_{\sigma_0}[\mathbb{S}_1] = \mathbb{E}_{\sigma_0}\|\mathbb{S}_1 - \mathbb{E}_{\sigma_0}[\mathbb{S}_1]\|_F^2 = O(n^{-2}).$$

We obtain the wanted result by applying Lemma 4.8. Moreover, by the large deviations theory, there exists $\gamma_0$ such that for $\sigma_0 \in \Sigma_F^{\rho_0}$ and $t \le e^{\gamma_0 n}$,

$$\mathbb{V}ar_{\sigma_0}[\mathbb{S}_t] = \mathbb{V}ar^\rho_{\sigma_0}[\mathbb{S}_t] + o(n^{-1}) = O(n^{-1}).$$

$\square$

4.5. **Drift of distance from the equilibrium state.** We analyze the distance between proportion matrix $\mathbb{S}_t$ and equiproportionality matrix $\mathbf{\Delta}$ defined in (2.1). We can observe that

$$\mathbb{E}\left[\langle \mathbb{S}_{t+1} - \mathbb{S}_t, \mathbb{S}_t - \mathbf{\Delta}\rangle_F | \mathcal{F}_t\right] = -\frac{m}{qn} - \frac{1}{n}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \frac{1}{n}\sum_{i=1}^m h^i(\mathbb{S}_t) + O(n^{-2}),$$

where $h^i(\mathbf{Z}) = \sum_{j=1}^q \mathbf{Z}^{ij} g_\beta^{(i,j)}(\mathbf{Z})$ in $\mathbf{Z} \in \mathcal{S}$, and

$$\mathbb{E}\big[\|\mathbb{S}_{t+1} - \mathbb{S}_t\|_F^2 | \mathcal{F}_t\big] = O(n^{-2}).$$

Combining the above results, we obtain that

$$\begin{aligned}\mathbb{E}\big[\|\mathbb{S}_{t+1} - \mathbf{\Delta}\|_F^2 | \mathcal{F}_t\big] &= \|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \mathbb{E}\big[\|\mathbb{S}_{t+1} - \mathbb{S}_t\|_F^2 | \mathcal{F}_t\big] + 2\mathbb{E}\big[\langle \mathbb{S}_{t+1} - \mathbb{S}_t, \mathbb{S}_t - \mathbf{\Delta}\rangle_F | \mathcal{F}_t\big] \\ &= \Big(1 - \frac{2}{n}\Big)\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + 2n^{-1}\sum_{i=1}^m \Big(h^i(\mathbb{S}_t) - \frac{1}{q}\Big) + O(n^{-2}).\end{aligned}$$

Now we research a function $h^i$ near $\mathbf{\Delta}$ using Taylor expansion. Since $h^i(\mathbf{Z}) = h^i(\mathbf{Z} - \mathbf{\Delta}) + 1/q$ and for $(i_1, j_1), (i_2, j_2) \in [1, m] \times [1, q]$,

$$\begin{aligned}&\frac{\partial h^i}{\partial \mathbf{Z}^{i_1 j_1}}(0) = \frac{\mathbb{1}_{\{i=i_1\}}}{q}, \\ &\frac{\partial^2 h^i}{\partial \mathbf{Z}^{i_1 j_1}\partial \mathbf{Z}^{i_2 j_2}}(0) = \frac{2\beta}{mq}\Big(\mathbb{1}_{\{j_1=j_2\}} - \frac{1}{q}\Big)\Big[2b\mathbb{1}_{\{i=i_1=i_2\}} + a(\mathbb{1}_{\{i=i_1\neq i_2\}} + \mathbb{1}_{\{i=i_2\neq i_1\}})\Big].\end{aligned}$$

Taylor expansion of $h^i$ is

$$h^i(\mathbf{Z}) = \frac{1}{q} + \frac{2\beta}{mq}\Big[(b-a)\langle \mathbf{Z}^{i*}, \mathbf{Q}\mathbf{Z}^{i*}\rangle + a\sum_{u=1}^{m}\langle \mathbf{Z}^{i*}, \mathbf{Q}\mathbf{Z}^{u*}\rangle\Big] + O(\|\mathbf{Z}\|_F^3),$$

where $\mathbf{Q} = \mathbf{I}_q - q^{-1}\mathbf{1}_{q\times q}$ is the orthogonal projection onto $(1,\cdots,1)^{\perp}$. Then,

$$\sum_{i=1}^{m}\Big(h^i(\mathbb{S}_t) - \frac{1}{q}\Big) = \frac{2\beta}{mq}\operatorname{Tr}((\mathbb{S}_t - \mathbf{\Delta})^{\intercal}\mathbf{K}(\mathbb{S}_t - \mathbf{\Delta})) + O(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^3).$$

Besides, through matrix calculation,

$$\begin{aligned}\operatorname{Tr}((\mathbb{S}_t - \mathbf{\Delta})^{\intercal}\mathbf{K}(\mathbb{S}_t - \mathbf{\Delta})) &= (b-a)\sum_{i=1}^{m}\sum_{j=1}^{q}\big((\mathbb{S}_t - \mathbf{\Delta})^{ij}\big)^2 + a\sum_{u,v=1}^{m}\sum_{j=1}^{q}(\mathbb{S}_t - \mathbf{\Delta})^{uj}(\mathbb{S}_t - \mathbf{\Delta})^{vj}\\ &\le (b-a)\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + a\|(\mathbb{S}_t - \mathbf{\Delta})(\mathbb{S}_t - \mathbf{\Delta})^{\intercal}\|_{(1,1)}\\ &\le (b-a)\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + am\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 = mJ\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2,\end{aligned}$$

where the last inequality holds because for $x \in \mathbb{R}^s$, $\|x\|_1 \le \sqrt{s}\|x\|_2$, and Frobenius norm is invariant under transpose operator and sub-multiplicative. Therefore,

$$\begin{aligned}\mathbb{E}\left[\|\mathbb{S}_{t+1} - \mathbf{\Delta}\|_F^2|\mathcal{F}_t\right] &\le (1-\frac{2}{n})\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \frac{2}{n}\frac{2\beta J}{q}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \frac{1}{n}O(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^3) + O(n^{-2})\\ &\le \Big(1 - \frac{2(1-2\beta J/q)}{n}\Big)\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \frac{1}{n}O(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^3) + O(n^{-2})\\ &\le p^2\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \frac{1}{n}O(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^3) + O(n^{-2}),\end{aligned}$$

where $p$ is defined in (3.2).

4.6. **Frobenius norm contraction.** Fix $\beta < \beta_c/J$, and suppose $\sigma_0 \in \Sigma_F^{\rho_0}$ as conditions of Proposition 4.9. Taking expectation on the above statement,

$$\mathbb{E}\|\mathbb{S}_{t+1} - \mathbf{\Delta}\|_F^2 \le p^2\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^3 \cdot O(n^{-1}) + O(n^{-2}).$$

Applying Taylor expansion to function $\cdot \to \|\cdot\|_F^3$ at $\mathbb{E}\left[\mathbb{S}_t - \mathbf{\Delta}\right]$,

$$\begin{aligned}\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^3 &= \|\mathbb{E}\left[\mathbb{S}_t - \mathbf{\Delta}\right]\|_F^3 + \Big\langle \nabla\|\cdot\|_F^3\big(\mathbb{E}\left[\mathbb{S}_t - \mathbf{\Delta}\right]\big), \mathbb{S}_t - \mathbf{\Delta} - \mathbb{E}\left[\mathbb{S}_t - \mathbf{\Delta}\right]\Big\rangle_F + O(n^{-1})\\ &= \|\mathbb{E}\left[\mathbb{S}_t - \mathbf{\Delta}\right]\|_F^3 + O(n^{-1}) = \big(\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + O(n^{-1})\big)^{3/2} + O(n^{-1})\\ &= \big(\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2\big)^{3/2} + O(n^{-1})\end{aligned}$$

via perspective of Proposition 4.9. Therefore,

$$\mathbb{E}\|\mathbb{S}_{t+1} - \mathbf{\Delta}\|_F^2 \le p^2\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \big(\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2\big)^{3/2}O(n^{-1}) + O(n^{-2}). \tag{4.2}$$

**Proposition 4.10.** *Fix $\beta < \beta_c/J$. Then, there exists $C > 0$ such that if $\rho \le \rho_0$ where $\rho_0$ is same as Proposition 4.9, there exists $\gamma(\rho) > 0$ such that:*

$$\mathbb{E}_{\sigma_0}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 = p^{2t}\left(\|\mathbb{S}_0 - \mathbf{\Delta}\|_F^2 + C\rho^3\right) + O(n^{-1}),$$

*uniformly in $\sigma_0 \in \Sigma_F^{\rho_0}$, and $t \le e^{\gamma(\rho)n}$.*

*Proof.* Define $\lambda_t := \mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2$, and let $\rho < \overline{\rho} < \rho_0$ where $\rho_0$ is defined in Proposition 4.9. There exists $\gamma(\rho, \overline{\rho}) > 0$ such that for uniformly $\sigma_0 \in \Sigma_F^{\rho}$ and $t \le e^{\gamma n}$,

$$\begin{aligned}\lambda_{t+1} &\le \lambda_t\big(p^2 + \lambda_t^{1/2} O(n^{-1})\big) + O(n^{-2})\\ &\le \lambda_t\big(p^2 + \overline{\rho} O(n^{-1})\big) + O(n^{-2}) = \lambda_t\big(p + \overline{\rho} O(n^{-1})\big)^2 + O(n^{-2}).\end{aligned}$$

We now use the following fact. If $(\lambda_t)_{t\ge 0}$ satisfies that

$$\lambda_{t+1} = p\lambda_t + ar^t + b,$$

for some $p \ne r$, $p \ne 1$, then;

$$\lambda_t = \lambda_0 p^t + a\,\frac{p^t - r^t}{p - r} + \frac{b}{1-p}(1 - p^t).$$

Using this fact, we obtain that for some $C > 0$,

$$\lambda_t \le \lambda_0\big(p + \overline{\rho} O(n^{-1})\big)^{2t} + \frac{1}{1 - \big(p + \overline{\rho} O(n^{-1})\big)^2} O(n^{-2}) \le C\overline{\rho}^2\big(p + \overline{\rho} O(n^{-1})\big)^{2t} + O(n^{-1}) \tag{4.3}$$

Plugging (4.3) into (4.2),

$$\lambda_{t+1} = p^2\lambda_t + \overline{\rho}^3\big(p + \overline{\rho} O(n^{-1})\big)^{3t} O(n^{-1}) + O(n^{-2}).$$

Again, by the inductive fact described above,

$$\begin{aligned}\lambda_t &\le \lambda_0 p^{2t} + \overline{\rho}^3 O(n^{-1}) \frac{p^{2t} - \big(p + \overline{\rho} O(n^{-1})\big)^{3t}}{p^2 - \big(p + \overline{\rho} O(n^{-1})\big)^3} + \frac{1}{1-p^2} O(n^{-2})\\ &= \big(\lambda_0 + O(\overline{\rho}^3)\big)p^{2t} + O(n^{-1}).\end{aligned}$$

Thus, we finish the proof. □

## 5. Mixing in the high-temperature regime

This section will prove the cutoff phenomena at $O(n \log n)$ time in the high-temperature regime. The cutoff time

$$t^{\xi}(n) = \xi n \log(n) \tag{5.1}$$

is determined by $\xi$ which was defined in (1.2). $t^{\xi}(n)$ is the crucial estimate of $O(n^{-1/2})$ contraction time for magnetization processes in the high-temperature region.

**Corollary 5.1.** *Fix $\beta < \beta_c/J$. Then, for all $r > 0$,*

$$\mathbb{P}_{\sigma_0}\left[\sigma_{t^\xi(n)} \notin \Sigma_F^{r/\sqrt{n}}\right] = O(r^{-1})$$

*uniformly in $\sigma_0 \in \Sigma_F^{\rho_0}$ where $\rho_0$ is defined in Proposition 4.9.*

*Proof.*

$$\begin{aligned}\mathbb{P}_{\sigma_0}\left[\sigma_{t^\xi(n)} \notin \Sigma_F^{r/\sqrt{n}}\right] &\le \mathbb{P}_{\sigma_0}\left[\|\mathbb{S}_{t^\xi(n)} - \mathbf{\Delta}\|_F \ge rn^{-1/2}\right] \\ &\le \frac{\mathbb{E}_{\sigma_0}\|\mathbb{S}_{t^\xi(n)} - \mathbf{\Delta}\|_F}{rn^{-1/2}} \le \frac{\left(\mathbb{E}_{\sigma_0}\|\mathbb{S}_{t^\xi(n)} - \mathbf{\Delta}\|_F^2\right)^{1/2}}{rn^{-1/2}} \le O(r^{-1}),\end{aligned}$$

where the last inequality holds by Proposition 4.10. □

5.1. **Proof of lower bound in Theorem 1.** In this subsection, we will prove the lower bound of cutoff in the high-temperature regime. Rigorously, we state that if $\beta < \beta_s/J$,

$$\lim_{\gamma\to-\infty} \liminf_{n\to\infty} \max_{\sigma_0\in\Sigma} \|\mathbb{P}_{\sigma_0}[\sigma_{t^\xi+\gamma n} \in \cdot] - \mu_*\|_{\mathrm{TV}} = 1,$$

where $t^\xi$ is defined in (5.1).

*Proof.* The analysis in this proof pertains to $\beta < \beta_c/J$. Fix $0 < \rho_2 < \rho_1 < \rho_0$, where $\rho_0$ is given in Proposition 4.9. Let $\sigma_0 \in \Sigma$ satisfy that $\rho_2 < \|\mathbb{S}_0 - \mathbf{\Delta}\|_F < \rho_1$. Then, if $t < t^\xi + \gamma n$, and $\rho_1$ is small enough, then by Proposition 4.10,

$$\mathbb{E}_{\sigma_0}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 \ge \frac{\rho_2^2}{2}\Big(1 - \frac{1-2\beta J/q}{n}\Big)^{2t^\xi+2\gamma n} + O(n^{-1})$$

for sufficiently large $-\gamma$ depending on $\rho_2$ and large enough $n$. Combined with the uniform variance bound given in Proposition 4.9, it follows that for large enough $n$,

$$\mathbb{E}_{\sigma_0}\|\mathbb{S}_t - \mathbf{\Delta}\|_F \ge \frac{C\rho_2}{\sqrt{n}} \exp\Big(-\frac{\gamma}{2\xi}\Big).$$

By Chebyshev's inequality, uniformly in all $r > 0$, $t \le t^\xi + \gamma n$, and $\sigma_0 \in \Sigma_F^{\rho_1} \setminus \Sigma_F^{\rho_2}$,

$$\begin{aligned}\mathbb{P}_{\sigma_0}\left[\|\mathbb{S}_t - \mathbf{\Delta}\|_F < \frac{r}{\sqrt{n}}\right] &\le \mathbb{P}_{\sigma_0}\left[\big|\|\mathbb{S}_t - \mathbf{\Delta}\|_F - \mathbb{E}_{\sigma_0}\|\mathbb{S}_t - \mathbf{\Delta}\|_F\big| > \frac{1}{\sqrt{n}}\big|C\rho_2 e^{-\gamma/2\xi} - r\big|\right] \\ &\le \frac{\mathbb{V}ar_{\sigma_0}\|\mathbb{S}_t - \mathbf{\Delta}\|_F}{n^{-1}\big|C\rho_2 e^{-\gamma/2\xi} - r\big|^2} \le O\Big(\big(C\rho_2 e^{-\gamma/2\xi} - r\big)^{-2}\Big),\end{aligned}$$

where the last inequality holds by the uniform variance bound. Therefore,

$$\lim_{\gamma\to-\infty} \limsup_{n\to\infty} \mathbb{P}_{\sigma_0}\left[\|\mathbb{S}_{t^\xi+\gamma n} - \mathbf{\Delta}\|_F < \frac{r}{\sqrt{n}}\right] = 0.$$

On the other side, with Chebyshev's inequality and Corollary 4.7, for $r > 0,\ t \ge 0$,

$$\mu_*\Big[\|\mathbb{S}_t - \mathbf{\Delta}\|_F \le \frac{r}{\sqrt{n}}\Big] = 1 - O(r^{-2}).$$

Combining two results, we obtain that

$$\lim_{\gamma\to-\infty}\liminf_{n\to\infty}\max_{\sigma_0\in\Sigma}\|\mathbb{P}_{\sigma_0}[\sigma_{t^\xi+\gamma n}\in\cdot]-\mu_*\|_{\mathrm{TV}}\geq 1-\frac{O(1)}{r^2}.$$

□

5.2. **Coordinatewise Coupling.** We obtain $O(n^{-1})$ from coalescence by constructing a more delicate coupling. Before the construction, we used the idea of a semi-independent coupling, which was discussed in [10].

**Definition 5.2.** [10, Subsection 4.3.] *Suppose that $\nu,\widetilde{\nu}$ be two probability measures on $\Omega=[1,m]$. Fix a non-empty subset $A\subset\Omega$. Then, $A$-semi-independent coupling of $\nu,\widetilde{\nu}$ is defined as a coupling of two random variables $X,\widetilde{X}$ constructed by the following process as;*

1. *Choose $U\in[0,1]$ uniformly.*
2. *If $U\leq\nu(A)\wedge\widetilde{\nu}(A)$, draw $X$ and $\widetilde{X}$ according to the optimal coupling of $(\nu|_A,\widetilde{\nu}|_A)$.*
3. *Otherwise, independently:*
   (a) *Draw $X$ according to $\nu|_A$ if $U<\nu(A)$ and according to $\nu|_{A^\complement}$ if $U\geq\nu(A)$.*
   (b) *Draw $\widetilde{X}$ according to $\widetilde{\nu}|_A$ if $U<\widetilde{\nu}(A)$ and according to $\widetilde{\nu}|_{A^\complement}$ if $U\geq\widetilde{\nu}(A)$.*

**Lemma 5.3.** [10, Proposition 4.2.] *Suppose that $X,\widetilde{X}$ are $A$-semi independent coupling of $\nu,\widetilde{\nu}$, where* A *is a non-empty subset of finite state space $\Omega$, and $\nu,\widetilde{\nu}$ are two probability distributions on $\Omega$. Then, the following statements hold;*

1. *$X,\widetilde{X}$ are distributed according to $\nu,\widetilde{\nu}$ respectively.*
2. $\mathbb{P}\big[\bigcup_{x\in A}\{X=x\}\boldsymbol{\Delta}\{\widetilde{X}=x\}\big]\leq\frac{3}{2}\sum_{x\in A}|\nu(x)-\widetilde{\nu}(x)|$.
3. $\forall x\notin A,\ \mathbb{P}\big[X=x,\widetilde{X}\neq x\big]\leq\nu(x)\widetilde{\nu}(A^c\setminus x),\ \mathbb{P}\big[\widetilde{X}=x,X\neq x\big]\leq\widetilde{\nu}(x)\nu(A^c\setminus x)$.

In this section, we define a coordinatewise coupling of $\sigma_t,\widetilde{\sigma}_t$. In a naive sense, we coalesce colors of two magnetizations one by one admitting slight fluctuation of previous colors. Formally, define for $u,r>0$, and $k\in[1,q]$,

$$\mathcal{H}_{u,r}^k=\left\{(\sigma,\widetilde{\sigma})\in\Sigma\times\Sigma:\|(\mathbb{S}-\widetilde{\mathbb{S}})^{*[1,k]}\|_{(1,1)}<\frac{u}{n},\|\mathbb{S}-\boldsymbol{\Delta}\|_F\vee\|\widetilde{\mathbb{S}}-\boldsymbol{\Delta}\|_F<\frac{r}{\sqrt{n}}\right\}$$

and set $\mathcal{H}_{u,r}=\mathcal{H}_{u,r}^q$.

We fix $\sigma_0,\widetilde{\sigma}_0\in\Sigma$, and $y_1,y_2,\cdots,y_{q-1}>0$. The coordinatewise coupling with parameters $y_1,y_2,\cdots,y_{q-1}$ and starting configurations $\sigma_0,\widetilde{\sigma}_0$ is defined as follows.

1. Set $T^{(0)}=0$, and $k=1$.
2. As long as $k\leq q-1$;
   (A) As long as $|\sum_{i=1}^m(\mathbb{S}_t-\widetilde{\mathbb{S}}_t)^{ik}|>\frac{y_k}{n}$;
      (a) Choose block number $l$ uniformly from $\{1,2,\cdots,m\}$.
      (b) Draw $I_{t+1},\widetilde{I}_{t+1}$, using $\{1,\cdots,k-1\}$-semi independent coupling of $\mathbb{S}_t^{l*},\widetilde{\mathbb{S}}_t^{l*}$.
      (c) Draw $J_{t+1},\widetilde{J}_{t+1}$, using $\{1,\cdots,k-1\}$-semi independent coupling of $g_\beta^{(l,*)}\left(\mathbb{S}_t-\frac{m}{n}e_{lI_{t+1}}\right),g_\beta^{(l,*)}\left(\widetilde{\mathbb{S}}_t-\frac{m}{n}e_{lI_{t+1}}\right)$.

(d) Change a uniformly chosen vertex in block $V_l$ of color $I_{t+1}$ in $\sigma_t$ to have color $J_{t+1}$ in $\sigma_{t+1}$.
(e) Change a uniformly chosen vertex in block $V_l$ of color $\widetilde{I}_{t+1}$ in $\widetilde{\sigma}_t$ to have color $\widetilde{J}_{t+1}$ in $\widetilde{\sigma}_{t+1}$.
(f) Set $t = t+1$.

(B) When $|\sum_{i=1}^m (\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{ik}| \leq \frac{y_k}{n}$, set $T^{(k)} = t - T^{(k-1)}$ and $k = k+1$.

(3) Set $T^{CC} = \sum_{k=1}^{q-1} T^{(k)}$.

We shall use $\mathbb{P}^{CC}_{\sigma_0,\widetilde{\sigma}_0}$ to denote the probability measure for this coupling and $\mathbb{P}^{CC(s)}_{\sigma_0,\widetilde{\sigma}_0}$ for the same coupling, only starting from the $s$-th stage. Notice that if the stopping condition at stage $s$ does not get satisfied forever, we set $T^{(s)} = T^{CC} = \infty$.

**Lemma 5.4.** *For $t \geq 0$, and $j \in [1,q]$,*

$$\mathbb{E}\Big[\sum_{i=1}^m (\mathbb{S}_{t+1} - \mathbb{S}_t)^{ij} \ \Big|\ \mathcal{F}_t\Big] = -\frac{1}{n}\Big(1 - \frac{2\beta J}{q}\Big)\sum_{i=1}^m (\mathbb{S}_t - \mathbf{\Delta})^{ij} + \frac{1}{n}O(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2) + O(n^{-2}).$$

*Proof.* Recalling (2.2),

$$\begin{aligned}
\mathbb{E}\big[\mathbb{S}^{ij}_{t+1} - \mathbb{S}^{ij}_t \mid \mathcal{F}_t\big] &= \frac{1}{n}\big(-\mathbb{S}^{ij}_t + g^{(i,j)}_\beta(\mathbb{S}_t)\big) + O(n^{-2}) \\
&= \frac{1}{n}\Bigg(-(\mathbb{S}_t - \mathbf{\Delta})^{ij} + \frac{2\beta}{mq}\Big[(b-a)(\mathbb{S}_t - \mathbf{\Delta})^{ij} + a\sum_{u=1}^m (\mathbb{S}_t - \mathbf{\Delta})^{uj}\Big] + O\big(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2\big)\Bigg) + O(n^{-2})
\end{aligned}$$

by Taylor expansion near equiproportionality matrix $\mathbf{\Delta}$. Thus,

$$\mathbb{E}\Big[\sum_{i=1}^m (\mathbb{S}^{ij}_{t+1} - \mathbb{S}^{ij}_t) \ \Big|\ \mathcal{F}_t\Big] = -\frac{1}{n}\Big(1 - \frac{2\beta J}{q}\Big)\sum_{i=1}^m (\mathbb{S}_t - \mathbf{\Delta})^{ij} + \frac{1}{n}O(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2) + O(n^{-2}).$$

□

**Proposition 5.5.** *Fix $\beta < q/(2J)$. Let $k \in [1, q-1]$. For all $\epsilon, u_{k-1}, r_{k-1} > 0$, there exist $y_k, u_k, r_k, \gamma_k > 0$, such that if $(\sigma_0, \widetilde{\sigma}_0) \in \mathcal{H}^{k-1}_{u_{k-1}, r_{k-1}}$, then*

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[T^{(k)} < \gamma_k n, (\sigma_{T^{(k)}}, \widetilde{\sigma}_{T^{(k)}}) \in \mathcal{H}^k_{u_k, r_k}\Big] \geq 1 - \epsilon.$$

*Proof.* Following all given assumptions, let small enough $\epsilon > 0$ be given. Define

$$\mathbb{W}_t = [W^1_t, W^2_t, \cdots, W^q_t]^\intercal$$

with its coordinate $W^j_t = \sum_{i=1}^m (\mathbb{S}^{ij}_t - \widetilde{\mathbb{S}}^{ij}_t)$ for each $j \in [1,q]$. Then, by Lemma 5.4,

$$\mathbb{E}[W^k_{t+1} - W^k_t \mid \mathcal{F}_t] = -\frac{1}{n}\Big(1 - \frac{2\beta J}{q}\Big)W^k_t + \frac{1}{n}O\big(\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 + \|\widetilde{\mathbb{S}}_t - \mathbf{\Delta}\|_F^2\big) + O(n^{-2}). \tag{5.2}$$

Since $\sigma_0, \widetilde{\sigma}_0 \in \Sigma_F^{r_{k-1}/\sqrt{n}}$,

$$\|\mathbb{S}_0 - \mathbf{\Delta}\|_F, \|\widetilde{\mathbb{S}}_0 - \mathbf{\Delta}\|_F = O(n^{-1/2}).$$

For some $r_k > 0$ to be chosen later, let

$$\tau^{(k)} = \inf\left\{t : \|\mathbb{S} - \mathbf{\Delta}\|_F \vee \|\widetilde{\mathbb{S}} - \mathbf{\Delta}\|_F \geq \frac{r_k}{\sqrt{n}}\right\},$$

and $\overline{W}_t^k = |W^k_{t\wedge\tau^{(k)}\wedge T^{(k)}}|$. We can take $y_k = y_k(r_k) > 0$ large enough such that

$$\begin{aligned}\mathbb{E}\Big[\overline{W}^k_{t+1} - \overline{W}^k_t \;\Big|\; \mathcal{F}_t\Big] &\leq -\frac{1-2\beta J/q}{n}\cdot \overline{W}^k_t + \frac{2r_k^2}{n^2} + O(n^{-2}) \\ &\leq -\frac{1-2\beta J/q}{n}\cdot\frac{y_k}{n} + O(n^{-2}) \leq 0.\end{aligned}$$

So, $\overline{W}^k_t$ is supermartingale with bounded drift because

$$\big|\overline{W}^k_{t+1} - \overline{W}^k_t\big| \leq \Big|\sum_{i=1}^m (\mathbb{S}_{t+1} - \mathbb{S}_t)^{ik}\Big| + \Big|\sum_{i=1}^m (\widetilde{\mathbb{S}}_{t+1} - \widetilde{\mathbb{S}}_t)^{ik}\Big| \leq \frac{2m}{n}.$$

If $t < \tau^{(k)} \wedge T^{(k)}$, then on event $\{l = u\}$,

$$\begin{aligned}&\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\overline{W}^k_{t+1} \neq \overline{W}^k_t \;\Big|\; \mathcal{F}_t\Big] \\ &\quad\geq \mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[I_{t+1} = k, J_{t+1}\neq k, \widetilde{I}_{t+1}\neq k\Big] \geq \mathbb{S}^{uk}_t\Big(\sum_{v=k+1}^q \widetilde{\mathbb{S}}^{uv}_t\Big)\Big[1 - g^{(u,k)}_\beta\Big(\mathbb{S}_t - \frac{m}{n}e_{uk}\Big)\Big] \\ &\quad\geq \mathbb{S}^{uk}_t\widetilde{\mathbb{S}}^{u(k+1)}_t\big[1 - g^{(u,k)}_\beta(\mathbb{S}_t) + O(n^{-1})\big] = \frac{q-1}{q^3} + O(n^{-1/2}),\end{aligned}$$

for each $u \in [1, m]$, where the last inequality holds by Taylor expansion near $\mathbf{\Delta}$. Therefore,

$$\begin{aligned}&\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\big|\overline{W}^k_{t+1} - \overline{W}^k_t\big|^2 \;\Big|\; \mathcal{F}_t\Big] \\ &\geq \sum_{i=1}^m \Big(\frac{m}{n}\Big)^2\cdot\frac{1}{m}\cdot\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\overline{W}^k_{t+1}\neq\overline{W}^k_t \mid l = i, \mathcal{F}_t\Big] \geq \frac{m^2}{n^2}\Big(\frac{q-1}{q^3} + O(n^{-1/2})\Big) \\ &\geq \frac{m^2}{n^2q^2}\Big(1 - \frac{1}{q}\Big) + O(n^{-5/2}) = \frac{m^2}{2q^2n^2} + O(n^{-5/2}),\end{aligned}$$

where the last inequality holds because $q > 2$. Moreover, since $\overline{W}^k_t \leq 2mr_k/\sqrt{n}$ and by viewpoint of (5.2), then

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\overline{W}^k_{t+1} - \overline{W}^k_t \;\Big|\; \mathcal{F}_t\Big] = O(n^{-3/2}).$$

Combining these two inequalities, there exists a constant $C_1 > 0$ being independent of $r_k$ and $y_k$ such that

$$\mathbb{V}ar^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\overline{W}^k_{t+1} \;\Big|\; \mathcal{F}_t\Big] \geq C_1 n^{-2},$$

uniformly in $n$. We employ Lemma 4.1 with setting $Z_t = \overline{W}^k_t$, and $N = \tau^{(k)} \wedge T^{(k)}$. Then, there exists large enough $\gamma_k > 0$ such that there exists a constant $C_2 > 0$ satisfying

$$\mathbb{P}_{\sigma_0,\widetilde{\sigma}_0}\Big[\gamma_k n < \tau^{(k)} \wedge T^{(k)}\Big] \leq \frac{C_2 r_{k-1}}{\sqrt{\gamma_k}}.$$

So we may choose $\gamma_k = \gamma_k(r_{k-1}, \epsilon)$, big enough to satisfy uniformly in $n > 0$,

$$\mathbb{P}_{\sigma_0,\widetilde{\sigma}_0}\left[\gamma_k n < \tau^{(k)} \wedge T^{(k)}\right] \leq \frac{\epsilon}{3}. \tag{5.3}$$

Besides by Proposition 4.3, there exists big enough $r_k > r_{k-1}$ such that

$$\mathbb{P}_{\sigma_0}\left[\exists 0 \leq s \leq \gamma_k n : \sigma_s \notin \Sigma_F^{r_k/\sqrt{n}}\right] \leq \frac{\epsilon}{6},\ \mathbb{P}_{\widetilde{\sigma}_0}\left[\exists 0 \leq s \leq \gamma_k n : \widetilde{\sigma}_s \notin \Sigma_F^{r_k/\sqrt{n}}\right] \leq \frac{\epsilon}{6}.$$

As a result, we obtain that

$$\mathbb{P}_{\sigma_0,\widetilde{\sigma}_0}\left[\tau^{(k)} < \gamma_k n\right] \leq \frac{\epsilon}{3}. \tag{5.4}$$

Applying Lemma 5.3, for each $u \in [1, m]$, on $\{l = u\}$,

$$\begin{aligned}
&\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\big(|\mathbb{S}^{us}_{t+1} - \widetilde{\mathbb{S}}^{us}_{t+1}|\big)_{s\in[1,k-1]} \neq \big(|\mathbb{S}^{us}_{t} - \widetilde{\mathbb{S}}^{us}_{t}|\big)_{s\in[1,k-1]} \;\Big|\; \mathcal{F}_t\Big] \\
&\leq \sum_{s=1}^{k-1} \mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\big(\{I_{t+1} = s\} \triangle \{\widetilde{I}_{t+1} = s\}\big) \cup \big(\{J_{t+1} = s\} \triangle \{\widetilde{J}_{t+1} = s\}\big) \;\Big|\; \mathcal{F}_t\Big] \\
&\leq \frac{3}{2}\sum_{s=1}^{k-1}\Big[|\mathbb{S}^{us}_t - \widetilde{\mathbb{S}}^{us}_t| + |g^{(u,s)}_\beta(\mathbb{S}_t) - g^{(u,s)}_\beta(\widetilde{\mathbb{S}}_t)|\Big] + O(n^{-1}) \\
&\leq C_3 \sum_{s=1}^{k-1} |\mathbb{S}^{us}_t - \widetilde{\mathbb{S}}^{us}_t| + O(r_k/n) + O(n^{-1})
\end{aligned}$$

with some constant $C_3 > 0$. Therefore, on $\{l = u\}$,

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\sum_{s=1}^{k-1}|\mathbb{S}^{us}_{t+1} - \widetilde{\mathbb{S}}^{us}_{t+1}| - \sum_{s=1}^{k-1}|\mathbb{S}^{us}_{t} - \widetilde{\mathbb{S}}^{us}_{t}| \;\Big|\; \mathcal{F}_t\Big] \leq \frac{C_4}{n}\sum_{s=1}^{k-1}|\mathbb{S}^{us}_t - \widetilde{\mathbb{S}}^{us}_t| + r_k O(n^{-2})$$

with some constant $C_4 > 0$. For fixed block number $u$,

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\sum_{u=1}^{m}\Big(\sum_{s=1}^{k-1}|\mathbb{S}^{us}_{t+1} - \widetilde{\mathbb{S}}^{us}_{t+1}| - \sum_{s=1}^{k-1}|\mathbb{S}^{us}_{t} - \widetilde{\mathbb{S}}^{us}_{t}|\Big) \;\Big|\; \mathcal{F}_t\Big] \leq \frac{C_4}{n}\sum_{u=1}^{m}\sum_{s=1}^{k-1}|\mathbb{S}^{us}_t - \widetilde{\mathbb{S}}^{us}_t| + O\Big(\frac{r_k}{n^2}\Big).$$

Hence we gain that,

$$\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\sum_{u=1}^{m}\sum_{s=1}^{k-1}|\mathbb{S}^{us}_{t+1} - \widetilde{\mathbb{S}}^{us}_{t+1}| \;\Big|\; \mathcal{F}_t\Big] \leq \Big(1 + \frac{C_4}{n}\Big)\sum_{i=1}^{m}\sum_{s=1}^{k-1}|\mathbb{S}^{is}_t - \widetilde{\mathbb{S}}^{is}_t| + O\Big(\frac{r_k}{n^2}\Big).$$

Let $\widehat{T}^{(k)} = \gamma_k n \wedge \tau^{(k)} \wedge T^{(k)}$. Inductively, by taking expectation and using assumption of $\sigma_0, \widetilde{\sigma}_0$,

$$\begin{aligned}
\mathbb{E}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left[\sum_{i=1}^{m}\big\|(\mathbb{S}_{\widehat{T}^{(k)}} - \widetilde{\mathbb{S}}_{\widehat{T}^{(k)}})^{(i,[1,k-1])}\big\|_{(1,1)}\right] &\leq \Big(1 + \frac{C}{n}\Big)^{\gamma_k n}\sum_{i=1}^{m}\big\|(\mathbb{S}_0 - \widetilde{\mathbb{S}}_0)^{(i,[1,k-1])}\big\|_{(1,1)} \\
&\leq \exp(\gamma_k C)\frac{u_{k-1}}{n}
\end{aligned}$$

for some $C > 0$. Taking appropriate $u_k,\ y_k > 0$,

$$\mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\|\overline{W}^{[1,k-1]}_{\gamma_k n}\|_1 > \frac{u_k - y_k}{n}\Big] = \mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\left[\sum_{j=1}^{k-1}\Big|\sum_{i=1}^{m}\big(\mathbb{S}^{ij}_{\widehat{T}^{(k)}} - \widetilde{\mathbb{S}}^{ij}_{\widehat{T}^{(k)}}\big)\Big| > \frac{u_k - y_k}{n}\right]$$
$$\leq \mathbb{P}^{CC(k)}_{\sigma_0,\widetilde{\sigma}_0}\Big[\Big\|\big(\mathbb{S}_{\widehat{T}^{(k)}} - \widetilde{\mathbb{S}}_{\widehat{T}^{(k)}}\big)\big)^{*[1,k-1]}\Big\|_{(1,1)} > \frac{u_k - y_k}{n}\Big] \leq C\frac{u_{k-1}}{u_k - y_k} < \frac{\epsilon}{3}.$$

Now, combining with (5.3), (5.4), the proof is done. ∎

Applying Proposition 5.5 repetitively, we can obtain the following result with $T^{CC} = \sum_{k=1}^{q-1} T^{CC(k)}$.

**Corollary 5.6.** *Fix $\beta < q/(2J)$. For $\epsilon, r > 0$, there exist $\gamma, u, r' > 0$ and $y_1, \cdots, y_{q-1} > 0$ such that for $\sigma_0, \widetilde{\sigma}_0 \in \Sigma_F^{r/\sqrt{n}}$, then*

$$\mathbb{P}^{CC}_{\sigma_0,\widetilde{\sigma}_0}\big[T^{CC} < \gamma n, (\sigma_{T^{CC}}, \widetilde{\sigma}_{T^{CC}}) \in \mathcal{H}_{u,r'}\big] \geq 1 - \epsilon.$$

5.3. **Coalescence of Proportion Matrix Chains.** After getting $O(n^{-1})$ coalescence of two Glauber dynamics, we prove the agreement of configurations in the perspective of magnetization operator $\mathbb{S}$.

**Proposition 5.7.** *Fix $\beta < q/(2J)$. For all $r, u, \epsilon > 0$, there exists $\gamma > 0$ such that, if $(\sigma_0, \widetilde{\sigma}_0) \in \mathcal{H}_{u,r}$, and $t \geq \gamma n$, under Synchronized Coupling,*

$$\mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left[\mathbb{S}_t \neq \widetilde{\mathbb{S}}_t\right] \leq \epsilon.$$

*Proof.* There exists big enough $\gamma > 0$ such that

$$\exp\Big(-\frac{\gamma}{2}\Big(1 - \frac{2\beta J}{q}\Big)\Big) < \frac{\epsilon}{u}.$$

Now, for $t \geq \gamma n$, by Corollary 4.5,

$$\mathbb{E}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left[\sum_{i=1}^{m}\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{i*}\|_1\right] \leq \Big(1 - \frac{1 - 2\beta J/q}{2n}\Big)^{\gamma n}\frac{u}{n}$$
$$\leq \exp\Big(-\frac{\gamma}{2}\Big(1 - \frac{2\beta J}{q}\Big)\Big)\frac{u}{n} < \frac{\epsilon}{n}.$$

Applying Markov inequality,

$$\mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left[\mathbb{S}_t \neq \widetilde{\mathbb{S}}_t\right] \leq \mathbb{P}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left[\sum_{i=1}^{m}\sum_{j=1}^{q}|\mathbb{S}^{ij}_t - \widetilde{\mathbb{S}}^{ij}_t| \geq \frac{m}{n}\right] \leq \frac{\mathbb{E}^{SC}_{\sigma_0,\widetilde{\sigma}_0}\left[\sum_{i=1}^{m}\|(\mathbb{S}_t - \widetilde{\mathbb{S}}_t)^{i*}\|_1\right]}{m/n} < \frac{\epsilon}{m}.$$

∎

5.4. **Basketwise Proportions Coalescence.** We construct small-sized partitions or baskets to improve the coalescence level from the proportion into the configuration.

Let $\mathfrak{B} = (\mathcal{B}_{li})_{l\in[1,q],i\in[1,m]}$ be a partition of $V$ satisfying that $V_i = \bigcup_{l\in[1,q]} \mathcal{B}_{li}$ for each $i \in [1, m]$. We say $\mathfrak{B}$ is $\lambda$-partition if,

$$|\mathcal{B}_{li}| > \lambda |V_i| \quad \forall l \in [1, q], i \in [1, m].$$

For given configuration $\sigma \in \Sigma$, we define a basketwise proportion matrix $\mathbf{S} = \mathbf{S}(\sigma)$ to be element of $\mathbb{R}^{q\times m\times q}$ whose $(l, i, j)$ entry is defined as;

$$\mathbf{S}^{lij}(\sigma) = \frac{1}{|\mathcal{B}_{li}|} \sum_{v\in\mathcal{B}_{li}} \mathbb{1}_{\{\sigma(v)=j\}}.$$

Then, $\mathbf{S}$ is an element in a manifold $\mathfrak{S} = \Pi_{l=1}^{q} \mathcal{S}$, and we define for $\rho > 0$ small,

$$\mathfrak{S}^\rho = \left\{ \mathbf{S} \in \mathfrak{S} : \left| \mathbf{S}^{lij} - \frac{1}{q} \right| < \rho \quad \forall i, l, j \right\}.$$

**Proposition 5.8.** *Suppose that $\mathfrak{B}$ is $\lambda$-partition for $\lambda > 0$. Then,*

(1) *For $l, j \in [1.q]$,*

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{m} (\mathbf{S}^{lij}_{t+1} - \mathbb{S}^{ij}_{t+1})\Big)^2 \;\Big|\; \mathcal{F}_t\Big] = \Big(1 - \frac{2}{n}\Big)\Big(\sum_{i=1}^{m} (\mathbf{S}^{lij}_{t} - \mathbb{S}^{ij}_{t})\Big)^2 + O(n^{-2}).$$

(2) *For $i_1 \neq i_2$ in $[1, m]$ and $l, j \in [1, q]$,*

$$\mathbb{E}\Big[(\mathbf{S}^{li_1j}_{t+1} - \mathbb{S}^{i_1j}_{t+1})(\mathbf{S}^{li_2j}_{t+1} - \mathbb{S}^{i_2j}_{t+1})\Big|\mathcal{F}_t\Big] = \Big(1 - \frac{2}{n}\Big)(\mathbf{S}^{li_1j}_{t} - \mathbb{S}^{i_1j}_{t})(\mathbf{S}^{li_2j}_{t} - \mathbb{S}^{i_2j}_{t}) + O(n^{-2}).$$

*Proof.* To prove (1), let $\lambda_{li} = |\mathcal{B}_{li}|/|V_i|$, and fix $l, j \in [1, q]$. Also, for time $t \geq 0$, we define a process $Q^{lj}_t = \sum_{i=1}^{m} m^{-1}(\mathbf{S}^{lij}_t - \mathbb{S}^{ij}_t)$. We investigate the four cases of $Q^{lj}_{t+1} \neq Q^{lj}_t$ as follow;

(A) $v_{t+1} \notin \mathcal{B}_{li}$, $\sigma_t(v_{t+1}) = j$, $\sigma_{t+1}(v_{t+1}) \neq j$,
(B) $v_{t+1} \notin \mathcal{B}_{li}$, $\sigma_t(v_{t+1}) \neq j$, $\sigma_{t+1}(v_{t+1}) = j$,
(C) $v_{t+1} \in \mathcal{B}_{li}$, $\sigma_t(v_{t+1}) = j$, $\sigma_{t+1}(v_{t+1}) \neq j$,
(D) $v_{t+1} \in \mathcal{B}_{li}$, $\sigma_t(v_{t+1}) \neq j$, $\sigma_{t+1}(v_{t+1}) = j$

with probabilities $p_A, p_B, p_C, p_D$ of each events, respectively. Therefore,

$$\begin{aligned}\mathbb{E}\big[(Q^{lj}_{t+1})^2 - (Q^{lj}_t)^2 \;\big|\; v_{t+1} \in V_i, \mathcal{F}_t\big] &= \frac{2}{n} Q^{lj}_t \left[(p_A - p_B) + \Big(1 - \frac{1}{\lambda_{li}}\Big)(p_C - p_D)\right] + O(n^{-2}) \\ &= \frac{2}{n} Q^{lj}_t (\mathbb{S}^{ij}_t - \mathbf{S}^{lij}_t) + O(n^{-2}).\end{aligned}$$

Taking expectation,

$$\mathbb{E}\big[(Q^{lj}_{t+1})^2 - (Q^{lj}_t)^2 \;\big|\; \mathcal{F}_t\big] = \frac{2}{n} Q^{lj}_t \sum_{i=1}^{m} m^{-1}(\mathbb{S}^{ij}_t - \mathbf{S}^{lij}_t) + O(n^{-2}) = -\frac{2}{n}(Q^{lj}_t)^2 + O(n^{-2}).$$

For (2), fix $i_1 \neq i_2$ in $[1, m]$, and $l, j \in [1, q]$. We now investigate $R_t = (\mathbf{S}^{li_1j}_t - \mathbb{S}^{i_1j}_t)(\mathbf{S}^{li_2j}_t - \mathbb{S}^{i_2j}_t)$. Then, $R_{t+1} \neq R_t$, if and only if, $v_{t+1} \in V_{i_1}$ or $v_{t+1} \in V_{i_2}$.

Assume that $v_{t+1} \in V_{i_1}$. With similar logic, we need to investigate four cases such as;

(a) $v_{t+1} \notin \mathcal{B}_{li_1}$, $\sigma_t(v_{t+1}) = j$, $\sigma_{t+1}(v_{t+1}) \neq j$,
(b) $v_{t+1} \notin \mathcal{B}_{li_1}$, $\sigma_t(v_{t+1}) \neq j$, $\sigma_{t+1}(v_{t+1}) = j$,
(c) $v_{t+1} \in \mathcal{B}_{li_1}$, $\sigma_t(v_{t+1}) = j$, $\sigma_{t+1}(v_{t+1}) \neq j$,
(d) $v_{t+1} \in \mathcal{B}_{li_1}$, $\sigma_t(v_{t+1}) \neq j$, $\sigma_{t+1}(v_{t+1}) = j$,

with probabilities $p_a, p_b, p_c, p_d$ of each events, respectively. Then,

$$\begin{aligned}&\mathbb{E}\big[(\mathbf{S}_{t+1}^{li_1j} - \mathbb{S}_{t+1}^{i_1j})(\mathbf{S}_{t+1}^{li_2j} - \mathbb{S}_{t+1}^{i_2j}) - (\mathbf{S}_{t}^{li_1j} - \mathbb{S}_{t}^{i_1j})(\mathbf{S}_{t}^{li_2j} - \mathbb{S}_{t}^{i_2j}) \bigm| v_{t+1} \in V_{i_1}, \mathcal{F}_t\big] \\ &= \frac{\mathbf{S}_t^{li_2j} - \mathbb{S}_t^{i_2j}}{n/m}\Big[(p_a - p_b) + \Big(1 - \frac{1}{\lambda_{li_1}}\Big)(p_c - p_d)\Big] = -\frac{1}{n/m}(\mathbf{S}_t^{li_2j} - \mathbb{S}_t^{i_2j})(\mathbf{S}_t^{li_1j} - \mathbb{S}_t^{i_1j}) + O(n^{-2}).\end{aligned}$$

Taking expectation,

$$\begin{aligned}&\mathbb{E}\big[(\mathbf{S}_{t+1}^{li_1j} - \mathbb{S}_{t+1}^{i_1j})(\mathbf{S}_{t+1}^{li_2j} - \mathbb{S}_{t+1}^{i_2j}) - (\mathbf{S}_{t}^{li_1j} - \mathbb{S}_{t}^{i_1j})(\mathbf{S}_{t}^{li_2j} - \mathbb{S}_{t}^{i_2j}) \bigm| \mathcal{F}_t\big] \\ &= -\frac{2}{n}(\mathbf{S}_t^{li_1j} - \mathbb{S}_t^{i_1j})(\mathbf{S}_t^{li_2j} - \mathbb{S}_t^{i_2j}) + O(n^{-2}).\end{aligned}$$

Therefore, we obtained the wanted results. □

**Corollary 5.9.** *Suppose that $\mathfrak{B}$ is $\lambda$-partition with $\lambda > 0$, and $l, j \in [1, q]$. Then,*

$$\mathbb{E}\Big[\sum_{i=1}^m \big(\mathbf{S}_{t+1}^{lij} - \mathbb{S}_{t+1}^{ij}\big)^2 \Bigm| \mathcal{F}_t\Big] = \Big(1 - \frac{2}{n}\Big)\sum_{i=1}^m \big(\mathbf{S}_t^{lij} - \mathbb{S}_t^{ij}\big)^2 + O(n^{-2}).$$

*Proof.* Expanding sum,

$$\big(\sum_{i=1}^m (\mathbf{S}_t^{lij} - \mathbb{S}_t^{ij})\big)^2 = \sum_{i=1}^m (\mathbf{S}_t^{lij} - \mathbb{S}_t^{ij})^2 + 2\sum_{i_1 \neq i_2}(\mathbf{S}_t^{li_1j} - \mathbb{S}_t^{i_1j})(\mathbf{S}_t^{li_2j} - \mathbb{S}_t^{i_2j})$$

and we apply Proposition 5.8. □

We enhance the coalescence level.

**Lemma 5.10.** *Suppose that $\mathfrak{B}$ is $\lambda$-partition for $0 < \lambda < 1$. Fix $\beta < q/(2J)$ and $\epsilon > 0$ be given. If either of the following holds;*

(1) *$\sigma_0 \in \Sigma_F^{\rho_0}$, and $t^\xi(n) \le t \le e^{\gamma(\rho_0)n}$, where $\rho_0, \gamma(\rho_0)$ are given in Proposition 4.9 and $t^\xi(n)$ is defined in (5.1),*
(2) *$\mathbf{S}_0 \in \mathfrak{S}^{r_0/\sqrt{n}}$ and $t \le \gamma_0 n$ for some $r_0, \gamma_0 > 0$,*

*then, as $r \to \infty$, uniformly in big enough $n > 0$,*

$$\mathbb{P}_{\sigma_0}[\mathbf{S}_t \notin \mathfrak{S}^{r/\sqrt{n}}] \le \epsilon.$$

*Proof.* We decompose $\mathbb{S}_0^{ij}$ into $\mathbf{S}_0^{lij}$'s;

$$\begin{aligned}\mathbb{S}_0^{ij} &= \frac{1}{n/m}\sum_{v \in V_i} \mathbb{1}_{\{\sigma_0(v) = j\}} \\ &= \frac{1}{n/m}\sum_{l=1}^q \sum_{v \in \mathcal{B}_{li}} \mathbb{1}_{\{\sigma_0(v) = j\}} = \frac{m}{n}\sum_{l=1}^q |\mathcal{B}_{li}|\mathbf{S}_0^{lij}.\end{aligned}$$

So, $|\mathbb{S}_0^{ij} - \frac{1}{q}| \leq \sum_{l=1}^q \lambda_{li}|\mathbf{S}_0^{lij} - \frac{1}{q}|$, and we define $\nu_t = \mathbb{E}\big[\sum_{i=1}^m (\mathbf{S}_t^{lij} - \mathbb{S}_t^{ij})^2\big]$. Then, by Corollary 5.9, we have the recurrence relation $\nu_{t+1} = (1-2/n)\nu_t + O(n^{-2})$. As a result, we obtain $\nu_t = (1-2/n)^t \nu_0 + O(n^{-1})$ for time $t \leq \gamma_0 n$. Now,

$$\mathbb{E}\Big[\sum_{i=1}^m (\mathbf{S}_t^{lij} - \mathbb{S}_t^{ij})^2\Big] = O(n^{-1}), \tag{5.5}$$

because for Case (1), with some constant $C > 0$,

$$\nu_t \leq C\left(1 - \frac{2}{n}\right)^{t^\xi} + O(n^{-1}) = O(n^{-1})$$

and for Case (2), $\nu_0 \leq \sum_{i=1}^m (2r_0/\sqrt{n})^2 = O(n^{-1})$. Using (5.5), we estimate the distance between proportion matrix $\mathbb{S}_t$ and basketwise proportion matrix $\mathbf{S}_t$.

$$\begin{aligned}\mathbb{P}_{\sigma_0}\Big[\sum_{l=1}^q \|\mathbf{S}_t^{l**} - \mathbb{S}_t\|_F \geq \frac{r}{\sqrt{n}}\Big] &\leq \frac{\mathbb{E}_{\sigma_0}\big[(\sum_{l=1}^q \|\mathbf{S}_t^{l**} - \mathbb{S}_t\|_F)^2\big]}{r^2/n} \leq \frac{qn}{r^2}\sum_{l=1}^q \mathbb{E}_{\sigma_0}\|\mathbf{S}_t^{l**} - \mathbb{S}_t\|_F^2 \\ &\leq \frac{qn}{r^2}\sum_{l=1}^q \mathbb{E}_{\sigma_0}\Big[\sum_{j=1}^q\sum_{i=1}^m (\mathbf{S}_t^{lij} - \mathbb{S}_t^{ij})^2\Big] = O(r^{-2}).\end{aligned}$$

In addition, we estimate the distance between $\mathbb{S}_t$ and $\mathbf{\Delta}$, claiming that for $r > 0$ big enough,

$$\mathbb{P}_{\sigma_0}\Big[\|\mathbb{S}_t - \mathbf{\Delta}\|_F > \frac{r}{\sqrt{n}}\Big] < \epsilon. \tag{5.6}$$

For Case 1, since $\sigma_0 \in \Sigma_F^{\rho_0}$, by Proposition 4.10,

$$\begin{aligned}\mathbb{E}\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2 &\leq \Big(1 - \frac{1-2\beta J/q}{n}\Big)^{2t^\xi}\big(\|\mathbb{S}_0 - \mathbf{\Delta}\|_F^2 + C\rho_0^3\big) + O(n^{-1}) \\ &\leq \exp(-\log n)C\rho_0^3 + O(n^{-1}) = O(n^{-1}).\end{aligned}$$

Applying Markov inequality,

$$\mathbb{P}_{\sigma_0}\Big[\|\mathbb{S}_t - \mathbf{\Delta}\|_F > \frac{r}{\sqrt{n}}\Big] = \frac{\mathbb{E}_{\sigma_0}\big[\|\mathbb{S}_t - \mathbf{\Delta}\|_F^2\big]}{r^2/n} = O(r^{-2}).$$

For Case 2, since $\sigma_0 \in \Sigma_F^{r_0/\sqrt{n}}$, then by Proposition 4.3, for $r > r_0$ big enough,

$$\mathbb{P}_{\sigma_0}\big[\|\mathbb{S}_t - \mathbf{\Delta}\|_F \geq r/\sqrt{n}\big] \leq \mathbb{P}_{\sigma_0}\big[\exists 0 \leq s \leq \gamma_0 n : \sigma_s \notin \Sigma_F^{r/\sqrt{n}}\big] \leq \epsilon.$$

Hence, for big enough $r$,

$$\mathbb{P}_{\sigma_0}\Big[\|\mathbb{S}_t - \mathbf{\Delta}\|_F \geq \frac{r}{\sqrt{n}}\Big] \leq \epsilon.$$

In conclusion, as $r \to \infty$,

$$\begin{aligned}&\mathbb{P}_{\sigma_0}\big[\mathbf{S}_t \notin \mathfrak{S}^{r/\sqrt{n}}\big] = \mathbb{P}_{\sigma_0}\Big[\exists(l,i,j) \text{ such that } \Big|\mathbf{S}_t^{lij} - \frac{1}{q}\Big| \geq \frac{r}{\sqrt{n}}\Big] \\ &\leq \mathbb{P}_{\sigma_0}\Big[\sum_{l=1}^q \|\mathbf{S}_t^{l**} - \mathbb{S}_t\|_F, \|\mathbb{S}_t - \mathbf{\Delta}\|_F < \frac{r}{2\sqrt{n}}, \text{ and } \exists(l,i,j) \text{ such that } \Big|\mathbf{S}_t^{lij} - \frac{1}{q}\Big| \geq \frac{r}{\sqrt{n}}\Big] + \epsilon < 2\epsilon.\end{aligned}$$

$\square$

Now we construct the basketwise coupling. Suppose that two initial configurations $\sigma_0, \widetilde{\sigma}_0$ have same proportion matrix as $\mathbb{S}_0 = \widetilde{\mathbb{S}}_0$. This coupling will lead to make $\mathbf{S}_t = \widetilde{\mathbf{S}}_t$ eventually. Once for basket number $li$, $\mathbf{S}^{li*} = \widetilde{\mathbf{S}}^{li*}$ is achieved, it would be lasted forever. Rigorously, using previous definitions of $\lambda$-basket $\mathfrak{B}$, the basketwise coupling is defined as follows;

1. Choose block number $i$ uniformly in $[1, m]$.
2. Set $t = 0, l = 1$.
   (1) As long as $l \leq q$;.
      A. As long as $\mathbf{S}_t^{li} \neq \widetilde{\mathbf{S}}_t^{li}$;
         (A) Choose old color $I_{t+1}$ according to $\mathbb{S}_t^{i*} = \widetilde{\mathbb{S}}_t^{i*}$.
         (B) Choose new color $J_{t+1}$ according to $g_\beta^{(i,*)}(\mathbb{S}_t - \frac{m}{n} e_{iI_{t+1}}) = g_\beta^{(i,*)}(\widetilde{\mathbb{S}}_t - \frac{m}{n} e_{iI_{t+1}})$.
         (C) Choose a vertex $v_{t+1}$ uniformly among all vertices in $V_i$ having color $I_{t+1}$ under $\sigma_t$.
         (D) Choose $\widetilde{v}_{t+1}$;
            (a) If $v_{t+1} \in \mathcal{B}_{l_0 i} \quad \exists l_0 < l$, then choose $\widetilde{v}_{t+1}$ uniformly among all vertices in $\mathcal{B}_{l_0 i}$ having color $I_{t+1}$ under $\widetilde{\sigma}_t$.
            (b) Else if $\mathbf{S}_t^{liI_{t+1}} \neq \widetilde{\mathbf{S}}_t^{liI_{t+1}}$ and $\mathbf{S}_t^{liJ_{t+1}} \neq \widetilde{\mathbf{S}}_t^{liJ_{t+1}}$, then choose $\widetilde{v}_{t+1}$ among all vertices in $\mathcal{B}_{[l,q]i}$ having color $I_{t+1}$ under $\widetilde{\sigma}_t$.
            (c) Otherwise, let $v^1, v^2, \cdots$ be an enumeration of vertices in $\mathcal{B}_{[l,q]i}$ having color $I_{t+1}$ ordered first by index of the basket they belong to and then by their index in $V_i$, and let $\widetilde{v}^1, \widetilde{v}^2, \cdots$ be the same for $\widetilde{\sigma}_t$. Then, set $\widetilde{v}_{t+1} = \widetilde{v}^\theta$, where $\theta$ is the index such that $v_{t+1} = v^\theta$.
         (E) Set $\sigma_{t+1}(v_{t+1}) = \widetilde{\sigma}_{t+1}(\widetilde{v}_{t+1}) = J_{t+1}$, and $t = t + 1$.
      B. Set $l = l + 1$.

And we denote the probability measure $\mathbb{P}^{BC}_{\sigma_0, \widetilde{\sigma}_0}$ for this basketwise coupling.

**Proposition 5.11.** *Fix $\beta < q/(2J)$. For $\lambda, r, \epsilon > 0$, there exists $\gamma = \gamma(\lambda, r, \epsilon) > 0$ such that for any $\lambda$-partition, and starting configurations $\sigma_0, \widetilde{\sigma}_0$ with $\mathbb{S}_0 = \widetilde{\mathbb{S}}_0$ and $\mathbf{S}_0, \widetilde{\mathbf{S}}_0 \in \mathfrak{S}^{r/\sqrt{n}}$, under Basketwise Coupling,*

$$\mathbb{P}^{BC}_{\sigma_0, \widetilde{\sigma}_0}[\mathbf{S}_{\gamma n} = \widetilde{\mathbf{S}}_{\gamma n}] \geq 1 - \epsilon.$$

*Proof.* According to the definition of basketwise coupling, once the proportion matrices of basket $\mathcal{B}_{li}$ have coalesced, they will remain forever. So it is enough to investigate coalescence time for some basket $\mathcal{B}_{li}$ for fixed $l \in [1, q]$ and $i \in [1, m]$. We may assume that $\mathbb{S}_t = \widetilde{\mathbb{S}}_t$ for any $t \geq 0$, and $\mathbf{S}_t^{[1,l-1]i*} = \widetilde{\mathbf{S}}_t^{[1,l-1]i*}$. Define $q \times q$ dimensional stochastic processes

$$\mathbf{W}_t = \mathbf{S}_t^i - \widetilde{\mathbf{S}}_t^i, \ W_t^l = \sum_{j=1}^q \left| \mathbf{W}_t^{lj} \right|,$$

where $\mathbf{W}_t^{lj} = \mathbf{S}_t^{lij} - \widetilde{\mathbf{S}}_t^{lij}$ for $l, j \in [1, q]$. We also define stopping times $\tau^{(0)} = 0$, and $\tau^{(l)} = \inf\{t \geq \tau^{(l-1)} : W_t^l = 0\}$ inductively for $l \in [1, q]$. Also, define $\tau_* = \inf\{t : \mathbf{S}_t \notin \mathfrak{S}^\rho \text{ or } \widetilde{\mathbf{S}}_t \notin \mathfrak{S}^\rho\}$ for sufficiently small $\rho > 0$, and let $\tau_*^{(l)} = \tau^{(l)} \wedge \tau_*$.

Assume the event $\{\tau^{(l-1)} \leq t < \tau^{(l)}, t < \tau_*\}$. By step 2-(1)-A-(D), we can divide into three cases as;

(a) Since $\mathbf{S}^{li}_{t+1} = \mathbf{S}^{li}_t$, $\widetilde{\mathbf{S}}^{li}_{t+1} = \widetilde{\mathbf{S}}^{li}_t$, then
$$W^l_{t+1} = W^l_t.$$

(b) Since $\mathbf{S}^{liI_{t+1}}_t \neq \widetilde{\mathbf{S}}^{liI_{t+1}}_t$, $\mathbf{S}^{liJ_{t+1}}_t \neq \widetilde{\mathbf{S}}^{liJ_{t+1}}_t$, then $\mathbf{W}^{lI_{t+1}}_{t+1}\mathbf{W}^{lI_{t+1}}_t \geq 0$, and $\mathbf{W}^{lJ_{t+1}}_{t+1}\mathbf{W}^{lJ_{t+1}}_t \geq 0$. Therefore,
$$\begin{aligned}\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[W^l_{t+1} - W^l_t \mid \mathcal{F}_t\big] =& \Big|\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\mathbf{W}^{lI_{t+1}}_{t+1} \mid \mathcal{F}_t\big]\Big| - \Big|\mathbf{W}^{lI_{t+1}}_t\Big| \\ &+ \Big|\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\mathbf{W}^{lJ_{t+1}}_{t+1} \mid \mathcal{F}_t\big]\Big| - \Big|\mathbf{W}^{lJ_{t+1}}_t\Big| \leq 0.\end{aligned}$$

(c) If $v_{t+1}, \widetilde{v}_{t+1} \in \mathcal{B}_{li}$ or $v_{t+1}, \widetilde{v}_{t+1} \notin \mathcal{B}_{li}$, then $W^l_{t+1} = W^l_t$. Otherwise if $v_{t+1} \in \mathcal{B}_{li}$ and $\widetilde{v}_{t+1} \notin \mathcal{B}_{li}$, then
$$|\mathbf{W}^{lI_{t+1}}_{t+1}| - |\mathbf{W}^{lI_{t+1}}_t| = -\frac{1}{|\mathcal{B}_{li}|} \text{ and } |\mathbf{W}^{lJ_{t+1}}_{t+1}| - |\mathbf{W}^{lJ_{t+1}}_t| < \frac{1}{|\mathcal{B}_{li}|}.$$
And the other case can be interpreted similarly by symmetry.

Combining all results from three cases, $\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[W^l_{t+1} - W^l_t \mid \mathcal{F}_t] \leq 0$. In other words, $(W^l_t)_{\tau^{(l-1)} \leq t < \tau^{(l)}, t < \tau_*}$ is a supermartingale. Besides, we can observe that the probability of the event (b) and variance of $W^l_{t+1}$ under event (b) are both uniformly bounded below, independent of $n$. Moreover, by Proposition 4.3, if there exists some time $t$ such that $\mathbf{S}_t, \widetilde{\mathbf{S}}_t \in \mathfrak{S}^{r/\sqrt{n}}$, then $\mathbf{S}_{t+\gamma' n}, \widetilde{\mathbf{S}}_{t+\gamma' n} \in \mathfrak{S}^{r'/\sqrt{n}}$ with probability $1 - \epsilon$ for some $\gamma', r'$. Since $|W^l_{t+1} - W^l_t| = O(n^{-1})$, we can apply Lemma 4.1 with $Z_t = W^l_t$, $N = \tau^{(l)}_*$. Therefore, there exists big enough $\gamma_l$ such that
$$\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\gamma_l n < \tau^{(l)} \wedge \tau_*\big] \leq \frac{C}{\sqrt{\gamma_l n}} < \frac{\epsilon}{2q},$$
uniformly in $n$. We define
$$B^{lj} = \bigcup_{t=1}^{\gamma n} \big\{|\mathbf{S}^{lij}_t - 1/q| \geq \rho\big\}, \quad Y^{lj} = \Big|\big\{1 \leq t \leq \gamma n : |\mathbf{S}^{lij}_t - 1/q| \geq \rho/2\big\}\Big|$$
and $\gamma := \sum_{l=1}^{q-1} \gamma_l$. Then,
$$\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[Y^{lj}] \leq \gamma n \mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[|\mathbf{S}^{lij}_t - 1/q| \geq \rho/2\big] \leq \gamma n \mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\mathbf{S}_t \notin \mathfrak{S}^{r/\sqrt{n}}\big] \leq \gamma n \epsilon$$
by taking $r = \rho\sqrt{n}/2$. Moreover,
$$B^{lj} \implies \exists 1 \leq t \leq \gamma n \ : |\mathbf{S}^{lij}_t - 1/q| \geq \rho \implies Y^{lj} \geq \rho n \lambda/(2m).$$
Therefore by Markov inequality,
$$\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[B^{lj}] \leq \mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[Y^{lj} \geq \rho n \lambda/(2m)\big] \leq \frac{\mathbb{E}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[Y^{lj}]}{\rho n \lambda/(2m)} = O(\epsilon).$$
So we obtain that $\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}[\tau_* \leq \gamma n] = O(\epsilon)$. Thus, for uniformly large enough $n$,
$$\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\tau^{(q)} > \gamma n\big] \leq 2\mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\gamma n \leq \tau^{(q)}_*\big] + \mathbb{P}^{BC}_{\sigma_0,\widetilde{\sigma}_0}\big[\tau_* \leq \gamma n\big] \leq \epsilon + O(\epsilon) = O(\epsilon).$$
□

5.5. **The Overall Coupling.** We now describe how previous couplings are combined to create the *overall coupling*. We define the overall coupling with parameters $\gamma_1, \gamma_2, \gamma_4, \gamma_5, \gamma_6$, $y_1, \cdots, y_{q-1} > 0$ , and initial configurations $\sigma_0, \widetilde{\sigma}_0 \in \Sigma$ by setting two chains $(\sigma_t)_{t\geq 0}$, $(\widetilde{\sigma}_t)_{t\geq 0}$ under probability measure $\mathbb{P}^{OC}$ as follows;

(1) Run $\sigma_t, \widetilde{\sigma}_t$ independently until the time $t^{(1)}(n) = \gamma_1 n$.
(2) Run $\sigma_t, \widetilde{\sigma}_t$ according to the *aggregate path coupling* until the time $t^{(2)}(n) = t^{(1)} + \gamma_2 n$.
  - Partition a vertex set $V$ into baskets $\mathfrak{B} = (\mathcal{B}_{li})_{l\in[1,q], i\in[1,m]}$ such that for each $l \in [1, q],\ i \in [1, m]$,
$$\mathcal{B}_{li} = \{v \in V_i : \sigma_{t^{(2)}}(v) = l\}.$$
(3) Run $\sigma_t,\ \widetilde{\sigma}_t$ independently until time $t^{(3)}(n) = t^{(2)} + t^{\xi}(n)$, where $t^{\xi}$ is defined in (5.1).
(4) Run $\sigma_t,\ \widetilde{\sigma}_t$ according to the *coordinatewise coupling* with parameters $y_1, \cdots y_{q-1} > 0$ until the time $t^{(4)}(n) = t^{(3)} + \gamma_4 n$.
(5) Run $\sigma_t,\ \widetilde{\sigma}_t$ according to the *synchronized coupling* until the time $t^{(5)}(n) = t^{(4)} + \gamma_5 n$.
(6) Run $\sigma_t,\ \widetilde{\sigma}_t$ according to the *basketwise coupling* until the time $t^{(6)}(n) = t^{(5)} + \gamma_6 n$ with defined baskets on the above.

5.6. **Proof of upper bound in Theorem 1.** In this subsection, we prove that if $\beta < \beta_s/J$,
$$\lim_{\gamma\to\infty} \limsup_{n\to\infty} \max_{\sigma_0\in\Sigma} \|\mathbb{P}_{\sigma_0}[\sigma_{t^{\xi}+\gamma n} \in \cdot] - \mu_*\|_{\mathrm{TV}} = 0,$$
where $t^{\xi}$ is defined in (5.1).

*Proof.* Fix $\beta < \beta_s/J$, and suppose small $\epsilon > 0$ be given. Let $\sigma_0$ be an arbitrary configuration in $\Sigma$, and $\widetilde{\sigma}_0$ be the stationary distribution $\mu_*$. Firstly, by (2.3), we know that for big $\gamma_1 > 0$,
$$\mathbb{P}^{OC}_{\sigma_0}\big[\|\widetilde{\mathbb{S}}_{t^{(1)}} - \mathbf{\Delta}\|_{(1,1)} \geq \epsilon'\big] \leq \exp\Big(-\frac{n}{\gamma}\min\big\{I(\mathbf{z}) : \|\mathbf{z} - \mathbf{\Delta}\|_{(1,1)} = \epsilon'\big\}\Big) < \epsilon.$$
Now we can apply Proposition 3.9, and so we can select big enough $\gamma_2$ to satisfy
$$\begin{aligned}\mathbb{E}^{OC}_{\sigma_0}\big[\|\mathbb{S}_{t^{(2)}} - \widetilde{\mathbb{S}}_{t^{(2)}}\|_F\big] &\leq \mathbb{E}^{OC}_{\sigma_0}\Big[\frac{m}{n} d(\sigma_{t^{(2)}}, \widetilde{\sigma}_{t^{(2)}})\Big] \\ &\leq \frac{m}{n}\Big(1 - \frac{1}{2\xi n}\Big)^{\gamma_2 n} d(\sigma_{t^{(1)}}, \widetilde{\sigma}_{t^{(2)}}) < m\exp\Big(-\frac{\gamma_2}{2\xi}\Big) < \epsilon,\end{aligned} \tag{5.7}$$
where the first inequality holds because $\|\mathbb{S}(\sigma) - \mathbb{S}(\widetilde{\sigma})\|_F \leq \|\mathbb{S}(\sigma) - \mathbb{S}(\widetilde{\sigma})\|_{(1,1)} < \frac{m}{n} d(\sigma, \widetilde{\sigma})$ for arbitrary configurations $\sigma, \widetilde{\sigma} \in \Sigma$. Using the large deviations result with big enough $\gamma_1, \gamma_2$, we may assume with high probability $1 - \epsilon$, $\widetilde{\sigma}_{t^{(2)}} \in \Sigma_F^{\rho}$ for some small enough $\rho > 0$. And by (5.7), we also may assume that $\sigma_{t^{(2)}} \in \Sigma_F^{\rho}$ with high probability $1 - \epsilon$. Assuming these events happen, by taking $\rho > 0$ small enough, defined $\mathfrak{B}$ is $(\frac{1}{q} - \rho)$-partition. Regarding these facts, since the conditions of Corollary 5.1 is satisfied, there exists big enough $r > 0$ such that
$$\sigma_{t^{(3)}} \in \Sigma_F^{r/\sqrt{n}},$$
of probability at least $1-\epsilon$. On the other hand, using the uniform variance of $\widetilde{\mathbb{S}}_t$ under stationary distribution $\mu_*$,
$$\widetilde{\sigma}_{t^{(3)}} \in \Sigma_F^{r/\sqrt{n}},$$

of probability at least $1-\epsilon$. In addition, by Corollary 5.6, there exists $u, r' > 0$ and $\gamma_4 > 0$ such that

$$\sigma_{t^{(4)}}, \widetilde{\sigma}_{t^{(4)}} \in \mathcal{H}_{u,r'},$$

of probability at least $1-\epsilon$. We may assume that $\sigma_{t^{(4)}}, \widetilde{\sigma}_{t^{(4)}} \in \Sigma_F^{r'/\sqrt{n}}$ uniformly for large enough $n$. According to Proposition 5.7, there exists $\gamma_5 > 0$ such that $\mathbb{S}_{t^{(5)}} = \widetilde{\mathbb{S}}_{t^{(5)}}$ of probability at least $1-\epsilon$. Now, since the first condition of Lemma 5.10 is satisfied, then $\mathbf{S}_{t^{(5)}}, \widetilde{\mathbf{S}}_{t^{(5)}} \in \mathfrak{S}^{r''/\sqrt{n}}$ of probability at least $1-\epsilon$ for some big enough $r'' > 0$. Moreover, by Proposition 5.11, we find $\gamma_6 > 0$ such that

$$\mathbf{S}_{t^{(6)}} = \widetilde{\mathbf{S}}_{t^{(6)}},$$

of probability at least $1-\epsilon$.

Considering symmetry, for time $t \geq t^{(2)}$, the conditional distribution of $\sigma_t$ via $\mathcal{F}_{t^{(2)}}$ is invariant under permutations of vertices in each baskets $\mathcal{B}_{li}$ and the same logic holds for the stationary distribution $\mu_*$. Therefore,

$$\begin{aligned} &\left\|\mathbb{P}^{OC}_{\sigma_0}\left[\sigma_{t^{(6)}} \in \cdot \,|\mathcal{F}_{t^{(2)}}, \sigma_{t^{(2)}} \in \Sigma_F^{\rho}\right] - \mu_*\right\|_{\mathrm{TV}} \\ = \quad & \left\|\mathbb{P}^{OC}_{\sigma_0}\left[\mathbf{S}_{t^{(6)}} \in \cdot \,|\mathcal{F}_{t^{(2)}}, \mathbb{S}_{t^{(2)}} \in \mathcal{S}^{\rho}\right] - \mu_* \circ \mathbf{S}^{-1}\right\|_{\mathrm{TV}} \\ \leq \quad & \mathbb{P}^{OC}_{\sigma_0}\left[\mathbf{S}_{t^{(6)}} \neq \widetilde{\mathbf{S}}_{t^{(6)}} |\mathcal{F}_{t^{(2)}}, \mathbb{S}_{t^{(2)}} \in \mathcal{S}^{\rho}\right] \quad \leq O(\epsilon), \end{aligned}$$

where $\mathcal{S}^{\rho} = \left\{\mathbf{z} \in \mathcal{S} : \left|\mathbf{z}^{ij} - \frac{1}{q}\right| < \rho, \ \forall i, j\right\}$. Using Jensen's inequality,

$$\begin{aligned} &\left\|\mathbb{P}^{OC}_{\sigma_0}[\sigma_{t^{(6)}} \in \cdot \,] - \mu_*\right\|_{\mathrm{TV}} \\ \leq \quad & \mathbb{E}^{OC}_{\sigma_0}\left[\|\mathbb{P}^{OC}_{\sigma_0}[\sigma_{t^{(6)}} \in \cdot \,|\mathcal{F}_{t^{(2)}}] - \mu_*\|_{\mathrm{TV}} \;\Big|\; \mathbb{S}_{t^{(2)}} \in \mathcal{S}^{\rho}\right] + \mathbb{P}^{OC}_{\sigma_0}\left[\mathbb{S}_{t^{(2)}} \notin \mathcal{S}^{\rho}\right] \quad \leq O(\epsilon). \end{aligned}$$

In conclusion,

$$\lim_{\widetilde{\gamma}\to\infty} \limsup_{n\to\infty} \max_{\sigma_0 \in \Sigma} \|\mathbb{P}_{\sigma_0}[\sigma_{t^{(6)}+\widetilde{\gamma}n} \in \cdot] - \mu_*\|_{\mathrm{TV}} \leq 1 - O(\epsilon),$$

by setting $t^{(6)}(n) = t^{\xi}(n) + \gamma n$, where $\gamma = \gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_6$. □

## 6. Mixing in the low-temperature regime

6.1. **Proof of Theorem 2.** In the low-temperature regime, the metastability makes exponentially slow mixing. Formally, if $\beta > \beta_s/J$,

$$t_{\mathrm{MIX}} \geq Ce^{cn}$$

with some constants $C, c > 0$ uniformly for large enough $n$.

*Proof.* We first prove the statement for $\beta_s/J < \beta < \beta_c/J$. Then, if $s^1 \in [1/q, 1-\delta]$ for some $\delta > 0$, conditional on $\{\mathbb{S}^{*1} = s^1\mathbf{1}_m\}$, $(\mathbb{S}^{ij}/(1-s^1) : 1 \leq i \leq m, \ 2 \leq j \leq q)$ is distributed as proportion matrix for the $(q-1)$-states on general Potts model with the interaction matrix $\mathbf{K} = \mathbf{K}_{a,b}$ on $(1-s^1)n$ number of vertices with

$$(\beta J)' = \beta J(1-s^1) \leq \beta J\Big(1 - \frac{1}{q}\Big) < \beta_c \cdot \Big(1 - \frac{1}{q}\Big) < \beta_c(q-1),$$

where $\beta_c = \beta_c(q)$ is a function in $q$. Then, since the proportion process concentrates on the equilibrium state $\mathbf{\Delta}$, for $\delta > 0$ small enough, and $n > 0$ large enough,

$$\mu_*\Big[\exists 1 \le i \le m,\ \exists 2 \le j \le q \text{ s.t. } \big|\mathbb{S}^{ij} - (1-s^1)/(q-1)\big| \ge \delta \ \Big|\ \mathbb{S}^{*1} = s^1\mathbf{1}_m\Big] = o(1).$$

Recall from (2.2),

$$\mathbb{E}\big[\mathbb{S}^{ij}_{t+1} - \mathbb{S}^{ij}_t \mid \mathbb{S}_t = s\big] = \frac{1}{n}\Big( - s^{ij} + g^{(i,j)}_\beta(s)\Big) + O(n^{-2}).$$

By setting

$$\mathbf{z} = \begin{bmatrix} s^1 & (1-s^1)/(q-1) & \cdots & (1-s^1)/(q-1) \\ \vdots & \vdots & \vdots & \vdots \\ s^1 & (1-s^1)/(q-1) & \cdots & (1-s^1)/(q-1) \end{bmatrix},$$

we obtain that

$$\begin{aligned}\mathbb{E}_{\mu_*}\big[\mathbb{S}^{i1}_{t+1} - \mathbb{S}^{i1}_t \mid \mathbb{S}^{*1}_t = s^1\mathbf{1}_m\big] &= \mathbb{E}_{\mu_*}\big[\mathbb{S}^{i1}_{t+1} - \mathbb{S}^{i1}_t \mid \mathbb{S}_t = \mathbf{z}\big] + o(1)\\ &= \frac{1}{n}\Big( - s^1 + \frac{\exp(2\beta J s^1)}{\exp(2\beta J s^1) + (q-1)\exp((2\beta J/(q-1))(1-s^1))}\Big) + O(n^{-2})\\ &= \frac{1}{n} D_{\beta J}(s^1) + O(n^{-2}),\end{aligned}$$

for any block number $i \in [1, m]$. From [10], we know that if $\beta > \beta_s/J$, then there exists $s^*(\beta) > 1/q$ and small enough $\delta_1 > 0$ such that $D_{\beta J}(s^1)$ is uniformly positive in $\delta_1$-neighborhood of $s^*(\beta)$. Therefore, there exists small enough $\epsilon > 0$ such that for any $i \in [1, m]$, $s^1 \in (s^* - \delta_1, s^* + \delta_1)$, and large enough $n > 0$,

$$\mathbb{E}_{\mu_*}\big[\mathbb{S}^{i1}_{t+1} - \mathbb{S}^{i1}_t \mid \mathbb{S}^{*1}_t = s^1\mathbf{1}_m\big] \ge \frac{\epsilon}{n}.$$

Also, by using continuity of probability measure, for each $l \in [1, m]$ and $j \in \{-1, 0, 1\}$,

$$\mathbb{P}_{\mu_*}\Big[\mathbb{S}^{*1}_{t+1} = s^1\mathbf{1}_m + \frac{m}{n} j e_l \ \Big|\ \mathbb{S}^{*1}_t = s^1\mathbf{1}_m\Big] - \mathbb{P}_{\mu_*}\Big[\mathbb{S}^{*1}_{t+1} = s^1\mathbf{1}_m + \frac{m}{n}e_l + \frac{m}{n} j e_l \ \Big|\ \mathbb{S}^{*1}_t = s^1\mathbf{1}_m + \frac{m}{n}e_l\Big] = o(1),$$

where $e_l$ is $l$-th standard Euclidean basis element in $\mathbb{R}^m$. Combining two facts, there exists $\lambda > 1$ such that for any $l \in [1, m]$ and $s^1 \in (s^* - \delta_1, s^* + \delta_1)$,

$$\mathbb{P}_{\mu_*}\Big[\mathbb{S}^{*1}_{t+1} = s^1\mathbf{1}_m + \frac{m}{n}e_l \ \Big|\ \mathbb{S}^{*1}_t = s^1\mathbf{1}_m\Big] \ge \lambda \mathbb{P}_{\mu_*}\Big[\mathbb{S}^{*1}_{t+1} = s^1\mathbf{1}_m \ \Big|\ \mathbb{S}^{*1}_t = s^1\mathbf{1}_m + \frac{m}{n}e_l\Big],$$

for $n$ large enough. Since $\mathbb{S}_t$ is reversible Markov process,

$$\mathbb{P}_{\mu_*}\Big[\mathbb{S}^{*1}_t = s^1\mathbf{1}_m + \frac{m}{n}e_l,\ \mathbb{S}^{*1}_t = s^1\mathbf{1}_m\Big] = \mathbb{P}_{\mu_*}\Big[\mathbb{S}^{*1}_t = s^1\mathbf{1}_m,\ \mathbb{S}^{*1}_t = s^1\mathbf{1}_m + \frac{m}{n}e_l\Big].$$

As a result, for any $l \in [1, m]$,

$$\mu_*\Big[\mathbb{S}^{*1} = s^1\mathbf{1}_m + \frac{m}{n}e_l\Big] \ge \lambda \mu_*\Big[\mathbf{S}^{*1} = s^1\mathbf{1}_m\Big].$$

Inductively, we obtain that

(6.1)
$$\mu_*\Big[\mathbb{S}^{*1} = (s^* - \delta_1, \cdots, s^* + \delta_1, \cdots, s^* - \delta_1)^\intercal\Big] \ge \lambda^{2\delta_1 n/m} \mu_*\Big[\mathbb{S}^{*1} = (s^* - \delta_1, \cdots, s^* - \delta_1)^\intercal\Big].$$

Define the bottleneck set

$$A = \{\mathbb{S} \in \mathcal{S} : \mathbb{S}^{*1} \geq (s^* - \delta_1)\mathbf{1}_m\},$$

where $v \geq w$ denotes that $v_i \geq w_i$ for all $i \in [1, m]$. By (6.1), we obtain that

$$\frac{\mu_*[\partial A]}{\mu_*[A]} \leq \lambda^{-\delta_1 n/m}.$$

Since $\beta < \beta_c/J$, and $s^*(\beta) - \delta_1 > 1/q$, then $\mu_*[A] = o(1)$. Therefore, Cheeger's inequality implies that

$$t_{\mathrm{MIX}}(n) \geq Ce^{cn}$$

for some constants $C, c > 0$.

Furthermore, if $\beta \geq \beta_c/J$, from the large deviations theory, with $\delta_2 > 0$ small enough, bottleneck set

$$A = \{\mathbb{S} \in \mathcal{S} : \|\mathbb{S} - m\nu^1(\beta J)\|_F < \delta_2\}$$

satisfies that

$$\limsup_{n\to\infty} \frac{\log(\mu_*[\partial A])}{n} < 0, \quad \liminf_{n\to\infty} \frac{\log(\mu_*[A])}{n} = 0,$$

where $\nu^1$ is defined in Proposition 2.2. And by symmetry, $\mu_*[A] \leq 1/q$ with sufficiently small $\delta_2 > 0$. With the same logic of Cheeger's inequality,

$$t_{\mathrm{MIX}}(n) \geq Ce^{cn}$$

for some constants $C, c > 0$. □

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