Kinetic theory of two-dimensional point vortices and fluctuation-dissipation theorem

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We complete the kinetic theory of two-dimensional (2D) point vortices initiated in previous works. We use a simpler and more physical formalism. We consider a system of 2D point vortices submitted to a small external stochastic perturbation and determine the response of the system to the perturbation. We derive the diffusion coefficient and the drift by polarization of a test vortex. We introduce a general Fokker-Planck equation involving a diffusion term and a drift term. When the drift by polarization can be neglected, we obtain a secular dressed diffusion (SDD) equation sourced by the external noise. When the external perturbation is created by a discrete collection of N point vortices, we obtain a Lenard-Balescu-like kinetic equation reducing to a Landau-like kinetic equation when collective effects are neglected. We consider a multi-species system of point vortices. We discuss the process of kinetic blocking in the single and multi-species cases. When the field vortices are at statistical equilibrium (thermal bath), we establish the proper expression of the fluctuation-dissipation theorem for 2D point vortices relating the power spectrum of the fluctuations to the response function of the system. In that case, the drift coefficient and the diffusion coefficient satisfy an Einstein-like relation and the Fokker-Planck equation reduces to a Smoluchowski-like equation. We mention the analogy between 2D point vortices and stellar systems. In particular, the drift of a point vortex in 2D hydrodynamics [P.H. Chavanis, Phys. Rev. E **58**, R1199 (1998)] is the counterpart of the Chandrasekhar dynamical friction in astrophysics. We also consider a gas of 2D Brownian point vortices described by N coupled stochastic Langevin equations and determine its mean and mesoscopic evolution. In the present paper, we treat the case of unidirectional flows but our results can be straightforwardly generalized to axisymmetric flows.

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I. INTRODUCTION

There exist remarkable analogies between 2D point vortices and stellar systems [1–3]. This is basically due to the fact that these systems have long-range interactions [4]. As a result, they self-organize into coherent structures such as large-scale vortices (e.g. Jupiter's Great Red spot) [2, 5] or globular clusters and galaxies [6]. However, the relaxation towards these organized states is nontrivial. Systems with long-range interactions experience two successive types of relaxation. There is first a violent collisionless relaxation to a metaequilibrium state on a very short timescale of the order of the dynamical time t_D. The collisionless evolution of stellar systems is governed by the Vlasov-Poisson equations [7, 8] and the metaequilibrium state resulting from violent relaxation can be predicted by the statistical theory of Lynden-Bell [9]. Similarly, the collisionless evolution of 2D point vortices is governed by the Euler-Poisson equations [10] and the metaequilibrium state can be predicted by the Miller-Robert-Sommeria (MRS) statistical theory [11, 12], which is the hydrodynamic analogue of the Lynden-Bell theory [1].[1] These theories rely on an assumption of ergodicity which is not always fulfilled in practice. This is the difficult problem of incomplete violent relaxation [13]. Then, on a much longer timescale, there is a slow (secular) collisional relaxation due to finite N effects (granularities) leading, for $t \to +\infty$, to the Boltzmann equilibrium distribution predicted by conventional statistical mechanics. For stellar systems, this equilibrium state has been considered by Ogorodnikov [14, 15], Antonov [16], and Lynden-Bell and Wood [17]. They showed that an equilibrium state does not always exist, even when the system is artificially enclosed within a box in order to prevent its evaporation. Indeed, self-gravitating systems may experience a gravothermal catastrophe [17]. For 2D point vortices, the Boltzmann equilibrium state has been considered by Joyce and Montgomery [18, 19], Kida [20], and Pointin and Lundgren [21, 22] following the pioneering work of Onsager [23].[2] The relaxation time towards the Boltzmann distribution diverges algebraically with N so that, when $N \to +\infty$, the system is always in the collisionless regime. Here, we focus on the secular evolution of the system for large but

[1] The MRS theory applies either to the 2D point vortex gas in the collisionless regime (when $N \to +\infty$ with $\gamma \sim 1/N$) or to continuous 2D incompressible flows in the inviscid regime (when $\nu \to 0$). The late time evolution of continuous 2D incompressible flows is dominated by viscous decay and does not relax towards an equilibrium state.

[2] In his seminal paper on the statistical mechanics of 2D point vortices, Onsager [23] related the formation of large-scale vortices to the existence of negative temperature states. Later on, in unpublished notes [24], he developed a mean field theory of 2D point vortices and derived the Boltzmann-Poisson equation several years before the authors of Refs. [18–22].

finite values of N.[3] We briefly review the case of systems of material particles with long-range interactions (stellar systems, plasmas, HMF model...) before considering the case of 2D point vortices.

Let us first consider the kinetic theory of spatially homogeneous systems with long-range interactions in dimension $d = 3$. The evolution of a test particle in a thermal bath is described by a Fokker-Planck equation of the Kramers form involving a diffusion term and a friction term. The friction and the diffusion coefficients satisfy the Einstein relation. At statistical equilibrium, the diffusion and the friction balance each other and the Boltzmann distribution is established. This approach was developed by Chandrasekhar [26–31] for stellar systems by analogy with the theory of Brownian motion [32]. The theory of stochastic gravitational fluctuations was studied in [33–39]. If we consider the evolution of the system as a whole, the kinetic evolution is described by the Landau [40] or Lenard-Balescu [41, 42] equation introduced in plasma physics. These equations describe the collisional evolution of the system at the order $1/N$ due to the development of two-body correlations. They conserve mass and energy, satisfy an H-theorem for the Boltzmann entropy, and relax towards the Boltzmann distribution. The relaxation time scales as Nt_D.[4] The Lenard-Balescu equation takes into account collective effects [41–46] that are neglected in the Landau approach. The Landau equation has also been applied to stellar systems by making a local approximation (which amounts to considering that the system is spatially homogeneous). In the thermal bath approximation, we recover the original Fokker-Planck (Kramers) equation of Chandrasekhar.[5] However, self-gravitating systems are spatially inhomogeneous and the local approximation leads to a logarithmic divergence at large scales. Furthermore, the Landau equation does not take into account collective effects. Recently, the Landau and Lenard-Balescu equations have been generalized to the case of spatially inhomogeneous systems by Heyvaerts [51] and Chavanis [52] using angle-action variables. For gravitational systems, the proper treatment of spatial inhomogeneity removes the logarithmic divergence at large scales. The inhomogeneous Landau and Lenard-Balescu equations conserve mass and energy and satisfy an H-theorem for the Boltzmann entropy. They usually relax towards the Boltzmann distribution except in the case of unconfined stellar systems where the relaxation is hampered by the phenomena of evaporation and gravothermal catastrophe (core collapse). The Landau and Lenard-Balescu equations are valid for all systems with long-range interactions in any dimension of space d. However, for spatially homogeneous systems in $d = 1$ (like the HMF model or spins with long-range interactions moving on a sphere), the Landau and Lenard-Balescu collision terms vanish identically [53, 54]. In that case, there is no kinetic evolution at the order $1/N$. This is a situation of kinetic blocking due to the absence of resonances. As a result, the system does not reach the Boltzmann distribution on a timescale Nt_D. We thus have to take into account three-body correlations and develop the kinetic theory at the order $1/N^2$. An explicit kinetic equation that is valid at the order $1/N^2$ has been obtained recently by Fouvry *et al.* [55, 56] for arbitrary homogeneous 1D systems with long-range interactions in the approximation where collective effects can be neglected. Remarkably, this equation satisfies an H-theorem and relaxes towards the Boltzmann distribution. This implies that the relaxation time scales as $N^2 t_D$ for homogeneous 1D systems with long-range interactions.

The evolution of a test vortex in a thermal bath is described by a Fokker-Planck equation of the Smoluchowski form involving a diffusion term and a drift term. The drift and the diffusion coefficients satisfy an Einstein-like relation. At statistical equilibrium, the diffusion and the drift balance each other and the Boltzmann distribution is established. This approach was developed by Chavanis [57, 58] by analogy with the theory of Chandrasekhar [26–28, 31] for stellar systems and the theory of Brownian motion [32]. The theory of stochastic fluctuations in the point vortex gas was studied in [59–61]. If we consider the evolution of the system as a whole, the kinetic evolution is more complicated. It is described by a Landau-like [58, 62–65] or a Lenard-Balescu-like [66–69] equation. These equations describe the collisional evolution of the system at the order $1/N$ due to two-body correlations. They conserve circulation and energy and satisfy an H-theorem for the Boltzmann entropy. The Lenard-Balescu-like equation takes into account collective effects that are ignored in the Landau-like equation. In the thermal bath approximation, we recover the original Fokker-Planck (Smoluchowski-like) equation of Chavanis [57, 58]. For general flows that are neither unidirectional nor axisymmetric, there is a collisional evolution at the order $1/N$. The corresponding kinetic equation [see Eq. (128) or Eq. (137) of [58]] approaches the Boltzmann distribution on a timescale Nt_D if there are sufficient "resonances" between the point vortices [58]. However, for unidirectional flows and for axisymmetric flows with a monotonic profile of angular velocity, the Landau and Lenard-Balescu-like collision terms vanish identically. Therefore, there is no

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[3] The kinetic theory of collisionless relaxation for systems with long-range interactions is discussed in [25] and references therein.

[4] In plasma physics and stellar dynamics the relaxation time scales as $(N/\ln N)t_D$ because of logarithmic corrections. In plasma physics, N represents the number of charges in the Debye sphere (usually denoted Λ). In stellar dynamics, N represents the number of stars in the Jeans sphere which corresponds to the typical size of the cluster.

[5] Chandrasekhar [26–28, 31] also considered the evolution of a test star experiencing gravitational encounters with field stars that are not necessarily at statistical equilibrium. His work was further developed by Rosenbluth *et al.* [47]. If we assume that the system evolves self-consistently [48, 49], the corresponding Fokker-Planck equation is equivalent to the Landau equation although it appears in a different form (see [50] for the correspondence between the Chandrasekhar and Landau equations).

kinetic evolution for unidirectional flows at the order $1/N$. For axisymmetric flows, the system evolves until the profile of angular velocity becomes monotonic and then stops evolving at the order $1/N$. This is a situation of kinetic blocking [62] due to the absence of resonances. As a result, the system does not reach the Boltzmann distribution on a timescale Nt_D. In order to describe the relaxation of the system towards the Boltzmann distribution it is necessary to take into account three-body correlations and develop the kinetic theory of 2D point vortices at the order $1/N^2$ like in the work of Fouvry *et al.* [55, 56].

In the present paper, we focus on the Landau and Lenard-Balescu equations for 2D point vortices at the order $1/N$. In previous works, these kinetic equations have been derived in different manners using the linear response theory [57], the projection operator formalism [58], the BBGKY hierarchy [63, 69], the Klimontovich equation [63, 66, 68], the Fokker-Planck equation [57, 58, 62, 68, 69], and the functional approach [65]. These derivations are rather formal and technical. In the present paper, we complement the kinetic theory of point vortices in the following manner:

(i) We provide a simpler and more physical derivation of the kinetic equation of 2D point vortices. We compute the diffusion coefficient D and the drift by polarization $V_{\rm pol}$ by a direct approach and substitute these expressions into the Fokker-Planck equation written in a suitable form in which the diffusion coefficient is "sandwiched" between the two gradients in position so that the drift by polarization appears naturally.

(ii) We derive the proper expression of the fluctuation-dissipation theorem for 2D point vortices.

(iii) We consider a multi-species system of point vortices while previous works were mostly restricted to point vortices with the same circulation.

(iv) We consider unidirectional flows[6] while previous works were developed for axisymmetric flows. We show that in the thermal bath approximation the kinetic equation becomes similar to the Smoluchowski equation of Brownian theory with a Rosen-Morse [71] (or Pöschl-Teller [72]) potential and a constant diffusion coefficient. This equation can be transformed into a Schrödinger-like equation (in imaginary time) which can be solved analytically [73].

(v) We consider a system of collisionless 2D point vortices (or a continuous vorticity field) submitted to a small external stochastic perturbation of arbitrary origin and derive a secular dressed diffusion (SDD) equation sourced by the external noise.

(vi) The Landau and Lenard-Balescu equations are associated with the microcanonical ensemble where the system of point vortices is isolated. In that case, the point vortices are fundamentally described by N-body Hamiltonian equations [10]. We compare these results with those obtained for a gas of 2D Brownian point vortices [74, 75] described by N-body stochastic Langevin equations. We establish a drift-diffusion equation governing their mean evolution as well as a stochastic partial differential equation governing their mesoscopic evolution. Similar equations are obtained from the stochastic damped 2D Euler equations.

(vii) Throughout the paper, we mention the numerous analogies between the kinetic theory of 2D point vortices and the kinetic theory of stellar systems (and other systems with long-range interactions).

The paper is organized as follows. In Sec. II, we present the basic equations describing a system of 2D point vortices submitted to a small external stochastic perturbation and introduce the quasilinear approximation. In Sec. III, we explain how the linearized equation for the perturbation can be analytically solved with Fourier transforms by making the Bogoliubov ansatz. In Sec. IV, we determine the linear response of the flow to a small external perturbation. In Sec. V, we relate the dressed power spectrum of the total fluctuating stream function to the correlation function of the external perturbation and consider the case where the external perturbation is due to a random distribution of N field vortices. In Sec. VI, we derive the fluctuation-dissipation theorem satisfied by an isolated system of point vortices at statistical equilibrium. In Sec. VII, we introduce the general Fokker-Planck equation adapted to a gas of point vortices. In Sec. VIII, we derive the diffusion coefficient of a test vortex experiencing an external stochastic perturbation and consider the case where the external perturbation is due to a random distribution of N field vortices. In Sec. IX, we derive the drift by polarization experienced by a test vortex traveling in a background flow possibly created by a smooth distribution of field vortices. In Sec. X, we consider the evolution of a test vortex is a sea of field vortices at statistical equilibrium and establish the appropriate form of Einstein relation between the drift and the diffusion. In Sec. XI, we derive the kinetic equation of 2D point vortices with an arbitrary velocity profile. In Sec. XII, we show how this kinetic equation simplifies itself when the velocity profile is monotonic. In Sec. XIII, we contrast the kinetic theory of an isolated Hamiltonian system of point vortices to the kinetic theory of a gas of 2D Brownian point vortices. In Sec. XIV, we derive the SDD equation describing the mean evolution of a continuous vorticity field submitted to a small external stochastic perturbation of arbitrary origin. In Sec. XV, we study the mean evolution and the mesoscopic evolution of a continuous vorticity field described by the stochastic damped 2D

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[6] A kinetic theory of Stewart point vortices moving on the background of a shear flow with uniform vorticity has been developed in [70]. It is valid in the case of short-range interactions between point vortices making the problem similar to the kinetic theory of gases. This is substantially different from the problem that we consider here where the evolution of the point vortices is due to long-range collisions and collective effects.

Euler equations. We recover by this approach the power spectrum and the diffusion coefficient of a gas of point vortices. In Sec. XVI, we apply the same approach to the case of stochastically forced 2D point vortices. In Sec. XVII, we summarise our results and compare the different kinetic equations obtained in this paper. The Appendices provide useful complements to the results established in the main text.

II. BASIC EQUATIONS

We consider a system of 2D point vortices of individual circulation γ (see Appendix A). We assume that the point vortices move under their own interactions and under the effect of an external stochastic incompressible velocity field $\mathbf{u}_e(\mathbf{r},t)$ (exterior perturbation) of zero mean. The equations of motion of the point vortices are

$$\frac{d\mathbf{r}_i}{dt} = -\mathbf{z}\times\nabla\psi_d(\mathbf{r}_i) - \mathbf{z}\times\nabla\psi_e(\mathbf{r}_i,t), \tag{1}$$

where $\psi_d(\mathbf{r}) = -(1/2\pi)\sum_j \gamma\ln|\mathbf{r}-\mathbf{r}_j|$ is the exact stream function produced by the point vortices. They can be written in Hamiltonian form as $\gamma d\mathbf{r}_i/dt = -\mathbf{z}\times\nabla(H_d+H_e)$, where $H_d = -(1/2\pi)\sum_{i<j}\gamma^2\ln|\mathbf{r}_i-\mathbf{r}_j|$ is the Hamiltonian of the point vortices and $H_e = \sum_i \gamma\psi_e(\mathbf{r}_i,t)$ is the Hamiltonian associated with the external flow. The discrete vorticity field $\omega_d(\mathbf{r},t) = \sum_i \gamma\delta(\mathbf{r}-\mathbf{r}_i(t))$ of the point vortex gas satisfies the equations

$$\frac{\partial\omega_d}{\partial t} + (\mathbf{u}_d+\mathbf{u}_e)\cdot\nabla\omega_d = 0, \tag{2}$$

$$\mathbf{u}_d = -\mathbf{z}\times\nabla\psi_d, \qquad \omega_d = -\Delta\psi_d, \tag{3}$$

$$\mathbf{u}_e = -\mathbf{z}\times\nabla\psi_e, \qquad \omega_e = -\Delta\psi_e, \tag{4}$$

where $\psi_d(\mathbf{r},t)$ is the stream function produced by the point vortices and $\psi_e(\mathbf{r},t)$ is the external stochastic stream function. These equations are similar to the 2D Euler-Poisson equations for an incompressible continuous flow but they apply here to a singular vorticity field which is a sum of Dirac distributions. The 2D Euler-Poisson equations for an incompressible continuous flow are the counterparts of the Vlasov-Poisson equations in stellar dynamics [7] and plasma physics [8] and the 2D Euler-Poisson equations for a singular system of point vortices are the counterparts of the Klimontovich equations [76] in plasma physics.

We introduce the mean vorticity $\omega(\mathbf{r},t) = \langle\omega_d(\mathbf{r},t)\rangle$ corresponding to an ensemble average of $\omega_d(\mathbf{r},t)$. We then write $\omega_d(\mathbf{r},t) = \omega(\mathbf{r},t)+\delta\omega(\mathbf{r},t)$ where $\delta\omega(\mathbf{r},t)$ denotes the fluctuations about the mean vorticity. Similarly, we write $\psi_d(\mathbf{r},t) = \psi(\mathbf{r},t)+\delta\psi(\mathbf{r},t)$ where $\delta\psi(\mathbf{r},t)$ denotes the fluctuations about the mean stream function $\psi(\mathbf{r},t) = \langle\psi_d(\mathbf{r},t)\rangle$. Substituting this decomposition into Eq. (2), we get

$$\frac{\partial\omega}{\partial t} + \frac{\partial\delta\omega}{\partial t} + (\mathbf{u}+\delta\mathbf{u}+\mathbf{u}_e)\cdot\nabla(\omega+\delta\omega) = 0, \tag{5}$$

$$\mathbf{u} = -\mathbf{z}\times\nabla\psi, \qquad \omega = -\Delta\psi, \tag{6}$$

$$\delta\mathbf{u} = -\mathbf{z}\times\nabla\delta\psi, \qquad \delta\omega = -\Delta\delta\psi. \tag{7}$$

If we introduce the total fluctuations $\delta\mathbf{u}_{\rm tot} = \delta\mathbf{u}+\mathbf{u}_e$, $\delta\psi_{\rm tot} = \delta\psi+\psi_e$ and $\delta\omega_{\rm tot} = \delta\omega+\omega_e$, which include the contribution of the external perturbation, we can rewrite the foregoing equations as

$$\frac{\partial\omega}{\partial t} + \frac{\partial\delta\omega}{\partial t} + (\mathbf{u}+\delta\mathbf{u}_{\rm tot})\cdot\nabla(\omega+\delta\omega) = 0, \tag{8}$$

$$\delta\mathbf{u}_{\rm tot} = -\mathbf{z}\times\nabla\delta\psi_{\rm tot}, \qquad \delta\omega_{\rm tot} = -\Delta\delta\psi_{\rm tot}. \tag{9}$$

Expanding the advection term in Eq. (8) we obtain

$$\frac{\partial\omega}{\partial t} + \frac{\partial\delta\omega}{\partial t} + \mathbf{u}\cdot\nabla\omega + \mathbf{u}\cdot\nabla\delta\omega + \delta\mathbf{u}_{\rm tot}\cdot\nabla\omega + \delta\mathbf{u}_{\rm tot}\cdot\nabla\delta\omega = 0. \tag{10}$$

Taking the ensemble average of Eq. (10) and subtracting the resulting equation from Eq. (10) we obtain the two coupled equations

$$\frac{\partial \omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=-\nabla\cdot\langle\delta\omega\delta\mathbf{u}_{\rm tot}\rangle, \tag{11}$$

$$\frac{\partial \delta\omega}{\partial t}+\mathbf{u}\cdot\nabla\delta\omega+\delta\mathbf{u}_{\rm tot}\cdot\nabla\omega=-\nabla\cdot(\delta\omega\delta\mathbf{u}_{\rm tot})+\nabla\cdot\langle\delta\omega\delta\mathbf{u}_{\rm tot}\rangle, \tag{12}$$

which govern the evolution of the mean flow and the fluctuations. To get the right hand side of Eqs. (11) and (12) we have used the incompressibility of the flow $\nabla\cdot\delta\mathbf{u}_{\rm tot}=0$ (see Appendix A). Equations (11) and (12) are exact in the sense that no approximation has been made for the moment. The right hand side of Eq. (11) can be interpreted as a "collision" term arising from the granularity of the system (finite N effects) and the correlations of the fluctuations due to the external stochastic perturbation (forcing).[7]

We now assume that the external velocity is weak and treat the stochastic stream function $\psi_e(\mathbf{r},t)$ as a small perturbation to the mean field dynamics. We also assume that the fluctuation of the stream function $\delta\psi(\mathbf{r},t)$ created by the point vortices is weak. Since the circulation of the point vortices scales as $\gamma\sim 1/N$ this approximation is valid when $N\gg 1$. If we ignore the external stochastic perturbation and the fluctuations of the stream function due to finite N effects altogether, the collision term vanishes and Eq. (11) reduces to the 2D Euler equation

$$\frac{\partial \omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=0. \tag{13}$$

The 2D Euler-Poisson equations (13) and (6) describe a self-consistent mean field dynamics. It is valid in the limit $\psi_e\rightarrow 0$ and in a proper thermodynamic limit $N\rightarrow+\infty$ with $\gamma\sim 1/N$. It is also valid for sufficiently short times.

We now take into account a small correction to the 2D Euler equation obtained by keeping the collision term on the right hand side of Eq. (11) but neglecting the quadratic terms on the right hand side of Eq. (12). We therefore obtain a set of two coupled equations

$$\frac{\partial \omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=-\nabla\cdot\langle\delta\omega\delta\mathbf{u}_{\rm tot}\rangle, \tag{14}$$

$$\frac{\partial \delta\omega}{\partial t}+\mathbf{u}\cdot\nabla\delta\omega+\delta\mathbf{u}_{\rm tot}\cdot\nabla\omega=0. \tag{15}$$

These equations form the starting point of the quasilinear theory of 2D point vortices which is valid in a weak coupling approximation ($\gamma\sim 1/N\ll 1$) and for a weak external stochastic perturbation ($\psi_e\ll 1$). Equation (14) describes the evolution of the mean vorticity sourced by the correlations of the fluctuations and Eq. (15) describes the evolution of the fluctuations due to the granularities of the system (finite N effects) and the external noise. These equations are valid at the order $1/N$ and to leading order in ψ_e.

If we restrict ourselves to unidirectional mean flows[8] and introduce a cartesian system of coordinates, we have

$$\mathbf{u}=U(y,t)\mathbf{x},\qquad \psi=\psi(y,t),\qquad \omega=\omega(y,t), \tag{16}$$

$$U=\frac{\partial\psi}{\partial y},\qquad \omega=-\frac{\partial^2\psi}{\partial y^2},\qquad \omega=-U'(y,t). \tag{17}$$

On the other hand, the two components of the fluctuating velocity field read

$$(\delta\mathbf{u}_{\rm tot})_x=\frac{\partial\delta\psi_{\rm tot}}{\partial y},\qquad (\delta\mathbf{u}_{\rm tot})_y=-\frac{\partial\delta\psi_{\rm tot}}{\partial x}. \tag{18}$$

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[7] We generically call it the "collision" term although it may have a more general meaning due to the contribution of the external perturbation. A more proper name could be the "correlational" term.

[8] In the following, we assume that the system remains unidirectional during the whole evolution. This may not always be the case. Even if we start from a unidirectional flow $\omega_0(y)$, the "collision" term (r.h.s. in Eq. (14)) will change it and induce a temporal evolution of the vorticity field $\omega(y,t)$. The system may become dynamically (Euler) unstable and undergo a dynamical phase transition from a unidirectional flow to a more complicated flow (e.g., a large scale vortex). We assume here that this transition does not take place or we consider a period of time preceding this transition.

As a result, Eqs. (14) and (15) become

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial y}\left\langle \delta\omega \frac{\partial \delta\psi_{\rm tot}}{\partial x}\right\rangle, \tag{19}$$

$$\frac{\partial \delta\omega}{\partial t} + U\frac{\partial \delta\omega}{\partial x} - \frac{\partial \delta\psi_{\rm tot}}{\partial x}\frac{\partial \omega}{\partial y} = 0. \tag{20}$$

For the simplicity of the presentation, we have assumed that the external velocity field $\mathbf{u}_e$ is of zero mean. If there is an external (unidirectional) mean flow U_e, it can be included in U by making the substitution $U \to U + U_e$. In other words, U represents the total mean flow including the mean flow produced by the system of point vortices and by the external perturbation.

Remark: Although we have introduced the above equations for a system of 2D point vortices, they are also valid for a continuous 2D incompressible flow forced by an external velocity field. In that case, Eqs. (2)-(4) are the 2D Euler equations for a continuous vorticity field ω_c and a continuous velocity field $\mathbf{u}_c$ replacing the discrete vorticity field ω_d and the discrete velocity field $\mathbf{u}_d$. If the continuous flow is submitted to an external stochastic perturbation $\mathbf{u}_e$, we can decompose the vorticity and the velocity into a mean component plus a fluctuation, writing $\omega_c = \omega + \delta\omega$ and $\mathbf{u}_c = \mathbf{u} + \delta\mathbf{u}$, and obtain the same equations as above with a different interpretation (see Sec. XIV).

III. BOGOLIUBOV ANSATZ

In order to solve Eq. (20) for the fluctuations, we resort to the Bogoliubov ansatz. We assume that there exist a timescale separation between a slow and a fast dynamics and we regard $U(y)$ and $\omega(y)$ in Eq. (20) as "frozen" (independent of time). This amounts to neglecting the temporal variation of the mean flow when we consider the evolution of the fluctuations. This is possible when the mean vorticity field evolves on a secular timescale that is long compared to the timescale over which the correlations of the fluctuations have their essential support. We can then introduce Fourier transforms in x and t for the vorticity fluctuations, writing

$$\delta\omega(x,y,t) = \int dk \int \frac{d\sigma}{2\pi}\, e^{i(kx-\sigma t)}\delta\hat{\omega}(k,y,\sigma), \tag{21}$$

$$\delta\hat{\omega}(k,y,\sigma) = \int \frac{dx}{2\pi}\int dt\, e^{-i(kx-\sigma t)}\delta\omega(x,y,t). \tag{22}$$

Similar expressions hold for the stream functions $\delta\psi(\mathbf{r},t)$ and $\psi_e(\mathbf{r},t)$. For future reference, we recall the Fourier representation of the Dirac δ-function

$$\delta(\sigma) = \int_{-\infty}^{+\infty} e^{i\sigma t}\,\frac{dt}{2\pi}. \tag{23}$$

Before going further, some comments about our procedure of derivation are required. In order to derive the Lenard-Balescu equation describing the mean evolution of a system of point vortices under discreteness effects ("collisions") at the order $1/N$ we usually take $\omega_e = 0$ in Eq. (2) and consider an initial value problem as described in Sec. 3 of [68] (see also Appendix J). In that case, Eq. (20) has to be solved by introducing a Fourier transform in space and a Laplace transform in time. This brings a term $\delta\hat{\omega}(k,y,0)$ related to the initial condition in the equation for the fluctuations [see Eq. (J3)]. This is how discreteness effects (granularities) are taken into account in this approach. Calculating the correlation function and substituting the result into Eq. (19), one obtains a Lenard-Balescu-like equation in which the diffusion and the drift terms appear simultaneously. This derivation involves, however, rather technical calculations. In the present paper, we shall derive the Lenard-Balescu equation differently by using a simpler and more physical (or more pedagogical) approach based on the Fokker-Planck equation (see Sec. VII). In this approach, discreteness effects are taken into account in the external perturbation ω_e. Indeed, we can regard ω_e either as having an arbitrary origin (see Sec. V A) or as being generated by a collection of N point vortices – the so-called field vortices (see Sec. V B). In the presence of an external perturbation, Eq. (20) can be solved by introducing Fourier transforms in space and time.[9] We can then derive the dressed power spectrum and the diffusion coefficient of a test vortex by taking

[9] In the absence of external perturbation, we need to introduce a Laplace transform in time yielding a term related to the initial condition. In our approach, the initial condition is rejected to the infinite past but we have to add a small imaginary term $i0^+$ in the pulsation σ of the Fourier transform to make the fluctuations vanish for $t \to -\infty$. In a sense, this procedure amouts to using a Laplace transform in time but neglecting the initial condition.

collective effects into account (see Sec. VIII). On the other hand, the drift by polarization of a test vortex can be obtained by determining the response of the background flow to the perturbation that it has caused (see Sec. IX). Substituting these coefficients into the Fokker-Planck equation, we obtain the Lenard-Balescu equation (see Secs. XI and XII). This formalism also allows us to treat situations in which the external perturbation ω_e is not necessarily due to a discrete distribution of point vortices. In that case, we can derive more general kinetic equations. When $N \to +\infty$, i.e. when the collisions between the point vortices are negligible, we obtain the secular dressed diffusion (SDD) equation involving a diffusion term due to the external perturbation (see Sec. XIV). For finite N, we obtain a mixed kinetic equation involving a SDD term and a Lenard-Balescu term (see Sec. XVII).

IV. RESPONSE FUNCTION

Let us determine the linear response of a 2D incompressible flow to a small external perturbation $\omega_e(x, y, t)$. In the present case, the perturbation is not necessarily stochastic. Since the perturbation is small, we can use the linearized 2D Euler equation (20). Taking the Fourier transform of this equation in x and t, we obtain

$$\delta\hat{\omega}(k, y, \sigma) = \frac{k\frac{\partial\omega}{\partial y}}{kU(y) - \sigma}\delta\hat{\psi}_{\rm tot}(k, y, \sigma). \tag{24}$$

On the other hand, according to Eqs. (4) and (7), we have

$$\Delta\delta\psi_{\rm tot} = -\delta\omega_{\rm tot} = -\delta\omega - \omega_e. \tag{25}$$

Writing this equation in Fourier space and combining the result with Eq. (24), we get

$$\left[\frac{d^2}{dy^2} - k^2 + \frac{k\frac{\partial\omega}{\partial y}}{kU(y) - \sigma}\right]\delta\hat{\psi}_{\rm tot} = -\hat{\omega}_e. \tag{26}$$

The formal solution of this differential equation is

$$\delta\hat{\psi}_{\rm tot}(k, y, \sigma) = \int G(k, y, y', \sigma)\hat{\omega}_e(k, y', \sigma)\, dy', \tag{27}$$

where $G(k, y, y', \sigma)$ is the Green function defined by

$$\left[\frac{d^2}{dy^2} - k^2 + \frac{k\frac{\partial\omega}{\partial y}}{kU(y) - \sigma}\right]G(k, y, y', \sigma) = -\delta(y - y'). \tag{28}$$

Although not explicitly written, we must use the Landau prescription $\sigma \to \sigma + i0^+$ in Eq. (28). As a result, $G(k, y, y', \sigma)$ is a complex function which plays the role of the response function (or dielectric function) in plasma physics and stellar dynamics [6, 77]. It determines the response of the system $\delta\hat{\psi}_{\rm tot}(k, y, \sigma)$ to an external perturbation $\hat{\omega}_e(k, y, \sigma)$ through Eq. (27). Assuming that the perturbation $\omega_e(x, y)$ is time-independent and taking $\sigma = 0$, Eq. (28) reduces to

$$\left[\frac{d^2}{dy^2} - k^2 + \frac{\frac{\partial\omega}{\partial y}}{U(y)}\right]G(k, y, y') = -\delta(y - y'). \tag{29}$$

This equation determines the static response function $G(k, y, y') \equiv G(k, y, y', \sigma = 0)$ of the system to a time-independent perturbation.

Without external perturbation, the flow would be (by assumption) purely unidirectional, described by $\omega(y)$. The external perturbation $\omega_e(x, y, t)$ creates a weak flow $\psi_e(x, y, t)$. This flow polarizes the system and induces through Eq. (20) a small change in the vorticity field $\delta\omega(x, y, t)$ producing in turn a weak flow $\delta\psi(x, y, t)$ through Eq. (7). As a result, the total stream function acting on a point vortex, sometimes called the dressed or effective stream function, is $\delta\psi_{\rm tot}(x, y, t) = \psi_e(x, y, t) + \delta\psi(x, y, t)$. This is the sum of the stream function due to the external perturbation plus the stream function induced by the system itself (i.e. the system's own response). Since $\delta\psi$ occurs in Eqs. (7) and (20), we have to solve a loop. The total stream function is related to the external perturbation $\omega_e(x, y, t)$ by Eq. (27). The dressed Green function $G(k, y, y', \sigma)$ takes into account the polarization of the system due to its self-interaction. This corresponds to the so-called "collective effects". The polarization cloud surrounding a point vortex may amplify or shield the action of the imposed external perturbation. Therefore, the stream function is modified by collective effects: $\delta\psi_{\rm tot} = \psi_e + \delta\psi \neq \psi_e$. This leads to the notion of "dressed" point vortices.[10] If we neglect collective effects ($\delta\omega = \delta\psi = 0$), we have $\delta\psi_{\rm tot} = \psi_e$ with

$$\hat{\psi}_e(k, y, \sigma) = \int G_{\rm bare}(k, y, y')\hat{\omega}_e(k, y', \sigma)\, dy', \tag{30}$$

where $G_{\rm bare}(k, y, y')$ satisfies the equation

$$\left(\frac{d^2}{dy^2} - k^2\right) G_{\rm bare}(k, y, y') = -\delta(y - y'). \tag{31}$$

Equations (30) and (31) define the bare stream function and the bare Green function. The bare Green function $G_{\rm bare}(k, y, y')$ is just the Fourier transform in x of the potential of interaction between the vortices (see Appendix B). Similarly, the Green function $G(k, y, y', \sigma)$ can be interpreted as a dressed potential of interaction between the vortices taking into account collective effects. Neglecting collective effects amounts to replacing the dressed stream function (or dressed Green function) by the bare stream function (or bare Green function).

Remark: When $\omega_e = 0$, Eq. (26) reduces to

$$\left[\frac{d^2}{dy^2} - k^2 - \frac{U''(y)}{U(y) - \sigma/k}\right] \delta\hat{\psi} = 0, \tag{32}$$

where we have used Eq. (17). This is the celebrated Rayleigh equation [81, 82] which determines the proper complex pulsations σ of the flow associated with the velocity field $U(y)$. It plays the role of the dispersion relation in plasma physics and stellar dynamics [6, 77]. It can be used to study the linear dynamical stability of unidirectional flows.

V. CORRELATION FUNCTION

We now assume that the external perturbation $\omega_e(x, y, t)$ is a stochastic process and we determine the dressed correlation function of the total stream function $\delta\psi_{\rm tot}(x, y, t)$ that it induces. We first give general results and then consider the case where the external perturbation is produced by a random distribution of N point vortices.

A. General results

We assume that the time evolution of the external vorticity field is a stationary stochastic process and write its auto-correlation function as

$$\langle \omega_e(x, y, t)\omega_e(x', y', t')\rangle = \delta(y - y')C(x - x', y, t - t'). \tag{33}$$

We assume that the fluctuations are δ-correlated in y but not necessarily in x and t. The function $C(x - x', y, t - t')$ describes a possibly colored noise. The Fourier transform in x and t of the correlation function of the external vorticity field is therefore

$$\langle \hat{\omega}_e(k, y, \sigma)\hat{\omega}_e(k', y', \sigma')\rangle = 2\pi\delta(k + k')\delta(\sigma + \sigma')\delta(y - y')\hat{C}(k, y, \sigma), \tag{34}$$

where $\hat{C}$ depends on k and σ (for a white noise it would be constant). Similarly, we define the correlation function $P(k, y, \sigma)$ of the total fluctuating stream function acting on the point vortices by

$$\langle \delta\hat{\psi}_{\rm tot}(k, y, \sigma)\delta\hat{\psi}_{\rm tot}(k', y, \sigma')\rangle = 2\pi\delta(k + k')\delta(\sigma + \sigma')P(k, y, \sigma). \tag{35}$$

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[10] The notion of "dressed" particles (or quasiparticles) has been introduced and developed by Hubbard [44–46] and Rostoker [78–80] in plasma physics.

We call it the power spectrum by analogy with plasma physics [45].[11] Using Eqs. (27) and (34), we get

$$\begin{aligned}\langle\delta\hat{\psi}_{\rm tot}(k,y,\sigma)\delta\hat{\psi}_{\rm tot}(k',y,\sigma')\rangle &= \int dy'dy''\, G(k,y,y',\sigma)G(k',y,y'',\sigma')\langle\hat{\omega}_e(k,y',\sigma)\hat{\omega}_e(k',y'',\sigma')\rangle\\ &= 2\pi\delta(k+k')\delta(\sigma+\sigma')\int dy'\, G(k,y,y',\sigma)G(-k,y,y',-\sigma)\hat{C}(k,y',\sigma)\\ &= 2\pi\delta(k+k')\delta(\sigma+\sigma')\int dy'\, |G(k,y,y',\sigma)|^2\hat{C}(k,y',\sigma), \end{aligned}\qquad (36)$$

where we have used Eq. (C6). Comparing Eq. (36) with Eq. (35), we obtain

$$P(k,y,\sigma)=\int dy'\, |G(k,y,y',\sigma)|^2\hat{C}(k,y',\sigma). \qquad (37)$$

This equation relates the power spectrum $P(k,y,\sigma)$ of the total fluctuating stream function acting on the point vortices to the correlation function $\hat{C}(k,y,\sigma)$ of the external stochastic perturbation. The "dressed" power spectrum $P(k,y,\sigma)$ takes into account collective effects. If we neglect collective effects, we just have to replace $G(k,y,y',\sigma)$ by $G_{\rm bare}(k,y,y')$ in Eq. (37). In that case, we find that the "bare" power spectrum $P_{\rm bare}(k,y,\sigma)$ defined by

$$\langle\hat{\psi}_e(k,y,\sigma)\hat{\psi}_e(k',y,\sigma')\rangle = 2\pi\delta(k+k')\delta(\sigma+\sigma')P_{\rm bare}(k,y,\sigma) \qquad (38)$$

is given by

$$P_{\rm bare}(k,y,\sigma)=\int dy'\, G_{\rm bare}(k,y,y')^2\hat{C}(k,y',\sigma). \qquad (39)$$

We note that $\hat{C}(k,y,\sigma)$, $P(k,y,\sigma)$ and $P_{\rm bare}(k,y,\sigma)$ are real and positive.

Remark: From Eq. (35) we easily obtain

$$\langle\delta\hat{\psi}_{\rm tot}(k,y,t)\delta\hat{\psi}_{\rm tot}(k',y,t')\rangle = \delta(k+k')\mathcal{P}(k,y,t-t'), \qquad (40)$$

where $\mathcal{P}(k,y,t)$ is the inverse Fourier transform in time of $P(k,y,\sigma)$. Therefore, the static power spectrum $P(k,y)=\mathcal{P}(k,y,0)$ is

$$P(k,y)=\int P(k,y,\sigma)\,\frac{d\sigma}{2\pi}. \qquad (41)$$

B. Correlation function created by a random distribution of N field vortices

We now assume that the external vorticity is created by a random distribution of N point vortices. We allow for different species of point vortices with circulations $\{\gamma_b\}$. The discrete vorticity of the field vortices is

$$\omega_e(x,y,t)=\sum_i \gamma_i\delta(x-x_i(t))\delta(y-y_i(t)). \qquad (42)$$

The initial positions (x_i,y_i) of the point vortices are assumed to be uncorrelated and randomly distributed. The distribution function of point vortices of species b, with circulation γ_b, is $P_1^{(b)}(y)$. The mean vorticity of species b is therefore $\omega_b(y)=N_b\gamma_b P_1^{(b)}(y)$, where N_b is the total number of point vortices of species b. When $N\rightarrow +\infty$ with $\gamma\sim 1/N$, the point vortices are advected by the mean flow $U(y)$ produced by the total vorticity $\omega(y)=\sum_b\omega_b(y)$. Their mean field trajectories are straight lines at constant y:

$$x_i(t)=x_i+U(y_i)t,\qquad y_i(t)=y_i. \qquad (43)$$

As a result, the external vorticity can be written as

$$\omega_e(x,y,t)=\sum_i \gamma_i\delta(x-x_i-U(y)t)\delta(y-y_i). \qquad (44)$$

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[11] More precisely, the power spectrum is $k^2P(k,y,\sigma)$.

Passing in Fourier space, we obtain

$$\begin{aligned}\hat{\omega}_e(k,y,\sigma) &= \sum_i \gamma_i \int \frac{dx}{2\pi}\int dt\, e^{-i(kx-\sigma t)}\delta(x-x_i-U(y)t)\delta(y-y_i)\\ &= \frac{1}{2\pi}\sum_i \gamma_i \int dt\, e^{-ikx_i}e^{-i(kU(y)-\sigma)t}\delta(y-y_i)\\ &= \sum_i \gamma_i e^{-ikx_i}\delta(kU(y)-\sigma)\delta(y-y_i). \end{aligned} \tag{45}$$

The correlation function of the external vorticity field in Fourier space is therefore

$$\begin{aligned}\langle \hat{\omega}_e(k,y,\sigma)\hat{\omega}_e(k',y',\sigma')\rangle &= \left\langle \sum_{ij}\gamma_i\gamma_j e^{-ikx_i}e^{-ik'x_j}\delta(kU(y)-\sigma)\delta(k'U(y')-\sigma')\delta(y-y_i)\delta(y'-y_j)\right\rangle\\ &= \left\langle \sum_i \gamma_i^2 e^{-i(k+k')x_i}\delta(kU(y)-\sigma)\delta(k'U(y')-\sigma')\delta(y-y_i)\delta(y'-y_i)\right\rangle\\ &= \sum_b N_b\gamma_b^2\int dx_1 dy_1\, e^{-i(k+k')x_1}\delta(kU(y)-\sigma)\delta(k'U(y')-\sigma')\delta(y-y_1)\delta(y'-y_1)P_1^{(b)}(y_1)\\ &= \sum_b \gamma_b\int dx_1 dy_1\, e^{-i(k+k')x_1}\delta(kU(y)-\sigma)\delta(k'U(y')-\sigma')\delta(y-y_1)\delta(y'-y_1)\omega_b(y_1)\\ &= \sum_b 2\pi\gamma_b\delta(k+k')\delta(\sigma+\sigma')\delta(y-y')\delta(kU(y)-\sigma)\omega_b(y). \end{aligned} \tag{46}$$

To get the second line, we have used the decomposition $\sum_{ij} = \sum_i + \sum_{i\neq j}$ and the fact that the terms involving different vortices ($i \neq j$) vanish in average since the point vortices are initially uncorrelated. To get the third line, we have used the fact that the point vortices of the same species are identical. Comparing Eqs. (34) and (46), we find that

$$\hat{C}(k,y,\sigma) = \sum_b \gamma_b \delta(kU(y)-\sigma)\omega_b(y). \tag{47}$$

This is the bare correlation function of the vorticity field created by the background point vortices. Using Eqs. (37) and (47), we obtain the dressed power spectrum of the total fluctuating stream function created by a random distribution of point vortices

$$P(k,y,\sigma) = \sum_b \gamma_b \int dy'\, |G(k,y,y',\sigma)|^2\delta(kU(y')-\sigma)\omega_b(y'). \tag{48}$$

This returns the expression (34) of the power spectrum given in Ref. [68] which was obtained from the Klimontovich formalism. The present approach provides an alternative, more physical, manner to derive this result. Using Eqs. (41) and (48), we obtain the static power spectrum

$$P(k,y) = \frac{1}{2\pi}\sum_b \gamma_b \int dy'\, |G(k,y,y',kU(y'))|^2\omega_b(y'). \tag{49}$$

If we neglect collective effects, we find that the bare correlation function of the total fluctuating stream function produced by a random distribution of field vortices is

$$P_{\rm bare}(k,y,\sigma) = \sum_b \gamma_b \int dy'\, G_{\rm bare}(k,y,y')^2\delta(kU(y')-\sigma)\omega_b(y'). \tag{50}$$

It can be obtained from Eq. (48) by replacing the dressed Green function (28) by the bare Green function (31). The same presciption can be used to obtain the bare static spectrum from Eq. (49).

C. Energy of fluctuations

The energy of fluctuations

$$\mathcal{E} = \frac{1}{2}\int \langle \delta\omega_{\rm tot}\delta\psi_{\rm tot}\rangle\, dy = \frac{1}{2}\int \langle (\nabla\delta\psi_{\rm tot})^2\rangle\, dy, \tag{51}$$

where $\delta\omega_{\rm tot} = \delta\omega + \omega_e$ and $\delta\psi_{\rm tot} = \delta\psi + \psi_e$ are the total fluctuations of vorticity and stream function, can be calculated as follows. Decompositing the fluctuations of stream function in Fourier modes, we get

$$\mathcal{E} = -\frac{1}{2}\int dy \int dk \int dk' \int \frac{d\sigma}{2\pi}\int \frac{d\sigma'}{2\pi} kk' e^{i(kx-\sigma t)} e^{i(k'x-\sigma' t)} \langle \delta\hat{\psi}_{\rm tot}(k,y,\sigma)\delta\hat{\psi}_{\rm tot}(k',y,\sigma')\rangle. \tag{52}$$

Introducing the power spectrum from Eq. (35) and integrating over k' and σ', we obtain

$$\mathcal{E} = \frac{1}{2}\int dy \int dk \int \frac{d\sigma}{2\pi}\, k^2 P(k,y,\sigma) = \frac{1}{2}\int dy \int dk\, k^2 P(k,y). \tag{53}$$

The energy of fluctuations is equal to the integral of $k^2 P(k,y)$ over the wavenumber k and over the position y. Using Eq. (37), we can rewrite the energy of fluctuations as

$$\mathcal{E} = \frac{1}{2}\int dy \int dk \int \frac{d\sigma}{2\pi}\int dy'\, k^2 |G(k,y,y',\sigma)|^2 \hat{C}(k,y',\sigma). \tag{54}$$

When the perturbation is due to a distribution of N point vortices, using Eq. (47), we obtain

$$\mathcal{E} = \frac{1}{4\pi}\sum_b \int dy \int dk \int dy'\, k^2 |G(k,y,y',kU(y'))|^2 \gamma_b \omega_b(y'). \tag{55}$$

We note that $\mathcal{E}$ is constant in time.

VI. FLUCTUATION-DISSIPATION THEOREM FOR AN ISOLATED SYSTEM OF POINT VORTICES AT STATISTICAL EQUILIBRIUM

A. Fluctuation-dissipation theorem

The fluctuation-dissipation theorem for a gas of point vortices can be written as

$$P(k,y,\sigma) = -\frac{1}{\pi\beta\sigma}{\rm Im}\, G(k,y,y,\sigma). \tag{56}$$

It relates the power spectrum $P(k,y,\sigma)$ of the fluctuations to the response function $G(k,y,y,\sigma)$ of a system of point vortices at statistical equilibrium with an inverse temperature β.[12] Although the fluctuation-dissipation theorem can be established from very general arguments [83–88], we shall derive Eq. (56) directly from the results obtained in the preceding sections.

We start from the general identity (see Appendix C)

$${\rm Im}\, G(k,y,y,\sigma) = \pi \int dy'\, |G(k,y,y',\sigma)|^2 \delta(\sigma - kU(y')) k \frac{\partial\omega'}{\partial y'}. \tag{57}$$

If the vorticity field is created by different species of point vortices, it can be rewritten as

$${\rm Im}\, G(k,y,y,\sigma) = \pi \sum_b \int dy'\, |G(k,y,y',\sigma)|^2 \delta(\sigma - kU(y')) k \frac{\partial\omega'_b}{\partial y'}. \tag{58}$$

[12] Note that Eq. (56) is valid at positive and negative temperatures.

If the point vortices are at statistical equilibrium with the Boltzmann distribution

$$\omega_b(y) = A_b\, e^{-\beta\gamma_b\psi(y)}, \tag{59}$$

we have the identity

$$\frac{\partial\omega_b}{\partial y} = -\beta\gamma_b\psi'(y)\omega_b(y). \tag{60}$$

Substituting Eq. (60) into Eq. (58), we obtain

$$\begin{aligned}
\mathrm{Im}\, G(k,y,y,\sigma) &= -\beta\pi\sum_b \gamma_b \int dy'\, |G(k,y,y',\sigma)|^2\delta(\sigma - kU(y'))k\frac{\partial\psi'}{\partial y'}\omega_b(y') \\
&= -\beta\pi\sum_b \gamma_b \int dy'\, |G(k,y,y',\sigma)|^2\delta(\sigma - kU(y'))kU(y')\omega_b(y') \\
&= -\beta\sigma\pi\sum_b \gamma_b \int dy'\, |G(k,y,y',\sigma)|^2\delta(\sigma - kU(y'))\omega_b(y') \\
&= -\beta\sigma\pi P(k,y,\sigma).
\end{aligned} \tag{61}$$

where we have used Eq. (17) to get the second line, the property of the δ-function to get the third line, and the expression (48) of the power spectrum to get the fourth line. This establishes Eq. (56).

B. Static power spectrum

At statistical equilibrium, we have the following relation

$$P(k,y) = -\frac{1}{\beta}G(k,y,y) \tag{62}$$

between the static power spectrum $P(k,y) = \frac{1}{2\pi}\int P(k,y,\sigma)\, d\sigma$ of the fluctuations (see Sec. V) and the static response function $G(k,y,y) = G(k,y,y,0)$ (see Sec. IV). This relation can be derived directly from the microcanonical distribution of point vortices at statistical equilibrium. It can also be recovered from the fluctuation-dissipation theorem (56) as follows.

Integrating Eq. (56) between $-\infty$ and $+\infty$, we get

$$P(k,y) = -\frac{1}{\beta}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\, G(k,y,y,\sigma)}{\pi\sigma}\, d\sigma. \tag{63}$$

On the other hand, applying at $\sigma = 0$ the Kramers-Kronig relation [89, 90]

$$G(k,y,y,\sigma) = \mathrm{P}\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\, [G(k,y,y,\sigma')]}{\sigma' - \sigma}\, d\sigma', \tag{64}$$

where P is the principal value, we get

$$G(k,y,y,0) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}\, [G(k,y,y,\sigma)]}{\sigma}\, d\sigma. \tag{65}$$

Comparing Eqs. (63) and (65) we obtain Eq. (62).

VII. FOKKER-PLANCK EQUATION

Here, we consider the evolution of a test vortex of circulation γ submitted to an external stochastic stream function $\psi_e(x,y,t)$. The equations of motion of the test vortex are (see Appendix A and Sec. II)

$$\frac{dx}{dt} = U(y,t) + \frac{\partial\delta\psi_{\rm tot}}{\partial y}(x,y,t), \qquad \frac{dy}{dt} = -\frac{\partial\delta\psi_{\rm tot}}{\partial x}(x,y,t), \tag{66}$$

where $\delta\psi_{\rm tot}(x,y,t)$ is the total fluctuation of the stream function. They can be written in Hamiltonian form as $\gamma d\mathbf{r}/dt=-\mathbf{z}\times\nabla(H+\delta H_{\rm tot})$, where $H=\gamma\int^y U(y')\,dy'$ is the mean Hamiltonian and $\delta H_{\rm tot}$ is the fluctuating Hamiltonian. The test vortex follows a rectilinear trajectory at constant velocity $U(y)$ on the line level y but it also experiences a small stochastic perturbation $\delta\psi_{\rm tot}=\psi_e+\delta\psi$ which is equal to the external stream function ψ_e plus the fluctuating stream function $\delta\psi$ produced by the system itself (collective effects). Eq. (66) can be formally integrated into

$$x(t)=x+\int_0^t U(y(t'),t')\,dt'+\int_0^t \frac{\partial\delta\psi_{\rm tot}}{\partial y}(x(t'),y(t'),t')\,dt', \qquad y(t)=y-\int_0^t \frac{\partial\delta\psi_{\rm tot}}{\partial x}(x(t'),y(t'),t')\,dt', \quad (67)$$

where we have assumed that, initially, the test vortex is at position (x,y). Since the fluctuations $\delta\psi_{\rm tot}$ of the stream function are small, the changes in the position of the test vortex in the y-direction are also small, and the dynamics of the test vortex can be represented by a stochastic process governed by a Fokker-Planck equation [91]. The Fokker-Planck equation can be derived from the Master equation by using the Kramers-Moyal expansion truncated at the level of the second moments of the position increment. If we denote by $P(y,t)$ the probability density that the test vortex is at y at time t, the general form of the Fokker-Planck equation is

$$\frac{\partial P}{\partial t}=\frac{\partial^2}{\partial y^2}(DP)-\frac{\partial}{\partial y}(PV_{\rm tot}). \quad (68)$$

The diffusion and drift coefficients are defined by

$$D(y)=\lim_{t\rightarrow +\infty}\frac{1}{2t}\langle (y(t)-y)^2\rangle=\frac{\langle (\Delta y)^2\rangle}{2\Delta t}, \quad (69)$$

$$V_{\rm tot}(y)=\lim_{t\rightarrow +\infty}\frac{1}{t}\langle y(t)-y\rangle=\frac{\langle \Delta y\rangle}{\Delta t}. \quad (70)$$

In writing these limits, we have implicitly assumed that the time t is long compared to the fluctuation time but short compared to the evolution time. As shown in our previous papers [63, 68], it is relevant to rewrite the Fokker-Planck equation in the alternative form

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial y}\left(D\frac{\partial P}{\partial y}-PV_{\rm pol}\right). \quad (71)$$

The total drift can be written as

$$V_{\rm tot}=V_{\rm pol}+\frac{\partial D}{\partial y}, \quad (72)$$

where $V_{\rm pol}$ is the drift by polarization (see Sec. IX of this paper and Sec. 4.3 of [68]) while the second term is due to the variation of the diffusion coefficient with y (see Sec. 4.4 of [53]). The drift by polarization $V_{\rm pol}$ arises from the retroaction (response) of the perturbation caused by the test vortex on the mean flow. It represents, however, only one component of the total drift $V_{\rm tot}$ experienced by the test vortex, the other component being $\partial D/\partial y$.

The two expressions (68) and (71) of the Fokker-Planck equation have their own interest. The expression (68), where the diffusion coefficient is placed after the second derivative $\partial^2(DP)$, involves the total drift $V_{\rm tot}$ and the expression (71), where the diffusion coefficient is placed between the derivatives $\partial D\partial P$, isolates the drift by polarization $V_{\rm pol}$. We shall see in Sec. XI that this second form is directly related to the Lenard-Balescu equation. It has therefore a clear physical meaning.

VIII. DIFFUSION COEFFICIENT

A. General expression of the diffusion coefficient

We now calculate the diffusion coefficient of the test vortex from Eq. (69) following the approach developed in [68]. The increment in position of the test vortex in the y-direction is

$$\Delta y=-\int_0^t \frac{\partial\delta\psi_{\rm tot}}{\partial x}(x(t'),y(t'),t')\,dt'. \quad (73)$$

Substituting Eq. (73) into Eq. (69) and assuming that the correlations of the fluctuating velocity persist for a time less than the time for the trajectory of the test vortex to be much altered, we can make a linear trajectory approximation

$$x(t') = x + U(y)t', \qquad y(t') = y, \tag{74}$$

and write

$$D = \lim_{t\to+\infty} \frac{1}{2t}\int_0^t dt' \int_0^t dt'' \left\langle \frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)t', y, t')\frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)t'', y, t'')\right\rangle. \tag{75}$$

Introducing the Fourier transform of the total fluctuating stream function, we obtain

$$\left\langle \frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)t', y, t')\frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)t'', y, t'')\right\rangle = \int dk \int \frac{d\sigma}{2\pi}\int dk' \int \frac{d\sigma'}{2\pi} \\ \times kk' e^{ik(x+U(y)t')}e^{-i\sigma t'}e^{ik'(x+U(y)t'')}e^{-i\sigma' t''}\langle \delta\hat{\psi}_{\rm tot}(k,y,\sigma)\delta\hat{\psi}_{\rm tot}(k',y,\sigma')\rangle. \tag{76}$$

Introducing the power spectrum from Eq. (35) and carrying out the integrals over k' and σ', we end up with the result

$$\left\langle \frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)t', y, t')\frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)t'', y, t'')\right\rangle = \int dk \int \frac{d\sigma}{2\pi}\, k^2 e^{i(kU(y)-\sigma)(t'-t'')}P(k,y,\sigma). \tag{77}$$

This expression shows that the auto-correlation function of the total fluctuating velocity appearing in Eq. (75) depends only on the difference of times $s = t'' - t'$. Using the identity [68]

$$\begin{aligned}\int_0^t dt' \int_0^t dt''\, f(t'-t'') &= 2\int_0^t dt' \int_{t'}^t dt''\, f(t'-t'') = 2\int_0^t dt' \int_0^{t-t'} ds\, f(s) \\ &= 2\int_0^t ds \int_0^{t-s} dt'\, f(s) = 2\int_0^t ds\,(t-s)f(s),\end{aligned} \tag{78}$$

and assuming that the autocorrelation function of the total fluctuating velocity $f(s)$ decreases more rapidly than s^{-1}, we find for $t\to+\infty$ that[13]

$$D = \int_0^{+\infty} \left\langle \frac{\partial \delta\psi_{\rm tot}}{\partial x}(x, y, 0)\frac{\partial \delta\psi_{\rm tot}}{\partial x}(x+U(y)s, y, s)\right\rangle ds. \tag{81}$$

Therefore, as in Brownian theory [32, 83, 84, 91, 92] and in fluid turbulence [93], the diffusion coefficient of a point vortex is equal to the integral of the temporal auto-correlation function $\langle V_y(0)V_y(t)\rangle$ of the fluctuating velocity felt by the point vortex [58]:

$$D = \int_0^{+\infty} \langle V_y(0)V_y(t)\rangle\, dt. \tag{82}$$

This is similar to the diffusion tensor of a star in a globular cluster which is given by $D_{ij} = \int_0^{+\infty}\langle F_i(0)F_j(t)\rangle\, dt$ where $\mathbf{F}(t)$ is the gravitational force by unit of mass experienced by the star [33, 39, 50, 94–98] (see also [44–46, 99, 100] for plasmas). Replacing the velocity auto-correlation function by its expression from Eq. (77), which can be written as

$$\langle V_y(0)V_y(t)\rangle = \int dk \int \frac{d\sigma}{2\pi}\, k^2 e^{i(kU(y)-\sigma)t}P(k,y,\sigma), \tag{83}$$

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[13] This formula can also be obtained by using the identity

$$D = \frac{1}{2}\frac{d}{dt}\langle(\Delta y)^2\rangle = \langle \dot{y}\Delta y\rangle = \int_0^t \left\langle V_y(x,y,0)V_y(x(t'),y(t'),t')\right\rangle dt', \tag{79}$$

where $V_y = (\delta u_{\rm tot})_y = -\partial\delta\psi_{\rm tot}/\partial x$. Making the approximations discussed above and taking the limit $t\to+\infty$, we obtain

$$D = \int_0^{+\infty} \langle V_y(x,y,0)V_y(x+U(y)s,y,s)\rangle\, ds, \tag{80}$$

which coincides with Eq. (81).

we obtain

$$D=\int_0^{+\infty}dt\int dk\int\frac{d\sigma}{2\pi}k^2e^{i(\sigma-kU(y))t}P(k,y,\sigma). \tag{84}$$

Making the change of variables $t\rightarrow -t$, $k\rightarrow -k$ and $\sigma\rightarrow-\sigma$, and using the fact that $P(-k,y,-\sigma)=P(k,y,\sigma)$, we see that we can replace $\int_0^{+\infty}dt$ by $(1/2)\int_{-\infty}^{+\infty}dt$. Therefore, we get

$$D=\frac{1}{2}\int_{-\infty}^{+\infty}dt\int dk\int\frac{d\sigma}{2\pi}k^2e^{i(\sigma-kU(y))t}P(k,y,\sigma). \tag{85}$$

Using the identity (23), we find that

$$D=\pi\int dk\int\frac{d\sigma}{2\pi}k^2\delta(\sigma-kU(y))P(k,y,\sigma). \tag{86}$$

The time integration has given a δ-function which creates a resonance condition for interaction. Integrating over the δ-function (resonance), we arrive at the following equation

$$D=\frac{1}{2}\int dk\,k^2P(k,y,kU(y)). \tag{87}$$

This equation expresses the diffusion coefficient of the test vortex in terms of the power spectrum of the fluctuations at the resonance $\sigma=kU(y)$. This is the general expression of the diffusion coefficient of a test vortex submitted to a stochastic perturbation. When collective effects are neglected, we get

$$D_{\rm bare}=\frac{1}{2}\int dk\,k^2P_{\rm bare}(k,y,kU(y)). \tag{88}$$

Using the relation between the power spectrum and the correlation function of the external perturbation [see Eq. (37)], we obtain

$$D=\frac{1}{2}\int dy'\int dk\,k^2|G(k,y,y',kU(y))|^2\hat{C}(k,y',kU(y)). \tag{89}$$

This expression shows that the diffusion coefficient of the test vortex depends on the correlation function of the external perturbation $\hat{C}(k,y',\sigma)$ and on the response function of the flow $G(k,y,y',\sigma)$ both evaluated at the resonance frequencies $\sigma=kU(y)$. As a result, the diffusion coefficient $D(y)$ depends on the position y of the test vortex, on the mean vorticity field $\omega(y)$ through the Green function $G(k,y,y',kU(y))$ defined by Eq. (28), and on the mean velocity $U(y)$. When collective effects are neglected, i.e., when we replace $G(k,y,y',kU(y))$ by $G_{\rm bare}(k,y,y')$ in Eq. (89), the diffusion coefficient reduces to

$$D_{\rm bare}=\frac{1}{2}\int dy'\int dk\,k^2G_{\rm bare}(k,y,y')^2\hat{C}(k,y',kU(y)). \tag{90}$$

Remark: Alternative derivations of the general expression of the diffusion coefficient of a test vortex are given in Appendices D 1 and D 2 (see also Appendix E where we compute the velocity auto-correlation function of the test vortex).

B. Expression of the diffusion coefficient due to N point vortices

We now assume that the external noise is due to a discrete collection of N point vortices. In that case $\hat{C}(k,y,\sigma)$ is given by Eq. (47) and we obtain

$$D=\frac{1}{2}\sum_b\gamma_b\int dy'\int dk\,k^2|G(k,y,y',kU(y))|^2\delta(kU(y')-kU(y))\omega_b(y'). \tag{91}$$

Using the identity

$$\delta(\lambda x)=\frac{1}{|\lambda|}\delta(x), \tag{92}$$

we can rewrite the foregoing equation as

$$D=\frac{1}{2}\sum_b \gamma_b \int dy' \int dk\, |k||G(k,y,y',kU(y))|^2 \delta(U(y')-U(y))\omega_b(y'). \tag{93}$$

This is the general expression of the diffusion coefficient of a test vortex of circulation γ produced by N field vortices of circulation $\{\gamma_b\}$. It returns Eq. (68) of [68]. The diffusion of the test vortex is due to the fluctuations of the field vortices induced by finite N effects (granularities). This is why it depends on $\{\gamma_b\}$ but not on γ.[14]

Introducing the function

$$\chi(y,y',U(y))=\frac{1}{2}\int |k||G(k,y,y',kU(y))|^2\, dk, \tag{94}$$

the diffusion coefficient (93) can be written in the more compact form

$$D=\sum_b \gamma_b \int dy' \chi(y,y',U(y))\delta(U(y')-U(y))\omega_b(y'). \tag{95}$$

Using the identity

$$\delta[g(x)]=\sum_j \frac{1}{|g'(x_j)|}\delta(x-x_j), \tag{96}$$

where the x_j are the simple zeros of the function $g(x)$ (i.e. $g(x_j)=0$ and $g'(x_j)\neq 0$), we can write the diffusion coefficient as

$$D=\sum_b\sum_r \gamma_b \frac{\chi(y,y_r,U(y))}{|U'(y_r)|}\omega_b(y_r), \tag{97}$$

where the y_r are the points that resonate with y, i.e., the points that satisfy $U(y_r)=U(y)$.

If the velocity profile is monotonic,[15] using the identity

$$\delta(U(y')-U(y))=\frac{1}{|U'(y)|}\delta(y-y'), \tag{98}$$

we find that

$$D=\sum_b \gamma_b \frac{\chi(y,y,U(y))}{|U'(y)|}\omega_b(y). \tag{99}$$

For a multispecies gas of field vortices the diffusion coefficient $D\propto 1/|U'(y)|$ decreases when the shear increases. If we consider a single species gas of field vortices with circulation γ_b, the foregoing expression reduces to

$$D=\gamma_b \frac{\chi(y,y,U(y))}{|U'(y)|}\omega_b(y)=|\gamma_b|\chi(y,y,U(y)). \tag{100}$$

To obtain the second equality, we have used the relation $\omega_b(y)=-U'(y)$ from Eq. (17).

Remark: An alternative derivation of the diffusion coefficient of a test vortex produced by N field vortices is given in Appendix D 3 (see also Appendix E).

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[14] Note, however, that some field vortices may have the circulation $\gamma_b=\gamma$, i.e., they belong to the same species as the test vortex.

[15] When the vorticity $\omega(y)$ is always of the same sign, the relation $\omega(y)=-U'(y)$ from Eq. (17) implies that the velocity profile $U(y)$ is monotonic. This is the case, in particular, when the circulations $\{\gamma_b\}$ of the point vortices have the same sign.

C. Expression of the diffusion coefficient due to N point vortices without collective effects

If we neglect collective effects, the previous results remain valid provided that $G(k,y,y',kU(y))$ is replaced by $G_{\rm bare}(k,y,y')$. The bare diffusion coefficient of a test vortex is

$$D_{\rm bare}=\frac{1}{2}\sum_b\gamma_b\int dy'\int dk\,|k|G_{\rm bare}(k,y,y')^2\delta(U(y')-U(y))\omega_b(y'). \tag{101}$$

Introducing the function

$$\chi_{\rm bare}(y,y')=\frac{1}{2}\int |k|G_{\rm bare}(k,y,y')^2\,dk, \tag{102}$$

it can be written as

$$\begin{aligned} D_{\rm bare} &= \sum_b\gamma_b\int dy'\chi_{\rm bare}(y,y')\delta(U(y')-U(y))\omega_b(y') \\ &= \sum_b\sum_r\gamma_b\frac{\chi_{\rm bare}(y,y_r)}{|U'(y_r)|}\omega_b(y_r). \end{aligned} \tag{103}$$

Explicit expressions of the function $\chi_{\rm bare}(y,y')$ are given in Appendix B. In the dominant approximation, we have $\chi_{\rm bare}(y,y')\simeq(1/4)\ln\Lambda$.

If the velocity profile is monotonic, we obtain

$$D_{\rm bare}=\sum_b\gamma_b\frac{\chi_{\rm bare}(y,y)}{|U'(y)|}\omega_b(y)=\frac{1}{4}\sum_b\gamma_b\frac{\ln\Lambda}{|U'(y)|}\omega_b(y), \tag{104}$$

where we have used Eq. (B12). If the field vortices have the same circulation, Eq. (104) becomes

$$D_{\rm bare}=\frac{1}{4}\gamma_b\frac{\ln\Lambda}{|U'(y)|}\omega_b(y)=\frac{1}{4}|\gamma_b|\ln\Lambda, \tag{105}$$

where we have used Eq. (17). In that case, the bare diffusion coefficient is constant.

IX. DRIFT BY POLARIZATION

Let us consider a 2D incompressible flow with a continuous vorticity profile $\omega(y)$ and let us introduce a test vortex with a small circulation γ in that flow. The vorticity profile $\omega(y)$ may be due to a collection of field vortices with circulations $\{\gamma_b\}$, in which case it represent their mean vorticity in the limit $N_b\rightarrow+\infty$ with $\gamma_b\sim 1/N_b$, but it can also have a more general origin. We want to determine the drift by polarization experienced by the test vortex due to the perturbation that it causes to the flow. We use the formalism of linear response theory developed in the previous sections and treat the perturbation induced by the test vortex as a small external perturbation $\omega_e(x,y,t)$ to the flow.

A. Drift by polarization with collective effects

The Fourier transform of the vorticity of the test vortex is [see Eq. (45)]

$$\hat{\omega}_e(k,y,\sigma)=\gamma e^{-ikx_0}\delta(kU(y)-\sigma)\delta(y-y_0), \tag{106}$$

where (x_0,y_0) denotes the initial position of the test vortex. According to Eqs. (27) and (106), the Fourier transform of the total stream function (including collective effects) created by the test vortex is given by

$$\begin{aligned} \delta\hat{\psi}_{\rm tot}(k,y,\sigma) &= \int G(k,y,y',\sigma)\hat{\omega}_e(k,y',\sigma)\,dy' \\ &= \gamma\int G(k,y,y',\sigma)e^{-ikx_0}\delta(kU(y')-\sigma)\delta(y'-y_0)\,dy' \\ &= \gamma G(k,y,y_0,\sigma)e^{-ikx_0}\delta(kU(y_0)-\sigma). \end{aligned} \tag{107}$$

The Fourier transform of the corresponding velocity field in the y-direction $V_y = (\delta u_{\rm tot})_y = -\partial\delta\psi_{\rm tot}/\partial x$ is

$$\begin{aligned}\hat{V}_y(k,y,\sigma) &= -ik\delta\hat{\psi}_{\rm tot}(k,y,\sigma)\\ &= -ik\gamma G(k,y,y_0,\sigma)e^{-ikx_0}\delta(kU(y_0)-\sigma).\end{aligned} \tag{108}$$

Returning to physical space, we get

$$\begin{aligned}V_y(x,y,t) &= -i\gamma\int dk\int\frac{d\sigma}{2\pi}\,e^{i(kx-\sigma t)}kG(k,y,y_0,\sigma)e^{-ikx_0}\delta(kU(y_0)-\sigma)\\ &= -i\frac{\gamma}{2\pi}\int dk\,e^{ik(x-x_0-U(y_0)t)}k\,G(k,y,y_0,kU(y_0)).\end{aligned} \tag{109}$$

The test vortex is submitted to the velocity field resulting from the perturbation that it has caused and, as a result, it experiences a systematic drift [57]. Applying Eq. (109) at the position of the test vortex at time t ($x = x_0 + U(y_0)t$, $y = y_0$), we obtain the drift by polarization

$$V_{\rm pol} = -i\frac{\gamma}{2\pi}\int dk\,k\,G(k,y,y,kU(y)). \tag{110}$$

Since $V_{\rm pol}$ is real, we can write

$$V_{\rm pol} = \frac{\gamma}{2\pi}\int dk\,k\,{\rm Im}\,G(k,y,y,kU(y)). \tag{111}$$

Using the identity from Eq. (C5) we can rewrite the foregoing equation as

$$V_{\rm pol} = \frac{\gamma}{2}\int dk\int dy'\,k^2|G(k,y,y',kU(y))|^2\delta(kU(y')-kU(y))\frac{\partial\omega'}{\partial y'}. \tag{112}$$

Finally, using the identity from Eq. (92), we obtain

$$V_{\rm pol} = \frac{\gamma}{2}\int dk\int dy'\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\frac{\partial\omega'}{\partial y'}. \tag{113}$$

This is the general expression of the drift by polarization of the test vortex. It returns Eq. (81) of [68].[16] The drift by polarization of the test vortex is due to the retroaction (response) of the perturbation that it has caused to the mean flow. This is why it is proportional to γ. The calculation of the polarization cloud created by the test vortex is detailed in Appendix F.

Introducing the function from Eq. (94), the drift by polarization can be written in the more compact form as

$$V_{\rm pol} = \gamma\int dy'\,\chi(y,y',U(y))\delta(U(y')-U(y))\frac{\partial\omega'}{\partial y'}. \tag{114}$$

Using the identity from Eq. (96) we get

$$V_{\rm pol} = \gamma\sum_r\frac{\chi(y,y_r,U(y))}{|U'(y_r)|}\frac{\partial\omega}{\partial y}(y_r). \tag{115}$$

If the velocity profile is monotonic, we obtain

$$V_{\rm pol} = \gamma\frac{\chi(y,y,U(y))}{|U'(y)|}\frac{\partial\omega}{\partial y}. \tag{116}$$

The drift by polarization is proportional to the vorticity gradient $\mathbf{V}_{\rm pol} \propto \gamma\nabla\omega$. If $\gamma > 0$ the test vortex ascends the vorticity gradient. If $\gamma < 0$ the test vortex descends the vorticity gradient [58].

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[16] The fact that the drift velocity calculated in this section corresponds to $V_{\rm pol}$ in Eq. (72) is justified in Ref. [68] by calculating $V_{\rm tot}$ directly from Eq. (70).

If the vorticity ω is due to a collection of N field vortices with circulation $\{\gamma_b\}$, the drift by polarization takes the form

$$V_{\rm pol}=\frac{\gamma}{2}\sum_b\int dk\int dy'\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\frac{\partial\omega_b'}{\partial y'}. \tag{117}$$

The other equations remain valid with $\omega=\sum_b\omega_b$. If the field vortices have the same circulation γ_b, the velocity field is monotonic (see footnote 15), and Eq. (116) reduces to

$$V_{\rm pol}=\gamma\frac{\chi(y,y,U(y))}{|U'(y)|}\frac{\partial\omega_b}{\partial y}. \tag{118}$$

We note the similarity between Eq. (117) and the expression (93) of the diffusion coefficient created by a collection of point vortices. The main difference is that the drift by polarization involves the gradient of the vorticity instead of the vorticity itself. In addition the drift by polarization is proportional to the circulation γ of the test vortex while the diffusion involves the circulations $\{\gamma_b\}$ of the field vortices.

Remark: The drift by polarization $V_{\rm pol}$ is just one component of the total drift $V_{\rm tot}$ of the test vortex which is given by Eq. (72). Substituting Eqs. (93) and (113) into Eq. (72) and making an integration by parts, we find that the total drift is

$$V_{\rm tot}=\sum_b\int dy'\,\omega_b(y')\left(\gamma_b\frac{\partial}{\partial y}-\gamma\frac{\partial}{\partial y'}\right)\chi(y,y',U(y))\delta(U(y')-U(y)). \tag{119}$$

B. Drift by polarization without collective effects

It is instructive to redo the calculation of the drift by polarization by neglecting collective effects from the start.[17] In that case, the change of the vorticity caused by the external perturbation is determined by the equation

$$\frac{\partial\delta\omega}{\partial t}+U\frac{\partial\delta\omega}{\partial x}-\frac{\partial\psi_e}{\partial x}\frac{\partial\omega}{\partial y}=0, \tag{120}$$

where we have neglected the term $\delta\psi$ in Eq. (20). Written in Fourier space, we get

$$\delta\hat\omega(k,y,\sigma)=\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}\hat\psi_e(k,y,\sigma). \tag{121}$$

Using Eqs. (30) and (106), the stream function created by the test vortex is

$$\hat\psi_e(k,y,\sigma)=\gamma G_{\rm bare}(k,y,y_0)e^{-ikx_0}\delta(kU(y_0)-\sigma). \tag{122}$$

Therefore, according to Eq. (121), the perturbed vorticity field is

$$\delta\hat\omega(k,y,\sigma)=\gamma\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}G_{\rm bare}(k,y,y_0)e^{-ikx_0}\delta(kU(y_0)-\sigma). \tag{123}$$

The Fourier transform of the stream function associated with the perturbed vorticity field is [see Eqs. (7) and (B2)]

$$\delta\hat\psi(k,y,\sigma)=\int G_{\rm bare}(k,y,y')\delta\hat\omega(k,y',\sigma)\,dy'. \tag{124}$$

Combining Eqs. (123) and (124), we get

$$\delta\hat\psi(k,y,\sigma)=\gamma\int dy'\,G_{\rm bare}(k,y,y')\frac{k\frac{\partial\omega'}{\partial y'}}{kU(y')-\sigma}G_{\rm bare}(k,y',y_0)e^{-ikx_0}\delta(kU(y_0)-\sigma). \tag{125}$$

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[17] We note that we cannot simply replace G by $G_{\rm bare}$ in Eq. (111) otherwise we would find $V_{\rm pol}=0$ since $G_{\rm bare}$ is real. We first have to use Eq. (C5), then replace G by $G_{\rm bare}$ in Eq. (113).

The Fourier transform of the corresponding velocity field in the y-direction $V_y = (\delta u)_y = -\partial\delta\psi/\partial x$ is

$$\begin{aligned}\hat{V}_y(k,y,\sigma) &= -ik\delta\hat{\psi}(k,y,\sigma)\\ &= -ik\gamma\int dy'\, G_{\rm bare}(k,y,y')\frac{k\frac{\partial\omega'}{\partial y'}}{kU(y')-\sigma}G_{\rm bare}(k,y',y_0)e^{-ikx_0}\delta(kU(y_0)-\sigma).\end{aligned}\tag{126}$$

Returning to physical space, we get

$$\begin{aligned}V_y(x,y,t) &= -i\gamma\int dk\int\frac{d\sigma}{2\pi}\,e^{i(kx-\sigma t)}\int dy'\, kG_{\rm bare}(k,y,y')\frac{k\frac{\partial\omega'}{\partial y'}}{kU(y')-\sigma}G_{\rm bare}(k,y',y_0)e^{-ikx_0}\delta(kU(y_0)-\sigma)\\ &= \pi\gamma\int dk\int\frac{d\sigma}{2\pi}\,e^{i(kx-\sigma t)}\int dy'\, kG_{\rm bare}(k,y,y')k\frac{\partial\omega'}{\partial y'}\delta(kU(y')-\sigma)G_{\rm bare}(k,y',y_0)e^{-ikx_0}\delta(kU(y_0)-\sigma)\\ &= \frac{\gamma}{2}\int dk\, e^{ik(x-x_0-U(y_0)t)}\int dy'\, kG_{\rm bare}(k,y,y')k\frac{\partial\omega'}{\partial y'}\delta(k(U(y')-U(y_0)))G_{\rm bare}(k,y',y_0)\\ &= \frac{\gamma}{2}\int dk\, e^{ik(x-x_0-U(y_0)t)}\int dy'\, |k|G_{\rm bare}(k,y,y')\frac{\partial\omega'}{\partial y'}\delta(U(y')-U(y_0))G_{\rm bare}(k,y',y_0),\end{aligned}\tag{127}$$

where we have used the Landau prescription $\sigma\to\sigma+i0^+$ and the Sokhotski-Plemelj [101, 102] formula (C4) to get the second line, and Eq. (92) to get the last line.

Applying Eq. (127) at the position of the test vortex at time t ($x = x_0 + U(y_0)t$, $y = y_0$) and using the identity $G_{\rm bare}(k,y',y) = G_{\rm bare}(k,y,y')$ (see Appendix B), we obtain the drift by polarization

$$V_{\rm pol} = \frac{\gamma}{2}\int dk\int dy'\, |k|G_{\rm bare}(k,y,y')^2\delta(U(y')-U(y))\frac{\partial\omega'}{\partial y'}.\tag{128}$$

This is the general expression of the drift by polarization when collective effects are neglected. It can be directly obtained from Eq. (113) by replacing the dressed Green function $G(k,y,y',kU(y))$ by the bare Green function $G_{\rm bare}(k,y,y')$ (see footnote 17). Introducing the function from Eq. (102), it can be written in the more compact form

$$\begin{aligned}V_{\rm pol} &= \gamma\int dy'\,\chi_{\rm bare}(y,y')\delta(U(y')-U(y))\frac{\partial\omega'}{\partial y'}\\ &= \gamma\sum_r\frac{\chi_{\rm bare}(y,y_r)}{|U'(y_r)|}\frac{\partial\omega}{\partial y}(y_r).\end{aligned}\tag{129}$$

Explicit expressions of the function $\chi_{\rm bare}(y,y')$ are given in Appendix B. In the dominant approximation, we have $\chi_{\rm bare}(y,y')\simeq(1/4)\ln\Lambda$. If the velocity profile is monotonic, we obtain

$$V_{\rm pol} = \gamma\frac{\chi_{\rm bare}(y,y)}{|U'(y)|}\frac{\partial\omega}{\partial y} = \frac{1}{4}\gamma\frac{\ln\Lambda}{|U'(y)|}\frac{\partial\omega}{\partial y},\tag{130}$$

where we have used Eq. (B12).

If the vorticity ω is due to a collection of field vortices, the drift by polarization takes the form

$$V_{\rm pol} = \frac{\gamma}{2}\sum_b\int dk\int dy'\, |k|G_{\rm bare}(k,y,y')^2\delta(U(y')-U(y))\frac{\partial\omega'_b}{\partial y'}.\tag{131}$$

The other equations remain valid with $\omega = \sum_b\omega_b$. If the field vortices have the same circulation γ_b, the velocity profile is monotonic (see footnote 15) and Eq. (131) reduces to

$$V_{\rm pol} = \frac{1}{4}\gamma\frac{\ln\Lambda}{|U'(y)|}\frac{\partial\omega_b}{\partial y}.\tag{132}$$

X. EINSTEIN RELATION

We consider here the evolution of a test vortex is a sea of field vortices and establish the Einstein relation for a thermal bath and its generalization for an out-of-equilibrium bath.

A. Einstein relation for a thermal bath

If the field vortices are at statistical equilibrium with the Boltzmann distribution (59), the drift by polarization from Eq. (117) can be rewritten as

$$\begin{aligned} V_{\rm pol}(y) &= -\beta\sum_b \gamma_b \frac{\gamma}{2}\int dk\int dy'\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\frac{d\psi'}{dy'}\omega_b(y') \\ &= -\beta\sum_b \gamma_b \frac{\gamma}{2}\int dk\int dy'\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))U(y')\omega_b(y') \\ &= -\beta\sum_b \gamma_b U(y)\frac{\gamma}{2}\int dk\int dy'\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\omega_b(y'). \end{aligned} \tag{133}$$

To get the second line, we have used Eq. (17) and to get the third line, we have used the properties of the δ-function. Recalling the expression (93) of the diffusion coefficient, we obtain

$$V_{\rm pol}(y) = -D\beta\gamma U(y) = -D\beta\gamma\frac{d\psi}{dy}. \tag{134}$$

We see that the drift by polarization $\mathbf{V}_{\rm pol} = -D\beta\gamma\nabla\psi$ is proportional to the gradient of the stream function (hence perpendicular to the mean field velocity $\mathbf{u} = -\mathbf{z}\times\nabla\psi$) and that the drift coefficient is given by a form of Einstein relation [57, 58]

$$\mu = D\beta\gamma, \tag{135}$$

like in the theory of Brownian motion [32].[18] The Einstein relation connecting the drift to the diffusion coefficient is a manifestation of the fluctuation-dissipation theorem (see the Remark below). We note that the Einstein relation is valid for the drift by polarization $V_{\rm pol}$, not for the total drift which has a more complicated expression due to the term $\partial D/\partial y$ [see Eqs. (72) and (119)]. We do not have this subtlety for the usual Brownian motion where the diffusion coefficient is constant. When $\beta < 0$ the drift by polarization is directed towards $y = 0$ so the point vortices tend to accumulate at the center of the domain. When $\beta > 0$ the drift by polarization is directed away from $y = 0$ so the point vortices tend to move to infinity. This is consistent with the results obtained by Onsager [23] from very general considerations. The drift by polarization therefore provides a mechanism for the self-organization of point vortices at negative temperatures. The existence of a systematic drift for 2D point vortices in a background vorticity gradient, and the analogy with Brownian theory, were first discussed by Chavanis [57, 58]. The drift experienced by a point vortex is the counterpart of the Chandrasekhar dynamical friction experienced by a star in a stellar system [27, 28, 31]. They both arise from a polarization process [57, 97]. The necessity of the drift of point vortices and the Einstein relation were discussed in Sec. III of [58] by using very general arguments similar to those given by Chandrasekhar for stellar systems [27, 28, 31].

Remark: These results can also be obtained by substituting the fluctuation-dissipation theorem (56) valid at statistical equilibrium into the expression (111) of the drift by polarization. This yields

$$V_{\rm pol}(y,t) = -\beta\frac{\gamma}{2}U(y)\int dk\, k^2 P(k,y,kU(y)). \tag{136}$$

Recalling the expression (87) of the diffusion coefficient, we recover Eq. (134). In this sense, the Einstein relation is another formulation of the fluctuation-dissipation theorem. On the other hand, combining Eqs. (82), (134) and (135) we obtain the relation

$$\mu = \beta\gamma\int_0^{+\infty}\langle V_y(0)V_y(t)\rangle\, dt. \tag{137}$$

This is a form of Green-Kubo [83, 84, 103] relation expressing the fluctuation-dissipation theorem. There is a similar relation in Brownian theory, fluids, plasmas and stellar systems relating the friction coefficient to the force auto-correlation function [39, 50, 83, 84, 96–98, 103–110].

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[18] We note that the temperature arising in this expression can be positive or negative. The temperature is negative in situations of most physical interest [23].

B. Generalized Einstein relation

We assume here that the vorticity field ω is due to a single population of point vortices with circulation γ_b that is not necessarily at statistical equilibrium. In that case, the velocity field is monotonic (see footnote 15) and the distribution of field vortices does not change on a timescale of the order $N\, t_D$ (see Sec. XII). It forms therefore an out-of-equilibrium bath. The diffusion coefficient and the drift by polarization are given by Eqs. (100) and (118). Combining these relations, we obtain

$$V_{\rm pol} = \frac{\gamma}{\gamma_b} D \frac{\partial \ln |\omega_b|}{\partial y}. \tag{138}$$

This equation can be seen as a generalized Einstein relation for an out-of-equilibrium bath. If the field vortices are at statistical equilibrium with the Boltzmann distribution from Eq. (59), we recover the expression of the drift by polarization from Eq. (134).

Remark: Eq. (138) can also be obtained by substituting into Eq. (111) the out-of-equilibrium fluctuation-dissipation theorem (H4) derived in Appendix H and using Eq. (87).

XI. KINETIC EQUATION WITH AN ARBITRARY VELOCITY PROFILE

The kinetic equation of 2D point vortices can be obtained by substituting the expressions of the diffusion coefficient (93) and drift by polarization (117) into the Fokker-Planck equation (71). This provides an alternative derivation of the Lenard-Balescu equation of 2D point vortices as compared to the one given in [68] which is based on the Klimontovich formalism. In this section, we study the general properties of this equation for an arbitrary velocity profile.

A. Multi-species systems

Substituting Eqs. (93) and (117) into the Fokker-Planck equation (71) and introducing the vorticity $\omega_a = N_a \gamma_a P_1^{(a)}$ of each species of point vortices, we obtain the integrodifferential equation

$$\frac{\partial \omega_a}{\partial t} = \frac{1}{2}\frac{\partial}{\partial y}\sum_b \int dy' \int dk\, |k| |G(k,y,y',kU(y))|^2 \delta(U(y')-U(y)) \left(\gamma_b \omega_b' \frac{\partial \omega_a}{\partial y} - \gamma_a \omega_a \frac{\partial \omega_b'}{\partial y'}\right), \tag{139}$$

where ω_a stands for $\omega_a(y,t)$ and $U(y)$ stands for $U(y,t)$. The mean velocity $U(y,t)$ is determined by the total vorticity $\omega(y,t) = \sum_a \omega_a(y,t)$. Equation (139) is the counterpart of the multi-species Lenard-Balescu equation in plasma physics. When collective effects are neglected, i.e., when $G(k,y,y',kU(y))$ is replaced by $G_{\rm bare}(k,y,y')$, it reduces to the multi-species Landau equation. It can be shown [62, 73] that the kinetic equation (139) conserves the total energy $E = \frac{1}{2}\int \omega\psi\, dy$, the total impulse $P = \int \omega y\, dy$ and the vortex number N_a (or the circulation $\Gamma_a = N_a \gamma_a = \int \omega_a\, dy$) of each species of point vortices, and that the Boltzmann entropy $S = -\sum_a \int \frac{\omega_a}{\gamma_a} \ln \frac{\omega_a}{\gamma_a}\, dy$ increases monotonically: $\dot S \geq 0$ (H-theorem). Furthermore, the multi-species Boltzmann distribution[19]

$$\omega_a^{\rm eq}(y) = A_a e^{-\beta\gamma_a \psi(y)}, \tag{140}$$

where the inverse temperature β is the same for all species of point vortices, is always a steady state of the kinetic equation (139). We note that the Boltzmann distribution of the different species of point vortices satisfies the relation

$$\omega_a^{\rm eq}(y) = C_{ab}[\omega_b^{\rm eq}(y)]^{\gamma_a/\gamma_b}, \tag{141}$$

where C_{ab} is a constant (independent of y). As discussed in Sec. XII, the kinetic equation (139) does not necessarily relax towards the Boltzmann distribution because of the phenomenon of kinetic blocking [62]. The Lenard-Balescu

[19] The Boltzmann distribution is the "most probable" distribution of point vortices. It can be obtained by maximizing the Boltzmann entropy S at fixed energy E, impulse P and vortex numbers N_a by introducing appropriate Lagrange multipliers β (inverse temperature), V (translation velocity) and μ_a (chemical potentials) [62, 73]. In the following, for simplicity, we shall work in a frame of reference where $V = 0$. More generally, we have to replace the stream function ψ by the relative stream function $\psi_{\rm eff} = \psi - Vy$ [111].

equation is valid at the order $1/N$ so it describes the evolution of the system on a timescale of the order Nt_D. It may be necessary to develop the kinetic theory at higher order to describe the relaxation of the system of point vortices towards the Boltzmann distribution.

Remark: Substituting Eqs. (87) and (111) into the Fokker-Planck equation (71) we obtain the kinetic equation

$$\frac{\partial \omega_a}{\partial t} = \frac{1}{2}\frac{\partial}{\partial y}\int dk\, k^2 \left[P(k,y,kU(y))\frac{\partial \omega_a}{\partial y} - \omega_a \frac{\gamma_a}{\pi k}\mathrm{Im}\, G(k,y,y,kU(y))\right]. \tag{142}$$

Using Eqs. (48) and (C5), we can check that Eq. (142) is equivalent to Eq. (139).

B. Moment equations

Introducing the notation from Eq. (94) the kinetic equation (139) can be written as

$$\frac{\partial \omega_a}{\partial t} = \frac{\partial}{\partial y}\sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\left(\gamma_b \omega_b' \frac{\partial \omega_a}{\partial y} - \gamma_a \omega_a \frac{\partial \omega_b'}{\partial y'}\right). \tag{143}$$

Using the identity from Eq. (96), we get

$$\frac{\partial \omega_a}{\partial t} = \frac{\partial}{\partial y}\sum_b\sum_r \frac{\chi(y,y_r,U(y))}{|U'(y_r)|}\left(\gamma_b \omega_b^r \frac{\partial \omega_a}{\partial y} - \gamma_a \omega_a \frac{\partial \omega_b^r}{\partial y_r}\right). \tag{144}$$

Equation (143) has the form of a Fokker-Planck equation

$$\frac{\partial \omega_a}{\partial t} = \frac{\partial}{\partial y}\left(D\frac{\partial \omega_a}{\partial y} - \omega_a V_{\rm pol}^{(a)}\right) \tag{145}$$

with a diffusion coefficient [see Eq. (95)]

$$D = \sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\gamma_b\omega_b' = \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\omega_2' \tag{146}$$

and a drift by polarization [see Eq. (117)]

$$V_{\rm pol}^{(a)} = \gamma_a \sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\frac{\partial \omega_b'}{\partial y'} = \gamma_a \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\frac{\partial \omega'}{\partial y'}. \tag{147}$$

In the second equalities of Eqs. (146) and (147) we have introduced the total vorticity $\omega = \sum_a \omega_a = \sum_a N_a\gamma_a P_1^{(a)}$ and the second moment $\omega_2 = \sum_a \gamma_a\omega_a = \sum_a N_a\gamma_a^2 P_1^{(a)}$ of the vorticity distribution. As explained previously, the diffusion coefficient D of a test vortex is due to the fluctuation of all the field vortices so it depends on $\{\gamma_b\}$ through ω_2. By contrast, the drift by polarization of a test vortex of species a is due to the retroaction (response) of the perturbation that this point vortex caused on the mean flow ω. As a result, $V_{\rm pol}^{(a)}$ is proportional to γ_a.

The equation for the total vorticity is

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial y}\int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\left(\omega_2' \frac{\partial \omega}{\partial y} - \omega_2 \frac{\partial \omega'}{\partial y'}\right). \tag{148}$$

This equation depends on the second moment ω_2. We can write down a hierarchy of equations for the moments $\omega_n = \sum_a \gamma_a^{n-1}\omega_a = \sum_a N_a\gamma_a^n P_1^{(a)}$. The generic term of this hierarchy is

$$\frac{\partial \omega_n}{\partial t} = \frac{\partial}{\partial y}\int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\left(\omega_2' \frac{\partial \omega_n}{\partial y} - \omega_{n+1} \frac{\partial \omega'}{\partial y'}\right). \tag{149}$$

This hierarchy is not closed since the equation for ω_n depends on ω_{n+1}.

Remark: If we neglect collective effects, the kinetic equation (143) reduces to

$$\frac{\partial \omega_a}{\partial t} = \frac{\partial}{\partial y}\sum_b \int dy'\, \chi_{\rm bare}(y,y')\delta[U(y')-U(y)]\left(\gamma_b \omega_b' \frac{\partial \omega_a}{\partial y} - \gamma_a \omega_a \frac{\partial \omega_b'}{\partial y'}\right), \tag{150}$$

where $\chi_{\rm bare}(y,y')$ is given in Appendix B. In the dominant approximation, we have $\chi_{\rm bare}(y,y') \simeq (1/4)\ln\Lambda$.

C. Thermal bath approximation

We consider a test vortex[20] of circulation γ in "collision" with field vortices of circulations $\{\gamma_b\}$. We assume that the vorticity ω_b of the field vortices is prescribed (the validity of this approximation is discussed below). The velocity profile $U(y)$, which is determined by $\omega = \sum_b \omega_b$, is also prescribed. This situation corresponds to the bath approximation in its general form. Under these conditions, the Lenard-Balescu equation (143) reduces to

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial y}\sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\left(\gamma_b\omega_b'\frac{\partial P}{\partial y} - \gamma P\frac{d\omega_b'}{dy'}\right). \tag{151}$$

Equation (151) governs the evolution of the probability density $P(y,t)$ of finding the test vortex of circulation γ in y at time t. It can be written under the form of a Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial y}\left(D\frac{\partial P}{\partial y} - PV_{\rm pol}\right) \tag{152}$$

with

$$D = \sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\gamma_b\omega_b' = \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\omega_2' \tag{153}$$

and

$$V_{\rm pol} = \gamma\sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\frac{d\omega_b'}{dy'} = \gamma\int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\frac{d\omega'}{dy'}. \tag{154}$$

In this manner, we have transformed an integrodifferential equation of the Lenard-Balescu or Landau form [see Eq. (143)] into a differential equation of the Fokker-Planck form [see Eq. (151)].

This bath approach is self-consistent in the general case only if the field vortices are at statistical equilibrium, otherwise their distribution ω_b evolves in time due to discreteness effects. If we assume that the field vortices are at statistical equilibrium with the Boltzmann distribution (59), corresponding to the thermal bath approximation, the Fokker-Planck equation (151) becomes

$$\begin{aligned}\frac{\partial P}{\partial t} &= \frac{\partial}{\partial y}\sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\left(\gamma_b\omega_b'\frac{\partial P}{\partial y} + \beta\gamma_b\gamma P U(y')\omega_b'\right)\\ &= \frac{\partial}{\partial y}\sum_b \int dy'\, \chi(y,y',U(y))\delta[U(y')-U(y)]\gamma_b\omega_b'\left(\frac{\partial P}{\partial y} + \beta\gamma P U(y)\right),\end{aligned} \tag{155}$$

where we have used Eqs. (17) and (59) to get the first equality, and the properties of the δ-function to get the second equality. Using Eq. (17) again, we can write the Fokker-Planck equation under the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial y}\left[D(y)\left(\frac{\partial P}{\partial y} + \beta\gamma P\frac{d\psi}{dy}\right)\right], \tag{156}$$

where D is given by Eq. (153) with the Boltzmann distribution (59). Equation (156) can also be directly obtained from Eq. (152) by using the expression (134) of the drift by polarization valid for a thermal bath (Einstein relation). The Fokker-Planck equation (156) conserves the normalization condition $\int P\, dy = 1$ and decreases the free energy $F = E - TS$ with $E = \int \gamma P\psi\, dy$ and $S = -\int P\ln P\, dy$ monotonically: $\dot F \le 0$ (canonical H-theorem). It relaxes towards the Boltzmann distribution

$$P_{\rm eq}(y) = A\, e^{-\beta\gamma\psi(y)}. \tag{157}$$

Since the Fokker-Planck equation (156) is valid at the order $1/N$, the relaxation time of a test vortex in a thermal bath scales as

$$t_{\rm R}^{\rm bath} \sim N t_D, \tag{158}$$

[20] This can be a single point vortex or an ensemble of noninteracting point vortices of the same species. In the second case, we take into account the collisions between the test vortices γ and the field vortices $\{\gamma_b\}$, but we ignore the collisions between the test vortices.

where t_D is the dynamical time. The Fokker-Planck equation (156) for a test vortex is similar to the Smoluchowski [112] equation describing the evolution of an overdamped Brownian particle submitted to an external potential ψ. The Smoluchowski equation for a test vortex evolving in a sea of field vortices is the counterpart of the Klein-Kramers-Chandrasekhar equation [27–32, 113, 114] for a test star evolving in a stellar system. The relaxation of a system of point vortices towards statistical equilibrium is due to a competition between diffusion and drift [57, 58]. Similarly, the relaxation of a stellar system is due to a competition between diffusion and friction [27, 28, 31]. We note that 2D point vortices do not have inertia. This is why they resemble overdamped Brownian particles described by the Smoluchowski equation in configuration space rather than inertial Brownian particles described by the Kramers equation in phase space.

Remark: In the thermal bath approximation, substituting the fluctuation-dissipation theorem (56) into the kinetic equation (142) and using Eq. (17), we obtain

$$\frac{\partial P}{\partial t}=\frac{1}{2}\frac{\partial}{\partial y}\int dk\, k^2 P(k,y,kU(y))\left(\frac{\partial P}{\partial y}+\beta\gamma P\frac{d\psi}{dy}\right). \tag{159}$$

This equation is equivalent to the Fokker-Planck equation (156). The diffusion coefficient D is given by Eqs. (48) and (87) which return Eq. (153).

D. Pure diffusion

When $\gamma \ll \gamma_b$, the drift by polarization can be neglected ($V_{\rm pol}=0$) and the test vortex has a purely diffusive evolution. In that case, the Fokker-Planck equation (152) reduces to

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial y}\left(D\frac{\partial P}{\partial y}\right), \tag{160}$$

where D is given by Eq. (153). We note that the diffusion coefficient D depends on the circulations $\{\gamma_b\}$ of the field vortices through the second moment ω_2. This reflects the discrete nature of the field vortices. On the other hand, the diffusion coefficient does not depend on the circulation γ of the test vortex.

Remark: Since the diffusion coefficient depends on y and since it is placed between the two spatial derivatives $\partial/\partial y$, Eq. (160) is not exactly a diffusion equation. It can be rewritten as Eq. (68) showing that the test vortex experiences a drift [see Eq. (72)]

$$V_{\rm tot}=\frac{\partial D}{\partial y}. \tag{161}$$

E. Pure drift

When $\gamma \gg \gamma_b$, the diffusion can be neglected ($D=0$) and the test vortex has a purely deterministic evolution. In that case, the Fokker-Planck equation (152) reduces to

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial y}\left(-PV_{\rm pol}\right), \tag{162}$$

where $V_{\rm pol}$ is given by Eq. (154). We note that the drift by polarization $V_{\rm pol}$ is proportional to the circulation γ of the test vortex. On the other hand, it depends only on the mean vorticity field ω, not on the individual circulations $\{\gamma_b\}$ of the field vortices reflecting their discrete nature. As a result, Eq. (162) remains valid when a test vortex of circulation γ evolves in a continuous vorticity field ω which is not necessarily due to a collection of point vortices.

Remark: The deterministic motion of the test vortex induced by the drift term can be written as

$$\frac{dy}{dt}=V_{\rm pol}(y,t), \tag{163}$$

and we have $V_{\rm tot}=V_{\rm pol}$ [see Eq. (72)]. This equation describes the drift of a point vortex in a mean flow. This is the counterpart of the sinking satellite problem in stellar dynamics [6]. The change of energy of the test vortex is $\dot\epsilon=\gamma U V_{\rm pol}$. This relation can be obtained by taking the time variation of $E=\gamma\int P\psi\, dy$, using Eq. (162), and integrating by parts. Using the general expression (112) of the drift term, we obtain

$$\dot\epsilon=\frac{\gamma^2}{2}\int dk\int dy'\, k[kU(y')]|G(k,y,y',kU(y))|^2\delta(kU(y')-kU(y))\frac{d\omega'}{dy'}. \tag{164}$$

For a stationary flow of the form $\omega=\omega(\psi)$, we find that

$$\dot{\epsilon}=\frac{\gamma^2}{2}\int dk\int dy'\,[kU(y')]^2|G(k,y,y',kU(y))|^2\delta(kU(y')-kU(y))\frac{d\omega'}{d\psi'}. \tag{165}$$

Therefore $\dot{\epsilon}$ is negative if $d\omega/d\psi<0$ and positive if $d\omega/d\psi>0$. In the first case, the test vortex loses energy to the flow and in the second case it gains energy from the flow. For the Boltzmann distribution (59), we have $d\omega/d\psi=-\beta\omega_2$ [111]. Therefore, $d\omega/d\psi<0$ corresponds to $\beta>0$ (positive temperature) and $d\omega/d\psi>0$ corresponds to $\beta<0$ (negative temperature).

XII. KINETIC EQUATION WITH A MONOTONIC VELOCITY PROFILE

In this section, we study the general properties of the kinetic equation of 2D point vortices when the velocity profile is monotonic.

A. Multi-species systems

If we velocity profile is monotonic, using identity (98), the kinetic equation (143) becomes[21]

$$\frac{\partial\omega_a}{\partial t}=\frac{\partial}{\partial y}\sum_b\frac{\chi(y,y,U(y))}{|U'(y)|}\left(\gamma_b\omega_b\frac{\partial\omega_a}{\partial y}-\gamma_a\omega_a\frac{\partial\omega_b}{\partial y}\right). \tag{166}$$

It can be written as a Fokker-Planck equation of the form of Eq. (145) with a diffusion coefficient [see Eq. (99)]

$$D=\sum_b\frac{\chi(y,y,U(y))}{|U'(y)|}\gamma_b\omega_b=\frac{\chi(y,y,U(y))}{|U'(y)|}\omega_2 \tag{167}$$

and a drift by polarization [see Eq. (116)]

$$V_{\rm pol}^{(a)}=\gamma_a\sum_b\frac{\chi(y,y,U(y))}{|U'(y)|}\frac{\partial\omega_b}{\partial y}=\gamma_a\frac{\chi(y,y,U(y))}{|U'(y)|}\frac{\partial\omega}{\partial y}. \tag{168}$$

Equation (166) conserves the energy, the impulse and the vortex number of each species, and satisfies an H-theorem for the Boltzmann entropy. At equilibrium, the currents J_a defined by $\partial\omega_a/\partial t=-\partial J_a/\partial y$ vanish (for each species), and we have the relation

$$\omega_a^{\rm eq}(y)=C_{ab}[\omega_b^{\rm eq}(y)]^{\gamma_a/\gamma_b}, \tag{169}$$

where C_{ab} is a constant (independent of y). This relation is similar to Eq. (141) which was obtained in the case where the equilibrium vorticity is given by the Boltzmann distribution (140). However, it is important to mention that Eq. (166) does *not* relax towards the Boltzmann distribution (see below). Therefore, the equilibrium vorticity distribution $\omega_a^{\rm eq}(y)$ in Eq. (169) is generally not given by Eq. (140).

Equation (148) for the total vorticity reduces to

$$\frac{\partial\omega}{\partial t}=0. \tag{170}$$

This equation is closed and shows that the average vorticity profile does not change (the total current vanishes). This kinetic blocking is discussed in more detail in the following section. We note, by contrast, that the vorticity ω_b of the different species evolves in time. Because of the drift term (see Sec. IX) the vortices with large positive (resp.

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[21] It is possible that, initially, the velocity profile is non-monotonic but that it becomes monotonic during the evolution. In that case, the evolution of the vorticity is first described by Eq. (143), then by Eq. (166).

negative) circulation tend to concentrate around maxima (resp. minima) of vorticity. The hierarchy of equations for the moments of the vorticity, Eq. (149), reduces to

$$\frac{\partial \omega_n}{\partial t} = \frac{\partial}{\partial y}\left[\frac{\chi(y,y,U(y))}{|U'(y)|}\left(\omega_2\frac{\partial \omega_n}{\partial y} - \omega_{n+1}\frac{\partial \omega}{\partial y}\right)\right]. \tag{171}$$

This hierarchy of moments is not closed (for $n \geq 2$) since the equation for ω_n depends on ω_{n+1}.

Remark: If we neglect collective effects, the function $\chi(y,y,U(y))$ can be replaced by $\chi_{\rm bare}(y,y) = \frac{1}{4}\ln\Lambda$ (see Appendix B) and the kinetic equation (166) reduces to

$$\frac{\partial \omega_a}{\partial t} = \frac{1}{4}\ln\Lambda\frac{\partial}{\partial y}\sum_b \frac{1}{|U'(y)|}\left(\gamma_b\omega_b\frac{\partial \omega_a}{\partial y} - \gamma_a\omega_a\frac{\partial \omega_b}{\partial y}\right). \tag{172}$$

The subsequent equations (167)-(171) can be simplified similarly.

B. Single-species systems

For a single species gas of point vortices, the Lenard-Balescu equation (143) reduces to

$$\frac{\partial \omega}{\partial t} = \gamma\frac{\partial}{\partial y}\int dy'\,\chi(y,y',kU(y))\delta(U(y')-U(y))\left(\omega'\frac{\partial \omega}{\partial y} - \omega\frac{\partial \omega'}{\partial y'}\right). \tag{173}$$

Since the velocity profile of a single species gas of point vortices is monotonic (see footnote 15) we find, using identity (98), that

$$\frac{\partial \omega}{\partial t} = \gamma\frac{\partial}{\partial y}\int dy'\,\chi(y,y',kU(y))\frac{1}{|U'(y)|}\delta(y'-y)\left(\omega'\frac{\partial \omega}{\partial y} - \omega\frac{\partial \omega'}{\partial y'}\right) = 0. \tag{174}$$

We recall that the Lenard-Balescu equation (173) is valid at the order $1/N$ so it describes the evolution of the average vorticity on a timescale Nt_D under the effect of two-body correlations. Equation (174) shows that the vorticity does not change on this timescale (the current vanishes at the order $1/N$). This is a situation of kinetic blocking due to the absence of resonance at the order $1/N$.[22] The vorticity may evolve on a longer timescale due to higher order correlations between the point vortices. For example, three-body correlations are of order $1/N^2$ and induce an evolution of the vorticity on a timescale $N^2\,t_D$.

Remark: The same situation of kinetic blocking at the order $1/N$ occurs for 1D homogeneous systems of material particles with long-range interactions such as 1D plasmas and the HMF model above the critical energy [53, 64]. In that case, an explicit kinetic equation has been derived at the order $1/N^2$ [55, 56]. This equation does not display kinetic blocking and relaxes towards the Boltzmann distribution. In that case, the relaxation time scales as $N^2 t_D$. The same results are expected to hold for unidirectional or axisymmetric distributions of 2D point vortices. By contrast, for more general flows that are neither unidirectional nor axisymmetric, the kinetic equation of 2D point vortices is given by Eq. (128) or Eq. (137) of [58] (see also Eq. (54) in [63] and Eq. (16) in [62]) and the collision term does not necessarily vanish.[23] In that case, the system may relax towards the Boltzmann distribution on a timescale Nt_D. The same situation holds for 1D inhomogeneous systems of material particles with long-range interactions such as 1D self-gravitating systems and for homogeneous or inhomogeneous systems of material particles with long-range interactions in $d \geq 2$ which relax towards the Boltzmann distribution on a timescale Nt_D [53, 64].

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[22] For an axisymmetric distribution of point vortices, the angular velocity profile is not necessarily monotonic, even in the single-species case. As long as the angular velocity profile is nonmonotonic there are resonances leading to a nonzero current ($J \neq 0$). However, the relaxation stops ($J = 0$) when the profile of angular velocity becomes monotonic even if the system has not reached the Boltzmann distribution of statistical equilibrium. This "kinetic blocking" for axisymmetric flows is illustrated in [62].

[23] This is because there are potentially more resonances at the order $1/N$ for complicated flows than for unidirectional and axisymmetric flows. Similarly, resonances appear for inhomogeneous 1D systems with long-range interactions that are not present for homogeneous 1D systems [115].

C. Out-of-equilibrium bath

The kinetic equation governing the evolution of a test vortex of circulation γ in a bath of field vortices with circulations $\{\gamma_b\}$ and vorticities $\{\omega_b\}$ is given by Eq. (151). If the velocity profile is monotonic, using identity (98), this Fokker-Planck equation becomes

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial y}\sum_b \frac{\chi(y,y,U(y))}{|U'(y)|}\left(\gamma_b\omega_b\frac{\partial P}{\partial y}-\gamma P\frac{d\omega_b}{dy}\right). \tag{175}$$

It can be written as Eq. (152) with

$$D=\sum_b \frac{\chi(y,y,U(y))}{|U'(y)|}\gamma_b\omega_b=\frac{\chi(y,y,U(y))}{|U'(y)|}\omega_2, \tag{176}$$

and

$$V_{\rm pol}=\gamma\sum_b \frac{\chi(y,y,U(y))}{|U'(y)|}\frac{d\omega_b}{dy}=\gamma\frac{\chi(y,y,U(y))}{|U'(y)|}\frac{d\omega}{dy}. \tag{177}$$

If we neglect collective effects, the function $\chi(y,y,U(y))$ can be replaced by $\chi_{\rm bare}(y,y)=\frac{1}{4}\ln\Lambda$ (see Appendix B). As explained in Sec. XI C, this approach is self-consistent in the multispecies case only if the field vortices are at statistical equilibrium with the Boltzmann distribution from Eq. (59) otherwise their distribution evolves under the effect of collisions (see Sec. XII A). However, if the field vortices have the same circulation, their distribution does not change on a timescale $N\,t_D$ (see Sec. XII B). They are in a blocked state. In that case, the bath approximation is justified (on this timescale) for an arbitrary vorticity field ω_b, not only for the Boltzmann distribution. The Fokker-Planck equation (175) can then be rewritten as

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial y}\left[D(y)\left(\frac{\partial P}{\partial y}-\frac{\gamma}{\gamma_b}P\frac{d\ln|\omega_b|}{dy}\right)\right] \tag{178}$$

with

$$D=\frac{\chi(y,y,U(y))}{|U'(y)|}\gamma_b\omega_b=\chi(y,y,U(y))|\gamma_b|. \tag{179}$$

This equation can also be directly obtained from the Fokker-Planck equation (152) by using the expression of the drift by polarization given by Eq. (138) and the diffusion coefficient from Eq. (100). The distribution of the test vortex relaxes towards the equilibrium distribution

$$P_{\rm eq}(y)=A|\omega_b|^{\gamma/\gamma_b} \tag{180}$$

on a relaxation time $t_R^{\rm bath}\sim Nt_D$.

Remark: In the bath approximation, substituting the out-of-equilibrium fluctuation-dissipation theorem (H4) into the kinetic equation (142), we obtain

$$\frac{\partial P}{\partial t}=\frac{1}{2}\frac{\partial}{\partial y}\int dk\,k^2 P(k,y,kU(y))\left(\frac{\partial P}{\partial y}-\frac{\gamma}{\gamma_b}P\frac{d\ln|\omega_b|}{dy}\right). \tag{181}$$

This equation is equivalent to the Fokker-Planck equation (178). The diffusion coefficient D is given by Eqs. (87) and (H3) which return Eq. (179).

D. Thermal bath

In the thermal bath approximation, using Eq. (59), the Fokker-Planck equation (178) reduces to

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial y}\left[D\left(\frac{\partial P}{\partial y}+\beta\gamma P\frac{d\psi}{dy}\right)\right], \tag{182}$$

in agreement with Eq. (156). When the field vortices have the same circulation γ_b, the equilibrium stream function can be calculated analytically (see Appendix G) yielding

$$\psi = -\frac{2}{|\beta|\gamma_b}\ln\left\{\cosh\left(\frac{|\beta|\gamma_b\Gamma_b y}{4}\right)\right\} \qquad (\beta<0). \tag{183}$$

The equilibrium distribution of the test vortex is given by [see Eq. (157) with Eq. (183)]

$$P_{\rm eq}(y) = \frac{A}{\cosh^{\frac{2\gamma}{\gamma_b}}\left(\frac{|\beta|\gamma_b\Gamma_b y}{4}\right)}, \tag{184}$$

where A is a constant determined by the normalization condition $\int_{-\infty}^{+\infty} P_{\rm eq}(y)\, dy = 1$. On the other hand, if we neglect collective effects, the diffusion coefficient (179) has a constant value given by

$$D = \chi_{\rm bare}(y,y)|\gamma_b| = \frac{1}{4}|\gamma_b|\ln\Lambda, \tag{185}$$

where we have used Eq. (B12). When the diffusion coefficient it constant, it is possible to transform the Smoluchowski equation (182) into a Schrödinger equation with imaginary time [73, 91]. Indeed, making the change of variables

$$P(y,t) = \Phi(y,t)e^{-\frac{1}{2}\beta\gamma\psi(y)}, \tag{186}$$

we obtain after simplification the Schrödinger-like equation

$$\frac{\partial\Phi}{\partial t} = D\frac{\partial^2\Phi}{\partial y^2} - V(y)\Phi \tag{187}$$

with the effective potential

$$V(y) = -\frac{1}{2}D\beta\gamma\frac{d^2\psi}{dy^2} + \frac{1}{4}D\beta^2\gamma^2\left(\frac{d\psi}{dy}\right)^2. \tag{188}$$

When the stream function is given by Eq. (183) we explicitly obtain

$$V(y) = \frac{1}{16}D\beta^2\gamma^2\Gamma_b^2\left[1 - \frac{1+\frac{\gamma_b}{\gamma}}{\cosh^2\left(\frac{|\beta|\gamma_b\Gamma_b y}{4}\right)}\right]. \tag{189}$$

This is a Rosen-Morse [71] (or Pöschl-Teller [72]) potential. Interestingly, the Smoluchowski equation (182) with the potential from Eq. (183) can be solved analytically [73].

XIII. 2D BROWNIAN POINT VORTICES

In the previous sections, we have considered an isolated system of 2D point vortices described by N-body Hamiltonian equations (see Appendix A). This is the Kirchhoff [116] model. It is associated with the microcanonical ensemble where the energy E of the system is conserved. When $N \to +\infty$ with $\gamma \sim 1/N$, the collisions between vortices are negligible and the evolution of the mean vorticity is described by the 2D Euler-Poisson equations (6) and (13). These equations generically experience a process of violent relaxation towards a quasistationary state (QSS) on a few dynamical times t_D.[24] On a longer timescale $\sim N t_D$, the evolution of the mean vorticity is governed by the Lenard-Balescu equation (139), which is valid at the order $1/N$. This equation takes into account the collisions between the vortices. For general flows that are not unidirectional or axisymmetric, the kinetic equation of point vortices [see Eq. (128) or Eq. (137) of [58]] is expected to relax towards the Boltzmann distribution with energy E on a timescale $t_R \sim N\, t_D$. For unidirectional flows and for axisymmetric flows with a monotonic profile of angular velocity, the Lenard-Balescu

[24] This metaequilibrium state, which is a stable steady state of the 2D Euler-Poisson equations, may be an unidirectional or axisymmetric flow (as considered in this paper) or have a more complicated structure.

equation (for a single species system of point vortices) reduces to $\partial\omega/\partial t = 0$. This leads to a situation of kinetic blocking [62]. In that case, the relaxation towards the Boltzmann distribution (59) with energy E is described by a kinetic equation valid at the order $1/N^2$ but the collision term of this equation is not explicitly known (see, however, Refs. [55, 56] in the case of material particles with long-range interactions). It is expected to relax towards the Boltzmann distribution on a timescale $t_R \sim N^2 t_D$.

In Ref. [75] we have formally introduced a system of 2D Brownian point vortices described by N-body stochastic Langevin equations[25]

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{z} \times \sum_j \frac{\gamma_j}{2\pi} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} + \beta D \gamma_i \sum_j \frac{\gamma_j}{2\pi} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} + \sqrt{2D}\mathbf{R}_i(t), \tag{190}$$

where $i = 1, ..., N$ label the point vortices and $\mathbf{R}_i(t)$ is a Gaussian white noise satisfying $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$ and $\langle R_i^\alpha(t) R_j^\beta(t') \rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$. As compared to the Kirchhoff model (first term), this Brownian model includes a drift velocity (second term) and a random velocity (third term). This corresponds to the canonical ensemble where the temperature T is fixed. This Brownian model could describe the motion of quantized 2D point vortices in Bose-Einstein condensates and superfluids, where the fluctuations and the dissipation (drift) are caused by impurities. When $N \to +\infty$ with $\gamma \sim 1/N$, the collisions between the vortices are negligible and the evolution of the mean vorticity is described by a mean field Fokker-Planck equation which has the form of a system of 2D Euler-Smoluchowski-Poisson equations

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nabla \cdot \left[D(\nabla \omega + \beta \gamma \omega \nabla \psi) \right], \tag{191}$$

$$\omega = -\Delta \psi. \tag{192}$$

These equations relax towards the Boltzmann distribution (59) with temperature T on a diffusive (Brownian) timescale $t_B \sim L^2/D$, where L is the system size. When $D \to 0$, these equations first experience a process of violent relaxation towards a QSS on a few dynamical times t_D before slowly relaxing towards the Boltzmann distribution (59), as discussed in [119] in the context of the BMF model.[26]

The 2D Euler-Smoluchowski-Poisson equations (191) and (192) associated with the canonical ensemble are structurally very different from the Lenard-Balescu equation (139) associated with the microcanonical ensemble. They are also very different from the Smoluchowski equation (156) describing the evolution of the mean vorticity of a system of noninteracting test vortices in a thermal bath of field vortices at statistical equilibrium. In that case, the stream function $\psi(y)$ is determined by the equilibrium distribution of the field vortices whereas in Eqs. (191) and (192) the stream function $\psi(\mathbf{r}, t)$ is self-consistently produced by the distribution of the point vortices itself. Furthermore, the 2D Euler-Smoluchowski-Poisson equations (191) and (192) are valid when $N \to +\infty$ while the Lenard-Balescu equation (139) and the Smoluchowski equation (156) are valid at the order $1/N$.

The 2D Boltzmann-Poisson equations may admit several equilibrium states.[27] The 2D Euler-Smoluchowski-Poisson equations describe the evolution of the mean vorticity towards one of these equilibrium states which is stable in the canonical ensemble. Following [75, 123], if we consider the evolution of 2D Brownian point vortices on a mesoscopic scale, we have to add a stochastic term in the kinetic equation. This noise term arises from finite N effects and takes fluctuations into account (note that we are still neglecting the collisions between the vortices). This leads to the

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[25] We have similarly introduced a system of self-gravitating Brownian particles in Ref. [117] and a system of Brownian particles with a cosine interaction, called the Brownian mean field (BMF) model, in Ref. [118].

[26] When N is finite and $N^2 \ll 1/D$, the system may first achieve a QSS on a timescale t_D, followed by a microcanonical equilibrium state at energy E on a timescale $t_R \sim N\, t_D$ or $t_R \sim N^2\, t_D$ (depending on the structure of the flow), itself followed by a canonical equilibrium state at temperature T on a timescale $t_B \sim L^2/D$ (see Ref. [119]). When the fluctuations are taken into account, the phenomenology is even richer as discussed at the end of this section and in Sec. XVI.

[27] This is the case in a shear layer where the statistical equilibrium state may be either unidirectional (jet) or have the form of a large scale vortex [120]. This is also the case when the point vortices have positive and negative circulations. In that case, the statistical equilibrium states have the form of monopoles, dipoles or even tripoles [121, 122]. In the present section, we write the equations for a single species gas of point vortices, but these equations can be straightforwardly generalized to a multi-species gas (see e.g. [75, 111] and Appendix I).

stochastic 2D Euler-Smoluchowski-Poisson equations [75][28]

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=\nabla\cdot\left[D(\nabla\omega+\beta\gamma\omega\nabla\psi)\right]+\nabla\cdot\left[\sqrt{2D\gamma\omega}\,\mathbf{R}(\mathbf{r},t)\right], \tag{193}$$

$$\omega=-\Delta\psi, \tag{194}$$

where $\mathbf{R}(\mathbf{r},t)$ is a Gaussian white noise such that $\langle\mathbf{R}(\mathbf{r},t)\rangle=\mathbf{0}$ and $\langle R_i(\mathbf{r},t)R_j(\mathbf{r}',t)\rangle=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$. When the 2D Boltzmann-Poisson equations admit different equilibrium states, Eqs. (193) and (194) can be used to study random transitions from one state to the other (see Ref. [124] in a similar context). The probability of transition is given by the Kramers formula, which can be established from the instanton theory associated with the Onsager-Machlup functional [124, 125].

The equation for the velocity field corresponding to Eqs. (193) and (194) reads [75]

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla P+D\Delta\mathbf{u}-\mu\omega\mathbf{u}+\sqrt{2D\gamma\omega}\,\mathbf{z}\times\mathbf{R}(\mathbf{r},t), \tag{195}$$

where $\mu=D\beta\gamma$ is the drift coefficient. Since $\beta<0$ in the most relevant situations, the term $-\mu\omega\mathbf{u}$ represents a nonlinear anti-friction (forcing) proportional to $\omega\mathbf{u}$ that opposes itself to the diffusion term $D\Delta\mathbf{u}$.

For unidirectional flows (possibly resulting from a process of violent relaxation), the stochastic Smoluchowski-Poisson equations (193) and (194) become

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial y}\left(D\frac{\partial\omega}{\partial y}+\mu\omega\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left[\sqrt{2D\gamma\omega}\,R(y,t)\right], \tag{196}$$

$$\omega=-\frac{\partial^2\psi}{\partial y^2}. \tag{197}$$

Using Eq. (17), they can be combined into a single equation for the velocity field

$$\frac{\partial U}{\partial t}=D\frac{\partial^2 U}{\partial y^2}+\mu U\frac{\partial U}{\partial y}-\sqrt{-2D\gamma U'(y)}R(y,t). \tag{198}$$

If we set $v=-\mu U$, the foregoing equation may be rewritten as

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}=D\frac{\partial^2 v}{\partial y^2}+\sqrt{2D\mu\gamma v'(y,t)}R(y,t). \tag{199}$$

It can be viewed as a stochastic (noisy) viscous Burgers equation for a pseudo "velocity" field $v(y,t)$ with a viscosity D. If we neglect the noise, it reduces to the viscous Burgers equation

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}=D\frac{\partial^2 v}{\partial y^2}. \tag{200}$$

This analogy allows us to solve the 1D Smoluchowski-Poisson equation analytically by using the Cole-Hopf transformation [73]. We note that the stationary solution of Eq. (200) is [see Eqs. (G7)]

$$v=\frac{1}{2}\mu\Gamma\tanh\left(\frac{|\beta|\gamma\Gamma y}{4}\right) \tag{201}$$

Remark: Equations (193) and (194) or Eqs. (196) and (197) take into account finite N effects ($\gamma\sim 1/N$) which are responsible for the noise term (fluctuations) but they ignore the collisions between the point vortices that would lead to the Lenard-Balescu collision term, as well as the correlations arising from the noise term. The correlations arising from the noise term induce an additional nonlinear diffusion of the mean vorticity (on a timescale Nt_D) which is discussed in Sec. XVI.

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[28] The evolution of the discrete (exact) vorticity field $\omega_d(\mathbf{r},t)=\sum_{i=1}^N\gamma_i\delta(\mathbf{r}-\mathbf{r}_i(t))$ is also determined by stochastic 2D Euler-Smoluchowski-Poisson equations of the form of Eqs. (193) and (194) with ω and ψ replaced by ω_d and ψ_d [75, 123]. Taking the ensemble average of these equations and making a mean field approximation ($N\to+\infty$ with $\gamma\sim 1/N$), we obtain Eqs. (191) and (192). Keeping track of the fluctuations at a mesoscopic scale, we obtain Eqs. (193) and (194).

XIV. SECULAR DRESSED DIFFUSION EQUATION

In this section, we consider the case where a continuous incompressible 2D flow[29] is submitted to an external stochastic velocity field $\mathbf{u}_e$ [see Eqs. (2)-(4)]. We consider a rather general situation where the external forcing is not necessarily induced by point vortices. We show that, under the effect of the external forcing, the evolution of the mean flow satisfies a SDD equation. We derive the SDD equation from the Klimontovich equation and from the Fokker-Planck equation and analyze its main properties.

A. From the Klimontovich equation

Under the assumptions of Sec. II, the basic equations governing the evolution of the mean vorticity $\omega(y,t)$ of a unidirectional flow[30] forced by an external perturbation are given by Eqs. (19) and (20). Introducing the Fourier transforms of the fluctuations of vorticity and stream function, these equations can be rewritten as

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial y}\int dk\int\frac{d\sigma}{2\pi}\int dk'\int\frac{d\sigma'}{2\pi}(ik')e^{i(kx-\sigma t)}e^{i(k'x-\sigma' t)}\langle\delta\hat{\omega}(k,y,\sigma)\delta\hat{\psi}_{\rm tot}(k',y,\sigma')\rangle,\tag{202}$$

$$\delta\hat{\omega}(k,y,\sigma)=\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}\delta\hat{\psi}_{\rm tot}(k,y,\sigma),\tag{203}$$

and they can be combined into

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial y}\int dk\int\frac{d\sigma}{2\pi}\int dk'\int\frac{d\sigma'}{2\pi}(ik')e^{i(kx-\sigma t)}e^{i(k'x-\sigma' t)}\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}\langle\delta\hat{\psi}_{\rm tot}(k,y,\sigma)\delta\hat{\psi}_{\rm tot}(k',y,\sigma')\rangle.\tag{204}$$

Introducing the power spectrum of the fluctuations from Eq. (35) we obtain

$$\frac{\partial\omega}{\partial t}=-i\frac{\partial}{\partial y}\int dk\int\frac{d\sigma}{2\pi}k\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}P(k,y,\sigma).\tag{205}$$

Recalling the Landau prescription $\sigma\to\sigma+i0^+$ and using the Sokhotski-Plemelj formula (C4), we can replace $1/(kU(y)-\sigma-i0^+)$ by $+i\pi\delta(kU(y)-\sigma)$. Accordingly,

$$\frac{\partial\omega}{\partial t}=\pi\frac{\partial}{\partial y}\int dk\int\frac{d\sigma}{2\pi}k^2\delta(kU(y)-\sigma)P(k,y,\sigma)\frac{\partial\omega}{\partial y}.\tag{206}$$

Integrating over the δ-function (resonance), we get

$$\frac{\partial\omega}{\partial t}=\frac{1}{2}\frac{\partial}{\partial y}\int dk\,k^2P(k,y,kU(y))\frac{\partial\omega}{\partial y}.\tag{207}$$

Therefore, the secular evolution of the mean vorticity $\omega(y,t)$ sourced by the external stochastic perturbation is governed by a nonlinear diffusion equation of the form

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial\omega}{\partial y}\right)\tag{208}$$

with a diffusion coefficient

$$D[y,\omega]=\frac{1}{2}\int dk\,k^2P(k,y,kU(y)).\tag{209}$$

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[29] This could be an intrinsically continuous flow (see the Remark at the end of Sec. II) or a gas of point vortices in the mean field limit $N\to+\infty$ with $\gamma\sim 1/N$ where the collisions between the point vortices are negligible.

[30] This configuration, which is a steady state of the 2D Euler equation, may be imposed initially or result from a process of violent relaxation.

Using Eq. (37), we can express the diffusion coefficient in terms of the correlation function of the external vorticity field as

$$D[y,\omega]=\frac{1}{2}\int dy'\int dk\,k^2|G(k,y,y',kU(y))|^2\hat{C}(k,y',kU(y)). \tag{210}$$

The diffusion coefficient depends on the correlation function of the external perturbation $\hat{C}(k,y',\sigma)$ and on the response function $G(k,y,y',\sigma)$ of the flow evaluated at the resonance frequencies $\sigma=kU(y)$. As a result, the diffusion coefficient $D[y,\omega]$ depends on the position y and on the vorticity $\omega(y,t)$ itself through the response function $G(k,y,y',kU(y))$ defined by Eq. (28). It also depends implicitly on $\omega(y,t)$ through $U(y,t)$. Equation (208) with the diffusion coefficient from Eq. (210) is therefore a complicated integrodifferential equation called the SDD equation. This is the counterpart of the SDD equation derived in Ref. [53] for homogeneous systems of material particles with long-range interactions forced by an external stochastic perturbation.[31] When collective effects are neglected, i.e., when we replace $G(k,y,y',kU(y))$ by $G_{\rm bare}(k,y,y')$ in Eqs. (210), we obtain

$$D_{\rm bare}[y,\omega]=\frac{1}{2}\int dy'\int dk\,k^2G_{\rm bare}(k,y,y')^2\hat{C}(k,y',kU(y)). \tag{211}$$

In that case, Eq. (208) is called the SBD equation.

Remark: If our system consists in a gas of point vortices with individual circulation $\gamma\sim 1/N$ and $N\gg 1$ (large but finite), the SDD equation describes the diffusive evolution of this near-equilibrium system caused by the fluctuations of the stream function induced by the external perturbation. The evolution timescale is intermediate between the violent collisionless relaxation time $\sim t_D$ and the collisional relaxation time $\sim Nt_D$ or $\sim N^2t_D$.

B. From the Fokker-Planck equation

The SDD equation (208) can also be derived from the Fokker-Planck equation (71). If our system is a continuous vorticity field or an ensemble of point vortices with circulation $\gamma\sim 1/N$ in the collisionless limit $N\rightarrow+\infty$, the drift by polarization, which is proportional to γ (see Sec. IX), vanishes

$$V_{\rm pol}=0. \tag{212}$$

Indeed, the perturbation on the system caused by the test particle is negligible. In that case, the Fokker-Planck equation (71) reduces to

$$\frac{\partial\omega}{\partial t}=\frac{\partial}{\partial y}\left(D\frac{\partial\omega}{\partial y}\right). \tag{213}$$

The diffusion coefficient can be calculated as in Sec. VIII A returning the expression from Eqs. (209)-(211). Therefore, the Klimontovich approach and the Fokker-Planck approach coincide.

Remark: We note that, in Eq. (213), the diffusion coefficient is "sandwiched" between the two spatial derivatives $\partial/\partial y$ in agreement with Eq. (208). As explained in Sec. VII, this is not the usual form of the Fokker-Planck equation which is given by Eq. (68). Therefore, the test vortex experiences a drift [see Eq. (72)]

$$V_{\rm tot}=\frac{\partial D}{\partial y}, \tag{214}$$

arising from the spatial inhomogeneity of the diffusion coefficient.[32] Using Eqs. (69) and (70), the relation (214) can be written as

$$\frac{\langle\Delta y\rangle}{\Delta t}=\frac{1}{2}\frac{\partial}{\partial y}\frac{\langle(\Delta y)^2\rangle}{\Delta t}. \tag{215}$$

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[31] A similar, but different, equation is derived in Refs. [126, 127]. See Ref. [128] for a comparison between these two approaches.
[32] This formula is established by a direct calculation in Sec. 4.4 of [68].

C. Properties of the SDD equation

Some general properties of the SDD equation (208) can be given. First of all, the circulation of the system $\Gamma = \int \omega\, dy$ is conserved since the right hand side of Eq. (208) is the divergence of a current. By contrast, the energy and the impulse of the system are not conserved, contrary to the case of the Lenard-Balescu equation [62], since the system is forced by an external medium. Taking the time derivative of the energy

$$E = \frac{1}{2}\int \omega\psi\, dy, \tag{216}$$

using Eq. (208), and integrating by parts, we get

$$\dot{E} = -\int D[y,\omega]\frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial y}\, dy. \tag{217}$$

In general, $\dot{E}$ has not a definite sign. However, for $\omega = \omega(\psi)$ we find that $\dot{E} = -\int \omega'(\psi)D(\partial\psi/\partial y)^2\, dy \geq 0$. When $\omega'(\psi) \leq 0$, the energy is injected in the system and when $\omega'(\psi) \geq 0$ the energy is absorbed from the system. For the impulse $P = \int \omega y\, dy$, we find that $\dot{P} = -\int D\frac{\partial\omega}{\partial y}\, dy$. Finally, introducing the H-functions

$$S = -\int C(\omega)\, dy, \tag{218}$$

where $C(\omega)$ is any convex function, we get

$$\dot{S} = \int C''(f)D[y,\omega]\left(\frac{\partial\omega}{\partial y}\right)^2\, dy. \tag{219}$$

Because of the convexity condition $C'' \geq 0$ and the fact that D is positive (see Sec. XIV A), we find that $\dot{S} \geq 0$. Therefore, all the H-functions increase monotonically with time. This is different from the case of the Lenard-Balescu equation where only the Boltzmann entropy increases monotonically [62].

D. Connection between the SDD equation and the multi-species Lenard-Balescu equation

Let us discuss the connection between the SDD equation (208) with Eq. (210) and the multi-species Lenard-Balescu equation (139). The Lenard-Balescu equation governs the evolution of the vorticity $\omega_a(y,t)$ of point vortices of species a under the effects of "collisions" with point vortices of all species "b" (including the point vortices of species a) with vorticity $\omega_b(y',t)$. The dressed Green function is given by Eq. (28) where $\omega(y,t)$ denotes the total vorticity $\sum_b \omega_b(y,t)$ and $U(y,t)$ is the corresponding velocity field. The set of equations (139), in which all the vorticities $\omega_a(y,t)$ evolve in a self-consistent manner, is closed.

We now make the following approximations to simplify these equations. The point vortices of circulation γ_a with vorticity $\omega_a(y,t)$ form our system. They will be called the test vortices. The vortices of circulation $\{\gamma_b\}_{b\neq a}$ with vorticities $\{\omega_b(y,t)\}_{b\neq a}$ form the external – background – medium. They will be called the field vortices. We take into account the collisions induced by the field vortices on the test vortices but we neglect the collisions induced by the test vortices on the field vortices and on themselves. This approximation is valid for very light test vortices $\gamma_a \ll \gamma_b$ or, more precisely, in the limit $N_a \to +\infty$ with $\gamma_a \sim 1/N_a$ (see Sec. XI D). Finally, the vorticities $\{\omega_b(y,t)\}_{b\neq a}$ of the field vortices are either assumed to be fixed (bath) or evolve according to their own dynamics (i.e. following equations that we do not write explicitly). Under these conditions, the kinetic equation (139) reduces to

$$\frac{\partial\omega_a}{\partial t} = \frac{1}{2}\frac{\partial}{\partial y}\sum_{b\neq a}\int dy'\int dk\, |k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\gamma_b\omega_b'\frac{\partial\omega_a}{\partial y}. \tag{220}$$

This equation can be interpreted as a nonlinear diffusion equation. The diffusion arises from the discrete distribution of the field vortices which creates a fluctuating velocity field (Poisson shot noise) acting on the test vortices. For that reason, the diffusion coefficient of the test vortices is proportional to the circulations $\{\gamma_b\}$ of the field vortices. The diffusion coefficient has no contribution from vortices of species a. The condition $\gamma_a \ll \gamma_b$ justifies neglecting the fluctuations induced by the test vortices on themselves. On the other hand, the drift by polarization vanishes ($V_{\rm pol} = 0$). Indeed, since the circulation γ_a of the test vortices is small, the test vortices do not significantly perturb the vorticity of the medium, so there is no drift by polarization (no retroaction). As a result, the circulation $\gamma_a \to 0$ of the vortices of species a does not appear in the kinetic equation (220).

Equation (220) can be written as a SDD equation

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial \omega}{\partial y}\right) \tag{221}$$

with a diffusion tensor

$$D[y,\omega] = \frac{1}{2}\sum_b \int dy' \int dk\, |k||G(k,y,y',kU(y))|^2 \delta(U(y') - U(y))\gamma_b \omega_b', \tag{222}$$

where we have droped the subscript a for clarity. The expression (222) of the diffusion coefficient is consistent with Eq. (210) where $\hat{C}(k,y,\sigma)$ is the bare correlation function of the vorticity field created by a discret collection of field vortices of circulations $\{\gamma_b\}$ given by Eq. (47). To exactly recover the SDD equation (208) with Eq. (210), we have to replace $\omega(y,t) + \sum_b \omega_b(y)$ by $\omega(y,t)$ in the dressed Green function. This assumes that the external medium – field vortices – is non polarizable (i.e. collective effects can be neglected) while our system – test vortices – is polarizable (i.e. collective effects must be taken into account).[33] As we have already mentioned, the SDD equation (221) is a nonlinear diffusion equation involving a diffusion coefficient which depends on the distribution function of the system $\omega(y,t)$ itself. It is therefore a complicated integrodifferential equation.

E. SDD equation with damping

Let us add a small linear damping term $-\alpha\omega$ on the right hand side of the SDD equation (213) in order to account for a possible dissipation. This yields

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial \omega}{\partial y}\right) - \alpha\omega. \tag{223}$$

The general behavior of this nonlinear equation is difficult to predict because it depends on the correlation function of the external potential. Furthermore, since the diffusion tensor is a functional of ω, the SDD equation presents a rich and complex behavior. It may relax towards a non-Boltzmannian steady state determined by

$$\frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial \omega}{\partial y}\right) - \alpha\omega = 0, \tag{224}$$

or exhibit a complicated (e.g. periodic) dynamics. Using Eq. (17), we can rewrite Eq. (223) in terms of the velocity field $U(y,t)$ as

$$\frac{\partial U}{\partial t} = D[y,U]\frac{\partial^2 U}{\partial y^2} - \alpha U. \tag{225}$$

The stationary solution of this equation is a jet profile $U(y)$ determined by

$$\frac{d^2 U}{dy^2} = \alpha \frac{U}{D[y,U]}. \tag{226}$$

Since D generically depends on U (the diffusion coefficient is a functional of U), this equation is a very nonlinear equation that has nontrivial solutions. If we assume that the external noise is due to N point vortices and if the velocity field is monotonic we have (see Sec. VIII B)

$$D = \sum_b \gamma_b \frac{\chi(y,y,U(y))}{|U'(y)|}\omega_b(y), \tag{227}$$

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[33] In the present case, Eq. (221) with Eq. (222) is more accurate than the SDD equation (208) with Eq. (210) because it takes into account the polarizability of the external medium.

where we have assumed that $U(y,t)$ is produced by $\omega(y,t)$ only. If we neglect collective effects and assume that $\omega_b(y)$ is uniform, we find that $D \sim 1/|U'(y)|$. On the other hand, when D is constant,[34] Eqs. (225) and (226) reduce to

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial y^2} - \alpha U \tag{228}$$

and

$$D\frac{\partial^2 U}{\partial y^2} - \alpha U = 0. \tag{229}$$

In that case, Eq. (229) leads to an exponential jet: $U(y) = e^{-|y|}$.

Remark: These results are similar to the results obtained for zonal jets in the context of forced 2D turbulence [129–140]. This connection is not unexpected since these approaches are based on the quasilinear theory of the 2D Euler equation (see an early work in Ref. [141] in the context of the theory of violent relaxation). The fact that the diffusion coefficient is generically inversely proportional to the local shear, $D \sim 1/|\Sigma|$, with $\Sigma = -U'(y)$ for unidirectional flows and $\Sigma = r(d/dr)(U_\theta/r)$ for axisymmetric flows, was pointed out in [57, 58]. Therefore, the diffusion is generically reduced as the shear increases. However, the forcing that we consider in this section is different from the forcing considered in Refs. [129–140]. Therefore, the approaches and the results are substantially distinct and complementary to each other.

F. SDD equation with drift

By analogy with the Smoluchowski equation (156) or (191), we can heuristically add a drift term on the right hand side of the SDD equation (213). This leads to an equation of the form

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial \omega}{\partial y} + \mu\omega\frac{\partial \psi}{\partial y}\right), \tag{230}$$

where ψ is either a given external stream function or the self-consistent stream function produced by the vorticity field ω. In this latter case, using Eq. (17), we can rewrite Eq. (230) in terms of the velocity field $U(y,t)$ alone as

$$\frac{\partial U}{\partial t} = D[y,U]\frac{\partial^2 U}{\partial y^2} + \mu U\frac{\partial U}{\partial y}. \tag{231}$$

If we set $v = -\mu U$, we get

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = D[y,v]\frac{\partial^2 v}{\partial y^2}. \tag{232}$$

This equation is similar to a viscous Burgers equation with a viscosity $D[y,v]$ which is a functional of the pseudo "velocity" $v(y,t)$.[35] The stationary solutions of Eqs. (230) and (231) satisfy

$$\frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial \omega}{\partial y} + \mu\omega\frac{\partial \psi}{\partial y}\right) = 0, \qquad D[y,U]\frac{\partial^2 U}{\partial y^2} + \mu U\frac{\partial U}{\partial y} = 0. \tag{233}$$

G. Stochastic SDD equation

The SDD equation (230), which is a deterministic partial differential equation, describes the evolution of the mean vorticity $\omega(y,t)$. If we take fluctuations into account, by analogy with the results presented in [75], we expect that the mesoscopic vorticity $\overline{\omega}(y,t)$ will satisfy a stochastic partial differential equation of the form

$$\frac{\partial \overline{\omega}}{\partial t} = \frac{\partial}{\partial y}\left(D[y,\overline{\omega}]\frac{\partial \overline{\omega}}{\partial y} + \mu\overline{\omega}\frac{\partial \overline{\psi}}{\partial y}\right) + \frac{\partial \zeta}{\partial y}(y,t), \tag{234}$$

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[34] This is the case, for example, when the external perturbation is created by a random distribution of point vortices with equal circulation γ_b, when collective effects are neglected, and when $U(y,t)$ in the diffusion coefficient is produced by $\omega_b(y,t)$ only (see Sec. VIII C).

[35] When $\psi(y)$ is a given external stream function, $v\partial_y v$ is replaced by $v_g\partial_y v$ with $v_g = -\mu d\psi/dy$.

where $\zeta(y,t)$ is a noise term with zero mean that generally depends on $\overline{\omega}(y,t)$. When D is constant and when the fluctuation-dissipation theorem is fulfilled so that $\mu = D\beta\gamma$, as in the case of 2D Brownian vortices (see Sec. XIII), the noise term is given by [75]

$$\zeta(y,t) = \sqrt{2D\gamma\overline{\omega}}\, Q(y,t), \tag{235}$$

where $Q(y,t)$ is a Gaussian white noise satisfying $\langle Q(y,t)\rangle = 0$ and $\langle Q(y,t)Q(y',t')\rangle = \delta(y-y')\delta(t-t')$. This expression can be obtained from an adaptation of the theory of fluctuating hydrodynamics [123]. When $D[\overline{\omega}]$ is a functional of $\overline{\omega}$, the noise term may be more complicated.[36] When the deterministic equation (230) admits several equilibrium states, the noise term in Eq. (234) can trigger random transitions from one state to the other (see, e.g., [124, 125, 127, 143–145] in different contexts).

Remark: Similarly, we can introduce a stochastic term (noise) in Eq. (223) to describe the evolution of the system on a mesoscopic scale. This leads to the stochastic SDD equation

$$\frac{\partial\overline{\omega}}{\partial t} = \frac{\partial}{\partial y}\left(D[y,\overline{\omega}]\frac{\partial\overline{\omega}}{\partial y}\right) - \alpha\overline{\omega} + \frac{\partial\zeta}{\partial y}(y,t). \tag{236}$$

Using Eq. (17), it can be written in terms of the velocity field as

$$\frac{\partial\overline{U}}{\partial t} = D[y,\overline{U}]\frac{\partial^2\overline{U}}{\partial y^2} - \alpha\overline{U} - \zeta(y,t). \tag{237}$$

The comments made previously also apply to these stochastic partial differential equations.

XV. STOCHASTIC DAMPED 2D EULER EQUATION

In this section, we consider the stochastic damped 2D Euler equation

$$\frac{\partial\omega_c}{\partial t} + \mathbf{u}_c\cdot\nabla\omega_c = -\alpha\omega_c + \sqrt{2\gamma\alpha\omega_c}\, Q(\mathbf{r},t), \tag{238}$$

where $\omega_c(\mathbf{r},t)$ is a continuous vorticity field, $Q(\mathbf{r},t)$ is a Gaussian white noise satisfying $\langle Q(\mathbf{r},t)\rangle = 0$ and $\langle Q(\mathbf{r},t)Q(\mathbf{r}',t')\rangle = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, α is a small damping coefficient, and γ has the dimension of a circulation (we will ultimately take the limit $\alpha\to 0$ and $\gamma\to 0$). We introduce this equation in an *ad hoc* manner but we will show below that the stochastic term generates a power spectrum that coincides with the power spectrum produced by an isolated distribution of point vortices of circulation $\gamma\sim 1/N \ll 1$. This approach therefore provides another manner to determine the power spectrum of a gas of point vortices. This is an additional motivation to study Eq. (238) in detail. The calculations of this section are inspired by similar calculations on fluctuating hydrodynamics performed in [123].

When $\alpha\to 0$, the mean vorticity $\omega(\mathbf{r},t) = \langle\omega_c(\mathbf{r},t)\rangle$ rapidly reaches a QSS which is a steady state of the 2D Euler-Poisson equations. This process of violent relaxation takes place on a few dynamical times. If the evolution is ergodic, the QSS can be determined by the MRS statistical theory [11, 12]. The kinetic theory of violent relaxation is discussed in [141, 146]. On a longer timescale, the mean vorticity evolves through a sequence of QSSs sourced by the noise. Adapting the procedure of Sec. II to the present context, and assuming that the mean flow (QSS) is unidirectional, we obtain the quasilinear equations

$$\frac{\partial\omega}{\partial t} = -\alpha\omega + \frac{\partial}{\partial y}\left\langle\delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle, \tag{239}$$

$$\frac{\partial\delta\omega}{\partial t} + U\frac{\partial\delta\omega}{\partial x} - \frac{\partial\delta\psi}{\partial x}\frac{\partial\omega}{\partial y} = -\alpha\delta\omega + \sqrt{2\gamma\alpha\omega}\, Q(x,y,t), \tag{240}$$

where $Q(x,y,t)$ is a Gaussian white noise satisfying $\langle Q(x,y,t)\rangle = 0$ and $\langle Q(x,y,t)Q(x',y',t')\rangle = \delta(x-x')\delta(y-y')\delta(t-t')$. Introducing the Fourier transforms of the fluctuations of vorticity and stream function, we can rewrite these equations as

$$\frac{\partial\omega}{\partial t} = -\alpha\omega + \frac{\partial}{\partial y}\int dk\int\frac{d\sigma}{2\pi}\int dk'\int\frac{d\sigma'}{2\pi}(ik')e^{i(kx-\sigma t)}e^{i(k'x-\sigma' t)}\langle\delta\hat{\omega}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle, \tag{241}$$

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[36] A general approach to obtain the noise term and the corresponding action is to use the theory of large deviations [142].

$$\delta\hat{\omega}(k,y,\sigma)=\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma-i\alpha}\delta\hat{\psi}(k,y,\sigma)-\frac{i\sqrt{2\gamma\alpha\omega(y)}}{kU(y)-\sigma-i\alpha}\hat{Q}(k,y,\sigma), \tag{242}$$

where $\hat{Q}(k,y,\sigma)$ is a Gaussian white noise satisfying $\langle\hat{Q}(k,y,\sigma)\rangle=0$ and $\langle\hat{Q}(k,y,\sigma)\hat{Q}(k',y',\sigma')\rangle=\delta(k+k')\delta(y-y')\delta(\sigma+\sigma')$. Substituting Eq. (242) into Eq. (241), we get

$$\frac{\partial\omega}{\partial t}=-\alpha\omega+\frac{\partial}{\partial y}\int dk\int\frac{d\sigma}{2\pi}\int dk'\int\frac{d\sigma'}{2\pi}(ik')e^{i(kx-\sigma t)}e^{i(k'x-\sigma' t)}\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma-i\alpha}\langle\delta\hat{\psi}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle. \tag{243}$$

In writing Eq. (243) we have only considered the contribution of the term $\langle\delta\hat{\psi}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle$ which leads to a diffusive evolution. The other terms will be investigated elsewhere [73].

Let us study Eq. (242) for the fluctuations and determine the power spectrum $P(k,y,\sigma)$ of the fluctuating stream function defined by

$$\langle\delta\hat{\psi}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle=2\pi\delta(k+k')\delta(\sigma+\sigma')P(k,y,\sigma). \tag{244}$$

The fluctuations of vorticity and stream function are related to each other by the Poisson equation

$$\Delta\delta\psi=-\delta\omega. \tag{245}$$

Writing this equation in Fourier space, we get

$$\left(\frac{d^2}{dy^2}-k^2\right)\delta\hat{\psi}=-\delta\hat{\omega}. \tag{246}$$

Substituting Eq. (242) into Eq. (246), we obtain

$$\left[\frac{d^2}{dy^2}-k^2+\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma-i\alpha}\right]\delta\hat{\psi}(k,y,\sigma)=\frac{i\sqrt{2\gamma\alpha\omega(y)}}{kU(y)-\sigma-i\alpha}\hat{Q}(k,y,\sigma). \tag{247}$$

The formal solution of this differential equation is

$$\delta\hat{\psi}(k,y,\sigma)=-\int dy'\,G(k,y,y',\sigma)\frac{i\sqrt{2\gamma\alpha\omega(y')}}{kU(y')-\sigma-i\alpha}\hat{Q}(k,y',\sigma), \tag{248}$$

where the Green function $G(k,y,y',\sigma)$ is defined in Eq. (28). The correlation function of the fluctuations of the stream function is therefore

$$\langle\delta\hat{\psi}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle=-\int dy'dy''\,G(k,y,y',\sigma)G(k',y,y'',\sigma')$$
$$\times\frac{2\gamma\alpha\sqrt{\omega(y')\omega(y'')}}{(kU(y')-\sigma-i\alpha)(k'U(y'')-\sigma'-i\alpha)}\langle\hat{Q}(k,y',\sigma)\hat{Q}(k',y'',\sigma')\rangle. \tag{249}$$

For a Gaussian white noise, we get

$$\langle\delta\hat{\psi}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle=\int dy'\,|G(k,y,y',\sigma)|^2\frac{2\gamma\alpha\omega(y')}{(kU(y')-\sigma)^2+\alpha^2}\delta(k+k')\delta(\sigma+\sigma'), \tag{250}$$

where we have used Eq. (C6) to simplify the expression. Taking the limit $\alpha\to 0$ and using the identity

$$\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}=\pi\,\delta(x), \tag{251}$$

we finally obtain

$$\langle\delta\hat{\psi}(k,y,\sigma)\delta\hat{\psi}(k',y,\sigma')\rangle=\pi\int dy'\,|G(k,y,y',\sigma)|^2 2\gamma\omega(y')\delta(kU(y')-\sigma)\delta(k+k')\delta(\sigma+\sigma'), \tag{252}$$

leading to

$$P(k,y,\sigma)=\gamma\int dy'\,|G(k,y,y',\sigma)|^2\delta(kU(y')-\sigma)\omega(y'). \tag{253}$$

This returns the power spectrum (48) produced by a random distribution of field vortices.

Substituting Eqs. (244) and (253) into Eq. (243) and repeating the calculations of Sec. XIV, we obtain the nonlinear diffusion equation

$$\frac{\partial \omega}{\partial t} = -\alpha\omega + \frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial \omega}{\partial y}\right) \tag{254}$$

with a diffusion coefficient

$$D[y,\omega] = \frac{1}{2}\int dk\, k^2 P(k,y,kU(y)) = \frac{1}{2}\gamma\int dy'\int dk\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\omega(y'), \tag{255}$$

which coincides with the diffusion coefficient of a gas of point vortices (see Sec. VIII). Using Eq. (17), we can write Eq. (254) in terms of $U(y,t)$ as in Eq. (225). If $\omega(y)$ is of constant sign, $U(y)$ is monotonic (see footnote 15) and we get

$$D[y,\omega] = \frac{1}{2}\gamma\int dk\,|k||G(k,y,y,kU(y))|^2\frac{\omega(y)}{|U'(y)|} = \frac{1}{2}|\gamma|\int dk\,|k||G(k,y,y,kU(y))|^2. \tag{256}$$

If we neglect collective effects, we find that the diffusion coefficient is constant (see Appendix B):

$$D = \frac{1}{2}|\gamma|\int dk\,|k|G_{\rm bare}(k,y,y)^2 = \frac{1}{4}|\gamma|\ln\Lambda. \tag{257}$$

In that case, the equation for $U(y,t)$ coincides with Eqs. (228) and (229). The term $-\alpha\omega$ describes the damping of the system on a timescale $1/\alpha$ and the diffusion coefficient $D[y,\omega]\sim 1/N$ describes its evolution on a timescale Nt_D.

On a mesoscopic scale, we can keep track of the fluctuations in the evolution of the vorticity and write

$$\frac{\partial \overline{\omega}}{\partial t} = -\alpha\overline{\omega} + \frac{\partial}{\partial y}\left(D[y,\overline{\omega}]\frac{\partial \overline{\omega}}{\partial y}\right) + \sqrt{2\alpha\gamma\overline{\omega}}\,Q(y,t), \tag{258}$$

where $Q(y,t)$ is a Gaussian white noise satisfying $\langle Q(y,t)\rangle = 0$ and $\langle Q(y,t)Q(y',t')\rangle = \delta(y-y')\delta(t-t')$. Equation (258) without the noise term may have several equilibrium states. The noise term allows the system to switch from one equilibrium state to another one through random transitions (see, e.g., [124, 125, 127, 143–145] in different contexts).

Remark: The power spectrum of a gas of point vortices can be obtained in different manners. It can be obtained from the linearized Klimontovich equation by solving an initial value problem (see Eq. (34) of Ref. [68]). It can also be obtained by considering the dressing of the bare correlation function of a random distribution of point vortices viewed as an external perturbation (see Eq. (48) of Sec. V B). Finally, in this section, we have determined the power spectrum [see Eq. (253)] directly from the stochastic damped 2D Euler equation (238) in the spirit of fluctuating hydrodynamics [123].

XVI. STOCHASTICALLY FORCED 2D POINT VORTICES

In this section, we consider a stochastic model of 2D point vortices described by the N coupled Langevin equations

$$\frac{d\mathbf{r}_i}{dt} = \frac{1}{2\pi}\mathbf{z}\times\sum_j \gamma_j\frac{\mathbf{r}_i-\mathbf{r}_j}{|\mathbf{r}_i-\mathbf{r}_j|^2} + \sqrt{2\nu}\mathbf{R}_i(t), \tag{259}$$

where $i=1,...,N$ label the point vortices and $\mathbf{R}_i(t)$ is a Gaussian white noise satisfying $\langle\mathbf{R}_i(t)\rangle = \mathbf{0}$ and $\langle R_i^\alpha(t)R_j^\beta(t')\rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$. The variable ν can be interpreted as a diffusion coefficient or as a small viscosity (we will ultimately take the limit $\nu\to 0$). This stochastic model of 2D point vortices was introduced by Marchioro and Pulvirenti [74]. It also corresponds to the model of 2D Brownian point vortices introduced in [75] with $\beta = 0$ (see Sec. XIII).

The exact equation satisfied by the discrete vorticity field $\omega_d(\mathbf{r},t) = \sum_{i=1}^N \gamma_i\delta(\mathbf{r}-\mathbf{r}_i(t))$ is [75]

$$\frac{\partial \omega_d}{\partial t} + \mathbf{u}_d\cdot\nabla\omega_d = \nu\Delta\omega_d + \nabla\cdot\left[\sqrt{2\nu\gamma\omega}\,\mathbf{R}(\mathbf{r},t)\right], \tag{260}$$

where $\mathbf{R}(\mathbf{r},t)$ is a Gaussian white noise such that $\langle\mathbf{R}(\mathbf{r},t)\rangle = \mathbf{0}$ and $\langle R_\alpha(\mathbf{r},t)R_\beta(\mathbf{r}',t)\rangle = \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$. For simplicity, we consider a single species gas of point vortices but the generalization to multiple species of point vortices is straightforward.

In this section, we neglect the collisions (finite N effects) between the point vortices that would lead to a Lenard-Balescu collision term (see Secs. XI and XII) and focus on the effect of the noise. In that case, the mean vorticity $\omega(\mathbf{r},t)=\langle\omega_d(\mathbf{r},t)\rangle$ satisfies the equation

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=\nu\Delta\omega \tag{261}$$

and the mesoscopic vorticity satisfies the equation

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=\nu\Delta\omega+\nabla\cdot\left[\sqrt{2\nu\gamma\omega}\,\mathbf{R}(\mathbf{r},t)\right]. \tag{262}$$

When $\nu\to 0$ the mean vorticity $\omega(\mathbf{r},t)$ rapidly reaches a QSS which is a steady state of the 2D Euler-Poisson equations. On a longer timescale, the mean vorticity evolves through a sequence of QSSs sourced by the noise. This is similar to the problem discussed in Sec. XV. Adapting the procedure of Sec. II to the present context, and assuming that the mean flow (QSS) is unidirectional, we obtain the quasilinear equations

$$\frac{\partial\omega}{\partial t}=\nu\frac{\partial^2\omega}{\partial y^2}+\frac{\partial}{\partial y}\left\langle\delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle, \tag{263}$$

$$\frac{\partial\delta\omega}{\partial t}+U\frac{\partial\delta\omega}{\partial x}-\frac{\partial\delta\psi}{\partial x}\frac{\partial\omega}{\partial y}=\nu\frac{\partial^2\delta\omega}{\partial x^2}+\frac{\partial}{\partial x}\left[\sqrt{2\nu\gamma\omega}\,Q(x,y,t)\right], \tag{264}$$

where $Q(x,y,t)$ is a Gaussian white noise satisfying $\langle Q(x,y,t)\rangle=0$ and $\langle Q(x,y,t)Q(x',y',t')\rangle=\delta(x-x')\delta(y-y')\delta(t-t')$. Repeating the calculations of Sec. XIV with only minor modifications, we obtain the nonlinear diffusion equation

$$\frac{\partial\omega}{\partial t}=\nu\frac{\partial^2\omega}{\partial y^2}+\frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial\omega}{\partial y}\right) \tag{265}$$

with the diffusion coefficient $D[y,\omega]$ from Eq. (255) which coincides with the diffusion coefficient of a gas of point vortices (see Sec. VIII). The diffusion coefficient ν describes the evolution of the system on a diffusive timescale L^2/ν (where L is the size of the system) and the diffusion coefficient $D[y,\omega]\sim 1/N$ describes its evolution on a timescale Nt_D. We should also take into account the collisions between the point vortices, leading to the Lenard-Balescu current from Eq. (139), which develop on the same timescale.

At a mesoscopic level,[37] we can keep track of the forcing in the evolution of the vorticity and write

$$\frac{\partial\omega}{\partial t}=\nu\frac{\partial^2\omega}{\partial y^2}+\frac{\partial}{\partial y}\left(D[y,\omega]\frac{\partial\omega}{\partial y}\right)+\frac{\partial}{\partial y}\left[\sqrt{2\nu\gamma\omega}\,Q(y,t)\right], \tag{266}$$

where $Q(y,t)$ is a Gaussian white noise satisfying $\langle Q(y,t)\rangle=0$ and $\langle Q(y,t)Q(y',t')\rangle=\delta(y-y')\delta(t-t')$. Using Eq. (17), we can rewrite Eq. (266) in terms of the velocity field $U(y,t)$ as

$$\frac{\partial U}{\partial t}=(\nu+D[y,U])\frac{\partial^2 U}{\partial y^2}-\sqrt{-2D\gamma U'(y)}R(y,t). \tag{267}$$

XVII. SUMMARY OF THE DIFFERENT KINETIC EQUATIONS

In this section, we recapitulate the different kinetic equations derived in our paper.

A. Lenard-Balescu equation

The Lenard-Balescu equation (139) governs the mean evolution of an isolated system of 2D point vortices due to discreteness effects ("collisions").

[37] We stress that there are several levels of description. The mesoscopic description leading to Eq. (266) takes place at a higher scale than the mesoscopic description leading to Eq. (262).

It can be derived from the Klimontovich formalism by taking $\mathbf{u}_e = 0$ in Eq. (2) and considering an initial value problem as explained in Sec. 3 of [68]. In that case, Eq. (20) with $\psi_e = 0$ has to be solved by using a Fourier transform in space and a Laplace transform in time. This introduces in Eq. (24) a term related to the initial condition [see Eq. (23) in [68]] instead of the term related to ψ_e. We can then compute the collision term in Eq. (19) with $\psi_e = 0$ as in Sec. 3 of [68] and obtain the Lenard-Balescu equation (139).

Another approach is to start from the Fokker-Planck equation (68) or (71) and compute the diffusion and drift coefficients individually.

(i) To compute the diffusion coefficient from Eq. (69) leading to Eq. (87), we have to evaluate the power spectrum of the stream function fluctuations created by a random distribution of field vortices. This can be done in two manners.

(a) The first possibility is to take $\psi_e = 0$ in Eq. (2) and solve Eq. (20) with $\psi_e = 0$ by using a Fourier transform in space and a Laplace transform in time as mentioned above. This leads to the expression (23) of [68] for the fluctuations, which involves the initial condition. The power spectrum is then given by Eq. (34) of [68] and the diffusion coefficient by Eq. (68) of [68].

(b) Another possibility is to introduce a stochastic perturbation ψ_e in Eq. (2) and solve Eq. (20) by introducing Fourier transforms in space and time. This leads to Eq. (24) for the fluctuations, which involves the external perturbation. The power spectrum is then given by Eq. (37). If we assume that the external perturbation is due to a collection of field vortices, we can use Eq. (47) to obtain the expression (48) of the power spectrum. The diffusion coefficient is then given by Eq. (93).

(ii) We can compute the drift in two manners.

(a) The first possibility is to compute the total drift (70) arising in Eq. (68) by proceeding like in Secs. 4.3 and 4.4 of [68]. This leads to Eq. (102) of [68]. We then find that the total drift splits in two terms: a term interpreted as a "drift by polarization" (see Sec. 4.3 of [68]) and another term related to the gradient of the diffusion coefficient (see Sec. 4.4 of [68]). Substituting the diffusion coefficient (Eq. (23) of [68]) and the total drift (Eq. (102) of [68]) in the ordinary expression (68) of the Fokker-Planck equation, and using an integration by parts, we obtain the Lenard-Balescu equation (139). This is the approach followed in Sec. 4.5 of [68].

(b) Alternatively, we can compute the drift by polarization arising in Eq. (71) by considering the response of the mean flow to the perturbation created by the test vortex (see Sec. IX). This leads to Eq. (113). This calculation convincingly shows that $V_{\rm pol}$ can be interpreted as a drift by polarization. Substituting the diffusion coefficient (93) and the drift by polarization (113) in the expression (71) of the Fokker-Planck equation, we obtain the Lenard-Balescu equation (139). This derivation is simpler (less technical) and more physical that the one given in [68].

B. SDD equation

The SDD equation (208) with Eqs. (209) and (210) governs the mean evolution of a gas of point vortices submitted to an external stochastic perturbation in the limit where the collisions between the point vortices are negligible, i.e., in the limit $N \to +\infty$ with $\gamma \sim 1/N \to 0$. In that case, we can solve Eq. (20) by using Fourier transforms in space and time. This leads to the expression (37) of the power spectrum. The SDD equation can be derived from the Klimontovich formalism (see Sec. XIV A) or from the Fokker-Planck formalism (see Sec. XIV B). Since there is no drift by polarization ($\gamma \to 0$), the Fokker-Planck equation (71) reduces to Eq. (213). Using Eq. (37), the diffusion coefficient (87) can be written as in Eq. (89). Substituting Eqs. (87) and (89) into Eq. (213), we obtain the SDD equation (208) with Eqs. (209) and (210). The SDD equation also describes the mean evolution of a continuous vorticity field submitted to an external stochastic perturbation (see the Remark at the end of Sec. II).

C. General kinetic equation

We now present a kinetic equation that generalizes the Lenard-Balescu equation (139) and the SDD equation (208)-(210). We consider a collection of point vortices of circulation γ submitted to a stochastic perturbation that can be internal or external to the system (or both). The test vortices experience a diffusion due to the stochastic perturbation and a drift by polarization due to retroaction (response) of the mean flow to the deterministic perturbation that they induce. The evolution of their density (mean vorticity) is thus governed by a general Fokker-Planck equation of the form of Eq. (71) where D is given by Eq. (87) and $V_{\rm pol}$ is given by Eq. (111). Explicitly,

$$\frac{\partial\omega}{\partial t} = \frac{1}{2}\frac{\partial}{\partial y}\int dk\, k^2 \left[P(k,y,kU(y))\frac{\partial\omega}{\partial y} - \omega\frac{\gamma}{\pi k}{\rm Im}\, G(k,y,y,kU(y))\right]. \tag{268}$$

Alternative expressions of this kinetic equation can be obtained by using Eqs. (89) and (113) instead of Eqs. (87) and (111). This kinetic equation is more general than the Lenard-Balescu equation (139) because the noise is not necessarily due to a discrete collection of point vortices. It is also more general than the SDD equation (208)-(210) because it takes into account the drift by polarization of the test vortices. If we neglect the drift by polarization (i.e. $\gamma \to 0$) we recover the SDD equation. If we assume that the external perturbation is only due to field vortices and use Eqs. (93) and (117), we recover the Lenard-Balescu equation. If we assume that a part of the stochastic perturbation is due to point vortices and another part is due to an external noise, we get an hybrid (mixed) kinetic equation with a Lenard-Balescu term and a SDD term (see Sec. XVII D).

D. Hybrid kinetic equation and its subcases

In order to be as general as possible, we consider a system of test vortices of circulation γ in "collision" with field vortices of circulations $\{\gamma_b\}$ and submitted in addition to an external noise. The evolution of the mean vorticity of the test vortices is governed by a mixed kinetic equation of the form[38]

$$\frac{\partial \omega}{\partial t} = C_{\rm LB} + C_{\rm SDD}, \tag{269}$$

with a Lenard-Balescu collision term $C_{\rm LB}$ due to the collisions between the point vortices (finite N effects) and a collision term $C_{\rm SDD}$ due to the external noise. This corresponds to Eq. (268) with a power spectrum $P = P_{\rm LB} + P_{\rm SDD}$.

Let us consider particular cases of this equation:

(i) When $\gamma \to 0$, we can neglect the drift by polarization and we get a diffusion equation with two terms of diffusion $D_{\rm LB}$ and $D_{\rm SDD}$.

(i-a) In the absence of external noise, we recover the diffusion equation (160) with Eq. (153).

(i-b) When $\gamma_b \to 0$ we can neglect the diffusion induced by the field vortices and we recover the SDD equation (208) with Eqs. (209) and (210).

(ii) When $\gamma_b \to 0$, we can neglect the diffusion induced by the field vortices and we get a Fokker-Planck equation of the form of Eq. (268) with a diffusion term $D_{\rm SDD}$ due to the external noise and a drift by polarization.

(ii-a) In the absence of external noise, we recover the deterministic equation (162) with Eq. (154).

(ii-b) When $\gamma \to 0$, we can neglect the drift by polarization and we recover the SDD equation (208) with Eqs. (209) and (210).

(ii-c) In the case where the field vortices are at statistical equilibrium, we can simplify the drift by polarization and we get an equation of the form of Eq. (230).

(iii) In the absence of external noise, we recover the Lenard-Balescu equation (139).

(iii-a) When $\gamma \to 0$, we can neglect the drift by polarization and we recover the diffusion equation (160) with Eq. (153).

(iii-b) When $\gamma_b \to 0$, we can neglect the diffusion induced by the field vortices and we recover the deterministic equation (162) with Eq. (154).

(iii-c) In the case where the field vortices are at statistical equilibrium we can simplify the drift by polarization and we recover the Smoluchowski equation (156).

XVIII. CONCLUSION

In this paper, we have completed the kinetic theory of 2D point vortices initiated in previous works. We have proposed a new and more physical derivation of the kinetic equation for a multispecies system of point vortices.

We started from the Fokker-Planck equation written in the form of Eq. (71) and we computed the diffusion coefficient D and the drift by polarization $V_{\rm pol}$ of a test vortex.

In order to take collective effects into account, we considered the response of the system to a small perturbation of arbitrary origin. We showed that the response function of the flow is determined by the dressed Green function $G(k, y, y', \sigma)$ defined by Eq. (28).

To derive the diffusion coefficient D, we assumed that the perturbation is a stationary stochastic process characterized by a bare correlation function $\hat{C}(k, y, \sigma)$ and we determined the dressed power spectrum $P(k, y, \sigma)$ of the total fluctuating stream function experienced by the test vortex [see Eq. (37)]. The diffusion coefficient is then given by Eqs. (87) and (89). We considered the case of an arbitrary external perturbation and the case of a perturbation

[38] For simplicity, we assume that the test vortices of circulation γ evolve in a bath of field vortices of circulations $\{\gamma_b\}$. We can then treat the test vortices as representing a particular species "a" and write self-consistent kinetic equations for all the species.

produced by a collection of N point vortices. In that latter case, we explicitly determined the bare correlation function [see Eq. (47)], the dressed power spectrum [see Eq. (48)], and the diffusion coefficient [see Eq. (93)].

To derive the drift by polarization $V_{\rm pol}$, we used the fact that the test vortex induces a small perturbation on the flow and we determined the response of the flow to that perturbation. The drift velocity of the test vortex then corresponds to the velocity produced by the perturbation that it has caused [see Eqs. (111) and (113)].

It is interesting to contrast the origin of the diffusion and drift by polarization of the test vortex. The diffusion of the test vortex is due to the stochastic perturbation caused by the field vortices or, more generally, by an external random potential. The drift by polarization of the test vortex is due to the retroaction (response) of the perturbation that it causes on the mean flow. This is a purely deterministic process. This is why the drift by polarization is proportional to the circulation γ of the test vortex while the diffusion coefficient is proportional to the circulations $\{\gamma_b\}$ of the field vortices.

Substituting the expressions of the diffusion coefficient and drift by polarization into the Fokker-Planck equation (71) we obtained the general kinetic equation (268). When the collisions between the point vortices are neglected ($N\rightarrow +\infty$), and when the system is subjected to an external perturbation, it reduces to the SDD equation (208) with Eqs. (209) and (210). When the system is isolated and the noise is due to the system of point vortices itself (finite N effects), it reduces to the Lenard-Balescu equation (139). In that latter case, we discussed the phenomenon of kinetic blocking that occurs when the velocity profile is monotonic. In the present paper, we considered unidirectional flows but similar results can be obtained for axisymmetric flows [62, 68, 69]. On the other hand, when collective effects are neglected, a general kinetic equation can be derived for an arbitrary distribution of point vortices [see Eq. (128) or Eq. (137) of [58]].

The previous results are valid for an isolated Hamiltonian system of point vortices. We also considered the case of 2D Brownian (stochastically forced) point vortices in the canonical ensemble and we established the mean field Fokker-Planck equation (191) and the stochastic Fokker-Planck equation (193). This last equation can describe random transitions between different equilibrium states caused by finite-N fluctuations. We showed furthermore that the fluctuations induce a nonlinear diffusion of the point vortices [see Eqs. (265) and (266)].

Another goal of the paper was to emphasize the fluctuation-dissipation theorem for 2D point vortices. The velocity of a test vortex moving in a sea of field vortices can be decomposed in two components. There is a mean field velocity due to the average distribution of point vortices and a "microscopic" velocity due to the discrete interaction between vortices (collisions). In turn, this microscopic velocity can be decomposed in two parts. There is a random part giving rise to a diffusion and a deterministic part giving rise to a systematic drift. The drift velocity and the random velocity must be related at statistical equilibrium because they both come from the same origin (finite N effects). This internal relationship between the systematic drift and the random part of the microscopic velocity is of a very general nature which is manifested in the so-called fluctuation-dissipation theorem [84]. A similar relationship between the friction and the random part of the microscopic force arises in the theory of Brownian motion and in the kinetic theory of systems with long-range interactions (self-gravitating systems, plasmas, HMF model...). The fluctuation-dissipation theorem states a general relationship between the response of a given system to an external perturbation and the internal fluctuations of the system in the absence of the perturbation. Specifically, it provides a relation between the response function of the system, the correlation function of the fluctuations, and the temperature. In the case of 2D point vortices at statistical equilibrium it takes the form of Eq. (56) between the power spectrum and the imaginary part of the Green function. This implies a relation between the drift velocity and the diffusion coefficient given by the Einstein relation (135) or by the Kubo formula (137). These equations involve the temperature of the point vortex gas which may be positive or negative.

In a companion paper [73], we shall study in more detail the kinetic equations derived in the present contribution and complement further the kinetic theory of 2D point vortices. Our results can also be exported to other systems with long-range interactions such as self-gravitating systems.

Appendix A: The point vortex model

We consider an incompressible and inviscid flow described by the Euler equations

$$\nabla\cdot\mathbf{u}=0,\qquad \frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla P. \tag{A1}$$

For a 2D flow, the incompressibility condition becomes $\partial_x u_x+\partial_y u_y=0$. In that case, we can introduce a stream function $\psi(\mathbf{r},t)$ such that $u_x=\partial_y\psi$ and $u_y=-\partial_x\psi$. The velocity field can be written as

$$\mathbf{u}=-\mathbf{z}\times\nabla\psi, \tag{A2}$$

where $\mathbf{z}$ is a unit vector normal to the plane of the flow. The vorticity is defined by $\nabla\times\mathbf{u}$. For a 2D incompressible flow, the vorticity is parallel to $\mathbf{z}$ and related to the stream function through the Poisson equation

$$\omega=-\Delta\psi. \tag{A3}$$

Using the identity $(\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla(\mathbf{u}^2/2)-\mathbf{u}\times(\nabla\times\mathbf{u})$ and taking the curl of Eq. (A1), we obtain

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega=0. \tag{A4}$$

This equation expresses the advection of the vorticity by the flow. It can be written as $D\omega/Dt=0$, where $D=\partial/\partial t+\mathbf{u}\cdot\nabla$ is the material derivative (Stokes operator). Equations (A3) and (A4) define the 2D Euler-Poisson system. In an infinite domain, the Poisson equation (A3) can be integrated into

$$\psi(\mathbf{r},t)=-\frac{1}{2\pi}\int\ln|\mathbf{r}-\mathbf{r}'|\omega(\mathbf{r}',t)\,d\mathbf{r}', \tag{A5}$$

leading to the velocity field

$$\mathbf{u}(\mathbf{r},t)=-\frac{1}{2\pi}\mathbf{z}\times\int\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^2}\omega(\mathbf{r}',t)\,d\mathbf{r}'. \tag{A6}$$

For a system of N point vortices with circulation γ_i the vorticity field can be written as

$$\omega(\mathbf{r},t)=\sum_i\gamma_i\delta(\mathbf{r}-\mathbf{r}_i(t)). \tag{A7}$$

The discrete vorticity is a sum of Dirac δ-functions. Substituting this expression into Eq. (A4) we find after straightforward manipulations (see below) that the velocity of a test vortex is given by

$$\mathbf{V}_i(t)=\frac{d\mathbf{r}_i}{dt}=\mathbf{u}(\mathbf{r}_i(t),t). \tag{A8}$$

From Eqs. (A5)-(A7), we get

$$\psi(\mathbf{r},t)=-\frac{1}{2\pi}\sum_i\gamma_i\ln|\mathbf{r}-\mathbf{r}_i|,\qquad \mathbf{u}(\mathbf{r},t)=\frac{1}{2\pi}\mathbf{z}\times\sum_i\gamma_i\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^2}, \tag{A9}$$

and

$$\mathbf{V}_i=\frac{d\mathbf{r}_i}{dt}=-\mathbf{z}\times\nabla\psi(\mathbf{r}_i)=\frac{1}{2\pi}\mathbf{z}\times\sum_j\gamma_j\frac{\mathbf{r}_i-\mathbf{r}_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}. \tag{A10}$$

The velocity of a point vortex is induced by the other point vortices. This is different from the case of material particles where the interaction between the particles produces an acceleration (or a force), not a velocity. In a sense, a point vortex does not have inertia. The equations of motion of the point vortices can be written in Hamiltonian form as

$$\gamma_i\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i},\qquad \gamma_i\frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}, \tag{A11}$$

with the Hamiltonian

$$H=-\frac{1}{2\pi}\sum_{i<j}\gamma_i\gamma_j\ln|\mathbf{r}_i-\mathbf{r}_j|. \tag{A12}$$

These are the so-called Kirchhoff equations [116]. We note that the coordinates (x,y) of the point vortices are canonically conjugate. We can also write the equations of motion of the point vortices under the form

$$\gamma_i\frac{d\mathbf{r}_i}{dt}=-\mathbf{z}\times\nabla H. \tag{A13}$$

Proof of Eq. (A8): From Eq. (A7) we have

$$\frac{\partial \omega}{\partial t} = -\sum_i \gamma_i \nabla \delta(\mathbf{r} - \mathbf{r}_i(t)) \cdot \frac{d\mathbf{r}_i}{dt}, \tag{A14}$$

and

$$\begin{aligned}
\mathbf{u} \cdot \nabla \omega &= \mathbf{u}(\mathbf{r}, t) \cdot \sum_i \gamma_i \nabla \delta(\mathbf{r} - \mathbf{r}_i(t)) \\
&= \sum_i \gamma_i \nabla \left(\delta(\mathbf{r} - \mathbf{r}_i(t)) \mathbf{u}(\mathbf{r}, t) \right) \\
&= \sum_i \gamma_i \nabla \left(\delta(\mathbf{r} - \mathbf{r}_i(t)) \mathbf{u}(\mathbf{r}_i(t), t) \right) \\
&= \sum_i \gamma_i \mathbf{u}(\mathbf{r}_i(t), t) \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_i(t)),
\end{aligned} \tag{A15}$$

where we have used the incompressibility of the flow to get the second line of Eq. (A15). Substituting these expressions into Eq. (A4), we obtain Eq. (A8) and the Kirchhoff equations (A11) and (A12). Inversely, starting from Eq. (A8) or from the Kirchhoff equations (A11) and (A12), we find that the discrete vorticity field defined by Eq. (A7) satisfies Eq. (A4). The discrete 2D Euler equation (A4) expressed in terms of δ-functions is the counterpart of the Klimontovich equation in plasma physics.

Appendix B: Green function without collective effects

The stream function ψ produced by the vorticity field ω is determined by the Poisson equation (A3). Introducing a system of cartesian coordinates and taking its Fourier transform in the x-direction, we obtain

$$\frac{d^2\hat{\psi}}{dy^2} - k^2\hat{\psi} = -\hat{\omega}. \tag{B1}$$

The general solution of this equation is given by

$$\hat{\psi}(k, y, \sigma) = \int G_{\rm bare}(k, y, y')\hat{\omega}(k, y', \sigma)\, dy', \tag{B2}$$

where $G_{\rm bare}(k, y, y')$ is the bare Green function determined by the differential equation

$$\frac{d^2 G_{\rm bare}}{dy^2} - k^2 G_{\rm bare} = -\delta(y - y'). \tag{B3}$$

In an unbounded domain, this equation can be solved analytically as follows. For $y \neq y'$, we have

$$\frac{d^2 G_{\rm bare}}{dy^2} - k^2 G_{\rm bare} = 0. \tag{B4}$$

This equation can be integrated into

$$G_{\rm bare}(k, y, y') = A e^{-|k||y-y'|}, \tag{B5}$$

where we have selected the solution that decays to zero at infinity. To determine the constant A we integrate Eq. (B3) between $-\infty$ and $+\infty$, giving

$$\int_{-\infty}^{+\infty} \frac{d^2 G_{\rm bare}}{dy^2}\, dy - k^2 \int_{-\infty}^{+\infty} G_{\rm bare}\, dy = -1. \tag{B6}$$

Since $G_{\rm bare}(k, y, y')$ and its derivatives vanish at infinity, the foregoing equation reduces to

$$2k^2 A \int_0^{+\infty} e^{-|k|y}\, dy = 1, \tag{B7}$$

yielding

$$A = \frac{1}{2|k|}. \tag{B8}$$

Therefore, the bare Green function in an infinite domain is given by

$$G_{\rm bare}(k,y,y') = \frac{1}{2|k|} e^{-|k||y-y'|}. \tag{B9}$$

Introducing the function

$$\chi_{\rm bare}(y,y') = \frac{1}{2}\int |k| G_{\rm bare}(k,y,y')^2\, dk, \tag{B10}$$

and using Eq. (B9), we get

$$\chi_{\rm bare}(y,y') = \int_0^{+\infty} \frac{1}{4k} e^{-2k|y-y'|}\, dk. \tag{B11}$$

When $y' = y$, this integral reduces to

$$\chi_{\rm bare}(y,y) = \frac{1}{4}\int_0^{+\infty} \frac{dk}{k} = \frac{1}{4}\ln\Lambda, \tag{B12}$$

where $\ln\Lambda = \int_0^{+\infty} dk/k = \ln(\lambda_{\rm max}/\lambda_{\rm min})$. We note that $\chi_{\rm bare}(y,y)$ involves an integral that diverges logarithmically at small and large scales. It can be regularized by introducing appropriate cut-offs (see Refs. [57, 58, 67] for more details) leading to a logarithmic factor $\ln\Lambda$ similar to the Coulomb logarithm in plasma physics. When $y' \neq y$, the integral from Eq. (B11) is convergent at small scales ($k \to +\infty$) but divergent at large scales ($k \to 0$). In the dominant approximation, we can write[39]

$$\chi_{\rm bare}(y,y') \simeq \frac{1}{4}\ln\Lambda. \tag{B13}$$

In order to regularize the large-scale divergence in Eq. (B11) we can replace the Poisson equation (A3) by the screened Poisson equation

$$\Delta\psi - k_R^2\psi = -\omega, \tag{B14}$$

and ultimately take the limit $k_R \to 0$ [58]. Equation (B14) can be introduced in an *ad hoc* manner but it is interesting to note that it also corresponds to the quasigeostrophic (QG) model describing geophysical flows [147]. In that context, k_R^{-1} is the so-called Rossby radius. The bare Green function corresponding to Eq. (B14) is obtained from Eq. (B9) by making the substitution $k^2 \to k^2 + k_R^2$. This yields

$$G_{\rm bare}(k,y,y') = \frac{1}{2\sqrt{k^2+k_R^2}} e^{-\sqrt{k^2+k_R^2}\,|y-y'|}. \tag{B15}$$

The function defined by Eq. (B10) takes the form

$$\chi_{\rm bare}(y,y') = \frac{1}{4}\int_0^{+\infty} \frac{k}{k^2+k_R^2} e^{-2\sqrt{k^2+k_R^2}\,|y-y'|}\, dk. \tag{B16}$$

It can be written as

$$\chi_{\rm bare}(y,y') = \frac{1}{4}E_1(2k_R|y-y'|), \tag{B17}$$

[39] Collective effects are usually negligible when $y' \to y$. In that case, $\chi(y,y,U(y))$ can be replaced by $\chi_{\rm bare}(y,y) = (1/4)\ln\Lambda$. More generally, in the dominant approximation, $\chi(y,y',U(y))$ may be replaced by $(1/4)\ln\Lambda$ with good accuracy.

where

$$E_1(x) = \int_x^{+\infty} \frac{e^{-t}}{t}\, dt \tag{B18}$$

is the exponential integral. For $x \to 0$, we have the expansion $E_1(x) = -\gamma_E - \ln x + ...$, where $\gamma_E = 0.57721$ is Euler's constant. Therefore, for $k_R \to 0$, we get

$$\chi_{\rm bare}(y, y') \simeq \frac{1}{4}\left[-\gamma_E - \ln(2k_R|y - y'|)\right], \tag{B19}$$

which is perfectly well-defined for $y' \neq y$. When $y' = y$, the integral (B16) converges at large scales ($k \to 0$) but diverges logarithmically at small scales ($k \to +\infty$).

Remark: The previous results can be generalized to an arbitrary potential of interaction of the form $u(|\mathbf{r}-\mathbf{r}'|)$ such that $\psi(\mathbf{r}) = \int u(|\mathbf{r}-\mathbf{r}'|)\omega(\mathbf{r}')\, d\mathbf{r}'$. For the 2D Euler equation in an infinite domain we have $u(|\mathbf{r}-\mathbf{r}'|) = -\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|$ and for the QG equations in an infinite domain we have $u(|\mathbf{r}-\mathbf{r}'|) = \frac{1}{2\pi}K_0(k_R|\mathbf{r}-\mathbf{r}'|)$, where $K_0(x)$ is the modified Bessel function of zeroth order. We note that $G_{\rm bare}(k, |y-y'|) = \hat{u}(k, |y-y'|)$ is the Fourier transform of the potential of interaction $u(|\mathbf{r}-\mathbf{r}'|)$ with respect to the variable x. Using Eqs. (B9) and (B15), we find that

$$\ln|\mathbf{r}-\mathbf{r}'| = -\pi \int e^{ik(x-x')}\frac{1}{|k|}e^{-|k||y-y'|}\, dk, \tag{B20}$$

$$K_0(k_R|\mathbf{r}-\mathbf{r}'|) = \pi \int e^{ik(x-x')}\frac{1}{\sqrt{k^2+k_R^2}}e^{-\sqrt{k^2+k_R^2}\,|y-y'|}\, dk. \tag{B21}$$

Appendix C: An important identity

The Green function $G(k, y, y', \sigma)$ introduced in Sec. IV is determined by the equation

$$\frac{d^2G}{dy^2} - k^2 G + \frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}G = -\delta(y-y') \tag{C1}$$

with the Landau prescription $\sigma \to \sigma + i0^+$. Multiplying Eq. (C1) by $G(k, y, y', \sigma)^*$ and integrating over y between $-\infty$ and $+\infty$, we get

$$-\int_{-\infty}^{+\infty}\left|\frac{dG}{dy}\right|^2 dy - k^2\int_{-\infty}^{+\infty}|G|^2\, dy + \int_{-\infty}^{+\infty}\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}|G|^2\, dy = -G(k, y', y', \sigma)^*, \tag{C2}$$

where we have integrated the first term by parts. Taking the imaginary part of this equation, we find that

$${\rm Im}\, G(k, y', y', \sigma) = {\rm Im}\int_{-\infty}^{+\infty}\frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}|G|^2(k, y, y', \sigma)\, dy. \tag{C3}$$

Using the Sokhotski-Plemelj formula

$$\frac{1}{x \pm i0^+} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x), \tag{C4}$$

we obtain the important identity

$${\rm Im}\, G(k, y, y, \sigma) = \pi\int_{-\infty}^{+\infty} k\frac{\partial\omega'}{\partial y'}\delta(kU(y')-\sigma)|G(k, y', y, \sigma)|^2\, dy'. \tag{C5}$$

We also mention the identity

$$G(-k, y, y', -\sigma) = G(k, y, y', \sigma)^*, \tag{C6}$$

which can be derived from Eq. (C1) by using the Landau prescription.

Appendix D: Alternative derivations of the diffusion coefficient

1. General expression of the diffusion coefficient using Fourier transforms in position and time

The change in position (in the y-direction) of a test vortex due to the total fluctuating stream function is

$$\frac{dy}{dt} = V_y = -\frac{\partial \delta\psi_{\rm tot}}{\partial x}(x, y, t). \tag{D1}$$

Integrating this equation between 0 and t, we obtain

$$\begin{aligned} \Delta y &= -\int_0^t \frac{\partial \delta\psi_{\rm tot}}{\partial x}(x(t'), y(t'), t')\, dt' \\ &= -\int_0^t \frac{\partial \delta\psi_{\rm tot}}{\partial x}(x + U(y)t', y, t')\, dt', \end{aligned} \tag{D2}$$

where we have used the unperturbed equation of motion (74) in the second equation (this accounts for the fact that the point vortex follows the mean field trajectory at leading order). Decomposing the stream function in Fourier modes, we get

$$\begin{aligned} \Delta y &= -\int_0^t dt' \frac{\partial}{\partial x} \int dk \int \frac{d\sigma}{2\pi} e^{ik(x+U(y)t')} e^{-i\sigma t'} \delta\hat{\psi}_{\rm tot}(k, y, \sigma) \\ &= -\int dk \int \frac{d\sigma}{2\pi} ike^{ikx} \delta\hat{\psi}_{\rm tot}(k, y, \sigma) \int_0^t e^{i(kU(y)-\sigma)t'}\, dt' \\ &= -\int dk \int \frac{d\sigma}{2\pi} ike^{ikx} \delta\hat{\psi}_{\rm tot}(k, y, \sigma) \frac{e^{i(kU(y)-\sigma)t} - 1}{i(kU(y) - \sigma)}. \end{aligned} \tag{D3}$$

The diffusion coefficient is defined by [see Eq. (69)]

$$D = \lim_{t\rightarrow+\infty} \frac{\langle (\Delta y)^2 \rangle}{2t}. \tag{D4}$$

Substituting Eq. (D3) into Eq. (D4), we obtain

$$D = -\lim_{t\rightarrow+\infty} \frac{1}{2t} \int dk \int dk' \int \frac{d\sigma}{2\pi} \int \frac{d\sigma'}{2\pi} kk' e^{i(k+k')x} \langle \delta\hat{\psi}_{\rm tot}(k, y, \sigma) \delta\hat{\psi}_{\rm tot}(k', y, \sigma') \rangle \frac{e^{i(kU(y)-\sigma)t} - 1}{i(kU(y) - \sigma)} \frac{e^{i(k'U(y)-\sigma')t} - 1}{i(k'U(y) - \sigma')}. \tag{D5}$$

Introducing the power spectrum from Eq. (35), the foregoing equations can be rewritten as

$$D = \lim_{t\rightarrow+\infty} \frac{1}{2t} \int dk \int \frac{d\sigma}{2\pi} k^2 P(k, y, \sigma) \frac{|e^{i(kU(y)-\sigma)t} - 1|^2}{(kU(y) - \sigma)^2}. \tag{D6}$$

Using the identity

$$\lim_{t\rightarrow+\infty} \frac{|e^{ixt} - 1|^2}{x^2 t} = 2\pi\delta(x), \tag{D7}$$

we find that

$$D = \pi \int dk \int \frac{d\sigma}{2\pi} k^2 P(k, y, \sigma) \delta(kU(y) - \sigma). \tag{D8}$$

Integrating over the δ-function (resonance), we get

$$D = \frac{1}{2} \int dk\, k^2 P(k, y, kU(y)), \tag{D9}$$

which returns Eq. (87). Then, using Eq. (37) we obtain Eq. (89).

Remark: If we do not take the limit $t \to +\infty$ in Eq. (D6), we obtain a time-dependent diffusion coefficient of the form

$$D(t) = \pi \int dk \int \frac{d\sigma}{2\pi} k^2 P(k, y, \sigma) \Delta(kU(y) - \sigma, t) \tag{D10}$$

with the regularized function

$$\Delta(x, t) = \frac{1}{2\pi t} \frac{|e^{ixt} - 1|^2}{x^2} = \frac{1 - \cos(xt)}{\pi t x^2}. \tag{D11}$$

When $t \to +\infty$, we can make the replacement $\Delta(x, t) \to \delta(x)$ corresponding to the diffusive regime. When $t \to 0$, we have $\Delta(x, t) \sim t/2\pi$ corresponding to the ballistic regime.

2. General expression of the diffusion coefficient using a Fourier transform in position

We can make the calculations of the previous section in a slightly different manner. In Eq. (D2) we decompose the total fluctuating stream function in Fourier modes in position but not in time. In that case, we get

$$\begin{aligned} \Delta y &= -\int_0^t dt' \frac{\partial}{\partial x} \int dk \, e^{ik(x+U(y)t')} \delta\hat{\psi}_{\rm tot}(k, y, t') \\ &= -\int_0^t dt' \int dk \, ik e^{ik(x+U(y)t')} \delta\hat{\psi}_{\rm tot}(k, y, t'). \end{aligned} \tag{D12}$$

Substituting Eq. (D12) into Eq. (D4), we obtain

$$D = -\lim_{t\to+\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \int dk \int dk' \, kk' e^{i(k+k')x} e^{ikU(y)t'} e^{ik'U(y)t''} \left\langle \delta\hat{\psi}_{\rm tot}(k, y, t') \delta\hat{\psi}_{\rm tot}(k', y, t'') \right\rangle. \tag{D13}$$

Introducing the inverse Fourier transform in time of the power spectrum from Eq. (40), we can rewrite the foregoing equation as

$$D = \lim_{t\to+\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \int dk \, k^2 e^{ikU(y)(t'-t'')} \mathcal{P}(k, y, t' - t''). \tag{D14}$$

Using the identity (78), we get

$$D = \lim_{t\to+\infty} \frac{1}{t} \int_0^t ds \, (t - s) \int dk \, k^2 e^{ikU(y)s} \mathcal{P}(k, y, s). \tag{D15}$$

Assuming that $\mathcal{P}(k, y, s)$ decreases more rapidly than s^{-1}, we obtain

$$D = \int_0^{+\infty} ds \int dk \, k^2 e^{ikU(y)s} \mathcal{P}(k, y, s). \tag{D16}$$

Making the change of variables $s \to -s$ and $k \to -k$, and using the fact that $\mathcal{P}(-k, y, -s) = \mathcal{P}(k, y, s)$, we see that we can replace $\int_0^{+\infty} ds$ by $(1/2) \int_{-\infty}^{+\infty} ds$. Therefore,

$$D = \frac{1}{2} \int_{-\infty}^{+\infty} ds \int dk \, k^2 e^{ikU(y)s} \mathcal{P}(k, y, s). \tag{D17}$$

Finally, taking the inverse Fourier transform in time of $\mathcal{P}(k, y, s)$ we find that

$$D = \frac{1}{2} \int dk \, k^2 P(k, y, kU(y)), \tag{D18}$$

which returns Eq. (87). Then, using Eq. (37) we obtain Eq. (89). We note that $\mathcal{P}(k, y, s)$ is complex while $P(k, y, \sigma)$ is real. They satisfy the identities $\mathcal{P}(-k, y, s) = \mathcal{P}(k, y, s)^* = \mathcal{P}(k, y, -s)$ and $P(k, y, \sigma) = P(k, y, \sigma)^* = P(-k, y, -\sigma)$ which can be directly obtained from the definition of $\mathcal{P}(k, y, s)$ and $P(k, y, \sigma)$ in Sec. V.

Remark: If we introduce the temporal Fourier transform of $\mathcal{P}(k, y, t)$ in Eq. (D14) we get

$$D = \lim_{t\to+\infty} \frac{1}{2t} \int_0^t dt' \int_0^t dt'' \int dk \int \frac{d\sigma}{2\pi} k^2 e^{ikU(y)(t'-t'')} e^{-i\sigma(t'-t'')} P(k, y, \sigma), \tag{D19}$$

which is equivalent to Eq. (75) with Eq. (77). If we integrate over t' and t'', we recover Eq. (D6).

3. Diffusion coefficient created by N point vortices

According to Eqs. (27) and (45), the total fluctuating stream function created by a collection of N point vortices is

$$\delta\hat{\psi}_{\rm tot}(k,y,\sigma)=\sum_i \gamma_i G(k,y,y_i,\sigma)e^{-ikx_i}\delta(\sigma-kU(y_i)). \tag{D20}$$

Substituting this expression into Eq. (D3) and integrating over σ, we obtain

$$\Delta y=-\frac{1}{2\pi}\sum_i\gamma_i\int dk\, ike^{ikx}\frac{e^{ik(U(y)-U(y_i))t}-1}{ik(U(y)-U(y_i))}G(k,y,y_i,kU(y_i))e^{-ikx_i}. \tag{D21}$$

The diffusion coefficient from Eq. (D4) is then given by

$$D=-\lim_{t\to+\infty}\frac{1}{2t}\Bigg\langle\frac{1}{4\pi^2}\sum_{ij}\gamma_i\gamma_j\int dk\int dk'\, kk'e^{i(k+k')x}\frac{e^{ik(U(y)-U(y_i))t}-1}{ik(U(y)-U(y_i))}\frac{e^{ik'(U(y)-U(y_j))t}-1}{ik'(U(y)-U(y_j))}$$
$$\times G(k,y,y_i,kU(y_i))G(k',y,y_j,k'U(y_j))e^{-ikx_i}e^{-ik'x_j}\Bigg\rangle. \tag{D22}$$

Since the point vortices are initially uncorrelated, and since the point vortices of the same species are identical, we get (see the similar steps detailed after Eq. (46) in Sec. V B)

$$D=-\lim_{t\to+\infty}\frac{1}{2t}\sum_b\frac{1}{4\pi^2}\int dx'\int dy'\int dk\int dk'\, kk'e^{i(k+k')x}\frac{e^{ik(U(y)-U(y'))t}-1}{ik(U(y)-U(y'))}\frac{e^{ik'(U(y)-U(y'))t}-1}{ik'(U(y)-U(y'))}$$
$$\times G(k,y,y',kU(y'))G(k',y,y',k'U(y'))e^{-i(k+k')x'}\gamma_b\omega_b(y'). \tag{D23}$$

Integrating over x', then over k', and using the identity from Eq. (C6), we get

$$D=\lim_{t\to+\infty}\frac{1}{2t}\sum_b\frac{1}{2\pi}\int dy'\int dk\, k^2\frac{|e^{ik(U(y)-U(y'))t}-1|^2}{k^2(U(y)-U(y'))^2}|G(k,y,y',kU(y'))|^2\gamma_b\omega_b(y'). \tag{D24}$$

Finally, using Eq. (D7), we obtain

$$D=\frac{1}{2}\sum_b\int dy'\int dk\, k^2\delta\left[k(U(y)-U(y'))\right]|G(k,y,y',kU(y'))|^2\gamma_b\omega_b(y'), \tag{D25}$$

which returns Eq. (91). Integrating over the δ-function (resonance) with the identity from Eq. (92), we recover Eq. (93).

Remark: If we do not take the limit $t\to+\infty$ in the foregoing equations, we obtain a time-dependent diffusion coefficient

$$D(t)=\frac{1}{2}\sum_b\int dy'\int dk\, k^2\Delta\left[k(U(y)-U(y')),t\right]|G(k,y,y',kU(y'))|^2\gamma_b\omega_b(y'), \tag{D26}$$

where the regularized function $\Delta(x,t)$ is defined in Eq. (D11).

Appendix E: Velocity auto-correlation function and diffusion coefficient with collective effects

The y-component of the velocity of a test vortex is

$$V_y=-\frac{\partial\delta\psi_{\rm tot}}{\partial x}, \tag{E1}$$

where $\delta\psi_{\rm tot}(x,y,t)$ is the total fluctuating stream function acting on the test vortex. Introducing the Fourier transform of the total stream function, the velocity auto-correlation function of the test vortex accounting for collective effects can be written as

$$\langle V_y(x,y,0)V_y(x+U(y)t,y,t)\rangle=-\int dk\int\frac{d\sigma}{2\pi}\int dk'\int\frac{d\sigma'}{2\pi}kk'e^{ikx}e^{ik'(x+U(y)t)}e^{-i\sigma't}\left\langle\delta\hat{\psi}_{\rm tot}(k,y,\sigma)\delta\hat{\psi}_{\rm tot}(k',y,\sigma')\right\rangle. \tag{E2}$$

To define the correlation function, we have used a Lagrangian point of view and we have used the fact that the test vortex follows the mean field trajectory from Eq. (74) at leading order. Using the expression (35) of the correlation function of the total fluctuating stream function (power spectrum), we get

$$\langle V_y(x,y,0)V_y(x+U(y)t,y,t)\rangle = \int dk \int \frac{d\sigma}{2\pi} k^2 e^{i(\sigma-kU(y))t} P(k,y,\sigma). \tag{E3}$$

Recalling the relation (37) between the dressed correlation function of the total fluctuating stream function and the bare correlation function of the external vorticity field, we obtain

$$\langle V_y(x,y,0)V_y(x+U(y)t,y,t)\rangle = \int dy' \int dk \int \frac{d\sigma}{2\pi}\, k^2 e^{i(\sigma-kU(y))t} |G(k,y,y',\sigma)|^2 \hat{C}(k,y',\sigma). \tag{E4}$$

If the external vorticity field is created by N point vortices then, using Eq. (47), the foregoing equation becomes

$$\begin{aligned}\langle V_y(x,y,0)V_y(x+U(y)t,y,t)\rangle &= \sum_b \gamma_b \int dy' \int dk \int \frac{d\sigma}{2\pi}\, k^2 e^{i(\sigma-kU(y))t} |G(k,y,y',\sigma)|^2 \delta(kU(y')-\sigma)\omega_b(y') \\ &= \sum_b \frac{\gamma_b}{2\pi} \int dy' \int dk\, k^2 e^{ik(U(y')-U(y))t} |G(k,y,y',kU(y'))|^2 \omega_b(y').\end{aligned} \tag{E5}$$

Explicit expressions of the velocity auto-correlation function of a point vortex are given in [58, 73]. Using Eq. (81), we find that the diffusion coefficient of the test vortex is

$$\begin{aligned} D &= \frac{1}{2}\sum_b \gamma_b \int dy' \int dk\, k^2 |G(k,y,y',kU(y))|^2 \delta(kU(y')-kU(y))\omega_b(y') \\ &= \frac{1}{2}\sum_b \gamma_b \int dy' \int dk\, |k| |G(k,y,y',kU(y))|^2 \delta(U(y')-U(y))\omega_b(y').\end{aligned} \tag{E6}$$

This returns the result from Eq. (93).

Appendix F: Polarization cloud

In this Appendix we determine the polarization cloud created by a test vortex moving in the flow.

1. With collective effects

If we take into account collective effects, the change of vorticity of the flow due to an external perturbation ω_e is given in Fourier space by Eqs. (24) and (27). We assume here that the perturbation is caused by a point vortex of circulation γ. The Fourier transform of the vorticity created by the point vortex is [see Eq. (45)]

$$\hat{\omega}_e(k,y,\sigma) = \gamma e^{-ikx_0} \delta(kU(y)-\sigma)\delta(y-y_0), \tag{F1}$$

where (x_0, y_0) denotes the initial position of the test vortex. From Eqs. (24), (27) and (F1), we get

$$\delta\hat{\psi}_{\rm tot}(k,y,\sigma) = \gamma\, G(k,y,y_0,\sigma) e^{-ikx_0} \delta(kU(y_0)-\sigma) \tag{F2}$$

and

$$\delta\hat{\omega}(k,y,\sigma) = \gamma \frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma} G(k,y,y_0,\sigma) e^{-ikx_0} \delta(kU(y_0)-\sigma). \tag{F3}$$

Returning to physical space, we obtain

$$\begin{aligned}\delta\omega(x,y,t) &= \gamma \int dk \int \frac{d\sigma}{2\pi}\, e^{i(kx-\sigma t)} \frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma} G(k,y,y_0,\sigma) e^{-ikx_0}\delta(kU(y_0)-\sigma) \\ &= \frac{\gamma}{2\pi} \frac{\frac{\partial\omega}{\partial y}}{U(y)-U(y_0)} \int dk\, e^{ik(x-x_0-U(y_0)t)} G(k,y,y_0,kU(y_0)).\end{aligned} \tag{F4}$$

If we measure the position x with respect to the position of the test vortex at the instant t, writing $X = x - x_0 - U(y_0)t$, we get

$$\delta\omega(X,y) = \frac{\gamma}{2\pi} \frac{\frac{\partial \omega}{\partial y}}{U(y) - U(y_0)} \int dk\, e^{ikX} G(k, y, y_0, kU(y_0)). \tag{F5}$$

If we neglect collective effects, we just have to replace the dressed Green function by the bare Green function. This yields

$$\delta\omega(X,y) = \frac{\gamma}{2\pi} \frac{\frac{\partial \omega}{\partial y}}{U(y) - U(y_0)} \int dk\, e^{ikX} G_{\rm bare}(k, y, y_0). \tag{F6}$$

Using the expression (B9) of the bare Green function in an infinite domain, we find that

$$\delta\omega(X,y) = \frac{\gamma}{2\pi} \frac{\frac{\partial \omega}{\partial y}}{U(y) - U(y_0)} \int dk\, e^{ikX} \frac{1}{2|k|} e^{-|k||y-y_0|}. \tag{F7}$$

The integral displays a logarithmic divergence when $k \to 0$. In the dominant approximation, we can write

$$\delta\omega(X,y) = \frac{\gamma}{2\pi} \frac{\frac{\partial \omega}{\partial y}}{U(y) - U(y_0)} \ln \Lambda. \tag{F8}$$

For $y \to y_0$, we obtain the equivalent

$$\delta\omega(X,y) \sim \frac{\gamma}{2\pi} \frac{\omega'(y_0)}{U'(y_0)(y-y_0)} \ln \Lambda, \tag{F9}$$

provided that $U'(y_0) \neq 0$.

2. Without collective effects

If we neglect collective effects from the start, the change of vorticity due to the external field is given by

$$\delta\hat{\omega}(k,y,\sigma) = \frac{k \frac{\partial \omega}{\partial y}}{kU(y) - \sigma} \hat{\psi}_e(k, y, \sigma) \tag{F10}$$

with Eq. (30). If the external vorticity is created by a point vortex, using Eqs. (30), (F1) and (F10), we obtain

$$\hat{\psi}_e(k,y,\sigma) = \gamma\, G_{\rm bare}(k, y, y_0) e^{-ikx_0} \delta(kU(y_0) - \sigma) \tag{F11}$$

and

$$\delta\hat{\omega}(k,y,\sigma) = \gamma \frac{k \frac{\partial \omega}{\partial y}}{kU(y) - \sigma} G_{\rm bare}(k, y, y_0) e^{-ikx_0} \delta(kU(y_0) - \sigma). \tag{F12}$$

Equations (F11) and (F12) correspond to Eqs. (F2) and (F3) with the dressed Green function replaced by the bare Green function. They finally lead to Eqs. (F6)-(F9).

Appendix G: Solution of the Boltzmann-Poisson equation

We consider a distribution of N point vortices with equal circulation γ at statistical equilibrium in an infinite domain. We assume that the mean flow is unidirectional. The equilibrium vorticity is given by the Boltzmann distribution (59) coupled to the Poisson equation (A3). This leads to the Boltzmann-Poisson equation

$$-\frac{d^2\psi}{dy^2} = \omega = A e^{-\beta\gamma\psi}. \tag{G1}$$

The constant A is determined by the circulation (or vortex number) $\Gamma = N\gamma$ and the inverse temperature β is determined by the energy of the flow E (see below). The vorticity can be written as

$$\omega = \omega_0 e^{-\phi} \quad \text{with} \quad \phi = \beta\gamma(\psi - \psi_0), \tag{G2}$$

where ω_0 and ψ_0 are the vorticity and the stream function at the origin $y = 0$. Equilibrium states exist in an infinite domain only for $\beta < 0$. Introducing the rescaled distance $\xi = (|\beta|\gamma\omega_0)^{1/2}y$, we can write the Boltzmann-Poisson equation (G1) as

$$\frac{d^2\phi}{d\xi^2} = e^{-\phi} \tag{G3}$$

with boundary conditions $\phi(0) = \phi'(0) = 0$. This equation is similar to the Emden equation in astrophysics [148, 149]. It is also similar to the equation of motion of a particle of unit mass moving in a potential $V(\phi) = e^{-\phi}$, where ϕ plays the role of the position and ξ the role of time. It has the analytical solution (see, e.g., [149])

$$e^{-\phi} = \frac{1}{\cosh^2\left(\frac{\xi}{\sqrt{2}}\right)}. \tag{G4}$$

Computing the total circulation $\Gamma = \int_{-\infty}^{+\infty} \omega(y)\, dy = -2\psi'(+\infty)$, we find that the central vorticity is given by $\omega_0 = |\beta|\gamma\Gamma^2/8$. We can then write the equilibrium vorticity profile as

$$\omega = \frac{|\beta|\gamma\Gamma^2}{8}\frac{1}{\cosh^2\left(\frac{|\beta|\gamma\Gamma y}{4}\right)}. \tag{G5}$$

Taking $\psi_0 = 0$ by convention, we obtain the equilibrium steam function

$$\psi = -\frac{2}{|\beta|\gamma}\ln\left\{\cosh\left(\frac{|\beta|\gamma\Gamma y}{4}\right)\right\}. \tag{G6}$$

The corresponding velocity field is

$$U(y) = -\frac{1}{2}\Gamma\tanh\left(\frac{|\beta|\gamma\Gamma y}{4}\right). \tag{G7}$$

Finally, the inverse temperature β is related to the energy by

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} \omega(y)\psi(y)\, dy = -\frac{\Gamma}{|\beta|\gamma}\int_0^{+\infty}\frac{\ln[\cosh(x)]}{\cosh^2(x)}\, dx = -\frac{\Gamma}{|\beta|\gamma}(1 - \ln 2). \tag{G8}$$

Appendix H: Out-of-equilibrium fluctuation-dissipation theorem

In this Appendix, we consider an arbitrary distribution of point vortices with a monotonic velocity profile (see Sec. XII) and we establish a form of out-of-equilibrium fluctuation-dissipation theorem.

If the velocity field is monotonic, we have the identity [see Eq. (96)]

$$\delta(kU(y') - \sigma) = \frac{1}{|kU'(y_*)|}\delta(y' - y_*), \tag{H1}$$

where $y_* = U^{-1}(\sigma/k)$ is the (unique) root of the equation $kU(y_*) = \sigma$. Substituting Eq. (H1) into Eq. (C5), we obtain

$$\mathrm{Im}\, G(k, y, y, \sigma) = \pi k \frac{\partial\omega}{\partial y}(y_*)\frac{1}{|kU'(y_*)|}|G(k, y_*, y, \sigma)|^2. \tag{H2}$$

Similarly, Eq. (48) can be written as

$$P(k, y, \sigma) = \sum_b \gamma_b |G(k, y, y_*, \sigma)|^2 \frac{\omega_b(y_*)}{|kU'(y_*)|}. \tag{H3}$$

For a single species system of point vortices with circulation γ_b, combining Eqs. (H2) and (H3), we obtain the out-of-equilibrium fluctuation-dissipation theorem

$$\mathrm{Im}\, G(k,y,y,\sigma) = \frac{\pi k}{\gamma_b}\frac{\partial \ln|\omega_b|}{\partial y}(y_*)P(k,y,\sigma). \tag{H4}$$

In the case where the field vortices are at statistical equilibrium with the Boltzmann distribution (59) we recover the usual fluctuation-dissipation theorem (56).

Appendix I: A simplified kinetic equation

In this Appendix, we propose a simplified kinetic equation that may approximately describe the dynamical evolution of a Hamiltonian system of 2D point vortices in certain cases. To obtain this equation, we make the thermal bath approximation in the Lenard-Balescu equation (143), leading to Eq. (156), but we assume that $\psi(y,t)$ evolves self-consistently with time, being determined by the total vorticity $\omega(y,t)=\sum_a \omega_a(y,t)$ through the Poisson equation (6) instead of being prescribed as in Sec. XI C.[40] This gives

$$\frac{\partial \omega_a}{\partial t} = \frac{\partial}{\partial y}\left[D\left(\frac{\partial \omega_a}{\partial y} + \beta\gamma_a\omega_a\frac{\partial \psi}{\partial y}\right)\right], \tag{I1}$$

$$\Delta\psi = -\sum_a \omega_a, \tag{I2}$$

where D is given by Eq. (153). This equation does not conserve the energy contrary to the Lenard-Balescu equation (143). However, following [111, 150], we can enforce the energy conservation by allowing β to depend on time in such a way that $\dot{E} = \int \psi \frac{\partial \omega}{\partial t}\, dy = 0$. This yields

$$\beta(t) = -\frac{\int D\frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial y}\, dy}{\int D\omega_2\left(\frac{\partial \psi}{\partial y}\right)^2 dy}. \tag{I3}$$

Equation (I1) with Eqs. (I2) and (I3) conserves the circulations of each species, the energy, and increases the entropy (H-theorem) [111].[41] It relaxes towards the Boltzmann distribution of statistical equilibrium on a timescale Nt_D. This equation is well-posed mathematically and interesting in its own right. It can be seen as a heuristic approximation of the Lenard-Balescu equation (143) providing a simplified kinetic equation for a Hamiltonian system of 2D point vortices. However, since the approximation leading to Eq. (I1) is uncontrolled, the solution of this equation may substantially differ from the solution of the Lenard-Balescu equation (143). For example, in the case where there is no resonance, Eq. (I1) gives a non-vanishing flux ($\partial\omega/\partial t \neq 0$) driving the system towards the Boltzmann distribution on a timescale Nt_D while the Lenard-Balescu flux vanishes ($\partial\omega_{\mathrm{LB}}/\partial t = 0$) and the Boltzmann distribution is reached on a longer timescale $N^2 t_D$.[42] More generally, the relevance of Eq. (I1) should be determined case by case by solving this equation numerically and comparing its solution with the solution of the Lenard-Balescu equation (143) or with direct numerical simulations of the N-point vortex system.

In principle, the diffusion coefficient D is a functional of ω_a but we shall assume $D = \mathrm{cst}$ for simplicity. We also take $\beta = \mathrm{cst}$ in Eq. (I1) like in the case of 2D Brownian vortices described by the canonical ensemble. In that case, Eqs. (I1) and (I2) conserve the circulations of the different species of point vortices and decrease the free energy $F = E - TS$. With this setting, we can take fluctuations due to finite N effects into account by adding a noise term in the kinetic equation like in Sec. XIII. This leads to[43]

$$\frac{\partial \omega_a}{\partial t} + \mathbf{u}\cdot\nabla\omega_a = \nabla\cdot\left[D\left(\nabla\omega_a + \beta\gamma_a\omega_a\nabla\psi\right)\right] + \nabla\cdot\left[\sqrt{2D\gamma_a\omega_a}\mathbf{R}_a(\mathbf{r},t)\right], \tag{I4}$$

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[40] In Sec. XI C, $\psi(y)$ is determined by the vorticity $\sum_b \omega_b(y)$ of the field vortices assumed to be independent of time.

[41] We can also conserve the linear impulse by introducing a relative stream function $\psi_{\mathrm{eff}} = \psi - V(t)y$ instead of ψ and proceed like in Ref. [111].

[42] This timescale discrepency could be corrected by empirically changing the value of D in Eq. (I1) to make it of order $1/N^2$.

[43] It is also possible to take into account fluctuations in the more general equations (I1)-(I3) and in the Lenard-Balescu equation (143) but the expression of the noise is more complicated [142].

$$\Delta\psi = -\sum_a \omega_a. \tag{I5}$$

Interestingly, Eqs. (I4) and (I5) apply to arbitrary flows (with the limitations about their validity mentioned previously). These equations are interesting in their own right. They could be used to describe random transitions between different equilibrium states as discussed in Sec. XIII.

Appendix J: Lenard-Balescu equation for 2D point vortices

We consider an isolated system of N point vortices with identical circulation γ. We assume that the mean flow is unidirectional. We want to determine the kinetic equation of 2D point vortices due to finite N effects by using the Klimontovich approach. The derivation is similar to the one given for axisymmetric flows in Refs. [66, 68]. We start from the quasilinear equations (19) and (20) without the external potential ($\psi_e = 0$) that we rewrite as

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial y}\left\langle \delta\omega \frac{\partial \delta\psi}{\partial x}\right\rangle, \tag{J1}$$

$$\frac{\partial \delta\omega}{\partial t} + U\frac{\partial \delta\omega}{\partial x} - \frac{\partial \delta\psi}{\partial x}\frac{\partial \omega}{\partial y} = 0. \tag{J2}$$

Taking the Fourier-Laplace transform of Eq. (J2), we find that

$$\delta\tilde{\omega}(k,y,\sigma) = \frac{k\frac{\partial\omega}{\partial y}}{kU-\sigma}\delta\tilde{\psi}(k,y,\sigma) + \frac{\delta\hat{\omega}(k,y,0)}{i(kU-\sigma)}, \tag{J3}$$

where $\delta\hat{\omega}(k,y,0)$ is the Fourier transform of the initial vorticity fluctuation caused by finite N effects. Combining this relation with the Poisson equation $\Delta\delta\psi = -\delta\omega$ written in Fourier space [see Eq. (B1)], we get

$$\left[\frac{d^2}{dy^2} - k^2 + \frac{k\frac{\partial\omega}{\partial y}}{kU(y)-\sigma}\right]\delta\tilde{\psi} = -\frac{\delta\hat{\omega}(k,y,0)}{i(kU-\sigma)}. \tag{J4}$$

The formal solution of this differential equation is

$$\delta\tilde{\psi}(k,y,\sigma) = \int G(k,y,y',\sigma)\frac{\delta\hat{\omega}(k,y',0)}{i(kU'-\sigma)}\,dy', \tag{J5}$$

where the Green function is defined in Eq. (28) and U' stands for $U(y')$. Taking the inverse Laplace transform of this equation, using the Cauchy residue theorem, and neglecting the contribution of the damped modes for sufficiently late times,[44] we obtain

$$\delta\hat{\psi}(k,y,t) = \int dy'\, G(k,y,y',kU')\delta\hat{\omega}(k,y',0)e^{-ikU't}. \tag{J6}$$

On the other hand, taking the Fourier transform of Eq. (J2), we find that

$$\frac{\partial \delta\hat{\omega}}{\partial t} + ikU\delta\hat{\omega} = ik\frac{\partial\omega}{\partial y}\delta\hat{\psi}. \tag{J7}$$

This first order differential equation in time can be solved with the method of the variation of the constant, giving

$$\delta\hat{\omega}(k,y,t) = \delta\hat{\omega}(k,y,0)e^{-ikUt} + ik\frac{\partial\omega}{\partial y}\int_0^t dt'\,\delta\hat{\psi}(k,y,t')e^{ikU(t'-t)}. \tag{J8}$$

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[44] We only consider the contribution of the pole $\sigma - kU'$ and ignore the contribution of the proper modes of the flow which are the solutions of the Rayleigh equation (32). See Ref. [151] for general considerations about the linear response theory of systems with long-range interactions.

Substituting Eq. (J6) into Eq. (J8), we obtain

$$\delta\hat{\omega}(k,y,t)=\delta\hat{\omega}(k,y,0)e^{-ikUt}+ik\frac{\partial\omega}{\partial y}\int dy'\,G(k,y,y',kU')\delta\hat{\omega}(k,y',0)e^{-ikUt}\int_0^t dt'\,e^{ik(U-U')t'}. \tag{J9}$$

Eqs. (J6) and (J9) relate $\delta\hat{\psi}(k,y,t)$ and $\delta\hat{\omega}(k,y,t)$ to the initial fluctuation $\delta\hat{\omega}(k,y,0)$.
We can now compute the flux

$$\left\langle \delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle=\int dkdk'\,ik'e^{ikx}e^{ik'x}\langle\delta\hat{\omega}(k,y,t)\delta\hat{\psi}(k',y,t)\rangle. \tag{J10}$$

From Eqs. (J6) and (J9) we get

$$\langle\delta\hat{\omega}(k,y,t)\delta\hat{\psi}(k',y,t)\rangle=\int dy'\,G(k',y,y',k'U')e^{-ik'U't}e^{-ikUt}\langle\delta\hat{\omega}(k,y,0)\delta\hat{\omega}(k',y',0)\rangle$$
$$+\int dy'\,G(k',y,y',k'U')e^{-ik'U't}ik\frac{\partial\omega}{\partial y}\int dy''\,G(k,y,y'',kU'')\langle\delta\hat{\omega}(k,y'',0)\delta\hat{\omega}(k',y',0)\rangle e^{-ikUt}\int_0^t dt'\,e^{ik(U-U'')t'}. \tag{J11}$$

The correlation function of the initial fluctuations in Fourier space is given by (see, e.g., Appendix D of [68])

$$\langle\delta\hat{\omega}(k,y,0)\delta\hat{\omega}(k',y',0)\rangle=\frac{\gamma}{2\pi}\delta(k+k')\delta(y-y')\omega(y). \tag{J12}$$

Eq. (J11) then reduces to

$$\langle\delta\hat{\omega}(k,y,t)\delta\hat{\psi}(k',y,t)\rangle=\frac{1}{2\pi}G(-k,y,y,-kU)\gamma\omega(y)\delta(k+k')$$
$$+\frac{1}{2\pi}\int dy'\,G(-k,y,y',-kU')ik\frac{\partial\omega}{\partial y}G(k,y,y',kU')\delta(k+k')\gamma\omega(y')\int_0^t ds\,e^{-ik(U-U')s}, \tag{J13}$$

where we have set $s=t-t'$. Substituting this relation into Eq. (J10), and taking the limit $t\to+\infty$, we obtain

$$\left\langle \delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle=-\frac{1}{2\pi}\int dk\,ikG(-k,y,y,-kU)\gamma\omega(y)$$
$$-\frac{1}{2\pi}\int dk\,ik\int dy'\,G(-k,y,y',-kU')ik\frac{\partial\omega}{\partial y}G(k,y,y',kU')\gamma\omega(y')\int_0^{+\infty}ds\,e^{-ik(U-U')s}. \tag{J14}$$

Making the transformations $s\to -s$ and $k\to -k$ we see that we can replace $\int_0^{+\infty}ds$ by $\frac{1}{2}\int_{-\infty}^{+\infty}ds$. We then get

$$\left\langle \delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle=-\frac{1}{2\pi}\int dk\,ikG(-k,y,y,-kU)\gamma\omega(y)$$
$$-\frac{1}{2\pi}\int dk\,ik\int dy'\,G(-k,y,y',-kU')ik\frac{\partial\omega}{\partial y}G(k,y,y',kU')\gamma\omega(y')\frac{1}{2}\int_{-\infty}^{+\infty}dt'\,e^{ik(U-U')t'}. \tag{J15}$$

Using the identities (23) and (92), we obtain

$$\left\langle \delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle=-\frac{1}{2\pi}\int dk\,ikG(-k,y,y,-kU)\gamma\omega(y)$$
$$+\frac{1}{2}\int dk\,|k|\int dy'\,G(-k,y,y',-kU')\frac{\partial\omega}{\partial y}G(k,y,y',kU')\gamma\omega(y')\delta(U-U'). \tag{J16}$$

Finally, using Eq. (C6), we can rewrite the foregoing equation as

$$\left\langle \delta\omega\frac{\partial\delta\psi}{\partial x}\right\rangle=-\frac{1}{2\pi}\int dk\,k\,\mathrm{Im}\,G(k,y,y,kU)\,\gamma\omega(y)+\frac{1}{2}\int dk\,|k|\int dy'\,\frac{\partial\omega}{\partial y}|G(k,y,y',kU')|^2\gamma\omega(y')\delta(U-U'). \tag{J17}$$

The first term is the drift term and the second term is the diffusion term. Using the identity (C5) and substituting the flux from Eq. (J17) into Eq. (J1), we obtain the Lenard-Balescu-like equation

$$\frac{\partial\omega}{\partial t}=\frac{\gamma}{2}\frac{\partial}{\partial y}\int dy'\int dk\,|k||G(k,y,y',kU(y))|^2\delta(U(y')-U(y))\left(\omega'\frac{\partial\omega}{\partial y}-\omega\frac{\partial\omega'}{\partial y'}\right). \tag{J18}$$

We recall that for a unidirectional flow made of a single species system of point vortices, the Lenard-Balescu flux vanishes (see Sec. XII). We can easily extend the derivation of the Lenard-Balescu equation to the multispecies case, leading to Eq. (139).

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