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Abstract

In this paper, we discuss the relations between the Jack polynomials, \hbar -dependent KP hierarchy and affine Yangian of $\mathfrak{gl}(1)$. We find that $\alpha = \hbar^2$ and $h_1 = \hbar$, $h_2 = -\hbar^{-1}$, where α is the parameter in Jack polynomials, and h_1 , h_2 are the parameters in affine Yangian of $\mathfrak{gl}(1)$. Then the vertex operators which are in Jack polynomials are the same with that in \hbar -KP hierarchy, and the Jack polynomials can be used to describe the tau functions of the \hbar -KP hierarchy.

Keywords: *ħ*-KP hierarchy, Affine Yangian, Jack polynomials, vertex operators, Boson-Fermion correspondence.

1 Introduction

The KP hierarchy is one of the most important integrable hierarchies and it arises in many different fields of mathematics and physics such as enumerative algebraic geometry, topological field and string theory[1, 2]. Meanwhile Young diagrams and symmetric functions are of interest to many researchers and have many applications in mathematics including combinatorics and representation theory of the symmetric and general linear group[3, 4, 5, 6]. Schur functions can be used to describe the tau functions of the KP hierarchy, and the vertex operators which realize the Schur functions have close relations with the Fermions in the KP hierarchy. In this paper, we generalize these to the case of Jack polynomials and \hbar -KP hierarchy.

In [7, 8], the authors K. Takasaki and T. Takebe defined the \hbar -dependent KP hierarchy (\hbar -KP hierarchy for short) by introduced a formal parameter \hbar . It is a generalization of the KP hierarchy in the sense that it becomes the KP hierarchy when $\hbar \to 1$. When $\hbar \to 0$, the \hbar -KP hierarchy becomes the dispersionless KP hierarchy. The \hbar -KP hierarchy was introduced to study the dispersionless KP hierarchy [7]. The \hbar -KP hierarchy is defined by the Lax representation

$$\hbar \frac{\partial L}{\partial x_j} = [B_j, L], \text{ with } B_j = (L^j)_+,$$

where the Lax operator L is the pseudodifferential operator of the following form

$$L = \hbar \partial + \sum_{j=1}^{\infty} f_j (\hbar \partial)^{-j}$$

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In [9], we introduce the vertex operators

$$X_{+}(z) = \sum_{n \in \mathbb{Z}} X_{n}^{+} z^{n} = \exp\left(\sum_{n \ge 1} \frac{x_{n}}{\sqrt{\alpha}} z^{n}\right) \exp\left(-\sum_{n \ge 1} \frac{\partial_{x_{n}}}{n} \sqrt{\alpha} z^{-n}\right)$$
$$X_{-}(z) = \sum_{n \in \mathbb{Z}} X_{n}^{-} z^{n} = \exp\left(-\sum_{n \ge 1} \frac{x_{n}}{\sqrt{\alpha}} z^{n}\right) \exp\left(\sum_{n \ge 1} \frac{\partial_{x_{n}}}{n} \sqrt{\alpha} z^{-n}\right).$$

They realize the Jack polynomials J_{λ} , and satisfy the Fermion relations. Then we define an integrable hierarchy by the bilinear relations:

$$\sum_{m+n=-1} X_m^- \tau \otimes X_n^+ \tau = 0.$$
⁽¹⁾

In this paper, we show that the integrable hierarchy above is exactly the \hbar -KP hierarchy by the Hirota form. Then we give the Boson-Fermion correspondence of \hbar -KP hierarchy, and describe the tau functions of \hbar -KP hierarchy by using the Jack polynomials \tilde{J}_{λ} and $S_{\lambda}\left(\frac{x}{\sqrt{\alpha}}\right)$.

The paper is organized as follows. In section 2, we recall the definition of \hbar -KP hierarchy. In section 3, we recall the definitions of the affine Yangian of $\mathfrak{gl}(1)$ and the Jack polynomials. Then we show the properties of the Jack polynomials and the vertex operators. In section 4, we use the Hirota equations to show that the \hbar -dependent KP hierarchy is exactly the integrable hierarchy defined in [9]. In section 5, we give the Boson-Fermion correspondence in the \hbar -KP hierarchy.

2 \hbar -dependent KP hierarchy

In [7, 8], the authors K. Takasaki and T. Takebe defined the \hbar -dependent KP hierarchy (\hbar -KP hierarchy for short) by introduced a formal parameter \hbar . When $\hbar \to 0$, the \hbar -KP hierarchy becomes the dispersionless KP hierarchy, and when $\hbar \to 1$, it becomes the KP hierarchy. The tau functions and the wave function in \hbar -KP hierarchy are functions of parameters x, t_1, t_2, \cdots , while x is only emerged in $x + t_1$. In this paper, we want the parameters are $x = (x_1, x_2, \cdots)$, which correspond to (t_1, t_2, \cdots) in [7, 8].

Consider a pseudodifferential operator

$$L = \hbar \partial + \sum_{j=1}^{\infty} f_j (\hbar \partial)^{-j}, \qquad (2)$$

and the corresponding eigenvalue problem

$$Lw = zw, (3)$$

where $\partial = \frac{\partial}{\partial x}$.

We consider a formal solution

$$w = e^{\sum_{j=1}^{\infty} \frac{x_j}{\hbar} z^j} \left(1 + \frac{w_1}{z} + \frac{w_2}{z^2} + \cdots \right)$$
(4)

$$= (1 + w_1(\hbar\partial)^{-1} + w_2(\hbar\partial)^{-2} + \cdots) e^{\sum_{j=1}^{\infty} \frac{x_j}{\hbar} z^j}.$$
 (5)

Then let

$$M = 1 + \sum_{j=1}^{\infty} w_j (\hbar \partial)^{-j}.$$
 (6)

Consider the linear system of equations

$$\hbar \frac{\partial w}{\partial x_j} = B_j w, \quad \text{with} \quad B_j = (L^j)_+, \tag{7}$$

where $(L^j)_+$ is the differential operator part of L^j , that is, $(L^j)_+$ includes the terms ∂^k , $k \ge 0$ in L^j .

The compatibility condition between (3) and (7) gives

$$\hbar \frac{\partial L}{\partial x_j} = [B_j, L]. \tag{8}$$

This is called the \hbar -dependent KP hierarchy, the \hbar -KP hierarchy for short. It is clear that when $\hbar \to 1$, it become the classical KP hierarchy.

Substituting (5) into (3), we get

$$L = M \cdot (\hbar\partial) \cdot M^{-1}. \tag{9}$$

This equation gives the relations between w_1, w_2, \cdots and f_1, f_2, \cdots . The compatibility condition (8) shows that these unknown functions can be written in terms of a single function $\tau(x)$ by the following relation

$$w = \frac{\tau(x - \hbar[z^{-1}])}{\tau(x)} e^{\sum_{j=1}^{\infty} \frac{x_j}{\hbar} z^j},$$
(10)

where

$$[z^{-1}] = (\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \cdots).$$

Then the \hbar -KP hierarchy is an infinite set of nonlinear differential equations in a function τ of infinitely many variables x_1, x_2, \cdots . As in the classical KP hierarchy, let

$$u = 2\hbar^2 \partial^2 \log \tau. \tag{11}$$

We get the first nonlinear differential equation in the \hbar -KP hierarchy

$$\frac{3}{4}\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x_3} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{\hbar^2}{4}\frac{\partial^3 u}{\partial x^3} \right).$$
(12)

This equation is called the \hbar -KP equation. When $\hbar \to 1$, it become the KP equation, and when $\hbar \to 0$, it become the dispersionless KP equation.

3 Affine Yangian of $\mathfrak{gl}(1)$ and Jack polynomials

In this section, we recall the definitions of the affine Yangian of $\mathfrak{gl}(1)$ and the Jack polynomials. Then we show the properties of the Jack polynomials according to the affine Yangian of $\mathfrak{gl}(1)$.

3.1 Affine Yangian of $\mathfrak{gl}(1)$

We almost copy this section from that in [9] since that is what we will use in the following of this paper. The affine Yangian \mathcal{Y} of $\widehat{\mathfrak{gl}}(1)$ is an associative algebra with generators

 e_j, f_j and $\psi_j, j = 0, 1, \ldots$ and the following relations[10, 11]

$$[\psi_j, \psi_k] = 0, \tag{13}$$

$$[e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] + \sigma_2[e_{j+2}, e_{k+3}] - \sigma_2[e_{j+2}, e_{k+3}] - \sigma_2[e_{j+2}, e_{k+3}] - \sigma_2[e_{j+3}, e_{k+3}]$$

$$+ \delta_{2} [c_{j+1}, c_{k}] - \delta_{2} [c_{j}, c_{k+1}] - \delta_{3} [c_{j}, c_{k}] = 0,$$

$$[f_{j+3}, f_{k}] - 3 [f_{j+2}, f_{k+1}] + 3 [f_{j+1}, f_{k+2}] - [f_{j}, f_{k+3}]$$

$$(14)$$

$$+\sigma_2 [f_{j+1}, f_k] - \sigma_2 [f_j, f_{k+1}] + \sigma_3 \{f_j, f_k\} = 0,$$
(15)

$$[e_j, f_k] = \psi_{j+k},\tag{16}$$

$$[\psi_{j+3}, e_k] - 3 [\psi_{j+2}, e_{k+1}] + 3 [\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] + \sigma_2 [\psi_{j+1}, e_k] - \sigma_2 [\psi_j, e_{k+1}] - \sigma_3 \{\psi_j, e_k\} = 0,$$
(17)

$$[\psi_{j+3}, f_k] - 3 [\psi_{j+2}, f_{k+1}] + 3 [\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}]$$

$$+\sigma_2 [\psi_{j+1}, f_k] - \sigma_2 [\psi_j, f_{k+1}] + \sigma_3 \{\psi_j, f_k\} = 0,$$
(18)

together with boundary conditions

$$[\psi_0, e_j] = 0, [\psi_1, e_j] = 0, [\psi_2, e_j] = 2e_j,$$
(19)

$$[\psi_0, f_j] = 0, [\psi_1, f_j] = 0, [\psi_2, f_j] = -2f_j,$$
(20)

and a generalization of Serre relations

$$\operatorname{Sym}_{(j_1, j_2, j_3)}[e_{j_1}, [e_{j_2}, e_{j_3+1}]] = 0, \tag{21}$$

$$\operatorname{Sym}_{(j_1, j_2, j_3)}[f_{j_1}, [f_{j_2}, f_{j_3+1}]] = 0,$$
(22)

where Sym is the complete symmetrization over all indicated indices which include 6 terms.

The notations σ_2 , σ_3 in the definition of affine Yangian are functions of three complex numbers h_1, h_2 and h_3 :

$$\sigma_1 = h_1 + h_2 + h_3 = 0, \tag{23}$$

$$\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3, \tag{24}$$

$$\sigma_3 = h_1 h_2 h_3. \tag{25}$$

The affine yangian \mathcal{Y} has a representation on the plane partitions. A plane partition π is a 2D Young diagram in the first quadrant of plane xOy filled with non-negative integers that form nonincreasing rows and columns [12, 13]. The number in the position (i, j) is denoted by $\pi_{i,j}$

$$\left(\begin{array}{ccc} \pi_{1,1} & \pi_{1,2} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \cdots \\ \cdots & \cdots & \cdots \end{array}\right)$$

The integers $\pi_{i,j}$ satisfy

$$\pi_{i,j} \ge \pi_{i+1,j}, \quad \pi_{i,j} \ge \pi_{i,j+1}, \quad \lim_{i \to \infty} \pi_{i,j} = \lim_{j \to \infty} \pi_{i,j} = 0$$

for all integers $i, j \ge 0$. Piling $\pi_{i,j}$ cubes over position (i, j) gives a 3D Young diagram. 3D Young diagrams arose naturally in the melting crystal model[13, 14]. We always identify 3D Young diagrams with plane partitions as explained above. For example, the 3D Young diagram \square can also be denoted by the plane partition (1, 1).

As in our paper [15], we use the following notations. For a 3D Young diagram π , the notation $\Box \in \pi^+$ means that this box is not in π and can be added to π . Here "can be added" means that when this box is added, it is still a 3D Young diagram. The notation

 $\Box \in \pi^-$ means that this box is in π and can be removed from π . Here "can be removed" means that when this box is removed, it is still a 3D Young diagram. For a box \Box , we let

$$h_{\Box} = h_1 y_{\Box} + h_2 x_{\Box} + h_3 z_{\Box}, \tag{26}$$

where $(x_{\Box}, y_{\Box}, z_{\Box})$ is the coordinate of box \Box in coordinate system O - xyz. Here we use the order $y_{\Box}, x_{\Box}, z_{\Box}$ to match that in paper [10].

Following [10, 11], we introduce the generating functions:

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}},$$

$$f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}},$$

$$\psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}},$$

(27)

where u is a parameter. Introduce

$$\psi_0(u) = \frac{u + \sigma_3 \psi_0}{u} \tag{28}$$

and

$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}.$$
(29)

For a 3D Young diagram π , define $\psi_{\pi}(u)$ by

$$\psi_{\pi}(u) = \psi_0(u) \prod_{\Box \in \pi} \varphi(u - h_{\Box}).$$
(30)

In the following, we recall the representation of the affine Yangian on 3D Young diagrams as in paper [10] by making a slight change. The representation of affine Yangian on 3D Young diagrams is given by

$$\psi(u)|\pi\rangle = \psi_{\pi}(u)|\pi\rangle,$$
(31)

$$e(u)|\pi\rangle = \sum_{\Box\in\pi^+} \frac{E(\pi\to\pi+\Box)}{u-h_{\Box}}|\pi+\Box\rangle, \qquad (32)$$

$$f(u)|\pi\rangle = \sum_{\square \in \pi^{-}} \frac{F(\pi \to \pi - \square)}{u - h_{\square}} |\pi - \square\rangle$$
(33)

where $|\pi\rangle$ means the state characterized by the 3D Young diagram π and the coefficients

$$E(\pi \to \pi + \Box) = -F(\pi + \Box \to \pi) = \sqrt{\frac{1}{\sigma_3} \operatorname{res}_{u \to h_\Box} \psi_\pi(u)}$$
(34)

Equations (32) and (33) mean generators e_j , f_j acting on the 3D Young diagram π by

$$e_j |\pi\rangle = \sum_{\Box \in \pi^+} h_{\Box}^j E(\pi \to \pi + \Box) |\pi + \Box\rangle,$$
(35)

$$f_j |\pi\rangle = \sum h_{\Box}^j F(\pi \to \pi - \Box) |\pi - \Box\rangle.$$
(36)

As in our paper [15], we use the following notations. 3D Young diagram π may have many ways to get by adding box, for example, there are two ways to get 3D Young diagram (2, 1), which are

$$(1) \to (1,1) \to (2,1),$$

 $(1) \to (2) \to (2,1),$

We denote the state corresponding to (2, 1) in the first equation of the two equations above by $|(2,1)\rangle_{h_1,h_3}$, and that in the second equation above by $|(2,1)\rangle_{h_3,h_1}$. We explain the subscripts: let $h_{\Box} = x_{\Box}h_2 + y_{\Box}h_1 + z_{\Box}h_3$ with $h_1 + h_2 + h_3 = 0$, we use h_{\Box} position to represent position $(x_{\Box}, y_{\Box}, z_{\Box})$ in coordinate system O - xyz. The notation " h_1, h_3 " means adding one box to \Box in h_1 -position first, then adding one box in h_3 position. Even though h_1 -position is not unique, for example, h_1 -position can be the positions $(1, 2, 1), (2, 3, 2), \cdots$ since $h_1 + h_2 + h_3 = 0$, but it is unique if we want to get a new 3D Young diagram after adding this box. Therefore, we can read the notation $|\begin{pmatrix} 1 & 1 \\ 1 \end{pmatrix}\rangle_{h_1,h_2}$, which means the 3D Young diagram $|\begin{pmatrix} 1 & 1 \\ 1 \end{pmatrix}\rangle$ is obtained from \Box by adding one box in h_1 -position first, then adding one box in h_2 -position. When there is no confusion, we will omit the subscripts.

The state corresponding to 3D Young diagram is related to its growth process, this is because we denote $E(\pi \rightarrow \pi + \Box)|\pi + \Box\rangle$ by $|\pi + \Box\rangle$ the the 3D Young diagram representation of affine Yangian of $\mathfrak{gl}(1)$. For example,

$$\begin{aligned} |(2,1)\rangle_{h_1,h_3} &= E((1) \to (1,1))E((1,1) \to (2,1))|(2,1)\rangle, \\ |(2,1)\rangle_{h_3,h_1} &= E((1) \to (2))E((2) \to (2,1))|(2,1)\rangle, \end{aligned}$$

then, $|(2,1)\rangle_{h_1,h_3} = \varphi(h_3 - h_1)|(2,1)\rangle_{h_3,h_1}$.

In the following subsection, we will discuss the Jack polynomials $\tilde{J}_{\lambda}(x)$, where we treat 2D Young diagrams as the special cases of 3D Young diagrams which have one layer in z-axis direction. The symmetric functions $\tilde{J}_{\lambda}(x)$ in the next subsection behave as the special case $\psi_0 = 1$, $h_1 = \sqrt{\alpha}$, $h_2 = -\sqrt{\alpha}^{-1}$ of the 3D Young diagrams in this subsection.

3.2 The Jack polynomials

The Jack polynomials we discussed here are denoted by \tilde{J}_{λ} or \tilde{J}_{λ} since it is well known that the notations J_{λ} are used in [4]. The Jack polynomials \tilde{J}_{λ} [16, 9] equal the Jack polynomials P_{λ}^{α} (defined in [4]) multiplied by a constant. We introduce the Jack polynomials \tilde{J}_{λ} since they behave the same as the Young diagrams in the last section. For example, it can be checked that $\langle \tilde{J}_{\lambda}, \tilde{J}_{\mu} \rangle$ in the following equals $\langle \lambda, \mu \rangle$ defined in the last section.

Let $p = (p_1, p_2, \cdots)$, the Jack polynomials \tilde{J}_{λ} are defined by [16]

$$\tilde{J}_{\lambda} := \frac{B_{\lambda}}{A_{\lambda}} P_{\lambda}^{\alpha} \tag{37}$$

where

$$A_{\lambda} = (\sqrt{\alpha})^{\lambda_{1}-\lambda_{2}-\dots-\lambda_{l}} \cdot \prod_{j=1}^{l-1} \prod_{i=0}^{\lambda_{l}-1} [(\lambda_{j}-i)\alpha + l - j] \cdot \prod_{i=1}^{l-1} \prod_{j=1}^{\lambda_{i+1}} [(\lambda_{i}-j)\alpha + 1]$$

$$\cdot \prod_{k=3}^{l} \prod_{j=0}^{\lambda_{k-1}-\lambda_{k}-1} \prod_{i=1}^{k-2} [(\lambda_{i}-\lambda_{k}-j)\alpha + k - i - 1], \qquad (38)$$
$$B_{\lambda} = \prod_{i=1}^{l-1} \prod_{j=0}^{\lambda_{i+1}-1} (\lambda_{i}-j) \cdot \prod_{i=1}^{l-1} \prod_{j=1}^{\lambda_{l}} [(\lambda_{i}-j)\alpha + l - i + 1]$$

$$l = \lambda_{k-1}-\lambda_{k} k-2$$

$$\cdot \prod_{k=3}^{l} \prod_{j=1}^{\lambda_{k-1}-\lambda_k} \prod_{i=1}^{k-2} \left[(\lambda_i - \lambda_k - j) \alpha + k - i \right].$$

$$(39)$$

for 2D Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

We change p_n in [4] to $\sqrt{\alpha}p_n$, then p_n in the following satisfies $\langle p_n, p_n \rangle = n$. Then Jack polynomials $\tilde{J}_{(n)}$ defined above satisfy

$$\exp\left(\sum_{n\geq 1}\frac{p_n}{n\sqrt{\alpha}}z^n\right) = \sum_{n\geq 0}\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)}\rangle}\frac{1}{\sqrt{\alpha}^n}\tilde{J}_{(n)}z^n.$$
(40)

Define the operator $\hat{\tilde{J}}_{(n)}$ by

$$\exp\left(\sum_{n\geq 1}\frac{\mathrm{ad}_{e_1}^{n-1}e_0}{n!\sqrt{\alpha}}z^n\right) = \sum_{n\geq 0}\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)}\rangle}\frac{1}{\sqrt{\alpha}^n}\hat{J}_{(n)}z^n.$$
(41)

The Pieri formula $\tilde{J}_{(n)}\tilde{J}_{\lambda}$ is defined by

$$\tilde{J}_{(n)}\tilde{J}_{\lambda} := \hat{\tilde{J}}_{(n)} \cdot \tilde{J}_{\lambda}.$$
(42)

Note that the actions of the generators e_k , f_k , ψ_k of affine Yangian of $\mathfrak{gl}(1)$ on \tilde{J}_{λ} are the same with that on λ . The expressions of \tilde{J}_{λ} for all Young diagrams λ can be determined by (41) and (42). Note that \tilde{J}_{λ} can not be expressed as the determinant of $\tilde{J}_{(n)}$, while Schur functions S_{λ} can be expressed as the determinant of $S_{(n)}$. Then we define $\tilde{\tilde{J}}_{\lambda}$ by

$$\tilde{\tilde{J}}_{\lambda} := \det\left(\frac{1}{\langle \tilde{J}_{\lambda_j - i + j}, \tilde{J}_{\lambda_j - i + j} \rangle \sqrt{\alpha}^{\lambda_j - i + j}} \tilde{J}_{\lambda_j - i + j}\right)_{1 \le i, j \le k},\tag{43}$$

which is slightly different from that in [9]. The transition matrix $M = (M_{\lambda\mu})$ from the set $\{\tilde{J}_{\lambda}\}$ to the set $\tilde{\tilde{J}}_{\lambda}$ is upper triangular[9], with the elements $M_{\lambda\lambda}$ in the diagonal equal

$$\frac{1}{\sqrt{\alpha^{|\lambda|}}} \frac{1}{\langle \tilde{J}_{\lambda}, \tilde{J}_{\lambda} \rangle} \prod_{j=2}^{k} \frac{(j\alpha)^{\lambda_j}}{(1+(j-1)\alpha)^{\lambda_j}}.$$
(44)

Introduce Bosons $a_n, n \in \mathbb{Z}, n \neq 0$ with the relations

$$[a_n, a_m] = n\delta_{n+m,0}.\tag{45}$$

In fact, for n > 0,[17]

$$a_{-n} = \frac{1}{(n-1)!} \operatorname{ad}_{e_1}^{n-1} e_0, \quad a_n = -\frac{1}{(n-1)!} \operatorname{ad}_{f_1}^{n-1} f_0.$$
(46)

On Jack polynomials \tilde{J}_{λ} , the Bosons a_n can be represented as

$$a_{-n} = p_n, \quad a_n = n\partial_{p_n}.$$
(47)

In the following of this subsection, we discuss the properties related to the Jack polynomials $\tilde{J}_{\lambda}(p)$ and $\tilde{\tilde{J}}_{\lambda}(p)$. Let $\partial_p = (\partial_{p_1}, 2\partial_{p_2}, 3\partial_{p_3}, \cdots)$. From (40), we have

$$\exp\left(\sum_{n=1}^{\infty} \frac{\partial_{p_n}}{\sqrt{\alpha}} z^n\right) = \sum_{n\geq 0} \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^n} \tilde{J}_n(\partial_p) z^n.$$
(48)

From

$$\exp\left(\sum_{n=1}^{\infty} \frac{\partial_{p_n}}{\sqrt{\alpha}} z^n\right) \exp\left(\sum_{n\geq 1} \frac{p_n}{n\sqrt{\alpha}} w^n\right)$$
$$\frac{1}{(1-zw)^{\frac{1}{\alpha}}} \exp\left(\sum_{n\geq 1} \frac{p_n}{n\sqrt{\alpha}} w^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{\partial_{p_n}}{\sqrt{\alpha}} z^n\right),\tag{49}$$

we get that the operators $\tilde{J}_{(n)}(\partial_p)$ and $\tilde{J}_{(m)}(p)$ satisfy

=

$$\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^{n}} \tilde{J}_{(n)}(\partial_{p}) \frac{1}{\langle \tilde{J}_{(m)}, \tilde{J}_{(m)} \rangle \sqrt{\alpha}^{m}} \tilde{J}_{(m)}(p) \tag{50}$$

$$= \sum_{k \geq 0} \binom{-1/\alpha}{k} (-1)^{k} \frac{1}{\langle \tilde{J}_{(m-k)}, \tilde{J}_{(m-k)} \rangle \sqrt{\alpha}^{m-k}} \tilde{J}_{(m-k)}(p) \frac{1}{\langle \tilde{J}_{(n-k)}, \tilde{J}_{(n-k)} \rangle \sqrt{\alpha}^{n-k}} \tilde{J}_{(n-k)}(\partial_{p}),$$

and the operators $\tilde{J}_{(n)}(\partial_p)$ acting on the polynomials $\tilde{J}_{(m)}(p)$ equals

$$\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^{n}} \tilde{J}_{(n)}(\partial_{p}) \frac{1}{\langle \tilde{J}_{(m)}, \tilde{J}_{(m)} \rangle \sqrt{\alpha}^{m}} \tilde{J}_{(m)}(p)
= \left(\frac{-1/\alpha}{n} \right) (-1)^{n} \frac{1}{\langle \tilde{J}_{(m-n)}, \tilde{J}_{(m-n)} \rangle \sqrt{\alpha}^{m-n}} \tilde{J}_{(m-n)}(p),$$
(51)

where we let $\tilde{J}_{(n)} = 0$ unless $n \ge 0$. The polynomials $\tilde{J}_{1^n}(p)$ satisfy

$$\exp\left(-\sum_{n\geq 1}\frac{p_n}{n}\sqrt{\alpha}z^n\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\alpha}^n}{\langle \tilde{J}_{(1^n)}, \tilde{J}_{(1^n)}\rangle}\tilde{J}_{(1^n)}(p)z^n.$$
(52)

Then

$$\exp\left(-\sum_{n\geq 1}\partial_{p_n}\sqrt{\alpha}z^n\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{\alpha}^n}{\langle \tilde{J}_{(1^n)}, \tilde{J}_{(1^n)}\rangle} \tilde{J}_{(1^n)}(\partial p)z^n.$$
(53)

From

$$\exp\left(-\sum_{n\geq 1}\partial_{p_n}\sqrt{\alpha}z^n\right)\exp\left(-\sum_{n\geq 1}\frac{p_n}{n}\sqrt{\alpha}w^n\right)$$
$$= \frac{1}{(1-zw)^{\alpha}}\exp\left(-\sum_{n\geq 1}\frac{p_n}{n}\sqrt{\alpha}w^n\right)\exp\left(-\sum_{n\geq 1}\partial_{p_n}\sqrt{\alpha}z^n\right),$$
(54)

we obtain that the operators $\tilde{J}_{1^n}(\partial_p)$ and $\tilde{J}_{1^m}(p)$ satisfy

$$\frac{(-1)^n \sqrt{\alpha}^n}{\langle \tilde{J}_{(1^n)}, \tilde{J}_{(1^n)} \rangle} \tilde{J}_{(1^n)}(\partial_p) \frac{(-1)^m \sqrt{\alpha}^m}{\langle \tilde{J}_{(1^m)}, \tilde{J}_{(1^m)} \rangle} \tilde{J}_{(1^m)}(p) \tag{55}$$

$$= \sum_{k \ge 0} \binom{-\alpha}{k} (-1)^k \frac{(-1)^{m-k} \sqrt{\alpha}^{m-k}}{\langle \tilde{J}_{(m-k)}, \tilde{J}_{(m-k)} \rangle} \tilde{J}_{(m-k)}(p) \frac{(-1)^{n-k} \sqrt{\alpha}^{n-k}}{\langle \tilde{J}_{(1^{n-k})}, \tilde{J}_{(1^{n-k})} \rangle} \tilde{J}_{(1^{n-k})}(\partial_p),$$

and the operators $\tilde{J}_{(1^n)}(\partial_p)$ acting on the polynomials $\tilde{J}_{(1^m)}(p)$ equals

$$\frac{\sqrt{\alpha}^{n}}{\langle \tilde{J}_{(1^{n})}, \tilde{J}_{(1^{n})} \rangle} \tilde{J}_{(1^{n})}(\partial_{p}) \frac{\sqrt{\alpha}^{m}}{\langle \tilde{J}_{(1^{m})}, \tilde{J}_{(1^{m})} \rangle \sqrt{\alpha}^{m}} \tilde{J}_{(m)}(p)$$

$$= \left(\begin{array}{c} -\alpha \\ n \end{array} \right) (-1)^{n} \frac{\sqrt{\alpha}^{m-n}}{\langle \tilde{J}_{(1^{m-n})}, \tilde{J}_{(1^{m-n})} \rangle} \tilde{J}_{(1^{m-n})}(p), \qquad (56)$$

where we let $\tilde{J}_{(1^n)} = 0$ unless $n \ge 0$. From

$$\exp\left(-\sum_{n\geq 1}\partial_{p_n}\sqrt{\alpha}z^n\right)\exp\left(\sum_{n\geq 1}\frac{p_n}{\sqrt{\alpha}}w^n\right) \tag{57}$$

$$= (1-zw) \exp\left(\sum_{n\geq 1} \frac{p_n}{\sqrt{\alpha}} w^n\right) \exp\left(-\sum_{n\geq 1} \partial_{p_n} \sqrt{\alpha} z^n\right),$$
(58)

we obtain the operators $\tilde{J}_{(1^n)}(\partial_p)$ and $\tilde{J}_{(m)}$ satisfy

$$\frac{1}{\langle \tilde{J}_{(1^{n})}, \tilde{J}_{(1^{n})} \rangle} \tilde{J}_{(1^{n})}(\partial_{p}) \frac{1}{\langle \tilde{J}_{(m)}, \tilde{J}_{(m)} \rangle} \tilde{J}_{(m)}(p)
= \frac{1}{\langle \tilde{J}_{(m)}, \tilde{J}_{(m)} \rangle} \tilde{J}_{(m)}(p) \frac{1}{\langle \tilde{J}_{(1^{n})}, \tilde{J}_{(1^{n})} \rangle} \tilde{J}_{(1^{n})}(\partial_{p})
+ \frac{1}{\langle \tilde{J}_{(m-1)}, \tilde{J}_{(m-1)} \rangle} \tilde{J}_{(m-1)}(p) \frac{1}{\langle \tilde{J}_{(1^{n-1})}, \tilde{J}_{(1^{n-1})} \rangle} \tilde{J}_{(1^{n-1})}(\partial_{p}).$$
(59)

From

$$\exp\left(-\sum_{n\geq 1}\partial_{p_n}\sqrt{\alpha}z^n\right)\exp\left(-\sum_{n\geq 1}\frac{p_n}{\sqrt{\alpha}}w^n\right) \tag{60}$$

$$= \frac{1}{1-zw} \exp\left(\sum_{n\geq 1} \frac{p_n}{\sqrt{\alpha}} w^n\right) \exp\left(-\sum_{n\geq 1} \partial_{p_n} \sqrt{\alpha} z^n\right),\tag{61}$$

we obtain the operators $\tilde{J}_{(1^n)}(\partial_p)$ and $\tilde{J}_{(m)}(p)$ satisfy

$$\frac{(-1)^n \sqrt{\alpha}^n}{\langle \tilde{J}_{(1^n)}, \tilde{J}_{(1^n)} \rangle} \tilde{J}_{(1^n)}(\partial_p) \frac{1}{\langle \tilde{J}_{(m)}, \tilde{J}_{(m)} \rangle \sqrt{\alpha}^m} \tilde{J}_{(m)}(p)$$

$$= \sum_{k \ge 0} \frac{1}{\langle \tilde{J}_{(m-k)}, \tilde{J}_{(m-k)} \rangle \sqrt{\alpha}^{m-k}} \tilde{J}_{(m-k)}(p) \frac{(-1)^{n-k} \sqrt{\alpha}^{n-k}}{\langle \tilde{J}_{(1^{n-k})}, \tilde{J}_{(1^{n-k})} \rangle} \tilde{J}_{(1^{n-k})}(\partial_p). \quad (62)$$

Introduce the vertex operators

$$X_{+}(z) = \sum_{n \in \mathbb{Z}} X_{n}^{+} z^{n} = \exp\left(\sum_{n \ge 1} \frac{p_{n}}{n\sqrt{\alpha}} z^{n}\right) \exp\left(-\sum_{n \ge 1} \partial_{p_{n}} \sqrt{\alpha} z^{-n}\right), \quad (63)$$

$$X_{-}(z) = \sum_{n \in \mathbb{Z}} X_{n}^{-} z^{n} = \exp\left(-\sum_{n \ge 1} \frac{p_{n}}{n\sqrt{\alpha}} z^{n}\right) \exp\left(\sum_{n \ge 1} \partial_{p_{n}} \sqrt{\alpha} z^{-n}\right).$$
(64)

The Jack polynomials $\tilde{J}_{\lambda}(p)$ have the vertex operator realization[9]

$$\tilde{\tilde{J}}_{\lambda} = X_{\lambda_1}^+ X_{\lambda_2}^+ \cdots X_{\lambda_k}^+ \cdot 1$$
(65)

for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$.

From (40), we know that

$$\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle} \frac{1}{\sqrt{\alpha}^n} \tilde{J}_{(n)} = S_{(n)} \left(\frac{p}{\sqrt{\alpha}}\right).$$
(66)

Then from the definition of $\tilde{\tilde{J}}_{\lambda}(p)$, we have

$$\tilde{\tilde{J}}_{\lambda}(p) = S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right).$$
(67)

In [9], an integrable hierarchy is defined to be the bilinear relations:

$$\sum_{m+n=-1} X_m^- \tau \otimes X_n^+ \tau = 0, \tag{68}$$

where $\tau = \tau(x)$ is an unknown function. In the next section, we will show that this hierarchy is exactly the \hbar -KP hierarchy defined in the last section.

The set $\{\tilde{J}_{\lambda}\}$ is an orthogonal basis, but $\{\tilde{J}_{\lambda}\}$ is not. From

$$\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle = \prod_{k=1}^{n} \frac{k}{1 + (k-1)\alpha},$$

we have

$$\langle S_{(n)}\left(\frac{p}{\sqrt{\alpha}}\right), S_{(n)}\left(\frac{p}{\sqrt{\alpha}}\right) \rangle = \prod_{k=1}^{n} \frac{1 + (k-1)\alpha}{k\alpha}.$$
 (69)

From

$$S_{(n,1)}\left(\frac{p}{\sqrt{\alpha}}\right) = \frac{1}{\sqrt{\alpha}^{n-1}} \frac{1}{\langle J_{(n,1)}, J_{(n,1)} \rangle} \frac{2}{1+\alpha} \tilde{J}_{(n,1)} + \frac{n(1-\alpha)}{1+n\alpha} S_{(n+1)}\left(\frac{p}{\sqrt{\alpha}}\right), (70)$$

where (n, 1) is the Young diagram obtained (1, 1) by adding (n - 1) box, we have

$$\langle S_{(n+1)}\left(\frac{p}{\sqrt{\alpha}}\right), S_{(n,1)}\left(\frac{p}{\sqrt{\alpha}}\right) \rangle = \frac{n(1-\alpha)}{1+n\alpha} \prod_{k=1}^{n+1} \frac{1+(k-1)\alpha}{k\alpha},\tag{71}$$

which shows that $S_{(n+1)}\left(\frac{p}{\sqrt{\alpha}}\right)$ and $S_{(n,1)}\left(\frac{p}{\sqrt{\alpha}}\right)$ is not orthogonal. The set $\{S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right)\}$ is still a basis since the transition matrix from the set $\{\tilde{J}_{\lambda}\}$ to $\{\tilde{J}_{\lambda}\}$ is upper triangular in the sense of Young diagram's reverse lexicographical order. For example,

$$\begin{split} S_{(n,1)}\left(\frac{p}{\sqrt{\alpha}}\right) &= \frac{1}{\langle \tilde{J}_{(1)}, \tilde{J}_{(1)} \rangle \sqrt{\alpha}} \tilde{J}_{(1)} \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^{n}} \tilde{J}_{(n)} - \frac{1}{\langle \tilde{J}_{(n+1)}, \tilde{J}_{(n+1)} \rangle \sqrt{\alpha}^{n+1}} \tilde{J}_{(n+1)} \\ &= \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^{n+1}} \tilde{J}_{(n,1)} + \frac{1}{\sqrt{\alpha}^{n+1}} \left(\frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)}} - \frac{1}{\langle \tilde{J}_{(n+1)}, \tilde{J}_{(n+1)} \rangle} \right) \tilde{J}_{n+1}, \end{split}$$

and

$$S_{(n,2)}\left(\frac{p}{\sqrt{\alpha}}\right) = \frac{1}{\langle \tilde{J}_{(2)}, \tilde{J}_{(2)} \rangle \sqrt{\alpha}^2} \tilde{J}_{(2)} \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^n} \tilde{J}_{(n)} - \frac{1}{\langle \tilde{J}_{(1)}, \tilde{J}_{(1)} \rangle \sqrt{\alpha}} \tilde{J}_{(1)} \frac{1}{\langle \tilde{J}_{(n+1)}, \tilde{J}_{(n+1)} \rangle \sqrt{\alpha}^{n+1}} \tilde{J}_{(n+1)}.$$

We show the actions of the Bosons on $S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right)$.

$$a_{-1}S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right) = p_{1}S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right) = \sqrt{\alpha}\sum_{\Box\in\lambda^{+}}S_{\lambda+\Box}\left(\frac{p}{\sqrt{\alpha}}\right),\tag{72}$$

$$a_{-2}S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right) = \sqrt{\alpha}\sum_{\square \models \lambda^{+}} S_{\lambda+\square}\left(\frac{p}{\sqrt{\alpha}}\right) - \sqrt{\alpha}\sum_{\square \models \lambda^{+}} S_{\lambda+\square}\left(\frac{p}{\sqrt{\alpha}}\right).$$
(73)

In fact, for n > 0, the Bosons a_{-n} acting on $S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right)$ here equals a_{-n} acting on $S_{\lambda}(p)$ in the KP hierarchy [4] multiplied by $\sqrt{\alpha}$, while the Bosons a_n acting on $S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right)$ here equals a_n acting on $S_{\lambda}(p)$ in the KP hierarchy [4] multiplied by $1/\sqrt{\alpha}$.

4 The Hirota equation and vertex operators

In this section, we will show that the \hbar -dependent KP hierarchy is exactly the integrable hierarchy defined in (68). The calculation is similar to that of KP hierarchy in [4].

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. The Hirota derivative $D_{x_k}^j$ is defined to be[4]

$$D_{x_k}^j f(x) \cdot g(x) = \partial_{y_k}^j f(x+y) g(x-y)|_{y=0}$$
(74)

The \hbar -KP equation (12) becomes the following Hirota equation

$$\hbar^2 D_{x_1}^4 \tau(x) \cdot \tau(x) - 4 D_{x_1} D_{x_3} \tau(x) \cdot \tau(x) + 3 D_{x_2}^2 \tau(x) \cdot \tau(x) = 0.$$
(75)

Set

$$P(k_1, k_2, k_3) = \hbar^2 k_1^4 + 3k_2^2 - 4k_1k_3.$$

The solutions of the equation $P(k_1, k_2, k_3) = 0$ are

$$(k_1, k_2, k_3) = (\frac{1}{\hbar}p - \frac{1}{\hbar}q, \frac{1}{\hbar}p^2 - \frac{1}{\hbar}q^2, \frac{1}{\hbar}p^3 - \frac{1}{\hbar}q^3)$$

for any z_1, z_2 . Take two solutions

$$(k_1, k_2, k_3) = (\frac{1}{\hbar}p_1 - \frac{1}{\hbar}q_1, \frac{1}{\hbar}p_1^2 - \frac{1}{\hbar}q_1^2, \frac{1}{\hbar}p_1^3 - \frac{1}{\hbar}q_1^3),$$

$$(k_1', k_2', k_3') = (\frac{1}{\hbar}p_2 - \frac{1}{\hbar}q_2, \frac{1}{\hbar}p_2^2 - \frac{1}{\hbar}q_2^2, \frac{1}{\hbar}p_2^3 - \frac{1}{\hbar}q_2^3),$$

we have

$$-\frac{P(k_1 - k_1', k_2 - k_2', k_3 - k_3')}{P(k_1 + k_1', k_2 + k_2', k_3 + k_3')} = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}$$

Suppose

$$\xi_{i} = \sum_{j=1}^{\infty} (p_{i}^{j} - q_{i}^{j}) \frac{x_{j}}{\hbar},$$

$$a_{ii'} = \frac{(p_{i} - p_{i'})(q_{i} - q_{i'})}{(p_{i} - q_{i'})(q_{i} - p_{i'})},$$

then for $I = \{1, 2, \cdots, n\},\$

$$\tau = \sum_{J \subset I} \left(\prod_{i \in J} c_i \right) \left(\prod_{i, i' \in J, i < i'} a_{ii'} \right) \exp\left(\sum_{i \in J} \xi_i \right)$$
(76)

gives the *n*-soliton solution of \hbar -KP hierarchy. Introduce the vertex operator

$$X(p,q) = \exp\left(\sum_{j=1}^{\infty} (p^j - q^j) \frac{x_j}{\hbar}\right) \exp\left(-\sum_{j=1}^{\infty} \frac{\hbar}{j} (p^{-j} - q^{-j}) \partial_{x_j}\right),\tag{77}$$

then the *n*-soliton solution τ above can be written as

$$\tau = e^{c_1 X(p_1, q_1)} \cdots e^{c_n X(p_n, q_n)} \cdot 1.$$
(78)

The tau functions satisfy the following bilinear identity. For any x and x', let

$$\xi = \sum_{j=1}^{\infty} \frac{x_j}{\hbar} z^j, \quad \xi' = \sum_{j=1}^{\infty} \frac{x'_j}{\hbar} z^j.$$

the bilinear identity holds:

$$\oint \frac{dz}{2\pi\sqrt{-1}} e^{\xi - \xi'} \tau(x - \hbar[z^{-1}]) \tau(x' + \hbar[z^{-1}]) = 0.$$
(79)

Introduce

$$w^{*}(x,z) = \frac{\tau(x+\hbar[z^{-1}])}{\tau(x)}e^{-\sum_{j=1}^{\infty}\frac{x_{j}}{\hbar}z^{j}},$$
(80)

it has the following form

$$w^*(x,z) = e^{-\sum_{j=1}^{\infty} \frac{x_j}{\hbar} z^j} \left(1 + \sum_{j=1}^{\infty} \frac{w_j^*}{z^j} \right).$$

Then the bilinear identity (79) becomes

$$\oint \frac{dz}{2\pi\sqrt{-1}}w(x,z)w^{*}(x',z) = 0.$$
(81)

From this relation, the linear system (7) can be obtained. Let

$$Q = \hbar \partial_{x_j} - (L^j)_+, \quad \tilde{w}(x,k) = Qw(x,k).$$

Then $\tilde{w}(x,k)$ has the following form

$$\tilde{w}(x,k) = e^{\sum_{j=1}^{\infty} \frac{x_j}{\hbar} z^j} \left(\sum_{j=1}^{\infty} \frac{\tilde{w}_j}{z^j} \right)$$

and satisfy

$$\oint \frac{dz}{2\pi\sqrt{-1}}\tilde{w}(x,z)w^*(x',z) = 0,$$

which show $\tilde{w}_j = 0$ for $j = 1, 2, \cdots$. Therefore, Qw(x, z) = 0, that is (7) is obtained. Let $x_n = p_n/n$ and $\alpha = \hbar^2$, we see that the bilinear identity (79) is the same with (68). Then the polynomials $\tilde{J}_{\lambda}(p) = S_{\lambda}\left(\frac{p}{\sqrt{\alpha}}\right)$ are all the solutions of the \hbar -KP hierarchy.

We have calculated that the first equation in the \hbar -KP hierarchy has the form[9]

$$\frac{4}{(1+\alpha)^{2}} \frac{1}{\langle \tilde{J}_{(2,2)}, \tilde{J}_{(2,2)} \rangle} \left(\tilde{J}_{(2,2)}(\partial x)\tau \right) \cdot \tau \\
+ \frac{6(\alpha-1)}{(2+\alpha)(3+\alpha)} \frac{1}{\langle \tilde{J}_{(2,1,1)}, \tilde{J}_{(2,1,1)} \rangle} \left(\tilde{J}_{(2,1,1)}(\partial x)\tau \right) \cdot \tau \\
+ \frac{2(\alpha-1)\alpha}{(2+\alpha)(3+\alpha)} \frac{1}{\langle \tilde{J}_{(1^{4})}, \tilde{J}_{(1^{4})} \rangle} \left(\tilde{J}_{(1^{4})}(\partial x)\tau \right) \cdot \tau \\
- \frac{2(\alpha-1)}{2+\alpha} \frac{1}{\langle \tilde{J}_{(1^{3})}, \tilde{J}_{(1^{3})} \rangle} \left(\tilde{J}_{(1^{3})}(\partial x)\tau \right) \cdot \frac{1}{\langle \tilde{J}_{(1)}, \tilde{J}_{(1)} \rangle} \left(\tilde{J}_{(1)}(\partial x)\tau \right) \\
- \frac{2}{1+\alpha} \frac{1}{\langle \tilde{J}_{(2,1)}, \tilde{J}_{(2,1)} \rangle} \left(\tilde{J}_{(2,1)}(\partial x)\tau \right) \cdot \frac{1}{\langle \tilde{J}_{(1)}, \tilde{J}_{(1)} \rangle} \left(\tilde{J}_{(1)}(\partial x)\tau \right) \\
+ \frac{2}{(1+\alpha)} \frac{1}{\langle \tilde{J}_{(2)}, \tilde{J}_{(2)} \rangle} \left(\tilde{J}_{(2)}(\partial x)\tau \right) \cdot \frac{1}{\langle \tilde{J}_{(1^{2})}, \tilde{J}_{(1^{2})} \rangle} \left(\tilde{J}_{(1^{2})}(\partial x)\tau \right) \\
+ \frac{(\alpha-1)}{1+\alpha} \frac{1}{\langle \tilde{J}_{(1^{2})}, \tilde{J}_{(1^{2})} \rangle} \left(\tilde{J}_{(1^{2})}(\partial x)\tau \right) \cdot \frac{1}{\langle \tilde{J}_{(1^{2})}, \tilde{J}_{(1^{2})} \rangle} \left(\tilde{J}_{(1^{2})}(\partial x)\tau \right) = 0. \quad (82)$$

If we substitute the expressions of Jack polynomials J_{λ} , this equation becomes (12) and (75). The equation (82) can also be written as

$$S_{(2,2)}(\hbar\partial x)\tau \cdot \tau - S_{(2,1)}(\hbar\partial x)\tau \cdot S_{(1)}(\hbar\partial x)\tau + S_{(2)}(\hbar\partial x)\tau \cdot S_{(1^2)}(\hbar\partial x)\tau = 0.$$
(83)

We write the tau functions τ of the forms

$$\tau = \sum_{\lambda} c_{\lambda}' \tilde{J}_{\lambda}(p) = \sum_{\lambda} c_{\lambda} S_{\lambda} \left(\frac{p}{\sqrt{\alpha}}\right).$$
(84)

If c_{λ} or c'_{λ} satisfy some relations (the Plücker relations), the tau functions τ are the solutions of the \hbar -KP hierarchy. For example,

$$c_{(2,2)}c_{\emptyset} - c_{(2,1)}c_{(1)} + c_{(2)}c_{(1,1)} = 0.$$
(85)

In fact, the coefficients c_{λ} satisfy the classical Plücker relations since $S_{\lambda}(\hbar \partial_p)S_{\mu}\left(\frac{p}{\hbar}\right)|_{p=0} = \delta_{\lambda\mu}$, which is the same with that in [18]. For c'_{λ} , since

$$\tilde{J}_{\lambda}(\partial_p)\tilde{J}_{\mu}(p)|_{p=0} = \delta_{\lambda\mu}\langle \tilde{J}_{\lambda}, \tilde{J}_{\lambda}\rangle,$$

we obtain the relations of c'_{λ} . For example,

$$\frac{4}{(1+\alpha)^2}c'_{(2,2)}c'_{\emptyset} + \frac{6(\alpha-1)}{(2+\alpha)(3+\alpha)}c'_{(2,1,1)}c'_{\emptyset} + \frac{2(\alpha-1)\alpha}{(2+\alpha)(3+\alpha)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)}{2+\alpha}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)}{(2+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)(3+\alpha)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)(3+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^3)}c'_{(1^4)}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^4)}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^4)}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^4)}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^4)}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{(1^4)}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset}c'_{\emptyset} - \frac{2(\alpha-1)\alpha}{(2+\alpha)}c'_{\emptyset}c'_$$

The set of coefficients $\{c_{\lambda}\}$ can be represented linearly by the set $\{c'_{\lambda}\}$ and vice versa. In fact, $\{c_{\lambda}\}$ satisfies the classical Plücker relations if and only if $\{c'_{\lambda}\}$ satisfies its Plücker relations. For example, $\{c_{\lambda}\}$ satisfies (85) if and only if $\{c'_{\lambda}\}$ satisfies (86).

5 The Boson-Fermion correspondence for \hbar -KP hierarchy

Let $p_n = nx_n$. For Schur functions $S_{\lambda}(x)$ and Jack polynomials $J_{\lambda}(x)$, we have

$$e^{\sum_{n=1}^{\infty} x_n z^n} = \sum_{n>0} S_{(n)}(x) z^n,$$
(87)

$$e^{-\sum_{n=1}^{\infty} x_n z^n} = \sum_{n\geq 0}^{-} (-1)^n S_{(1^n)}(x) z^n.$$
(88)

Then we have

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{\sqrt{\alpha}} z^n\right) = \sum_{n\geq 0} S_{(n)}\left(\frac{x}{\sqrt{\alpha}}\right) z^n = \frac{1}{\langle \tilde{J}_{(n)}, \tilde{J}_{(n)} \rangle \sqrt{\alpha}^n} \tilde{J}_{(n)}(x) z^n, \quad (89)$$
$$\exp\left(-\sum_{n\geq 0} x_n \sqrt{\alpha} z^n\right) = \sum_{n\geq 0} (-1)^n S_{1^n}(\sqrt{\alpha} x) z^n = \sum_{n\geq 0} \frac{(-1)^n \sqrt{\alpha}^n}{\langle z^n \rangle} \tilde{J}_{(1^n)}(p) z^n. (90)$$

$$\exp\left(-\sum_{n\geq 1}x_n\sqrt{\alpha}z^n\right) = \sum_{n\geq 0}(-1)^n S_{1^n}(\sqrt{\alpha}x)z^n = \sum_{n=0}\frac{(-1)^n\sqrt{\alpha}}{\langle \tilde{J}_{(1^n)}, \tilde{J}_{(1^n)}\rangle}\tilde{J}_{(1^n)}$$

、

The Cauchy formula is

$$\exp\left(\sum_{n\geq 1} \frac{p_n p'_n}{n}\right) = \exp(\sum_{n\geq 1} n x_n x'_n)$$
$$= \sum_{\lambda} S_{\lambda}(x) S_{\lambda}(x') = \sum_{\lambda} \frac{1}{\langle \tilde{J}_{\lambda}, \tilde{J}_{\lambda} \rangle} \tilde{J}_{\lambda}(x) \tilde{J}_{\lambda}(x'). \tag{91}$$

,

Note that $\tilde{J}_{\lambda}(x)$ is not equal to $S_{\lambda}(x)$ (or multiplied by a constant). The Fermions ψ_j and ψ_j^* are defined as usual. For $j \in \mathbb{Z} + \frac{1}{2}$, ψ_j and ψ_j^* satisfy[2]

$$[\psi_i, \psi_j]_+ = 0, \ [\psi_i^*, \psi_j^*] = 0, \ [\psi_i, \psi_j^*] = \delta_{i+j,0},$$
(92)

where $[A, B]_+ = AB - BA$. Particularly,

$$\psi_j^2 = 0, \quad \psi_j^{*2} = 0.$$

The Fermionic Fock space \mathcal{F} is the space of Maya diagrams[2]. A Maya diagram can be discribed as an increasing sequence of half-integers

$$|\mathbf{u}\rangle = |u_1, u_2, \cdots \rangle$$
, with $u_1 < u_2 < \cdots$,

and $u_{j+1} = u_j + 1$ for all sufficiently large j.

The actions of Fermions ψ_j, ψ_j^* on Maya diagrams are determined by

$$\psi_j |\mathbf{u}\rangle = \begin{cases} (-1)^{i-1} |\cdots, u_{i-1}, u_{i+1}, \cdots \rangle & \text{if } u_i = -j \text{ for some } i, \\ 0 & \text{otherwise,} \end{cases}$$
(93)

$$\psi_j^* |\mathbf{u}\rangle = \begin{cases} (-1)^i | \cdots, u_i, j, u_{i+1}, \cdots \rangle & \text{if } u_i < j < u_{i+1} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$
(94)

The generating functions of Fermions are

$$\psi(z) = \sum_{j \in \mathbb{Z} + 1/2} \psi_j z^{-j-1/2}, \ \psi^*(z) = \sum_{j \in \mathbb{Z} + 1/2} \psi_j^* z^{-j-1/2}.$$

The normal order is defined as usual. For Maya diagrams $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$, the pair $\langle \mathbf{v}|\mathbf{u}\rangle$ is defined by the formula

$$\langle \mathbf{v} | \mathbf{u} \rangle = \delta_{v_1 + u_1, 0} \delta_{v_2 + u_2, 0} \cdots$$

Let

$$H_n = \sum_{j \in \mathbb{Z} + 1/2} : \psi_{-j} \psi_{j+n}^* : .$$
(95)

It satisfy[2]

$$[H_n, \psi_j] = \psi_{n+j}, \ [H_n, \psi_j^*] = -\psi_{n+j}^*,$$

and

$$[H_n, H_m] = n\delta_{n+m,0}$$

Then we show the Boson-Fermion correspondence in the $\hbar\text{-KP}$ hierarchy. Define

$$H(x) = \sum_{n \ge 1} \frac{x_n}{\sqrt{\alpha}} H_n.$$
(96)

For any element $|\mathbf{u}\rangle \in \mathcal{F}$, define the map

$$\Phi(|u\rangle) = \sum_{l \in \mathbb{Z}} z^l \langle l | e^{H(x)} | \mathbf{u} \rangle.$$
(97)

Then $\Phi(z)$ is in the space $\mathbb{C}(\alpha)[z, z^{-1}, x_1, x_2, \cdots]$, and the correspondence Φ is an isomorphism of the vector spaces \mathcal{F} over $\mathbb{C}(\alpha)$ and $\mathbb{C}(\alpha)[z, z^{-1}, t_1, t_2, \cdots]$. Moreover, for n > 0,

$$\Phi(H_n|\mathbf{u}\rangle) = \sqrt{\alpha}\partial_{x_n}\Phi(|\mathbf{u}\rangle), \quad \Phi(H_{-n}|\mathbf{u}\rangle) = n\frac{x_n}{\sqrt{\alpha}}\Phi(|\mathbf{u}\rangle). \tag{98}$$

From the commutation relations above, we get

$$[H(x),\psi(z)] = \left(\sum_{n\geq 1}\frac{x_n}{\sqrt{\alpha}}z^n\right)\psi(z),\tag{99}$$

and

$$[H(x),\psi^*(z)] = \left(-\sum_{n\geq 1}\frac{x_n}{\sqrt{\alpha}}z^n\right)\psi^*(z),\tag{100}$$

which show

$$e^{H(x)}\psi(z)e^{-H(x)} = e^{\sum_{n\geq 1}\frac{x_n}{\sqrt{\alpha}}z^n}\psi(z),$$
(101)

$$e^{H(x)}\psi(z)^*e^{-H(x)} = e^{-\sum_{n\geq 1}\frac{x_n}{\sqrt{\alpha}}z^n}\psi^*(z).$$
 (102)

Then

$$e^{H(x)}\psi_j e^{-H(x)} = \sum_{n=1}^{\infty} \psi_{j+n} S_{(n)}\left(\frac{x}{\sqrt{\alpha}}\right), \qquad (103)$$

$$e^{H(x)}\psi_{j}^{*}e^{-H(x)} = \sum_{n=1}^{\infty}\psi_{j+n}S_{(n)}\left(-\frac{x}{\sqrt{\alpha}}\right).$$
 (104)

From these formulas, we polynomials $\Phi(|\mathbf{u}\rangle)$ can be determined. For example, let $|\mathbf{u}\rangle = \psi_{-5/2}|\text{vac}\rangle$,

$$\begin{split} \Phi(\psi_{-5/2}|\text{vac}\rangle) &= z\langle 1|e^{H(x)}\psi_{-5/2}|\text{vac}\rangle\\ &= z\langle \text{vac}|\psi_{1/2}^*e^{H(x)}\psi_{-5/2}e^{-H(x)}|\text{vac}\rangle\\ &= zS_{(2)}\left(\frac{x}{\sqrt{\alpha}}\right)\\ &= z\frac{1}{\langle \tilde{J}_{(2)},\tilde{J}_{(2)}\rangle\sqrt{\alpha}^2}\tilde{J}_{(2)}(x). \end{split}$$

Let the operators z^{H_0} and e^K are defined the same as that in [2]. Then define

$$\Psi(z) = \sum_{j \in \mathbb{Z} + \frac{1}{2}} \Psi_j z^{-j-1/2} = e^{\left(\sum_{n \ge 1} \frac{x_n}{\sqrt{\alpha}} z^n\right)} e^{\left(-\sum_{n \ge 1} \partial_{p_n} \sqrt{\alpha} z^{-n}\right)} e^K z^{H_0}, \quad (105)$$

$$\Psi^{*}(z) = \sum_{j \in \mathbb{Z} + \frac{1}{2}} \Psi_{j}^{*} z^{-j-1/2} = e^{\left(-\sum_{n \ge 1} \frac{x_{n}}{\sqrt{\alpha}} z^{n}\right)} e^{\left(\sum_{n \ge 1} \partial_{p_{n}} \sqrt{\alpha} z^{-n}\right)} e^{-K} z^{-H_{0}}.$$
 (106)

They give the realization of the Fermioinic generating functions $\psi(z)$ and $\psi^*(z)$ in the Bosonic Fock space $\mathbb{C}(\alpha)[z, z^{-1}, t_1, t_2, \cdots]$, that is, we have

$$\Phi(\psi(z)|\mathbf{u}\rangle) = \Psi(z)\Phi(|\mathbf{u}\rangle), \ \Phi(\psi^*(z)|\mathbf{u}\rangle) = \Psi^*(z)\Phi(|\mathbf{u}\rangle).$$
(107)

For example,

$$\Psi_{-5/2} \cdot 1 = \sum_{n-m=2} S_{(n)}\left(\frac{x}{\sqrt{\alpha}}\right) (-1)^m S_{(1^m)}(\sqrt{\alpha}\partial_x) \cdot 1 = S_{(2)}\left(\frac{x}{\sqrt{\alpha}}\right),$$

which equals $\Phi(\psi_{-5/2}|\text{vac}\rangle)$. Then the \hbar -KP hierarchy can be written in the Fermion form

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}}\psi_j^*\tau\otimes\psi_{-j}\tau=0.$$
(108)

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declaration of interest statement

The authors declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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