INVERSE SCATTERING METHOD FOR NONLINEAR KLEIN–GORDON EQUATION COUPLED WITH A SCALAR FIELD

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Abstract. In the present work, a novel class of negative order Ablowitz-Kaup- Newell-Segur (AKNS) nonlinear evolution equations are obtained by applying the Lax hierarchy of the generalized Zakharov-Shabat (ZS) system. The inverse scattering problem on the whole axis is examined in the case where the ZS system consists of two equations and admits a real symmetric potential. Referring to these results, the N-soliton solutions for the integro-differential version of the nonlinear Klein–Gordon equation coupled with a scalar field are obtained by using the inverse scattering method.

1. Introduction

A completely integrable nonlinear equation of mathematical physics is one which has a Lax representation, or, more precisely, can be solved via a linear integral equation of Gerlfand - Levitan - Marchenko (GLM) type, the classic examples being the Korteweg–de Vries, sine-Gordon and nonlinear Schrödinger equations, [1, 2, 3]. It is applied the AKNS hierarchy to derive soliton solutions of these integrable models by the inverse scattering method. Recently, many integrable hierarchies of soliton equations have been extended to hierarchies of a negative order AKNS equation by many authors, [16, 4, 10]. This gives an useful necessary extension for complete integrability, which is applied to investigate the integrability of certain generalizations of the Klein–Gordon equations, some model nonlinear wave equations of nonlinear Klein–Gordon equation coupled with a scalar field.

Consider the nonlinear Klein–Gordon equation coupled with a field v, in the form [11]:

$$\left\{ \begin{array}{c} u_{\varkappa\varkappa} - u_{\tau\tau} - u + 2u^3 + 2vu = 0, \\ v_{\varkappa} - v_{\tau} - 4uu_{\tau} = 0. \end{array} \right. \tag{1.1}$$

In the case $v \neq 0$, this equation is integrable since it admits the same bilinear form with the well-known sine-Gordon equation, [11].

The coupled nonlinear Klein–Gordon equations are analyzed for their integrability properties in [12] where the Hirota bilinear form is identified, from which one-soliton solutions are derived. Then, the results are generalized to the two, three and N-coupled Klein–Gordon equations in [14, 15]. Another direct method

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for traveling wave solutions of coupled nonlinear Klein–Gordon equations is employed in [13].

The equation (1.1) becomes the following negative first order equation:

$$r_{tx} = r\left(1 - \partial_x^{-1}\left[\left(r^2\right)_t\right]\right) \tag{1.2}$$

by the change of variables $\varkappa = \frac{4x+t}{8}$, $\tau = \frac{t-4x}{8}$, elimination of ν in second equation under the assumption that the scalar field v tends to zero at infinity and by the substitution $r = -\frac{1}{\sqrt{2}}u$, where $\partial_x^{-1} = \int_x^{\infty} dx$ is indefinite integral with respect to x.

It is our aim in this paper to find the soliton solutions of (1.2) by the inverse scattering method. The inverse scattering method is the most important discovery in the theory of soliton. It provides us alternatively show the complete integrability of the nonlinear evolution equation. This method also enables to solve the initial value problem for nonlinear evolution equation (1.2). Shortly we call the equation (1.2) the CKG equation in future.

The another nonlinear Klein - Gordon equations which the nonlinear term includes the first order derivative by time or by spatial variable are the nonlinear σ-model [9] and short puls equation [17] both are integrable and therefore they have soliton solutions.

The brief outline of the paper is the followings. In Section 2, we find that the CKG possesses a Lax pair of the negative order AKNS equation. It is shown that the auxiliary systems corresponding to CKG is classical ZS system with real and symmetric potential. Then, in Section 3, we recall the necessary result on the inverse scattering problem for the ZS equation on the whole line. In Section 4, we show how the scattering data evolves when coefficients of ZS system satisfies the CKG equation In this section, the N-soliton solutions of the CKG equation are obtained by inverse scattering method via the GLM equation.

2. Negative first Order AKNS Equations

Consider the spectral problem for the generalized Zakharov-Shabat (ZS) system (is called also Manakov system, [5]])

$$\begin{bmatrix} \varphi_{1x} \\ \varphi_{2x} \\ \varphi_{3x} \end{bmatrix} = X(p,q) \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}, \tag{2.1}$$

where $X(p,q) = \begin{bmatrix} i\alpha_1\lambda & ip_1 & ip_2 \\ iq_1 & i\alpha_2\lambda & 0 \\ iq_2 & 0 & i\alpha_2\lambda \end{bmatrix}$ with λ is a nonzero eigenvalue, φ_1, φ_2 and φ_3 are linearly independent eigenfunctions, $i^2 = -1$, α_1 and α_2 are real constants, $p_1 = p_1(x,t)$, $p_2 = p_2(x,t)$, $q_1 = q_1(x,t)$ and $q_2 = q_2(x,t)$ are the rapidly decreasing at infinity complex valued coefficients.

The auxiliary spectral problem described as follows:

$$\begin{bmatrix} \varphi_{1t} \\ \varphi_{2t} \\ \varphi_{3t} \end{bmatrix} = T(p,q) \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}, \tag{2.2}$$

where $T(p,q) = \begin{bmatrix} a & b & c \\ d & e & f \\ k & l & m \end{bmatrix}$ and a,b,c,d,e,f,k,l and m are scalar functions, independent of φ_1, φ_2 and φ_3.

From (2.1) and (2.2), the zero curvature equation $X_t - T_x + [X,T] = 0$ yields

$$\begin{array}{ll} a_x = ip_1 d + ip_2 k - iq_1 b - iq_2 c, & f_x = iq_1 c - ip_2 d, \\ b_x = i\alpha\lambda b - ip_1\alpha + ip_1 e + ip_2 l + ip_{1t}, & l_x = iq_2 b - ip_1 k, \\ c_x = i\alpha\lambda c + ip_1 f - ip_2\alpha + ip_2 m + ip_{2t}, & m_x = iq_2 c - ip_2 k, \end{array}$$

(2.3)

$$\begin{array}{c} d_x = -i\alpha\lambda d + iq_1 a - iq_1 e - iq_2 f + iq_{1t}, \\ k_x = -i\alpha\lambda k - iq_1 l + iq_2 a - iq_2 m + iq_{2t}, \\ e_x = iq_1 b - ip_1 d, \end{array}$$

where $\alpha = \alpha_1 - \alpha_2$. Let the following transformations be applied to the system (2.3):

$$\begin{array}{rcl} a & = & \dfrac{A(x,t)}{\lambda}, b = \dfrac{B(x,t)}{\lambda}, c = \dfrac{C(x,t)}{\lambda}, \\ d & = & \dfrac{D(x,t)}{\lambda}, e = \dfrac{E(x,t)}{\lambda}, f = \dfrac{F(x,t)}{\lambda}, \\ k & = & \dfrac{K(x,t)}{\lambda}, l = \dfrac{L(x,t)}{\lambda}, m = \dfrac{M(x,t)}{\lambda}. \end{array}$$

As a result the following equations are obtained:

$$\begin{array}{ll} A_x = ip_1 D + ip_2 K - iq_1 B - iq_2 C, & E_x = iq_1 B - ip_1 D, \\ B_x = ip_1 E - ip_1 A + ip_2 L,\ B = -\frac{1}{\alpha} p_{1t}, & L_x = iq_2 B - ip_1 K, \\ C_x = ip_1 F + ip_2 M - ip_2 A,\ C = -\frac{1}{\alpha} p_{2t}, & M_x = iq_2 C - ip_2 K, \end{array}$$

(2.4)

$$\begin{array}{c} D_x = iq_1 A - iq_1 E - iq_2 F,\ D = \frac{1}{\alpha} q_{1t}, \\ K_x = -iq_1 L + iq_2 A - iq_2 M,\ K = \frac{1}{\alpha} q_{2t}, \\ F_x = iq_1 C - ip_2 D, \end{array}$$

The following negative first order AKNS equations are obtained for important cases of spectral problem (2.1).

Proposition 1. *If the coefficients of (2.1) satisfies the properties $p_1 = q_1$ and $p_2 = q_2$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= p_1 \left[2\partial_x^{-1} \left(p_1^2\right)_t + \partial_x^{-1} \left(p_2^2\right)_t \right] + p_2 \partial_x^{-1} \left(p_1 p_2\right)_t, \\ p_{2tx} &= p_2 \left[\partial_x^{-1} \left(p_1^2\right)_t + 2\partial_x^{-1} \left(p_2^2\right)_t \right] + p_1 \partial_x^{-1} \left(p_1 p_2\right)_t. \end{aligned} \tag{2.5}$$

Proof. Let's consider the system (2.1) the case $p_1 = q_1$ and $p_2 = q_2$. It is clearly seen that $A = -E - M, B = -D, C = -K$ and $F = L$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha} p_{1t},\ C = -\frac{1}{\alpha} p_{2t},$$

$$E_x = -\frac{i}{\alpha}\left(p_1^2\right)_t,\ F_x = -\frac{i}{\alpha}\left(p_1 p_2\right)_t,\ M_x = -\frac{i}{\alpha}\left(p_2^2\right)_t,$$

$$B_x = ip_1\left(2E + M\right) + ip_2 F,$$

$$C_x = ip_1 F + ip_2\left(E + 2M\right).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.5), where $\partial_x^{-1} = \int\limits_x^{\infty} dx$ is indefinite integral with respect to x. □

Proposition 2. *If the coefficients of (2.1) satisfies the properties $p_1 = -q_1$ and $p_2 = -q_2$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= -p_1\left[2\partial_x^{-1}\left(p_1^2\right)_t + \partial_x^{-1}\left(p_2^2\right)_t\right] - p_2\partial_x^{-1}\left(p_1p_2\right)_t, \\ p_{2tx} &= -p_2\left[\partial_x^{-1}\left(p_1^2\right)_t + 2\partial_x^{-1}\left(p_2^2\right)_t\right] - p_1\partial_x^{-1}\left(p_1p_2\right)_t. \end{aligned} \tag{2.6}$$

Proof. Let's consider the system (2.1) the case $p_1 = -q_1$ and $p_2 = -q_2$. It is clearly seen that $A = -E - M, B = D, C = K$ and $F = L$ in the system (2.4).This system becomes

$$B = -\frac{1}{\alpha}p_{1t},\ C = -\frac{1}{\alpha}p_{2t},$$

$$E_x = \frac{i}{\alpha}\left(p_1^2\right)_t,\ F_x = \frac{i}{\alpha}\left(p_1 p_2\right)_t,\ M_x = \frac{i}{\alpha}\left(p_2^2\right)_t,$$

$$B_x = ip_1\left(2E + M\right) + ip_2 F,$$

$$C_x = ip_1 F + ip_2\left(E + 2M\right).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.6). □

Proposition 3. *If the coefficients of (2.1) satisfies the properties $p_1 = q_1^*$ and $p_2 = q_2^*$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= p_1\left[2\partial_x^{-1}\left|p_1\right|_t^2 + \partial_x^{-1}\left|p_2\right|_t^2\right] - p_2\partial_x^{-1}\left(p_1p_2^*\right)_t, \\ p_{2tx} &= p_2\left[\partial_x^{-1}\left|p_1\right|_t^2 + 2\partial_x^{-1}\left|p_2\right|_t^2\right] + p_1\partial_x^{-1}\left(p_1^*p_2\right)_t, \end{aligned} \tag{2.7}$$

where q_1^* and q_2^* are the complex conjugates of q_1 and q_2, respectively.

Proof. Let's consider the system (2.1) the case $p_1 = q_1^*$ and $p_2 = q_2^*$. It is clearly seen that $A = E^* + M^*, B = -D^*, C = -K^*$ and $F = -L^*$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha}p_{1t},\ C = -\frac{1}{\alpha}p_{2t},$$

$$E_x = -\frac{i}{\alpha}\left|p_1\right|_t^2,\ F_x = -\frac{i}{\alpha}\left(p_1^* p_2\right)_t,\ M_x = -\frac{i}{\alpha}\left|p_2\right|_t^2,$$

$$B_x = ip_1 (2E + M) - ip_2 F^*,$$

$$C_x = ip_1 F + ip_2 (E + 2M).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.7). □

Proposition 4. *If the coefficients of (2.1) satisfies the properties $p_1 = -q_1^*$ and $p_2 = -q_2^*$ then the system of equations (2.4) becomes the following negative order pair of equations:*

$$\begin{aligned} p_{1tx} &= -p_1 \left[2\partial_x^{-1} |p_1|_t^2 + \partial_x^{-1} |p_2|_t^2 \right] + p_2 \partial_x^{-1} (p_1 p_2^*)_t, \\ p_{2tx} &= -p_2 \left[\partial_x^{-1} |p_1|_t^2 + 2\partial_x^{-1} |p_2|_t^2 \right] - p_1 \partial_x^{-1} (p_1^* p_2)_t, \end{aligned} \tag{2.8}$$

where q_1^* and q_2^* are the complex conjugates of q_1 and q_2, respectively.

Proof. Let's consider the system (2.1) the case $p_1 = -q_1^*$ and $p_2 = -q_2^*$. It is clearly seen that $A = E^* + M^*, B = D^*, C = K^*$ and $F = -L^*$ in the system (2.4). This system becomes

$$B = -\frac{1}{\alpha} p_{1t}, \ C = -\frac{1}{\alpha} p_{2t},$$

$$E_x = \frac{i}{\alpha} |p_1|_t^2, \ F_x = \frac{i}{\alpha} (p_1^* p_2)_t, \ M_x = \frac{i}{\alpha} |p_2|_t^2,$$

$$B_x = ip_1 (2E + M) - ip_2 F^*,$$

$$C_x = ip_1 F + ip_2 (E + 2M).$$

For the compatibility of these equations the functions p_1 and p_2 must be satisfied the system (2.8). □

The following corollary of Proposition 1 is valid.

Corollary 1. *In the case $p_1 = p_2$ the nonlinear evolution equation (2.5) has the form*

$$p_{tx} = 4p \int_x^{+\infty} \left(p^2\right)_t dx + p. \tag{2.9}$$

The equation (2.9) becomes (1.2) by the substitution $p = -\frac{ir}{2}$.

3. Zakharov-Shabat System with Real and Symmetric Potential

In this section, we recall the necessary results from [6, 7, 8] on inverse scattering problem for classical Zakharov-Shabat system with real coefficient $r = r(x)$:

$$\begin{cases} u_{1x} = -i\mu u_1 + ru_2, \\ u_{2x} = ru_1 + i\mu u_2. \end{cases} \tag{3.1}$$

It is clear from Corollary 1 that this system is one of the Lax pairs for negative order equation (2.9). Really, if $p_1 = p_2 = q_1 = q_2 = p$ are taken in the system (2.6), we obtain

$$\begin{cases} \varphi_{1x} = i\alpha_1\lambda\varphi_1 + ip(\varphi_2 + \varphi_3), \\ \varphi_{2x} = ip\varphi_1 + i\alpha_2\lambda\varphi_2, \\ \varphi_{3x} = ip\varphi_1 + i\alpha_2\lambda\varphi_3. \end{cases}$$

This system becomes

$$\begin{cases} v_{1x} = i\alpha_1\lambda v_1 + ipv_2, \\ v_{2x} = 2ipv_1 + i\alpha_2\lambda v_2, \end{cases}$$

by the substitution $\varphi_1 = v_1, \varphi_2 + \varphi_3 = v_2$. The substitutions $\sqrt{2}v_1 = u_1, v_2 = u_2$, $\alpha_1 = \alpha_2 = \beta$, $\mu = \beta\lambda$ and $p = -\frac{ir}{\sqrt{2}}$ transforms this system to classical ZS system (3.1).

Let the eigenfunctions Φ, Ψ, $\overline{\Phi}$ and $\overline{\Psi}$ be defined with the following boundary conditions for the eigenvalue μ in system (3.1)

$$\begin{array}{ccc} \Phi & \sim & \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mu x}, \; \Psi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\mu x}, \\ x & \to & -\infty \qquad\qquad\quad x \to +\infty \end{array}$$

(3.2)

$$\begin{array}{ccc} \overline{\Phi} & \sim & \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\mu x}, \; \overline{\Psi} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mu x}. \\ x & \to & -\infty \qquad\qquad\quad x \to +\infty \end{array}$$

For the eigenfunctions (3.2), $W(\Phi, \overline{\Phi}) = -1$ and $W(\Psi, \overline{\Psi}) = -1$, where W is the Wronskian. Therefore, the eigenfunctions Ψ and $\overline{\Psi}$ are linearly independent. Hence the functions Φ and $\overline{\Phi}$ can be written as

$$\Phi = a(\mu)\overline{\Psi} + b(\mu)\Psi, \tag{3.3a}$$

$$\overline{\Phi} = -\overline{a}(\mu)\Psi + \overline{b}(\mu)\overline{\Psi}. \tag{3.3b}$$

The scattering matrix is usually defined as

$$S = \begin{pmatrix} a & b \\ \overline{b} & -\overline{a} \end{pmatrix}. \tag{3.3c}$$

Using the (3.3) system and $W(\Phi, \overline{\Phi}) = -1$ equation,

$$a(\mu)\overline{a}(\mu) + b(\mu)\overline{b}(\mu) = 1,$$

is obtained.

Now let's create the analytical properties of the scattering data. When $r \in L_1$, $e^{i\mu x}\Phi$ and $e^{-i\mu x}\Psi$ are analytical in the upper half plane and $e^{-i\mu x}\overline{\Phi}$ and $e^{i\mu x}\overline{\Psi}$ are analytical in the lower half plane. Therefore, $a = W(\Phi, \Psi) = \Phi_1\Psi_2 - \Psi_1\Phi_2$ is analytical in the upper half plane and $\overline{a} = W(\overline{\Phi}, \overline{\Psi})$ is analytical in the lower half plane. Generally, $b = -W(\Phi, \overline{\Psi})$ and $\overline{b} = W(\overline{\Phi}, \Psi)$ need not be analytical in any region. Usually, to use these properties, the scattering problem is converted into an integral equation. For example, system (3.1) for Φ provides the equations

$$\Phi_1(x,\mu)e^{i\mu x} = 1 + \int_{-\infty}^{x} r(y)dy \int_{-\infty}^{y} r(z)e^{2i\mu(y-z)}\Phi_1(z,\mu)e^{i\mu z}dz,$$

or

$$\Phi_1(x,\mu)e^{i\mu x} = 1 + \int_{-\infty}^{x} M(x,y,\mu)\Phi_1(y,\mu)e^{i\mu y}dy,$$

$$\Phi_2(x,\mu)e^{i\mu x} = \int_{-\infty}^{x} e^{2i\mu(x-y)}r(y)\Phi_1(y,\mu)e^{i\mu y}dy$$

where

$$M(x,y,\mu) = r(y)\int_{y}^{x} e^{2i\mu(z-y)}r(z)dz.$$

As long as r is not very small, system (3.1) can have discrete eigenvalues. This occurs when $a(\mu)$ has a zero in the upper half plane or $\overline{a}(\mu)$ has a zero in the lower half plane. If the zeros of $a(\mu)$ are called $\mu_k, k = 1, 2, ..., N$ then at $\mu = \mu_k$, Φ and Ψ proportional such that

$$\Phi = c_k\Psi.$$

Similarly, if the zeros of $\overline{a}(\mu)$ are called $\overline{\mu}_k, k = 1, 2, ..., \overline{N}$ then at $\mu = \overline{\mu}_k$, $\overline{\Phi}$ and $\overline{\Psi}$ proportional such that

$$\overline{\Phi} = \overline{c_k}\overline{\Psi}.$$

For $|x| \to \infty$, if r decreases rapidly, $a, b, \overline{a}$ and $\overline{b}$ become complete functions. In this case, b and $\overline{b}$ can be expanded to $c_k = b(\mu_k)$ and $\overline{c_k} = \overline{b}(\overline{\mu}_k)$. In this case, $a(\mu)$ and $\overline{a}(\mu)$ are analytic on the real axis, and are also analytic in the upper half plane and the lower half plane. This means that $a(\mu)$ has only a finite number of zeros for $\mathrm{Im}(\mu) \geqq 0$. From system (3.3), we have symmetry relations

$$\overline{\Psi}(x,\mu) = \begin{pmatrix} \Psi_2(x,-\mu) \\ \Psi_1(x,-\mu) \end{pmatrix}, \overline{\Phi}(x,\mu) = \begin{pmatrix} -\Phi_2(x,-\mu) \\ -\Phi_1(x,-\mu) \end{pmatrix},$$

which imply

$$\overline{a}(\mu) = a(-\mu),\ \overline{b}(\mu) = -b(-\mu) \tag{3.4a}$$

and consequently

(3.4b) $$\overline{N} = N, \overline{\mu_k} = -\mu_k, \overline{c_k} = -c_k.$$

Now let's examine the inverse scattering problem. Assuming that the scattering data $a, \overline{a}, b$ and $\overline{b}$ are complete functions, derive the inverse scattering formulas. For this to hold, it is sufficient to assume r decay faster than any exponential as $|x| \to \infty$.

Let the Ψ and $\overline{\Psi}$ functions be expressed with the integral equations

(3.5) $$\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\mu x} + \int_x^{+\infty} K(x,s) e^{i\mu s} ds,$$

$$\overline{\Psi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mu x} + \int_x^{+\infty} \overline{K}(x,s) e^{-i\mu s} ds,$$

where $\mu = \varsigma + i\eta, \eta \geqq 0$ and $K(x,s) = \begin{pmatrix} K_1(x,s) \\ K_2(x,s) \end{pmatrix}, \overline{K}(x,s) = \begin{pmatrix} \overline{K_1}(x,s) \\ \overline{K_2}(x,s) \end{pmatrix}$.

The integral terms containing the kernels K and $\overline{K}$ represent the difference between the limit values at $x = \infty$ and the true eigenfunctions. Also, these kernels are independent of the μ eigenvalue. For proof, for example, substitute equation (3.5) in system (3.1)

$$\int_x^{+\infty} e^{i\mu s} \left[(\partial_x - \partial_s) K_1(x,s) - r(x) K_2(x,s) \right] ds - \left[r(x) + 2K_1(x,x) \right] e^{i\mu x} + \lim_{s\to\infty} \left[K_1(x,s) e^{i\mu s} \right] = 0,$$

$$\int_x^{+\infty} e^{i\mu s} \left[(\partial_x + \partial_s) K_2(x,s) - r(x) K_1(x,s) \right] ds - \lim_{s\to\infty} \left[K_2(x,s) e^{i\mu s} \right] = 0.$$

It is necessary and sufficient to have

$$(\partial_x - \partial_s) K_1(x,s) - r(x) K_2(x,s) = 0,$$

$$(\partial_x + \partial_s) K_2(x,s) - r(x) K_1(x,s) = 0,$$

subject to the boundary conditions

(3.6) $$K_1(x,x) = -\frac{1}{2} r(x),$$

$$\lim_{s\to\infty} K_2(x,s) = 0.$$

Now derive the linear integral equation of the inverse scattering, Gel'fand-Levitan-Marchenko (GLM) equation. Consider a complex plane μ on a contour $\acute{C}$, starting at $\mu = -\infty + i0^+$, passing through all zeros of $a(\mu)$, and ending at $\mu = +\infty + i0^+$. Since there is a strong decay on r, expansion towards the upper half plane can be made such that

$$\frac{\Phi(x,\mu)}{a(\mu)} = \overline{\Psi}(x,\mu) + \frac{b(\mu)}{a(\mu)}\Psi(x,\mu). \tag{3.7}$$

Substitute (3.5) into (3.7) to find

$$\frac{\Phi(x,\mu)}{a(\mu)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mu x} + \int_x^{+\infty} \overline{K}(x,s)e^{-i\mu s}ds + \frac{b(\mu)}{a(\mu)}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\mu x} + \int_x^{+\infty} K(x,s)e^{i\mu s}ds\right).$$

If $\delta(x) = \left(\frac{1}{2\pi}\right)\int_{\acute{C}} e^{i\mu x}d\lambda$ for $y > x$ is taken ($\delta(x)$ is the Dirac delta function),

$$I = \overline{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^{+\infty} K(x,s)F(s+y)\,ds$$

is obtained, where

$$F(x) \equiv \left(\frac{1}{2\pi}\right)\int_{\acute{C}} \frac{b(\mu)}{a(\mu)} e^{i\mu x}d\mu, \tag{3.8}$$

$$I \equiv \left(\frac{1}{2\pi}\right)\int_{\acute{C}} \frac{\Phi(x,\mu)}{a(\mu)} e^{i\mu y}d\mu.$$

Since $\Phi e^{i\mu x}$ is analytic in the upper half plane, $y > x$, and the contour $\acute{C}$ passes over all the zeros a, we have that $I = 0$. Hence we have

$$\overline{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^{+\infty} K(x,s)F(s+y)ds = 0. \tag{3.9}$$

Similarly, by analytical expansion in the lower half plane,

$$K(x,y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{F}(x+y) - \int_x^{+\infty} \overline{K}(x,s)\overline{F}(s+y)ds = 0 \tag{3.10a}$$

is obtained, where

$$\overline{F}(x) \equiv \left(\frac{1}{2\pi}\right)\int_{\overline{\acute{C}}} \frac{\overline{b}(\mu)}{\overline{a}(\mu)} e^{-i\mu x}d\mu. \tag{3.10b}$$

$\overline{\acute{C}}$ is a contour that passes under all zeros of $\overline{a}(\mu)$. Contour integrations in (3.8) and (3.10b) give

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(\mu)}{a(\mu)} e^{i\mu x}d\lambda - i\sum_{j=1}^{N} c_j e^{i\mu_j x},$$

$$\overline{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\overline{b}(\mu)}{\overline{a}(\mu)}e^{-i\mu x}d\lambda+i\sum_{j=1}^{\overline{N}}\overline{c_j}e^{-i\mu_j x},$$

where

$$c_j=\frac{b(\mu_j)}{a'(\mu_j)},\overline{c_j}=\frac{\overline{b}(\overline{\mu_j})}{\overline{a}'(\overline{\mu_j})}.$$

Integral equations (3.9) and (3.10a) can be expressed with matrices

$$\widetilde{K}=\begin{pmatrix}\overline{K_1} & K_1\\ \overline{K_2} & K_2\end{pmatrix},\widetilde{F}=\begin{pmatrix}0 & -\overline{F}\\ F & 0\end{pmatrix},$$

whereby we have

$$\widetilde{K}(x,y)+\widetilde{F}(x+y)+\int_{x}^{+\infty}\widetilde{K}(x,s)\widetilde{F}(s+y)ds=0. \tag{3.11}$$

Equation (3.11) is Gel'fand-Levitan-Marchenko (GLM) equation. With (3.4a) and (3.4b) symmetry conditions

$$\overline{F}(x)=-F(x),$$

$$\overline{K}(x,y)=\begin{pmatrix}K_2(x,y)\\ K_1(x,y)\end{pmatrix},$$

are obtained. The GLM equation (3.11) becomes

$$K_1\left(x,y\right)-F\left(x+y\right)+\int_{x}^{+\infty}\int_{x}^{+\infty}K_1\left(x,z\right)F\left(z+s\right)F\left(s+y\right)dsdz=0,$$

with the above mentioned conditions. Also, using (3.6) the potential is found as

$$r(x)=-2K_1(x,x).$$

4. N-Soliton Solutions of Coupled Klein - Gordon Equation

The N-soliton solutions of Klein-Gordon equation (1.2) coupled with a scalar field will be studied by using the inverse scattering method. The Gel'fand-Levitan-Marchenko (GLM) equation corresponding to the Zakharov-Shabat system (3.1) with real and symmetric potential will be applied.

As is shown in previous section, under the substitution that $p_1=p_2=q_1=q_2=p$, $\varphi_1=\frac{u_1}{\sqrt{2}},\varphi_2+\varphi_3=u_2$, $\alpha_1=\alpha_2=\beta$, $\mu=\beta\lambda$ and $p=-\frac{ir}{\sqrt{2}}$ the system (2.1) becomes ZS system (3.1) with real and symmetric potential, the another component of Lax pair comes to form:

$$\begin{aligned}A_x &= r(C-B),\\ B_x+2i\mu B &= r_t-2Ar,\\ C_x-2i\mu C &= r_t+2Ar.\end{aligned} \tag{4.1}$$

.

It is our close-up aim to study evolution of scattering data $\left\{\frac{a(\mu)}{b(\mu)};\ \mu_n,\ c_n,\ n=1,2,...,N\right\}$ for the system (3.1) when the potential $r(x,t)$ satisfies the equation (1.2).

Theorem 1. *Let $r(x,t)$ be a coefficient of the system (3.1) satisfying the equation (1.2), then the evolution of the scattering data of this system (3.1) is the following form:*

$$a(\mu,t)=a(\mu,0),$$

$$b(\mu,t)=b(\mu,0)e^{\frac{t}{i\mu}}, \tag{4.2}$$

$$\mu_n(t)=\mu_n;\ c_n(t)=c_{n,0}e^{\frac{t}{i\mu_n}},$$

where $c_{n,0}=c_n(t=0)$.

Proof. Let the time dependent $\Phi,\Psi.\overline{\Phi}$ and $\overline{\Psi}$ eigenfunctions be defined for $|x|\to\infty$ such that $A\to A_{-}(\mu), D\to -A_{-}(\mu),\ B,\ C\to 0$,

$$\begin{aligned}\Phi^{(t)} &= \Phi e^{A_{-}(t)},\ \Psi^{(t)}=\Psi e^{-A_{-}(t)},\\ \overline{\Phi}^{(t)} &= \overline{\Phi}e^{-A_{-}(t)},\ \overline{\Psi}^{(t)}=\overline{\Psi}e^{A_{-}(t)}.\end{aligned}$$

Here $\Phi,\Psi.\overline{\Phi}$ and $\overline{\Psi}$ have boundary conditions and satisfy system (3.1). The time evolution of $\Phi^{(t)}$ becomes

$$\frac{d\Phi^{(t)}}{dt}=\begin{pmatrix} A & B\\ C & -A\end{pmatrix}\Phi^{(t)},$$

and the function Φ satisfies the equation

$$\frac{d\Phi}{dt}=\begin{pmatrix} A-A_{-}(\mu) & B\\ C & -A-A_{-}(\mu)\end{pmatrix}\Phi.$$

If

$$\Phi=a\overline{\Psi}+b\Psi\sim a\begin{pmatrix}1\\0\end{pmatrix}e^{-i\mu x}+b\begin{pmatrix}0\\1\end{pmatrix}e^{i\mu x}\ (x\sim\infty)$$

relations are used system (3.3) becomes

$$\begin{pmatrix} a_t e^{-i\mu x}\\ b_t e^{i\mu x}\end{pmatrix}=\begin{pmatrix} 0\\ -2A_{-}(\mu)be^{i\mu x}\end{pmatrix}.$$

Therefore, the following equations are obtained

$$a(\mu,t)=a(\mu,0),$$

$$b(\mu,t)=b(\mu,0)e^{-2A_{-}(\mu)t}, \tag{4.3}$$

$$c_n(t)=c_{n,0}e^{-2A_{-}(\mu_n)t}, n=1,2,...,N.$$

Here $c_n(t)$ is obtained using the definition of normalized coefficients. Taking into account (4.3) the following equations are obtained

$$
\begin{aligned}
a(\mu,t) &= a(\mu,0), \\
b(\mu,t) &= b(\mu,0)e^{-2\frac{c_0}{\mu}t}, \\
c_n(t) &= c_{n,0}e^{-2\frac{c_0}{\mu_n}t}.
\end{aligned} \tag{4.4}
$$

If $c_0 = -\frac{1}{2i}$ is chosen in system (4.4), system (4.2) is obtained. $\square$

If $\mu_n = i\kappa_n$ is chosen in system (4.2) the more suitable formulas

$$
\begin{aligned}
a(\mu,t) &= a(\mu,0), \\
b(\mu,t) &= b(\mu,0)e^{\frac{t}{i\mu}}, \\
c_n(t) &= c_{n,0}e^{-\frac{t}{\kappa_n}}
\end{aligned} \tag{4.5}
$$

are obtained.

Let $r(x,t)$ be a coefficient of the ZS system with real and symmetric potential (system (3.1)) which satisfies the KGC equation ((1.2)) and also this system has only discrete spectrum, then the corresponding GLM equation for this system is as follows

$$
K(x,y;t) - F(x+y;t) + \int_x^{+\infty}\int_x^{+\infty} K(x,z;t)\,F(z+s;t)\,F(s+y;t)\,ds dz = 0, \tag{4.6}
$$

where

$$
F(x+y;t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(\mu,0)}{a(\mu,0)} e^{i\mu(x+y)-2A_{-}(\mu)t} d\zeta - i\sum_{j=1}^{N} c_{j,0} e^{i\mu_j(x+y)-2A_{-}(\mu_j)t}. \tag{4.7}
$$

Also, the coefficient system (3.1) becomes

$$
r(x,t) = -2K(x,x;t).
$$

If scattering data (4.5) is substituted in equation (4.7), and $-ic_n(t) = \omega_n^2(t), \omega_n(t) = \omega_n(0)e^{-\frac{t}{2\kappa_n}}$ are chosen,

$$
F(x,y;t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(\mu,0)}{a(\mu,0)} e^{i\mu(x+y)-\frac{t}{i\mu}} d\mu + \sum_{n=1}^{N} \omega_n^2(t) e^{-\kappa_n(x+y)}
$$

is obtained.

Since the system has only discrete spectrum

$$
F_D(x,y;t) = \sum_{n=1}^{N} \omega_n^2(t) e^{-\kappa_n(x+y)} \tag{4.8}
$$

is taken. Because of equation (4.8),

$$
F(x,y;t) = \sum_{n=1}^{N} \omega_n^2(0) e^{-\kappa_n(x+y)-\frac{t}{\kappa_n}},
$$

is obtained. By adding it in (4.6), the following equation

$$K(x,y;t)+\int_x^{+\infty}\int_x^{+\infty} K(x,z;t)\sum_{n=1}^{N}\omega_n^4(0)e^{-\kappa_n(x+s)--\kappa_n(s+y)-\frac{2t}{\kappa_n}}dsdz=\sum_{n=1}^{N}\omega_n^2(0)e^{-\kappa_n(x+y)-\frac{t}{\kappa_n}}$$

is obtained. If this equation is arranged

(4.9)
$$K\left(x,y;t\right)+\omega_n^4(0)\sum_{n=1}^{N}\frac{e^{-\kappa_n y-2\kappa_n x-\frac{2t}{\kappa_n}}}{2\kappa_n}\int_x^{+\infty}K(x,z;t)e^{-\kappa_n z}dz=\sum_{n=1}^{N}\omega_n^2(0)e^{-\kappa_n(x+y)-\frac{t}{\kappa_n}},$$

is obtained.

Theorem 2. *Let $r(x,t)$ be a coefficient of the system (3.1) satisfying the equation (1.2) and also system (3.1) has only discrete spectrum. Then the solution of the inverse problem for this system is*

$$r(x,t)=-2\Delta^{-1}\sum_{m=1}^{N}\sum_{n=1}^{N}\omega_n^2(0)e^{-(\kappa_m+\kappa_n)x-\frac{t}{\kappa_n}}Q_{mn}, \tag{4.10}$$

where $\Delta=\det(I+\Lambda)$; I is the $N\times N$ identity matrix;

$$\Lambda_{mn}=\omega_n^4(0)\frac{e^{-2\kappa_n x-\frac{2t}{\kappa_n}}}{2\kappa_n}\frac{e^{-(\kappa_m+\kappa_n)x}}{(\kappa_m+\kappa_n)}, \tag{4.11}$$

is $N\times N$ matrix whose components are m,n and Q_{mn} are cofactors of $I+\Lambda$; $\mu_n=i\kappa_n$ are eigenfunctions and $c_{n,0}=i\omega_n^2(0)$ are normalized numbers of the ZS system (3.1).

Proof. Let $r(x,t)$ be a coefficient of the ZS system with real and symmetric potential (system (3.1)) which satisfies the KGC equation ((1.2)) and also this system has only discrete spectrum. If

$$K(x,y;t)=\sum_{n=1}^{N}\omega_n(0)e^{-\kappa_n y}K_n(x,t) \tag{4.12}$$

is chosen as the natural form in equation (4.9),

$$K_n(x,t)+\omega_n^4(0)\sum_{m,n=1}^{N}\frac{e^{-2\kappa_n x-\frac{2t}{\kappa_n}}}{2\kappa_n}\frac{e^{-(\kappa_m+\kappa_n)x}}{(\kappa_m+\kappa_n)}K_m\left(x,t\right)=\sum_{n=1}^{N}\omega_n(0)e^{-\kappa_n x-\frac{t}{\kappa_n}}$$

is obtained. This equation can be written in the form below

(4.13)
$$\sum_{m,n=1}^{N}\left[\delta_{mn}+\omega_n^4(0)\frac{e^{-2\kappa_n x-\frac{2t}{\kappa_n}}}{2\kappa_n}\frac{e^{-(\kappa_m+\kappa_n)x}}{(\kappa_m+\kappa_n)}\right]K_n\left(x,t\right)=\sum_{n=1}^{N}\omega_n(0)e^{-\kappa_n x-\frac{t}{\kappa_n}},$$

where

$$\delta_{mn} = \begin{cases} 1, if\ m = n \\ 0, if\ m \neq n \end{cases}.$$

Let E and K are column vectors with n th components $\omega_n(0)e^{-\kappa_n x - \frac{t}{\kappa_n}}$ and K_n, respectively, and Λ an $N \times N$ matrix with its m, n entry (4.11); then equation (4.13), in matrix form, is given by

$$(I + \Lambda) K = E, \tag{4.14}$$

where I is the $N \times N$ identity matrix. Since Λ is positive definite here, it can be assured that equation (4.14) has a solution K. A solution of equation (4.14) with Cramer's method and standard expansion of determinants is as follows

$$K_n(x, t) = \Delta^{-1} \sum_{n=1}^{N} \omega_n(0) e^{-\kappa_n x - \frac{t}{\kappa_n}} Q_{mn}, \tag{4.15}$$

where $\Delta = \det(I + \Lambda)$ and Q_{mn} matrices are cofactors of $I + \Lambda$. In equation (4.15), expansion was made along the n th column. If $y = x$ taken equation (4.13) and equation (4.15) is substituted in (4.12),

$$K(x, x; t) = \Delta^{-1} \sum_{m=1}^{N} \sum_{n=1}^{N} \omega_n^2(0) e^{-(\kappa_m + \kappa_n)x - \frac{t}{\kappa_n}} Q_{mn}$$

is obtained. Due to $r(x, t) = -2K(x, x; t)$, the $r(x, t)$ coefficient is found as equation (4.10). □

5. Conclusion

In this paper, we study the soliton solutions of the coupled Klein–Gordon (CKG) equation coupled with a scalar field which shares the same bilinear form with the sine-Gordon equation. We found a Lax pair of the CKG, of the negative order AKNS type. The spectral problem is the ZS system with real and symmetric potential. This makes possible to use the inverse scattering method's technique for obtaining and analyzing the soliton solutions of the CKG. This method provides us to show the complete integrability of the coupled Klein–Gordon equation. On the other side the various extensions and generalizations of the inverse scattering method have been discovered, it seems that many different integrable Klein - Gordon type equations still remains to be found.

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