

Hamiltonian representation of isomonodromic deformations of general rational connections on $\mathfrak{gl}_2(\mathbb{C})$

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Abstract: In this paper, we study and build the Hamiltonian system attached to any $\mathfrak{gl}_2(\mathbb{C})$ meromorphic connection with arbitrary number of non-ramified poles of arbitrary degrees. In particular, we propose the Lax pairs and Hamiltonian evolutions expressed in terms of irregular times and monodromies associated to the poles as well as g Darboux coordinates defined as the apparent singularities arising in the oper gauge. Moreover, we also provide a reduction of the isomonodromic deformations to a subset of g non-trivial isomonodromic deformations. This reduction is equivalent to a map reducing the set of irregular times to only g non-trivial isomonodromic times. We apply our construction to all cases where the associated spectral curve has genus 1 and recover the standard Painlevé equations. We finally make the connection with the topological recursion and the quantization of classical spectral curve from this perspective.

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1 Introduction

The isomonodromic deformations of meromorphic connections on vector bundles over Riemann surfaces still present a vast subject of modern mathematics and have known many developments since the pioneer works of E. Picard, L. Fuchs, R. Fuchs, P. Painlevé, R. Garnier and L. Schlesinger [68, 29, 34, 67, 32, 69]. Fuchsian singularities in $\mathfrak{gl}_2(\mathbb{C})$ were first studied by R. Fuchs and lead to the analysis of the Painlevé 6 equation. These works, their links with the Painlevé property and with isomonodromic deformations of rational connections on $\mathfrak{gl}_2(\mathbb{C})$ with only simple poles were later pursued by R. Garnier and K. Okamoto [64, 65] for arbitrary Fuchsian systems in $\mathfrak{gl}_2(\mathbb{C})$, i.e. to Garnier systems in their scalar version or to Schlesinger systems [69] in their matrix form. Nowadays, connections with Fuchsian singularities and their underlying Hamiltonian systems are well-understood and we refer to Chap. 3 of [43] for a review on the subject. Another significant result obtained by R. Garnier is the generalization to all other five Painlevé equations proving that these equations may be obtained by completely integrability conditions. But it was J. Malmquist [51] who pointed out for the first time that all Painlevé equations can be written as Hamiltonian systems, while the relations with isomonodromic deformations of linear ordinary differential equations with irregular singularities was given in [63]. Other interesting works related to irregular Garnier systems, obtained by confluence of simple poles, can also be found in the literature [48, 46, 47, 24, 49, 58]. However, the case of general isomonodromic deformations of linear ordinary differential equations with irregular singularities is much more complicated than the case of Fuchsian singularities and is still actively studied. For example, seminal contributions to the theory were made by the Japanese school of M. Jimbo, T. Miwa, and K. Ueno [45, 44] in the case of generic (in the sense that the leading term of the connections at each pole has distinct eigenvalues) singularity structures with arbitrary order poles on arbitrary rank bundles. The vast family of nonlinear differential equations resulting from the isomonodromic deformations is the largest known to have the “Painlevé property”. Of special interest are the Painlevé equations that have been studied in various ways: these equations are required to preserve a set of generalized monodromy data, which is obtained by considering Stokes data around the irregular singularities in addition to the usual monodromies. The space of deformations itself needs to be extended beyond the space of complex structures of the Riemann surface by including the irregular type of the monodromy considered [12, 13, 15, 18, 9]. There are currently many ways to view and to approach the problem. Indeed, it is known since the works of K. Okamoto [66] in the 1980s that isomonodromy equations may be rewritten using Hamiltonian formulations. They may also be regarded as a natural extension of the symplectic structures investigated by Narasimhan and Seshadri [60], Atiyah and Bott [4], Goldman [35] and Fock and Rosly [28] on spaces of flat connections on Riemann surfaces to the world of isomonodromic systems - a perspective pursued by P. Boalch [13, 14, 17]. There exists another description in terms of moment maps to (central extensions of) loop algebras, a viewpoint introduced by J. Harnad [37], J. Hurtubise [3] and the Montréal school. These works were extended to the case of general singularity structures by N. Woodhouse [71]. By considering data associated with families of Riemann surfaces branched over the singularities, one may also consider the isomonodromy equations as non-Abelian Gauss-Manin connections. This observation was first made by Y. Manin in [54, 53] for the Painlevé 6 equation and significantly extended by P. Boalch [13] leading to the notion of wild non-Abelian Hodge theory [10, 52] of central importance in Seiberg-Witten theory and theoretical physics and providing a modern viewpoint on isomonodromic equations as Gauss-Manin connections for families of wild Riemann surfaces [74, 16, 15, 18, 17].

Due to the many possible perspectives and the relations with the Painlevé property and

transcendents, isomonodromic deformations have an extremely diverse range of applications in mathematical physics. Without being exhaustive, let us mention that they play an important role in the study of random matrix theory and provide generating functions for moduli spaces of two-dimensional topological quantum field theories and the study of quantum cohomology and Gromov-Witten invariants. In addition, a renewed interest on the study of these isomonodromic systems in the mathematical physics community was motivated by the discovery of a correspondence between conformal blocks of conformal field theories and isomonodromic tau functions [31, 19] and irregular Higgs bundles. Moreover, the recent development of exact WKB theory [42, 40] in relation with the Painlevé equations and topological recursion [26] also offers new insights on the subject.

The non-autonomous Hamiltonian nature of isomonodromic deformations has been uncovered using the spectral invariants of a corresponding Hitchin system [37, 38, 13, 50]. If a geometrical understanding of the Hamiltonian representation of the isomonodromic equations for a generic meromorphic connection is now well understood [39, 7], an explicit expression for the Hamiltonians was up to now only derived on a case by case basis. It is believed that any autonomous Hamiltonian encountered in the Hitchin system, hence corresponding to an isospectral deformation, can be turned into a Hamiltonian corresponding to an isomonodromic deformation by making it dependent explicitly on the Casimirs of the system. However, this procedure is easy to implement on simple examples such as Painlevé equations or Fuchsian systems but finding how to de-autonomize any Hamiltonian for generic irregular connections is a more complicated task. Recently Gaiur, Mazzocco and Rubtsov [30] solved this problem for isomonodromic deformations arising from confluences of isomonodromic deformations of Fuchsian systems and we shall recover their results in a specific gauge as discussed in Remark 7.2.

In this article, we contribute to the subject of isomonodromic deformations by proposing an explicit expression for the Hamiltonian systems in terms of Darboux coordinates corresponding to apparent singularities. Our construction is valid for an arbitrary number of pole singularities and more interestingly for regular or irregular unramified poles (of arbitrary degree) in $\mathfrak{gl}_2(\mathbb{C})$. We also give an explicit expression of the corresponding Lax pairs whose compatibility equations provide the Hamiltonian systems. In other words, this article may be seen as a generalization of the famous six Lax pairs proposed by Jimbo-Miwa (Appendix C of [44]) in the case of arbitrary $\mathfrak{gl}_2(\mathbb{C})$ meromorphic connections with unramified poles or the generalization of Schlesinger Hamiltonians for non-Fuchsian singularities in $\mathfrak{gl}_2(\mathbb{C})$. Our approach consists in using the geometric knowledge at each pole, described by the irregular times (sometimes also referred to as “KP times” or “spectral times”) to provide a first natural space of isomonodromic deformations. Building the corresponding Lax pairs and writing the compatibility equations in terms of the spectral Darboux coordinates given by the apparent singularities $(q_i)_{1 \leq i \leq g}$ and their dual coordinates $(p_i)_{1 \leq i \leq g}$ allows to derive the evolutions relatively to this set of coordinates and prove that these evolutions are indeed Hamiltonian (Theorem 5.1). It should be noted that this result complements [30] by considering any isomonodromic deformation while the space of deformations obtained by confluence of simple poles has lower dimension. In the second part of the paper, we show that the initial space of isomonodromic deformations can be reduced to only g non-trivial deformations thus giving a minimal set of non-trivial isomonodromic times and isomonodromic deformations providing a Liouville-integrable Hamiltonian system. This reduction of the tangent space provides an explicit map between the irregular times and the non-trivial isomonodromic times as well as simplifications for the Hamiltonians (Theorem 7.1) for our choice of non-trivial isomonodromic times. Finally, in order to illustrate our method, we present several examples: the Painlevé equations and the second element of the Painlevé 2

hierarchy recovering the standard results of Jimbo-Miwa [44] and H. Chiba [21].

Recent works of P. Boalch and D. Yamakawa [13, 18, 72, 73] are also closely related to the present article. Indeed, in [13] P. Boalch proved that the isomonodromy equations of Jimbo-Miwa-Ueno constitute a flat Ehresmann connection on a non-linear fibre bundle and then constructed a symplectic structure on each fibre to prove afterwards that the parallel transport preserves the symplectic structure of the fibres providing a symplectic connection. Moreover, in [72], the author gives a complete flat symplectic Ehresmann connection on the total space of deformation parameters and provides a completely integrable non-autonomous Hamiltonian system associated to isomonodromic deformations. This result is complemented (in the much more general context of non-resonant isomonodromic deformations of meromorphic connections on the trivial principal G -bundle over $\mathbb{P}^1$, where G is any complex reductive group) in [73] with an explicit description of the fundamental two-forms for isomonodromy equations. As we will see below, we shall recover many features of these results for $G = GL_2(\mathbb{C})$. In particular, we shall provide explicit formulas for the Hamiltonians associated to general isomonodromic deformations and provide a decomposition of the deformation space in which the fundamental two-form may be reduced (Theorem 6.3). In other words, we propose (Cf. Theorem 5.2) an explicit birational map between the Jimbo-Miwa-Ueno/Boalch symplectic isomonodromy connection and the symplectic Ehresmann connection built from our Hamiltonians by providing explicit formulas for the time-dependent Hamiltonian systems characterizing the symplectic Ehresmann connection and explicit formulas for the Lax pairs characterizing the Jimbo-Miwa-Ueno/Boalch connection. Our construction recovers the Jimbo-Miwa-Ueno/Boalch connection since our Lax matrix $\tilde{L} \in \hat{F}_{\mathcal{R}, \mathbf{r}}$ is built from Birkhoff local diagonalizations at each pole (characterizing the base B of times $\mathbf{t}$) and solving the isomonodromy compatibility equations $\partial_{t_k} \tilde{L} = \partial_\lambda \tilde{A}_{t_k} + [\tilde{L}, \tilde{A}_{t_k}]$ using a specific set of Darboux coordinates $(\mathbf{q}, \mathbf{p})$ corresponding to apparent singularities $\mathbf{q}$ and their dual $\mathbf{p}$ on the spectral curve ($\det(p_i I_2 - \tilde{L}(q_i)) = 0$). In particular, we provide (See Definition 5.1) the explicit expression of the associated fundamental two-form Ω defined by Yamakawa [73]

$$\Omega = \sum_i dq_i \wedge dp_i + \sum_k d\text{Ham}_k(\mathbf{q}, \mathbf{p}, \mathbf{t}) \wedge dt_k$$

through the explicit expression of the time-dependent Hamiltonians in Theorem 5.1.

Note that our results differ from those of [72, 73] since the strategy is different. Indeed, D. Yamakawa uses the local diagonal gauge to derive his formulas. This strategy is natural from the geometric interpretation of isomonodromic deformations developed by P. Boalch [13] or to make connections with isospectral deformations. More precisely, D. Yamakawa first defines the isospectral Hamiltonians using the standard intrinsic residue-trace formulas at each pole. He then imposes some conditions on the exterior derivatives relatively to irregular times (Lemma 4.1 of [72]), i.e. implicitly selects special Darboux coordinates, so that the isospectral Hamiltonians match with the isomonodromic Hamiltonians. This strategy consists in using special Darboux coordinates to identify the isospectral invariants with the isomonodromic Hamiltonians for non-Fuchsian singularities has also been used in [1, 7, 21]. However, if the construction is geometrically very interesting and efficient, it does not provide explicit Darboux coordinates nor expressions for the Lax matrices and it requires substantial computations to apply (because one needs to compute all special local gauge matrices and connect them together) on examples. In our case, the existence of local diagonalizations remains the starting point and provides the natural framework for isomonodromic deformations. However, our strategy consists in going to the oper gauge (i.e. companion-like in $\mathfrak{gl}_2(\mathbb{C})$) which is uniquely defined for any meromorphic connection. It turns out that there are natural Darboux coordinates in this gauge, namely the

apparent singularities and their dual partners on the spectral curve, that we shall use to express the Lax matrices and the Hamiltonian evolutions following the standard and historical approach of Garnier and Jimbo-Miwa-Ueno [33, 34, 45]. In fact, the two strategies may be seen as solutions to the same problem, namely that for non-Fuchsian singularities isospectral invariants do not generally match with isomonodromic Hamiltonians. Therefore a first strategy is to select the Darboux coordinates specifically so that the identification remains valid [7, 72]. A second strategy, used in this paper and in [30], is to take the natural Darboux coordinates using apparent singularities and to relate explicitly the isomonodromic Hamiltonians to the isospectral invariants (Theorem 7.1). Let us finally mention that these two strategies have been merged for the $\mathfrak{gl}_2$ case in [56] during the final redaction of this paper by providing a time-dependent and birational change of coordinates relating both sets of Darboux coordinates. Up to this change of coordinates, the non-autonomous Hamiltonians and the fundamental symplectic two form (Theorems 6.3 and 7.1) computed in this paper should identify with those of Yamakawa [72, 73].

Note also that the oper gauge in which computations are carried out in this article, is equivalent to the so-called “quantum curve” or scalar form associated to the meromorphic connection and is currently a key feature associated to the quantization of moduli spaces. It can be seen as a generalization of the reduction of Schlesinger systems to Garnier systems in the presence of irregular singularities. The choice of apparent singularities as (half of) Darboux coordinates is also canonical since it extends the works of Jimbo-Miwa-Ueno and H. Chiba. Finally, our formulas allow for an immediate use in practice without additional computation and we are able to recover the Painlevé cases in a few lines. In the end, we believe that the two strategies are complementary and have their own interests. Although it is beyond the scope of this article to compare both approaches in details, we shall discuss about this question in Section 9.

Moreover, let us also stress that one of the main interests of the method developed in this article is that it bypasses the isospectral deformations setup that was used in previous papers to handle isomonodromic deformations. Nevertheless, our main results (Theorem 7.1) combined with those of [7] indicate that both formalisms are related in a very explicit way that has been very recently investigated in [56].

To sum up, the main results obtained in this article are the following:

- A general expression of the Lax pairs in terms of apparent singularities $(q_i)_{1 \leq i \leq g}$ and their dual coordinates $(p_i)_{1 \leq i \leq g}$ in Propositions 2.4, 4.2 and 4.3.
- A general expression of the evolutions of the Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ under any irregular time or any position of the poles in Theorem 5.1 and a proof that these evolutions are Hamiltonian with explicit expressions for the corresponding Hamiltonians. This provides an explicit expression of the fundamental symplectic two-form Ω in a canonical form in Definition 5.1.
- The explicit expressions of the Lax matrices and the Hamiltonians gives a birational map between the symplectic Ehresmann connection and the Jimbo-Miwa-Ueno/Boalch symplectic isomonodromy connection in Theorem 5.2.
- A reduction of the total isomonodromic deformations space to a subspace of non-trivial deformations (of dimension g) preserving the fundamental symplectic two-form Ω (Theorem 6.3). This reduction is equivalent to a map between the set of irregular times and location of the poles towards a set of trivial and non-trivial isomonodromic times that is provided in Definitions 6.2, 6.3, 6.4, 6.5 depending on the degree of the pole at infinity and the number of poles. In particular, this reduction provides a Liouville-integrable Hamiltonian system since we get as many Hamiltonians as non-trivial isomonodromic times.

- Simple formulas of the Hamiltonians in terms of the non-trivial isomonodromic times after a canonical choice of the trivial times are provided in Section 7. In particular, we obtain that the Hamiltonians are (time-dependent) linear combinations of the isospectral Hamiltonians (that are independent of the deformation) in Theorem 7.1. Coefficients of the linear combinations are explicit and recover recent results of [30].
- Application of the general construction to the Painlevé equations and Fuchsian systems is presented in Section 8. Moreover, the case of the second element of the Painlevé 2 hierarchy (which is a case of genus $g = 2$) is also presented in Section 8.7 and recovers results of H. Chiba [21].
- The connection with the quantization of classical spectral curves via topological recursion is presented as a by-product in Section 3.

The paper is meant to be self-contained and all details of each proof are presented in appendices for completeness. Maple files for each of the examples are available at <http://math.univ-lyon1.fr/~marchal/AdditionalRessources/index.html> and can be used as a check for the general formulas presented in this article.

2 Meromorphic connections, gauges and Darboux coordinates

2.1 Meromorphic connections and irregular times

The space of $\mathfrak{gl}_2(\mathbb{C})$ meromorphic connections has been studied from many different perspectives. In the present article, we shall mainly follow the point of view of the Montréal group [1, 2] together with some insights from the work of P. Boalch [13]. Let us first define the space we shall study.

Definition 2.1 (Space of rational connections). Let $n \in \mathbb{N}$ be a non-negative integer and $\{X_i\}_{i=1}^n$ be n distinct points in the complex plane. Let us denote $\mathcal{R} := \{\infty, X_1, \dots, X_n\}$ and $\mathcal{R}_0 := \{X_1, \dots, X_n\}$. For any $\mathbf{r} := (r_\infty, r_1, \dots, r_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$, let us define

$$F_{\mathcal{R}, \mathbf{r}} := \left\{ \hat{L}(\lambda) = \sum_{k=1}^{r_\infty-1} \hat{L}^{[\infty, k]} \lambda^{k-1} + \sum_{s=1}^n \sum_{k=0}^{r_s-1} \frac{\hat{L}^{[X_s, k]}}{(\lambda - X_s)^{k+1}} \text{ with } \{\hat{L}^{[p, k]}\} \in (\mathfrak{gl}_2)^{r-1} \right\} / \text{GL}_2(\mathbb{C}) \quad (2-1)$$

where $r = r_\infty + \sum_{s=1}^n r_s$ and $\text{GL}_2(\mathbb{C})$ acts simultaneously by conjugation on all coefficients $\{\hat{L}^{[p, k]}\}_{p, k}$.

Let us denote $\hat{F}_{\mathcal{R}, \mathbf{r}}$ the subspace in $F_{\mathcal{R}, \mathbf{r}}$ composed of elements with coefficients $\{\hat{L}^{[\infty, k]}\}_{0 \leq k \leq r_\infty-1} \cup \{\hat{L}^{[X_s, k]}\}_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ having distinct eigenvalues.⁴

Remark 2.1. In the present article, we shall assume that ∞ is always a pole following the standard convention. Of course, one may always use a change of coordinates in order to remove such assumption.

⁴The present work may easily be adapted to the case where some of the coefficients $\{\hat{L}^{[p, k]}\}_{p \in \mathcal{R}}$ are assumed to have a double eigenvalue with the additional assumption that $\{\hat{L}^{[p, r_p-1]}\}_{p \in \mathcal{R}}$ remains diagonalizable. In this case, the deformation space of Definition 4.1 is of lower dimension and the size of the corresponding upcoming matrices $(M_s)_{1 \leq s \leq n}$ or M_∞ shall be reduced. Definitions of trivial/isomonodromic times of Section 6.2 also require modifications left as an easy exercise.

$F_{\mathcal{R},\mathbf{r}}$ can be given a Poisson structure inherited from the Poisson structure of a corresponding loop algebra [37, 3, 71]. It is a Poisson space of dimension

$$\dim F_{\mathcal{R},\mathbf{r}} = 4r - 7. \quad (2-2)$$

The space $F_{\mathcal{R},\mathbf{r}}$ has been intensively studied from the point of view of isospectral and isomonodromic deformations. Following the works of P. Boalch [13, 14, 17] and of D. Yamakawa [72, 73], one can use the Poisson structure on $F_{\mathcal{R},\mathbf{r}}$ in order to describe it as a bundle whose fibers are symplectic leaves obtained by fixing the irregular type and monodromies of $\hat{L}(\lambda)$. Let us briefly review this perspective and use it to define local coordinates on $F_{\mathcal{R},\mathbf{r}}$ trivializing the fibration.

In this article, **we shall restrict to $\hat{F}_{\mathcal{R},\mathbf{r}}$ to simplify the presentation** but the present setup may surely be extended to $F_{\mathcal{R},\mathbf{r}}$.⁵

For any pole $p \in \mathcal{R}$, let us define a local coordinate

$$\forall p \in \mathcal{R}, z_p(\lambda) := \begin{cases} (\lambda - p) & \text{if } p \in \{X_1, \dots, X_n\} \\ \lambda^{-1} & \text{if } p = \infty. \end{cases} \quad (2-3)$$

Given $\hat{L}(\lambda)$ in an orbit of $\hat{F}_{\mathcal{R},\mathbf{r}}$, let $\hat{\Psi}(\lambda)$ be a wave matrix solution to the linear differential system

$$\partial_\lambda \hat{\Psi}(\lambda) = \hat{L}(\lambda) \hat{\Psi}(\lambda).$$

Then, for any pole $p \in \mathcal{R}$, there exists a gauge matrix $G_p \in \text{GL}_2[[z_p(\lambda)]]$ holomorphic at $\lambda = p$, which might be seen as a formal bundle automorphism, such that the gauge transformation $\Psi_p = G_p \hat{\Psi}$ provides

$$\Psi_p(\lambda) = \Psi_p^{(\text{reg})}(\lambda) \text{diag} \left(\exp \left(- \sum_{k=1}^{r_p-1} \frac{t_{p^{(1)},k}}{k z_p(\lambda)^k} + t_{p^{(1)},0} \ln z_p(\lambda) \right), \exp \left(- \sum_{k=1}^{r_p-1} \frac{t_{p^{(2)},k}}{k z_p(\lambda)^k} + t_{p^{(2)},0} \ln z_p(\lambda) \right) \right) \quad (2-4)$$

where $\Psi_p^{(\text{reg})}(\lambda)$ is regular at $\lambda = p$. It corresponds to a Lax matrix $L_p = G_p \hat{L}(\lambda) G_p^{-1} + (\partial_\lambda G_p) G_p^{-1}$ satisfying

$$G_p \hat{L}(\lambda) d\lambda G_p^{-1} + (\partial_\lambda G_p) G_p^{-1} d\lambda = dQ_p(z_p) + \Lambda_p \frac{dz_p}{z_p} \text{ where } Q_p(z_p) = \sum_{k=1}^{r_p-1} \frac{Q_{p,k}}{z_p^k} \quad (2-5)$$

with

$$Q_{p,k} = \text{diag} \left(-\frac{t_{p^{(1)},k}}{k}, -\frac{t_{p^{(2)},k}}{k} \right) \text{ and } \Lambda_p = \text{diag}(t_{p^{(1)},0}, t_{p^{(2)},0}), \forall p \in \mathcal{R} \quad (2-6)$$

for some complex numbers $(t_{p^{(1)},k}, t_{p^{(2)},k})_{p \in \mathcal{R}, 0 \leq k \leq r_p-1}$. $Q_p(z_p)$ is called the irregular type of $\hat{L}$ at p and Λ_p its residue (also called exponent of formal monodromy). Equation (2-4) is known as the Birkhoff factorization or formal normal solution or Turrutin-Levelt fundamental form [11, 70]. We shall denote $\mathbf{t} = (t_{p^{(i)},k})_{1 \leq i \leq 2, p \in \mathcal{R}, 1 \leq k \leq r_p-1}$ the irregular times while the residues $\mathbf{t}_0 := (t_{p^{(i)},0})_{1 \leq i \leq 2, p \in \mathcal{R}}$ will be referred to as monodromies by abuse of language.

⁵In particular, the case of the Painlevé 1 equation and hierarchy for which $\check{L}^{[\infty, r_\infty-1]}$ cannot be diagonalized has been investigated in [55] and provides results similar to the ones presented in this article, in other words, this article deals with the *generic* case only.

Remark 2.2. The relation of the present setup to the study of moduli spaces, seen as a set of isomorphism classes consisting of a generic connection over a trivialisable holomorphic vector bundle with a compatible framing, may be easily seen. However we shall not insist on this point since it is beyond the scope of this article and is left for future investigations.

Remark 2.3. In the literature, the set of irregular times $\mathbf{t} = \left(t_{p^{(i)},k}\right)_{1 \leq i \leq 2, p \in \mathcal{R}, 1 \leq k \leq r_p-1}$ is referred to as “spectral times” or “KP times”. This terminology originates from the study of isospectral systems and do not include the monodromy parameters $\left(t_{p^{(i)},0}\right)_{1 \leq i \leq 2, p \in \mathcal{R}}$.

2.2 Representative normalized at infinity

Fixing the irregular times $\mathbf{t}$ and monodromies $\mathbf{t}_0$ of $\hat{L}(\lambda)$ does not fix it uniquely. Indeed (Cf. [72]), in each orbit in $\hat{F}_{\mathcal{R},\mathbf{r}}$, there exists a unique element such that $\hat{L}^{[\infty, r_\infty-1]}$ is diagonal and such that the subleading order at $\lambda \rightarrow \infty$ is of the form

$$\operatorname{Res}_{\lambda \rightarrow \infty} \hat{L}(\lambda) \lambda^{-(r_\infty-2)} = - \begin{pmatrix} \beta_{r_\infty-2} & 1 \\ \delta_{r_\infty-2} & \gamma_{r_\infty-2} \end{pmatrix}. \quad (2-7)$$

In particular, since the Poisson structure is independent of the choice of representative in the orbit ([37, 3, 71, 72]), one can identify $\hat{F}_{\mathcal{R},\mathbf{r}}$ with the space of such representatives:

- If $r_\infty \geq 3$:

$$\begin{aligned} \hat{F}_{\mathcal{R},\mathbf{r}} := & \left\{ \tilde{L}(\lambda) = \sum_{k=1}^{r_\infty-1} \tilde{L}^{[\infty,k]} \lambda^{k-1} + \sum_{s=1}^n \sum_{k=0}^{r_s-1} \frac{\tilde{L}^{[X_s,k]}}{(\lambda - X_s)^{k+1}} \text{ such that} \right. \\ & \left\{ \tilde{L}^{[\infty,k]} \right\}_{1 \leq k \leq r_\infty-1} \cup \left\{ \tilde{L}^{[X_s,k]} \right\}_{1 \leq s \leq n, 0 \leq k \leq r_s-1} \in (\mathfrak{gl}_2(\mathbb{C}))^{r-1} \text{ have distinct eigenvalues and} \\ & \tilde{L}^{[\infty, r_\infty-1]} = \operatorname{diag}(-t_{\infty(1), r_\infty-1}, -t_{\infty(2), r_\infty-1}) \text{ and} \\ & \tilde{L}^{[\infty, r_\infty-2]} = \begin{pmatrix} \beta_{r_\infty-2} & 1 \\ \delta_{r_\infty-2} & \gamma_{r_\infty-2} \end{pmatrix}, (\beta_{r_\infty-2}, \delta_{r_\infty-2}, \gamma_{r_\infty-2}) \in \mathbb{C}^3 \left. \right\} \end{aligned} \quad (2-8)$$

- If $r_\infty = 2$:

$$\begin{aligned} \hat{F}_{\mathcal{R},\mathbf{r}} := & \left\{ \tilde{L}(\lambda) = \tilde{L}^{[\infty,1]} + \sum_{s=1}^n \sum_{k=0}^{r_s-1} \frac{\tilde{L}^{[X_s,k]}}{(\lambda - X_s)^{k+1}} \text{ such that} \right. \\ & \left\{ \tilde{L}^{[\infty,1]} \right\} \cup \left\{ \tilde{L}^{[X_s,k]} \right\}_{1 \leq s \leq n, 0 \leq k \leq r_s-1} \in (\mathfrak{gl}_2(\mathbb{C}))^{r-1} \text{ have distinct eigenvalues and} \\ & \tilde{L}^{[\infty,1]} = \operatorname{diag}(-t_{\infty(1),1}, -t_{\infty(2),1}) \text{ and } \sum_{s=1}^n \tilde{L}^{[X_s,0]} = \begin{pmatrix} \beta_0 & 1 \\ \delta_0 & \gamma_0 \end{pmatrix}, (\beta_0, \delta_0, \gamma_0) \in \mathbb{C}^3 \left. \right\} \end{aligned} \quad (2-9)$$

- If $r_\infty = 1$:

$$\begin{aligned} \hat{F}_{\mathcal{R},\mathbf{r}} := & \left\{ \tilde{L}(\lambda) = \sum_{s=1}^n \sum_{k=0}^{r_s-1} \frac{\tilde{L}^{[X_s,k]}}{(\lambda - X_s)^{k+1}} \text{ such that} \right. \\ & \left\{ \tilde{L}^{[X_s,k]} \right\}_{1 \leq s \leq n, 0 \leq k \leq r_s-1} \in (\mathfrak{gl}_2(\mathbb{C}))^{r-1} \text{ have distinct eigenvalues and} \\ & \sum_{s=1}^n \tilde{L}^{[X_s,0]} = \operatorname{diag}(-t_{\infty(1),0}, -t_{\infty(2),0}) \text{ and} \\ & \sum_{s=1}^n \tilde{L}^{[X_s,1]} + \sum_{s=1}^n X_s \tilde{L}^{[X_s,0]} = \begin{pmatrix} \beta_{-1} & 1 \\ \delta_{-1} & \gamma_{-1} \end{pmatrix}, (\beta_{-1}, \delta_{-1}, \gamma_{-1}) \in \mathbb{C}^3 \left. \right\} \end{aligned} \quad (2-10)$$

In the following, we shall use the notation $\tilde{L}(\lambda)$ whenever we consider such a representative and we shall call it a representative “normalized at infinity” to stress that the $\text{GL}_2(\mathbb{C})$ global conjugation action has been used to select a representative of the orbit specifically normalized at infinity.

Remark 2.4. The choice of normalization at infinity, implies that coefficients $\beta_{r_\infty-2}$ and $\gamma_{r_\infty-2}$ are directly connected to the irregular times $-t_{\infty(1),r_\infty-2}$ and $-t_{\infty(2),r_\infty-2}$ for $r_\infty \geq 2$. Indeed, let us note that the diagonalization of the singular part given by (2-5) implies that

$$\begin{aligned} \det(\tilde{L}(\lambda) + G_\infty^{-1} \partial_\lambda G_\infty) &\stackrel{\lambda \rightarrow \infty}{=} \left(\sum_{k=0}^{r_\infty-1} t_{\infty(1),k} \lambda^{k-1} + O(\lambda^{-2}) \right) \left(\sum_{k=0}^{r_\infty-1} t_{\infty(2),k} \lambda^{k-1} + O(\lambda^{-2}) \right) \\ &= t_{\infty(1),r_\infty-1} t_{\infty(2),r_\infty-1} \lambda^{2r_\infty-4} + (t_{\infty(1),r_\infty-1} t_{\infty(2),r_\infty-2} + t_{\infty(2),r_\infty-1} t_{\infty(1),r_\infty-2}) \lambda^{2r_\infty-5} \\ &\quad + O(\lambda^{2r_\infty-6}). \end{aligned} \tag{2-11}$$

The l.h.s. is of the form:

- If $r_\infty \geq 3$:

$$\begin{aligned} \det(\tilde{L}(\lambda) + G_\infty^{-1} \partial_\lambda G_\infty) &\stackrel{\lambda \rightarrow \infty}{=} \begin{vmatrix} -t_{\infty(1),r_\infty-1} \lambda^{r_\infty-2} + \beta_{r_\infty-2} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) & \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) \\ \delta_{r_\infty-2} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) & -t_{\infty(2),r_\infty-1} \lambda^{r_\infty-2} + \gamma_{r_\infty-2} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}) \end{vmatrix} \\ &= t_{\infty(1),r_\infty-1} t_{\infty(2),r_\infty-1} \lambda^{2r_\infty-4} - (\gamma_{r_\infty-2} t_{\infty(1),r_\infty-1} + \beta_{r_\infty-2} t_{\infty(2),r_\infty-1}) \lambda^{2r_\infty-5} + O(\lambda^{2r_\infty-6}). \end{aligned} \tag{2-12}$$

Moreover, the diagonalization of the singular part also implies that for $r_\infty \geq 3$:

$$\text{Tr} \tilde{L}(\lambda) \stackrel{\lambda \rightarrow \infty}{=} (t_{\infty(1),r_\infty-1} + t_{\infty(2),r_\infty-1}) \lambda^{r_\infty-2} + (t_{\infty(1),r_\infty-2} + t_{\infty(2),r_\infty-2}) \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}). \tag{2-13}$$

Identifying (2-12) with the r.h.s. of (2-11) and using (2-13), we get

$$\beta_{r_\infty-2} = -t_{\infty(1),r_\infty-2} \quad \text{and} \quad \gamma_{r_\infty-2} = -t_{\infty(2),r_\infty-2}. \tag{2-14}$$

- If $r_\infty = 2$, we observe that the matrix $G_\infty(\lambda)$ is of the form $G_\infty(\lambda) = G_0 + G_1 \lambda^{-1} + O(\lambda^{-2})$ with G_0 diagonal (in order to preserve the fact that leading order is already diagonal). Consequently $G_\infty^{-1} \partial_\lambda G_\infty = O(\lambda^{-2})$. The diagonalization (2-5) implies

$$\det(\tilde{L}(\lambda) + G_\infty^{-1} \partial_\lambda G_\infty) \stackrel{\lambda \rightarrow \infty}{=} t_{\infty(1),1} t_{\infty(2),1} + (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0}) \lambda^{-1} + O(\lambda^{-2}). \tag{2-15}$$

Since $G_\infty^{-1} \partial_\lambda G_\infty = O(\lambda^{-2})$, the l.h.s. is of the form

$$\begin{aligned} \det(\tilde{L}(\lambda) + G_\infty^{-1} \partial_\lambda G_\infty) &\stackrel{\lambda \rightarrow \infty}{=} \begin{vmatrix} -t_{\infty(1),1} + \beta_0 \lambda^{-1} + O(\lambda^{-2}) & \lambda^{-1} + O(\lambda^{-2}) \\ \delta_0 \lambda^{-1} + O(\lambda^{-2}) & -t_{\infty(2),1} + \gamma_0 \lambda^{-1} + O(\lambda^{-2}) \end{vmatrix} \\ &= t_{\infty(1),1} t_{\infty(2),1} + (\gamma_0 t_{\infty(1),1} + \beta_0 t_{\infty(2),1}) \lambda^{-1} + O(\lambda^{-2}). \end{aligned} \tag{2-16}$$

Moreover, the diagonalization of the singular part also implies that for $r_\infty = 2$:

$$\text{Tr} \tilde{L}(\lambda) \stackrel{\lambda \rightarrow \infty}{=} (t_{\infty(1),1} + t_{\infty(2),1}) + (t_{\infty(1),0} + t_{\infty(2),0}) \lambda^{-1} + O(\lambda^{-2}). \tag{2-17}$$

Identifying (2-16) with the r.h.s. of (2-15) and using (2-17) we get

$$\beta_0 = -t_{\infty(1),0} \quad \text{and} \quad \gamma_0 = -t_{\infty(2),0}. \tag{2-18}$$

- If $r_\infty = 1$ we simply have

$$\beta_{-1} = \sum_{s=1}^n \left[\tilde{L}^{[X_s,1]} + X_s \tilde{L}^{[X_s,0]} \right]_{1,1}, \quad \gamma_{-1} = \sum_{s=1}^n \left[\tilde{L}^{[X_s,1]} + X_s \tilde{L}^{[X_s,0]} \right]_{2,2}. \tag{2-19}$$

2.3 Darboux coordinates and general isomonodromic deformations

Let $\hat{F}_{\mathcal{R},\mathbf{r},\mathbf{t}_0} \subset \hat{F}_{\mathcal{R},\mathbf{r}}$ be the subset of $\hat{F}_{\mathcal{R},\mathbf{r}}$ composed of $\hat{L}(\lambda)$ with fixed monodromies $\mathbf{t}_0$.

Let us first recall [37, 3, 71, 14] that the set

$$\hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t}_0} := \left\{ \hat{L}(\lambda) \in \hat{F}_{\mathcal{R},\mathbf{r}} / \hat{L}(\lambda) \text{ has irregular times } \mathbf{t} \text{ and monodromies } \mathbf{t}_0 \right\} \quad (2-20)$$

is a symplectic manifold of dimension

$$\dim \hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t}_0} = 4r - 7 - (2r - 1) = 2g \quad \text{where } g := r - 3 \quad (2-21)$$

is the genus of the spectral curve defined by the algebraic equation $\det(yI_2 - \hat{L}(\lambda)) = 0$. **In the rest of the paper we shall assume that $g > 0$ corresponding to a non-trivial symplectic structure.** For a generic value of the irregular times $\mathbf{t}$ and monodromies $\mathbf{t}_0$, the Montréal group introduced a set of local Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ on $\hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t}_0}$ which can be obtained in practice as follows.

Let $\tilde{L}(\lambda) \in \hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t}_0}$ be a representative of the form described in Section 2.2. The entry $\left[\tilde{L}(\lambda) \right]_{1,2}$ is a rational function of λ with g zeros denoted $(q_i)_{1 \leq i \leq g}$:

$$\forall i \in \llbracket 1, g \rrbracket : \left[\tilde{L}(q_i) \right]_{1,2} = 0. \quad (2-22)$$

This defines half of the spectral Darboux coordinates. The second half is obtained by evaluating the entry $\left[\tilde{L}(\lambda) \right]_{1,1}$ at $\lambda = q_i$:

$$\forall i \in \llbracket 1, g \rrbracket : p_i := \left[\tilde{L}(q_i) \right]_{1,1}. \quad (2-23)$$

Let us remark that, by definition, the pair (q_i, p_i) defines a point on the spectral curve for any $i \in \llbracket 1, g \rrbracket$:

$$\forall i \in \llbracket 1, g \rrbracket : \det(p_i I_2 - \tilde{L}(q_i)) = 0. \quad (2-24)$$

Thus, we have obtained a local description of the space $\hat{F}_{\mathcal{R},\mathbf{r},\mathbf{t}_0}$ as a trivial bundle $\hat{F}_{\mathcal{R},\mathbf{r},\mathbf{t}_0} \rightarrow B$ where the base B is the set of irregular times satisfying the condition

$$B := \left\{ \mathbf{t} \in \mathbb{C}^{2(r-n-1)} / \forall p \in \mathcal{R}, \forall k \in \llbracket 1, r_p - 1 \rrbracket, t_{p^{(1)},k} \neq t_{p^{(2)},k} \right\}. \quad (2-25)$$

In particular, the fiber above a point $\mathbf{t} \in B$ is $\hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t}_0}$ which can be equipped with spectral Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$.

The space B is a space of isomonodromic deformations meaning that any vector field $\partial_t \in T_{\mathbf{t}}B$ gives rise to a deformation of $\tilde{L}(\lambda)$ preserving its generalized monodromy data. In addition to deformations relatively to a vector $\partial_t \in T_{\mathbf{t}}B$ we shall also consider the standard deformations relatively to the position $(X_i)_{1 \leq i \leq n}$ of the finite poles.

There exist different equivalent ways to characterize the property of being an isomonodromic vector field. One of them [3] is the existence of a compatible system, referred to as a Lax pair, of the form

$$\begin{cases} \partial_\lambda \tilde{\Psi}(\lambda, \mathbf{t}) = \tilde{L}(\lambda) \tilde{\Psi}(\lambda, \mathbf{t}) \\ \partial_t \tilde{\Psi}(\lambda, \mathbf{t}) = \tilde{A}_t(\lambda) \tilde{\Psi}(\lambda, \mathbf{t}) \end{cases} \quad (2-26)$$

where $\tilde{A}_t(\lambda)$ is a rational function of λ with poles dominated by the poles of $\tilde{L}(\lambda)$. In the rest of the paper, we shall extensively study the compatibility condition (zero-curvature equation)

$$\partial_\lambda \tilde{A}_t(\lambda) - \partial_t \tilde{L}(\lambda) + \left[\tilde{L}(\lambda), \tilde{A}_t(\lambda) \right] = 0. \quad (2-27)$$

obtained from (2-26).

2.4 Scalar differential equation, oper gauge and local diagonal gauges

Let us now consider an orbit in $\hat{F}_{\mathcal{R}, \mathbf{r}}$ and a representative $\tilde{L}(\lambda)$ of this orbit normalized at infinity as described above. Let $\tilde{\Psi}(\lambda)$ be a wave matrix solution to the linear system

$$\partial_\lambda \tilde{\Psi}(\lambda) = \tilde{L}(\lambda) \tilde{\Psi}(\lambda). \quad (2-28)$$

With the notations above, the Birkhoff factorization implies that there exist local holomorphic gauge transformations $\Psi_\infty = G_\infty \tilde{\Psi}$ and for all $s \in \llbracket 1, n \rrbracket$: $\Psi_{X_s} = G_{X_s} \tilde{\Psi}$ in which the corresponding Lax matrices $\tilde{L}_\infty$ and $(\tilde{L}_{X_s})_{1 \leq s \leq n}$ have their singular part diagonalized:

$$\begin{aligned} \tilde{L}_\infty &\stackrel{\lambda \rightarrow \infty}{=} \text{diag} \left(- \sum_{k=0}^{r_\infty-1} t_{\infty(1),k} \lambda^{k-1}, - \sum_{k=0}^{r_\infty-1} t_{\infty(2),k} \lambda^{k-1} \right) + o(\lambda^{-1}), \\ \tilde{L}_{X_s} &\stackrel{\lambda \rightarrow X_s}{=} \text{diag} \left(- \sum_{k=0}^{r_s-1} t_{X_s(1),k} (\lambda - X_s)^{-k-1}, - \sum_{k=0}^{r_s-1} t_{X_s(2),k} (\lambda - X_s)^{-k-1} \right) + o(1) \end{aligned} \quad (2-29)$$

and that the corresponding wave matrices are taken into their fundamental normal form according to (2-4). Note that the assumption that the poles are non-ramified is equivalent to the fact that the singular part is diagonalizable with distinct eigenvalues.

Moreover, the differential system $\partial_\lambda \tilde{\Psi}(\lambda) = \tilde{L}(\lambda) \tilde{\Psi}(\lambda)$ may be written as a scalar differential equation for $\tilde{\Psi}_{1,1}$ that is equivalent to a companion-like (oper) matrix system. More precisely, defining

$$\Psi(\lambda) = G(\lambda) \tilde{\Psi}(\lambda) \text{ with } G(\lambda) = \begin{pmatrix} 1 & 0 \\ \tilde{L}_{1,1} & \tilde{L}_{1,2} \end{pmatrix} \quad (2-30)$$

we end up with the fact that Ψ is a solution of the companion-like system

$$\partial_\lambda \Psi(\lambda) = L(\lambda) \Psi(\lambda) \text{ with } L(\lambda) = \begin{pmatrix} 0 & 1 \\ L_{2,1} & L_{2,2} \end{pmatrix} \quad (2-31)$$

given by

$$\begin{aligned} L_{2,1} &= -\det \tilde{L} + \partial_\lambda \tilde{L}_{1,1} - \tilde{L}_{1,1} \frac{\partial_\lambda \tilde{L}_{1,2}}{\tilde{L}_{1,2}}, \\ L_{2,2} &= \text{Tr } \tilde{L} + \frac{\partial_\lambda \tilde{L}_{1,2}}{\tilde{L}_{1,2}}. \end{aligned} \quad (2-32)$$

Note in particular that the first line of Ψ and $\tilde{\Psi}$ is obviously the same: $\Psi_{1,1} = \tilde{\Psi}_{1,1} \stackrel{\text{def}}{=} \psi_1$ and $\Psi_{1,2} = \tilde{\Psi}_{1,2} \stackrel{\text{def}}{=} \psi_2$. Hence, we immediately obtain that

$$\Psi(\lambda) = \begin{pmatrix} \tilde{\Psi}_{1,1}(\lambda) & \tilde{\Psi}_{1,2}(\lambda) \\ \partial_\lambda \tilde{\Psi}_{1,1}(\lambda) & \partial_\lambda \tilde{\Psi}_{1,2}(\lambda) \end{pmatrix} = \begin{pmatrix} \psi_1(\lambda) & \psi_2(\lambda) \\ \partial_\lambda \psi_1(\lambda) & \partial_\lambda \psi_2(\lambda) \end{pmatrix}. \quad (2-33)$$

The companion-like system (2-31) is equivalent to say that ψ_1 and ψ_2 satisfy the linear ODE:

$$([\partial_\lambda]^2 - L_{2,2}(\lambda) \partial_\lambda - L_{2,1}(\lambda)) \psi_i = 0, \quad i \in \{1, 2\} \quad (2-34)$$

which is sometimes referred to as the “quantum curve”. The companion gauge is also referred to as the “oper gauge” in part of the literature.

2.5 Introduction of a scaling parameter $\hbar$

In order to make the connection with formal $\hbar$ -transseries appearing in the quantization of classical spectral curves via topological recursion of [26], we shall also introduce a formal $\hbar$ parameter by a simple rescaling of the location of the poles, monodromies and irregular times. We shall perform the following rescaling⁶:

$$\begin{aligned} t_{\infty^{(i)},k} &\rightarrow \hbar^{k-1} t_{\infty^{(i)},k}, \quad \forall (i,k) \in \llbracket 1, 2 \rrbracket \times \llbracket 0, r_\infty - 1 \rrbracket, \\ t_{X_s^{(i)},k} &\rightarrow \hbar^{-1-k} t_{X_s^{(i)},k}, \quad \forall (i,s,k) \in \llbracket 1, 2 \rrbracket \times \llbracket 1, n \rrbracket \times \llbracket 0, r_s - 1 \rrbracket, \\ X_s &\rightarrow \hbar^{-1} X_s, \quad \forall s \in \llbracket 1, n \rrbracket, \\ \lambda &\rightarrow \hbar^{-1} \lambda \\ \Psi &\rightarrow \text{diag} \left(\hbar^{-\frac{r_\infty-3}{2}}, \hbar^{\frac{r_\infty-3}{2}} \right) \Psi. \end{aligned} \quad (2-35)$$

In particular the differential system now reads

$$\hbar \partial_\lambda \tilde{\Psi}(\lambda, \hbar) = \tilde{L}(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar). \quad (2-36)$$

Such differential systems are known as $\hbar$ -connections. The problem of Abelianisation of $\hbar$ -connections is presently a very active domain in relation with exact WKB, Borel resummation, etc. This provides the main motivation to introduce the formal $\hbar$ parameter in the rest of the paper. However, for readers uneasy with this additional $\hbar$ parameter, **we stress that $\hbar$ may be fixed to 1 in the rest of the paper** except for Section 3.

2.6 Explicit expression for the gauge transformations and Lax matrices

In order to relate the matrices Ψ to $\tilde{\Psi}$, we shall introduce an intermediate wave matrix $\tilde{\Psi}$. Let us define the following gauge transformations.

Proposition 2.1. *The matrices are related by the gauge transformations*

$$\begin{aligned} \tilde{\Psi}(\lambda, \hbar) &= G_1(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar) \quad \text{with} \quad G_1(\lambda, \hbar) = \begin{pmatrix} 1 & 0 \\ t_{\infty^{(1)}, r_\infty-1} \lambda + \eta_0 & 1 \end{pmatrix}, \\ \tilde{\Psi}(\lambda, \hbar) &= J(\lambda, \hbar) \Psi(\lambda, \hbar) \quad \text{with} \quad J(\lambda, \hbar) = \begin{pmatrix} 1 & 0 \\ \frac{Q(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - q_j)} & \frac{\prod_{s=1}^n (\lambda - X_s)^{r_s}}{\prod_{j=1}^g (\lambda - q_j)} \end{pmatrix} \end{aligned} \quad (2-37)$$

where Q is the unique polynomial in λ of degree $g-1$ such that (with the convention that empty products are set to 1)

$$Q(q_i, \hbar) = -p_i \prod_{s=1}^n (q_i - X_s)^{r_s}, \quad \forall i \in \llbracket 1, g \rrbracket, \quad (2-38)$$

i.e.

$$Q(\lambda, \hbar) = - \sum_{i=1}^g p_i \prod_{s=1}^n (q_i - X_s)^{r_s} \prod_{j \neq i} \frac{\lambda - q_j}{q_i - q_j}, \quad (2-39)$$

and η_0 is given by

$$\eta_0 = t_{\infty^{(1)}, r_\infty-2} + t_{\infty^{(1)}, r_\infty-1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) \quad \text{if } r_\infty \geq 2,$$

⁶The change on Ψ is made so that the normalization at infinity is preserved by the rescaling.

$$\eta_0 = t_{\infty(1),0} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) - \sum_{s=1}^n \left[X_s \tilde{L}^{[X_s,0]} + \tilde{L}^{[X_s,1]} \right]_{1,1} \quad \text{if } r_\infty = 1. \quad (2-40)$$

Proof. The proof is based on the fact that

$$\tilde{G}(\lambda) = (G_1(\lambda, \hbar) J(\lambda, \hbar))^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{-Q(\lambda) - (t_{\infty(1),r_\infty-1} \lambda + \eta_0) \prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} & \frac{\prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \end{pmatrix} \quad (2-41)$$

recovers the matrix (2-30). It is done in details in Appendix A. $\square$

We shall also introduce a matrix Ψ_{GMR} in order to connect (See Proposition 2.2) the present work with those of Gaiur, Mazzocco and Rubtsov [30] that obtained similar results using confluences of simple poles.

$$\Psi_{\text{GMR}}(\lambda, \hbar) := G_{\text{GMR}}(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar) \quad \text{with} \quad G_{\text{GMR}}(\lambda, \hbar) := \begin{pmatrix} 1 & 0 \\ -\frac{Q(\lambda, \hbar)}{\prod_{i=1}^n (\lambda - q_i)} & 1 \end{pmatrix}. \quad (2-42)$$

The gauge transformation is such that $\tilde{\Psi}(\lambda, \hbar)$, $\check{\Psi}(\lambda, \hbar)$, $\Psi(\lambda, \hbar)$ and $\Psi_{\text{GMR}}(\lambda, \hbar)$ satisfy the Lax systems,

$$\begin{aligned} \hbar \partial_\lambda \tilde{\Psi}(\lambda, \hbar) &= \tilde{L}(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar), \\ \hbar \partial_\lambda \check{\Psi}(\lambda, \hbar) &= \check{L}(\lambda, \hbar) \check{\Psi}(\lambda, \hbar), \\ \hbar \partial_\lambda \Psi(\lambda, \hbar) &= L(\lambda, \hbar) \Psi(\lambda, \hbar), \\ \hbar \partial_\lambda \Psi_{\text{GMR}}(\lambda, \hbar) &= L_{\text{GMR}}(\lambda, \hbar) \Psi_{\text{GMR}}(\lambda, \hbar). \end{aligned} \quad (2-43)$$

Remark 2.5. The intermediate matrices $\tilde{\Psi}$ and $\check{L}$ have been used in [57, 25]. In this gauge there are no apparent singularities at $\lambda = q_i$ but the normalization at infinity does not match with the choice of the representative defined in Section 2.2, hence the necessity of the additional gauge transformation G_1 .

Note that by definition, the entries of $\check{L}$ are related to those of L by

$$\begin{aligned} \check{L}_{1,1}(\lambda, \hbar) &= -\frac{Q(\lambda, \hbar)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}}, \\ \check{L}_{1,2}(\lambda, \hbar) &= \frac{\prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}}, \\ \check{L}_{2,2}(\lambda, \hbar) &= P_1(\lambda) + \frac{Q(\lambda, \hbar)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}}, \\ \check{L}_{2,1}(\lambda, \hbar) &= \hbar \frac{\partial \left(\frac{Q(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - q_j)} \right)}{\partial \lambda} + L_{2,1}(\lambda, \hbar) \frac{\prod_{s=1}^n (\lambda - X_s)^{r_s}}{\prod_{j=1}^g (\lambda - q_j)} \end{aligned}$$

$$-P_1(\lambda) \frac{Q(\lambda, \hbar)}{\prod_{j=1}^g (\lambda - q_j)} - \frac{Q(\lambda, \hbar)^2}{\prod_{j=1}^g (\lambda - q_j) \prod_{s=1}^n (\lambda - X_s)^{r_s}} \quad (2-44)$$

where we have defined

$$\begin{aligned} P_{\infty, k}^{(1)} &= -(t_{\infty^{(1)}, k+1} + t_{\infty^{(2)}, k+1}), \quad \forall k \in \llbracket 0, r_{\infty} - 2 \rrbracket, \\ P_{X_s, k}^{(1)} &= t_{X_s^{(1)}, k-1} + t_{X_s^{(2)}, k-1}, \quad \forall s \in \llbracket 1, n \rrbracket, \quad k \in \llbracket 1, r_s \rrbracket \end{aligned} \quad (2-45)$$

and regrouped them into the rational function P_1 :

$$P_1(\lambda) = \sum_{j=0}^{r_{\infty}-2} P_{\infty, j}^{(1)} \lambda^j + \sum_{s=1}^n \sum_{j=1}^{r_s} P_{X_s, j}^{(1)} (\lambda - X_s)^{-j}. \quad (2-46)$$

Similarly, the entries of $\tilde{L}$ are related to those of $\check{L}$ by

$$\begin{aligned} \tilde{L}_{1,1}(\lambda, \hbar) &= \check{L}_{1,1}(\lambda, \hbar) - (t_{\infty^{(1)}, r_{\infty}-1} \lambda + \eta_0) \check{L}_{1,2}(\lambda, \hbar) \\ \tilde{L}_{1,2}(\lambda, \hbar) &= \check{L}_{1,2}(\lambda, \hbar) \\ \tilde{L}_{2,1}(\lambda, \hbar) &= \check{L}_{2,1}(\lambda, \hbar) - (t_{\infty^{(1)}, r_{\infty}-1} \lambda + \eta_0)^2 \check{L}_{1,2}(\lambda, \hbar) \\ &\quad + (t_{\infty^{(1)}, r_{\infty}-1} \lambda + \eta_0) (\check{L}_{1,1}(\lambda, \hbar) - \check{L}_{2,2}(\lambda, \hbar)) + \hbar t_{\infty^{(1)}, r_{\infty}-1} \\ \tilde{L}_{2,2}(\lambda, \hbar) &= \check{L}_{2,2}(\lambda, \hbar) + (t_{\infty^{(1)}, r_{\infty}-1} \lambda + \eta_0) \check{L}_{1,2}(\lambda, \hbar). \end{aligned} \quad (2-47)$$

Finally, let us note that the previous results imply the following proposition

Proposition 2.2. *We have*

$$L_{GMR}(\lambda, \hbar) = \begin{pmatrix} 0 & \frac{\prod_{i=1}^n (\lambda - q_i)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \\ \frac{\prod_{s=1}^n (\lambda - X_s)^{r_s}}{\prod_{i=1}^n (\lambda - q_i)} L_{2,1}(\lambda, \hbar) & P_1(\lambda) \end{pmatrix} \quad (2-48)$$

so that

$$L_{2,1}(\lambda, \hbar) = -\det(L_{GMR}(\lambda, \hbar)) = \frac{1}{2} \text{Tr}(L_{GMR}(\lambda, \hbar)^2) - \frac{1}{2} P_1(\lambda)^2. \quad (2-49)$$

Proof. The proof follows from direct computations. $\square$

We shall also define the corresponding Wronskians and obtain their explicit expressions:

Definition 2.2 (Wronskians). Let us define $W(\lambda, \hbar) = \det \Psi(\lambda, \hbar)$, $\check{W}(\lambda, \hbar) = \det \check{\Psi}(\lambda, \hbar)$ and $\tilde{W}(\lambda, \hbar) = \det \tilde{\Psi}(\lambda, \hbar)$ the Wronskians associated to the corresponding wave matrices. They are given by

$$\begin{aligned} \tilde{W}(\lambda) &= \tilde{W}_0 \exp \left(\frac{1}{\hbar} \int_0^\lambda P_1(\lambda) d\lambda \right), \\ \check{W}(\lambda) &= \check{W}(\lambda) = \check{W}_0 \exp \left(\frac{1}{\hbar} \int_0^\lambda P_1(\lambda) d\lambda \right) \\ W(\lambda) &= W_0 \frac{\prod_{i=1}^g (\lambda - q_i)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \exp \left(\frac{1}{\hbar} \int_0^\lambda P_1(\tilde{\lambda}) d\tilde{\lambda} \right). \end{aligned} \quad (2-50)$$

Combining the gauge transformations $(G_p)_{p \in \mathcal{R}}$, G_1 and J , we get the following proposition.

Proposition 2.3. *The scalar wave functions $\psi_1 = \Psi_{1,1} = \tilde{\Psi}_{1,1} = \check{\Psi}_{1,1}$ and $\psi_2 = \Psi_{1,2} = \tilde{\Psi}_{1,2} = \check{\Psi}_{1,2}$ have the following expansions around each pole of $\mathcal{R}$.*

$$\begin{aligned}
\psi_1(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_\infty-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(1)},0} \ln \lambda + A_{\infty^{(1)},0} + O(\lambda^{-1}) \right), \\
\psi_2(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_\infty-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(2)},0} \ln \lambda - \ln \lambda + A_{\infty^{(2)},0} + O(\lambda^{-1}) \right), \\
\psi_1(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_s-1} \frac{t_{X_s^{(1)},k}}{k} (\lambda - X_s)^{-k} + \frac{1}{\hbar} t_{X_s^{(1)},0} \ln(\lambda - X_s) + A_{X_s^{(1)},0} + O(\lambda - X_s) \right), \\
\psi_2(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_s-1} \frac{t_{X_s^{(2)},k}}{k} (\lambda - X_s)^{-k} + \frac{1}{\hbar} t_{X_s^{(2)},0} \ln(\lambda - X_s) + A_{X_s^{(2)},0} + O(\lambda - X_s) \right)
\end{aligned} \tag{2-51}$$

for all $s \in \llbracket 1, n \rrbracket$.

It is important to remark that the logarithmic terms in the expansions around ∞ differ in ψ_1 and ψ_2 by a shift $+\hbar$ due to the gauge transformations. The reason is presented in the proof done in Appendix B.

Remark 2.6. Note that we recover the shift by $\hbar$ in the asymptotic expansion of ψ_2 at infinity that was arising in [25]. In [25], the shift arises from the choice of $\infty^{(1)}$ as a base point for integrating the Eynard-Orantin differentials. The present setup shows that this choice maps directly to the choice of normalization of the Lax matrix at infinity, i.e. a choice of representative of the orbit in $\hat{F}_{\mathcal{R},\mathbf{r}}$.

2.7 Explicit expression for the Lax matrix L

We shall now write an explicit formula for the Lax matrix L . In order to do so, we introduce the following quantities.

Definition 2.3. We define:

$$\begin{aligned}
P_{\infty, 2r_\infty-4-k}^{(2)} &= \sum_{j=0}^k t_{\infty^{(1)}, r_\infty-1-j} t_{\infty^{(2)}, r_\infty-1-(k-j)}, \quad \forall k \in \llbracket 0, r_\infty-1 \rrbracket, \\
P_{X_s, 2r_s-k}^{(2)} &= \sum_{j=0}^k t_{X_s^{(1)}, r_s-1-j} t_{X_s^{(2)}, r_s-1-(k-j)}, \quad \forall s \in \llbracket 1, n \rrbracket, \quad \forall k \in \llbracket 0, r_s-1 \rrbracket
\end{aligned} \tag{2-52}$$

and regroup the previous quantities to define rational functions P_2 and $\tilde{P}_2$ by

$$\begin{aligned}
P_2(\lambda) &= \sum_{j=0}^{2r_\infty-4} P_{\infty,j}^{(2)} \lambda^j + \sum_{s=1}^n \sum_{j=1}^{2r_s} \frac{P_{X_s,j}^{(2)}}{(\lambda - X_s)^j}, \\
\tilde{P}_2(\lambda) &= \sum_{j=\max(0, r_\infty-3)}^{2r_\infty-4} P_{\infty,j}^{(2)} \lambda^j + \sum_{s=1}^n \sum_{j=r_s+1}^{2r_s} \frac{P_{X_s,j}^{(2)}}{(\lambda - X_s)^j}.
\end{aligned} \tag{2-53}$$

Remark 2.7. Note that the coefficients $\left(P_{\infty,k}^{(2)}\right)_{0 \leq k \leq r_\infty-4}$ and $\left(P_{X_s,k}^{(2)}\right)_{1 \leq k \leq r_s}$ for all $s \in \llbracket 1, n \rrbracket$ remain undetermined at this stage. As we will see below, they shall correspond to the Hamiltonians. On the contrary, the coefficients of $\tilde{P}_2$, that are completely determined by the monodromies and irregular times, are often referred to as the Casimirs to remind their origin in the associated isospectral system.

Studying the asymptotics at each pole using (2-51), we obtain the general form of the Lax matrix $L(\lambda, \hbar)$.

Proposition 2.4 (General form of the Lax matrix). *The Lax matrix has the following entries*

$$\begin{aligned} L_{1,1}(\lambda) &= 0, \\ L_{1,2}(\lambda) &= 1, \\ L_{2,1}(\lambda) &= -\tilde{P}_2(\lambda) + \sum_{j=0}^{r_\infty-4} H_{\infty,j} \lambda^j + \sum_{s=1}^n \sum_{j=1}^{r_s} H_{X_s,j} (\lambda - X_s)^{-j} \\ &\quad - \hbar t_{\infty^{(1)}, r_\infty-1} \lambda^{r_\infty-3} \delta_{r_\infty \geq 3} - \sum_{j=1}^g \frac{\hbar p_j}{\lambda - q_j}, \\ L_{2,2}(\lambda) &= P_1(\lambda) + \sum_{j=1}^g \frac{\hbar}{\lambda - q_j} - \sum_{s=1}^n \frac{\hbar r_s}{\lambda - X_s}. \end{aligned} \quad (2-54)$$

Moreover for $r_\infty = 2$, we have from Remark C.1,

$$\begin{aligned} P_{\infty,0}^{(2)} &= t_{\infty^{(1)},1} t_{\infty^{(2)},1}, \\ \sum_{s=1}^n H_{X_s,1} &= \hbar \sum_{j=1}^g p_j - (t_{\infty^{(1)},1} t_{\infty^{(2)},0} + t_{\infty^{(2)},1} t_{\infty^{(1)},0} + \hbar t_{\infty^{(1)},1}) \end{aligned} \quad (2-55)$$

while, for $r_\infty = 1$, we have

$$\begin{aligned} \sum_{s=1}^n H_{X_s,1} &= \hbar \sum_{j=1}^g p_j, \\ \sum_{s=1}^n X_s H_{X_s,1} + \sum_{s=1}^n H_{X_s,2} \delta_{r_s \geq 2} - \sum_{s=1}^n (t_{X_s,0})^2 \delta_{r_s=1} &= \hbar \sum_{j=1}^g q_j p_j - t_{\infty^{(1)},0} (t_{\infty^{(2)},0} + \hbar). \end{aligned} \quad (2-56)$$

Remark 2.8. For $r_\infty = 1$, the behavior at infinity implies some additional relations.

$$\begin{aligned} 0 &= \sum_{s=1}^n P_{X_s,1}^{(2)}, \\ t_{\infty^{(1)},0} t_{\infty^{(2)},0} &= \sum_{s=1}^n P_{X_s,2}^{(2)} \delta_{r_s \geq 2} + \sum_{s=1}^n X_s P_{X_s,1}^{(2)}. \end{aligned} \quad (2-57)$$

For $r_\infty = 2$, we have the additional constraint

$$\sum_{s=1}^n P_{X_s,1}^{(2)} = t_{\infty^{(1)},1} t_{\infty^{(2)},0} + t_{\infty^{(2)},1} t_{\infty^{(1)},0}. \quad (2-58)$$

Proof. Details of the computation at each pole are presented in Appendix C. $\square$

Proposition 2.4 combined with Proposition 2.2 immediately implies that the coefficients $(H_{\infty,j})_{0 \leq j \leq r_\infty - 4}$ and $(H_{X_s,j})_{1 \leq s \leq n, 1 \leq j \leq r_s}$ may be recovered as suitable residues of $\text{Tr}(L_{\text{GMR}}^2)$.

Proposition 2.5. *We have*

$$\begin{aligned} H_{\infty,j} &= -\text{Res}_{\lambda \rightarrow \infty} \left(\frac{1}{2} \text{Tr}(L_{\text{GMR}}(\lambda, \hbar)^2) - \frac{1}{2} P_1(\lambda)^2 \right) \lambda^{-j-1}, \quad \forall j \in \llbracket 0, r_\infty - 4 \rrbracket \\ H_{X_s,j} &= \text{Res}_{\lambda \rightarrow X_s} \left(\frac{1}{2} \text{Tr}(L_{\text{GMR}}(\lambda, \hbar)^2) - \frac{1}{2} P_1(\lambda)^2 \right) (\lambda - X_s)^{j-1}, \quad \forall (s, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s \rrbracket. \end{aligned} \quad (2-59)$$

We shall discuss the importance of this result and the connection with results of [30] in Remark 7.2.

In addition, we get an alternative expression for the coefficient η_0 used in the gauge transformation (2-37).

Proposition 2.6. *The coefficient η_0 is also given by*

$$\begin{aligned} \eta_0 &= t_{\infty^{(1)}, r_\infty - 2} + t_{\infty^{(1)}, r_\infty - 1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right), \quad \text{if } r_\infty \geq 2 \\ \eta_0 &= \frac{1}{t_{\infty^{(1)}, 0} - t_{\infty^{(2)}, 0}} \left[- \sum_{s=1}^n (2X_s P_{X_s, 2}^{(2)} \delta_{r_s=1} + P_{X_s, 3}^{(2)} \delta_{r_s=2}) \right. \\ &\quad + \sum_{s=1}^n (X_s^2 H_{X_s, 1} + 2X_s H_{X_s, 2} \delta_{r_s \geq 2} + H_{X_s, 3} \delta_{r_s \geq 3}) - \hbar \sum_{j=1}^g p_j q_j^2 \\ &\quad - t_{\infty^{(1)}, 0} \sum_{s=1}^n \left((t_{X_s^{(1)}, 1} + t_{X_s^{(2)}, 1}) \delta_{r_s \geq 2} + X_s (t_{X_s^{(1)}, 0} + t_{X_s^{(2)}, 0}) \right) \\ &\quad \left. + t_{\infty^{(1)}, 0} (t_{\infty^{(1)}, 0} - t_{\infty^{(2)}, 0} - \hbar) \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) \right] \quad \text{if } r_\infty = 1. \end{aligned} \quad (2-60)$$

Proof. The proof is done in Appendix D. $\square$

Finally, let us make a comment about gauge choices that shall be used when considering general isomonodromic deformations.

Remark 2.9. Fixing the matrix $\tilde{L}$ does not determine uniquely the gauge. Indeed, gauge transformations of the form $\hat{\Psi} = G \tilde{\Psi}$ with $G = u(\mathbf{t}) I_2$ (i.e. a gauge matrix independent of λ and proportional to the identity matrix) do not change neither $\tilde{L}$ nor L . On the contrary, these gauge transformations may change the auxiliary matrices $\tilde{A}_t$ defined in Section 4 because of the term $\hbar \partial_t [G] G^{-1} = \hbar \partial_t [u(\mathbf{t})] u(\mathbf{t})^{-1}$. This gauge freedom is discussed in Remark 6.13 where it is proven that this gauge freedom is irrelevant regarding isomonodromic deformations.

2.8 Summary of the construction

Using the geometric considerations and the existence of the local gauge transformations we have been able to study the asymptotics of the wave functions ψ_1 and ψ_2 around each pole (Proposition 2.3) and deduce from it the general form of the associated Lax matrices in terms of spectral Darboux coordinates, monodromies and irregular times. In particular, we have introduced 3 different gauges with explicit expressions of the gauge matrices connecting them. Their main features are

- The gauge $\tilde{\Psi}$ corresponds to the gauge in which $\tilde{L}$ is the unique representative of $\hat{F}_{\mathcal{R},\mathbf{r}}$. In particular, $\tilde{L}$ has only pole singularities in $\mathcal{R}$ and is properly normalized at infinity.
- The gauge $\check{\Psi}$ is an intermediate gauge for which $\check{L}$ has only pole singularities in $\mathcal{R}$ but is not normalized in a specific way at infinity. It is the gauge used in [57, 25] and may be used as a starting point for other choice of normalizations.
- The companion gauge Ψ in which L is companion-like. In this gauge, L has pole singularities in $\mathcal{R}$ but also apparent singularities at $\lambda \in \{(q_i)_{1 \leq i \leq g}\}$. However, since L is companion-like, there are only two non-trivial entries and their general form is given by Proposition 2.4. As we shall see below, this gauge (which is directly equivalent to the quantum curve satisfied by ψ_1 and ψ_2) is very convenient for the computation of the compatibility equations.

3 Classical spectral curve and connection with topological recursion

Before turning to isomonodromic deformations, let us briefly mention the connection of the present setup with the classical spectral curve and the topological recursion. Although the connection with topological recursion is interesting for applications in mathematical physics, one does not need it to obtain the isomonodromic deformations and Hamiltonian systems that shall be built below. Thus, readers with no interest in topological recursion or in WKB expansions may skip the content of this section.

Let us first recall how one may obtain the classical spectral curve from a Lax system. When dealing with a Lax system of the form

$$\hbar \partial_\lambda \Psi(\lambda, \hbar) = L(\lambda, \hbar) \Psi(\lambda, \hbar), \quad (3-1)$$

it is standard to define the “classical spectral curve” as $\lim_{\hbar \rightarrow 0} \det(yI_2 - L(\lambda, \hbar)) = 0$. It is important to note that the classical spectral curve is unaffected by the gauge transformations $\Psi(\lambda, \hbar) \rightarrow G(\lambda, \hbar) \Psi(\lambda, \hbar)$ with $G(\lambda, \hbar)$ regular in $\hbar$. Indeed, the conjugation of the Lax matrix does not change the characteristic polynomial and the additional term $\hbar(\partial_\lambda G)G^{-1}$ disappears in the limit $\hbar \rightarrow 0$. In other words,

$$\lim_{\hbar \rightarrow 0} \det(yI_2 - L(\lambda, \hbar)) = \lim_{\hbar \rightarrow 0} \det(yI_2 - \check{L}(\lambda, \hbar)) = \lim_{\hbar \rightarrow 0} \det(yI_2 - \tilde{L}(\lambda, \hbar)). \quad (3-2)$$

In our case, the general expression of the matrix $L(\lambda, \hbar)$ implies that the classical spectral curve is

$$y^2 - P_1(\lambda, \hbar = 0)y + P_2(\lambda, \hbar = 0) = 0. \quad (3-3)$$

It defines a Riemann surface Σ of genus $g = r_\infty - 3 + \sum_{s=1}^n r_s$ whose coefficients are determined by (2-45) and (2-52). Note that only g coefficients remained undetermined at this stage and can be mapped with the so-called filling fractions $(\epsilon_i)_{1 \leq i \leq g}$ naturally associated to some period integrals on this Riemann surface. The asymptotic expansions of the differential form ydx at each pole is in direct relation with the asymptotics of the wave functions (C-1) since we have

$$y(z) \stackrel{z \rightarrow \infty^{(i)}}{\sim} - \sum_{k=0}^{r_\infty-1} t_{\infty^{(i)},k} x(z)^{k-1} + O((x(z))^{-2}),$$

$$y(z) \stackrel{z \rightarrow X_s^{(i)}}{=} \sum_{k=0}^{r_s-1} t_{X_s^{(i)},k} (x(z) - X_s)^{-k-1} + O(1). \quad (3-4)$$

Finally, remark that the shift by $\hbar$ of $t_{\infty(2),0}$ vanishes in the limit $\hbar \rightarrow 0$. In particular, the study of the residues of the classical spectral curve implies that

$$0 = t_{\infty(1),0} + t_{\infty(2),0} + \sum_{s=1}^n (t_{X_s^{(1)},0} + t_{X_s^{(2)},0}). \quad (3-5)$$

Let us now discuss the connection of the present work with the Chekhov-Eynard-Orantin topological recursion [20, 26, 27]. Recent works [57, 25] have shown how to quantize the classical spectral curve using topological recursion. Indeed, applying the topological recursion to the classical spectral curve (3-3) generates Eynard-Orantin differentials $(\omega_{h,n})_{h \geq 0, n \geq 0}$ that can be regrouped into formal $\hbar$ -transseries to define formal wave functions $(\psi_1^{\text{TR}}, \psi_2^{\text{TR}})$ that satisfy a quantum curve, i.e. a linear ODE of degree 2 with pole singularities in $\mathcal{R}$ and apparent singularities at $\lambda = q_i$ and whose $\hbar \rightarrow 0$ limit recovers the classical spectral curve. In particular, the construction presented in [57, 25] implies that this ODE is the same as the one defined by the Lax matrix $L(\lambda, \hbar)$ of the present paper so that we get

$$\Psi(\lambda, \hbar) = C \begin{pmatrix} \psi_1^{\text{TR}}(\lambda, \hbar) & \psi_2^{\text{TR}}(\lambda, \hbar) \\ \hbar \partial_\lambda \psi_1^{\text{TR}}(\lambda, \hbar) & \hbar \partial_\lambda \psi_2^{\text{TR}}(\lambda, \hbar) \end{pmatrix} \quad (3-6)$$

where C is a constant (independent of λ) matrix. In other words, the topological recursion reconstructs our wave functions making the classical spectral curve the only necessary object to build the full Lax system. However the price to pay in this point of view is the mandatory introduction of the formal parameter $\hbar$ to define the formal $\hbar$ -transseries and then $(\psi_1^{\text{TR}}, \psi_2^{\text{TR}})$. As explained in Section 2.5, this formal parameter can be removed by proper rescaling at the level of the Lax system but it is unclear how the topological recursion wave functions may be defined after this rescaling since there is no more formal parameter to define the series. This issue is in deep relation with the analytical meaning that might be given to the formal $\hbar$ -transseries. In particular, it is presently unclear how to resum analytically the $\hbar$ -transseries to obtain non-formal quantities but current works are in progress to tackle this problem. In particular, the main issue at stake for the content of this paper is the following: even if the formal $\hbar$ -transseries wave functions may have some analytical meanings in some neighborhoods of $\hbar = 0$ (using for example works of N. Nikolaev [62, 61] or works of O. Costin and R.D. Costin [22, 23]), it is unknown if one may extend these analytical objects up to $\hbar = 1$ that corresponds to the natural value of the parameter from the geometric perspective.

Remark 3.1 (Topological Type property). The construction of the Lax pairs and the Hamiltonian systems presented in this paper is independent of the type of solutions that one may look for. According to [25], the most general solutions of the Lax system are expected to be $\hbar$ -transseries. However, one may look for simpler solutions. Of particular interests are formal power series solutions of the Hamiltonian systems:

$$\hat{q}_i(\tau, \hbar) = \sum_{k=0}^{\infty} q_i^{(k)}(\tau) \hbar^k, \quad \hat{p}_i(\tau, \hbar) = \sum_{k=0}^{\infty} p_i^{(k)}(\tau) \hbar^k, \quad \forall i \in \llbracket 1, g \rrbracket \quad (3-7)$$

that equivalently correspond to formal WKB solutions of the wave functions

$$\hat{\Psi}(\lambda, \tau, \hbar) = \exp \left(\sum_{k=-1}^{\infty} \Psi_k(\lambda, \tau) \hbar^k \right) \quad (3-8)$$

of the Lax system. In [57], the authors proved that, in this formal WKB solutions setup, the Lax systems arising from general isomonodromic deformations always satisfy the so-called “topological type property” of [5]. In particular, the central argument (section 4.2 of [57]) to prove the topological type property is the existence of an isomonodromic time τ for which the corresponding auxiliary matrix $\tilde{A}_{\alpha\tau}(\lambda, \hbar)$ is of the form $\tilde{A}_{\alpha\tau}(\lambda, \hbar) = \frac{B_1\lambda+B_0}{p(\lambda)}$ where B_0 and B_1 are independent of λ and p is a polynomial. Our formalism shall provide a similar result without using isospectral deformations. Indeed, upcoming results of Section 7 indicate that any of the isomonodromic times $\tau_{\infty, r_{\infty}-3}$, or τ_{X_s, r_s-1} or $\tilde{X}_s$ for $s \in \llbracket 1, n \rrbracket$ provides an auxiliary matrix satisfying the form required in [57]. Thus, in the context of formal WKB solutions, the geometric Lax pairs constructed in the present paper always satisfy the topological type property. Consequently, one may reconstruct the formal correlation functions built from “determinantal formulas” (see [6] for definitions) of the differential system $\hbar\partial_{\lambda}\tilde{\Psi}(\lambda, \hbar) = \tilde{L}(\lambda, \hbar)\tilde{\Psi}(\lambda, \hbar)$ using the Eynard-Orantin differentials $(\omega_{k,n})_{k \geq 0, n \geq 0}$ produced by the topological recursion on the classical spectral curve (that always reduces in this formal WKB setup to a genus 0 curve). Moreover, the formal Jimbo-Miwa-Ueno τ -function τ_{JMU} [45, 8] is reconstructed by the free energies $(\omega_{k,0})_{k \geq 0}$ (Corollary 5.1 of [57]).

4 General isomonodromic deformations and auxiliary matrices

4.1 Definition of general isomonodromic deformations

Section 2 provides a natural set of parameters for which we may consider deformations, namely the irregular times $(t_{\infty(i),k})_{1 \leq k \leq r_{\infty}-1, 1 \leq i \leq 2}$ and $(t_{X_s^{(i)},k})_{0 \leq i \leq 2, 1 \leq s \leq n, 1 \leq k \leq r_s-1}$ and the location of the poles $(X_s)_{1 \leq s \leq n}$. In order to study deformations relatively to these parameters we introduce the following definition.

Definition 4.1. We define the following general deformation operators.

$$\mathcal{L}_{\alpha} = \hbar \sum_{i=1}^2 \sum_{k=1}^{r_{\infty}-1} \alpha_{\infty(i),k} \partial_{t_{\infty(i),k}} + \hbar \sum_{i=1}^2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} \alpha_{X_s^{(i)},k} \partial_{t_{X_s^{(i)},k}} + \hbar \sum_{s=1}^n \alpha_{X_s} \partial_{X_s} \quad (4-1)$$

where we define the vector $\alpha \in \mathbb{C}^{2r_{\infty}-2+2\sum_{s=1}^n r_s-2n+n} = \mathbb{C}^{2g+4-n}$ by

$$\alpha = \sum_{i=1}^2 \sum_{k=1}^{r_{\infty}-1} \alpha_{\infty(i),k} \mathbf{e}_{\infty(i),k} + \sum_{i=1}^2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} \alpha_{X_s^{(i)},k} \mathbf{e}_{X_s^{(i)},k} + \sum_{s=1}^n \alpha_{X_s} \mathbf{e}_{X_s}. \quad (4-2)$$

It is important to notice that we do not consider deformations relatively to neither $(t_{\infty(i),0})_{1 \leq i \leq 2}$ nor $(t_{X_s^{(i)},0})_{1 \leq i \leq 2, 1 \leq s \leq n}$ since that would affect the monodromy data at each pole. Thus, deformations defined by Definition 4.1 shall be seen as “general isomonodromic deformations” in $\hat{F}_{\mathcal{R},r}$ ([72]). Moreover, we stress that the coefficients of the vector α are allowed to depend on the position of the poles X_s and on all the coefficients of the singular parts of the wave functions (including $(t_{\infty(i),0})_{1 \leq i \leq 2}$ and $(t_{X_s^{(i)},0})_{1 \leq i \leq 2, 1 \leq s \leq n}$).

Associated to a vector α are general auxiliary Lax matrices $\tilde{A}_{\alpha}(\lambda)$, $\check{A}_{\alpha}(\lambda)$ and $A_{\alpha}(\lambda)$ defined by

$$\tilde{A}_{\alpha}(\lambda) = \mathcal{L}_{\alpha}[\tilde{\Psi}(\lambda)]\tilde{\Psi}^{-1}(\lambda) \quad \Leftrightarrow \quad \mathcal{L}_{\alpha}[\tilde{\Psi}(\lambda)] = \tilde{A}_{\alpha}(\lambda)\tilde{\Psi}(\lambda)$$

$$\begin{aligned}\check{A}_\alpha(\lambda) = \mathcal{L}_\alpha[\check{\Psi}(\lambda)]\check{\Psi}^{-1}(\lambda) &\Leftrightarrow \mathcal{L}_\alpha[\check{\Psi}(\lambda)] = \check{A}_\alpha(\lambda)\check{\Psi}(\lambda) \\ A_\alpha(\lambda) = \mathcal{L}_\alpha[\Psi(\lambda)]\Psi^{-1}(\lambda) &\Leftrightarrow \mathcal{L}_\alpha[\Psi(\lambda)] = A_\alpha(\lambda)\Psi(\lambda).\end{aligned}\tag{4-3}$$

In particular, $\check{A}_\alpha(\lambda)$ and $\tilde{A}_\alpha(\lambda)$ are rational functions of λ with only possible poles in $\mathcal{R}$ while $A(\lambda)$ may also have additional poles at $\{q_1, \dots, q_g\}$ ([39, 36]). Note that $(L(\lambda), A_\alpha(\lambda))$, $(\check{L}(\lambda), \check{A}_\alpha(\lambda))$ and $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$ provide equivalent Lax pairs (i.e. corresponding to the same isomonodromic deformations and providing the same Hamiltonian system) but expressed in three different gauges. The corresponding compatibility equations are

$$\begin{aligned}\mathcal{L}_\alpha[L] &= [A_\alpha, L] + \hbar \partial_\lambda A_\alpha \\ \mathcal{L}_\alpha[\check{L}] &= [\check{A}_\alpha, \check{L}] + \hbar \partial_\lambda \check{A}_\alpha \\ \mathcal{L}_\alpha[\tilde{L}] &= [\tilde{A}_\alpha, \tilde{L}] + \hbar \partial_\lambda \tilde{A}_\alpha.\end{aligned}\tag{4-4}$$

We shall now use the asymptotic expansions of the wave matrices in order to obtain information on the general form of the auxiliary matrices. Then, we shall use the compatibility equations in order to determine the evolutions of the Darboux coordinates under general isomonodromic deformations and prove that these evolutions are Hamiltonian.

4.2 General form of the auxiliary matrix $A_\alpha(\lambda, \hbar)$

Using compatibility equations one may easily obtain two of the entries of $A_\alpha(\lambda)$. Indeed, since L is a companion-like matrix, compatibility equations (4-4) imply that

$$\begin{aligned}[A_\alpha(\lambda)]_{2,1} &= \hbar \partial_\lambda [A_\alpha(\lambda)]_{1,1} + [A_\alpha(\lambda)]_{1,2} L_{2,1}(\lambda), \\ [A_\alpha(\lambda)]_{2,2} &= \hbar \partial_\lambda [A_\alpha(\lambda)]_{1,2} + [A_\alpha(\lambda)]_{1,1} L_{2,2}(\lambda),\end{aligned}\tag{4-5}$$

so that only the first line of $A_\alpha(\lambda)$ remains unknown at this stage. The other two entries of the compatibility equation (4-4) leads to

$$\begin{aligned}\mathcal{L}_\alpha[L_{2,1}(\lambda)] &= \hbar^2 \frac{\partial^2 [A_\alpha(\lambda)]_{1,1}}{\partial \lambda^2} + 2\hbar L_{2,1}(\lambda) \partial_\lambda [A_\alpha(\lambda)]_{1,2} + \hbar [A_\alpha(\lambda)]_{1,2} \partial_\lambda L_{2,1}(\lambda) \\ &\quad - \hbar L_{2,2}(\lambda) \partial_\lambda [A_\alpha(\lambda)]_{1,1}, \\ \mathcal{L}_\alpha[L_{2,2}(\lambda)] &= \hbar^2 \frac{\partial^2 [A_\alpha(\lambda)]_{1,2}}{\partial \lambda^2} + 2\hbar \partial_\lambda [A_\alpha(\lambda)]_{1,1} + \hbar L_{2,2}(\lambda) \partial_\lambda [A_\alpha(\lambda)]_{1,2} \\ &\quad + \hbar [A_\alpha(\lambda)]_{1,2} \partial_\lambda L_{2,2}(\lambda)\end{aligned}\tag{4-6}$$

that shall be used later to determine the evolution equations for $(q_i, p_i)_{1 \leq i \leq g}$. Before studying the compatibility equations, let us observe that the asymptotic expansions of the wave matrix Ψ at each pole allows to determine the general form of the auxiliary matrix $A_\alpha(\lambda, \hbar)$. Indeed, we get the following results.

Proposition 4.1. *The asymptotic expansions of entry $[A_\alpha(\lambda)]_{1,2}$ at each pole are given by*

$$\begin{aligned}[A_\alpha(\lambda)]_{1,2} &\stackrel{\lambda \rightarrow \infty}{=} \sum_{i=-1}^{r_\infty-3} \frac{\nu_{\infty,i}^{(\alpha)}}{\lambda^i} + O\left(\lambda^{-(r_\infty-2)}\right), \\ [A_\alpha(\lambda)]_{1,2} &\stackrel{\lambda \rightarrow X_s}{=} \sum_{i=0}^{r_s-1} \nu_{X_s,i}^{(\alpha)} (\lambda - X_s)^i + O((\lambda - X_s)^{r_s})\end{aligned}\tag{4-7}$$

Coefficients $\left(\nu_{\infty,k}^{(\alpha)}\right)_{-1 \leq k \leq r_\infty-3}$ and $\left(\nu_{X_s,i}^{(\alpha)}\right)_{1 \leq s \leq n, 0 \leq i \leq r_s-1}$ are determined by

$$\forall s \in \llbracket 1, n \rrbracket : \nu_{X_s,0}^{(\alpha)} = -\alpha_{X_s} \text{ and } M_s \begin{pmatrix} \nu_{X_s,1}^{(\alpha)} \\ \vdots \\ \nu_{X_s,r_s-1}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} -\frac{\alpha_{X_s^{(1)},r_s-1} - \alpha_{X_s^{(2)},r_s-1}}{r_s-1} \\ \vdots \\ -\frac{\alpha_{X_s^{(1)},1} - \alpha_{X_s^{(2)},1}}{1} \end{pmatrix} \quad (4-8)$$

where $(M_s)_{1 \leq s \leq n}$ are lower triangular Toeplitz matrices independent of the deformation vector α :

$$M_s = \begin{pmatrix} (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) & 0 & \dots & \dots & 0 \\ (t_{X_s^{(1)},r_s-2} - t_{X_s^{(2)},r_s-2}) & (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (t_{X_s^{(1)},2} - t_{X_s^{(2)},2}) & \ddots & \ddots & \ddots & 0 \\ (t_{X_s^{(1)},1} - t_{X_s^{(2)},1}) & (t_{X_s^{(1)},2} - t_{X_s^{(2)},2}) & \dots & (t_{X_s^{(1)},r_s-2} - t_{X_s^{(2)},r_s-2}) & (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) \end{pmatrix}. \quad (4-9)$$

The situation at ∞ is similar but depends on the value of r_∞ .

- For $r_\infty \geq 3$ we have

$$M_\infty \begin{pmatrix} \nu_{\infty,-1}^{(\alpha)} \\ \nu_{\infty,0}^{(\alpha)} \\ \vdots \\ \nu_{\infty,r_\infty-3}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_{\infty^{(1)},r_\infty-1} - \alpha_{\infty^{(2)},r_\infty-1}}{r_\infty-1} \\ \frac{\alpha_{\infty^{(1)},r_\infty-2} - \alpha_{\infty^{(2)},r_\infty-2}}{r_\infty-2} \\ \vdots \\ \frac{\alpha_{\infty^{(1)},1} - \alpha_{\infty^{(2)},1}}{1} \end{pmatrix} \quad (4-10)$$

where M_∞ is a lower triangular Toeplitz matrix of size $(r_\infty - 1) \times (r_\infty - 1)$ independent of the deformation α ,

$$M_\infty = \begin{pmatrix} (t_{\infty^{(1)},r_\infty-1} - t_{\infty^{(2)},r_\infty-1}) & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ (t_{\infty^{(1)},2} - t_{\infty^{(2)},2}) & \dots & (t_{\infty^{(1)},r_\infty-1} - t_{\infty^{(2)},r_\infty-1}) & & 0 \\ (t_{\infty^{(1)},1} - t_{\infty^{(2)},1}) & \dots & (t_{\infty^{(1)},r_\infty-2} - t_{\infty^{(2)},r_\infty-2}) & (t_{\infty^{(1)},r_\infty-1} - t_{\infty^{(2)},r_\infty-1}) \end{pmatrix}. \quad (4-11)$$

- For $r_\infty = 2$, M_∞ is a 1×1 matrix and

$$M_\infty \nu_{\infty,-1}^{(\alpha)} = (t_{\infty^{(1)},1} - t_{\infty^{(2)},1}) \nu_{\infty,-1}^{(\alpha)} = \alpha_{\infty^{(1)},1} - \alpha_{\infty^{(2)},1}. \quad (4-12)$$

Note in particular that $\nu_{\infty,0}^{(\alpha)}$ is not determined.

- For $r_\infty = 1$, M_∞ is not defined and neither $\nu_{\infty,-1}^{(\alpha)}$ nor $\nu_{\infty,0}^{(\alpha)}$ is determined.

Proof. The proof is done in Appendix E. □

The previous proposition may be used to determine the general form of the entry $[A_{\alpha}(\lambda)]_{1,2}$.

Proposition 4.2. *Entry $[A_{\alpha}(\lambda)]_{1,2}$ is given by*

$$[A_{\alpha}(\lambda)]_{1,2} = \nu_{\infty,-1}^{(\alpha)} \lambda + \nu_{\infty,0}^{(\alpha)} + \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{\lambda - q_j}. \quad (4-13)$$

Coefficients $\left(\mu_j^{(\alpha)}\right)_{1 \leq j \leq g}$ are determined by the linear system

$$\mathbf{V} \begin{pmatrix} \mu_1^{(\alpha)} \\ \vdots \\ \mu_g^{(\alpha)} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} V_{\infty} \\ V_1 \\ \vdots \\ V_n \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha)} \\ \vdots \\ \mu_g^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \nu_{\infty}^{(\alpha)} \\ \nu_{X_1}^{(\alpha)} \\ \vdots \\ \nu_{X_n}^{(\alpha)} \end{pmatrix} \quad (4-14)$$

where V_{∞} is a $(r_{\infty} - 3) \times g$ matrix and $(V_s)_{1 \leq s \leq n}$ are $r_s \times g$ matrices given by

$$V_{\infty} = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ q_1 & q_2 & \dots & \dots & q_g \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ q_1^{r_{\infty}-4} & q_2^{r_{\infty}-4} & \dots & \dots & q_g^{r_{\infty}-4} \end{pmatrix}, \quad V_s = \begin{pmatrix} \frac{1}{q_1 - X_s} & \dots & \dots & \frac{1}{q_g - X_s} \\ \frac{1}{(q_1 - X_s)^2} & \dots & \dots & \frac{1}{(q_g - X_s)^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{(q_1 - X_s)^{r_s}} & \dots & \dots & \frac{1}{(q_g - X_s)^{r_s}} \end{pmatrix} \quad (4-15)$$

and $\nu_{\infty}^{(\alpha)} \in \mathbb{C}^{r_{\infty}-3}$, $\nu_{X_s}^{(\alpha)} \in \mathbb{C}^{r_s}$ are vectors given by

$$\nu_{\infty}^{(\alpha)} = \begin{pmatrix} \nu_{\infty,1}^{(\alpha)} \\ \nu_{\infty,2}^{(\alpha)} \\ \vdots \\ \vdots \\ \nu_{\infty,r_{\infty}-3}^{(\alpha)} \end{pmatrix}, \quad \nu_{X_s}^{(\alpha)} = \begin{pmatrix} -\nu_{X_s,0}^{(\alpha)} + \nu_{\infty,0}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)} X_s \\ -\nu_{X_s,1}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)} \\ -\nu_{X_s,2}^{(\alpha)} \\ \vdots \\ -\nu_{X_s,r_s-1}^{(\alpha)} \end{pmatrix}. \quad (4-16)$$

Proof. The proof is done in Appendix F. □

Note that for $r_{\infty} \leq 3$, V_{∞} is not defined and shall not be written in (4-14). Thus, for $r_{\infty} \leq 2$, the previous linear system may look over-determined since there are more lines than columns. This is not the case because of the following remark.

Remark 4.1. For $r_{\infty} \leq 2$, one has to remember that the quantities $\nu_{\infty,-1}^{(\alpha)}$ and $\nu_{\infty,0}^{(\alpha)}$ may not be determined by Proposition 4.1. For $r_{\infty} = 2$, only $\nu_{\infty,-1}^{(\alpha)}$ is determined by Proposition 4.1 while for $r_{\infty} = 1$ neither $\nu_{\infty,-1}^{(\alpha)}$ nor $\nu_{\infty,0}^{(\alpha)}$ is determined by Proposition 4.1. In these cases, one must use the extra lines of (4-14) to determine these additional unknown coefficients and then use the information to obtain $\left(\mu_j^{(\alpha)}\right)_{1 \leq j \leq g}$.

Remark 4.2. Note that the determinant of $\mathbf{V}$ is given by

$$\det \mathbf{V} = (-1)^n \frac{\prod_{1 \leq i < j \leq g} (q_i - q_j)}{\prod_{i=1}^g \prod_{s=1}^n (q_i - X_s)^{r_s}} \prod_{1 \leq s' < s \leq n} (X_s - X_{s'})^{r_s r_{s'}}. \quad (4-17)$$

In particular, it is non-zero as soon as $(q_1, \dots, q_g, X_1, \dots, X_n)$ are all distinct.

Finally, one can obtain the general form of entry $[A_{\alpha}(\lambda)]_{1,1}$. We get the following proposition.

Proposition 4.3. *The entry $[A_{\alpha}(\lambda)]_{1,1}$ is given by*

$$[A_{\alpha}(\lambda)]_{1,1} = \sum_{i=0}^{r_{\infty}-1} c_{\infty,i}^{(\alpha)} \lambda^i + \sum_{s=1}^n \sum_{i=1}^{r_s-1} c_{X_s,i}^{(\alpha)} (\lambda - X_s)^{-i} + \sum_{j=1}^g \frac{\rho_j^{(\alpha)}}{\lambda - q_j}. \quad (4-18)$$

with

$$\forall j \in \llbracket 1, n \rrbracket : \rho_j^{(\alpha)} = -\mu_j^{(\alpha)} p_j \quad (4-19)$$

Coefficients $\left(c_{\infty,k}^{(\alpha)}\right)_{1 \leq k \leq r_{\infty}-1}$ and $\left(c_{X_s,k}^{(\alpha)}\right)_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ are determined by

$$M_{\infty} \begin{pmatrix} c_{\infty,r_{\infty}-1}^{(\alpha)} \\ c_{\infty,r_{\infty}-2}^{(\alpha)} \\ \vdots \\ c_{\infty,i}^{(\alpha)} \\ \vdots \\ c_{\infty,1}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{t_{\infty(2),r_{\infty}-1}^{\alpha} t_{\infty(1),r_{\infty}-1}^{-t} t_{\infty(2),r_{\infty}-1}^{\alpha} t_{\infty(1),r_{\infty}-1}^{-t}}{r_{\infty}-2} + \frac{t_{\infty(2),r_{\infty}-2}^{\alpha} t_{\infty(1),r_{\infty}-2}^{-t} t_{\infty(2),r_{\infty}-2}^{\alpha} t_{\infty(1),r_{\infty}-2}^{-t}}{r_{\infty}-1} \\ \vdots \\ \sum_{k=r_{\infty}-i}^{r_{\infty}-1} \frac{t_{\infty(2),2r_{\infty}-1-i-k}^{\alpha} t_{\infty(1),k}^{-t} t_{\infty(2),2r_{\infty}-1-i-k}^{\alpha} t_{\infty(1),k}^{-t}}{k} \\ \vdots \\ \frac{t_{\infty(2),r_{\infty}-1}^{\alpha} t_{\infty(1),1}^{-t} t_{\infty(2),r_{\infty}-1}^{\alpha} t_{\infty(1),1}^{-t}}{1} + \dots + \frac{t_{\infty(2),1}^{\alpha} t_{\infty(1),r_{\infty}-1}^{-t} t_{\infty(2),1}^{\alpha} t_{\infty(1),r_{\infty}-1}^{-t}}{r_{\infty}-1} \end{pmatrix} \quad (4-20)$$

with the matrix M_{∞} given by (4-11). Similarly, for all $s \in \llbracket 1, n \rrbracket$,

$$M_s \begin{pmatrix} c_{X_s,r_s-1}^{(\alpha)} \\ c_{X_s,r_s-2}^{(\alpha)} \\ \vdots \\ c_{X_s,i}^{(\alpha)} \\ \vdots \\ c_{X_s,1}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{t_{X_s(2),r_s-1}^{\alpha} t_{X_s(1),r_s-1}^{-t} t_{X_s(2),r_s-1}^{\alpha} t_{X_s(1),r_s-1}^{-t}}{r_s-2} + \frac{t_{X_s(2),r_s-2}^{\alpha} t_{X_s(1),r_s-2}^{-t} t_{X_s(2),r_s-2}^{\alpha} t_{X_s(1),r_s-2}^{-t}}{r_s-1} \\ \vdots \\ \sum_{k=r_s-i}^{r_s-1} \frac{t_{X_s(2),2r_s-1-i-k}^{\alpha} t_{X_s(1),k}^{-t} t_{X_s(2),2r_s-1-i-k}^{\alpha} t_{X_s(1),k}^{-t}}{k} \\ \vdots \\ \frac{t_{X_s(2),r_s-1}^{\alpha} t_{X_s(1),1}^{-t} t_{X_s(2),r_s-1}^{\alpha} t_{X_s(1),1}^{-t}}{1} + \dots + \frac{t_{X_s(2),1}^{\alpha} t_{X_s(1),r_s-1}^{-t} t_{X_s(2),1}^{\alpha} t_{X_s(1),r_s-1}^{-t}}{r_s-1} \end{pmatrix} \quad (4-21)$$

with the matrices $(M_s)_{1 \leq s \leq n}$ given by (4-9).

Proof. The proof is done in Appendix G. $\square$

In the previous propositions, one may easily observe that quantities like M_{∞} , $(M_s)_{1 \leq s \leq n}$, $\left(\nu_{\infty,k}^{(\alpha)}\right)_{1 \leq k \leq r_{\infty}-3}$, $\left(\nu_{X_s,k}^{(\alpha)}\right)_{0 \leq s \leq n, 1 \leq k \leq r_s-1}$, $\left(c_{X_s,k}^{(\alpha)}\right)_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$, $\left(c_{\infty,k}^{(\alpha)}\right)_{1 \leq k \leq r_{\infty}-1}$ are independent of Darboux coordinates and are only determined by the monodromies, the irregular times and the coefficients of the deformation α . On the contrary, quantities like $\left(\mu_j^{(\alpha)}\right)_{1 \leq j \leq g}$, $\left(\rho_j^{(\alpha)}\right)_{1 \leq j \leq g}$, V_{∞} and $(V_s)_{1 \leq s \leq n}$ depends on the Darboux coordinates. The situation is more complicated for $\nu_{\infty,-1}^{(\alpha)}$ and $\nu_{\infty,0}^{(\alpha)}$ since they do not depend on the Darboux coordinates for $r_{\infty} \geq 3$ but depend on them for $r_{\infty} \leq 2$ according to Remark 4.1.

Remark 4.3. The coefficient $c_{\infty,0}^{(\alpha)}$ is not determined in the expression of $[A_{\alpha}(\lambda)]_{1,1}$. This coefficient is irrelevant in the determination of the Hamiltonian system because it disappears in the compatibility equations. In fact, this coefficient is directly related to the choice of normalization of $\tilde{L}_{1,2}(\lambda)$. In the present setup where $\tilde{L}_{1,2}(\lambda) = \lambda^{r_{\infty}-3} + O(\lambda^{r_{\infty}-4})$, we find $c_{\infty,0}^{(\alpha)} = \frac{1}{2}\nu_{\infty,-1}^{(\alpha)}$.

5 General Hamiltonian evolutions

The previous sections provide the general form of the matrix $L(\lambda, \hbar)$ and $A_{\alpha}(\lambda)$ through Propositions 2.4, 4.1, 4.2, 4.3 and equation (4-5). As we shall see below, inserting this previous knowledge into the compatibility equations (4-6) provides the evolutions of the Darboux coordinates.

The first step is to look at order $(\lambda - q_j)^{-2}$ in $\mathcal{L}_{\alpha}[L_{2,2}(\lambda)]$. We obtain, for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_{\alpha}[q_j] = 2\mu_j^{(\alpha)} \left(p_j - \frac{1}{2}P_1(q_j) + \frac{1}{2} \sum_{s=1}^n \frac{\hbar r_s}{q_j - X_s} \right) - \hbar \nu_{\infty,0}^{(\alpha)} - \hbar \nu_{\infty,-1}^{(\alpha)} q_j - \hbar \sum_{i \neq j} \frac{\mu_j^{(\alpha)} + \mu_i^{(\alpha)}}{q_j - q_i}. \quad (5-1)$$

The next step is to determine the coefficients $(H_{\infty,j})_{0 \leq j \leq r_{\infty}-4}$ and $(H_{X_s,j})_{1 \leq s \leq n, 1 \leq j \leq r_s}$ that remain unknown in $L_{2,1}(\lambda)$. To achieve this task, we look at order $(\lambda - q_j)^{-2}$ in $\mathcal{L}_{\alpha}[L_{2,1}(\lambda)]$ using (4-6). We obtain

$$\begin{aligned} -\hbar p_j \mathcal{L}_{\alpha}[q_j] &= -2\hbar \mu_j^{(\alpha)} \left(-\tilde{P}_2(q_j) + \sum_{k=0}^{r_{\infty}-4} H_{\infty,k} q_j^k + \sum_{s=1}^n \sum_{k=1}^{r_s} H_{X_s,k} (q_j - X_s)^{-k} - \hbar t_{\infty(1),r_{\infty}-1} q_j^{r_{\infty}-3} \delta_{r_{\infty} \geq 3} - \sum_{i \neq j} \frac{\hbar p_i}{q_j - q_i} \right) \\ &\quad + \hbar^2 p_j \left(\nu_{\infty,-1}^{(\alpha)} q_j + \nu_{\infty,0}^{(\alpha)} + \sum_{i \neq j} \frac{\mu_i^{(\alpha)}}{q_j - q_i} \right) - \hbar \mu_j^{(\alpha)} p_j \left(P_1(q_j) - \sum_{s=1}^n \frac{\hbar r_s}{q_j - X_s} + \sum_{i \neq j} \frac{\hbar}{q_j - q_i} \right). \end{aligned} \quad (5-2)$$

Inserting (5-1) provides, for all $j \in \llbracket 1, g \rrbracket$,

$$\begin{aligned} \sum_{k=0}^{r_{\infty}-4} H_{\infty,k} q_j^k + \sum_{s=1}^n \sum_{k=1}^{r_s} H_{X_s,k} (q_j - X_s)^{-k} &= p_j^2 - P_1(q_j) p_j - p_j \sum_{s=1}^n \frac{\hbar r_s}{q_j - X_s} + \tilde{P}_2(q_j) \\ &\quad + \hbar \sum_{i \neq j} \frac{p_i - p_j}{q_j - q_i} + \hbar t_{\infty(1),r_{\infty}-1} q_j^{r_{\infty}-3} \delta_{r_{\infty} \geq 3} \end{aligned} \quad (5-3)$$

where it is obvious that the r.h.s. is independent of the deformation vector α . The last relation can be rewritten into a matrix form.

Proposition 5.1. *We have*

$$(V_{\infty}^t \quad V_1^t \quad \dots \quad V_n^t) \begin{pmatrix} \mathbf{H}_{\infty} \\ \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} = \begin{pmatrix} p_1^2 - P_1(q_1) p_1 + p_1 \sum_{s=1}^n \frac{\hbar r_s}{q_1 - X_s} + \tilde{P}_2(q_1) + \hbar \sum_{i \neq 1} \frac{p_i - p_1}{q_1 - q_i} + \hbar t_{\infty(1),r_{\infty}-1} q_1^{r_{\infty}-3} \delta_{r_{\infty} \geq 3} \\ \vdots \\ p_g^2 - P_1(q_g) p_g + p_g \sum_{s=1}^n \frac{\hbar r_s}{q_g - X_s} + \tilde{P}_2(q_g) + \hbar \sum_{i \neq g} \frac{p_i - p_g}{q_g - q_i} + \hbar t_{\infty(1),r_{\infty}-1} q_g^{r_{\infty}-3} \delta_{r_{\infty} \geq 3} \end{pmatrix} \quad (5-4)$$

with $\mathbf{H}_{\infty} = (H_{\infty,0}, \dots, H_{\infty,r_{\infty}-4})^t$ (null is $r_{\infty} \leq 3$), and, for all $s \in \llbracket 1, n \rrbracket$, $\mathbf{H}_{X_s} = (H_{X_s,1}, \dots, H_{X_s,r_s})^t$. We recall here that the matrices V_{∞} and $(V_s)_{1 \leq s \leq n}$ are defined by (4-15).

For $r_\infty = 2$, the previous linear system has to be complemented with the additional relation (2-55)

$$\sum_{s=1}^n H_{X_s,1} = \hbar \sum_{j=1}^g p_j - (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}). \quad (5-5)$$

For $r_\infty = 1$, the previous linear system has to be complemented with the two additional relations (2-56)

$$\begin{aligned} \sum_{s=1}^n X_s H_{X_s,1} + \sum_{s=1}^n H_{X_s,2} \delta_{r_s \geq 2} &= \hbar \sum_{j=1}^g q_j p_j + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty(1),0} (t_{\infty(2),0} + \hbar), \\ \sum_{s=1}^n H_{X_s,1} &= \hbar \sum_{j=1}^g p_j. \end{aligned} \quad (5-6)$$

Finally, in order to obtain the evolution equation for p_j we look at order $(\lambda - q_j)^{-1}$ of the entry $\mathcal{L}_\alpha[L_{2,1}(\lambda)]$. We get, for all $j \in \llbracket 1, g \rrbracket$,

$$\begin{aligned} \mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} \\ &+ \mu_j^{(\alpha)} \left(p_j P_1'(q_j) + p_j \sum_{s=1}^n \frac{\hbar r_s}{(q_j - X_s)^2} - \tilde{P}_2'(q_j) + \sum_{k=1}^{r_\infty-4} k H_{\infty,k} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s} k H_{X_s,k} (q_j - X_s)^{-k-1} \right. \\ &\left. - \hbar(r_\infty - 3) t_{\infty(1),r_\infty-1} q_j^{r_\infty-4} \delta_{r_\infty \geq 3} \right) \\ &+ \hbar \nu_{\infty,-1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty-1} k c_{\infty,k}^{(\alpha)} q_j^{k-1} - \hbar \sum_{s=1}^n \sum_{k=1}^{r_s-1} k c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k-1}. \end{aligned} \quad (5-7)$$

Thus, we have obtained the general evolutions for $(p_j, q_j)_{1 \leq j \leq g}$ through (5-1) and (5-7). It turns out that these evolutions are Hamiltonian.

Theorem 5.1 (Hamiltonian evolution). *Defining*

$$\begin{aligned} Ham^{(\alpha)}(\mathbf{q}, \mathbf{p}) &= \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\alpha)} H_{\infty,k} - \sum_{s=1}^n \sum_{k=2}^{r_s} \nu_{X_s,k-1}^{(\alpha)} H_{X_s,k} + \sum_{s=1}^n \alpha_{X_s}^{(\alpha)} H_{X_s,1} \\ &- \hbar \sum_{j=1}^g \left[\sum_{k=0}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k + \sum_{s=1}^n \sum_{k=1}^{r_s-1} c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k} \right] \\ &+ \nu_{\infty,-1}^{(\alpha)} \sum_{s=1}^n (X_s H_{X_s,1} + H_{X_s,2} \delta_{r_s \geq 2}) + \nu_{\infty,0}^{(\alpha)} \sum_{s=1}^n H_{X_s,1} \\ &- \delta_{r_\infty \in \{1,2\}} \left(\sum_{s=1}^n H_{X_s,1} - \hbar \sum_{j=1}^g p_j \right) \nu_{\infty,0}^{(\alpha)} \\ &- \delta_{r_\infty=1} \left(\sum_{s=1}^n X_s H_{X_s,1} + \sum_{s=1}^n H_{X_s,2} \delta_{r_s \geq 2} - \hbar \sum_{j=1}^g q_j p_j \right) \nu_{\infty,-1}^{(\alpha)} \\ &- \hbar \nu_{\infty,0}^{(\alpha)} \sum_{j=1}^g p_j - \hbar \nu_{\infty,-1}^{(\alpha)} \sum_{j=1}^g q_j p_j, \end{aligned} \quad (5-8)$$

the evolutions for $j \in \llbracket 1, g \rrbracket$,

$$\mathcal{L}_\alpha[q_j] = 2\mu_j^{(\alpha)} \left(p_j - \frac{1}{2} P_1(q_j) + \frac{1}{2} \sum_{s=1}^n \frac{\hbar r_s}{q_j - X_s} \right) - \hbar \nu_{\infty,0}^{(\alpha)} - \hbar \nu_{\infty,-1}^{(\alpha)} q_j - \hbar \sum_{i \neq j} \frac{\mu_j^{(\alpha)} + \mu_i^{(\alpha)}}{q_j - q_i},$$

$$\begin{aligned}
\mathcal{L}_\alpha[p_j] = & \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} \\
& + \mu_j^{(\alpha)} \left(p_j P_1'(q_j) + p_j \sum_{s=1}^n \frac{\hbar r_s}{(q_j - X_s)^2} - \tilde{P}_2'(q_j) + \sum_{k=1}^{r_\infty-4} k H_{\infty,k} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s} k H_{X_s,k} (q_j - X_s)^{-k-1} \right. \\
& \left. - \hbar(r_\infty - 3) t_{\infty^{(1)}, r_\infty-1} q_j^{r_\infty-4} \delta_{r_\infty \geq 3} \right) \\
& + \hbar \nu_{\infty,-1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty-1} k c_{\infty,k}^{(\alpha)} q_j^{k-1} - \hbar \sum_{s=1}^n \sum_{k=1}^{r_s-1} k c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k-1}
\end{aligned} \tag{5-9}$$

are Hamiltonian in the sense that

$$\forall j \in \llbracket 1, g \rrbracket : \mathcal{L}_\alpha[q_j] = \frac{\partial \text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial p_j} \text{ and } \mathcal{L}_\alpha[p_j] = - \frac{\partial \text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial q_j}. \tag{5-10}$$

Quantities involved in the Hamiltonian evolution are defined by Propositions 4.1, 4.2, 4.3 and 5.1.

Proof. Proof is done in Appendix H. $\square$

Remark 5.1. The Hamiltonian may also be rewritten as

$$\begin{aligned}
\text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p}) = & -\frac{\hbar}{2} \sum_{\substack{(i,j) \in \llbracket 1, g \rrbracket^2 \\ i \neq j}} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{q_i - q_j} - \hbar \sum_{j=1}^g (\nu_{\infty,0}^{(\alpha)} p_j + \nu_{\infty,-1}^{(\alpha)} q_j p_j) \\
& + \sum_{j=1}^g \mu_j^{(\alpha)} p_j \sum_{s=1}^n \frac{\hbar r_s}{q_j - X_s} \\
& + \sum_{j=1}^g \mu_j^{(\alpha)} \left[p_j^2 - P_1(q_j) p_j + \tilde{P}_2(q_j) + \hbar t_{\infty^{(1)}, r_\infty-1} q_j^{r_\infty-3} \delta_{r_\infty \geq 3} \right] \\
& - \hbar \sum_{j=1}^g \left[\sum_{k=1}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k + \sum_{s=1}^n \sum_{k=1}^{r_s-1} c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k} \right] \\
& + \delta_{r_\infty=2} (t_{\infty^{(1)},1} t_{\infty^{(2)},0} + t_{\infty^{(2)},1} t_{\infty^{(1)},0} + \hbar t_{\infty^{(1)},1}) \nu_{\infty,0}^{(\alpha)} \\
& - \delta_{r_\infty=1} \left(\sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty^{(1)},0} (t_{\infty^{(2)},0} + \hbar) \right) \nu_{\infty,-1}^{(\alpha)}. \tag{5-11}
\end{aligned}$$

Remark 5.2. It is important to notice that the coefficients $(H_{\infty,k})_{0 \leq k \leq r_\infty-4}$ and $(H_{X_s,k})_{1 \leq s \leq n, 1 \leq k \leq r_s}$ determined by Proposition 5.1 do not depend on the kind of deformations that is considered. This is coherent with the fact that these quantities are coefficients of the Lax matrix $L(\lambda)$ and thus should not depend on deformations. Consequently, the dependence of the Hamiltonian regarding deformations only appears in the coefficients $(\nu_{\infty,k}^{(\alpha)})_{-1 \leq k \leq r_\infty-3}$ and $(\nu_{X_s,k}^{(\alpha)})_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ and thus in the r.h.s. of Proposition 4.1.

Theorem 5.1 recovers the fact (proved for example in [72, 73]) that $\hat{\mathcal{F}}_{\mathcal{R}, \mathbf{r}, \mathbf{t}_0}$ has an underlying symplectic structure. In fact, we may define the fundamental two form in the following way.

Definition 5.1 (Fundamental symplectic two-form on $\hat{\mathcal{F}}_{\mathcal{R}, \mathbf{r}, \mathbf{t}_0}$). Let us define

$$\begin{aligned} \Omega := & \hbar \sum_{j=1}^g dq_j \wedge dp_j - \sum_{k=1}^{r_\infty-1} \sum_{i=1}^2 dt_{\infty^{(i)}, k} \wedge d\text{Ham}^{(\mathbf{e}_{\infty^{(i)}, k})} \\ & - \sum_{s=1}^n \sum_{k=1}^{r_s-1} \sum_{i=1}^2 dt_{X_s^{(i)}, k} \wedge d\text{Ham}^{(\mathbf{e}_{X_s^{(i)}, k})} - \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(\mathbf{e}_{X_s})}. \end{aligned} \quad (5-12)$$

Then Ω is a symplectic two-form on $\hat{\mathcal{F}}_{\mathcal{R}, \mathbf{r}, \mathbf{t}_0}$ that we shall refer to as “the fundamental symplectic two-form” following [73].

One may observe that Ω is a symplectic form of dimension $2(g + 2r - 2 - n) = 2g + (r_\infty + 1) + \sum_{s=1}^n (r_s - 1) > 2g$. The purpose of the next section is to prove that it is in fact equal to a symplectic form of dimension $2g$. This is achieved in Theorem 6.3 after defining a shift of the Darboux coordinates and an appropriate decomposition of the tangent space corresponding to trivial and non-trivial isomonodromic times at the level of coordinates. However let us observe that the combination of the results of the previous sections implies the following theorem.

Theorem 5.2. *There exists a birational map between the symplectic Ehresmann connection and the Jimbo-Miwa-Ueno/Boalch symplectic isomonodromy connection.*

Proof. The symplectic Ehresmann connection is characterized by the explicit time-dependent Hamiltonians given by Theorem 5.1 where coefficients are determined as rational functions in terms of irregular times, monodromies, positions of finite poles and Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ in Propositions 4.1, 4.2, 4.3 and 5.1. On the other hand, the entries of the Lax matrices in the oper gauge $(L(\lambda), A_\alpha(\lambda))$ are determined by (4-5), Propositions 2.4, 4.2, 4.3 and 5.1 and are also rational functions of the irregular times, monodromies, positions of finite poles and Darboux coordinates. Finally, one may obtain the entries of the Lax matrices in the geometric gauge $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$ using the previous expressions and the explicit expression of the gauge transformation $\tilde{\Psi}(\lambda) = G(\lambda)\Psi(\lambda)$ given in Proposition 2.1 (and also Proposition 2.6 for the value of η_0) using the standard formula

$$\begin{aligned} \tilde{L}(\lambda) &= G(\lambda)L(\lambda)G(\lambda)^{-1} + \hbar(\partial_\lambda G(\lambda))G(\lambda)^{-1}, \\ \tilde{A}_\alpha(\lambda) &= G(\lambda)A_\alpha(\lambda)G(\lambda)^{-1} + (\mathcal{L}_\alpha G(\lambda))G(\lambda)^{-1}. \end{aligned}$$

Since the gauge matrix is rational in λ , irregular times, monodromies, position of finite poles and Darboux coordinates and because the time evolution of Darboux coordinates is rational from Theorem 5.1, we obtain that the Lax matrices $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$, characterizing the Jimbo-Miwa-Ueno/Boalch symplectic isomonodromy connection are also rational functions of the irregular times, monodromies, positions of finite poles and Darboux coordinates. Combining both sides we obtain the explicit birational map between the symplectic Ehresmann connection and the Jimbo-Miwa-Ueno/Boalch symplectic isomonodromy connection. $\square$

6 Decomposition and reduction of the space of isomonodromic deformations

6.1 Subspaces of trivial and non-trivial deformations

So far we have considered general isomonodromic deformations relatively to all irregular times by considering $\mathcal{L}_\alpha$ characterized by a general vector $\alpha \in \mathbb{C}^{2g+4-n}$. However, as we will see

below, there exists a subspace of deformations of dimension $g + 4 - n$ for which the evolutions of the Darboux coordinates are trivial, leaving thus only a non-trivial subspace of deformations of dimension g . These non-trivial deformations shall later be mapped to g isomonodromic times whose expressions will be explicit in terms of the initial irregular times providing a Liouville-integrable Hamiltonian system. On the contrary, the trivial deformations correspond to the fact that only differences $\left(t_{p^{(1)},k} - t_{p^{(2)},k}\right)_{p \in \mathcal{R}, k \geq 0}$ are relevant whereas sums $\left(t_{p^{(1)},k} + t_{p^{(2)},k}\right)_{p \in \mathcal{R}, k \geq 0}$ do not appear in the Hamiltonians. This recovers the standard result that considering meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ or in $\mathfrak{sl}_2(\mathbb{C})$ is essentially the same at the level of the Hamiltonian systems because one may remove the trace of the Lax matrices by normalizing the wave matrix using $\tilde{\Psi} \rightarrow \tilde{\Psi}(\det \tilde{\Psi})^{-\frac{1}{2}}$.⁷ This remark provides a subspace of trivial deformations of dimension $g + 2 - n$. Finally the remaining two trivial deformations correspond to the remaining degrees of freedom in the action of the Möbius transformations (keeping in mind that we assumed that ∞ is always a pole, thus fixing automatically one degree of freedom). Indeed, one may rescale $\lambda \rightarrow T_2 \lambda + T_1$ without changing the symplectic structure. However, this rescaling is non-trivial for our choice of Darboux coordinates nor on the irregular times and one needs to keep track of this rescaling to obtain invariant quantities. These two degrees of freedom may be used to either fixing leading coefficients at infinity or some positions of the finite poles depending on the values of r_∞ and n . More precisely, according to the expression of the Hamiltonian in Theorem 5.1, the most convenient choice is **to fix the leading coefficients at infinity first and then only some locations of some finite poles when $r_\infty \leq 2$** .⁸ In the present paper, we choose

- For $r_\infty \geq 3$, we fix the leading coefficient $t_{\infty^{(1)}, r_\infty - 1}$ (conventionally set to 1) and sub-leading coefficient $t_{\infty^{(1)}, r_\infty - 2}$ (conventionally set to 0).
- For $r_\infty = 2$, we fix the leading coefficient $t_{\infty^{(1)}, 1}$ (conventionally set to 1) and the location of a finite pole X_1 (conventionally set to 0)
- For $r_\infty = 1$ and $n \geq 2$, we fix the location of the finite poles X_1 and X_2 (conventionally set to 0 and 1)
- For $r_\infty = 1$ and $n = 1$, we fix the location of the finite pole X_1 (conventionally set to 0) and the leading coefficient $t_{X_1^{(1)}, 1}$ (conventionally set to 1)

Let us stress that this symplectic reduction was already known from the early works of Fuchs and Garnier (See [43]) in the case of Fuchsian singularities. However, if the strategy remains the same, we need to keep track of the transformations $\lambda \rightarrow a\lambda + b$ and $\tilde{\Psi} \rightarrow \tilde{\Psi}(\det \tilde{\Psi})^{\frac{1}{2}}$ on all quantities involved not only in the Hamiltonian systems but also in the entries of the Lax matrices. In order to decompose the space of isomonodromic deformations, we introduce the following vectors.

Definition 6.1. We define the following vectors and their corresponding deformations.

⁷Note however that if there is no difference in the Hamiltonian system, the Lax matrices have explicit dependence in $(t_{p^{(1)},k} + t_{p^{(2)},k})_{p \in \mathcal{R}, k \geq 0}$ and that this dependence may be non-trivial depending on the normalization at infinity of $\tilde{L}$.

⁸Note that this is not the convention used by Jimbo-Miwa in the case of the Painlevé equations [44] since they chose to fix the location of finite poles first and then only leading coefficients at infinity if needed (when $n \leq 1$). This will provide a different Lax pair for the Painlevé 4 equation but for completeness, we shall also provide the corresponding choice to obtain the standard Jimbo-Miwa Lax pair.

- For all $k \in \llbracket 1, r_\infty - 1 \rrbracket$, the vectors $\mathbf{v}_{\infty, k}$,

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\infty, k}} &= \hbar \partial_{t_{\infty(1), k}} + \hbar \partial_{t_{\infty(2), k}} \Leftrightarrow \\ &\alpha_{\infty(i), r} = \delta_{r, k}, \alpha_{X_s^{(i)}, m} = 0, \alpha_{X_s} = 0, \\ &\forall s \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_\infty - 1 \rrbracket, m \in \llbracket 1, r_s \rrbracket. \end{aligned} \quad (6-1)$$

- For all $s \in \llbracket 1, n \rrbracket$ and all $k \in \llbracket 1, r_s - 1 \rrbracket$, the vectors $\mathbf{v}_{X_s, k}$,

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{X_s, k}} &= \hbar \partial_{t_{X_s^{(1)}, k}} + \hbar \partial_{t_{X_s^{(2)}, k}} \Leftrightarrow \\ &\alpha_{\infty(i), r} = 0, \alpha_{X_{s'}^{(i)}, m} = \delta_{s', s} \delta_{m, k}, \alpha_{X_{s'}} = 0, \\ &\forall s' \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_{s'} - 1 \rrbracket, m \in \llbracket 1, r_{s'} \rrbracket. \end{aligned} \quad (6-2)$$

- For all $k \in \llbracket 1, r_\infty - 1 \rrbracket$, the vectors $\mathbf{u}_{\infty, k}$,

$$\begin{aligned} \mathcal{L}_{\mathbf{u}_{\infty, k}} &= \hbar \sum_{j=1}^k j (t_{\infty(1), r_\infty - 1 - k + j} \partial_{t_{\infty(1), j}} + t_{\infty(2), r_\infty - 1 - k + j} \partial_{t_{\infty(2), j}}) \Leftrightarrow \\ &\alpha_{\infty(i), r} = r t_{\infty(i), r_\infty - 1 - k + r} \delta_{1 \leq r \leq k}, \alpha_{X_s^{(i)}, m} = 0, \alpha_{X_s} = 0, \\ &\forall s \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_\infty - 1 \rrbracket, m \in \llbracket 1, r_s \rrbracket. \end{aligned} \quad (6-3)$$

- For all $s \in \llbracket 1, n \rrbracket$ and all $k \in \llbracket 1, r_s - 1 \rrbracket$, the vectors $\mathbf{u}_{X_s, k}$,

$$\begin{aligned} \mathcal{L}_{\mathbf{u}_{X_s, k}} &= \hbar \sum_{j=1}^k j (t_{X_s^{(1)}, r_s - 1 - k + j} \partial_{t_{X_s^{(1)}, j}} + t_{X_s^{(2)}, r_s - 1 - k + j} \partial_{t_{X_s^{(2)}, j}}) \Leftrightarrow \\ &\alpha_{X_{s'}^{(i)}, r} = r t_{X_{s'}^{(i)}, r_{s'} - 1 - k + r} \delta_{s', s} \delta_{1 \leq r \leq k}, \alpha_{\infty(i), r} = 0, \alpha_{X_s} = 0, \\ &\forall s' \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_\infty - 1 \rrbracket, m \in \llbracket 1, r_{s'} \rrbracket. \end{aligned} \quad (6-4)$$

- The vector $\mathbf{a}$,

$$\begin{aligned} \mathcal{L}_{\mathbf{a}} &= \mathcal{L}_{\mathbf{u}_{\infty, r_\infty - 1}} - \sum_{s=1}^n \mathcal{L}_{\mathbf{u}_{X_s, r_s - 1}} - \hbar \sum_{s=1}^n X_s \partial_{X_s} \\ &= \hbar \sum_{r=1}^{r_\infty - 1} r (t_{\infty(1), r} \partial_{t_{\infty(1), r}} + t_{\infty(2), r} \partial_{t_{\infty(2), r}}) \\ &\quad - \hbar \sum_{s=1}^n \sum_{r=1}^{r_s - 1} r (t_{X_s^{(1)}, r} \partial_{t_{X_s^{(1)}, r}} + t_{X_s^{(2)}, r} \partial_{t_{X_s^{(2)}, r}}) - \hbar \sum_{s=1}^n X_s \partial_{X_s} \\ &\Leftrightarrow \alpha_{X_s^{(i)}, m} = -r t_{X_s^{(i)}, m}, \alpha_{\infty(i), r} = r t_{\infty(i), r}, \alpha_{X_s} = -X_s, \\ &\forall s \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_\infty - 1 \rrbracket, m \in \llbracket 1, r_s \rrbracket. \end{aligned} \quad (6-5)$$

- The vector $\mathbf{b}$,

$$\begin{aligned} \mathcal{L}_{\mathbf{b}} &= \mathcal{L}_{\mathbf{u}_{\infty, r_\infty - 2}} - \hbar \sum_{s=1}^n \partial_{X_s} \\ &= \hbar \sum_{r=1}^{r_\infty - 2} r (t_{\infty(1), r+1} \partial_{t_{\infty(1), r}} + t_{\infty(2), r+1} \partial_{t_{\infty(2), r}}) - \hbar \sum_{s=1}^n \partial_{X_s} \\ &\Leftrightarrow \alpha_{X_s^{(i)}, k} = 0, \alpha_{\infty(i), r} = r t_{\infty(i), r+1} \delta_{1 \leq r \leq r_\infty - 2}, \alpha_{X_s} = -1, \\ &\forall s \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_\infty - 1 \rrbracket, m \in \llbracket 1, r_s \rrbracket. \end{aligned} \quad (6-6)$$

- For all $s \in \llbracket 1, n \rrbracket$, the vectors $\mathbf{w}_s$,

$$\begin{aligned} \mathcal{L}_{\mathbf{w}_s} &= \hbar \partial_{X_s} \Leftrightarrow \\ \alpha_{X_{s'}} &= \delta_{s',s}, \alpha_{X_{s'},m}^{(i)} = 0, \alpha_{\infty,r}^{(i)} = 0, \\ \forall s' \in \llbracket 1, n \rrbracket, i \in \llbracket 1, 2 \rrbracket, r \in \llbracket 1, r_\infty - 1 \rrbracket, m \in \llbracket 1, r_{s'} \rrbracket. \end{aligned} \quad (6-7)$$

Note that the vectors $(\mathbf{u}_{\infty,k})_{1 \leq k \leq r_\infty - 3}$, $(\mathbf{u}_{s,k})_{1 \leq s \leq n, 1 \leq k \leq r_s - 1}$, $(\mathbf{v}_{\infty,k})_{1 \leq k \leq r_\infty - 1}$, $(\mathbf{v}_{s,k})_{1 \leq s \leq n, 1 \leq k \leq r_s - 1}$, $\mathbf{a}$, $\mathbf{b}$, $(\mathbf{w}_s)_{1 \leq s \leq n}$ are a basis of the deformation space (of dimension $2g + 4 - n$), since we have $r_\infty - 3 + \sum_{s=1}^n r_s - n + r_\infty - 1 + \sum_{s=1}^n r_s - n + 2 + n = 2r_\infty + 2 \sum_{s=1}^n r_s - n - 2 = 2g + 4 - n$ vectors that are obviously linearly independent.

Remark 6.1. The previous definitions are chosen so that the vector fields $(\mathcal{L}_{\mathbf{v}_{\infty,k}})_{1 \leq k \leq r_\infty - 1}$ and $(\mathcal{L}_{\mathbf{v}_{X_s,k}})_{s \leq n, 1 \leq k \leq r_s - 1}$ corresponds to the transformation $\tilde{\Psi} \rightarrow \tilde{\Psi}(\det \tilde{\Psi})^{\frac{1}{2}}$, i.e. to go from connections on $\mathfrak{gl}_2(\mathbb{C})$ to connections on $\mathfrak{sl}_2(\mathbb{C})$. On the contrary, the vector field $\mathcal{L}_{\mathbf{a}}$ corresponds to keep track of the dilatation $\lambda \rightarrow c\lambda$ while $\mathcal{L}_{\mathbf{b}}$ corresponds to keep track of the translation $\lambda \rightarrow \lambda + c$.

The previous vectors satisfy the following proposition.

Proposition 6.1. *For a given $j \in \llbracket 1, r_\infty - 1 \rrbracket$, we have*

$$\begin{aligned} 0 &= \nu_{\infty,k}^{(\mathbf{v}_{\infty,j})}, \forall k \in \llbracket -1, r_\infty - 3 \rrbracket, \\ 0 &= \nu_{X_s,k}^{(\mathbf{v}_{\infty,j})}, \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ -\frac{1}{j} \delta_{k,j} &= c_{\infty,k}^{(\mathbf{v}_{\infty,j})}, \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= c_{X_s,k}^{(\mathbf{v}_{\infty,j})}, \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket. \end{aligned} \quad (6-8)$$

Let $s \in \llbracket 1, n \rrbracket$ and $j \in \llbracket 1, r_s - 1 \rrbracket$, then we have

$$\begin{aligned} 0 &= \nu_{\infty,k}^{(\mathbf{v}_{X_s,j})}, \forall k \in \llbracket -1, r_\infty - 3 \rrbracket, \\ 0 &= \nu_{X_{s'},k}^{(\mathbf{v}_{X_s,j})}, \forall (s',k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_{s'} - 1 \rrbracket, \\ 0 &= c_{\infty,k}^{(\mathbf{v}_{X_s,j})}, \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ -\frac{1}{j} \delta_{s',s} \delta_{k,j} &= c_{X_{s'},k}^{(\mathbf{v}_{X_s,j})}, \forall (s',k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_{s'} - 1 \rrbracket. \end{aligned} \quad (6-9)$$

Proof. The proof is done in Appendix I. □

We also have,

Proposition 6.2. *For a given $j \in \llbracket 1, r_\infty - 1 \rrbracket$, we have*

$$\begin{aligned} \delta_{r_\infty - 2 - j, k} &= \nu_{\infty,k}^{(\mathbf{u}_{\infty,j})}, \forall k \in \llbracket -1, r_\infty - 3 \rrbracket, \\ 0 &= \nu_{X_s,k}^{(\mathbf{u}_{\infty,j})}, \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ 0 &= c_{\infty,k}^{(\mathbf{u}_{\infty,j})}, \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= c_{X_s,k}^{(\mathbf{u}_{\infty,j})}, \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket. \end{aligned} \quad (6-10)$$

Let $s \in \llbracket 1, n \rrbracket$ and $j \in \llbracket 1, r_s - 1 \rrbracket$, then we have

$$\begin{aligned}
0 &= \nu_{\infty, k}^{(\mathbf{u}_{X_s, j})}, \forall k \in \llbracket -1, r_\infty - 3 \rrbracket, \\
-\delta_{s', s} \delta_{k, r_s - j} &= \nu_{X_{s'}, k}^{(\mathbf{u}_{X_s, j})}, \forall (s', k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_{s'} - 1 \rrbracket, \\
0 &= c_{\infty, k}^{(\mathbf{u}_{X_s, j})}, \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\
0 &= c_{X_{s'}, k}^{(\mathbf{u}_{X_s, j})}, \forall (s', k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_{s'} - 1 \rrbracket.
\end{aligned} \tag{6-11}$$

Proof. The proof is done in Appendix J. $\square$

Finally, vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy,

Proposition 6.3. *The vectors $\mathbf{a}$ and $\mathbf{b}$ are chosen so that*

$$\begin{aligned}
\nu_{X_s, 0}^{(\mathbf{a})} &= X_s, \forall s \in \llbracket 1, n \rrbracket, \\
\nu_{X_s, k}^{(\mathbf{a})} &= \delta_{k, 1}, \forall s \in \llbracket 1, n \rrbracket, k \in \llbracket 1, r_s - 1 \rrbracket, \\
\nu_{\infty, k}^{(\mathbf{a})} &= \delta_{k, -1}, \forall k \in \llbracket -1, r_\infty - 3 \rrbracket, \\
c_{\infty, k}^{(\mathbf{a})} &= 0, \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\
c_{X_s, k}^{(\mathbf{a})} &= 0, \forall s \in \llbracket 1, n \rrbracket, k \in \llbracket 1, r_s - 1 \rrbracket, \\
\nu_{X_s, 0}^{(\mathbf{b})} &= 1, \forall s \in \llbracket 1, n \rrbracket, \\
\nu_{\infty, k}^{(\mathbf{b})} &= \delta_{k, 0}, \forall k \in \llbracket -1, r_\infty - 3 \rrbracket, \\
\nu_{X_s, k}^{(\mathbf{b})} &= 0, \forall s \in \llbracket 1, n \rrbracket, k \in \llbracket 1, r_s - 1 \rrbracket, \\
c_{\infty, k}^{(\mathbf{b})} &= 0, \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\
c_{X_s, k}^{(\mathbf{b})} &= 0, \forall s \in \llbracket 1, n \rrbracket, k \in \llbracket 1, r_s - 1 \rrbracket.
\end{aligned} \tag{6-12}$$

Proof. The proof mainly follows from Proposition 6.2. For completeness, we detail it in Appendix K. $\square$

The previous propositions imply that we get trivial deformations along vectors $(\mathbf{v}_{\infty, k})_{1 \leq k \leq r_\infty - 1}$, $(\mathbf{v}_{X_s, k})_{1 \leq s \leq n, 1 \leq k \leq r_s - 1}$, $\mathbf{a}$ and $\mathbf{b}$.

Theorem 6.1. *We have the following trivial deformations.*

$$\begin{aligned}
0 &= \mathcal{L}_{\mathbf{v}_{\infty, k}}[q_j], \forall (j, k) \in \llbracket 1, g \rrbracket \times \llbracket 1, r_\infty - 1 \rrbracket, \\
-\hbar q_j^{k-1} &= \mathcal{L}_{\mathbf{v}_{\infty, k}}[p_j], \forall (j, k) \in \llbracket 1, g \rrbracket \times \llbracket 1, r_\infty - 1 \rrbracket, \\
0 &= \mathcal{L}_{\mathbf{v}_{X_s, k}}[q_j], \forall (j, s, k) \in \llbracket 1, g \rrbracket \times \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
\hbar(q_j - X_s)^{k-1} &= \mathcal{L}_{\mathbf{v}_{X_s, k}}[p_j], \forall (j, s, k) \in \llbracket 1, g \rrbracket \times \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
-\hbar q_j &= \mathcal{L}_{\mathbf{a}}[q_j], \forall j \in \llbracket 1, g \rrbracket, \\
-\hbar &= \mathcal{L}_{\mathbf{b}}[q_j], \forall j \in \llbracket 1, g \rrbracket, \\
\hbar p_j &= \mathcal{L}_{\mathbf{a}}[p_j], \forall j \in \llbracket 1, g \rrbracket, \\
0 &= \mathcal{L}_{\mathbf{b}}[p_j], \forall j \in \llbracket 1, g \rrbracket.
\end{aligned} \tag{6-13}$$

Proof. The proof is done in Appendix L. $\square$

Theorem 6.1 is coherent with the action on the Darboux coordinates of the transformations equivalent to the vector fields mentioned in Remark 6.1. Note that the subspace of trivial deformations is of dimension $r_\infty - 1 + \sum_{s=1}^n r_s - n + 2 = g - n + 4$ leaving a subspace of non-trivial deformations of dimension g .

6.2 Definition of trivial times and isomonodromic times

The last section indicates that there are only g non-trivial deformations given by $(\mathcal{L}_{\mathbf{u}_{\infty,k}})_{1 \leq k \leq r_{\infty}-3}$, $(\mathcal{L}_{\mathbf{u}_{X_s,k}})_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ and $(\mathcal{L}_{\mathbf{x}_s})_{1 \leq s \leq n}$. The split in the tangent space may be translated at the level of coordinates. This corresponds to choosing $g + 4 - n$ trivial times and g non-trivial times so that the Darboux coordinates are independent of the trivial times. In particular, one may choose these trivial times to take any arbitrary values without changing the Hamiltonian evolutions. However, the choice of trivial times and isomonodromic times is not unique since, for example, one may use any arbitrary combination of isomonodromic times to provide a new one. As we will see below, the symplectic reduction allows to take $P_1 = 0$, i.e. to remove the trace of $\tilde{L}$, which is a canonical choice in the context of integrable systems. However, the situation is more involved for translations and dilatations because the canonical choice for the corresponding trivial times depend on the value of r_{∞} and of the number of finite poles. However, it is possible to choose them wisely (in the next four subsections) so that they share the same main properties.

6.2.1 The case $r_{\infty} \geq 3$

As explained above, when $r_{\infty} \geq 3$, one may fix the two leading coefficients at infinity. This corresponds to choosing the following set of times.

Definition 6.2 (Trivial and isomonodromic times). For $r_{\infty} \geq 3$, we define the following set of trivial coordinates denoted $\mathcal{T}_{\text{trivial}}$.

$$\begin{aligned} T_{\infty,k} &= t_{\infty^{(1)},k} + t_{\infty^{(2)},k}, \quad \forall k \in \llbracket 1, r_{\infty} - 1 \rrbracket, \\ T_{X_s,k} &= t_{X_s^{(1)},k} + t_{X_s^{(2)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ T_1 &= \frac{t_{\infty^{(1)},r_{\infty}-2} - t_{\infty^{(2)},r_{\infty}-2}}{2^{\frac{1}{r_{\infty}-1}} (r_{\infty} - 2) (t_{\infty^{(1)},r_{\infty}-1} - t_{\infty^{(2)},r_{\infty}-1})^{\frac{r_{\infty}-2}{r_{\infty}-1}}}, \\ T_2 &= \left(\frac{t_{\infty^{(1)},r_{\infty}-1} - t_{\infty^{(2)},r_{\infty}-1}}{2} \right)^{\frac{1}{r_{\infty}-1}}. \end{aligned} \quad (6-14)$$

Moreover, the set of isomonodromic times, denoted $\mathcal{T}_{\text{iso}}$, contains

- For all $j \in \llbracket 1, r_{\infty} - 3 \rrbracket$:

$$\begin{aligned} \tau_{\infty,j} &= 2^{\frac{j}{r_{\infty}-1}} \left[\sum_{i=0}^{r_{\infty}-j-3} \frac{(-1)^i (j+i-1)!}{i! (j-1)! (r_{\infty}-2)^i} \frac{(t_{\infty^{(1)},r_{\infty}-2} - t_{\infty^{(2)},r_{\infty}-2})^i (t_{\infty^{(1)},j+i} - t_{\infty^{(2)},j+i})}{(t_{\infty^{(1)},r_{\infty}-1} - t_{\infty^{(2)},r_{\infty}-1})^{\frac{i(r_{\infty}-1)+j}{r_{\infty}-1}}} \right. \\ &\quad \left. + \frac{(-1)^{r_{\infty}-j-2} (r_{\infty}-3)!}{(r_{\infty}-1-j)(r_{\infty}-j-3)! (j-1)! (r_{\infty}-2)^{r_{\infty}-j-2}} \frac{(t_{\infty^{(1)},r_{\infty}-2} - t_{\infty^{(2)},r_{\infty}-2})^{r_{\infty}-1-j}}{(t_{\infty^{(1)},r_{\infty}-1} - t_{\infty^{(2)},r_{\infty}-1})^{\frac{(r_{\infty}-2)(r_{\infty}-1-j)}{r_{\infty}-1}}} \right]. \end{aligned} \quad (6-15)$$

- For all $s \in \llbracket 1, n \rrbracket$: $(\tau_{X_s,k})_{1 \leq k \leq r_s-1}$ are defined by:

$$\tau_{X_s,k} = (t_{X_s^{(1)},k} - t_{X_s^{(2)},k}) \left(\frac{t_{\infty^{(1)},r_{\infty}-1} - t_{\infty^{(2)},r_{\infty}-1}}{2} \right)^{\frac{k}{r_{\infty}-1}}, \quad \forall k \in \llbracket 1, r_s - 1 \rrbracket. \quad (6-16)$$

- For all $s \in \llbracket 1, n \rrbracket$, the times $(\tilde{X}_s)_{3 \leq s \leq n}$ are defined by

$$\tilde{X}_s = X_s \left(\frac{t_{\infty^{(1)}, r_{\infty}-1} - t_{\infty^{(2)}, r_{\infty}-1}}{2} \right)^{\frac{1}{r_{\infty}-1}} + \frac{\left(t_{\infty^{(1)}, r_{\infty}-2} - t_{\infty^{(2)}, r_{\infty}-2} \right)}{2^{\frac{1}{r_{\infty}-1}} (r_{\infty}-2) \left(t_{\infty^{(1)}, r_{\infty}-1} - t_{\infty^{(2)}, r_{\infty}-1} \right)^{\frac{r_{\infty}-2}{r_{\infty}-1}}}. \quad (6-17)$$

We have the inverse relations

Proposition 6.4. *Irregular times and location of the poles are related to the set of trivial and isomonodromic times by the relations*

- For all $k \in \llbracket 1, r_{\infty}-3 \rrbracket$, $i \in \llbracket 1, 2 \rrbracket$:

$$\begin{aligned} t_{\infty^{(i)}, r_{\infty}-1} &= \frac{1}{2} T_{\infty, r_{\infty}-1} + (-1)^{i+1} T_2^{r_{\infty}-1}, \\ t_{\infty^{(i)}, r_{\infty}-2} &= \frac{1}{2} T_{\infty, r_{\infty}-2} + (-1)^{i+1} (r_{\infty}-2) T_1 T_2^{r_{\infty}-2}, \\ t_{\infty^{(i)}, k} &= \frac{1}{2} T_{\infty, k} + \frac{(-1)^{i+1}}{2} T_2^k \left(\frac{2(r_{\infty}-2)!}{(k-1)!(r_{\infty}-1-k)!} T_1^{r_{\infty}-1-k} \right. \\ &\quad \left. + \sum_{j=2}^{r_{\infty}-1-k} \frac{(r_{\infty}-2-j)!}{(k-1)!(r_{\infty}-1-k-j)!} T_1^{r_{\infty}-1-j-k} \tau_{\infty, r_{\infty}-1-j} \right). \end{aligned} \quad (6-18)$$

- For all $i \in \llbracket 1, 2 \rrbracket$, $s \in \llbracket 1, n \rrbracket$, for all $k \in \llbracket 1, r_s-1 \rrbracket$:

$$t_{X_s^{(i)}, k} = \frac{1}{2} T_{X_s, k} + \frac{(-1)^{i+1}}{2} T_2^{-k} \tau_{X_s, k}. \quad (6-19)$$

- The position of the poles are given by

$$X_s = \frac{\tilde{X}_s - T_1}{T_2}, \quad \forall s \in \llbracket 1, n \rrbracket. \quad (6-20)$$

Proof. The proof is presented in Appendix M. □

The inverse relations provide the following proposition.

Proposition 6.5 (Deformations dual to the isomonodromic times for $r_{\infty} \geq 3$). *For $r_{\infty} \geq 3$, we have, for all $r \in \llbracket 1, r_{\infty}-3 \rrbracket$,*

$$\partial_{\tau_{\infty, r}} = \frac{1}{2} \sum_{k=1}^r T_2^k \frac{(r-1)!}{(k-1)!(r-k)!} T_1^{r-k} \left(\partial_{t_{\infty^{(1)}, k}} - \partial_{t_{\infty^{(2)}, k}} \right), \quad \forall r \in \llbracket 1, r_{\infty}-3 \rrbracket \quad (6-21)$$

and for all $s \in \llbracket 1, n \rrbracket$:

$$\partial_{\tau_{X_s, k}} = \frac{1}{2} T_2^{-k} \left(\partial_{t_{X_s^{(1)}, k}} - \partial_{t_{X_s^{(2)}, k}} \right), \quad \forall k \in \llbracket 1, r_s-1 \rrbracket \quad (6-22)$$

and

$$\partial_{\tilde{X}_s} = \frac{1}{T_2} \partial_{X_s}, \quad \forall s \in \llbracket 1, n \rrbracket. \quad (6-23)$$

6.2.2 The case $r_\infty = 2$

In the case $r_\infty = 2$ we necessarily have $n \geq 1$ in order to have $g > 0$. We may thus fix the coefficient $t_{\infty^{(1)},1}$ and the location of the pole X_1 .

Definition 6.3 (Trivial and isomonodromic times). For $r_\infty = 2$, we define the following set of trivial coordinates denoted $\mathcal{T}_{\text{trivial}}$.

$$\begin{aligned} T_{\infty,1} &= t_{\infty^{(1)},1} + t_{\infty^{(2)},1}, \\ T_{X_s,k} &= t_{X_s^{(1)},k} + t_{X_s^{(2)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ T_1 &= -X_1 \left(\frac{t_{\infty^{(1)},1} - t_{\infty^{(2)},1}}{2} \right), \\ T_2 &= \left(\frac{t_{\infty^{(1)},1} - t_{\infty^{(2)},1}}{2} \right). \end{aligned} \tag{6-24}$$

Moreover, the set of isomonodromic times, denoted $\mathcal{T}_{\text{iso}}$, contains

- For all $s \in \llbracket 1, n \rrbracket$: $(\tau_{X_s,k})_{1 \leq k \leq r_s - 1}$ are defined by:

$$\tau_{X_s,k} = (t_{X_s^{(1)},k} - t_{X_s^{(2)},k}) \left(\frac{t_{\infty^{(1)},1} - t_{\infty^{(2)},1}}{2} \right)^k, \quad \forall k \in \llbracket 1, r_s - 1 \rrbracket. \tag{6-25}$$

- For all $s \in \llbracket 2, n \rrbracket$, the times $(\tilde{X}_s)_{2 \leq s \leq n}$ are defined by

$$\tilde{X}_s = (X_s - X_1) \left(\frac{t_{\infty^{(1)},1} - t_{\infty^{(2)},1}}{2} \right). \tag{6-26}$$

We have the inverse relations

Proposition 6.6. *Irregular times and location of the poles are related to the set of trivial and isomonodromic times by the relations*

- For all $i \in \llbracket 1, 2 \rrbracket$:

$$t_{\infty^{(i)},1} = \frac{1}{2} T_{\infty,1} + (-1)^{i+1} T_2. \tag{6-27}$$

- For all $i \in \llbracket 1, 2 \rrbracket$, $s \in \llbracket 1, n \rrbracket$, for all $k \in \llbracket 1, r_s - 1 \rrbracket$:

$$t_{X_s^{(i)},k} = \frac{1}{2} T_{X_s,k} + \frac{(-1)^{i+1}}{2} T_2^{-k} \tau_{X_s,k}. \tag{6-28}$$

- The position of the poles are given by

$$\begin{aligned} X_1 &= -\frac{T_1}{T_2}, \\ X_s &= \frac{\tilde{X}_s - T_1}{T_2}, \quad \forall s \in \llbracket 2, n \rrbracket. \end{aligned} \tag{6-29}$$

Proof. The proof is obvious. □

The inverse relations provide the following proposition.

Proposition 6.7 (Deformations dual to the isomonodromic times for $r_\infty = 2$). *For $r_\infty = 2$, for all $s \in \llbracket 1, n \rrbracket$:*

$$\partial_{\tau_{X_s, k}} = \frac{1}{2} T_2^{-k} \left(\partial_{t_{X_s^{(1)}, k}} - \partial_{t_{X_s^{(2)}, k}} \right), \quad \forall k \in \llbracket 1, r_s - 1 \rrbracket \quad (6-30)$$

and for all $s \in \llbracket 2, n \rrbracket$:

$$\partial_{\tilde{X}_s} = \frac{1}{T_2} \partial_{X_s}, \quad \forall s \in \llbracket 1, n \rrbracket. \quad (6-31)$$

6.2.3 The case $r_\infty = 1$ and $n \geq 2$

In this section we consider $r_\infty = 1$ and $n \geq 2$ and we fix the position of two finite poles. This corresponds to the following set of trivial and isomonodromic times.

Definition 6.4 (Trivial and isomonodromic times). For $r_\infty = 1$ and $n \geq 2$, we define the following set of trivial coordinates denoted $\mathcal{T}_{\text{trivial}}$.

$$\begin{aligned} T_{X_s, k} &= t_{X_s^{(1)}, k} + t_{X_s^{(2)}, k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ T_1 &= -\frac{X_1}{X_2 - X_1}, \\ T_2 &= (X_2 - X_1)^{-1}. \end{aligned} \quad (6-32)$$

Moreover, the set of isomonodromic times, denoted $\mathcal{T}_{\text{iso}}$, contains

- For all $s \in \llbracket 1, n \rrbracket$: $(\tau_{X_s, k})_{1 \leq k \leq r_s - 1}$ are defined by:

$$\tau_{X_s, k} = (t_{X_s^{(1)}, k} - t_{X_s^{(2)}, k}) (X_2 - X_1)^{-k}, \quad \forall k \in \llbracket 1, r_s - 1 \rrbracket. \quad (6-33)$$

- For all $s \in \llbracket 3, n \rrbracket$, the times $(\tilde{X}_s)_{3 \leq s \leq n}$ are defined by

$$\tilde{X}_s = \frac{X_s - X_1}{X_2 - X_1}. \quad (6-34)$$

The inverse relations provide the following proposition.

Proposition 6.8. *Irregular times and location of the poles are related to the set of trivial and isomonodromic times by the relations*

- For all $i \in \llbracket 1, 2 \rrbracket$, $s \in \llbracket 1, n \rrbracket$, for all $k \in \llbracket 1, r_s - 1 \rrbracket$:

$$t_{X_s^{(i)}, k} = \frac{1}{2} T_{X_s, k} + \frac{(-1)^{i+1}}{2} T_2^{-k} \tau_{X_s, k}. \quad (6-35)$$

- The position of the poles are given by

$$\begin{aligned} X_1 &= -\frac{T_1}{T_2}, \\ X_2 &= \frac{1 - T_1}{T_2}, \\ X_s &= \frac{\tilde{X}_s - T_1}{T_2}, \quad \forall s \in \llbracket 3, n \rrbracket. \end{aligned} \quad (6-36)$$

Proof. The proof is obvious. $\square$

The inverse relations provide the following proposition.

Proposition 6.9 (Deformations dual to the isomonodromic times for $r_\infty = 1$ and $n \geq 2$). *For $r_\infty = 1$ and $n \geq 2$, for all $s \in \llbracket 1, n \rrbracket$:*

$$\partial_{\tau_{X_s, k}} = \frac{1}{2} T_2^{-k} \left(\partial_{t_{X_s^{(1)}, k}} - \partial_{t_{X_s^{(2)}, k}} \right), \quad \forall k \in \llbracket 1, r_s - 1 \rrbracket \quad (6-37)$$

and for all $s \in \llbracket 3, n \rrbracket$:

$$\partial_{\tilde{X}_s} = \frac{1}{T_2} \partial_{X_s}, \quad \forall s \in \llbracket 1, n \rrbracket. \quad (6-38)$$

6.2.4 The case $r_\infty = 1$ and $n = 1$

In this section we consider $r_\infty = 1$ and $n = 1$ (hence $r_1 \geq 2$ to have $g > 0$). In this case, we may fix the position of the finite pole and its leading coefficient. It corresponds to the following definition of trivial and isomonodromic times.

Definition 6.5 (Trivial and isomonodromic times). For $r_\infty = 1$ and $n = 1$, we define the following set of trivial coordinates denoted $\mathcal{T}_{\text{trivial}}$.

$$\begin{aligned} T_{X_1, k} &= t_{X_1^{(1)}, k} + t_{X_1^{(2)}, k}, \quad \forall k \in \llbracket 1, r_1 - 1 \rrbracket, \\ T_1 &= -X_1 \left(\frac{t_{X_1^{(1)}, r_1-1} - t_{X_1^{(2)}, r_1-1}}{2} \right)^{-\frac{1}{r_1-1}}, \\ T_2 &= \left(\frac{t_{X_1^{(1)}, r_1-1} - t_{X_1^{(2)}, r_1-1}}{2} \right)^{-\frac{1}{r_1-1}}. \end{aligned} \quad (6-39)$$

Moreover, the set of isomonodromic times, denoted $\mathcal{T}_{\text{iso}}$, contains

- $(\tau_{X_1, k})_{1 \leq k \leq r_1-2}$ are defined by:

$$\tau_{X_1, k} = (t_{X_1^{(1)}, k} - t_{X_1^{(2)}, k}) \left(\frac{t_{X_1^{(1)}, r_1-1} - t_{X_1^{(2)}, r_1-1}}{2} \right)^{-\frac{k}{r_1-1}}, \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket. \quad (6-40)$$

The inverse relations are given by the following proposition.

Proposition 6.10. *Irregular times and location of the poles are related to the set of trivial and isomonodromic times by the relations*

- For all $i \in \llbracket 1, 2 \rrbracket$:

$$\begin{aligned} t_{X_1^{(i)}, r_1-1} &= \frac{1}{2} T_{X_1, r_1-1} + (-1)^{i+1} T_2^{-(r_1-1)}, \\ t_{X_1^{(i)}, k} &= \frac{1}{2} T_{X_s, k} + \frac{(-1)^{i+1}}{2} T_2^{-k} \tau_{X_1, k}, \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket. \end{aligned} \quad (6-41)$$

- The position of the finite pole are given by

$$X_1 = -\frac{T_1}{T_2}. \quad (6-42)$$

Proof. The proof is obvious. $\square$

The inverse relations provide the following proposition.

Proposition 6.11 (Deformations dual to the isomonodromic times for $r_\infty = 1$ and $n = 1$). *For $r_\infty = 1$ and $n = 1$, we have:*

$$\partial_{\tau_{X_1,k}} = \frac{1}{2} T_2^{-k} \left(\partial_{t_{X_1^{(1)},k}} - \partial_{t_{X_1^{(2)},k}} \right), \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket. \quad (6-43)$$

6.3 Properties of trivial and isomonodromic times

We first note that in each case, we always have exactly g isomonodromic times that are complemented by $r_\infty + 1 + \sum_{s=1}^n r_s - n$ trivial times so that we have in total $2r_\infty + 2 \sum_{s=1}^n r_s - n - 2$ new coordinates corresponding to the dimension of the initial deformation space. It is also straightforward to see that the set of new coordinates $\mathcal{T} := \mathcal{T}_{\text{trivial}} \cup \mathcal{T}_{\text{iso}}$ is in one-to-one correspondence with the set of initial coordinates $\{(t_{\infty^{(i)},k})_{1 \leq i \leq 2, 1 \leq k \leq r_\infty - 1}\} \cup \{(t_{X_s^{(i)},k})_{1 \leq i \leq 2, 1 \leq s \leq n, 1 \leq k \leq r_s - 1}\} \cup \{X_s\}_{1 \leq s \leq n}$ since we could exhibit the inverse relations in Propositions 6.4, 6.6 and 6.8.

Let us first observe that the trivial times T_1 and T_2 are chosen to satisfy the following proposition.

Proposition 6.12. *For any $s \geq 0$, the trivial times T_1 and T_2 satisfy*

$$\begin{aligned} 0 &= \mathcal{L}_{\mathbf{v}_{\infty,k}}[T_2], \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= \mathcal{L}_{\mathbf{v}_{X_s,k}}[T_2], \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_\infty - 1 \rrbracket, \\ \hbar T_2 &= \mathcal{L}_{\mathbf{a}}[T_2], \\ 0 &= \mathcal{L}_{\mathbf{b}}[T_2], \\ 0 &= \mathcal{L}_{\mathbf{v}_{\infty,k}}[T_1], \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= \mathcal{L}_{\mathbf{v}_{X_s,k}}[T_1], \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= \mathcal{L}_{\mathbf{a}}[T_1], \\ \hbar T_2 &= \mathcal{L}_{\mathbf{b}}[T_1]. \end{aligned} \quad (6-44)$$

Proof. The proof is done in Appendix N. $\square$

Moreover, the isomonodromic times are defined so that the following theorem holds:

Theorem 6.2. *The general solutions $f(X_s, t_{\infty^{(i)},k}, t_{X_s^{(i)},k})$ of the partial differential equations*

$$\begin{aligned} 0 &= \mathcal{L}_{v_{\infty,k}}[f], \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= \mathcal{L}_{v_{X_s,k}}[f], \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ 0 &= \mathcal{L}_{\mathbf{a}}[f], \\ 0 &= \mathcal{L}_{\mathbf{b}}[f] \end{aligned} \quad (6-45)$$

are arbitrary functions of the isomonodromic times. Note in particular that the isomonodromic times are themselves solutions of (6-45).

Proof. The proof is done in Appendix O. $\square$

6.4 Shifted Darboux coordinates and symplectic reduction

Theorem 6.1 indicates that deformations $\mathcal{L}_a$ and $\mathcal{L}_b$ do not act trivially on the Darboux coordinates $(q_j, p_j)_{1 \leq j \leq g}$. However, since the action is very simple, we may easily perform a symplectic transformation on the Darboux coordinates to obtain “shifted Darboux coordinates” for which the action of $\mathcal{L}_a$ and $\mathcal{L}_b$ becomes trivial.

Definition 6.6. For any $j \in \llbracket 1, g \rrbracket$, we define

$$\begin{aligned}\check{q}_j &= T_2 q_j + T_1, \\ \check{p}_j &= T_2^{-1} \left(p_j - \frac{1}{2} P_1(q_j) \right).\end{aligned}\tag{6-46}$$

that we call “shifted Darboux coordinates”.

The definition of the shifted Darboux coordinates implies the following main result.

Theorem 6.3. *[Symplectic reduction] The fundamental symplectic two-form Ω defined in Definition 5.1 reduces to*

$$\begin{aligned}\Omega &= \hbar \sum_{j=1}^g dq_j \wedge dp_j - \sum_{s=1}^n \sum_{i=1}^2 \sum_{k=1}^{r_s-1} dt_{X_s^{(i)},k} \wedge d\text{Ham}^{(\mathbf{e}_{X_s^{(i)},k})} \\ &\quad - \sum_{i=1}^2 \sum_{k=1}^{r_\infty-1} dt_{\infty^{(i)},k} \wedge d\text{Ham}^{(\mathbf{e}_{\infty^{(i)},k})} - \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(\mathbf{e}_{X_s})} \\ &= \hbar \sum_{j=1}^g d\check{q}_j \wedge d\check{p}_j - \sum_{\tau \in T_{iso}} d\tau \wedge d\text{Ham}^{(\alpha_\tau)}.\end{aligned}\tag{6-47}$$

Proof. The proof is done by direct computation of the symplectic form using the definition of the trivial and isomonodromic times. It is done for each case in Appendix P. $\square$

Theorem 6.3 (or Theorem 6.2) implies the following corollary:

Corollary 6.1. *The shifted Darboux coordinates $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ are independent of the trivial times and thus may only depend on the isomonodromic times.*

Moreover, we may complement the previous result with the following theorem regarding the dependence on the monodromy parameters:

Theorem 6.4. *The shifted Darboux coordinates $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ depend only on the differences $\{t_{\infty^{(1)},0} - t_{\infty^{(2)},0}, t_{X_1^{(1)},0} - t_{X_1^{(2)},0}, \dots, t_{X_n^{(1)},0} - t_{X_n^{(2)},0}\}$ but not on $\{t_{\infty^{(1)},0} + t_{\infty^{(2)},0}, t_{X_1^{(1)},0} + t_{X_1^{(2)},0}, \dots, t_{X_n^{(1)},0} + t_{X_n^{(2)},0}\}$. In other words, $T_{\infty,0} := t_{\infty^{(1)},0} + t_{\infty^{(2)},0}$ and $T_{X_s,0} := t_{X_s^{(1)},0} + t_{X_s^{(2)},0}$ for $s \in \llbracket 1, n \rrbracket$ may also be considered as additional trivial times.*

Proof. The proof is done in Appendix Q. $\square$

Remark 6.2. Corollary 6.1 implies that the shifted Darboux coordinates $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$ are solutions of the partial differential equations (6-45).

Finally let us mention the following observation.

Proposition 6.13. *For any isomonodromic deformations $(\tau_j)_{1 \leq j \leq g}$, associated to vectors α_{τ_j} , the trace of the corresponding matrices $\check{A}_{\alpha_{\tau_j}}$ and $\tilde{A}_{\alpha_{\tau_j}}$ are independent of λ because of the compatibility equations. Moreover, the matrices $(\check{A}_{\alpha_{\tau_j}})_{1 \leq j \leq g}$ (resp. $(\tilde{A}_{\alpha_{\tau_j}})_{1 \leq j \leq g}$) can be set traceless simultaneously by the additional gauge transformation $\check{\Psi}_n = \check{G}\check{\Psi}$ (resp. $\tilde{\Psi}_n = \tilde{G}\tilde{\Psi}$) with*

$$\begin{aligned}\check{G} &= \exp \left(-\frac{1}{2} \sum_{j=1}^g \int^{\tau_j} \text{Tr}(\check{A}_{\alpha_{\tau_j}}(s)) ds \right) I_2, \\ \tilde{G} &= \exp \left(-\frac{1}{2} \sum_{j=1}^g \int^{\tau_j} \text{Tr}(\tilde{A}_{\alpha_{\tau_j}}(s)) ds \right) I_2.\end{aligned}\tag{6-48}$$

Note that these additional gauge transformations do not change neither $\check{L}$ nor $\tilde{L}$.

Proof. For any isomonodromic deformation τ we have $\hbar \partial_\tau[P_1] = 0$ because the coefficients of P_1 , given by (2-45), are precisely trivial times. From the expression (2-44), we get that $\text{Tr}\check{L} = P_1(\lambda)$ and then by the gauge transformation (2-37), we get $\text{Tr}\tilde{L} = P_1(\lambda)$. Note also that the gauge transformation (2-37) implies that $\text{Tr}\check{A}_{\alpha_\tau} = \text{Tr}\check{A}_{\alpha_{\tau,1}} = \text{Tr}\check{A}_{\alpha_{\tau,2}}$. Thus, we get that $\partial_\tau[\text{Tr}\check{L}] = \partial_\tau[\text{Tr}\tilde{L}] = 0$. The compatibility equation (4-4) implies that $\partial_\lambda \text{Tr}\check{A}_{\alpha_\tau} = 0$. Moreover, for $g \geq 2$, if we denote $(\tau_i)_{1 \leq i \leq g}$ a set of isomonodromic times, then the compatibility of the Lax system also gives

$$\partial_{\tau_j}[\check{A}_{\alpha_{\tau_i}}] = \partial_{\tau_i}[\check{A}_{\alpha_{\tau_j}}] + [\check{A}_{\alpha_{\tau_j}}, \check{A}_{\alpha_{\tau_i}}], \quad \forall i \neq j.\tag{6-49}$$

In particular, we get that $\partial_{\tau_j}[\text{Tr}\check{A}_{\alpha_{\tau_i}}] = \partial_{\tau_i}[\text{Tr}\check{A}_{\alpha_{\tau_j}}]$. It is obvious that the additional gauge transformation $\check{\Psi}_{n,1} = \check{G}^{(1)}\check{\Psi}$ with $\check{G}^{(1)} = \exp(-\frac{1}{2} \int^{\tau_1} \text{Tr}(\check{A}_{\alpha_{\tau_1}}(s)) ds) I_2$ defines a gauge in which the corresponding $\check{A}_{\alpha_{\tau_1}}$ is traceless. In this new gauge, (6-49) implies that $\partial_{\tau_1}[\text{Tr}\check{A}_{\alpha_{\tau_i}}^{(1)}] = 0$ for all $i \geq 2$. In particular a new gauge transformation $\check{\Psi}_{n,2} = \check{G}^{(2)}\check{\Psi}_{n,1}$ with $\check{G}^{(2)} = \exp(-\frac{1}{2} \int^{\tau_2} \text{Tr}(\check{A}_{\alpha_{\tau_2}}^{(1)}(s)) ds) I_2$ does not change the value of $\check{A}_{\alpha_{\tau_1}}^{(1)} = \check{A}_{\alpha_{\tau_1}}^{(2)}$ so that by induction we get the result. Finally, it is obvious that a gauge transformation independent of λ and proportional to I_2 does not change neither $\check{L}$ nor $\tilde{L}$. $\square$

The last proposition shall be useful when $\check{L}$ and $\tilde{L}$ are traceless. In this case, it is interesting to perform this additional gauge transformation in order to obtain a Lax pair that belongs to $\mathfrak{sl}_2(\mathbb{C})$ rather than $\mathfrak{gl}_2(\mathbb{C})$. In particular, this is always possible for the canonical choice of trivial times proposed in Section 7.

7 Canonical choice of trivial times and simplification of the Hamiltonian systems

The aim of this section is to combine the general expressions of the Hamiltonian systems given by Theorem 5.1 and the reduction of the deformation space developed in Section 6. The purpose is thus to obtain the non-trivial isomonodromic evolutions for the shifted Darboux coordinates $(\partial_\tau \check{q}_j, \partial_\tau \check{p}_j)_{1 \leq j \leq g}$ for any isomonodromic time τ in a simpler form, using the fact that these quantities are independent of the trivial times. Indeed, since the shifted Darboux coordinates $(\partial_\tau \check{q}_j, \partial_\tau \check{p}_j)_{1 \leq j \leq g}$ are independent of the trivial times from Theorems 6.1 and 6.4, the evolutions $(\partial_\tau \check{q}_j, \partial_\tau \check{p}_j)_{1 \leq j \leq g}$ do not depend on the trivial times for any isomonodromic time τ . Thus,

one may obtain these evolutions by taking any value of the trivial times. For computational purposes, there exists a canonical choice of the trivial times for which formulas get simpler. But we stress again that these evolution equations would be exactly the same for any other choice of the trivial times.

7.1 Canonical choice of the trivial times and main theorem

In the rest of Section 7, we set the trivial times to the following particular values.

Definition 7.1 (Canonical choice of trivial times). We shall call “canonical choice of the trivial times” the choice of the following values:

$$\begin{aligned} T_{\infty,k} &= 0, \quad \forall k \in \llbracket 0, r_\infty - 1 \rrbracket, \\ T_{X_s,k} &= 0, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 0, r_s - 1 \rrbracket, \\ T_1 &= 0, \\ T_2 &= 1. \end{aligned} \tag{7-1}$$

In particular, the canonical choice of trivial times implies that

$$\begin{aligned} t_{\infty(2),k} &= -t_{\infty(1),k}, \quad \forall k \in \llbracket 0, r_\infty - 1 \rrbracket, \\ \alpha_{\infty(2),k} &= -\alpha_{\infty(1),k}, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ t_{X_s^{(2)},k} &= -t_{X_s^{(1)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 0, r_s - 1 \rrbracket, \\ \alpha_{X_s^{(2)},k} &= -\alpha_{X_s^{(1)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket. \end{aligned} \tag{7-2}$$

Moreover, we always have under this canonical choice of trivial times:

$$P_1(\lambda) = 0 \quad \text{and} \quad (q_j, p_j) = (\check{q}_j, \check{p}_j), \quad \forall j \in \llbracket 1, g \rrbracket. \tag{7-3}$$

In particular, **under this canonical choice of trivial times (7-1), the initial Darboux coordinates $(q_j, p_j)_{1 \leq j \leq g}$ identify with the shifted Darboux coordinates $(\check{q}_j, \check{p}_j)_{1 \leq j \leq g}$.** Note that we also get from (4-20) and (4-21) that for this specific choice of trivial times and for any isomonodromic time τ ,

$$\begin{aligned} c_{\infty,k}^{(\alpha_\tau)} &= 0, \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ c_{X_s,k}^{(\alpha_\tau)} &= 0, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_\infty - 1 \rrbracket. \end{aligned} \tag{7-4}$$

Indeed, the right-hand-sides of (4-20) and (4-21) are always vanishing by obvious symmetry.

Finally, note that since $P_1 = 0$, $\tilde{L}$ and $\tilde{L}$ are traceless. Hence, Proposition 6.13 implies that under a potential additional trivial gauge transformation, we may choose a gauge in which $\tilde{L}$, $\tilde{A}_{\alpha_\tau}$ and $\tilde{A}_{\alpha_\tau}$ are traceless for any isomonodromic time τ .

We shall now apply Theorem 5.1 for the canonical values of the trivial times given by (7-1). In particular, we get the very nice simplification:

Theorem 7.1 (Hamiltonian representation for the canonical choice of trivial times). *The canonical choice of the trivial times given by (7-1) and the definitions of trivial times made in Definitions 6.2, 6.3, 6.4 and 6.5 imply that for any non-trivial isomonodromic time $\tau \in \mathcal{T}_{iso}$:*

$$Ham^{(\alpha_\tau)}(\check{\mathbf{q}}, \check{\mathbf{p}}) = \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\alpha_\tau)} H_{\infty,k} - \sum_{s=1}^n \sum_{k=2}^{r_s} \nu_{X_s,k-1}^{(\alpha_\tau)} H_{X_s,k} + \sum_{s=1}^n \alpha_{X_s}^{(\alpha_\tau)} H_{X_s,1}. \tag{7-5}$$

In other words, we get that the Hamiltonians $\{(Ham^{(\alpha_{\tau_{\infty},k)})}_{0 \leq k \leq r_{\infty}-4}, (Ham^{(\alpha_{\tau_{X_s},k)})}_{1 \leq s \leq n, 1 \leq k \leq r_s}\}$ are (time-dependent) linear combinations of the isospectral Hamiltonians $\{(H_{\infty,k})_{0 \leq k \leq r_{\infty}-4}, (H_{X_s,k})_{1 \leq s \leq n, 1 \leq k \leq r_s}\}$:

$$\begin{aligned} \begin{pmatrix} Ham^{(\alpha_{\tau_{\infty},1})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \\ \vdots \\ (r_{\infty}-3)Ham^{(\alpha_{\tau_{\infty},r_{\infty}-3})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \end{pmatrix} &= (\tilde{M}_{\infty})^{-1} \begin{pmatrix} H_{\infty,r_{\infty}-4} \\ \vdots \\ H_{\infty,0} \end{pmatrix}, \\ \begin{pmatrix} Ham^{(\alpha_{\tau_{X_s},1})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \\ \vdots \\ (r_s-1)Ham^{(\alpha_{\tau_{X_s},r_s-1})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \end{pmatrix} &= (M_s)^{-1} \begin{pmatrix} H_{X_s,r_s} \\ \vdots \\ H_{X_s,2} \end{pmatrix}, \quad \forall s \in \llbracket 1, n \rrbracket, \\ \begin{pmatrix} Ham^{(\alpha_{\tau_{\tilde{X}_1})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \\ \vdots \\ Ham^{(\alpha_{\tau_{\tilde{X}_n})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \end{pmatrix} &= \begin{pmatrix} H_{X_1,1} \\ \vdots \\ H_{X_n,1} \end{pmatrix}, \end{aligned} \quad (7-6)$$

where

$$\tilde{M}_{\infty} = \begin{pmatrix} 2 & 0 & \dots & \dots & 0 \\ 0 & 2 & 0 & & \vdots \\ \tau_{\infty,r_{\infty}-3} & 0 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 & \ddots & \vdots \\ \tau_{\infty,4} & \ddots & \ddots & \ddots & \ddots & 2 & 0 \\ \tau_{\infty,3} & \tau_{\infty,4} & \dots & \tau_{\infty,r_{\infty}-3} & 0 & 2 \end{pmatrix} \in \mathcal{M}_{r_{\infty}-3}(\mathbb{C}) \quad (7-7)$$

and

$$M_s = \begin{pmatrix} \tau_{X_s,r_s-1} & 0 & \dots & \dots & 0 \\ \tau_{X_s,r_s-2} & \tau_{X_s,r_s-1} & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \tau_{X_s,2} & \ddots & \ddots & \ddots & \tau_{X_s,r_s-1} & 0 \\ \tau_{X_s,1} & \tau_{X_s,2} & \dots & \tau_{X_s,r_s-2} & \tau_{X_s,r_s-1} \end{pmatrix}, \quad \forall s \in \llbracket 1, n \rrbracket. \quad (7-8)$$

Note that only the coefficients of the linear combination depend on the deformation, since the isospectral Hamiltonians are independent of it and are determined by Proposition 5.1.

Proof. The proof is presented in Appendix R. $\square$

Remark 7.1. Note that the previous proof is only valid for the choice of trivial times made in Definitions 6.2, 6.3, 6.4 and 6.5. In particular, in order to obtain this simplification, one needs to use the additional degrees of freedom to first fix coefficients at infinity rather than location of the poles. Fixing the location of the poles first would imply that for $r_{\infty} \geq 2$, terms proportional to $\nu_{\infty,-1}^{(\alpha_{\tau})}$ or $\nu_{\infty,0}^{(\alpha_{\tau})}$ do not vanish and thus, one should keep the full formula of Theorem 5.1 creating unnecessary complications for interpretation.

Remark 7.2. One of the main advantages of the reduced form of Theorem 7.1 is that we may combine it with Proposition 2.5 to make some connections with results of [30]. Indeed, under the canonical choice of trivial times (7-1), Proposition 2.5 reduces to

$$\begin{aligned} H_{\infty,j} &= -\frac{1}{2} \operatorname{Res}_{\lambda \rightarrow \infty} \operatorname{Tr}(L_{\text{GMR}}(\lambda, \hbar)^2) \lambda^{-j-1}, \quad \forall j \in \llbracket 0, r_{\infty} - 4 \rrbracket, \\ H_{X_s,j} &= \frac{1}{2} \operatorname{Res}_{\lambda \rightarrow X_s} \operatorname{Tr}(L_{\text{GMR}}(\lambda, \hbar)^2) (\lambda - X_s)^{j-1}, \quad \forall (s, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s \rrbracket. \end{aligned} \quad (7-9)$$

The Toeplitz matrices $\tilde{M}_{\infty}$ and $(M_s)_{1 \leq s \leq n}$ appearing in (7-6) are equivalent (up to the explicit expression of the inverse of the Toeplitz matrices) to Theorem 0.7 and equation (11) of [30]. In particular, it implies that the Lax matrix $A(\lambda)$ of [30] is equal to our $L_{\text{GMR}}(\lambda)$ matrix. An important point to notice is that the simple pole confluence approach of [30] does not produce the matrix $\tilde{L}$ but rather L_{GMR} and that $\tilde{L}$ would not satisfy (7-9) since we have instead (under the canonical choice of trivial times (7-1))

$$\frac{1}{2} \operatorname{Tr}(\tilde{L}^2) = \frac{1}{2} \operatorname{Tr}(\tilde{L}_{\text{GMR}}^2) - \hbar \frac{\prod_{i=1}^g (\lambda - q_i)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \left[\partial_{\lambda} \left(\frac{Q(\lambda, \hbar)}{\prod_{i=1}^g (\lambda - q_i)} \right) + t_{\infty(1), r_{\infty}-1} \right]. \quad (7-10)$$

At the geometric level, it means that the natural spectral Hamiltonians $\left(-\frac{1}{2} \operatorname{Res}_{\lambda \rightarrow \infty} \operatorname{Tr}(\tilde{L}(\lambda, \hbar)^2) \lambda^{-j-1} \right)_{j \in \llbracket 0, r_{\infty}-4 \rrbracket}$ and $\left(\frac{1}{2} \operatorname{Res}_{\lambda \rightarrow X_s} \operatorname{Tr}(\tilde{L}(\lambda, \hbar)^2) (\lambda - X_s)^{j-1} \right)_{(s,j) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s \rrbracket}$ are not the isospectral Hamiltonians but that the gauge transformation (2-42) is necessary. Surprisingly, in the simple pole confluence approach of [30], the construction seems to automatically preserve the gauge Ψ_{GMR} in which spectral Hamiltonians are equal to the isomonodromic Hamiltonians. However, the construction of [30] does not preserve the geometric choice of a representative that would correspond to $\tilde{L}$.

Theorem 7.1 implies that the Hamiltonians are explicit time-dependent linear combinations of the isospectral Hamiltonians that are determined by Proposition 5.1. Under our canonical choice of trivial times, quantities involved in Proposition 5.1 gets simplified and we propose the corresponding expressions depending on the value of r_{∞} . In particular, we believe that this shall be useful for readers focused on immediate use for applications.

7.2 The case $r_{\infty} \geq 3$

7.2.1 General case $n \geq 0$

For $r_{\infty} \geq 3$, the canonical choice of trivial times (7-1) implies that

$$t_{\infty(1), r_{\infty}-1} = 1 \quad \text{and} \quad t_{\infty(1), r_{\infty}-2} = 0. \quad (7-11)$$

In particular, isomonodromic times simplify to

$$\begin{aligned} \tau_{\infty,j} &= 2t_{\infty(1),j}, \quad \forall j \in \llbracket 1, r_{\infty} - 3 \rrbracket, \\ \tau_{X_s,k} &= 2t_{X_s^{(1)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ \tilde{X}_s &= X_s, \quad \forall s \in \llbracket 1, n \rrbracket. \end{aligned} \quad (7-12)$$

The expression for $\tilde{P}_2$ (eq. (2-53)) simplifies to

$$\begin{aligned}
\tilde{P}_2(\lambda) = & -\lambda^{2r_\infty-4} - \sum_{r=r_\infty-2}^{2r_\infty-6} \left(\tau_{\infty,r-r_\infty+3} + \frac{1}{4} \sum_{j=2}^{2r_\infty-6-r} \tau_{\infty,r_\infty-1-j} \tau_{\infty,j+r-r_\infty+3} \right) \lambda^r \\
& - \left(2t_{\infty(1),0} + \frac{1}{4} \sum_{j=2}^{r_\infty-3} \tau_{\infty,r_\infty-1-j} \tau_{\infty,j} \right) \lambda^{r_\infty-3} \\
& - \frac{1}{4} \sum_{s=1}^n \sum_{j=r_s+2}^{2r_s} \sum_{i=0}^{2r_s-j} \tau_{X_s,r_s-1-i} \tau_{X_s,i+j-r_s-1} (\lambda - \tilde{X}_s)^{-j} \\
& - \frac{1}{4} \sum_{s=1}^n \sum_{i=1}^{r_s-2} \tau_{X_s,r_s-1-i} \tau_{X_s,i} (\lambda - \tilde{X}_s)^{-r_s-1} \\
& - \sum_{s=1}^n \left(t_{X_s^{(1)},0} \tau_{X_s,r_s-1} \delta_{r_s \geq 2} + (t_{X_s^{(1)},0})^2 \delta_{r_s=1} \right) (\lambda - \tilde{X}_s)^{-r_s-1}. \tag{7-13}
\end{aligned}$$

Matrices $(V_s)_{1 \leq s \leq n}$ and V_∞ , defined by (4-15) reduce to

$$V_\infty = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ \check{q}_1 & \check{q}_2 & \dots & \dots & \check{q}_g \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \check{q}_1^{r_\infty-4} & \check{q}_2^{r_\infty-4} & \dots & \dots & \check{q}_g^{r_\infty-4} \end{pmatrix}, \quad V_s = \begin{pmatrix} \frac{1}{\check{q}_1 - \tilde{X}_s} & \dots & \dots & \frac{1}{\check{q}_g - \tilde{X}_s} \\ \frac{1}{(\check{q}_1 - \tilde{X}_s)^2} & \dots & \dots & \frac{1}{(\check{q}_g - \tilde{X}_s)^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{(\check{q}_1 - \tilde{X}_s)^{r_s}} & \dots & \dots & \frac{1}{(\check{q}_g - \tilde{X}_s)^{r_s}} \end{pmatrix} \tag{7-14}$$

and we have

$$(V_\infty^t \quad V_1^t \quad \dots \quad V_n^t) \begin{pmatrix} \mathbf{H}_\infty \\ \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} = \begin{pmatrix} \check{p}_1^2 + \check{p}_1 \sum_{s=1}^n \frac{hr_s}{\check{q}_1 - \tilde{X}_s} + \tilde{P}_2(\check{q}_1) + \hbar \sum_{i \neq 1} \frac{\check{p}_i - \check{p}_1}{\check{q}_1 - \check{q}_i} + \hbar \check{q}_1^{r_\infty-3} \\ \vdots \\ \check{p}_g^2 + \check{p}_g \sum_{s=1}^n \frac{hr_s}{\check{q}_g - \tilde{X}_s} + \tilde{P}_2(\check{q}_g) + \hbar \sum_{i \neq g} \frac{\check{p}_i - \check{p}_g}{\check{q}_g - \check{q}_i} + \hbar \check{q}_g^{r_\infty-3} \end{pmatrix}. \tag{7-15}$$

Remark 7.3. Note also that coefficients $(\mu_j^{(\alpha_\tau)})_{1 \leq j \leq q}$ are given by Proposition 4.2 using the simplified expressions (7-14) for the matrices V_∞ and $(V_s)_{1 \leq s \leq n}$ and $\nu_{X_s}^{(\alpha_\tau)} = (\alpha_{X_s}, -\nu_{X_s,1}^{(\alpha_\tau)}, \dots, -\nu_{X_s,r_s-1}^{(\alpha_\tau)})^t$.

7.2.2 The sub-case $n = 0$: the Painlevé 2 hierarchy

Let us assume that $r_\infty \geq 4$ and $n = 0$. In other words we deal with a member of Painlevé 2 hierarchy. The Hamiltonian structure of the Painlevé 2 hierarchy was presented in [59] in a very different way. Our approach thus provides another proof of the Hamiltonian structure of the Painlevé 2 hierarchy with the additional explicit expressions. In this case, Theorem 7.1 takes a particularly simple form. Indeed we have:

$$\tilde{P}_2(\lambda) = -\lambda^{2r_\infty-4} - \sum_{r=r_\infty-2}^{2r_\infty-6} \left(\tau_{\infty,r-r_\infty+3} + \frac{1}{4} \sum_{j=2}^{2r_\infty-6-r} \tau_{\infty,r_\infty-1-j} \tau_{\infty,j+r-r_\infty+3} \right) \lambda^r$$

$$- \left(2t_{\infty(1),0} + \frac{1}{4} \sum_{j=2}^{r_\infty-3} \tau_{\infty,r_\infty-1-j} \tau_{\infty,j} \right) \lambda^{r_\infty-3}. \quad (7-16)$$

Coefficients $(H_{\infty,k})_{1 \leq k \leq r_\infty-4}$ are determined by

$$\begin{pmatrix} 1 & \dots & \dots & 1 \\ \check{q}_1 & \dots & \dots & \check{q}_g \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \check{q}_1^{r_\infty-4} & \dots & \dots & \check{q}_g^{r_\infty-4} \end{pmatrix}^t \begin{pmatrix} H_{\infty,0} \\ \vdots \\ H_{\infty,r_\infty-4} \end{pmatrix} = \begin{pmatrix} \check{p}_1^2 + \tilde{P}_2(\check{q}_1) + \hbar \sum_{i \neq 1} \frac{\check{p}_i - \check{p}_1}{\check{q}_1 - \check{q}_i} + \hbar \check{q}_1^{r_\infty-3} \\ \vdots \\ \check{p}_g^2 + \tilde{P}_2(\check{q}_g) + \hbar \sum_{i \neq g} \frac{\check{p}_i - \check{p}_g}{\check{q}_g - \check{q}_i} + \hbar \check{q}_g^{r_\infty-3} \end{pmatrix}. \quad (7-17)$$

The isomonodromic deformations reduce to $\partial_{\tau_{\infty,j}} = \frac{1}{2} \partial_{t_{\infty(1),j}}$ for $j \in \llbracket 1, r_\infty - 3 \rrbracket$ and the Hamiltonians are given by

$$\begin{pmatrix} \text{Ham}^{(\alpha_{\tau_{\infty,1}})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \\ \vdots \\ (r_\infty - 3) \text{Ham}^{(\alpha_{\tau_{\infty,r_\infty-3}})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \end{pmatrix} = \tilde{M}_\infty^{-1} \begin{pmatrix} H_{\infty,r_\infty-4} \\ \vdots \\ H_{\infty,0} \end{pmatrix}. \quad (7-18)$$

7.3 The case $r_\infty = 2$

For $r_\infty = 2$ (thus $n \geq 1$ in order to have $g > 0$), the canonical choice of trivial times (7-1) implies that

$$t_{\infty(1),1} = 1 \quad \text{and} \quad X_1 = 0. \quad (7-19)$$

In particular, isomonodromic times simplify to

$$\begin{aligned} \tau_{X_s,k} &= 2t_{X_s^{(1)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ \tilde{X}_s &= X_s, \quad \forall s \in \llbracket 2, n \rrbracket. \end{aligned} \quad (7-20)$$

The expression for $\tilde{P}_2$ reduces to

$$\begin{aligned} \tilde{P}_2(\lambda) &= -1 - \frac{1}{4} \sum_{j=r_1+2}^{2r_1} \sum_{i=0}^{2r_1-j} \tau_{X_1,r_1-1-i} \tau_{X_1,i+j-r_1-1} \lambda^{-j} \\ &\quad - \frac{1}{4} \sum_{i=1}^{r_1-2} \tau_{X_1,r_1-1-i} \tau_{X_1,i} \lambda^{-r_1-1} \\ &\quad - \left(t_{X_1^{(1)},0} \tau_{X_1,r_1-1} \delta_{r_1 \geq 2} + (t_{X_1^{(1)},0})^2 \delta_{r_1=1} \right) \lambda^{-r_1-1} \\ &\quad - \frac{1}{4} \sum_{s=2}^n \sum_{j=r_s+2}^{2r_s} \sum_{i=0}^{2r_s-j} \tau_{X_s,r_s-1-i} \tau_{X_s,i+j-r_s-1} (\lambda - \tilde{X}_s)^{-j} \\ &\quad - \frac{1}{4} \sum_{s=2}^n \sum_{i=1}^{r_s-2} \tau_{X_s,r_s-1-i} \tau_{X_s,i} (\lambda - \tilde{X}_s)^{-r_s-1} \\ &\quad - \sum_{s=2}^n \left(t_{X_s^{(1)},0} \tau_{X_s,r_s-1} \delta_{r_s \geq 2} + (t_{X_s^{(1)},0})^2 \delta_{r_s=1} \right) (\lambda - \tilde{X}_s)^{-r_s-1}. \end{aligned} \quad (7-21)$$

Matrices $(V_s)_{1 \leq s \leq n}$, defined by (4-15), reduce to

$$V_1 = \begin{pmatrix} \frac{1}{\check{q}_1} & \cdots & \cdots & \frac{1}{\check{q}_g} \\ \frac{1}{\check{q}_1^2} & \cdots & \cdots & \frac{1}{\check{q}_g^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{\check{q}_1^{r_s}} & \cdots & \cdots & \frac{1}{\check{q}_g^{r_s}} \end{pmatrix}, \quad V_s = \begin{pmatrix} \frac{1}{\check{q}_1 - \check{X}_s} & \cdots & \cdots & \frac{1}{\check{q}_g - \check{X}_s} \\ \frac{1}{(\check{q}_1 - \check{X}_s)^2} & \cdots & \cdots & \frac{1}{(\check{q}_g - \check{X}_s)^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{(\check{q}_1 - \check{X}_s)^{r_s}} & \cdots & \cdots & \frac{1}{(\check{q}_g - \check{X}_s)^{r_s}} \end{pmatrix}, \quad \forall s \in \llbracket 2, n \rrbracket. \quad (7-22)$$

We have

$$\begin{pmatrix} \mathbf{e}_1^t & \mathbf{e}_1^t & \cdots & \mathbf{e}_1^t \\ V_1^t & V_2^t & \cdots & V_n^t \end{pmatrix} \begin{pmatrix} \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} = \begin{pmatrix} \hbar \sum_{j=1}^g \check{p}_j + 2t_{\infty(1),0} - \hbar \\ \check{p}_1^2 + \check{p}_1 \left(\frac{\hbar r_1}{\check{q}_1} + \sum_{s=2}^n \frac{\hbar r_s}{\check{q}_1 - \check{X}_s} \right) + \tilde{P}_2(\check{q}_1) + \hbar \sum_{i \neq 1} \frac{\check{p}_i - \check{p}_1}{\check{q}_1 - \check{q}_i} \\ \vdots \\ \check{p}_g^2 + \check{p}_g \left(\frac{\hbar r_1}{\check{q}_g} + \sum_{s=2}^n \frac{\hbar r_s}{\check{q}_g - \check{X}_s} \right) + \tilde{P}_2(\check{q}_g) + \hbar \sum_{i \neq g} \frac{\check{p}_i - \check{p}_g}{\check{q}_g - \check{q}_i} \end{pmatrix} \quad (7-23)$$

Note that one may obtain the coefficients $(\mu_j^{(\alpha_\tau)})_{1 \leq j \leq g} \cup \{\nu_{\infty,0}^{(\alpha_\tau)}\}$ for any $\tau \in \mathcal{T}_{\text{iso}}$ by solving

$$\begin{pmatrix} V_1 \\ \vdots \\ \vdots \\ V_n \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha_\tau)} \\ \vdots \\ \vdots \\ \mu_g^{(\alpha_\tau)} \end{pmatrix} = \begin{pmatrix} \nu_{X_1}^{(\alpha_\tau)} \\ \vdots \\ \vdots \\ \nu_{X_n}^{(\alpha_\tau)} \end{pmatrix} \quad \text{with} \quad \nu_{X_s}^{(\alpha_\tau)} = \begin{pmatrix} \alpha_{X_s} + \nu_{\infty,0}^{(\alpha_\tau)} \\ -\nu_{X_s,1}^{(\alpha_\tau)} \\ -\nu_{X_s,2}^{(\alpha_\tau)} \\ \vdots \\ -\nu_{X_s,r_s-1}^{(\alpha_\tau)} \end{pmatrix}, \quad \forall s \in \llbracket 1, n \rrbracket. \quad (7-24)$$

7.4 The case $r_\infty = 1$ and $n \geq 2$

For $r_\infty = 1$ with $n \geq 2$, the canonical choice of trivial times (7-1) implies that

$$X_1 = 0 \quad \text{and} \quad X_2 = 1. \quad (7-25)$$

In particular, isomonodromic times simplify to

$$\begin{aligned} \tau_{X_s,k} &= 2t_{X_s^{(1)},k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ \check{X}_s &= X_s, \quad \forall s \in \llbracket 3, n \rrbracket. \end{aligned} \quad (7-26)$$

The expression for $\tilde{P}_2$ reduces to

$$\begin{aligned} \tilde{P}_2(\lambda) &= -\frac{1}{4} \sum_{j=r_1+2}^{2r_1} \sum_{i=0}^{2r_1-j} \tau_{X_1,r_1-1-i} \tau_{X_1,i+j-r_1-1} \lambda^{-j} \\ &\quad - \frac{1}{4} \sum_{i=1}^{r_1-2} \tau_{X_1,r_1-1-i} \tau_{X_1,i} \lambda^{-r_1-1} \end{aligned}$$

$$\begin{aligned}
& - \left(t_{X_1^{(1)},0} \tau_{X_1,r_1-1} \delta_{r_1 \geq 2} + (t_{X_1^{(1)},0})^2 \delta_{r_1=1} \right) \lambda^{-r_1-1} \\
& - \frac{1}{4} \sum_{j=r_2+2}^{2r_2} \sum_{i=0}^{2r_2-j} \tau_{X_2,r_2-1-i} \tau_{X_2,i+j-r_2-1} (\lambda-1)^{-j} \\
& - \frac{1}{4} \sum_{i=1}^{r_2-2} \tau_{X_1,r_2-1-i} \tau_{X_1,i} (\lambda-1)^{-r_2-1} \\
& - \left(t_{X_2^{(1)},0} \tau_{X_2,r_2-1} \delta_{r_2 \geq 2} + (t_{X_2^{(1)},0})^2 \delta_{r_2=1} \right) \lambda^{-r_2-1} \\
& - \frac{1}{4} \sum_{s=3}^n \sum_{j=r_s+2}^{2r_s} \sum_{i=0}^{2r_s-j} \tau_{X_s,r_s-1-i} \tau_{X_s,i+j-r_s-1} (\lambda - \tilde{X}_s)^{-j} \\
& - \frac{1}{4} \sum_{s=3}^n \sum_{i=1}^{r_s-2} \tau_{X_s,r_s-1-i} \tau_{X_s,i} (\lambda - \tilde{X}_s)^{-r_s-1} \\
& - \sum_{s=3}^n \left(t_{X_s^{(1)},0} \tau_{X_s,r_s-1} \delta_{r_s \geq 2} + (t_{X_s^{(1)},0})^2 \delta_{r_s=1} \right) (\lambda - \tilde{X}_s)^{-r_s-1}. \tag{7-27}
\end{aligned}$$

Matrices $(V_s)_{1 \leq s \leq n}$, defined by (4-15), reduce to

$$V_1 = \begin{pmatrix} \frac{1}{\check{q}_1} & \cdots & \cdots & \frac{1}{\check{q}_g} \\ \frac{1}{\check{q}_1^2} & \cdots & \cdots & \frac{1}{\check{q}_g^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{\check{q}_1^{r_s}} & \cdots & \cdots & \frac{1}{\check{q}_g^{r_s}} \end{pmatrix}, \quad V_2 = \begin{pmatrix} \frac{1}{(\check{q}_1-1)} & \cdots & \cdots & \frac{1}{(\check{q}_g-1)} \\ \frac{1}{(\check{q}_1-1)^2} & \cdots & \cdots & \frac{1}{(\check{q}_g-1)^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{(\check{q}_1-1)^{r_s}} & \cdots & \cdots & \frac{1}{(\check{q}_g-1)^{r_s}} \end{pmatrix} \tag{7-28}$$

and

$$\forall s \in \llbracket 3, n \rrbracket : V_s = \begin{pmatrix} \frac{1}{\check{q}_1 - \tilde{X}_s} & \cdots & \cdots & \frac{1}{\check{q}_g - \tilde{X}_s} \\ \frac{1}{(\check{q}_1 - \tilde{X}_s)^2} & \cdots & \cdots & \frac{1}{(\check{q}_g - \tilde{X}_s)^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{(\check{q}_1 - \tilde{X}_s)^{r_s}} & \cdots & \cdots & \frac{1}{(\check{q}_g - \tilde{X}_s)^{r_s}} \end{pmatrix}, \quad \forall s \in \llbracket 2, n \rrbracket. \tag{7-29}$$

We have

$$\begin{aligned}
& \begin{pmatrix} \mathbf{e}_1^t & \mathbf{e}_1^t & \cdots & \cdots & \mathbf{e}_1^t \\ \mathbf{e}_2^t \delta_{r_1 \geq 2} & \mathbf{e}_2^t \delta_{r_2 \geq 2} + \mathbf{e}_1^t & \mathbf{e}_2^t \delta_{r_3 \geq 2} + \tilde{X}_3 \mathbf{e}_1^t & \cdots & \mathbf{e}_2^t \delta_{r_n \geq 2} + \tilde{X}_n \mathbf{e}_1^t \\ V_1^t & V_2^t & V_3^t & \cdots & V_n^t \end{pmatrix} \begin{pmatrix} \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} \\
& = \begin{pmatrix} \hbar \sum_{j=1}^g \check{p}_j \\ \hbar \sum_{j=1}^g \check{q}_j \check{p}_j - \sum_{s=1}^n (t_{X_s^{(1)},0})^2 \delta_{r_s=1} + t_{\infty^{(1)},0} (t_{\infty^{(1)},0} - \hbar) \\ \check{p}_1^2 + \check{p}_1 \left(\frac{\hbar r_1}{\check{q}_1} + \frac{\hbar r_2}{\check{q}_1-1} + \sum_{s=3}^n \frac{\hbar r_s}{\check{q}_1 - \tilde{X}_s} \right) + \tilde{P}_2(\check{q}_1) + \hbar \sum_{i \neq 1} \frac{\check{p}_i - \check{p}_1}{\check{q}_1 - \check{q}_i} \\ \vdots \\ \check{p}_g^2 + \check{p}_g \left(\frac{\hbar r_1}{\check{q}_g} + \frac{\hbar r_2}{\check{q}_g-1} + \sum_{s=3}^n \frac{\hbar r_s}{\check{q}_g - \tilde{X}_s} \right) + \tilde{P}_2(\check{q}_g) + \hbar \sum_{i \neq g} \frac{\check{p}_i - \check{p}_g}{\check{q}_g - \check{q}_i} \end{pmatrix}.
\end{aligned}$$

(7-30)

Note that one may obtain the coefficients $\left(\mu_j^{(\alpha_\tau)}\right)_{1 \leq j \leq g} \cup \left\{\nu_{\infty,-1}^{(\alpha_\tau)}, \nu_{\infty,0}^{(\alpha_\tau)}\right\}$ for any $\tau \in \mathcal{T}_{\text{iso}}$ by solving

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha_\tau)} \\ \vdots \\ \vdots \\ \mu_g^{(\alpha_\tau)} \end{pmatrix} = \begin{pmatrix} \nu_{X_1}^{(\alpha_\tau)} \\ \vdots \\ \vdots \\ \nu_{X_n}^{(\alpha_\tau)} \end{pmatrix} \quad \text{with} \quad \nu_{X_s}^{(\alpha_\tau)} = \begin{pmatrix} \alpha_{X_s} + \nu_{\infty,0}^{(\alpha_\tau)} + \nu_{\infty,-1}^{(\alpha_\tau)} \tilde{X}_s \\ -\nu_{X_s,1}^{(\alpha_\tau)} + \nu_{\infty,-1}^{(\alpha_\tau)} \\ -\nu_{X_s,2}^{(\alpha_\tau)} \\ \vdots \\ -\nu_{X_s,r_s-1}^{(\alpha_\tau)} \end{pmatrix}. \quad (7-31)$$

7.5 The case $r_\infty = 1$ and $n = 1$

For $r_\infty = 1$ and $n = 1$, ($r_1 \geq 3$ in order to have $g > 0$) the canonical choice of trivial times (7-1) implies that

$$X_1 = 0 \quad \text{and} \quad t_{X_1^{(1)}, r_1-1} = 1. \quad (7-32)$$

In particular, isomonodromic times simplify to

$$\tau_{X_1,k} = 2t_{X_1^{(1)},k}, \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket. \quad (7-33)$$

The expression for $\tilde{P}_2$ reduces to

$$\begin{aligned} \tilde{P}_2(\lambda) = & -\lambda^{-2r_1} - \frac{1}{2} \sum_{j=r_1+2}^{2r_1-1} \tau_{X_1,j-r_1-1} \lambda^{-j} - \frac{1}{4} \sum_{j=r_1+2}^{2r_1} \sum_{i=1}^{2r_1-j} \tau_{X_1,r_1-1-i} \tau_{X_1,i+j-r_1-1} \lambda^{-j} \\ & - 2t_{X_1^{(1)},0} \lambda^{-r_1-1}. \end{aligned} \quad (7-34)$$

Matrix V_1 is defined by

$$V_1 = \begin{pmatrix} \frac{1}{\tilde{q}_1} & \cdots & \cdots & \frac{1}{\tilde{q}_g} \\ \frac{1}{\tilde{q}_1^2} & \cdots & \cdots & \frac{1}{\tilde{q}_g^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{\tilde{q}_1^{r_1}} & \cdots & \cdots & \frac{1}{\tilde{q}_g^{r_1}} \end{pmatrix}. \quad (7-35)$$

We have

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & (V_1)^t & & \end{pmatrix} \begin{pmatrix} H_{X_1,1} \\ \vdots \\ H_{X_1,r_1} \end{pmatrix} = \begin{pmatrix} \hbar \sum_{j=1}^g \tilde{p}_j \\ \hbar \sum_{j=1}^g \tilde{q}_j \tilde{p}_j + t_{\infty^{(1)},0} (t_{\infty^{(1)},0} - \hbar) \\ \tilde{p}_1^2 + \tilde{p}_1 \left(\frac{\hbar r_1}{\tilde{q}_1} \right) + \tilde{P}_2(\tilde{q}_1) + \hbar \sum_{i \neq 1} \frac{\tilde{p}_i - \tilde{p}_1}{\tilde{q}_1 - \tilde{q}_i} \\ \vdots \\ \vdots \\ \tilde{p}_g^2 + \tilde{p}_g \left(\frac{\hbar r_1}{\tilde{q}_g} \right) + \tilde{P}_2(\tilde{q}_g) + \hbar \sum_{i \neq g} \frac{\tilde{p}_i - \tilde{p}_g}{\tilde{q}_g - \tilde{q}_i} \end{pmatrix}. \quad (7-36)$$

Note that in this case ($t_{X_1^{(1)}, r_1-1} = 1 = \frac{1}{2}\tau_{X_1, r_1-1}$), the matrix M_1 reduce to:

$$M_1 = \begin{pmatrix} 2 & 0 & \dots & \dots & 0 \\ \tau_{X_1, r_1-2} & 2 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \tau_{X_1, 2} & \ddots & \ddots & \ddots & 2 \\ \tau_{X_1, 1} & \tau_{X_1, 2} & \dots & \tau_{X_1, r_1-2} & 2 \end{pmatrix}. \quad (7-37)$$

Note that one could obtain the coefficients $(\mu_j^{(\alpha_\tau)})_{1 \leq j \leq g} \cup \{\nu_{\infty, -1}^{(\alpha_\tau)}, \nu_{\infty, 0}^{(\alpha_\tau)}\}$ for any $\tau \in \mathcal{T}_{\text{iso}}$ by solving

$$V_1 \begin{pmatrix} \mu_1^{(\alpha_\tau)} \\ \vdots \\ \vdots \\ \mu_g^{(\alpha_\tau)} \end{pmatrix} = \nu_{X_1}^{(\alpha_\tau)} \quad \text{with} \quad \nu_{X_1}^{(\alpha)} = \begin{pmatrix} \nu_{\infty, 0}^{(\alpha_\tau)} \\ \nu_{\infty, -1}^{(\alpha_\tau)} \\ -\nu_{X_1, 2}^{(\alpha)} \\ \vdots \\ -\nu_{X_1, r_1-1}^{(\alpha)} \end{pmatrix}. \quad (7-38)$$

8 Examples: Painlevé equations and Fuchsian systems

We shall apply the general theory developed in this article to the examples of the Painlevé equations (giving all possibilities covered by the present work to have $g = 1$), the Painlevé 2 hierarchy and Fuchsian systems. In each case, we shall write explicitly the evolution equations for $(\check{\mathbf{q}}, \check{\mathbf{p}})$, the corresponding Hamiltonians and the associated Lax pairs $(\tilde{L}(\lambda, \hbar), \tilde{A}_{\alpha_\tau}(\lambda, \hbar))$. Note that the Painlevé 1 equation is not covered by the present work since it corresponds to the case where ∞ is ramified.⁹ These examples have been studied in the literature so this section is meant as a test to show that the present work correctly recovers these known cases with almost no computation. It may also help the reader to understand how to apply concretely the various formulas obtained in the general setting.

8.1 Painlevé 2 case

The Painlevé 2 case corresponds to $n = 0$ and $r_\infty = 4$ and is a direct application of Section 7.2.2. It provides a genus 1 curve. The only isomonodromic time is $t = \tau_{\infty, 1} = 2t_{\infty^{(1)}, 1}$. Proposition 4.1 gives $\nu_{\infty, -1}^{(\alpha_t)} = 0$, $\nu_{\infty, 0}^{(\alpha_t)} = 0$ and equation (R-3) gives $\nu_{\infty, 1}^{(\alpha_t)} = \frac{1}{2}$ so that from Remark 7.3 $\mu_1^{(\alpha_t)} = \nu_{\infty, 1}^{(\alpha_t)} = \frac{1}{2}$. Using (7-16) we have,

$$\tilde{P}_2(\lambda) = -\lambda^4 - t\lambda^2 - 2t_{\infty^{(1)}, 0}\lambda. \quad (8-1)$$

Thus, we get from (7-17) that the Hamiltonian is given by

$$\text{Ham}^{(\alpha_t)}(\check{q}, \check{p}) = \frac{1}{2}H_{\infty, 0} = \frac{1}{2}\check{p}^2 - \frac{1}{2}\check{q}^4 - \frac{t}{2}\check{q}^2 - \frac{\check{q}}{2}(2t_{\infty^{(1)}, 0} - \hbar). \quad (8-2)$$

⁹The Painlevé 1 hierarchy is done in details in [55] using similar techniques.

Moreover, Theorem 5.1 provides

$$\begin{aligned}\hbar\partial_t[\check{q}] &= \check{p}, \\ \hbar\partial_t[\check{p}] &= \frac{1}{2}\left(4\check{q}^3 + 2t\check{q} + 2t_{\infty(1),0} - \hbar\right) = 2\check{q}^3 + t\check{q} + t_{\infty(1),0} - \frac{\hbar}{2}.\end{aligned}\quad (8-3)$$

In particular, $\check{q}$ satisfies the ODE

$$\hbar^2 \frac{\partial^2 \check{q}}{\partial t^2} = 2\check{q}^3 + t\check{q} + t_{\infty(1),0} - \frac{\hbar}{2}, \quad (8-4)$$

i.e. the standard Painlevé 2 equation with parameter $\alpha = t_{\infty(1),0} - \frac{\hbar}{2} = \frac{1}{2}\left(t_{\infty(1),0} - t_{\infty(2),0} - \hbar\right)$. The associated Lax pair in the gauge without apparent singularities, normalized at infinity according to (2-8) and under the choice of trivial times (7-1) (i.e. $t_{\infty(2),k} = -t_{\infty(1),k}$ for all $k \in \llbracket 0, 3 \rrbracket$ and $t_{\infty(1),3} = 1$, $t_{\infty(1),2} = 0$ and $t_{\infty(1),1} = \frac{t}{2}$) is

$$\begin{aligned}\tilde{L}(\lambda, \hbar) &= \begin{pmatrix} -\lambda^2 + \check{q}^2 + \check{p} & \lambda - \check{q} \\ (2\check{q}^2 + 2\check{p} + t)\lambda + 2\check{q}^3 + (2\check{p} + t)\check{q} + 2t_{\infty(1),0} & \lambda^2 - \check{q}^2 - \check{p} \end{pmatrix} \\ \tilde{A}_{\alpha_t}(\lambda, \hbar) &= \begin{pmatrix} -\frac{\lambda + \check{q}}{2} & \frac{1}{2} \\ \check{q}^2 + \check{p} + \frac{t}{2} & \frac{\lambda + \check{q}}{2} \end{pmatrix}.\end{aligned}\quad (8-5)$$

One may also recover the standard Jimbo-Miwa Lax pair [44] by defining $(Q, P) = (-\check{q}, -\check{p} - \check{q}^2 - \frac{t}{2})$ in which case the Hamiltonian is

$$\text{Ham}^{(\alpha_t)}(Q, P) = \frac{1}{2}P^2 + \left(Q^2 + \frac{t}{2}\right)P + \frac{t^2}{8} + \frac{Q}{2}\left(2t_{\infty(1),0} - \hbar\right). \quad (8-6)$$

8.2 Painlevé 3 case

The Painlevé 3 case corresponds to $n = 1$ with $r_1 = 2$ and $r_\infty = 2$ and thus provides a spectral curve of genus $g = 1$. It is a special case of Section 7.3. The canonical choice of trivial times provides $X_1 = 0$, $t_{\infty(1),1} = 1 = -t_{\infty(2),1}$ and the isomonodromic time is $t = \tau_{X_1,1} = 2t_{X_1^{(1)},1}$. In particular from (R-12), we get that

$$\nu_{\infty,-1}^{(\alpha_t)} = 0 \text{ and } \nu_{X_1,0}^{(\alpha_t)} = 0 \text{ and } \nu_{X_1,1}^{(\alpha_t)} = -\frac{1}{t}. \quad (8-7)$$

In this case, we have

$$\tilde{P}_2(\lambda) = -1 - \frac{t_{X_1^{(1)},0}t}{\lambda^3} - \frac{t^2}{4\lambda^4}. \quad (8-8)$$

Since $r_\infty = 2$, the coefficients $(\nu_{\infty,0}^{(\alpha_t)}, \mu_1^{(\alpha_t)})$ are determined by (7-24):

$$\left(\frac{1}{\check{q}}\right)\mu_1^{(\alpha_t)} = \begin{pmatrix} -\nu_{X_1,0}^{(\alpha_t)} + \nu_{\infty,0}^{(\alpha_t)} \\ -\nu_{X_1,1}^{(\alpha_t)} + \nu_{\infty,-1}^{(\alpha_t)} \end{pmatrix} = \begin{pmatrix} \nu_{\infty,0}^{(\alpha_t)} \\ \frac{1}{t} \end{pmatrix} \quad (8-9)$$

which is equivalent to

$$\nu_{\infty,0}^{(\alpha_t)} = \frac{\check{q}}{t}, \mu_1^{(\alpha_t)} = \frac{\check{q}^2}{t}. \quad (8-10)$$

The constants $H_{X_1,1}$ and $H_{X_1,2}$ are determined by Proposition (7-23),

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{\check{q}} & \frac{1}{\check{q}^2} \end{pmatrix} \begin{pmatrix} H_{X_1,1} \\ H_{X_1,2} \end{pmatrix} = \begin{pmatrix} \hbar\check{p} + (2t_{\infty(1),0} - \hbar) \\ \check{p}^2 + \frac{2\hbar}{\check{q}}\check{p} - 1 - \frac{t_{X_1(1),0}t}{\check{q}^3} - \frac{t^2}{4\check{q}^4} \end{pmatrix}, \quad (8-11)$$

i.e.

$$\begin{aligned} H_{X_1,1} &= \hbar\check{p} + 2t_{\infty(1),0} - \hbar, \\ H_{X_1,2} &= \check{q}^2\check{p}^2 + \hbar\check{q}\check{p} - \frac{t^2}{4\check{q}^2} - \frac{t_{X_1(1),0}t}{\check{q}} - \check{q}^2 - (2t_{\infty(1),0} - \hbar)\check{q}. \end{aligned} \quad (8-12)$$

The corresponding Hamiltonian is

$$\begin{aligned} \text{Ham}^{(\alpha_t)}(\check{q}, \check{p}) &= -\nu_{X_1,0}^{(\alpha_t)} H_{X_1,1} - \nu_{X_1,1}^{(\alpha_t)} H_{X_1,2} = \frac{1}{t} H_{X_1,2} \\ &= \frac{1}{t} \left(\check{q}^2\check{p}^2 + \hbar\check{q}\check{p} - \check{q}^2 - (2t_{\infty(1),0} - \hbar)\check{q} - \frac{t_{X_1(1),0}t}{\check{q}} - \frac{t^2}{4\check{q}^2} \right). \end{aligned} \quad (8-13)$$

We also get from Theorem 5.1:

$$\begin{aligned} \hbar\partial_t\check{q} &= \frac{2\check{q}^2}{t}\check{p} + \hbar\frac{\check{q}}{t}, \\ \hbar\partial_t\check{p} &= -\frac{2\check{q}}{t}\check{p}^2 - \hbar\frac{\check{p}}{t} - \frac{t}{2\check{q}^3} - \frac{t_{X_1(1),0}}{\check{q}^2} + \frac{2\check{q}}{t} + \frac{2t_{\infty(1),0} - \hbar}{t}. \end{aligned} \quad (8-14)$$

Finally $\check{q}$ satisfies the Painlevé 3 equation:

$$\hbar^2\check{q} = \frac{(\hbar\check{q})^2}{\check{q}} - \frac{\hbar^2\check{q}}{t} + \frac{\alpha\check{q}^2 + \gamma\check{q}^3}{t^2} + \frac{\beta}{t} + \frac{\delta}{\check{q}}, \quad (8-15)$$

with

$$\alpha = 2(2t_{\infty(1),0} - \hbar), \beta = -2t_{X_1(1),0}, \gamma = 4, \delta = -1. \quad (8-16)$$

The associated Lax pair in the gauge without apparent singularities, normalized at infinity according to (2-8) and under the canonical choice of trivial times (7-1) (i.e. $t_{\infty(2),k} = -t_{\infty(1),k}$ and $t_{X_1(2),k} = -t_{X_1(1),k}$ for all $k \in \llbracket 0, 1 \rrbracket$, $X_1 = 0$, $t_{\infty(1),1} = 1$ and $t_{X_1(1),1} = \frac{t}{2}$) is

$$\begin{aligned} \tilde{L}(\lambda, \hbar) &= \text{diag}(-1, 1) + \frac{\tilde{L}_1}{\lambda} + \frac{\tilde{L}_2}{\lambda^2} \text{ with} \\ \tilde{L}_1 &= \begin{pmatrix} -t_{\infty(1),0} & 1 \\ 4\check{q}^4 - 4t_{\infty(1),0}^2\check{q}^2 - 4t_{X_1(1),0}t\check{q} - t^2 & t_{\infty(1),0} \end{pmatrix}, \\ \tilde{L}_2 &= \begin{pmatrix} \check{q}^2\check{p} + \check{q}^2 + t_{\infty(1),0}\check{q} & -\check{q} \\ \check{q}^3\check{p}^2 + 2\check{q}^2(\check{q} + t_{\infty(1),0})\check{p} + \frac{(2\check{q}^2 + 2\check{q}t_{\infty(1),0})^2 - t^2}{4\check{q}} & -(\check{q}^2\check{p} + \check{q}^2 + t_{\infty(1),0}\check{q}) \end{pmatrix}. \end{aligned} \quad (8-17)$$

The associated auxiliary deformation matrix is

$$\begin{aligned} \tilde{A}_{\alpha_t}(\lambda, \hbar) &= \text{diag}\left(-\frac{\check{q}}{t}, \frac{\check{q}}{t}\right) \\ &+ \frac{1}{\lambda} \begin{pmatrix} -\frac{\check{q}^2\check{p}}{t} - \frac{\check{q}(\check{q} + t_{\infty(1),0})}{t} & \frac{\check{q}}{t} \\ -\frac{((\check{p}+1)\check{q}^2 + t_{\infty(1),0}\check{q} + \frac{t}{2})((\check{p}+1)\check{q}^2 + t_{\infty(1),0}\check{q} - \frac{t}{2})}{t\check{q}} & -\left(-\frac{\check{q}^2\check{p}}{t} - \frac{\check{q}(\check{q} + t_{\infty(1),0})}{t}\right) \end{pmatrix}. \end{aligned} \quad (8-18)$$

8.3 Painlevé 4 case

8.3.1 Painlevé 4 case in the canonical choice of trivial times

The Painlevé 4 case corresponds to $n = 1$ with $r_1 = 1$ and $r_\infty = 3$. It is a special case of Section 7.2.1. The canonical choice of trivial times (7-1) provides $t_{\infty(1),2} = 1$ and $t_{\infty(1),1} = 0$ and the only non-trivial isomonodromic time is $t = \tilde{X}_1 = X_1$. First, we observe from (7-13) that

$$\tilde{P}_2(\lambda) = -\frac{t_{X_1^{(1)},0}^2}{(\lambda - t)^2} - \lambda^2 - 2t_{\infty(1),0}. \quad (8-19)$$

Since the only isomonodromic time is a position of a pole we get

$$\nu_{\infty,-1}^{(\alpha_t)} = 0, \quad \nu_{\infty,0}^{(\alpha_t)} = 0, \quad \nu_{X_1,0}^{(\alpha_t)} = 0 \quad \text{and} \quad \nu_{X_1,1}^{(\alpha_t)} = 0. \quad (8-20)$$

The constant $H_{X_1,1}$ is determined by Proposition 5.1,

$$V_1^t H_{X_1,1} = \check{p}^2 + \frac{\hbar \check{p}}{\check{q} - t} - \frac{t_{X_1^{(1)},0}^2}{(\check{q}^2 - t)^2} - \check{q}^2 - 2t_{\infty(1),0} + \hbar, \quad (8-21)$$

i.e.

$$H_{X_1,1} = \check{p}^2(\check{q} - t) + \hbar \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q} - t} - (\check{q}^2 + 2t_{\infty(1),0} - \hbar)(\check{q} - t). \quad (8-22)$$

The Hamiltonian is given by (7-6)

$$\begin{aligned} \text{Ham}^{(\alpha_t)}(\check{q}, \check{p}) &= H_{X_1,1} \\ &= \check{p}^2(\check{q} - t) + \hbar \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q} - t} - (\check{q}^2 + 2t_{\infty(1),0} - \hbar)(\check{q} - t) \\ &= (\check{q} - t)\check{p}^2 + \hbar \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q} - t} - \check{q}^3 + t\check{q}^2 - (2t_{\infty(1),0} - \hbar)\check{q} + (2t_{\infty(1),0} - \hbar)t \\ &= (\check{q} - t)\check{p}^2 + \hbar \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q} - t} - (\check{q} - t)^3 - 2t(\check{q} - t)^2 - t^2(\check{q} - t) \\ &\quad - (2t_{\infty(1),0} - \hbar)(\check{q} - t). \end{aligned} \quad (8-23)$$

Coefficient $\mu_1^{(\alpha_t)}$ is determined by Remark 7.3: $\mu_1^{(\alpha_t)} = \check{q} - t$ so that the evolution equations of Theorem 5.1 reads:

$$\begin{aligned} \hbar \partial_t \check{q} &= 2(\check{q} - t) \left(\check{p} + \frac{\hbar}{2(\check{q} - t)} \right) = 2\check{p}(\check{q} - t) + \hbar, \\ \hbar \partial_t \check{p} &= (\check{q} - t) \left(\frac{\hbar \check{p}}{(\check{q} - t)^2} - \frac{2t_{X_1^{(1)},0}^2}{(\check{q} - t)^3} + 2\check{q} - \frac{H_{X_1,1}}{(\check{q} - t)^2} \right) \\ &= -\check{p}^2 - \frac{t_{X_1^{(1)},0}^2}{(\check{q} - t)^2} + 3\check{q}^2 - 2t\check{q} + 2t_{\infty(1),0} - \hbar \\ &= -\check{p}^2 - \frac{t_{X_1^{(1)},0}^2}{(\check{q} - t)^2} + 3(\check{q} - t)^2 + 4t(\check{q} - t) + t^2 + 2t_{\infty(1),0} - \hbar. \end{aligned} \quad (8-24)$$

Thus, $\check{q}(t)$ satisfies the differential equation:

$$\hbar^2(\check{q}-t)\ddot{\check{q}} = \frac{1}{2}(\hbar\dot{\check{q}})^2 - \hbar^2\dot{\check{q}} + 6(\check{q}-t)^4 + 8t(\check{q}-t)^3 + 2(t^2 - 2t_{\infty(1),0} - \hbar)(\check{q}-t)^2 - 2t_{X_1^{(1)},0} + \frac{\hbar^2}{2}. \quad (8-25)$$

This equation can be transformed into the usual Painlevé 4 equation by the change of coordinates $(u, \check{p}) = (\check{q} - t, \check{p})$ for which $u(t)$ satisfies (8-34). Moreover, for this change of coordinates, the Hamiltonian (8-23) and evolution equations (8-24) also recover the corresponding upcoming (8-33) and (8-32).

The associated Lax pair is

$$\begin{aligned} \tilde{L}(\lambda, \hbar) &= \begin{pmatrix} -\lambda + \frac{(\check{q}-t)(\check{p}+\check{q})}{\lambda-t} & \frac{\lambda-\check{q}}{\lambda-t} \\ -\frac{(t_{X_1^{(1)},0})^2}{(\check{q}-t)(\lambda-t)} + \frac{(\check{q}-t)(\check{p}+\check{q})^2}{\lambda-t} + 2(\check{q}-t)(\check{p}+\check{q}) + 2t_{\infty(1),0} & \lambda - \frac{(\check{q}-t)(\check{p}+\check{q})}{\lambda-t} \end{pmatrix}, \\ \tilde{A}_{\alpha_t}(\lambda, \hbar) &= \begin{pmatrix} -(\check{q}-t)\left(\frac{\check{p}+\check{q}}{\lambda-t} + 1\right) & \frac{\check{q}-t}{\lambda-t} \\ \frac{(t_{X_1^{(1)},0})^2}{(\check{q}-t)(\lambda-t)} - \frac{(\check{q}-t)(\check{p}+\check{q})^2}{\lambda-t} & (\check{q}-t)\left(\frac{\check{p}+\check{q}}{\lambda-t} + 1\right) \end{pmatrix}. \end{aligned} \quad (8-26)$$

8.3.2 Painlevé 4 case in the Jimbo-Miwa setup

The Painlevé 4 case may also be recovered by another choice of trivial times. Indeed, since it corresponds to $n = 1$ with $r_1 = 1$ and $r_\infty = 3$ one may decide to fix $t_{\infty(1),2} = 1 = -t_{\infty(2),2}$ and $X_1 = 0$ and take $t = t_{\infty(1),1}$ as the only non-trivial isomonodromic time. This is the standard setup used by the Japanese school and in the works of Jimbo-Miwa [44] rather than fixing the two leading coefficients at infinity. In this setup, Theorem 5.1 remains valid and we propose for completeness to apply it with this choice of trivial times. First, we observe that

$$\tilde{P}_2(\lambda) = -\frac{t_{X_1^{(1)},0}^2}{\lambda^2} - \lambda^2 - 2t\lambda - t^2 - 2t_{\infty(1),0}. \quad (8-27)$$

Since $r_\infty = 3$, coefficients $(\nu_{\infty,k}^{(\alpha_t)})_{-1 \leq k \leq 0}$ are determined by Proposition 4.1,

$$\begin{pmatrix} 2 & 0 \\ 2t & 2 \end{pmatrix} \begin{pmatrix} \nu_{\infty,-1}^{(\alpha_t)} \\ \nu_{\infty,0}^{(\alpha_t)} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Leftrightarrow \nu_{\infty,-1}^{(\alpha_t)} = 0 \text{ and } \nu_{\infty,0}^{(\alpha_t)} = 1. \quad (8-28)$$

Proposition 4.2 implies that $\mu_1^{(\alpha_t)}$ is determined by

$$V_1 \mu_1^{(\alpha_t)} = \frac{1}{\check{q}} \mu_1^{(\alpha_t)} = -\nu_{X_1,0}^{(\alpha_t)} + \nu_{\infty,0}^{(\alpha_t)} + \nu_{\infty,-1}^{(\alpha_t)} X_1 = \nu_{\infty,0}^{(\alpha_t)} \Leftrightarrow \mu_1^{(\alpha_t)} = \check{q}. \quad (8-29)$$

The constant $H_{X_1,1}$ is determined by Proposition 5.1,

$$V_1^t H_{X_1,1} = \check{p}^2 + \frac{\hbar \check{p}}{\check{q}} - \frac{t_{X_1^{(1)},0}^2}{\check{q}^2} - \check{q}^2 - 2t\check{q} - t^2 - 2t_{\infty(1),0} + \hbar, \quad (8-30)$$

i.e.

$$H_{X_1,1} = \check{p}^2 \check{q} + \hbar \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q}} - \check{q}^3 - 2t\check{q}^2 - (t^2 + 2t_{\infty(1),0} - \hbar)\check{q}. \quad (8-31)$$

Thus, we get from Theorem 5.1:

$$\begin{aligned} \hbar \partial_t \check{q} &= 2\check{p}\check{q}, \\ \hbar \partial_t \check{p} &= -\check{p}^2 - \frac{t_{X_1^{(1)},0}^2}{\check{q}^2} + (t^2 - \hbar + 2t_{\infty^{(1)},0}) + 4t\check{q} + 3\check{q}^2. \end{aligned} \quad (8-32)$$

The corresponding Hamiltonian is

$$\begin{aligned} \text{Ham}^{(\alpha_t)}(\check{q}, \check{p}) &= \nu_{\infty,0}^{(\alpha_t)} H_{X_1,1} - \hbar \nu_{\infty,0}^{(\alpha_t)} \check{p} - \hbar \nu_{\infty,-1}^{(\alpha_t)} \check{q}\check{p} \stackrel{(8-28)}{=} H_{X_1,1} - \hbar \check{p} \\ &\stackrel{(8-31)}{=} \check{q}\check{p}^2 - \check{q}^3 - 2t\check{q}^2 - (t^2 + 2t_{\infty^{(1)},0} - \hbar)\check{q} - \frac{t_{X_1^{(1)},0}^2}{\check{q}}. \end{aligned} \quad (8-33)$$

In particular, $\check{q}(t)$ satisfies the Painlevé 4 equation,

$$\hbar^2 \check{q} \ddot{\check{q}} = \frac{1}{2}(\hbar \dot{\check{q}})^2 + 6\check{q}^4 + 8t\check{q}^3 + 2(t^2 + 2t_{\infty^{(1)},0} - \hbar)\check{q}^2 - 2t_{X_1^{(1)},0}^2. \quad (8-34)$$

This is equivalent to say that $u(t) := 2\check{q}(t)$ satisfies the normalized Gambier Painlevé 4 equation

$$\hbar^2 u \ddot{u} = \frac{1}{2}(\hbar \dot{u})^2 + \frac{3}{2}u^4 + 4tu^3 + 2(t^2 - \alpha)u^2 + \beta \quad \text{with } \alpha = -(2t_{\infty^{(1)},0} - \hbar), \beta = -8t_{X_1^{(1)},0}^2. \quad (8-35)$$

Remark 8.1. From our work, it appears that the natural form of the Painlevé 4 equation is (8-34) and not the normalized Gambier Painlevé 4 equation (8-35). Indeed, the natural coordinates are the zeros of the Wronskian given by $\check{q}$ and not $2\check{q}$. This observation was already made in [41, 57].

The associated Lax pair in the gauge without apparent singularities, normalized at infinity according to (2-8) and under the special choice of trivial times $X_1 = 0$, $t_{\infty^{(1)},2} = 1 = -t_{\infty^{(2)},2}$, $t = t_{\infty^{(1)},1} = -t_{\infty^{(2)},1}$, $t_{\infty^{(1)},0} = -t_{\infty^{(2)},0}$ and $t_{X_1^{(1)},0} = -t_{X_1^{(2)},0}$) is

$$\begin{aligned} \left[\tilde{L}(\lambda, \hbar) \right]_{1,1} &= -\lambda - t + \frac{\check{q}(\check{p} + \check{q} - t) - t\check{p}}{\lambda}, \\ \left[\tilde{L}(\lambda, \hbar) \right]_{1,2} &= 1 + \frac{t - \check{q}}{\lambda}, \\ \left[\tilde{L}(\lambda, \hbar) \right]_{2,1} &= 2\check{q}^2 + 2(\check{p} - t)\check{q} - 2t\check{p} + 2t_{\infty^{(1)},0} \\ &\quad + \frac{(\check{q}^2 + (\check{p} - t)\check{q} - t\check{p} + t_{X_1^{(1)},0})(-\check{q}^2 + (t - \check{p})\check{q} + t\check{p} + t_{X_1^{(1)},0})}{(t - \check{q})\lambda}, \\ \left[\tilde{L}(\lambda, \hbar) \right]_{2,2} &= -\left[\tilde{L}(\lambda, \hbar) \right]_{1,1}, \\ \tilde{A}_{\alpha_t}(\lambda, \hbar) &= \begin{pmatrix} -\lambda - \check{q} & 1 \\ 2(\check{q}^2 - t\check{q} + \check{p}\check{q} - t\check{p} + t_{\infty^{(1)},0}) & \lambda + \check{q} \end{pmatrix}. \end{aligned} \quad (8-36)$$

8.4 Painlevé 5 case

The Painlevé 5 case corresponds to $r_{\infty} = 1$ with $n = 2$, $r_1 = 1$ and $r_2 = 2$. It is a special case of Section 7.4. The canonical choice of trivial times (7-1), the isomonodromic time is $t = \tau_{X_2,1} = 2t_{X_2^{(1)},1}$. Equation (R-12) provides $\nu_{X_2,1}^{(\alpha_t)} = -\frac{1}{t}$. We also have

$$\tilde{P}_2(\lambda) = -\frac{t^2}{4(\lambda - 1)^4} - \frac{t_{X_2^{(1)},0}t}{(\lambda - 1)^3} - \frac{t_{X_1^{(1)},0}^2}{\lambda^2}. \quad (8-37)$$

Since $r_\infty = 1$, the coefficients $\nu_{\infty,-1}^{(\alpha_t)}$, $\nu_{\infty,0}^{(\alpha_t)}$ and $\mu_1^{(\alpha_t)}$ are determined by (7-31):

$$\begin{pmatrix} \frac{1}{\check{q}} \\ \frac{1}{\check{q}-1} \\ \frac{1}{(\check{q}-1)^2} \end{pmatrix} \mu_1^{(\alpha_t)} = \begin{pmatrix} \nu_{\infty,0}^{(\alpha_t)} \\ \nu_{\infty,0}^{(\alpha_t)} + \nu_{\infty,-1}^{(\alpha_t)} \\ -\nu_{X_{1,1}}^{(\alpha_t)} + \nu_{\infty,-1}^{(\alpha_t)} \end{pmatrix} = \begin{pmatrix} \nu_{\infty,0}^{(\alpha_t)} \\ \nu_{\infty,0}^{(\alpha_t)} + \nu_{\infty,-1}^{(\alpha_t)} \\ \frac{1}{t} + \nu_{\infty,-1}^{(\alpha_t)} \end{pmatrix} \quad (8-38)$$

which is equivalent to

$$\nu_{\infty,-1}^{(\alpha_t)} = \frac{\check{q}-1}{t}, \nu_{\infty,0}^{(\alpha_t)} = \frac{(\check{q}-1)^2}{t}, \mu_1^{(\alpha_t)} = \frac{\check{q}(\check{q}-1)^2}{t}. \quad (8-39)$$

Coefficients $H_{X_{1,1}}$, $H_{X_{2,1}}$ and $H_{X_{2,2}}$ are determined by (7-30):

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \frac{1}{\check{q}} & \frac{1}{\check{q}-1} & \frac{1}{(\check{q}-1)^2} \end{pmatrix} \begin{pmatrix} H_{X_{1,1}} \\ H_{X_{2,1}} \\ H_{X_{2,2}} \end{pmatrix} = \begin{pmatrix} \hbar \check{p} \\ \hbar \check{p} \check{q} - t_{X_1^{(1)},0}^2 - t_{\infty^{(1)},0}(t_{\infty^{(1)},0} + \hbar) \\ \check{p}^2 + \left(\frac{\hbar}{\check{q}} + \frac{2\hbar}{\check{q}-1}\right) \check{p} + \tilde{P}_2(\check{q}) \end{pmatrix} \quad (8-40)$$

i.e.

$$\begin{aligned} H_{X_{1,1}} &= \check{q}(\check{q}-1)^2 \check{p}^2 + (\check{q}-1)^2 \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q}} \\ &\quad - \frac{t^2}{4(\check{q}-1)^2} - \frac{(4t_{X_2^{(1)},0} + t)t}{4(\check{q}-1)} + t_{\infty^{(1)},0}(-t_{\infty^{(1)},0} + \hbar)\check{q} - t_{X_2^{(1)},0}t - 2t_{X_1^{(1)},0}^2, \\ H_{X_{2,1}} &= -\check{q}(\check{q}-1)^2 \check{p}^2 - \hbar \check{q}(\check{q}-2)\check{p} + \frac{t_{X_1^{(1)},0}^2}{\check{q}} \\ &\quad + \frac{t^2}{4(\check{q}-1)^2} + \frac{(4t_{X_2^{(1)},0} + t)t}{4(\check{q}-1)} - t_{\infty^{(1)},0}(-t_{\infty^{(1)},0} + \hbar)\check{q} + t_{X_2^{(1)},0}t - 2t_{X_1^{(1)},0}^2, \\ H_{X_{2,2}} &= \check{q}(\check{q}-1)^2 \check{p}^2 + \hbar \check{q}(\check{q}-1)\check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q}} \\ &\quad - \frac{t^2}{4(\check{q}-1)^2} - \frac{(4t_{X_2^{(1)},0} + t)t}{4(\check{q}-1)} + t_{\infty^{(1)},0}(-t_{\infty^{(1)},0} + \hbar)(\check{q}-1) - t_{X_2^{(1)},0}t + t_{X_1^{(1)},0}^2. \end{aligned} \quad (8-41)$$

Thus, we get from Theorem 5.1

$$\begin{aligned} \hbar \partial_t \check{q} &= \frac{2\check{q}(\check{q}-1)^2 \check{p}}{t} + \hbar \frac{\check{q}(\check{q}-1)}{t}, \\ \hbar \partial_t \check{p} &= -\frac{(3\check{q}-1)(\check{q}-1)}{t} \check{p}^2 - \frac{\hbar(2\check{q}-1)}{t} \check{p} \\ &\quad - \frac{t_{X_1^{(1)},0}^2}{t\check{q}^2} - \frac{t}{2(\check{q}-1)^3} - \frac{4t_{X_2^{(1)},0} + t}{4(\check{q}-1)^2} + \frac{t_{\infty^{(1)},0}(t_{\infty^{(1)},0} - \hbar)}{t}. \end{aligned} \quad (8-42)$$

In particular, $\check{q}$ satisfies the Painlevé 5 equation

$$\hbar^2 \ddot{\check{q}} = \left(\frac{1}{2\check{q}} + \frac{1}{\check{q}-1} \right) (\hbar \dot{\check{q}})^2 - \frac{\hbar^2}{t} \dot{\check{q}} + \frac{(\check{q}-1)^2}{t^2} \left(\alpha \check{q} + \frac{\beta}{\check{q}} \right) + \gamma \frac{\check{q}}{t} + \delta \frac{\check{q}(\check{q}+1)}{\check{q}-1} \quad (8-43)$$

with

$$\alpha = \frac{(2t_{\infty^{(1)},0} - \hbar)^2}{2}, \beta = -2t_{X_1^{(1)},0}^2, \gamma = -2t_{X_2^{(1)},0}, \delta = -\frac{1}{2}. \quad (8-44)$$

The corresponding Hamiltonian is given by Theorem 7.1:

$$\begin{aligned}
\text{Ham}^{(\alpha_t)}(\check{q}, \check{p}) &= -\nu_{X_{2,1}}^{(\alpha_t)} H_{X_{2,2}} \\
&= \frac{1}{t} \left[\check{q}(\check{q}-1)^2 \check{p}^2 + \hbar \check{q}(\check{q}-1) \check{p} - \frac{t_{X_1^{(1)},0}^2}{\check{q}} \right. \\
&\quad - \frac{t^2}{4(\check{q}-1)^2} - \frac{(4t_{X_2^{(1)},0} + t)t}{4(\check{q}-1)} + t_{\infty^{(1)},0}(-t_{\infty^{(1)},0} + \hbar)(\check{q}-1) \\
&\quad \left. - t_{X_2^{(1)},0}t + t_{X_1^{(1)},0}^2 \right]. \tag{8-45}
\end{aligned}$$

The associated Lax pair in the gauge without apparent singularities, normalized at infinity according to (2-8) and under the choice of trivial times (7-1) (i.e. $X_1 = 0$, $t_{\infty^{(1)},2} = 1 = -t_{\infty^{(2)},2}$, $t = t_{\infty^{(1)},1} = -t_{\infty^{(2)},1}$ and $t_{\infty^{(1)},0} = -t_{\infty^{(2)},0}$ and $t_{X_1^{(1)},0} = -t_{X_1^{(2)},0}$) is

$$\begin{aligned}
\tilde{L}_{1,1}(\lambda, \hbar) &= \frac{(\check{q}-1) \left(\eta_0 + \check{p}\check{q}(\check{q}-1) + t_{\infty^{(1)},0} \right)}{(\lambda-1)^2} - \frac{\check{q}\eta_0 + \check{p}\check{q}(\check{q}-1)^2 + t_{\infty^{(1)},0}}{\lambda-1} \\
&\quad + \frac{\check{q}\eta_0 + \check{q}(\check{q}-1)^2 \check{p}}{\lambda}, \\
\tilde{L}_{2,2}(\lambda, \hbar) &= -\tilde{L}_{1,1}(\lambda, \hbar), \\
\tilde{L}_{1,2}(\lambda, \hbar) &= \frac{\lambda - \check{q}}{\lambda(\lambda-1)^2}, \\
\tilde{L}_{2,1}(\lambda, \hbar) &= \frac{(\check{q}-1) \left((\eta_0 + \check{p}\check{q}(\check{q}-1) + t_{\infty^{(1)},0})^2 - \frac{t^2}{4(\check{q}-1)^2} \right)}{(\lambda-1)^2} \\
&\quad - \frac{\check{q} \left((\eta_0 + \check{p}(\check{q}-1)^2)^2 - \frac{t_{X_1^{(1)},0}^2}{\check{q}^2} \right)}{\lambda-1} \\
&\quad + \frac{\check{q} \left((\eta_0 + \check{p}(\check{q}-1)^2)^2 - \frac{t_{X_1^{(1)},0}^2}{\check{q}^2} \right)}{\lambda} \tag{8-46}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}_{1,1}(\lambda, \hbar) &= -\frac{(\check{q}-1)(\check{p}\check{q}(\check{q}-1) + \eta_0 + t_{\infty^{(1)},0})}{t(\lambda-1)} - \frac{t_{\infty^{(1)},0}(\check{q}-1)}{t}, \\
\tilde{A}_{2,2}(\lambda, \hbar) &= \frac{(\check{q}-1)(\check{p}\check{q}(\check{q}-1) + \eta_0 + t_{\infty^{(1)},0})}{t(\lambda-1)} + \frac{(t_{\infty^{(1)},0} - \hbar)(\check{q}-1)}{t}, \\
\tilde{A}_{1,2}(\lambda, \hbar) &= \frac{\check{q}-1}{t(\lambda-1)}, \\
\tilde{A}_{2,1}(\lambda, \hbar) &= -\frac{\frac{(\check{q}-1)}{t} \left((\eta_0 + \check{p}\check{q}(\check{q}-1) + t_{\infty^{(1)},0})^2 - \frac{t^2}{4} \right) + \frac{t\check{q}(\check{q}-2)}{4(\check{q}-1)}}{\lambda-1} \tag{8-47}
\end{aligned}$$

where we have defined following (2-60)

$$\eta_0 = \frac{1}{2t_{\infty^{(1)},0}} \left[\check{q}(\check{q}-1)^2 \check{p}^2 - \frac{t_{X_1^{(1)},0}^2}{\check{q}} - \frac{t^2}{4(\check{q}-1)^2} - \frac{t(t + 4t_{X_2^{(1)},0})}{4(\check{q}-1)} + t_{\infty^{(1)},0}^2(\check{q}-2) \right]. \tag{8-48}$$

Note that both matrices $\tilde{L}(\lambda, \hbar)$ and $\tilde{A}(\lambda, \hbar)$ may be set traceless by the additional trivial gauge transformation defined by $G = u(t)I_2$ with $u(t) = \exp\left(\int^t \frac{\tilde{q}(s)-1}{2s} ds\right)$ following Remark 6.13.

Remark 8.2. The Painlevé 5 case may also be recovered by setting $n = 2$, $r_1 = 1$, $r_2 = 1$ and $r_\infty = 2$ that corresponds to a Möbius transformation exchanging only $\lambda = X_2$ with $\lambda = \infty$. In this case, the standard approach is use a choice of trivial times so that it fixes the position of the two finite poles at $X_1 = 0$ and $X_2 = 1$ so that the only non-trivial isomonodromic time is $t = \tilde{X}_2 = t_{\infty(1),1}$. In this setup, the Darboux coordinate $\tilde{q}$ satisfies the ODE

$$\begin{aligned} \hbar^2 \ddot{\tilde{q}} &= \frac{(2\tilde{q}-1)}{2\tilde{q}(\tilde{q}-1)}(\hbar\dot{\tilde{q}})^2 - \frac{\hbar^2}{t}\dot{\tilde{q}} + \frac{2t_{X_1^{(1)},0}^2(\tilde{q}-1)}{t^2\tilde{q}} - \frac{(t_{X_2^{(1)},0} - t_{X_2^{(2)},0})^2\tilde{q}}{2t^2(\tilde{q}-1)} \\ &\quad + 2\tilde{q}(\tilde{q}-1)(2\tilde{q}-1) + \frac{2\tilde{q}(\tilde{q}-1)}{t}(2t_{\infty(1),0} - \hbar) \end{aligned} \quad (8-49)$$

It is then straightforward to check that $u = \frac{\tilde{q}}{\tilde{q}-1}$ satisfies the Painlevé 5 equation given by (8-43). This indirect approach was used for example in [44, 57] or in articles in which infinity is always assumed to be one of the poles of highest order.

8.5 Painlevé 6 case

The Painlevé 6 case corresponds to $n = 3$ with $r_1 = 1$, $r_2 = 1$, $r_3 = 1$ and $r_\infty = 1$ and thus is a direct application of Section 7.4. The isomonodromic time is $t = \tilde{X}_3$. The corresponding derivative under the canonical choice of trivial times is $\partial_{\tilde{X}_3} = \partial_{X_3}$ corresponding to $\alpha_{X_3} = 1$ while all other coefficients vanish. Coefficient $\left(\nu_{\infty,-1}^{(\alpha_{\tilde{X}_3})}, \nu_{\infty,0}^{(\alpha_{\tilde{X}_3})}, \mu_1^{(\alpha_{\tilde{X}_3})}\right)$ are determined by Proposition 4.2

$$\left(\frac{\frac{1}{\tilde{q}}}{\frac{1}{\tilde{q}-1}}\right) \mu_1^{(\alpha_{\tilde{X}_3})} = \begin{pmatrix} \nu_{\infty,0}^{(\alpha_{\tilde{X}_3})} \\ \nu_{\infty,0}^{(\alpha_{\tilde{X}_3})} + \nu_{\infty,-1}^{(\alpha_{\tilde{X}_3})} \\ 1 + \nu_{\infty,0}^{(\alpha_{\tilde{X}_3})} + \nu_{\infty,-1}^{(\alpha_{\tilde{X}_3})} t \end{pmatrix} \quad (8-50)$$

i.e.

$$\nu_{\infty,0}^{(\alpha_{\tilde{X}_3})} = \frac{\mu_1^{(\alpha_{\tilde{X}_3})}}{\tilde{q}}, \nu_{\infty,-1}^{(\alpha_{\tilde{X}_3})} = \frac{\mu_1^{(\alpha_{\tilde{X}_3})}}{\tilde{q}(\tilde{q}-1)}, \left(\frac{1}{\tilde{q}-t} - \frac{1}{\tilde{q}} - \frac{t}{\tilde{q}(\tilde{q}-1)}\right) \mu_1^{(\alpha_{\tilde{X}_3})} = 1 \quad (8-51)$$

so that

$$\mu_1^{(\alpha_{\tilde{X}_3})} = \frac{\tilde{q}(\tilde{q}-1)(\tilde{q}-t)}{t(t-1)}. \quad (8-52)$$

Coefficients $(H_{X_1,1}, H_{X_2,1}, H_{X_3,1})$ are determined by (7-30):

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & t \\ \frac{1}{\tilde{q}} & \frac{1}{\tilde{q}-1} & \frac{1}{\tilde{q}-t} \end{pmatrix} \begin{pmatrix} H_{X_1,1} \\ H_{X_2,1} \\ H_{X_3,1} \end{pmatrix} = \begin{pmatrix} \hbar\tilde{p} \\ \hbar\tilde{p}\tilde{q} - t_{X_1^{(1)},0}^2 - t_{X_2^{(1)},0}^2 - t_{X_3^{(1)},0}^2 + t_{\infty(1),0}(t_{\infty(1),0} - \hbar) \\ \tilde{p}^2 + \hbar\tilde{p}\left(\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}-1} + \frac{1}{\tilde{q}-t}\right) - \frac{t_{X_1^{(1)},0}^2}{\tilde{q}^2} - \frac{t_{X_1^{(1)},0}^2}{(\tilde{q}-1)^2} - \frac{t_{X_3^{(1)},0}^2}{(\tilde{q}-t)^2} \end{pmatrix} \quad (8-53)$$

so that we get from Theorem 5.1

$$\hbar\partial_t \tilde{q} = \frac{2\tilde{q}(\tilde{q}-1)(\tilde{q}-t)}{t(t-1)}\tilde{p} + \hbar\frac{(\tilde{q}-1)\tilde{q}}{t(t-1)},$$

$$\begin{aligned}
\hbar \partial_t \check{p} &= \mu_1^{(\alpha_{\check{X}_3})} \left[-\tilde{P}'_2(\check{q}) + \check{p} \left(\frac{\hbar}{\check{q}^2} + \frac{\hbar}{(\check{q}-1)^2} + \frac{\hbar}{(\check{q}-t)^2} \right) - \frac{H_{X_1,1}}{\check{q}^2} - \frac{H_{X_2,1}}{(\check{q}-1)^2} - \frac{H_{X_3,1}}{(\check{q}-t)^2} \right] + \hbar \nu_{\infty,-1}^{(\alpha_{\check{X}_3})} \check{p} \\
&= -\frac{3\check{q}^2 - 2t\check{q} - 2\check{q} + t}{t(t-1)} \check{p}^2 - \hbar \frac{(2\check{q}-1)\check{p}}{t(t-1)} - \frac{t_{X_1^{(1)},0}^2}{(t-1)\check{q}^2} + \frac{t_{X_2^{(1)},0}^2}{t(\check{q}-1)^2} - \frac{t_{X_3^{(1)},0}^2}{(\check{q}-t)^2} + \frac{t_{\infty^{(1)},0}(t_{\infty^{(1)},0} - \hbar)}{t(t-1)}.
\end{aligned} \tag{8-54}$$

Finally, we obtain that $\check{q}$ satisfies the Painlevé 6 equation

$$\begin{aligned}
\hbar^2 \ddot{\check{q}} &= \frac{1}{2} \left(\frac{1}{\check{q}} + \frac{1}{\check{q}-1} + \frac{1}{\check{q}-t} \right) (\hbar \dot{\check{q}})^2 - \hbar \dot{\check{q}} \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\check{q}-t} \right) \\
&\quad + \frac{\check{q}(\check{q}-1)(\check{q}-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{\check{q}^2} + \gamma \frac{t-1}{(\check{q}-1)^2} + \delta \frac{t(t-1)}{(\check{q}-t)^2} \right)
\end{aligned} \tag{8-55}$$

with $\alpha = \frac{(2t_{\infty^{(1)},0} - \hbar)^2}{2}$, $\beta = -2t_{X_1^{(1)},0}^2$, $\gamma = 2t_{X_2^{(1)},0}^2$ and $\delta = -\left(2t_{X_3^{(1)},0}^2 - \frac{\hbar^2}{2}\right)$.

The corresponding Hamiltonian is

$$\begin{aligned}
\text{Ham}^{(\alpha_t)}(\check{q}, \check{p}) &= H_{X_3,1} \\
&= \frac{\check{q}(\check{q}-1)(\check{q}-t)}{t(t-1)} \check{p}^2 + \hbar \frac{\check{q}(\check{q}-1)}{t(t-1)} \check{p} - \frac{t_{X_1^{(1)},0}^2}{(t-1)\check{q}} + \frac{t_{X_2^{(1)},0}^2}{t(\check{q}-1)} - \frac{t_{X_3^{(1)},0}^2}{(\check{q}-t)} \\
&\quad - \frac{t_{\infty^{(1)},0}(t_{\infty^{(1)},0} - \hbar)}{t(t-1)}
\end{aligned} \tag{8-56}$$

The associated Lax pair in the gauge without apparent singularities, normalized at infinity according to (2-8) and under the choice of trivial times (7-1) (i.e. $X_1 = 0$, $X_2 = 1$, $X_3 = t$ and $t_{\infty^{(2)},0} = -t_{\infty^{(1)},0}$, $t_{X_j^{(2)},0} = -t_{X_j^{(1)},0}$ for all $1 \leq j \leq 3$) is

$$\tilde{L}(\lambda, \hbar) = \frac{\tilde{L}_0}{\lambda} + \frac{\tilde{L}_1}{\lambda-1} + \frac{\tilde{L}_t}{\lambda-t}, \quad \tilde{A}_{\alpha_t}(\lambda, \hbar) = -\frac{\tilde{A}_t}{\lambda-t} - \tilde{A}_{\infty} \tag{8-57}$$

with

$$\begin{aligned}
\tilde{L}_0 &= \begin{pmatrix} \frac{\check{q}}{t}(\eta_0 + (\check{q}-1)(\check{q}-t)\check{p}) & -\frac{\check{q}}{t} \\ \frac{\check{q}}{t} \left((\eta_0 + \check{p}(\check{q}-1)(\check{q}-t))^2 - \frac{t_{X_1^{(1)},0}^2}{\check{q}^2} \right) & -\frac{\check{q}}{t}(\eta_0 + (\check{q}-1)(\check{q}-t)\check{p}) \end{pmatrix}, \\
\tilde{L}_1 &= \begin{pmatrix} -\frac{(\check{q}-1)}{(t-1)}(\eta_0 + \check{q}(\check{q}-t)\check{p} + t_{\infty^{(1)},0}) & \frac{\check{q}-1}{t-1} \\ -\frac{\check{q}-1}{t-1} \left((\eta_0 + \check{q}(\check{q}-t)\check{p} + t_{\infty^{(1)},0})^2 - \frac{t_{X_2^{(1)},0}^2(t-1)^2}{(\check{q}-1)^2} \right) & \frac{(\check{q}-1)}{(t-1)}(\eta_0 + \check{q}(\check{q}-t)\check{p} + t_{\infty^{(1)},0}) \end{pmatrix}, \\
\tilde{L}_t &= \begin{pmatrix} \frac{(\check{q}-t)}{t(t-1)}(\eta_0 + \check{q}(\check{q}-1)\check{p} + t_{\infty^{(1)},0}t) & -\frac{\check{q}-t}{t(t-1)} \\ \frac{\check{q}-t}{t(t-1)} \left((\eta_0 + \check{q}(\check{q}-1)\check{p} + t_{\infty^{(1)},0}t)^2 - \frac{t_{X_3^{(1)},0}^2 t^2(t-1)^2}{(\check{q}-t)^2} \right) & -\frac{(\check{q}-t)}{t(t-1)}(\eta_0 + \check{q}(\check{q}-1)\check{p} + t_{\infty^{(1)},0}t) \end{pmatrix}
\end{aligned} \tag{8-58}$$

satisfying

$$\tilde{L}_{\infty} := \tilde{L}_0 + \tilde{L}_1 + \tilde{L}_t = \begin{pmatrix} -t_{\infty^{(1)},0} & 0 \\ 0 & t_{\infty^{(1)},0} \end{pmatrix} \tag{8-59}$$

and

$$\tilde{A}_t = \begin{pmatrix} -\frac{(\eta_0 + \check{p}\check{q}(\check{q}-1) + t_{\infty^{(1)},0}t)(\check{q}-t)}{t(t-1)} & \frac{\check{q}-t}{t(t-1)} \\ -\frac{(\check{q}-t)}{t(t-1)} \left((\eta_0 + \check{q}(\check{q}-1)\check{p} + t_{\infty^{(1)},0}t)^2 - \frac{t_{X_3^{(1)},0}^2 t^2(t-1)^2}{(\check{q}-t)^2} \right) & \frac{(\eta_0 + \check{p}\check{q}(\check{q}-1) + t_{\infty^{(1)},0}t)(\check{q}-t)}{t(t-1)} \end{pmatrix},$$

$$\tilde{A}_\infty = \begin{pmatrix} -\frac{t_{\infty(1),0}(\tilde{q}-t)}{t(t-1)} & 0 \\ 0 & \frac{(t_{\infty(1),0}-h)(\tilde{q}-t)}{t(t-1)} \end{pmatrix} \quad (8-60)$$

where we have defined following (2-60):

$$\eta_0 = \frac{1}{2t_{\infty(1),0}} \left[\tilde{q}(\tilde{q}-1)(\tilde{q}-t)\tilde{p}^2 - \frac{t_{X_1^{(1)},0}^2 t}{\tilde{q}} + \frac{t_{X_2^{(1)},0}^2 (t-1)}{\tilde{q}-1} - \frac{t_{X_3^{(1)},0}^2 t(t-1)}{\tilde{q}-t} + t_{\infty(1),0}^2 (\tilde{q}-1-t) \right]. \quad (8-61)$$

Note that both matrices $\tilde{L}(\lambda, \hbar)$ and $\tilde{A}_{\alpha_i}(\lambda, \hbar)$ may be set traceless by the additional trivial gauge transformation defined by $G = u(t)I_2$ with $u(t) = \exp\left(\int^t \frac{(\tilde{q}(s)-s)}{2s(s-1)} ds\right)$ following Remark 6.13.

8.6 Fuchsian systems

Fuchsian systems correspond to the case $r_\infty = 1$ and $n \geq 3$ with $r_i = 1$ for all $i \in \llbracket 1, n \rrbracket$. It provides a spectral curve of genus $g = n - 2$. It is a special case of Section 7.4. The canonical choice of trivial times (7-1) implies that $X_1 = 0$, $X_2 = 1$ and that the isomonodromic times are $\tilde{X}_s = X_s$ for $s \geq 3$. We get from (7-27):

$$\tilde{P}_2(\lambda) = -\frac{t_{X_1^{(1)},0}^2}{\lambda^2} - \frac{t_{X_2^{(1)},0}^2}{(\lambda-1)^2} - \sum_{s=3}^n \frac{t_{X_s^{(1)},0}^2}{(\lambda - \tilde{X}_s)^2}. \quad (8-62)$$

Coefficients $(H_{X_s,1})_{1 \leq s \leq n}$ are given by (7-30)

$$\begin{pmatrix} \frac{1}{0} & \frac{1}{\frac{1}{q_1}} & \frac{1}{\frac{1}{q_2}} & \dots & \frac{1}{\frac{1}{q_g}} \\ \frac{1}{\frac{1}{q_1}-1} & \frac{1}{\frac{1}{q_1}-\tilde{X}_3} & \frac{1}{\frac{1}{q_2}-\tilde{X}_3} & \dots & \frac{1}{\frac{1}{q_g}-\tilde{X}_3} \\ \frac{1}{\frac{1}{q_1}-\tilde{X}_3} & \frac{1}{\frac{1}{q_1}-\tilde{X}_n} & \frac{1}{\frac{1}{q_2}-\tilde{X}_n} & \dots & \frac{1}{\frac{1}{q_g}-\tilde{X}_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\frac{1}{q_1}-\tilde{X}_n} & \frac{1}{\frac{1}{q_1}-\tilde{X}_n} & \frac{1}{\frac{1}{q_2}-\tilde{X}_n} & \dots & \frac{1}{\frac{1}{q_g}-\tilde{X}_n} \end{pmatrix} \begin{pmatrix} H_{X_1,1} \\ \vdots \\ \vdots \\ \vdots \\ H_{X_n,1} \end{pmatrix} = \begin{pmatrix} h \sum_{j=1}^g \tilde{p}_j \\ h \sum_{j=1}^g \tilde{q}_j \tilde{p}_j - \sum_{s=1}^n t_{X_s^{(1)},0}^2 + t_{\infty(1),0}^2 (t_{\infty(1),0} - h) \\ \tilde{p}_1^2 + h \tilde{p}_1 \left(\frac{1}{\tilde{q}_1} + \frac{1}{\tilde{q}_1-1} + \sum_{s=3}^n \frac{1}{\tilde{q}_1-\tilde{X}_s} \right) - \frac{t_{X_1^{(1)},0}^2}{\tilde{q}_1^2} - \frac{t_{X_2^{(1)},0}^2}{(\tilde{q}_1-1)^2} - \sum_{s=3}^n \frac{t_{X_s^{(1)},0}^2}{(\tilde{q}_1-\tilde{X}_s)^2} - h \sum_{i \neq 1} \frac{\tilde{p}_1 - \tilde{p}_i}{\tilde{q}_1 - \tilde{q}_i} \\ \vdots \\ \tilde{p}_g^2 + h \tilde{p}_g \left(\frac{1}{\tilde{q}_g} + \frac{1}{\tilde{q}_g-1} + \sum_{s=3}^n \frac{1}{\tilde{q}_g-\tilde{X}_s} \right) - \frac{t_{X_1^{(1)},0}^2}{\tilde{q}_g^2} - \frac{t_{X_2^{(1)},0}^2}{(\tilde{q}_g-1)^2} - \sum_{s=3}^n \frac{t_{X_s^{(1)},0}^2}{(\tilde{q}_g-\tilde{X}_s)^2} - h \sum_{i \neq g} \frac{\tilde{p}_g - \tilde{p}_i}{\tilde{q}_g - \tilde{q}_i} \end{pmatrix}. \quad (8-63)$$

The Hamiltonian corresponding to a deformation relatively to $\tilde{X}_s$ is

$$\text{Ham}^{(\alpha_{\tilde{X}_s})}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = H_{X_s,1} \quad (8-64)$$

that is obtained from inverting (8-63).

For any $s \in \llbracket 3, n \rrbracket$, the deformation relatively to $\tilde{X}_s = X_s$ provides the relation

$$\begin{pmatrix} \frac{1}{\frac{1}{q_1}} & \dots & \dots & \frac{1}{\frac{1}{q_g}} \\ \frac{1}{\frac{1}{q_1}-1} & \dots & \dots & \frac{1}{\frac{1}{q_g}-1} \\ \frac{1}{\frac{1}{q_1}-\tilde{X}_3} & \dots & \dots & \frac{1}{\frac{1}{q_g}-\tilde{X}_3} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\frac{1}{q_1}-\tilde{X}_n} & \dots & \dots & \frac{1}{\frac{1}{q_g}-\tilde{X}_n} \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha_{\tilde{X}_s})} \\ \vdots \\ \vdots \\ \mu_g^{(\alpha_{\tilde{X}_s})} \end{pmatrix} = \begin{pmatrix} \nu_{\infty,0}^{(\alpha_{\tilde{X}_s})} \\ \nu_{\infty,0}^{(\alpha_{\tilde{X}_s})} + \nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})} \\ \delta_{s,3} + \nu_{\infty,0}^{(\alpha_{\tilde{X}_s})} + \tilde{X}_3 \nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})} \\ \vdots \\ \delta_{s,n} + \nu_{\infty,0}^{(\alpha_{\tilde{X}_s})} + \tilde{X}_n \nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})} \end{pmatrix} \quad (8-65)$$

that determines $(\nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})})$, $\nu_{\infty,0}^{(\alpha_{\tilde{X}_s})}$ and all $(\mu_j^{(\alpha_{\tilde{X}_s})})_{1 \leq j \leq g}$. The evolution equations are given by

$$\begin{aligned} \hbar \partial_{\tilde{X}_s} \check{q}_j &= 2\mu_j^{(\alpha_{\tilde{X}_s})} \left(\check{p}_j + \frac{1}{2} \left(\frac{\hbar}{\check{q}_j} + \frac{\hbar}{\check{q}_j - 1} + \sum_{s=3}^n \frac{\hbar}{\check{q}_j - \tilde{X}_s} \right) \right) - \hbar \nu_{\infty,0}^{(\alpha_{\tilde{X}_s})} - \hbar \nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})} \check{q}_j \\ &\quad - \hbar \sum_{i \neq j} \frac{\mu_j^{(\alpha_{\tilde{X}_s})} + \mu_i^{(\alpha_{\tilde{X}_s})}}{\check{q}_j - \check{q}_i}, \\ \hbar \partial_{\tilde{X}_s} \check{p}_j &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha_{\tilde{X}_s})} + \mu_j^{(\alpha_{\tilde{X}_s})})(\check{p}_i - \check{p}_j)}{(\check{q}_j - \check{q}_i)^2} + \mu_j^{(\alpha_{\tilde{X}_s})} \left[\check{p}_j \left(\frac{\hbar}{\check{q}_j^2} + \frac{\hbar}{(\check{q}_j - 1)^2} + \sum_{s=3}^n \frac{\hbar}{(\check{q}_j - \tilde{X}_s)^2} \right) \right. \\ &\quad \left. + \frac{t_{X_1^{(1)},0}^2}{\check{q}_j^2} + \frac{t_{X_2^{(1)},0}^2}{(\check{q}_j - 1)^2} + \sum_{s=3}^n \frac{t_{X_s^{(1)},0}^2}{(\check{q}_j - \tilde{X}_s)^2} - \sum_{s=1}^n H_{X_s,1}(\check{q}_j - \tilde{X}_s)^{-2} \right] + \hbar \nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})} \check{p}_j. \end{aligned} \quad (8-66)$$

The Lax pair in the companion-like gauge is given by

$$L(\lambda, \hbar) = \begin{pmatrix} 0 & 1 \\ -\frac{t_{X_1^{(1)},0}^2}{\lambda^2} - \frac{t_{X_2^{(1)},0}^2}{(\lambda-1)^2} - \sum_{s=3}^n \frac{t_{X_s^{(1)},0}^2}{(\lambda-\tilde{X}_s)^2} + \frac{H_{X_1,1}}{\lambda} + \frac{H_{X_2,1}}{\lambda-1} + \sum_{s=3}^n \frac{H_{X_s,1}}{\lambda-\tilde{X}_s} - \sum_{j=1}^g \frac{\hbar p_j}{\lambda - q_j} & \sum_{j=1}^g \frac{\hbar}{\lambda - \check{q}_j} - \frac{\hbar}{\lambda} - \frac{\hbar}{\lambda-1} - \sum_{s=3}^n \frac{\hbar}{\lambda - \tilde{X}_s} \end{pmatrix} \quad (8-67)$$

and

$$\begin{aligned} [A_{\alpha_{\tilde{X}_s}}]_{1,1}(\lambda, \hbar) &= - \sum_{j=1}^g \frac{\mu_j^{(\alpha_{\tilde{X}_s})} \check{p}_j}{\lambda - \check{q}_j}, \\ [A_{\alpha_{\tilde{X}_s}}]_{1,2}(\lambda, \hbar) &= \nu_{\infty,-1}^{(\alpha_{\tilde{X}_s})} \lambda + \nu_{\infty,0}^{(\alpha_{\tilde{X}_s})} + \sum_{j=1}^g \frac{\mu_j^{(\alpha_{\tilde{X}_s})}}{\lambda - \check{q}_j}. \end{aligned} \quad (8-68)$$

Using (4-5), one may obtain the other two entries of $A_{\alpha_{\tilde{X}_s}}(\lambda, \hbar)$ and use (2-44) and (2-47) in order to obtain $\tilde{L}(\lambda, \hbar)$ and $\tilde{A}_{\alpha_{\tilde{X}_s}}(\lambda, \hbar)$.

8.7 Second element of the Painlevé 2 hierarchy

In this section, we propose the application of our general results to a genus 2 case, namely the second element of the Painlevé 2 hierarchy. It corresponds to $n = 0$ and $r_\infty = 5$. This example is a direct application of Section 7.2.2. The choice of trivial times (7-1) corresponds to

$$\begin{aligned} t_{\infty^{(1)},4} &= 1 = -t_{\infty^{(2)},4}, \quad t_{\infty^{(1)},3} = 0 = -t_{\infty^{(2)},3}, \\ t_{\infty^{(1)},2} &= \frac{1}{2} \tau_{\infty,2} = -t_{\infty^{(2)},2}, \quad t_{\infty^{(1)},1} = \frac{1}{2} \tau_{\infty,1} = -t_{\infty^{(2)},1}, \\ t_{\infty^{(2)},0} &= 1 = -t_{\infty^{(1)},0}. \end{aligned} \quad (8-69)$$

We get from (7-16):

$$\tilde{P}_2(\lambda) = -\lambda^6 - \tau_{\infty,2} \lambda^4 - \tau_{\infty,1} \lambda^3 - \left(2t_{\infty^{(1)},0} + \frac{1}{4} \tau_{\infty,2}^2 \right) \lambda^2. \quad (8-70)$$

Coefficients $(H_{\infty,0}, H_{\infty,1})$ are determined by (7-17):

$$\begin{pmatrix} 1 & \check{q}_1 \\ 1 & \check{q}_2 \end{pmatrix} \begin{pmatrix} H_{\infty,0} \\ H_{\infty,1} \end{pmatrix} = \begin{pmatrix} \check{p}_1^2 + \tilde{P}_2(\check{q}_1) - \hbar \frac{\check{p}_2 - \check{p}_1}{\check{q}_2 - \check{q}_1} + \hbar \check{q}_1^2 \\ \check{p}_2^2 + \tilde{P}_2(\check{q}_2) - \hbar \frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} + \hbar \check{q}_2^2 \end{pmatrix} \quad (8-71)$$

i.e.

$$\begin{aligned}
H_{\infty,0} &= \frac{\check{q}_1 \check{p}_2^2 - \check{q}_2 \check{p}_1^2}{\check{q}_1 - \check{q}_2} - \hbar \frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} \\
&\quad + \check{q}_1 \check{q}_2 \left(\check{q}_1^4 + \check{q}_2^4 + \check{q}_1^3 \check{q}_2 + \check{q}_2^3 \check{q}_1 + \check{q}_1^2 \check{q}_2^2 + (\check{q}_1^2 + \check{q}_1 \check{q}_2 + \check{q}_2^2) \tau_{\infty,2} + (\check{q}_1 + \check{q}_2) \tau_{\infty,1} + \frac{\tau_{\infty,2}^2}{4} + 2t_{\infty(1),0} - \hbar \right), \\
H_{\infty,1} &= \frac{\check{p}_1^2 - \check{p}_2^2}{\check{q}_1 - \check{q}_2} - (\check{q}_1 + \check{q}_2) \left((\check{q}_1^2 + \check{q}_2^2 + \frac{1}{2} \tau_{\infty,2})^2 - \check{q}_1^2 \check{q}_2^2 + 2t_{\infty(1),0} - \hbar \right) - (\check{q}_1^2 + \check{q}_1 \check{q}_2 + \check{q}_2^2) \tau_{\infty,1}.
\end{aligned} \tag{8-72}$$

Coefficients $(\nu_{\infty,1}^{(\alpha_{\infty,1})}, \nu_{\infty,2}^{(\alpha_{\infty,1})}, \nu_{\infty,1}^{(\alpha_{\infty,2})}, \nu_{\infty,2}^{(\alpha_{\infty,2})})$ are determined by Proposition 4.2 that trivially gives

$$\nu_{\infty,1}^{(\alpha_{\infty,1})} = 0, \quad \nu_{\infty,2}^{(\alpha_{\infty,1})} = \frac{1}{2}, \quad \nu_{\infty,1}^{(\alpha_{\infty,2})} = \frac{1}{4}, \quad \nu_{\infty,2}^{(\alpha_{\infty,2})} = 0. \tag{8-73}$$

According to (7-18), it provides the Hamiltonians:

$$\begin{aligned}
\text{Ham}^{(\alpha_{\infty,1})}(\check{q}_1, \check{q}_2, \check{p}_1, \check{p}_2) &= \frac{1}{2} H_{\infty,1} \\
&= \frac{\check{p}_1^2 - \check{p}_2^2}{2(\check{q}_1 - \check{q}_2)} - \frac{(\check{q}_1 + \check{q}_2)((\check{q}_1^2 + \check{q}_2^2)^2 - \check{q}_1^2 \check{q}_2^2)}{2} - \frac{\check{q}_1^2 + \check{q}_1 \check{q}_2 + \check{q}_2^2}{2} \tau_{\infty,1} \\
&\quad - \frac{(\check{q}_1 + \check{q}_2)(\check{q}_1^2 + \check{q}_2^2)}{2} \tau_{\infty,2} - \frac{\check{q}_1 + \check{q}_2}{8} \tau_{\infty,2}^2 - \frac{(\check{q}_1 + \check{q}_2)(2t_{\infty(1),0} - \hbar)}{2} \\
\text{Ham}^{(\alpha_{\infty,2})}(\check{q}_1, \check{q}_2, \check{p}_1, \check{p}_2) &= \frac{1}{4} H_{\infty,0} \\
&= \frac{\check{q}_1 \check{p}_2^2 - \check{q}_2 \check{p}_1^2 - \hbar(\check{p}_1 - \check{p}_2)}{4(\check{q}_1 - \check{q}_2)} + \frac{(\check{q}_1^4 + \check{q}_1^3 \check{q}_2 + \check{q}_1^2 \check{q}_2^2 + \check{q}_1 \check{q}_2^3 + \check{q}_2^4) \check{q}_1 \check{q}_2}{4} \\
&\quad + \frac{(\check{q}_1 + \check{q}_2) \check{q}_1 \check{q}_2}{4} \tau_{\infty,1} + \frac{(\check{q}_1^2 + \check{q}_1 \check{q}_2 + \check{q}_2^2) \check{q}_1 \check{q}_2}{4} \tau_{\infty,2} + \frac{\check{q}_1 \check{q}_2}{16} \tau_{\infty,2}^2 + \frac{(2t_{\infty(1),0} - \hbar) \check{q}_1 \check{q}_2}{4}.
\end{aligned} \tag{8-74}$$

Finally coefficients $(\mu_1^{(\alpha_{\infty,1})}, \mu_2^{(\alpha_{\infty,1})}, \mu_1^{(\alpha_{\infty,2})}, \mu_2^{(\alpha_{\infty,2})})$ are determined by

$$\begin{pmatrix} 1 & 1 \\ \check{q}_1 & \check{q}_2 \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha_{\infty,1})} \\ \mu_2^{(\alpha_{\infty,1})} \end{pmatrix} = \begin{pmatrix} \nu_{\infty,1}^{(\alpha_{\infty,1})} \\ \nu_{\infty,2}^{(\alpha_{\infty,1})} \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ \check{q}_1 & \check{q}_2 \end{pmatrix} \begin{pmatrix} \mu_1^{(\alpha_{\infty,2})} \\ \mu_2^{(\alpha_{\infty,2})} \end{pmatrix} = \begin{pmatrix} \nu_{\infty,1}^{(\alpha_{\infty,2})} \\ \nu_{\infty,2}^{(\alpha_{\infty,2})} \end{pmatrix} \tag{8-75}$$

i.e.

$$\mu_1^{(\alpha_{\infty,1})} = \frac{1}{2(\check{q}_1 - \check{q}_2)}, \quad \mu_2^{(\alpha_{\infty,1})} = -\frac{1}{2(\check{q}_1 - \check{q}_2)}, \quad \mu_1^{(\alpha_{\infty,2})} = -\frac{\check{q}_2}{4(\check{q}_1 - \check{q}_2)}, \quad \mu_2^{(\alpha_{\infty,2})} = \frac{\check{q}_1}{4(\check{q}_1 - \check{q}_2)}. \tag{8-76}$$

The corresponding evolution equations are

$$\begin{aligned}
\hbar \partial_{\tau_{\infty,1}} \check{q}_1 &= \frac{\check{p}_1}{\check{q}_1 - \check{q}_2}, \\
\hbar \partial_{\tau_{\infty,1}} \check{q}_2 &= -\frac{\check{p}_2}{\check{q}_1 - \check{q}_2}, \\
\hbar \partial_{\tau_{\infty,1}} \check{p}_1 &= \frac{\check{p}_1^2 - \check{p}_2^2}{2(\check{q}_1 - \check{q}_2)^2} + \frac{1}{2} (5\check{q}_1^4 + 4\check{q}_1^3 \check{q}_2 + 3\check{q}_1^2 \check{q}_2^2 + 2\check{q}_1 \check{q}_2^3 + \check{q}_2^4) + \left(\check{q}_1 + \frac{\check{q}_2}{2} \right) \tau_{\infty,1} \\
&\quad + \frac{1}{2} (3\check{q}_1^2 + 2\check{q}_1 \check{q}_2 + \check{q}_2^2) \tau_{\infty,2} + \frac{1}{8} \tau_{\infty,2}^2 + t_{\infty(1),0} - \frac{\hbar}{2}, \\
\hbar \partial_{\tau_{\infty,1}} \check{p}_2 &= -\frac{\check{p}_1^2 - \check{p}_2^2}{2(\check{q}_1 - \check{q}_2)^2} + \frac{1}{2} (5\check{q}_2^4 + 4\check{q}_2^3 \check{q}_1 + 3\check{q}_2^2 \check{q}_1^2 + 2\check{q}_2 \check{q}_1^3 + \check{q}_1^4) + \left(\check{q}_2 + \frac{\check{q}_1}{2} \right) \tau_{\infty,1}
\end{aligned}$$

$$+\frac{1}{2} (3\check{q}_2^2 + 2\check{q}_2\check{q}_1 + \check{q}_1^2) \tau_{\infty,2} + \frac{1}{8} \tau_{\infty,2}^2 + t_{\infty^{(1)},0} - \frac{\hbar}{2} \quad (8-77)$$

and

$$\begin{aligned} \hbar \partial_{\tau_{\infty,2}} \check{q}_1 &= -\frac{\check{p}_1 \check{q}_2}{2(\check{q}_1 - \check{q}_2)} - \frac{\hbar}{4(\check{q}_1 - \check{q}_2)}, \\ \hbar \partial_{\tau_{\infty,2}} \check{q}_2 &= \frac{\check{p}_2 \check{q}_1}{2(\check{q}_1 - \check{q}_2)} + \frac{\hbar}{4(\check{q}_1 - \check{q}_2)}, \\ \hbar \partial_{\tau_{\infty,2}} \check{p}_1 &= -\frac{\check{q}_2(\check{p}_1^2 - \check{p}_2^2)}{4(\check{q}_1 - \check{q}_2)^2} - \frac{\hbar(\check{p}_1 - \check{p}_2)}{4(\check{q}_1 - \check{q}_2)^2} - \frac{\check{q}_2(5\check{q}_1^4 + 4\check{q}_1^3\check{q}_2 + 3\check{q}_1^2\check{q}_2^2 + 2\check{q}_1\check{q}_2^3 + \check{q}_2^4)}{4} \\ &\quad - \frac{\check{q}_2(2\check{q}_1 + \check{q}_2)}{4} \tau_{\infty,1} - \frac{\check{q}_2(3\check{q}_1^2 + 2\check{q}_1\check{q}_2 + \check{q}_2^2)}{4} \tau_{\infty,2} - \frac{\check{q}_2}{16} \tau_{\infty,2}^2 - \frac{\check{q}_2(2t_{\infty^{(1)},0} - \hbar)}{4}, \\ \hbar \partial_{\tau_{\infty,2}} \check{p}_2 &= \frac{\check{q}_1(\check{p}_1^2 - \check{p}_2^2)}{4(\check{q}_1 - \check{q}_2)^2} + \frac{\hbar(\check{p}_1 - \check{p}_2)}{4(\check{q}_1 - \check{q}_2)^2} - \frac{\check{q}_1(5\check{q}_2^4 + 4\check{q}_2^3\check{q}_1 + 3\check{q}_2^2\check{q}_1^2 + 2\check{q}_2\check{q}_1^3 + \check{q}_1^4)}{4} \\ &\quad - \frac{\check{q}_1(2\check{q}_2 + \check{q}_1)}{4} \tau_{\infty,1} - \frac{\check{q}_1(3\check{q}_2^2 + 2\check{q}_2\check{q}_1 + \check{q}_1^2)}{4} \tau_{\infty,2} - \frac{\check{q}_1}{16} \tau_{\infty,2}^2 - \frac{\check{q}_1(2t_{\infty^{(1)},0} - \hbar)}{4}. \end{aligned} \quad (8-78)$$

Moreover, we get the Lax matrices normalized at infinity according to (2-8) are

$$\begin{aligned} \tilde{L}_{1,1} &= -\lambda^3 + \left(\frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} + \check{q}_1^2 + \check{q}_2^2 + \check{q}_1\check{q}_2 \right) \lambda + \frac{\check{p}_2\check{q}_1 - \check{p}_1\check{q}_2}{\check{q}_1 - \check{q}_2} - \check{q}_1\check{q}_2(\check{q}_1 + \check{q}_2), \\ \tilde{L}_{1,2} &= (\lambda - \check{q}_1)(\lambda - \check{q}_2), \\ \tilde{L}_{2,2} &= -\tilde{L}_{1,1}, \\ \tilde{L}_{2,1} &= 2 \left(\frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} + \check{q}_1^2 + \check{q}_1\check{q}_2 + \check{q}_2^2 + \frac{\tau_{\infty,2}}{2} \right) \lambda^2 + 2 \left(\frac{\check{p}_1\check{q}_1 - \check{p}_2\check{q}_2}{\check{q}_1 - \check{q}_2} + (\check{q}_1 + \check{q}_2)(\check{q}_1^2 + \check{q}_2^2 + \frac{\tau_{\infty,2}}{2}) + \frac{\tau_{\infty,1}}{2} \right) \lambda \\ &\quad - \frac{(\check{p}_1 - \check{p}_2)^2}{(\check{q}_1 - \check{q}_2)^2} - \frac{2(\check{p}_1\check{q}_2 - \check{p}_2\check{q}_1)(\check{q}_1 + \check{q}_2)}{\check{q}_1 - \check{q}_2} + \check{q}_1^4 + \check{q}_2^4 - \check{q}_1^2\check{q}_2^2 + (\check{q}_1^2 + \check{q}_1\check{q}_2 + \check{q}_2^2)\tau_{\infty,2} \\ &\quad + (\check{q}_1 + \check{q}_2)\tau_{\infty,1} + \frac{\tau_{\infty,2}^2}{4} + 2t_{\infty^{(1)},0} \end{aligned} \quad (8-79)$$

and

$$\tilde{A}_{\alpha_{\tau_{\infty,1}}}(\lambda) = \begin{pmatrix} -\frac{\lambda + \check{q}_1 + \check{q}_2}{2} & \frac{1}{2} \\ \frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} + \check{q}_1^2 + \check{q}_1\check{q}_2 + \check{q}_2^2 + \frac{\tau_{\infty,2}}{2} & \frac{\lambda + \check{q}_1 + \check{q}_2}{2} \end{pmatrix} \quad (8-80)$$

as well as

$$\begin{aligned} [\tilde{A}_{\alpha_{\tau_{\infty,2}}}]_{1,1} &= -\frac{\lambda^2}{4} + \frac{\check{p}_1 - \check{p}_2}{4(\check{q}_1 - \check{q}_2)} + \frac{(\check{q}_1 + \check{q}_2)^2}{4}, \\ [\tilde{A}_{\alpha_{\tau_{\infty,2}}}]_{2,2} &= -[\tilde{A}_{\alpha_{\tau_{\infty,2}}}]_{1,1}, \\ [\tilde{A}_{\alpha_{\tau_{\infty,2}}}]_{1,2} &= \frac{\lambda - (\check{q}_1 + \check{q}_2)}{4}, \\ [\tilde{A}_{\alpha_{\tau_{\infty,2}}}]_{2,1} &= \frac{1}{2} \left(\frac{\check{p}_1 - \check{p}_2}{\check{q}_1 - \check{q}_2} + \check{q}_1^2 + \check{q}_1\check{q}_2 + \check{q}_2^2 + \frac{\tau_{\infty,2}}{2} \right) \lambda \\ &\quad + \frac{1}{2} \left(\frac{\check{p}_1\check{q}_1 - \check{p}_2\check{q}_2}{\check{q}_1 - \check{q}_2} + (\check{q}_1 + \check{q}_2)(\check{q}_1^2 + \check{q}_2^2 + \frac{\tau_{\infty,2}}{2}) + \frac{\tau_{\infty,1}}{2} \right). \end{aligned} \quad (8-81)$$

Note that these Lax matrices are the same as the one given by H. Chiba in [21] (Section 4.1, with the identification $t_1 = \frac{1}{2}\tau_{\infty,1}$, $t_2 = \frac{1}{4}\tau_{\infty,2}$, $W_1 = -(\check{q}_1 + \check{q}_2)$, $\tilde{W}_2 = \check{q}_1\check{q}_2 - t_2$, $\alpha_4 = 2t_{\infty^{(1)},0}$, $V_2 = \frac{2(\check{p}_1 - \check{p}_2)}{\check{q}_1 - \check{q}_2} + 2\check{q}_1^2 + \check{q}_1\check{q}_2 + 2\check{q}_2^2 + 4t_2$, $V_3 = \frac{2(\check{p}_1\check{q}_1 + \check{p}_2\check{q}_2)}{\check{q}_1 - \check{q}_2} + 2\check{q}_1^3 + 2\check{q}_1^2\check{q}_2 + 2\check{q}_1\check{q}_2^2 + 2\check{q}_2^3 + 4(\check{q}_1 + \check{q}_2)t_2 + 2t_1$).

Following the works of [21], it is natural to perform a symplectic change of variables

$$\begin{aligned}
Q_1 &= -(\check{q}_1 + \check{q}_2), \\
Q_2 &= \check{q}_1 \check{q}_2 - \frac{1}{4} \tau_{\infty,2}, \\
P_1 &= -\frac{(\check{p}_1 \check{q}_1 + \check{p}_2 \check{q}_2)}{\check{q}_1 - \check{q}_2} + \check{q}_1^3 + \check{q}_1^2 \check{q}_2 + \check{q}_1 \check{q}_2^2 + \check{q}_2^3 + \frac{1}{2}(\check{q}_1 + \check{q}_2) \tau_{\infty,2} + \frac{1}{2} \tau_{\infty,1}, \\
P_2 &= -\frac{(\check{p}_1 - \check{p}_2)}{\check{q}_1 - \check{q}_2} + \check{q}_1^2 + \frac{1}{2} \check{q}_1 \check{q}_2 + \check{q}_2^2 + \frac{1}{2} \tau_{\infty,2}
\end{aligned} \tag{8-82}$$

for which Hamiltonian evolutions are polynomial:

$$\begin{aligned}
\hbar \partial_{\tau_{\infty,1}} Q_1 &= P_2 - Q_1^2 + Q_2 - \frac{1}{4} \tau_{\infty,2}, \\
\hbar \partial_{\tau_{\infty,1}} Q_2 &= P_2 Q_1 - Q_1 Q_2 + P_1 - \frac{1}{4} \tau_{\infty,2} Q_1 - \frac{1}{2} \tau_{\infty,1}, \\
\hbar \partial_{\tau_{\infty,1}} P_1 &= -\frac{1}{2} P_2^2 + Q_2 P_2 + 2 P_1 Q_1 + \frac{1}{4} \tau_{\infty,2} P_2 - t_{\infty(1),0} + \hbar, \\
\hbar \partial_{\tau_{\infty,1}} P_2 &= P_2 Q_1 - P_1
\end{aligned} \tag{8-83}$$

whose Hamiltonian is

$$\begin{aligned}
\text{Ham}^{(\alpha_{\tau_{\infty,1}})} &= -Q_1 Q_2 P_2 - P_1 Q_1^2 + \frac{1}{2} P_2^2 Q_1 + P_1 P_2 + P_1 Q_2 - \frac{1}{4} \tau_{\infty,2} P_2 Q_1 - \frac{1}{4} \tau_{\infty,2} P_1 \\
&\quad - \frac{1}{2} \tau_{\infty,1} P_2 + (t_{\infty(1),0} - \hbar) Q_1
\end{aligned} \tag{8-84}$$

and

$$\begin{aligned}
\hbar \partial_{\tau_{\infty,2}} Q_1 &= \frac{1}{2} P_2 Q_1 - \frac{1}{2} Q_1 Q_2 - \frac{1}{8} \tau_{\infty,2} Q_1 + \frac{1}{2} P_1 - \frac{1}{4} \tau_{\infty,1}, \\
\hbar \partial_{\tau_{\infty,2}} Q_2 &= \frac{1}{2} Q_1^2 P_2 - \frac{1}{2} Q_2 P_2 + \frac{1}{2} P_1 Q_1 - \frac{1}{2} Q_2^2 - \frac{1}{8} \tau_{\infty,2} P_2 - \frac{1}{4} \tau_{\infty,1} Q_1 + \frac{1}{32} \tau_{\infty,2}^2, \\
\hbar \partial_{\tau_{\infty,2}} P_1 &= -\frac{1}{2} P_2^2 Q_1 - \frac{1}{2} P_1 P_2 + \frac{1}{2} Q_2 P_1 + \frac{1}{8} \tau_{\infty,2} P_1 + \frac{1}{4} \tau_{\infty,1} P_2, \\
\hbar \partial_{\tau_{\infty,2}} P_2 &= \frac{1}{4} P_2^2 + P_2 Q_2 + \frac{1}{2} P_1 Q_1 - \frac{1}{2} (t_{\infty(1),0} - \hbar)
\end{aligned} \tag{8-85}$$

whose Hamiltonian is

$$\begin{aligned}
\text{Ham}^{(\alpha_{\tau_{\infty,2}})} &= \frac{1}{4} (Q_1^2 - Q_2 - \frac{1}{4} \tau_{\infty,2}) P_2^2 + \frac{1}{2} (P_1 Q_1 - \frac{1}{2} \tau_{\infty,1} Q_1 - Q_2^2 + \frac{1}{16} \tau_{\infty,2}) P_2 \\
&\quad + \frac{1}{4} P_1^2 - \frac{1}{2} (Q_1 Q_2 + \frac{1}{4} \tau_{\infty,2} Q_1 + \frac{1}{2} \tau_{\infty,1}) P_1 + \frac{1}{2} (t_{\infty(1),0} - \hbar) Q_2.
\end{aligned} \tag{8-86}$$

Note in particular that Q_1 satisfies the ODE

$$\begin{aligned}
0 &= 8 Q_1^2 \frac{\partial^4 Q_1}{\partial \tau_{\infty,1}^4} - 16 Q_1 \frac{\partial Q_1}{\partial \tau_{\infty,1}} \frac{\partial^3 Q_1}{\partial \tau_{\infty,1}^3} - 12 Q_1 \left(\frac{\partial^2 Q_1}{\partial \tau_{\infty,1}^2} \right)^2 + 16 \left(\frac{\partial Q_1}{\partial \tau_{\infty,1}} \right)^2 \frac{\partial^2 Q_1}{\partial \tau_{\infty,1}^2} \\
&\quad - 8 Q_1 (\tau_{\infty,1} + 5 Q_1^3) \frac{\partial^2 Q_1}{\partial \tau_{\infty,1}^2} + (8 \tau_{\infty,1} - 20 Q_1^3) \left(\frac{\partial Q_1}{\partial \tau_{\infty,1}} \right)^2 - 8 Q_1 \frac{\partial Q_1}{\partial \tau_{\infty,1}} \\
&\quad + 20 Q_1^7 + 12 \tau_{\infty,2} Q_1^5 + 8 \tau_{\infty,1} Q_1^4 + (\tau_{\infty,2}^2 - 24 t_{\infty(1),0} + 12 \hbar) Q_1^3 - \tau_{\infty,1}^2 Q_1
\end{aligned} \tag{8-87}$$

that automatically verifies the Painlevé property by construction.

9 Outlooks

In this article, we proved that the evolutions of Darboux coordinates relatively to isomonodromic deformations of $\mathfrak{gl}_2(\mathbb{C})$ meromorphic connections are Hamiltonian and provided explicit expressions for these evolutions and Hamiltonians. In addition, we presented a split of the tangent space into trivial and non-trivial deformations with their associated dual trivial and non-trivial times. Using a canonical choice of trivial times, we simplified the general Hamiltonian systems in order to obtain a standard symplectic space of dimension $2g$ and the corresponding Lax pairs. The method developed in the present article opens the way to several generalizations.

- In this article, we assumed that all coefficients $\hat{L}^{[p,k]}$ had distinct eigenvalues. A natural problem is thus to extend the present setup in the case where such assumption is released corresponding to twisted connections. In particular, the Painlevé 1 case and the Painlevé 1 hierarchy would be natural cases to study and results of [55] indicate that the present construction can be adapted to these twisted cases.
- In this article, we restricted to the case of meromorphic connections in $\mathfrak{gl}_2(\mathbb{C})$ so a natural question is to know if the present setup extends to $\mathfrak{gl}_d(\mathbb{C})$ with $d \geq 2$. In principle, a similar strategy shall be used but it is unclear if all technical issues might be overcome when the rank is arbitrary. We let this very interesting question for future works. In particular, as discussed in the introduction a possible strategy could be to make the connection with the works of Yamakawa [72, 73] more explicit and, since they are valid in $\mathfrak{gl}_d$, to use them to generalise the present article. Let us also mention that one may even generalize the present work to the case of any connected reductive group G over the complex plane admitting a distinguished Borel subgroup which is the natural framework for opers.
- As mentioned in the introduction, the isospectral approach [72, 7] and the isomonodromic approach using apparent singularities as Darboux coordinates have recently been merged in [56]. In particular, [56] complements the present article with the explicit expression of the Lax matrices $(\tilde{L}(\lambda), \tilde{A}_\alpha(\lambda))$ using those of $(L(\lambda), A_\alpha(\lambda))$ and the explicit gauge transformation derived in this paper. Moreover, it provides the change of coordinates between the set of Darboux coordinates $(q_i, p_i)_{1 \leq i \leq g}$ used in this paper and the set of “isospectral Darboux coordinates” for which isospectral invariants identify with the isomonodromic Hamiltonians. Using this change of coordinates one should thus be able to identify the Hamiltonians and the fundamental two-form defined by Yamakawa [72, 73] with the ones obtained in this article. This observation may also open the way to the generalization of the present article to $\mathfrak{gl}_d(\mathbb{C})$ and would definitely deserve additional investigations.

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A Proof of Proposition 2.1

The proof of Proposition 2.1 consists in observing that

$$\tilde{G}(\lambda) = (G_1(\lambda, \hbar)J(\lambda, \hbar))^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{-Q(\lambda) - (t_{\infty(1), r_{\infty}-1} \lambda + \eta_0) \prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} & \frac{\prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \end{pmatrix} \quad (\text{A-1})$$

recovers the matrix (2-30). Indeed, we first have that $\tilde{G}_{2,2}(\lambda)$ is rational in λ with poles in $\mathcal{R}$ of order compatible with $F_{\mathcal{R}, \mathbf{r}}$. Moreover at $\lambda \rightarrow \infty$ it behaves like $\tilde{G}_{2,2}(\lambda) = \lambda^{r_{\infty}-3} + O(\lambda^{r_{\infty}-4})$. Finally its zeros are precisely given by $(q_i)_{1 \leq i \leq g}$ as required for (2-22). Hence $\tilde{G}_{2,2}(\lambda) = \tilde{L}_{1,2}(\lambda)$. Similarly, the entry $\tilde{G}_{2,1}(\lambda)$ is rational in λ with poles only in $\mathcal{R}$. The orders at finite poles are compatible with $F_{\mathcal{R}, \mathbf{r}}$. Moreover, it satisfies $\tilde{G}_{2,1}(q_i) = p_i$ for all $i \in \llbracket 1, g \rrbracket$ as required for (2-23). At infinity, we get that

$$\tilde{G}_{2,1}(\lambda) = -t_{\infty(1), r_{\infty}-1} \lambda^{r_{\infty}-2} - \left(\eta_0 - t_{\infty(1), r_{\infty}-1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) \right) \lambda^{r_{\infty}-3} + O(\lambda^{r_{\infty}-4}) \quad (\text{A-2})$$

so that the normalization of $\hat{F}_{\mathcal{R}, \mathbf{r}}$ and Remark 2.4 is verified by taking for $r_{\infty} \geq 2$

$$\eta_0 = t_{\infty(1), r_{\infty}-2} + t_{\infty(1), r_{\infty}-1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right). \quad (\text{A-3})$$

Similarly, for $r_{\infty} = 1$, we have that

$$\left[\tilde{L}(\lambda) \right]_{1,1} = -t_{\infty(1), 0} \lambda^{-1} + \left(\sum_{s=1}^n \left[X_s \tilde{L}^{[X_s, 0]} + \tilde{L}^{[X_s, 1]} \right]_{1,1} \right) \lambda^{-2} + O(\lambda^{-3}) \quad (\text{A-4})$$

so that we should take

$$\eta_0 = t_{\infty(1), 0} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) - \sum_{s=1}^n \left[X_s \tilde{L}^{[X_s, 0]} + \tilde{L}^{[X_s, 1]} \right]_{1,1}. \quad (\text{A-5})$$

Thus, $\tilde{G}_{2,1}(\lambda) = \tilde{L}_{1,1}(\lambda)$ ending the proof.

B Proof of Proposition 2.3

Only the case of infinity is non-trivial and shall be detailed. As mentioned in equation (2-4) there exists a gauge transformation G_{∞} holomorphic at $\lambda = \infty$ such that $\Psi_{\infty} = G_{\infty} \tilde{\Psi}$ with Ψ_{∞} of the form given by (2-4). Isolating the singular part of Ψ_{∞} at infinity provides the existence of a matrix $R_{\infty}^{(\text{reg})} = G_{\infty}^{-1} \Psi_{\infty}^{(\text{reg})}$ regular at infinity such that

$$\tilde{\Psi} = R_{\infty}^{(\text{reg})} \begin{pmatrix} \exp \left(- \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty(1), k}}{k} \lambda^k - t_{\infty(1), 0} \ln \lambda \right) & 0 \\ 0 & \exp \left(- \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty(2), k}}{k} \lambda^k - t_{\infty(2), 0} \ln \lambda \right) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{bmatrix} R_{\infty}^{(\text{reg})} \end{bmatrix}_{1,1} \exp \left(- \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - t_{\infty^{(1)},0} \ln \lambda \right) & \begin{bmatrix} R_{\infty}^{(\text{reg})} \end{bmatrix}_{1,2} \exp \left(- \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - t_{\infty^{(2)},0} \ln \lambda \right) \\ \begin{bmatrix} R_{\infty}^{(\text{reg})} \end{bmatrix}_{2,1} \exp \left(- \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - t_{\infty^{(1)},0} \ln \lambda \right) & \begin{bmatrix} R_{\infty}^{(\text{reg})} \end{bmatrix}_{2,2} \exp \left(- \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - t_{\infty^{(2)},0} \ln \lambda \right) \end{pmatrix}. \quad (\text{B-1})$$

The first line is compatible with the fact that $\tilde{\Psi}_{1,1} = \Psi_{1,1} = \psi_1$ and $\tilde{\Psi}_{1,2} = \Psi_{1,2} = \psi_2$. Moreover, since Ψ is solution to a companion-like system we have also

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \hbar \partial_{\lambda} \psi_1 & \hbar \partial_{\lambda} \psi_2 \end{pmatrix}. \quad (\text{B-2})$$

In particular (2-51) implies that

$$\begin{aligned} \Psi_{2,1}(\lambda) &= \left(- \sum_{k=1}^{r_{\infty}-1} t_{\infty^{(1)},k} \lambda^k - \frac{t_{\infty^{(1)},0}}{\lambda} \right) \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(1)},0} \ln \lambda + A_{\infty^{(1)},0} + O(\lambda^{-1}) \right), \\ \Psi_{2,2}(\lambda) &= \left(- \sum_{k=1}^{r_{\infty}-1} t_{\infty^{(2)},k} \lambda^k - \frac{t_{\infty^{(2)},0} + \hbar}{\lambda} \right) \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(2)},0} \ln \lambda - \ln \lambda + A_{\infty^{(2)},0} + O(\lambda^{-1}) \right), \end{aligned} \quad (\text{B-3})$$

so that $\tilde{\Psi} = G_1 J \Psi$ should behave like

$$\begin{aligned} \tilde{\Psi}_{2,1}(\lambda) &= ((t_{\infty^{(1)},r_{\infty}-1} - t_{\infty^{(1)},r_{\infty}-1})\lambda + O(1)) \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(1)},0} \ln \lambda + O(\lambda^{-1}) \right) \\ &= O(1) \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(1)},0} \ln \lambda + O(\lambda^{-1}) \right), \\ \tilde{\Psi}_{2,2}(\lambda) &= ((t_{\infty^{(2)},r_{\infty}-1} - t_{\infty^{(1)},r_{\infty}-1})\lambda + O(1)) \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(2)},0} \ln \lambda - \ln \lambda + O(\lambda^{-1}) \right) \\ &= O(1) \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - \frac{1}{\hbar} t_{\infty^{(2)},0} \ln \lambda + O(\lambda^{-1}) \right), \end{aligned} \quad (\text{B-4})$$

in accordance with (B-1).

C Proof of Proposition 2.4

Let us recall that the wave functions Ψ_1 and Ψ_2 are such that for all $s \in \llbracket 1, n \rrbracket$:

$$\begin{aligned} \Psi_1(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(1)},k}}{k} \lambda^k - \frac{t_{\infty^{(1)},0}}{\hbar} \ln \lambda + A_{\infty^{(1)},0} + O(\lambda^{-1}) \right) \\ \Psi_2(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty^{(2)},k}}{k} \lambda^k - \frac{t_{\infty^{(2)},0}}{\hbar} \ln \lambda - \ln \lambda + A_{\infty^{(2)},0} + O(\lambda^{-1}) \right) \\ \Psi_1(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_s-1} \frac{t_{X_s^{(1)},k}}{k} (\lambda - X_s)^{-k} + \frac{t_{X_s^{(1)},0}}{\hbar} \ln(\lambda - X_s) + A_{X_s^{(1)},0} + O(\lambda - X_s) \right) \\ \Psi_2(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} \exp \left(- \frac{1}{\hbar} \sum_{k=1}^{r_s-1} \frac{t_{X_s^{(2)},k}}{k} (\lambda - X_s)^{-k} + \frac{t_{X_s^{(2)},0}}{\hbar} \ln(\lambda - X_s) + A_{X_s^{(2)},0} + O(\lambda - X_s) \right). \end{aligned} \quad (\text{C-1})$$

We consider the Wronskian $W(\lambda) = \hbar(\Psi_1 \partial_{\lambda} \Psi_2 - \Psi_2 \partial_{\lambda} \Psi_1)$. It is well-known that W is a rational function of λ with only possible poles at $\lambda \in \mathcal{R}$. From the behavior of the wave functions

at each pole we get that:

$$\begin{aligned}
W(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \left(\sum_{k=1}^{r_\infty-1} (t_{\infty^{(1)},k} - t_{\infty^{(2)},k}) \lambda^{k-1} + \frac{(t_{\infty^{(1)},0} - t_{\infty^{(2)},0} - \hbar)}{\lambda} + O(\lambda^{-2}) \right) \lambda^{-1} \\
&\exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_\infty-1} \frac{(t_{\infty^{(1)},k} + t_{\infty^{(2)},k})}{k} \lambda^k - \frac{(t_{\infty^{(1)},0} + t_{\infty^{(2)},0})}{\hbar} \ln \lambda + O(1) \right) \\
&\stackrel{\lambda \rightarrow \infty}{=} \left(\sum_{k=1}^{r_\infty-1} (t_{\infty^{(1)},k} - t_{\infty^{(2)},k}) \lambda^k + (t_{\infty^{(1)},0} - t_{\infty^{(2)},0} - \hbar) \right) \lambda^{-1} \\
&\exp \left(\frac{1}{\hbar} \int_0^\lambda \sum_{j=0}^{r_\infty-2} P_{\infty,j}^{(1)} \tilde{\lambda}^j d\tilde{\lambda} - \frac{(t_{\infty^{(1)},0} + t_{\infty^{(2)},0})}{\hbar} \ln \lambda \right) \kappa_\infty (1 + O(\lambda^{-1}))
\end{aligned} \tag{C-2}$$

and for all $s \in \llbracket 1, n \rrbracket$:

$$\begin{aligned}
W(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} - \left(\sum_{k=1}^{r_s-1} (t_{X_s^{(1)},k} - t_{X_s^{(2)},k}) (\lambda - X_s)^{-k-1} + \frac{t_{X_s^{(1)},0} - t_{X_s^{(2)},0}}{(\lambda - X_s)} + O((\lambda - X_s)^{-2}) \right) \\
&\exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_s-1} \frac{(t_{X_s^{(1)},k} + t_{X_s^{(2)},k})}{k} (\lambda - X_s)^{-k} + \frac{t_{X_s^{(1)},0} + t_{X_s^{(2)},0}}{\hbar} \ln(\lambda - X_s) + O(1) \right) \\
&\stackrel{\lambda \rightarrow X_s}{=} - \sum_{k=0}^{r_s-1} (t_{X_s^{(1)},k} - t_{X_s^{(2)},k}) (\lambda - X_s)^{-k-1} \exp \left(\frac{1}{\hbar} \int_0^\lambda \sum_{j=1}^{r_s} \frac{P_{X_s,j}^{(1)}}{(\tilde{\lambda} - X_s)^j} d\tilde{\lambda} \right) \kappa_s ((1 + O(\lambda - X_s))).
\end{aligned} \tag{C-3}$$

Since the previous asymptotic expansions imply that W admits a pole at infinity of order $\lambda^{r_\infty-3}$ and a pole at X_s of order r_s , we finally get:

$$W(\lambda) = \kappa \frac{\prod_{i=1}^g (\lambda - q_i)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \exp \left(\frac{1}{\hbar} \int_0^\lambda P_1(\tilde{\lambda}) d\tilde{\lambda} \right) \tag{C-4}$$

where we have denoted $(q_j)_{1 \leq j \leq g}$ the zeros of W and an unknown constant κ . Note that the additional condition (3-5) is necessary for the previous formula to hold around $\lambda \rightarrow \infty$.

Since L is the companion matrix attached to Ψ_1 and Ψ_2 , it is a straightforward computation to show that

$$\begin{aligned}
L_{2,2}(\lambda) &= \hbar \frac{\partial_\lambda W(\lambda)}{W(\lambda)}, \\
L_{2,1}(\lambda) &= -Y_1(\lambda) Y_2(\lambda) + \hbar \frac{Y_2(\lambda) \partial_\lambda Y_1(\lambda) - Y_1(\lambda) \partial_\lambda Y_2(\lambda)}{Y_2(\lambda) - Y_1(\lambda)},
\end{aligned} \tag{C-5}$$

where we have defined $Y_i(\lambda) = \hbar \frac{1}{\Psi_i(\lambda)} \frac{\partial \Psi_i(\lambda)}{\partial \lambda}$ for $i \in \{1, 2\}$. Using the relation between $L_{2,2}$ and the Wronskian, we get:

$$L_{2,2}(\lambda) = P_1(\lambda) + \sum_{j=1}^g \frac{\hbar}{\lambda - q_j} - \sum_{s=1}^n \frac{\hbar r_s}{\lambda - X_s}. \tag{C-6}$$

Then, using (C-1) we have for $r_\infty \geq 3$:

$$Y_1(\lambda) \stackrel{\lambda \rightarrow \infty}{=} - \sum_{k=1}^{r_\infty-1} t_{\infty^{(1)},k} \lambda^{k-1} - \frac{t_{\infty^{(1)},0}}{\lambda} + O(\lambda^{-2}),$$

$$\begin{aligned}
Y_2(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} - \sum_{k=1}^{r_\infty-1} t_{\infty^{(2)},k} \lambda^{k-1} - \frac{t_{\infty^{(2)},0} + \hbar}{\lambda} + O(\lambda^{-2}), \\
Y_i(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} \sum_{k=1}^{r_s-1} t_{X_s^{(i)},k} (\lambda - X_s)^{-k-1} + \frac{t_{X_s^{(i)},0}}{\lambda - X_s} + O((\lambda - X_s)^{-2}),
\end{aligned} \tag{C-7}$$

so that we get:

$$\begin{aligned}
\frac{Y_2(\lambda) \partial_\lambda Y_1(\lambda) - Y_1(\lambda) \partial_\lambda Y_2(\lambda)}{Y_2(\lambda) - Y_1(\lambda)} &\stackrel{\lambda \rightarrow \infty}{=} O(\lambda^{r_\infty-4}), \\
\frac{Y_2(\lambda) \partial_\lambda Y_1(\lambda) - Y_1(\lambda) \partial_\lambda Y_2(\lambda)}{Y_2(\lambda) - Y_1(\lambda)} &\stackrel{\lambda \rightarrow X_s}{=} O((\lambda - X_s)^{-r_s}),
\end{aligned} \tag{C-8}$$

from which we immediately obtain using (2-52) that

$$\begin{aligned}
L_{2,1}(\lambda) &\stackrel{\lambda \rightarrow \infty}{=} \sum_{j=r_\infty-3}^{2r_\infty-4} P_{\infty,j}^{(2)} \lambda^j - \hbar t_{\infty^{(1)},r_\infty-1} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}), \\
L_{2,1}(\lambda) &\stackrel{\lambda \rightarrow X_s}{=} \sum_{j=r_s+1}^{2r_s} P_{X_s,j}^{(2)} (\lambda - X_s)^{-j} + O((\lambda - X_s)^{-r_s}), \\
L_{2,1}(\lambda) &\stackrel{\lambda \rightarrow q_j}{=} O(\lambda - q_j)^{-1}.
\end{aligned} \tag{C-9}$$

Remark C.1. For $r_\infty = 1$, similar computations lead to

$$L_{2,1}(\lambda) \stackrel{\lambda \rightarrow \infty}{=} -\frac{t_{\infty^{(1)},0}(t_{\infty^{(2)},0} + \hbar)}{\lambda^2} + O(\lambda^{-3}) \tag{C-10}$$

while for $r_\infty = 2$ we obtain:

$$L_{2,1}(\lambda) \stackrel{\lambda \rightarrow \infty}{=} -t_{\infty^{(1)},1} t_{\infty^{(2)},1} - \frac{t_{\infty^{(1)},1} t_{\infty^{(2)},0} + t_{\infty^{(2)},1} t_{\infty^{(1)},0} + \hbar t_{\infty^{(1)},1}}{\lambda} + O(\lambda^{-2}). \tag{C-11}$$

Combining the asymptotic behavior at each pole and the fact that L is rational with poles only in $\mathcal{R}$ and simple poles at $\lambda \in \{q_1, \dots, q_g\}$, we get Proposition 2.4.

D Proof of Proposition 2.6

We first observe that from (2-44):

$$\begin{aligned}
\check{L}_{1,1}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \check{L}_{1,1}^{(0)} \lambda^{r_\infty-4} + O(\lambda^{r_\infty-5}), \\
\check{L}_{1,2}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \lambda^{r_\infty-3} + \check{L}_{1,2}^{(1)} \lambda^{r_\infty-4} + O(\lambda^{r_\infty-5}), \\
\check{L}_{2,1}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \check{L}_{2,1}^{(0)} \lambda^{r_\infty-1} + \check{L}_{2,1}^{(1)} \lambda^{r_\infty-2} + O(\lambda^{r_\infty-3}), \\
\check{L}_{2,2}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \check{L}_{2,2}^{(0)} \lambda^{r_\infty-2} + \check{L}_{2,2}^{(1)} \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}).
\end{aligned} \tag{D-1}$$

These asymptotics are valid for any value of $r_\infty \geq 1$. Note also that the terms $\check{L}_{2,1}^{(0)}$ and $\check{L}_{2,1}^{(1)}$ of the entries $\check{L}_{2,1}$ are only determined by the $L_{2,1}(\lambda, \hbar) \frac{\prod_{s=1}^n (\lambda - X_s)^{r_s}}{\prod_{j=1}^g (\lambda - q_j)}$ since the other terms are all of

lower orders when $\lambda \rightarrow \infty$. It gives, using the fact that $\check{L}_{1,2}^{(1)} = \frac{\prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}}$ that:

$$\begin{aligned}\check{L}_{2,1}^{(0)} &= L_{2,1}^{(0)}, \\ \check{L}_{2,1}^{(1)} &= L_{2,1}^{(1)} - L_{2,1}^{(0)} \check{L}_{1,2}^{(1)}\end{aligned}\tag{D-2}$$

where we have denoted

$$L_{2,1}(\lambda, \hbar) \stackrel{\lambda \rightarrow \infty}{=} L_{2,1}^{(0)} \lambda^{2r_\infty - 4} + L_{2,1}^{(1)} \lambda^{2r_\infty - 5} + O(\lambda^{2r_\infty - 6}).\tag{D-3}$$

From the fact that $\check{L}_{1,2}^{(1)} = \frac{\prod_{j=1}^g (\lambda - q_j)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}}$, it is easy to obtain:

$$\check{L}_{1,2}^{(1)} = \sum_{s=1}^n r_s X_s - \sum_{j=1}^g q_j.\tag{D-4}$$

We perform the following gauge transformation:

$$\tilde{\Psi}(\lambda, \hbar) = G(\lambda, \hbar) \check{\Psi}(\lambda, \hbar) \quad \text{with } G(\lambda, \hbar) = \begin{pmatrix} 1 & 0 \\ \eta_1 \lambda + \eta_0 & 1 \end{pmatrix}.\tag{D-5}$$

It satisfies:

$$\hbar \partial_\lambda \tilde{\Psi}(\lambda, \hbar) = \tilde{L}(\lambda, \hbar) \tilde{\Psi}(\lambda, \hbar),\tag{D-6}$$

with

$$\begin{aligned}\tilde{L}(\lambda, \hbar) &= G(\lambda, \hbar) \check{L}(\lambda, \hbar) G^{-1}(\lambda, \hbar) + \hbar (\partial_\lambda G(\lambda, \hbar)) G^{-1}(\lambda, \hbar) \\ &= \begin{pmatrix} \check{L}_{1,1}(\lambda, \hbar) + (\eta_1 \lambda + \eta_0) \check{L}_{1,2}(\lambda, \hbar) & \check{L}_{1,2}(\lambda, \hbar) \\ (\eta_1 \lambda + \eta_0)^2 \check{L}_{1,2}(\lambda, \hbar) + \check{L}_{2,1}(\lambda, \hbar) - (\eta_1 \lambda + \eta_0) (\check{L}_{1,1}(\lambda, \hbar) - \check{L}_{2,2}(\lambda, \hbar)) + \hbar & \check{L}_{2,2}(\lambda, \hbar) - (\eta_1 \lambda + \eta_0) \check{L}_{1,2}(\lambda, \hbar) \end{pmatrix}.\end{aligned}\tag{D-7}$$

Thus, it is clear that the conditions

$$\tilde{L}(\lambda, \hbar) \stackrel{\lambda \rightarrow \infty}{=} \text{diag}(X_1, X_2) \lambda^{r_\infty - 3} + \begin{pmatrix} X & 1 \\ X & X \end{pmatrix} \lambda^{r_\infty - 4} + O(\lambda^{r_\infty - 5})\tag{D-8}$$

are equivalent to (note that entry $\check{L}_{1,2}(\lambda, \hbar)$ is already properly normalized at subleading order at infinity)

$$\begin{aligned}0 &= \check{L}_{2,1}^{(0)} - \eta_1 (\eta_1 + \check{L}_{2,2}^{(0)}) + \hbar \eta_1 \delta_{r_\infty=1}, \\ 0 &= \check{L}_{2,1}^{(1)} - (\eta_1 + \check{L}_{2,2}^{(0)}) \eta_0 - (\check{L}_{2,1}^{(1)} \eta_1 + \eta_0 + \check{L}_{2,2}^{(1)}) \eta_1 + \hbar \eta_1 \delta_{r_\infty=2}.\end{aligned}\tag{D-9}$$

Using (D-2), this is equivalent to say that the coefficients (η_1, η_0) are completely determined by $(L_{2,1}^{(0)}, L_{2,1}^{(1)}, \check{L}_{2,2}^{(0)}, \check{L}_{2,2}^{(1)})$. Since these coefficients depend on the value of r_∞ , we need to split the computations between the three cases $r_\infty = 1$, $r_\infty = 2$ and $r_\infty \geq 3$.

D.1 The case $r_\infty \geq 3$

Let us assume that $r_\infty \geq 3$. In this case the two leading orders at infinity of $\check{L}_{2,2}(\lambda, \hbar)$ are given by $P_1(\lambda)$. Thus, we get:

$$\begin{aligned}\check{L}_{2,2}^{(0)} &= P_{\infty, r_\infty-2}^{(1)} = -(t_{\infty(1), r_\infty-1} + t_{\infty(2), r_\infty-1}), \\ \check{L}_{2,2}^{(1)} &= P_{\infty, r_\infty-3}^{(1)} = -(t_{\infty(1), r_\infty-2} + t_{\infty(2), r_\infty-2}), \\ L_{2,1}^{(0)} &= -P_{\infty, 2r_\infty-4}^{(2)} = -t_{\infty(1), r_\infty-1} t_{\infty(2), r_\infty-1}, \\ L_{2,1}^{(1)} &= -P_{\infty, 2r_\infty-5}^{(2)} = -(t_{\infty(1), r_\infty-1} t_{\infty(2), r_\infty-2} + t_{\infty(2), r_\infty-1} t_{\infty(1), r_\infty-2}).\end{aligned}\tag{D-10}$$

Thus, conditions (D-9) are equivalent to two sets of solutions (η_1, η_0) given by

$$\begin{aligned}\eta_1 &= t_{\infty(1), r_\infty-1}, \\ \eta_0 &= t_{\infty(1), r_\infty-2} + t_{\infty(1), r_\infty-1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right), \\ \check{L}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \text{diag}(-t_{\infty(1), r_\infty-1}, -t_{\infty(2), r_\infty-1}) \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}),\end{aligned}\tag{D-11}$$

or

$$\begin{aligned}\eta_1 &= t_{\infty(2), r_\infty-1}, \\ \eta_0 &= t_{\infty(2), r_\infty-2} + t_{\infty(2), r_\infty-1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right), \\ \check{L}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \text{diag}(-t_{\infty(2), r_\infty-1}, -t_{\infty(1), r_\infty-1}) \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}),\end{aligned}\tag{D-12}$$

but whose asymptotics does not fit the one required for $\check{L}$ at infinity so that only (D-11) is valid.

D.2 The case $r_\infty = 2$

Let us assume that $r_\infty = 2$. In this case, the leading orders at infinity of $L_{2,1}(\lambda, \hbar)$ are given by Remark C.1. Moreover, the leading order of $\check{L}_{2,2}(\lambda, \hbar)$ are given by (2-52) with the additional residue condition (3-5).

$$\begin{aligned}\check{L}_{2,2}^{(0)} &= P_{\infty, 0}^{(1)} = -(t_{\infty(1), 1} + t_{\infty(2), 1}), \\ \check{L}_{2,2}^{(1)} &= -(t_{\infty(1), 0} + t_{\infty(2), 0}), \\ L_{2,1}^{(0)} &= -P_{\infty, 0}^{(2)} = -t_{\infty(1), 1} t_{\infty(2), 1}, \\ L_{2,1}^{(1)} &= -(t_{\infty(1), 1} t_{\infty(2), 0} + t_{\infty(2), 1} t_{\infty(1), 0} + \hbar t_{\infty(1), 1}).\end{aligned}\tag{D-13}$$

Thus, conditions (D-9) are equivalent to two sets of solutions (η_1, η_0) given by

$$\begin{aligned}\eta_1 &= t_{\infty(1), 1}, \\ \eta_0 &= t_{\infty(1), 0} + t_{\infty(1), 1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right), \\ \check{L}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \text{diag}(-t_{\infty(1), 1}, -t_{\infty(2), 1}) \lambda^{r_\infty-3} + O(\lambda^{r_\infty-4}),\end{aligned}\tag{D-14}$$

or

$$\begin{aligned}\eta_1 &= t_{\infty(2),1}, \\ \eta_0 &= t_{\infty(2),0} + \hbar + t_{\infty(2),1} \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right), \\ \tilde{L}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \text{diag}(-t_{\infty(2),1}, -t_{\infty(1),1}) \lambda^{r_\infty - 3} + O(\lambda^{r_\infty - 4}),\end{aligned}\tag{D-15}$$

but whose asymptotics does not fit the one required for $\tilde{L}$ at infinity so that only (D-14) is valid.

D.3 The case $r_\infty = 1$

Let us assume that $r_\infty = 1$. In this case, the situation is more complicated to obtain the sub-leading orders of $L_{2,1}(\lambda, \hbar)$ and $\tilde{L}_{2,2}(\lambda, \hbar)$ at infinity. We have:

$$\tilde{L}_{2,2}(\lambda, \hbar) = P_1(\lambda) + O(\lambda^{-3})\tag{D-16}$$

with P_1 given by (2-45):

$$P_1(\lambda) = \sum_{s=1}^n \sum_{j=1}^{r_s} \frac{P_{X_s,j}^{(1)}}{(\lambda - X_s)^j} \stackrel{\lambda \rightarrow \infty}{=} \frac{\sum_{s=1}^n P_{X_s,1}^{(1)}}{\lambda} + \frac{\sum_{s=1}^n (P_{X_s,2}^{(1)} \delta_{r_s \geq 2} + X_s P_{X_s,1}^{(1)})}{\lambda^2} + O(\lambda^{-3}).\tag{D-17}$$

The residue condition (3-5) implies that $\sum_{s=1}^n P_{X_s,1}^{(1)} = -(t_{\infty(1),0} + t_{\infty(2),0})$ but the next term cannot be simplified:

$$\sum_{s=1}^n (P_{X_s,2}^{(1)} \delta_{r_s \geq 2} + X_s P_{X_s,1}^{(1)}) = \sum_{s=1}^n ((t_{X_s^{(1)},1} + t_{X_s^{(2)},1}) \delta_{r_s \geq 2} + X_s (t_{X_s^{(1)},0} + t_{X_s^{(2)},0})).\tag{D-18}$$

Thus, we finally get:

$$\begin{aligned}\tilde{L}_{2,2}^{(0)} &= -(t_{\infty(1),0} + t_{\infty(2),0}), \\ \tilde{L}_{2,2}^{(1)} &= \sum_{s=1}^n ((t_{X_s^{(1)},1} + t_{X_s^{(2)},1}) \delta_{r_s \geq 2} + X_s (t_{X_s^{(1)},0} + t_{X_s^{(2)},0})).\end{aligned}\tag{D-19}$$

Similarly, the leading order at infinity of the asymptotic expansion of $L_{2,1}(\lambda, \hbar)$ is given by Remark C.1 while the sub-leading order has to be computed through the definition given in Proposition 2.4:

$$L_{2,1}^{(0)} = -t_{\infty(1),0}(t_{\infty(2),0} + \hbar)\tag{D-20}$$

and

$$\begin{aligned}L_{2,1}(\lambda, \hbar) &= -\sum_{s=1}^n \sum_{j=r_s+1}^{2r_s} P_{X_s,j}^{(2)} (\lambda - X_s)^{-j} + \sum_{s=1}^n \sum_{j=1}^{r_s} H_{X_s,j} (\lambda - X_s)^{-j} - \sum_{j=1}^g \frac{\hbar p_j}{\lambda - q_j} \\ &\stackrel{\lambda \rightarrow \infty}{=} -\frac{t_{\infty(1),0}(t_{\infty(2),0} + \hbar)}{\lambda^2} + \frac{L_{2,1}^{(1)}}{\lambda^3} + O(\lambda^{-4})\end{aligned}\tag{D-21}$$

with

$$L_{2,1}^{(1)} = -\sum_{s=1}^n (2X_s P_{X_s,2}^{(2)} \delta_{r_s=1} + P_{X_s,3}^{(2)} \delta_{r_s=2})$$

$$+ \sum_{s=1}^n (X_s^2 H_{X_s,1} + 2X_s H_{X_s,2} \delta_{r_s \geq 2} + H_{X_s,3} \delta_{r_s \geq 3}) - \hbar \sum_{j=1}^g p_j q_j^2. \quad (\text{D-22})$$

Thus, conditions (D-9) are equivalent to two sets of solutions (η_1, η_0) given by

$$\begin{aligned} \eta_1 &= t_{\infty^{(1)},0}, \\ \eta_0 &= \frac{1}{t_{\infty^{(1)},0} - t_{\infty^{(2)},0}} \left[\right. \\ &\quad - \sum_{s=1}^n (2X_s P_{X_s,2}^{(2)} \delta_{r_s=1} + P_{X_s,3}^{(2)} \delta_{r_s=2}) \\ &\quad + \sum_{s=1}^n (X_s^2 H_{X_s,1} + 2X_s H_{X_s,2} \delta_{r_s \geq 2} + H_{X_s,3} \delta_{r_s \geq 3}) - \hbar \sum_{j=1}^g p_j q_j^2 \\ &\quad - t_{\infty^{(1)},0} \sum_{s=1}^n \left((t_{X_s^{(1)},1} + t_{X_s^{(2)},1}) \delta_{r_s \geq 2} + X_s (t_{X_s^{(1)},0} + t_{X_s^{(2)},0}) \right) \\ &\quad \left. + t_{\infty^{(1)},0} (t_{\infty^{(1)},0} - t_{\infty^{(2)},0} - \hbar) \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) \right], \\ \tilde{L}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \text{diag}(-t_{\infty^{(1)},0}, -t_{\infty^{(2)},0}) \lambda^{-2} + O(\lambda^{-3}), \end{aligned} \quad (\text{D-23})$$

or

$$\begin{aligned} \eta_1 &= t_{\infty^{(2)},0} + \hbar, \\ \eta_0 &= -\frac{1}{t_{\infty^{(1)},0} - t_{\infty^{(2)},0} - 2\hbar} \left[- \sum_{s=1}^n (2X_s P_{X_s,2}^{(2)} \delta_{r_s=1} + P_{X_s,3}^{(2)} \delta_{r_s=2}) \right. \\ &\quad + \sum_{s=1}^n (X_s^2 H_{X_s,1} + 2X_s H_{X_s,2} \delta_{r_s \geq 2} + H_{X_s,3} \delta_{r_s \geq 3}) - \hbar \sum_{j=1}^g p_j q_j^2 \\ &\quad - (t_{\infty^{(2)},0} + \hbar) \sum_{s=1}^n \left((t_{X_s^{(1)},1} + t_{X_s^{(2)},1}) \delta_{r_s \geq 2} + X_s (t_{X_s^{(1)},0} + t_{X_s^{(2)},0}) \right) \\ &\quad \left. - (t_{\infty^{(2)},0} + \hbar) (t_{\infty^{(1)},0} - t_{\infty^{(2)},0} - \hbar) \left(\sum_{j=1}^g q_j - \sum_{s=1}^n r_s X_s \right) \right], \\ \tilde{L}(\lambda, \hbar) &\stackrel{\lambda \rightarrow \infty}{=} \text{diag}(-t_{\infty^{(2)},0} - \hbar, -t_{\infty^{(1)},0} + \hbar) \lambda^{-2} + O(\lambda^{-3}), \end{aligned} \quad (\text{D-24})$$

but whose asymptotics does not fit the one required for $\tilde{L}$ at infinity so that only (D-23) is valid.

E Proof of Proposition 4.1

In order to prove Proposition 4.1, we shall observe that the first line of $A_{\alpha}(\lambda)$ is given by

$$\begin{aligned} [A_{\alpha}(\lambda)]_{1,2} &= \frac{W_{\alpha}(\lambda)}{W(\lambda)} = \frac{Z_{\alpha,2}(\lambda) - Z_{\alpha,1}(\lambda)}{Y_2(\lambda) - Y_1(\lambda)}, \\ [A_{\alpha}(\lambda)]_{1,1} &= \frac{Z_{\alpha,1}(\lambda) Y_2(\lambda) - Z_{\alpha,2}(\lambda) Y_1(\lambda)}{Y_2(\lambda) - Y_1(\lambda)}, \end{aligned} \quad (\text{E-1})$$

where we have defined

$$\begin{aligned} Z_{\alpha,i}(\lambda) &= \frac{\mathcal{L}_{\alpha}[\psi_i(\lambda)]}{\psi_i(\lambda)}, \quad \forall i \in \llbracket 1, 2 \rrbracket, \\ W_{\alpha}(\lambda) &= \mathcal{L}_{\alpha}[\psi_2(\lambda)]\psi_1(\lambda) - \mathcal{L}_{\alpha}[\psi_1(\lambda)]\psi_2(\lambda). \end{aligned} \quad (\text{E-2})$$

Using (C-1) we get that $\mathcal{L}_{\alpha}[\Psi_i(\lambda, \hbar)]$ has the following local expansions.

$$\begin{aligned} \mathcal{L}_{\alpha}[\psi_1(\lambda, \hbar)] &\stackrel{\lambda \rightarrow \infty}{=} - \left(\sum_{k=1}^{r_{\infty}-1} \frac{\alpha_{\infty(1),k}}{k} \lambda^k + O(1) \right) \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty(1),k}}{k} \lambda^k - \frac{t_{\infty(1),0}}{\hbar} \ln \lambda + A_{\infty(1),0} + O(\lambda^{-1}) \right), \\ \mathcal{L}_{\alpha}[\psi_2(\lambda, \hbar)] &\stackrel{\lambda \rightarrow \infty}{=} - \left(\sum_{k=1}^{r_{\infty}-1} \frac{\alpha_{\infty(2),k}}{k} \lambda^k + O(1) \right) \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_{\infty}-1} \frac{t_{\infty(2),k}}{k} \lambda^k - \frac{t_{\infty(2),0}}{\hbar} \ln \lambda - \ln \lambda + A_{\infty(2),0} + O(\lambda^{-1}) \right), \\ \mathcal{L}_{\alpha}[\psi_i(\lambda, \hbar)] &\stackrel{\lambda \rightarrow X_s}{=} \left(-\alpha_{X_s} \sum_{k=0}^{r_s-1} \frac{t_{X_s^{(i)},k}}{k} (\lambda - X_s)^{-k-1} - \sum_{k=1}^{r_s-1} \frac{\alpha_{X_s^{(i)},k}}{k} (\lambda - X_s)^{-k} + O(1) \right) \\ &\quad \exp \left(-\frac{1}{\hbar} \sum_{k=1}^{r_s-1} \frac{t_{X_s^{(i)},k}}{k} (\lambda - X_s)^{-k} + \frac{t_{X_s^{(i)},0}}{\hbar} \ln(\lambda - X_s) + A_{X_s^{(i)},0} + O(\lambda - X_s) \right). \end{aligned} \quad (\text{E-3})$$

Thus, from (E-1) and the asymptotic expansions (C-7) and (E-3), we deduce that

$$\begin{aligned} [A_{\alpha}(\lambda)]_{1,2} &\stackrel{\lambda \rightarrow \infty}{=} \sum_{i=-1}^{r_{\infty}-3} \frac{\nu_{\infty,i}^{(\alpha)}}{\lambda^i} + O(\lambda^{-(r_{\infty}-2)}), \\ [A_{\alpha}(\lambda)]_{1,2} &\stackrel{\lambda \rightarrow X_s}{=} \sum_{i=0}^{r_s-1} \nu_{X_s,i}^{(\alpha)} (\lambda - X_s)^i + O((\lambda - X_s)^{r_s}) \end{aligned} \quad (\text{E-4})$$

where $(\nu_{\infty,i}^{(\alpha)})_{-1 \leq i \leq r_{\infty}-3}$ are defined recursively by

$$\begin{aligned} \nu_{\infty,-1}^{(\alpha)} &= \frac{\alpha_{\infty(1),r_{\infty}-1} - \alpha_{\infty(2),r_{\infty}-1}}{(r_{\infty}-1)(t_{\infty(1),r_{\infty}-1} - t_{\infty(2),r_{\infty}-1})} \\ \nu_{\infty,0}^{(\alpha)} &= -\frac{(t_{\infty(1),r_{\infty}-2} - t_{\infty(2),r_{\infty}-2})}{(r_{\infty}-1)(t_{\infty(1),r_{\infty}-1} - t_{\infty(2),r_{\infty}-1})^2} (\alpha_{\infty(1),r_{\infty}-1} - \alpha_{\infty(2),r_{\infty}-1}) \\ &\quad + \frac{\alpha_{\infty(1),r_{\infty}-2} - \alpha_{\infty(2),r_{\infty}-2}}{(r_{\infty}-2)(t_{\infty(1),r_{\infty}-1} - t_{\infty(2),r_{\infty}-1})} \\ \nu_{\infty,r_{\infty}-2-k}^{(\alpha)} &= \frac{1}{(t_{\infty(1),r_{\infty}-1} - t_{\infty(2),r_{\infty}-1})} \left(\frac{\alpha_{\infty(1),k} - \alpha_{\infty(2),k}}{k} - \sum_{i=-1}^{r_{\infty}-3-k} (t_{\infty(1),k+i+1} - t_{\infty(2),k+i+1}) \nu_{\infty,i}^{(\alpha)} \right) \end{aligned} \quad (\text{E-5})$$

for $k \in \llbracket 1, r_{\infty}-1 \rrbracket$. Similarly, $(\nu_{X_s,i}^{(\alpha)})_{1 \leq i \leq r_s-1}$ are defined recursively by

$$\begin{aligned} \nu_{X_s,0}^{(\alpha)} &= -\alpha_{X_s} \\ (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) \nu_{X_s,r_s-k}^{(\alpha)} &= -\frac{\alpha_{X_s^{(1)},k} - \alpha_{X_s^{(2)},k}}{k} - \alpha_{X_s} (t_{X_s^{(1)},k-1} - t_{X_s^{(2)},k-1}) \\ &\quad - \sum_{i=0}^{r_s-1-k} (t_{X_s^{(1)},k+i-1} - t_{X_s^{(2)},k+i-1}) \nu_{X_s,i}^{(\alpha)} \\ &= -\frac{\alpha_{X_s^{(1)},k} - \alpha_{X_s^{(2)},k}}{k} - \sum_{i=1}^{r_s-1-k} (t_{X_s^{(1)},k+i-1} - t_{X_s^{(2)},k+i-1}) \nu_{X_s,i}^{(\alpha)} \end{aligned} \quad (\text{E-6})$$

for $k \in \llbracket 1, r_s-1 \rrbracket$.

In particular, equations (E-5) and (E-6) may be rewritten in a matrix form giving Proposition 4.1.

F Proof of Proposition 4.2

Since $[A_\alpha(\lambda)]_{1,2}$ is a rational function of λ with only possible poles in $\mathcal{R} \cup \{q_1, \dots, q_g\}$, the asymptotic expansion at each pole provided by Proposition 4.1 implies that

$$[A_\alpha(\lambda)]_{1,2} = \nu_{\infty,-1}^{(\alpha)} \lambda + \nu_{\infty,0}^{(\alpha)} + \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{\lambda - q_j}. \quad (\text{F-1})$$

One then observes that the coefficients $\left(\mu_j^{(\alpha)}\right)_{1 \leq j \leq g}$ are related to $\left(\nu_{\infty,i}^{(\alpha)}\right)_{-1 \leq i \leq r_\infty-3}$ and $\left(\nu_{X_s,i}^{(\alpha)}\right)_{1 \leq i \leq r_s-1}$ through (E-5) and (E-6). We thus get

$$\begin{aligned} [A_\alpha(\lambda)]_{1,2} &= \nu_{\infty,-1}^{(\alpha)} \lambda + \nu_{\infty,0}^{(\alpha)} + \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{\lambda - q_j} \\ &\stackrel{\lambda \rightarrow \infty}{=} \nu_{\infty,-1}^{(\alpha)} \lambda + \nu_{\infty,0}^{(\alpha)} + \sum_{k=1}^{\infty} \sum_{j=1}^g \frac{\mu_j^{(\alpha)} q_j^{k-1}}{\lambda^k} \\ &\stackrel{\lambda \rightarrow X_s}{=} \nu_{\infty,-1}^{(\alpha)} (\lambda - X_s) + \nu_{\infty,-1}^{(\alpha)} X_s + \nu_{\infty,0}^{(\alpha)} + \sum_{k=0}^{\infty} \sum_{j=1}^g \frac{\mu_j^{(\alpha)} (-1)^k}{(X_s - q_j)^{k+1}} (\lambda - X_s)^k \\ &\stackrel{\lambda \rightarrow X_s}{=} \nu_{\infty,-1}^{(\alpha)} X_s + \nu_{\infty,0}^{(\alpha)} - \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{q_j - X_s} + \left(\nu_{\infty,-1}^{(\alpha)} - \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^2} \right) (\lambda - X_s) \\ &\quad - \sum_{k=2}^{\infty} \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^{k+1}} (\lambda - X_s)^k \end{aligned} \quad (\text{F-2})$$

so that, for all $s \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} \forall k \in \llbracket 1, r_\infty - 3 \rrbracket : \sum_{j=1}^g \mu_j^{(\alpha)} q_j^{k-1} &= \nu_{\infty,k}^{(\alpha)}, \\ \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{q_j - X_s} &= -\nu_{X_s,0}^{(\alpha)} + \nu_{\infty,0}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)} X_s, \\ \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^2} &= -\nu_{X_s,1}^{(\alpha)} + \nu_{\infty,-1}^{(\alpha)}, \\ \forall k \in \llbracket 2, r_s - 1 \rrbracket : \sum_{j=1}^g \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^{k+1}} &= -\nu_{X_s,k}^{(\alpha)}. \end{aligned} \quad (\text{F-3})$$

This provides $r_\infty - 3 + \sum_{s=1}^n r_s = g$ relations that fully determine the g unknown coefficients $\left(\mu_j^{(\alpha)}\right)_{1 \leq j \leq g}$. The former relations may be rewritten using a $g \times g$ matrix $\mathbf{V}$ as given in Proposition 4.2.

G Proof of Proposition 4.3

We may perform the same kind of computations for $[A_\alpha(\lambda)]_{1,1}$ starting from (E-1). Note in particular that terms proportional to α_{X_s} cancel in the numerator because of the antisymmetry. We get that

$$[A_\alpha(\lambda)]_{1,1} = \sum_{i=0}^{r_\infty-1} c_{\infty,i}^{(\alpha)} \lambda^i + \sum_{s=1}^n \sum_{i=1}^{r_s-1} c_{X_s,i}^{(\alpha)} (\lambda - X_s)^{-i} + \sum_{j=1}^g \frac{\rho_j^{(\alpha)}}{\lambda - q_j}. \quad (\text{G-1})$$

The coefficients $(c_{\infty,i}^{(\alpha)})_{1 \leq i \leq r_\infty-1}$ are defined recursively by

$$\begin{aligned} c_{\infty,r_\infty-1}^{(\alpha)} &= \frac{\alpha_{\infty(1),r_\infty-1} t_{\infty(2),r_\infty-1} - \alpha_{\infty(2),r_\infty-1} t_{\infty(1),r_\infty-1}}{(r_\infty - 1)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})}, \\ c_{\infty,r_\infty-2}^{(\alpha)} &= \frac{\alpha_{\infty(1),r_\infty-2} t_{\infty(2),r_\infty-1} - \alpha_{\infty(2),r_\infty-2} t_{\infty(1),r_\infty-1}}{(r_\infty - 2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} \\ &\quad - \frac{t_{\infty(2),r_\infty-1} t_{\infty(1),r_\infty-2} - t_{\infty(1),r_\infty-1} t_{\infty(2),r_\infty-2}}{(r_\infty - 1)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^2} (\alpha_{\infty(1),r_\infty-1} - \alpha_{\infty(2),r_\infty-1}) \\ (t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1}) c_{\infty,r_\infty-1-j}^{(\alpha)} &= \sum_{i=j+1}^{r_\infty-1} \frac{t_{\infty(2),r_\infty+j-i} \alpha_{\infty(1),i} - t_{\infty(1),r_\infty+j-i} \alpha_{\infty(2),i}}{i} \\ &\quad - \sum_{k=1}^j (t_{\infty(1),r_\infty-1-k} - t_{\infty(2),r_\infty-1-k}) c_{\infty,r_\infty-1-j+k}^{(\alpha)}, \quad \forall j \in \llbracket 1, r_\infty - 2 \rrbracket \end{aligned} \quad (\text{G-2})$$

and, for all $s \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} c_{X_s,r_s-1}^{(\alpha)} &= \frac{\alpha_{X_s^{(1)},r_s-1} t_{X_s^{(2)},r_s-1} - \alpha_{X_s^{(2)},r_s-1} t_{X_s^{(1)},r_s-1}}{(r_s - 1)(t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1})}, \\ c_{X_s,r_s-2}^{(\alpha)} &= \frac{\alpha_{X_s^{(1)},r_s-2} t_{X_s^{(2)},r_s-1} - \alpha_{X_s^{(2)},r_s-2} t_{X_s^{(1)},r_s-1}}{(r_s - 2)(t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1})} \\ &\quad - \frac{t_{X_s^{(2)},r_s-1} t_{X_s^{(1)},r_s-2} - t_{X_s^{(1)},r_s-1} t_{X_s^{(2)},r_s-2}}{(r_s - 1)(t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1})^2} (\alpha_{X_s^{(1)},r_s-1} - \alpha_{X_s^{(2)},r_s-1}), \\ (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) c_{X_s,r_s-1-j}^{(\alpha)} &= \sum_{i=j+1}^{r_s-1} \frac{t_{X_s^{(2)},r_s+j-i} \alpha_{X_s^{(1)},i} - t_{X_s^{(1)},r_s+j-i} \alpha_{X_s^{(2)},i}}{i} \\ &\quad - \sum_{k=1}^j (t_{X_s^{(1)},r_s-1-k} - t_{X_s^{(2)},r_s-1-k}) c_{X_s,r_s-1-j+k}^{(\alpha)}, \quad \forall j \in \llbracket 1, r_s - 2 \rrbracket. \end{aligned} \quad (\text{G-3})$$

Note that $c_{\infty,0}^{(\alpha)}$ is not determined but will play no role in the rest of the paper. The previous recursive relations may be rewritten in a matrix form giving Proposition 4.3. Finally, the coefficients $(\rho_j^{(\alpha)})_{1 \leq j \leq g}$ are obtained by looking at order $(\lambda - q_j)^{-3}$ of $\mathcal{L}_\alpha[L_{2,1}(\lambda)]$.

H Proof of Theorem 5.1

In this section we prove Theorem 5.1.

H.1 Preliminary results

We postpone the proof and start with the following lemma:

Lemma H.1. *For all $j \in \llbracket 1, g \rrbracket$:*

$$\begin{aligned}
& \sum_{k=0}^{r_\infty-4} \sum_{i=1}^g H_{\infty,k} q_i^k \partial_{q_j} \mu_i^{(\alpha)} + \sum_{s=1}^n \sum_{k=1}^{r_s} \sum_{i=1}^g H_{X_s,k} (q_i - X_s)^{-k} \partial_{q_j} \mu_i^{(\alpha)} \\
&= -\mu_j^{(\alpha)} \left(\sum_{k=0}^{r_\infty-4} k H_{\infty,k} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s} k H_{X_s,k} (q_j - X_s)^{-k} \right) \\
&+ \delta_{r_\infty=2} \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty^{(1)},1} t_{\infty^{(2)},0} + t_{\infty^{(2)},1} t_{\infty^{(1)},0} + \hbar t_{\infty^{(1),1}}) \right) \\
&+ \delta_{r_\infty=1} \left[\left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty^{(1)},0} (t_{\infty^{(2)},0} + \hbar) \right) + \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right]. \tag{H-1}
\end{aligned}$$

Proof. The proof follows from the expression relating $\left(\nu_{p,k}^{(\alpha)} \right)_{p,k}$ and $\left(\mu_j^{(\alpha)} \right)_{1 \leq j \leq g}$ given by (F-3). Taking the derivative relatively to q_j and using the fact that the $\left(\nu_{p,k}^{(\alpha)} \right)$'s are independent of q_j , except for $\nu_{\infty,-1}^{(\alpha)}$ and $\nu_{\infty,0}^{(\alpha)}$ when $r_\infty \leq 2$, gives:

$$\begin{aligned}
\forall k \in \llbracket 0, r_\infty - 4 \rrbracket : \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) q_i^k &= -k \mu_j^{(\alpha)} q_j^{k-1}, \\
\forall k \in \llbracket 1, r_s \rrbracket : \sum_{i=1}^g \frac{\partial_{q_j} \mu_i^{(\alpha)}}{(q_i - X_s)^k} &= k \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^{k+1}} + \left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \delta_{k=2} \\
&+ \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} + X_s \partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \delta_{k=1}. \tag{H-2}
\end{aligned}$$

Thus

$$\begin{aligned}
& \sum_{k=0}^{r_\infty-4} \sum_{i=1}^g H_{\infty,k} q_i^k \partial_{q_j} \mu_i^{(\alpha)} + \sum_{s=1}^n \sum_{k=1}^{r_s} \sum_{i=1}^g H_{X_s,k} (q_i - X_s)^{-k} \partial_{q_j} \mu_i^{(\alpha)} \\
&= -\mu_j^{(\alpha)} \left(\sum_{k=0}^{r_\infty-4} k H_{\infty,k} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s} k H_{X_s,k} \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^{k+1}} \right) \\
&+ \left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \left(\sum_{s=1}^n H_{X_s,2} \delta_{r_s \geq 2} + X_s H_{X_s,1} \right) + \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\sum_{s=1}^n H_{X_s,1} \right). \tag{H-3}
\end{aligned}$$

We now recall that $\nu_{\infty,-1}^{(\alpha)}$ and $\nu_{\infty,0}^{(\alpha)}$ only depend on $(q_j)_{1 \leq j \leq g}$ when $r_\infty \leq 2$. In fact for $r_\infty = 2$ only $\nu_{\infty,0}^{(\alpha)}$ depends on $(q_j)_{1 \leq j \leq g}$. In these cases, the sums $\sum_{s=1}^n H_{X_s,2} \delta_{r_s \geq 2} + X_s H_{X_s,1}$ and $\sum_{s=1}^n H_{X_s,1}$ are determined by (5-5) or (5-6). Thus we get:

$$\begin{aligned}
& \sum_{k=0}^{r_\infty-4} \sum_{i=1}^g H_{\infty,k} q_i^k \partial_{q_j} \mu_i^{(\alpha)} + \sum_{s=1}^n \sum_{k=1}^{r_s} \sum_{i=1}^g H_{X_s,k} (q_i - X_s)^{-k} \partial_{q_j} \mu_i^{(\alpha)} \\
&= -\mu_j^{(\alpha)} \left(\sum_{k=0}^{r_\infty-4} k H_{\infty,k} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s} k H_{X_s,k} \frac{\mu_j^{(\alpha)}}{(q_j - X_s)^{k+1}} \right)
\end{aligned}$$

$$\begin{aligned}
& +\delta_{r_\infty=2} \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}) \right) \\
& +\delta_{r_\infty=1} \left[\left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty(1),0} (t_{\infty(2),0} + \hbar) \right) \right. \\
& \left. + \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right]
\end{aligned} \tag{H-4}$$

so that the lemma is proved. $\square$

We may now provide an alternative expression for $\mathcal{L}_\alpha[p_j]$:

Proposition H.1. *Let $j \in \llbracket 1, g \rrbracket$, we have an alternative expression for $\mathcal{L}_\alpha[p_j]$:*

$$\begin{aligned}
\mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} + \frac{\hbar}{2} \sum_{\substack{(r,s) \in \llbracket 1, g \rrbracket^2 \\ r \neq s}} \frac{(p_s - p_r)(\partial_{q_j} \mu_r^{(\alpha)} + \partial_{q_j} \mu_s^{(\alpha)})}{q_s - q_r} \\
& - \mu_j^{(\alpha)} \left(\tilde{P}'_2(q_j) - p_j P'_1(q_j) - \hbar p_j \sum_{s=1}^n \frac{r_s}{(q_j - X_s)^2} + \hbar(r_\infty - 3) t_{\infty(1), r_\infty-1} q_j^{r_\infty-4} \delta_{r_\infty \geq 3} \right) \\
& + \hbar \nu_{\infty,-1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty-1} k c_{\infty,k}^{(\alpha)} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} k c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k-1} \\
& - \sum_{r=1}^g (\partial_{q_j} \mu_r^{(\alpha)}) \left(\tilde{P}_2(q_r) + p_r^2 - P_1(q_r) p_r + 2 \hbar p_r \sum_{s=1}^n \frac{r_s}{q_r - X_s} + \hbar t_{\infty(1), r_\infty-1} q_j^{r_\infty-3} \delta_{r_\infty \geq 3} \right) \\
& + \delta_{r_\infty=2} \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}) \right) \\
& + \delta_{r_\infty=1} \left[\left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty(1),0} (t_{\infty(2),0} + \hbar) \right) \right. \\
& \left. + \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right].
\end{aligned} \tag{H-5}$$

Proof. Using Lemma H.1 we get that the expression (5-7) for $\mathcal{L}[p_j]$ becomes:

$$\begin{aligned}
\mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} \\
& + \mu_j^{(\alpha)} \left(p_j P'_1(q_j) + \hbar p_j \sum_{s=1}^n \frac{r_s}{(q_j - X_s)^2} - \tilde{P}'_2(q_j) - \hbar(r_\infty - 3) t_{\infty(1), r_\infty-1} q_j^{r_\infty-4} \delta_{r_\infty \geq 3} \right) \\
& + \hbar \nu_{\infty,-1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty-1} k c_{\infty,k}^{(\alpha)} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} k c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k-1} \\
& - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left(\sum_{k=0}^{r_\infty-4} H_{\infty,k} q_i^k + \sum_{s=1}^n \sum_{k=1}^{r_s} \sum_{i=1}^g H_{X_s,k} (q_i - X_s)^{-k} \right) \\
& + \delta_{r_\infty=2} \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}) \right) \\
& + \delta_{r_\infty=1} \left[\left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty(1),0} (t_{\infty(2),0} + \hbar) \right) \right. \\
& \left. + \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right].
\end{aligned} \tag{H-6}$$

We now use (5-3) to get

$$\begin{aligned}
\mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} \\
&\quad + \mu_j^{(\alpha)} \left(p_j P_1'(q_j) + \hbar p_j \sum_{s=1}^n \frac{r_s}{(q_j - X_s)^2} - \tilde{P}_2'(q_j) - \hbar(r_\infty - 3)t_{\infty^{(1)}, r_\infty - 1} q_j^{r_\infty - 4} \delta_{r_\infty \geq 3} \right) \\
&\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s - 1} k c_{X_s, k}^{(\alpha)} (q_j - X_s)^{-k-1} \\
&\quad - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left[p_i^2 - P_1(q_i) p_i + \hbar p_i \sum_{s=1}^n \frac{r_s}{q_i - X_s} + \tilde{P}_2(q_i) \right. \\
&\quad \left. + \hbar \sum_{r \neq i} \frac{p_r - p_i}{q_i - q_r} + \hbar t_{\infty^{(1)}, r_\infty - 1} q_i^{r_\infty - 3} \delta_{r_\infty \geq 3} \right] \\
&\quad + \delta_{r_\infty = 2} \left(\partial_{q_j} \nu_{\infty, 0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty^{(1)}, 1} t_{\infty^{(2)}, 0} + t_{\infty^{(2)}, 1} t_{\infty^{(1)}, 0} + \hbar t_{\infty^{(1)}, 1}) \right) \\
&\quad + \delta_{r_\infty = 1} \left[\left(\partial_{q_j} \nu_{\infty, -1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)}, 0} t_{X_s^{(2)}, 0} \delta_{r_s = 1} - t_{\infty^{(1)}, 0} (t_{\infty^{(2)}, 0} + \hbar) \right) \right. \\
&\quad \left. + \left(\partial_{q_j} \nu_{\infty, 0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right] \\
&= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} \\
&\quad + \mu_j^{(\alpha)} \left(p_j P_1'(q_j) + \hbar p_j \sum_{s=1}^n \frac{r_s}{(q_j - X_s)^2} - \tilde{P}_2'(q_j) - \hbar(r_\infty - 3)t_{\infty^{(1)}, r_\infty - 1} q_j^{r_\infty - 4} \delta_{r_\infty \geq 3} \right) \\
&\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s - 1} k c_{X_s, k}^{(\alpha)} (q_j - X_s)^{-k-1} \\
&\quad - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left(p_i^2 - P_1(q_i) p_i + \hbar p_i \sum_{s=1}^n \frac{r_s}{q_i - X_s} + \tilde{P}_2(q_i) + \hbar t_{\infty^{(1)}, r_\infty - 1} q_i^{r_\infty - 3} \delta_{r_\infty \geq 3} \right) \\
&\quad + \delta_{r_\infty = 2} \left(\partial_{q_j} \nu_{\infty, 0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty^{(1)}, 1} t_{\infty^{(2)}, 0} + t_{\infty^{(2)}, 1} t_{\infty^{(1)}, 0} + \hbar t_{\infty^{(1)}, 1}) \right) \\
&\quad + \delta_{r_\infty = 1} \left[\left(\partial_{q_j} \nu_{\infty, -1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)}, 0} t_{X_s^{(2)}, 0} \delta_{r_s = 1} - t_{\infty^{(1)}, 0} (t_{\infty^{(2)}, 0} + \hbar) \right) \right. \\
&\quad \left. + \left(\partial_{q_j} \nu_{\infty, 0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right] \\
&\quad + \hbar \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \sum_{r \neq i} \frac{p_r - p_i}{q_r - q_i}. \tag{H-7}
\end{aligned}$$

The last sums may be split into a symmetric and anti-symmetric case: $\partial_{q_j} \mu_i^{(\alpha)} = \frac{1}{2}(\partial_{q_j} \mu_i^{(\alpha)} - \partial_{q_j} \mu_i^{(\alpha)} + \frac{1}{2}(\partial_{q_j} \mu_i^{(\alpha)} + \partial_{q_j} \mu_i^{(\alpha)})$. The term involving $\partial_{q_j} \mu_i^{(\alpha)} - \partial_{q_j} \mu_i^{(\alpha)}$ is trivially zero because the sum is anti-symmetric so that we end up with

$$\begin{aligned}
\mathcal{L}_\alpha[p_j] &= \hbar \sum_{i \neq j} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_j - q_i)^2} \\
&\quad - \mu_j^{(\alpha)} \left(\tilde{P}_2'(q_j) - p_j P_1'(q_j) - \hbar p_j \sum_{s=1}^n \frac{r_s}{(q_j - X_s)^2} + \hbar(r_\infty - 3)t_{\infty^{(1)}, r_\infty - 1} q_j^{r_\infty - 4} \delta_{r_\infty \geq 3} \right) \\
&\quad + \hbar \nu_{\infty, -1}^{(\alpha)} p_j + \hbar \sum_{k=1}^{r_\infty - 1} k c_{\infty, k}^{(\alpha)} q_j^{k-1} - \hbar \sum_{s=1}^n \sum_{k=1}^{r_s - 1} k c_{X_s, k}^{(\alpha)} (q_j - X_s)^{-k-1} \\
&\quad - \sum_{i=1}^g (\partial_{q_j} \mu_i^{(\alpha)}) \left(p_i^2 - P_1(q_i) p_i + \hbar p_i \sum_{s=1}^n \frac{r_s}{q_i - X_s} + \tilde{P}_2(q_i) + \hbar t_{\infty^{(1)}, r_\infty - 1} q_i^{r_\infty - 3} \delta_{r_\infty \geq 3} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\hbar}{2} \sum_{i=1}^g \sum_{r \neq i} \frac{(p_r - p_i)(\partial_{q_j} \mu_i^{(\alpha)} + \partial_{q_j} \mu_r^{(\alpha)})}{q_r - q_i} \\
& + \delta_{r_\infty=2} \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i - (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}) \right) \\
& + \delta_{r_\infty=1} \left[\left(\partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g q_i p_i + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty(1),0} (t_{\infty(2),0} + \hbar) \right) \right. \\
& \left. + \left(\partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right) \left(\hbar \sum_{i=1}^g p_i \right) \right]. \tag{H-8}
\end{aligned}$$

□

H.2 Proof of the Theorem 5.1

We may now proceed to the proof of Theorem 5.1. We recall that the Hamiltonian is given by:

$$\begin{aligned}
\text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p}) &= -\frac{\hbar}{2} \sum_{\substack{(i,j) \in \llbracket 1,g \rrbracket^2 \\ i \neq j}} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{q_i - q_j} - \hbar \sum_{j=1}^g (\nu_{\infty,0}^{(\alpha)} p_j + \nu_{\infty,-1}^{(\alpha)} q_j p_j) \\
&+ \sum_{j=1}^g \mu_j^{(\alpha)} \left[p_j^2 - P_1(q_j) p_j + \hbar p_j \sum_{s=1}^n \frac{r_s}{q_j - X_s} + \tilde{P}_2(q_j) + \hbar t_{\infty(1),r_\infty-1} q_j^{r_\infty-3} \delta_{r_\infty \geq 3} \right] \\
&- \hbar \sum_{j=1}^g \left[\sum_{k=0}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k + \sum_{s=1}^n \sum_{k=1}^{r_s-1} c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k} \right] \\
&+ \delta_{r_\infty=2} (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}) \nu_{\infty,0}^{(\alpha)} \\
&- \delta_{r_\infty=1} \left(\sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty(1),0} (t_{\infty(2),0} + \hbar) \right) \nu_{\infty,-1}^{(\alpha)}. \tag{H-9}
\end{aligned}$$

It is a straightforward computation from (5-8) and from the fact that the $(\nu_{p,k}^{(\alpha)}, c_{p,k}^{(\alpha)})_{p,k}$ are independent of q_j (except for $r_\infty = 2$ where $\nu_{\infty,0}^{(\alpha)}$ depends on $(q_j)_{1 \leq j \leq g}$ and for $r_\infty = 1$ where both $\nu_{\infty,0}^{(\alpha)}$ and $\nu_{\infty,-1}^{(\alpha)}$ depends on $(q_j)_{1 \leq j \leq g}$) to get that

$$\begin{aligned}
-\frac{\partial \text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial q_j} &= \hbar \sum_{\substack{i \in \llbracket 1,g \rrbracket \\ i \neq j}} \frac{(\mu_i^{(\alpha)} + \mu_j^{(\alpha)})(p_i - p_j)}{(q_i - q_j)^2} \\
&+ \frac{\hbar}{2} \sum_{\substack{(r,s) \in \llbracket 1,g \rrbracket^2 \\ r \neq s}} \frac{(\partial_{q_j} \mu_r^{(\alpha)} + \partial_{q_j} \mu_s^{(\alpha)})(p_r - p_s)}{q_r - q_s} + \hbar \nu_{\infty,-1}^{(\alpha)} p_j \\
&- \sum_{i=1}^g \partial_{q_j} (\mu_i^{(\alpha)}) \left[p_i^2 - P_1(q_i) p_i + \hbar p_i \sum_{s=1}^n \frac{r_s}{q_i - X_s} + \tilde{P}_2(q_i) + \hbar t_{\infty(1),r_\infty-1} q_i^{r_\infty-3} \delta_{r_\infty \geq 3} \right] \\
&- \mu_j^{(\alpha)} \left[\tilde{P}_2'(q_j) - p_j P_1'(q_j) - \hbar p_j \sum_{s=1}^n \frac{r_s}{(q_j - X_s)^2} + \hbar t_{1,r_\infty-1} (r_\infty - 3) q_j^{r_\infty-4} \delta_{r_\infty \geq 3} \right] \\
&+ \hbar \left[\sum_{k=1}^{r_\infty-1} k c_{\infty,k}^{(\alpha)} q_j^{k-1} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} k c_{X_s,k}^{(\alpha)} (q_i - X_s)^{-k-1} \right] \\
&+ \delta_{r_\infty=2} \left(\hbar \sum_{i=1}^g p_i - (t_{\infty(1),1} t_{\infty(2),0} + t_{\infty(2),1} t_{\infty(1),0} + \hbar t_{\infty(1),1}) \right) \partial_{q_j} \nu_{\infty,0}^{(\alpha)}
\end{aligned}$$

$$\begin{aligned}
& +\delta_{r_\infty=1} \left[\left(\hbar \sum_{i=1}^n q_i p_i + \sum_{s=1}^n t_{X_s^{(1)},0} t_{X_s^{(2)},0} \delta_{r_s=1} - t_{\infty^{(1)},0} (t_{\infty^{(2)},0} + \hbar) \right) \partial_{q_j} \nu_{\infty,-1}^{(\alpha)} \right. \\
& \left. + \hbar \left(\sum_{i=1}^g p_i \right) \partial_{q_j} \nu_{\infty,0}^{(\alpha)} \right] \\
& \stackrel{(H.1)}{=} \mathcal{L}_\alpha[p_j].
\end{aligned} \tag{H-10}$$

Similarly a straightforward computation using the fact that $\left(\nu_{p,k}^{(\alpha)}\right)_{p,k}$ (including $\nu_{\infty,-1}^{(\alpha)}$ and $\nu_{\infty,0}^{(\alpha)}$ when $r_\infty \leq 2$) and $\left(c_{p,k}^{(\alpha)}\right)_{p,k}$ and $\left(\mu_i^{(\alpha)}\right)_{1 \leq i \leq g}$ are independent of p_j gives:

$$\frac{\partial \text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p})}{\partial p_j} = -\hbar \sum_{\substack{i \in \llbracket 1, g \rrbracket \\ i \neq j}} \frac{\mu_i^{(\alpha)} + \mu_j^{(\alpha)}}{q_j - q_i} - \hbar \nu_{\infty,0}^{(\alpha)} - \hbar \nu_{\infty,-1}^{(\alpha)} q_j + \mu_j^{(\alpha)} \left(2p_j - P_1(q_j) + \hbar \sum_{s=1}^n \frac{r_s}{q_j - X_s} \right) \tag{H-11}$$

which is exactly $\mathcal{L}_\alpha[q_j]$ given by (5-1).

The last step is to verify that from Propositions 4.2 and 5.1:

$$\begin{aligned}
& \sum_{j=1}^g \mu_j^{(\alpha)} \left(p_j^2 - P_1(q_j) p_j + p_j \sum_{s=1}^n \frac{\hbar r_s}{q_j - X_s} + \tilde{P}_2(q_j) + \hbar \sum_{i \neq j} \frac{p_i - p_j}{q_j - q_i} + \hbar t_{\infty^{(1)}, r_\infty-1} q_j^{r_\infty-3} \delta_{r_\infty \geq 3} \right) \\
& = \left(\mu_1^{(\alpha)}, \dots, \mu_g^{(\alpha)} \right) \begin{pmatrix} p_1^2 - P_1(q_1) p_1 + p_1 \sum_{s=1}^n \frac{\hbar r_s}{q_1 - X_s} + \tilde{P}_2(q_1) + \hbar \sum_{i \neq 1} \frac{p_i - p_1}{q_1 - q_i} + \hbar t_{\infty^{(1)}, r_\infty-1} q_1^{r_\infty-3} \delta_{r_\infty \geq 3} \\ \vdots \\ p_g^2 - P_1(q_g) p_g + p_g \sum_{s=1}^n \frac{\hbar r_s}{q_g - X_s} + \tilde{P}_2(q_g) + \hbar \sum_{i \neq g} \frac{p_i - p_g}{q_g - q_i} + \hbar t_{\infty^{(1)}, r_\infty-1} q_g^{r_\infty-3} \delta_{r_\infty \geq 3} \end{pmatrix} \\
& = \left(\mu_1^{(\alpha)}, \dots, \mu_g^{(\alpha)} \right) \mathbf{V}^t \begin{pmatrix} \mathbf{H}_\infty \\ \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} = \left((\nu_\infty^{(\alpha)})^t, (\nu_{X_1}^{(\alpha)})^t, \dots, (\nu_{X_n}^{(\alpha)})^t \right) \begin{pmatrix} \mathbf{H}_\infty \\ \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} \\
& = \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\alpha)} H_{\infty,k} - \sum_{s=1}^n \sum_{k=2}^{r_s} \nu_{X_s,k-1}^{(\alpha)} H_{X_s,k} - \sum_{s=1}^n \nu_{X_s,0}^{(\alpha)} H_{X_s,1} \\
& + \nu_{\infty,-1}^{(\alpha)} \sum_{s=1}^n (X_s H_{X_s,1} + H_{X_s,2} \delta_{r_s \geq 2}) + \nu_{\infty,0}^{(\alpha)} \sum_{s=1}^n H_{X_s,1}
\end{aligned} \tag{H-12}$$

so that (H-9) becomes (using (2-55) and (2-56))

$$\begin{aligned}
\text{Ham}^{(\alpha)}(\mathbf{q}, \mathbf{p}) & = \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\alpha)} H_{\infty,k} - \sum_{s=1}^n \sum_{k=2}^{r_s} \nu_{X_s,k-1}^{(\alpha)} H_{X_s,k} + \sum_{s=1}^n \alpha_{X_s}^{(\alpha)} H_{X_s,1} \\
& - \hbar \sum_{j=1}^g \left[\sum_{k=0}^{r_\infty-1} c_{\infty,k}^{(\alpha)} q_j^k + \sum_{s=1}^n \sum_{k=1}^{r_s-1} c_{X_s,k}^{(\alpha)} (q_j - X_s)^{-k} \right] \\
& + \nu_{\infty,-1}^{(\alpha)} \sum_{s=1}^n (X_s H_{X_s,1} + H_{X_s,2} \delta_{r_s \geq 2}) + \nu_{\infty,0}^{(\alpha)} \sum_{s=1}^n H_{X_s,1} \\
& - \delta_{r_\infty \in \{1,2\}} \left(\sum_{s=1}^n H_{X_s,1} - \hbar \sum_{j=1}^g p_j \right) \nu_{\infty,0}^{(\alpha)}
\end{aligned}$$

$$\begin{aligned}
& -\delta_{r_\infty=1} \left(\sum_{s=1}^n X_s H_{X_s,1} + \sum_{s=1}^n H_{X_s,2} \delta_{r_s \geq 2} - \hbar \sum_{j=1}^g q_j p_j \right) \nu_{\infty,-1}^{(\alpha)} \\
& - \hbar \nu_{\infty,0}^{(\alpha)} \sum_{j=1}^g p_j - \hbar \nu_{\infty,-1}^{(\alpha)} \sum_{j=1}^g q_j p_j.
\end{aligned} \tag{H-13}$$

I Proof of Proposition 6.1

Let us take $j \in \llbracket 1, r_\infty - 1 \rrbracket$ and consider $\mathcal{L}_{\mathbf{v}_\infty, j}$. The r.h.s. of (4-10) is null and the r.h.s of (4-8) is trivially null too. Thus, all $(\nu_{\infty, k}^{(\mathbf{v}_\infty, j)})_{-1 \leq k \leq r_\infty - 3}$ and all $(\nu_{X_s, k}^{(\mathbf{v}_\infty, j)})_{1 \leq s \leq n, 1 \leq k \leq r_s - 1}$ are vanishing. In the same way, the r.h.s. of (4-21) is trivially vanishing so that for all $s \in \llbracket 1, n \rrbracket$, all $(c_{X_s, k}^{(\mathbf{v}_\infty, j)})_{1 \leq k \leq r_s - 1}$ are vanishing. On the contrary, the r.h.s. of (4-20) is non-zero. More precisely, the r.h.s. of (4-20) is $\left(0, \dots, 0, \frac{t_{\infty(2), r_\infty - 1} - t_{\infty(1), r_\infty - 1}}{j}, \dots, \frac{t_{\infty(2), r_\infty - j} - t_{\infty(1), r_\infty - j}}{j} \right)^t$ which is precisely $-\frac{1}{j}$ times the $(r_\infty - j)^{\text{th}}$ column of M_∞ given by (4-11). Hence, $(c_{\infty, r_\infty - 1}^{(\mathbf{v}_\infty, j)}, \dots, c_{\infty, 1}^{(\mathbf{v}_\infty, j)})^t = -\frac{1}{j} \mathbf{e}_{r_\infty - j}$, i.e. $c_{\infty, i}^{(\mathbf{v}_\infty, j)} = -\frac{1}{j} \delta_{i, j}$ for all $i \in \llbracket 1, r_\infty - 1 \rrbracket$.

Let us now take $s_0 \in \llbracket 1, n \rrbracket$ and $j \in \llbracket 1, r_{s_0} - 1 \rrbracket$ and consider $\mathcal{L}_{\mathbf{v}_{X_{s_0}}, j}$. The r.h.s. of (4-10) and (4-8) are trivially null too. Thus, all $(\nu_{\infty, k}^{(\mathbf{v}_{X_{s_0}}, j)})_{-1 \leq k \leq r_\infty - 3}$ and all $(\nu_{X_s, k}^{(\mathbf{v}_{X_{s_0}}, j)})_{1 \leq s \leq n, 1 \leq k \leq r_s - 1}$ are vanishing. In the same way, the r.h.s. of (4-20) is vanishing so that all $(c_{\infty, k}^{(\mathbf{v}_{X_{s_0}}, j)})_{1 \leq k \leq r_\infty - 1}$ are vanishing. Similarly, the r.h.s. of (4-21) is vanishing for $s \neq s_0$ so that all $(c_{X_s, k}^{(\mathbf{v}_{X_{s_0}}, j)})_{1 \leq k \leq r_s - 1}$ are vanishing for $s \neq s_0$. On the contrary, the r.h.s. of (4-21) is non zero for $s = s_0$ and is given by $\left(0, \dots, 0, \frac{t_{X_{s_0}^{(2)}, r_{s_0} - 1} - t_{X_{s_0}^{(1)}, r_{s_0} - 1}}{j}, \dots, \frac{t_{X_{s_0}^{(2)}, r_{s_0} - j} - t_{X_{s_0}^{(1)}, r_{s_0} - j}}{j} \right)^t$ which is precisely $-\frac{1}{j}$ times the $(r_{s_0} - j)^{\text{th}}$ column of M_{s_0} given by (4-9). Hence, $\left(c_{X_{s_0}, r_{s_0} - 1}^{(\mathbf{v}_{X_{s_0}}, j)}, \dots, c_{X_{s_0}, 1}^{(\mathbf{v}_{X_{s_0}}, j)} \right)^t = -\frac{1}{j} \mathbf{e}_{r_{s_0} - j}$, i.e. $c_{X_{s_0}, i}^{(\mathbf{v}_{X_{s_0}}, j)} = -\frac{1}{j} \delta_{i, j}$ for all $i \in \llbracket 1, r_{s_0} - 1 \rrbracket$.

J Proof of Proposition 6.2

J.1 The case of $\mathbf{u}_{\infty, j}$

Let $j \in \llbracket 1, r_\infty - 1 \rrbracket$ and we consider $\mathcal{L}_{\mathbf{u}_{\infty}, j}$. For $i \in \llbracket 1, 2 \rrbracket$, it corresponds to

$$\left(\alpha_{\infty(i), 1}, \dots, \alpha_{\infty(i), r_\infty - 1} \right)^t = \left(t_{\infty(i), r_\infty - j}, \dots, j t_{\infty(i), r_\infty - 1}, 0, \dots, 0 \right)^t \tag{J-1}$$

so that the r.h.s. of (4-10) is $\left(0, \dots, 0, (t_{\infty(1), r_\infty - 1} - t_{\infty(2), r_\infty - 1}), \dots, (t_{\infty(1), r_\infty - j} - t_{\infty(2), r_\infty - j}) \right)^t$ which is precisely the $(r_\infty - j)^{\text{th}}$ column of M_∞ . Since M_∞ is invertible, we necessarily have $(\nu^{(\mathbf{u}_{\infty}, j)}_{\infty, -1}, \dots, \nu^{(\mathbf{u}_{\infty}, j)}_{\infty, r_\infty - 3})^t = \mathbf{e}_{r_\infty - j}$, i.e.

$$\nu_{X_k}^{(\mathbf{u}_{\infty}, j)} = \delta_{k, r_\infty - j - 2}, \quad \forall k \in \llbracket -1, r_\infty - 3 \rrbracket. \tag{J-2}$$

It is then obvious that the r.h.s. of (4-8) is null so that we get

$$\nu_{X_s, k}^{(\mathbf{u}_{\infty}, j)} = 0, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket. \tag{J-3}$$

In the same way, it is obvious that the r.h.s. of (4-21) is null so that we get

$$c_{X_s,k}^{(\mathbf{u}_{\infty,j})} = 0, \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket. \quad (\text{J-4})$$

Finally, let us compute the r.h.s. of (4-20). The i^{th} line, denoted RHS_i is given by

$$\text{RHS}_i = \sum_{r=r_{\infty}-i}^{r_{\infty}-1} \frac{t_{\infty(2),2r_{\infty}-1-i-r} \alpha_{\infty(1),r} - t_{\infty(1),2r_{\infty}-1-i-r} \alpha_{\infty(2),r}}{r}. \quad (\text{J-5})$$

However, by definition $\alpha_{\infty(1),r} = r t_{\infty(1),r_{\infty}-1-j+r} \delta_{1 \leq r \leq j}$ and $\alpha_{\infty(2),r} = r t_{\infty(2),r_{\infty}-1-j+r} \delta_{1 \leq r \leq j}$ so that

$$\begin{aligned} \text{RHS}_i &= \sum_{r=r_{\infty}-i}^{r_{\infty}-1} (t_{\infty(2),2r_{\infty}-1-i-r} t_{\infty(1),r_{\infty}-1-j+r} - t_{\infty(1),2r_{\infty}-1-i-r} t_{\infty(2),r_{\infty}-1-j+r}) \delta_{1 \leq r \leq j} \\ &= \sum_{r=r_{\infty}-i}^{r_{\infty}-1} t_{\infty(2),2r_{\infty}-1-i-r} t_{\infty(1),r_{\infty}-1-j+r} \delta_{1 \leq r \leq j} \\ &\quad - \sum_{r=r_{\infty}-i}^{r_{\infty}-1} t_{\infty(1),2r_{\infty}-1-i-r} t_{\infty(2),r_{\infty}-1-j+r} \delta_{1 \leq r \leq j} \\ &= \sum_{r=r_{\infty}-i}^j t_{\infty(2),2r_{\infty}-1-i-r} t_{\infty(1),r_{\infty}-1-j+r} - \sum_{r=r_{\infty}-i}^j t_{\infty(1),2r_{\infty}-1-i-r} t_{\infty(2),r_{\infty}-1-j+r} \\ &\stackrel{s=r_{\infty}+j-i-r}{=} \sum_{r=r_{\infty}-i}^j t_{\infty(2),2r_{\infty}-1-i-r} t_{\infty(1),r_{\infty}-1-j+r} - \sum_{s=r_{\infty}-i}^j t_{\infty(1),r_{\infty}-1-j+s} t_{\infty(2),2r_{\infty}-1-i-s} \\ &= 0. \end{aligned} \quad (\text{J-6})$$

Thus, the r.h.s. of (4-20) is null and so are all $\left(c_{\infty,k}^{(\mathbf{u}_{\infty,j})} \right)_{1 \leq k \leq r_{\infty}-1}$.

J.2 The case of $\mathbf{u}_{X_s,j}$

Let $s \in \llbracket 1, n \rrbracket$ and $j \in \llbracket 1, r_{\infty} - 1 \rrbracket$ and we consider $\mathcal{L}_{\mathbf{u}_{X_s,j}}$. The only non-zero deformation parameters are

$$\alpha_{X_{s'},r}^{(i)} = r t_{X_{s'},r_{s'}-1-j+r} \delta_{s',s} \delta_{1 \leq r \leq j}. \quad (\text{J-7})$$

Thus, it is obvious that the r.h.s. of (4-10) and (4-8) are null for $s' \neq s$ so that

$$\begin{aligned} 0 &= \nu_{\infty,k}^{(\mathbf{u}_{X_s,j})}, \forall k \in \llbracket -1, r_{\infty} - 3 \rrbracket, \\ 0 &= \nu_{X_{s'},k}^{(\mathbf{u}_{X_s,j})}, \forall (s',k) \in (\llbracket 1, n \rrbracket \setminus \{s\}) \times \llbracket 1, r_{s'} - 1 \rrbracket. \end{aligned} \quad (\text{J-8})$$

Similarly, the r.h.s. of (4-20) and (4-21) are null for $s' \neq s$ so that

$$\begin{aligned} 0 &= c_{\infty,k}^{(\mathbf{u}_{X_s,j})}, \forall k \in \llbracket 1, r_{\infty} - 1 \rrbracket, \\ 0 &= c_{X_{s'},k}^{(\mathbf{u}_{X_s,j})}, \forall (s',k) \in (\llbracket 1, n \rrbracket \setminus \{s\}) \times \llbracket 1, r_{s'} - 1 \rrbracket. \end{aligned} \quad (\text{J-9})$$

Let us now turn to the r.h.s. of (4-8) for $s' = s$. By definition we have

$$\left(\alpha_{X_s,1}^{(i)}, \dots, \alpha_{X_s,r_s-1}^{(i)} \right)^t = \left(t_{X_s^{(i)},r_s-j}, \dots, j t_{X_s^{(i)},r_s-1}, 0, \dots, 0 \right)^t. \quad (\text{J-10})$$

Since $\alpha_{X_s} = 0$, we get that the r.h.s. of (4-8) is

$$\left(0, \dots, 0, -(t_{X_s^{(1)}, r_s-1} - t_{X_s^{(2)}, r_s-1}), \dots, -(t_{X_s^{(1)}, r_s-j} - t_{X_s^{(2)}, r_s-j})\right)^t$$

which is precisely the opposite of $(r_s - j)^{\text{th}}$ column of M_s . Since M_s is invertible, we necessarily have $\left(\nu^{(\mathbf{u}_{X_s, j})_{X_1}}, \dots, \nu^{(\mathbf{u}_{X_s, j})_{X_{r_s-1}}}\right)^t = -\mathbf{e}_{r_s-j}$, i.e.

$$\nu_{X_k}^{(\mathbf{u}_{X_s, j})} = -\delta_{k, r_s-j}, \quad \forall k \in \llbracket 1, r_s - 1 \rrbracket. \quad (\text{J-11})$$

Finally, let us look at the r.h.s. of (4-21) for $s' = s$. The i^{th} line, denoted RHS_i is given by

$$\text{RHS}_i = \sum_{k=r_s-i}^{r_s-1} \frac{t_{X_s^{(2)}, 2r_s-1-i-k} \alpha_{X_s^{(1)}, k} - t_{X_s^{(1)}, 2r_s-1-i-k} \alpha_{X_s^{(2)}, k}}{k}. \quad (\text{J-12})$$

However, by definition $\alpha_{X_s^{(1)}, r} = r t_{X_s^{(1)}, r_s-1-j+r} \delta_{1 \leq r \leq j}$ and $\alpha_{X_s^{(2)}, r} = r t_{X_s^{(2)}, r_s-1-j+r} \delta_{1 \leq r \leq j}$ so that

$$\begin{aligned} \text{RHS}_i &= \sum_{r=r_s-i}^{r_s-1} (t_{X_s^{(2)}, 2r_s-1-i-r} t_{X_s^{(1)}, r_s-1-j+r} - t_{X_s^{(1)}, 2r_s-1-i-r} t_{X_s^{(2)}, r_s-1-j+r}) \delta_{1 \leq r \leq j} \\ &= \sum_{r=r_s-i}^{r_s-1} t_{X_s^{(2)}, 2r_s-1-i-r} t_{X_s^{(1)}, r_s-1-j+r} \delta_{1 \leq r \leq j} \\ &\quad - \sum_{r=r_s-i}^{r_s-1} t_{X_s^{(1)}, 2r_s-1-i-r} t_{X_s^{(2)}, r_s-1-j+r} \delta_{1 \leq r \leq j} \\ &= \sum_{r=r_\infty-i}^j t_{X_s^{(2)}, 2r_s-1-i-r} t_{X_s^{(1)}, r_s-1-j+r} - \sum_{r=r_\infty-i}^j t_{X_s^{(1)}, 2r_s-1-i-r} t_{X_s^{(2)}, r_s-1-j+r} \\ &\stackrel{u=r_\infty+j-i-r}{=} \sum_{r=r_s-i}^j t_{X_s^{(2)}, 2r_s-1-i-r} t_{X_s^{(1)}, r_s-1-j+r} - \sum_{u=r_s-i}^j t_{X_s^{(1)}, r_s-1-j+u} t_{X_s^{(2)}, 2r_s-1-i-u} \\ &= 0. \end{aligned} \quad (\text{J-13})$$

Thus, the r.h.s. of (4-21) is null and so are all $\left(c_{X_s, k}^{(\mathbf{u}_{X_s, j})}\right)_{1 \leq k \leq r_s-1}$.

K Proof of Proposition 6.3

K.1 The case of $\mathcal{L}_a$

The main idea of the proof is that the connections between $\left(\nu_{\infty, k}^{(\mathbf{a})}\right)_{-1 \leq k \leq r_\infty-3}$, $\left(\nu_{X_s, k}^{(\mathbf{a})}\right)_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ and $\left(\alpha_{\infty^{(i)}, k}\right)_{1 \leq k \leq r_\infty-1}$, $\left(\alpha_{X_s^{(i)}, k}\right)_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ given by (E-5) and (E-6) are independent of α_{X_s} . The same is true for the connection between $\left(c_{\infty, k}^{(\mathbf{a})}\right)_{-1 \leq k \leq r_\infty-3}$, $\left(c_{X_s, k}^{(\mathbf{a})}\right)_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ and $\left(\alpha_{\infty^{(i)}, k}\right)_{1 \leq k \leq r_\infty-1}$, $\left(\alpha_{X_s^{(i)}, k}\right)_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ given by (4-20) and (4-21) are independent of α_{X_s} .

More precisely, we observe from (E-5) that $\left(\nu_{\infty,k}^{(\mathbf{a})}\right)_{-1 \leq k \leq r_\infty-3}$ are only determined by the term involving $\mathcal{L}_{\mathbf{u}_{\infty,r_\infty-1}}$. Hence, from Proposition 6.2, we get that $\nu_{\infty,k}^{(\mathbf{a})} = \delta_{k,-1}$ for all $k \in \llbracket -1, r_\infty - 3 \rrbracket$. Similarly from (4-20) we have $c_{\infty,k}^{(\mathbf{a})} = 0$ for all $k \in \llbracket 1, r_\infty - 1 \rrbracket$.

Similarly, from (E-6) we observe that for any $s \in \llbracket 1, n \rrbracket$, $\left(\nu_{X_s,k}^{(\mathbf{a})}\right)_{1 \leq k \leq r_s-1}$ are only determined by the term involving $\mathcal{L}_{\mathbf{u}_{X_s,r_s-1}}$. Hence, from Proposition 6.2, we get that $\nu_{X_s,k}^{(\mathbf{a})} = \delta_{k,1}$ for all $k \in \llbracket 1, r_s - 1 \rrbracket$. Similarly from (4-21) we have $c_{X_s,k}^{(\mathbf{a})} = 0$ for all $k \in \llbracket 1, r_s - 1 \rrbracket$.

Finally from (E-6), we get that for all $s \in \llbracket 1, n \rrbracket$, $\nu_{X_s,0}^{(\mathbf{a})} = X_s$.

K.2 The case of $\mathcal{L}_{\mathbf{b}}$

Similarly, we observe from (E-5) that $\left(\nu_{\infty,k}^{(\mathbf{b})}\right)_{-1 \leq k \leq r_\infty-3}$ are only determined by the term involving $\mathcal{L}_{\mathbf{u}_{\infty,r_\infty-2}}$. Hence, from Proposition 6.2, we get that $\nu_{\infty,k}^{(\mathbf{b})} = \delta_{k,0}$ for all $k \in \llbracket -1, r_\infty - 3 \rrbracket$. In the same way, from (4-20) we have $c_{\infty,k}^{(\mathbf{b})} = 0$ for all $k \in \llbracket 1, r_\infty - 1 \rrbracket$.

Similarly, from (E-6) and (4-21) it is trivial that $\nu_{X_s,k}^{(\mathbf{b})} = 0$ and $c_{X_s,k}^{(\mathbf{b})} = 0$ for all $k \in \llbracket 1, r_s - 1 \rrbracket$.

Finally from (E-6), we get that for all $s \in \llbracket 1, n \rrbracket$, $\nu_{X_s,0}^{(\mathbf{b})} = 1$.

L Proof of Theorem 6.1

L.1 The case of $\mathcal{L}_{\mathbf{v}_{\infty,k}}$

Let $k \in \llbracket 1, r_\infty - 1 \rrbracket$. $\mathcal{L}_{\mathbf{v}_{\infty,k}}$ gives vanishing $\left(\nu_{\infty,m}^{(\mathbf{v}_{\infty,k})}\right)_{1 \leq m \leq r_\infty-1}$ and vanishing $\left(\nu_{X_s,m}^{(\mathbf{v}_{\infty,k})}\right)_{1 \leq s \leq n, 1 \leq m \leq r_s-1}$ from Proposition 6.1. Moreover, it also gives vanishing $\left(\nu_{X_s,0}^{(\mathbf{v}_{\infty,k})}\right)_{1 \leq s \leq n}$ because $\alpha_{X_s} = 0$ for all $s \in \llbracket 1, n \rrbracket$. Thus, the r.h.s. of (4-14) given by (4-16) is vanishing so that all $\left(\mu_j^{(\mathbf{v}_{\infty,k})}\right)_{1 \leq j \leq g}$ are null. Thus we trivially get $\mathcal{L}_{\mathbf{v}_{\infty,k}}[q_j] = 0$ for all $j \in \llbracket 1, g \rrbracket$ from (5-1).

Moreover, $\mathcal{L}_{\mathbf{v}_{\infty,k}}[p_j]$ given by (5-7) reduces to $\hbar \sum_{i=1}^{r_\infty-1} i c_{\infty,i}^{(\mathbf{v}_{\infty,k})} q_j^{i-1} - \sum_{s=1}^n \sum_{i=1}^{r_s-1} i c_{X_s,i}^{(\mathbf{v}_{\infty,k})} (q_j - X_s)^{-i-1}$.

But Proposition 6.1 implies that this reduces only to $-\hbar q_j^{k-1}$. Thus, we end up with

$$\mathcal{L}_{\mathbf{v}_{\infty,k}}[q_j] = 0, \quad \mathcal{L}_{\mathbf{v}_{\infty,k}}[p_j] = -\hbar q_j^{k-1}. \quad (\text{L-1})$$

L.2 The case of $\mathcal{L}_{\mathbf{v}_{X_s,k}}$

Let $s_0 \in \llbracket 1, n \rrbracket$ and $k \in \llbracket 1, r_{s_0} \rrbracket$. $\mathcal{L}_{\mathbf{v}_{X_{s_0},k}}$ gives vanishing $\left(\nu_{\infty,m}^{(\mathbf{v}_{X_{s_0},k})}\right)_{1 \leq m \leq r_\infty-1}$ and vanishing $\left(\nu_{X_s,m}^{(\mathbf{v}_{X_{s_0},k})}\right)_{1 \leq s \leq n, 1 \leq m \leq r_s-1}$ from Proposition 6.1. Moreover, it also gives vanishing $\left(\nu_{X_s,0}^{(\mathbf{v}_{X_{s_0},k})}\right)_{1 \leq s \leq n}$ because $\alpha_{X_s} = 0$ for all $s \in \llbracket 1, n \rrbracket$. Thus, the r.h.s. of (4-14) given by (4-16) is vanishing so that all $\left(\mu_j^{(\mathbf{v}_{X_{s_0},k})}\right)_{1 \leq j \leq g}$ are null. Thus we trivially get $\mathcal{L}_{\mathbf{v}_{X_{s_0},k}}[q_j] = 0$ for all $j \in \llbracket 1, g \rrbracket$ from (5-1). Moreover, $\mathcal{L}_{\mathbf{v}_{X_{s_0},k}}[p_j]$ given by (5-7) reduces to

$\hbar \sum_{i=1}^{r_\infty-1} i c_{\infty,i}^{(\mathbf{v}_{X_{s_0},k})} q_j^{i-1} - \sum_{s=1}^n \sum_{i=1}^{r_s-1} i c_{X_s,i}^{(\mathbf{v}_{X_{s_0},k})} (q_j - X_s)^{-i-1}$. But Proposition 6.1 implies that this reduces only to $\hbar(q_j - X_{s_0})^{-k-1}$. Thus, we end up with

$$\mathcal{L}_{\mathbf{v}_{X_{s_0},k}}[q_j] = 0, \quad \mathcal{L}_{\mathbf{v}_{X_{s_0},k}}[p_j] = \hbar(q_j - X_{s_0})^{-k-1}. \quad (\text{L-2})$$

L.3 The case of $\mathcal{L}_{\mathbf{a}}$

We use results of Proposition 6.3 into (4-16). We get:

$$\boldsymbol{\nu}_{\infty}^{(\mathbf{a})} = \mathbf{0}, \quad \boldsymbol{\nu}_{X_s}^{(\mathbf{a})} = \begin{pmatrix} -\nu_{X_s,0}^{(\mathbf{a})} + \nu_{\infty,0}^{(\mathbf{a})} + \nu_{\infty,-1}^{(\mathbf{a})} X_s \\ -\nu_{X_s,1}^{(\mathbf{a})} + \nu_{\infty,-1}^{(\mathbf{a})} \\ -\nu_{X_s,2}^{(\mathbf{a})} \\ \vdots \\ -\nu_{X_s,r_s-1}^{(\mathbf{a})} \end{pmatrix} = \mathbf{0}. \quad (\text{L-3})$$

Hence from (4-14) we trivially get,

$$\forall j \in \llbracket 1, g \rrbracket : \mu_j^{(\mathbf{a})} = 0. \quad (\text{L-4})$$

Finally, we get from (5-1), (5-7) that for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_{\mathbf{a}}[q_j] = -\hbar q_j, \quad \mathcal{L}_{\mathbf{a}}[p_j] = \hbar p_j. \quad (\text{L-5})$$

L.4 The case of $\mathcal{L}_{\mathbf{b}}$

We use results of Proposition 6.3 into (4-16). We get:

$$\boldsymbol{\nu}_{\infty}^{(\mathbf{b})} = \mathbf{0}, \quad \boldsymbol{\nu}_{X_s}^{(\mathbf{b})} = \begin{pmatrix} -\nu_{X_s,0}^{(\mathbf{b})} + \nu_{\infty,0}^{(\mathbf{b})} + \nu_{\infty,-1}^{(\mathbf{b})} X_s \\ -\nu_{X_s,1}^{(\mathbf{b})} + \nu_{\infty,-1}^{(\mathbf{b})} \\ -\nu_{X_s,2}^{(\mathbf{b})} \\ \vdots \\ -\nu_{X_s,r_s-1}^{(\mathbf{b})} \end{pmatrix} = \mathbf{0}. \quad (\text{L-6})$$

Hence from (4-14) we trivially get,

$$\forall j \in \llbracket 1, g \rrbracket : \mu_j^{(\mathbf{b})} = 0. \quad (\text{L-7})$$

Finally, we get from (5-1), (5-7) that for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_{\mathbf{b}}[q_j] = -\hbar, \quad \mathcal{L}_{\mathbf{b}}[p_j] = 0. \quad (\text{L-8})$$

M Proof of Proposition 6.5

From the definition of the trivial and isomonodromic times given in each of the three cases (Definitions 6.2, 6.3 and 6.4), one may obtain the irregular times by inverting the change of coordinates. In fact, it is obvious that the only non-trivial cases are $(t_{\infty(1),k} - t_{\infty(2),k})_{1 \leq k \leq r_\infty-1}$, that we may combine with $T_{\infty,k}$ in order to obtain $t_{\infty(1),k}$ and $t_{\infty(2),k}$.

Let us prove the following Lemma:

Lemma M.1. For complex numbers A and B and any $r \in \llbracket 2, r_\infty - 2 \rrbracket$:

$$\begin{aligned}
& \sum_{j=2}^r \frac{(r_\infty - 2 - j)!}{(r_\infty - 2 - r)!(r - j)!} A^{r_\infty - 1 - j} B^{r - j} \\
& \left(\sum_{i=0}^{j-2} \frac{(-1)^i (r_\infty - 2 - j + i)!}{i!(r_\infty - 2 - j)!} \frac{B^i (t_{\infty(1), r_\infty - 1 - j + i} - t_{\infty(2), r_\infty - 1 - j + i})}{A^{r_\infty - 1 - j}} + \frac{(-1)^{j-1} (r_\infty - 2)!}{j(j-2)!(r_\infty - 2 - j)!} B^j A^j \right) \\
& = -\frac{(r_\infty - 2)!}{(r_\infty - 2 - r)!r!} B^r A^{r_\infty - 1} + (t_{\infty(1), r_\infty - 1 - r} - t_{\infty(2), r_\infty - 1 - r}).
\end{aligned} \tag{M-1}$$

Proof. We have

$$\begin{aligned}
& \sum_{j=2}^r \frac{(r_\infty - 2 - j)!}{(r_\infty - 2 - r)!(r - j)!} A^{r_\infty - 1 - j} B^{r - j} \sum_{i=0}^{j-2} \frac{(-1)^i (r_\infty - 2 - j + i)!}{i!(r_\infty - 2 - j)!} \frac{B^i (t_{\infty(1), r_\infty - 1 - j + i} - t_{\infty(2), r_\infty - 1 - j + i})}{A^{r_\infty - 1 - j}} \\
& + \sum_{j=2}^r \frac{(r_\infty - 2 - j)!}{(r_\infty - 2 - r)!(r - j)!} A^{r_\infty - 1 - j} B^{r - j} \frac{(-1)^{j-1} (r_\infty - 2)!}{j(j-2)!(r_\infty - 2 - j)!} B^j A^j \\
& = \sum_{j=2}^r \frac{(-1)^{j-1} (r_\infty - 2)!}{(r_\infty - 2 - r)!(r - j)!j(j-2)!} B^r A^{r_\infty - 1} \\
& + \sum_{j=2}^r \sum_{i=0}^{j-2} \frac{(-1)^i (r_\infty - 2 - j + i)!}{(r_\infty - 2 - r)!(r - j)!i!} B^{r+i-j} (t_{\infty(1), r_\infty - 1 - j + i} - t_{\infty(2), r_\infty - 1 - j + i}) \\
& \stackrel{i=j-s}{=} \frac{(r_\infty - 2)!}{(r_\infty - 2 - r)!} B^r A^{r_\infty - 1} \sum_{j=2}^r \frac{(-1)^{j-1}}{(r - j)!j(j-2)!} \\
& + \sum_{j=2}^r \sum_{s=2}^j \frac{(-1)^{j-s} (r_\infty - 2 - s)!}{(r_\infty - 2 - r)!(r - j)!(j - s)!} B^{r-s} (t_{\infty(1), r_\infty - 1 - s} - t_{\infty(2), r_\infty - 1 - s}) \\
& = \frac{(r_\infty - 2)!}{(r_\infty - 2 - r)!} B^r A^{r_\infty - 1} \left(-\frac{1}{r!} \right) + \sum_{s=2}^r \sum_{j=s}^r \frac{(-1)^{j-s} (r_\infty - 2 - s)!}{(r_\infty - 2 - r)!(r - j)!(j - s)!} B^{r-s} (t_{\infty(1), r_\infty - 1 - s} - t_{\infty(2), r_\infty - 1 - s}) \\
& \stackrel{p=j-s}{=} -\frac{(r_\infty - 2)!}{(r_\infty - 2 - r)!r!} B^r A^{r_\infty - 1} + \sum_{s=2}^r \left(\sum_{p=0}^{r-s} \frac{(-1)^p}{(r - s - p)!p!} \right) \frac{(r_\infty - 2 - s)!}{(r_\infty - 2 - r)!} B^{r-s} (t_{\infty(1), r_\infty - 1 - s} - t_{\infty(2), r_\infty - 1 - s}) \\
& = -\frac{(r_\infty - 2)!}{(r_\infty - 2 - r)!r!} B^r A^{r_\infty - 1} + \sum_{s=2}^r (\delta_{r-s=0}) \frac{(r_\infty - 2 - s)!}{(r_\infty - 2 - r)!} B^{r-s} (t_{\infty(1), r_\infty - 1 - s} - t_{\infty(2), r_\infty - 1 - s}) \\
& = -\frac{(r_\infty - 2)!}{(r_\infty - 2 - r)!r!} B^r A^{r_\infty - 1} + (t_{\infty(1), r_\infty - 1 - r} - t_{\infty(2), r_\infty - 1 - r}).
\end{aligned} \tag{M-2}$$

□

Lemma M.1 implies that for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$\begin{aligned}
& t_{\infty(1), k} - t_{\infty(2), k} = \frac{(r_\infty - 2)!}{(k-1)!(r_\infty - 1 - k)!} B^{r_\infty - 1 - k} A^{r_\infty - 1} \\
& + \sum_{j=2}^{r_\infty - 1 - k} \frac{(r_\infty - 2 - j)!}{(k-1)!(r_\infty - 1 - k - j)!} A^{r_\infty - 1 - j} B^{r_\infty - 1 - k - j} \\
& \left(\sum_{i=0}^{j-2} \frac{(-1)^i (r_\infty - 2 - j + i)!}{i!(r_\infty - 2 - j)!} \frac{B^i (t_{\infty(1), r_\infty - 1 - j + i} - t_{\infty(2), r_\infty - 1 - j + i})}{A^{r_\infty - 1 - j}} \right. \\
& \left. + \frac{(-1)^{j-1} (r_\infty - 2)!}{j(j-2)!(r_\infty - 2 - j)!} B^j A^j \right).
\end{aligned} \tag{M-3}$$

Let us now observe that in the case $r_\infty \geq 3$ the expression for $(\tau_{\infty,k})_{1 \leq k \leq r_\infty-3}$ is given by

$$2^{-\frac{k}{r_\infty-1}} \tau_{\infty,k} = \sum_{i=0}^{r_\infty-k-3} \frac{(-1)^i (k+i-1)!}{i! (k-1)! (r_\infty-2)^i} \frac{(t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2})^i (t_{\infty(1),k+i} - t_{\infty(2),k+i})}{(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{i(r_\infty-1)+k}{r_\infty-1}}} \\ + \frac{(-1)^{r_\infty-k-2} (r_\infty-3)!}{(r_\infty-1-k)(r_\infty-k-3)!(k-1)!(r_\infty-2)^{r_\infty-k-2}} \frac{(t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2})^{r_\infty-1-k}}{(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{(r_\infty-2)(r_\infty-1-k)}{r_\infty-1}}} \quad (\text{M-4})$$

so that performing the change of variables $k = r_\infty - 1 - j$ leads to the formula for all $j \in \llbracket 2, r_\infty - 2 \rrbracket$:

$$2^{-\frac{r_\infty-1-j}{r_\infty-1}} \tau_{\infty,r_\infty-1-j} = \sum_{i=0}^{j-2} \frac{(-1)^i (r_\infty-2-j+i)!}{i! (r_\infty-2-j)!} \left(\frac{(t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2})}{(r_\infty-2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} \right)^i \\ \frac{1}{\left((t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{1}{r_\infty-1}} \right)^{r_\infty-1-j}} (t_{\infty(1),r_\infty-1-j+i} - t_{\infty(2),r_\infty-1-j+i}) \\ + \frac{(-1)^{j-1} (r_\infty-2)!}{j(j-2)!(r_\infty-2-j)!} \left(\frac{t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2}}{(r_\infty-2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} \right)^j \left((t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{1}{r_\infty-1}} \right)^j. \quad (\text{M-5})$$

Thus, applying Lemma M.1 with

$$B = \frac{t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2}}{(r_\infty-2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} \\ A = (t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{1}{r_\infty-1}} \quad (\text{M-6})$$

we get from (M-3) for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$t_{\infty(1),k} - t_{\infty(2),k} = \frac{(r_\infty-2)!}{(k-1)!(r_\infty-1-k)!} \left((t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{1}{r_\infty-1}} \right)^{r_\infty-1} \\ \left(\frac{t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2}}{(r_\infty-2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} \right)^{r_\infty-1-k} \\ + \sum_{j=2}^{r_\infty-1-k} \frac{(r_\infty-2-j)!}{(k-1)!(r_\infty-1-k-j)!} \left(\frac{t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2}}{(r_\infty-2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} \right)^{r_\infty-1-k-j} \\ \left((t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{1}{r_\infty-1}} \right)^{r_\infty-1-j} 2^{-\frac{r_\infty-1-j}{r_\infty-1}} \tau_{\infty,r_\infty-1-j}. \quad (\text{M-7})$$

The definition of T_1 and T_2 implies that

$$t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1} = 2T_2^{r_\infty-1}, \\ t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2} = 2(r_\infty-2)T_1T_2^{r_\infty-2}, \quad (\text{M-8})$$

so that

$$B = \frac{t_{\infty(1),r_\infty-2} - t_{\infty(2),r_\infty-2}}{(r_\infty-2)(t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})} = \frac{(r_\infty-2)T_1T_2^{r_\infty-2}}{(r_\infty-2)T_2^{r_\infty-1}} = \frac{T_1}{T_2} \\ A = (t_{\infty(1),r_\infty-1} - t_{\infty(2),r_\infty-1})^{\frac{1}{r_\infty-1}} = 2^{\frac{1}{r_\infty-1}} T_2. \quad (\text{M-9})$$

Hence from (M-7) we obtain for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$t_{\infty(1),k} - t_{\infty(2),k} = \frac{2(r_\infty-2)!}{(k-1)!(r_\infty-1-k)!} T_2^{r_\infty-1} \left(\frac{T_1}{T_2} \right)^{r_\infty-1-k}$$

$$\begin{aligned}
& + \sum_{j=2}^{r_\infty-1-k} 2^{\frac{r_\infty-1-j}{r_\infty-1}} \frac{(r_\infty-2-j)!}{(k-1)!(r_\infty-1-k-j)!} \left(\frac{T_1}{T_2}\right)^{r_\infty-1-j-k} T_2^{r_\infty-1-j} \tau_{\infty, r_\infty-1-j} \\
& = \frac{2(r_\infty-2)!}{(k-1)!(r_\infty-1-k)!} T_2^k T_1^{r_\infty-1-k} \\
& + T_2^k \sum_{j=2}^{r_\infty-1-k} 2^{\frac{r_\infty-1-j}{r_\infty-1}} \frac{(r_\infty-2-j)!}{(k-1)!(r_\infty-1-k-j)!} T_1^{r_\infty-1-j-k} 2^{-\frac{r_\infty-1-j}{r_\infty-1}} \tau_{\infty, r_\infty-1-j} \\
& = \frac{2(r_\infty-2)!}{(k-1)!(r_\infty-1-k)!} T_2^k T_1^{r_\infty-1-k} + T_2^k \sum_{j=2}^{r_\infty-1-k} \frac{(r_\infty-2-j)!}{(k-1)!(r_\infty-1-k-j)!} T_1^{r_\infty-1-j-k} \tau_{\infty, r_\infty-1-j}.
\end{aligned} \tag{M-10}$$

N Proof of Proposition 6.12

Let us start with $r_\infty \geq 3$. In this case $T_1 = \frac{t_{\infty(1), r_\infty-2} - t_{\infty(2), r_\infty-2}}{2^{\frac{1}{r_\infty-1}} (r_\infty-2) (t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1})^{\frac{r_\infty-2}{r_\infty-1}}}$ and $T_2 = \left(\frac{t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1}}{2}\right)^{\frac{1}{r_\infty-1}}$. By symmetry, it is obvious that $(\mathcal{L}_{\mathbf{v}_{\infty, k}})_{1 \leq k \leq r_\infty-1}$ and $(\mathcal{L}_{\mathbf{v}_{X_s, k}})_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ acts trivially on T_1 and T_2 . We also have

$$\begin{aligned}
\mathcal{L}_{\mathbf{a}}[T_1] &= \hbar \left((r_\infty-2) \frac{t_{\infty(1), r_\infty-2} - t_{\infty(2), r_\infty-2}}{2^{\frac{1}{r_\infty-1}} (r_\infty-2) (t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1})^{\frac{r_\infty-2}{r_\infty-1}}} \right. \\
&\quad \left. - (r_\infty-1) \frac{r_\infty-2}{r_\infty-1} \frac{t_{\infty(1), r_\infty-2} - t_{\infty(2), r_\infty-2}}{2^{\frac{1}{r_\infty-1}} (r_\infty-2) (t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1})^{\frac{r_\infty-2}{r_\infty-1}}} \right) \\
&= 0, \\
\mathcal{L}_{\mathbf{b}}[T_1] &= \hbar \frac{(r_\infty-2) (t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1})}{2^{\frac{1}{r_\infty-1}} (r_\infty-2) (t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1})^{\frac{r_\infty-2}{r_\infty-1}}} \\
&= \hbar T_2, \\
\mathcal{L}_{\mathbf{a}}[T_2] &= \hbar \left(\frac{t_{\infty(1), r_\infty-1} - t_{\infty(2), r_\infty-1}}{2} \right)^{\frac{1}{r_\infty-1}} = \hbar T_2, \\
\mathcal{L}_{\mathbf{b}}[T_2] &= 0.
\end{aligned} \tag{N-1}$$

Let us continue with the case $r_\infty = 2$. In this case, we have $T_1 = -X_1 \left(\frac{t_{\infty(1), 1} - t_{\infty(2), 1}}{2} \right)$. and $T_2 = \left(\frac{t_{\infty(1), 1} - t_{\infty(2), 1}}{2} \right)$. Since

$$\begin{aligned}
\mathcal{L}_{\mathbf{a}} &= \hbar (t_{\infty(1), 1} \partial_{t_{\infty(1), 1}} + t_{\infty(2), 1} \partial_{t_{\infty(2), 1}}) - \hbar \sum_{s=1}^n X_s \partial_{X_s} \\
&\quad - \hbar \sum_{s=1}^n \sum_{r=1}^{r_s-1} r (t_{X_s^{(1)}, r} \partial_{t_{X_s^{(1)}, r}} + t_{X_s^{(2)}, 1} \partial_{t_{X_s^{(2)}, 1}}), \\
\mathcal{L}_{\mathbf{b}} &= -\hbar \sum_{s=1}^n \partial_{X_s},
\end{aligned} \tag{N-2}$$

we immediately get that $\mathcal{L}_{\mathbf{a}}[T_1] = 0$, $\mathcal{L}_{\mathbf{b}}[T_1] = \hbar T_2$, $\mathcal{L}_{\mathbf{a}}[T_2] = \hbar T_2$ and $\mathcal{L}_{\mathbf{b}}[T_2] = 0$.

Let us now consider $r_\infty = 1$ and $n \geq 2$. In this case we have $T_1 = -\frac{X_1}{X_2 - X_1}$ and $T_2 = \frac{1}{X_2 - X_1}$. The action of $\mathcal{L}_\mathbf{a}$ and $\mathcal{L}_\mathbf{b}$ on these two quantities respectively reduces only to $-\hbar \sum_{s=1}^n X_s \partial_{X_s}$ and $-\hbar \sum_{s=1}^n \partial_{X_s}$ so that the result is trivial.

Finally, let us consider $r_\infty = 1$ and $n = 1$. In this case we have $T_1 = -X_1 \left(\frac{t_{X_1^{(1)}, r_1-1} - t_{X_1^{(2)}, r_1-1}}{2} \right)^{-\frac{1}{r_1-1}}$ and $T_2 = \left(\frac{t_{X_1^{(1)}, r_1-1} - t_{X_1^{(2)}, r_1-1}}{2} \right)^{-\frac{1}{r_1-1}}$. The action of $\mathcal{L}_\mathbf{b}$ is immediate and the action of $\mathcal{L}_\mathbf{a}$ provides $\mathcal{L}_\mathbf{a}[T_1] = 0$ and $\mathcal{L}_\mathbf{a}[T_2] = \hbar T_2$.

O Proof of Theorem 6.2

Let us look at solutions $f(X_s, t_{\infty^{(i)}, k}, t_{X_s^{(i)}, k})$ of the set of partial differential equations:

$$\begin{aligned} 0 &= \mathcal{L}_{v_{\infty, k}}[f], \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\ 0 &= \mathcal{L}_{v_{X_s, k}}[f], \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ 0 &= \mathcal{L}_\mathbf{a}[f], \\ 0 &= \mathcal{L}_\mathbf{b}[f]. \end{aligned} \tag{O-1}$$

The first two lines are equivalent to $(\partial_{t_{\infty^{(1)}, k}} + \partial_{t_{\infty^{(2)}, k}})[f] = 0$ and $(\partial_{t_{X_s^{(1)}, k}} + \partial_{t_{X_s^{(2)}, k}})[f] = 0$. This is equivalent to say that f may only be a function of the differences $\tilde{t}_{\infty, k} := t_{\infty^{(1)}, k} - t_{\infty^{(2)}, k}$ and of the sums $\tilde{t}_{s, k} := t_{X_s^{(1)}, k} + t_{X_s^{(2)}, k}$. Let us now recall that

$$\begin{aligned} \mathcal{L}_\mathbf{a} &= \hbar \sum_{r=1}^{r_\infty-1} r(t_{\infty^{(1)}, r} \partial_{t_{\infty^{(1)}, r}} + t_{\infty^{(2)}, r} \partial_{t_{\infty^{(2)}, r}}) \\ &\quad - \hbar \sum_{s=1}^n \sum_{r=1}^{r_s-1} r(t_{X_s^{(1)}, r} \partial_{t_{X_s^{(1)}, r}} + t_{X_s^{(2)}, r} \partial_{t_{X_s^{(2)}, r}}) - \hbar \sum_{s=1}^n X_s \partial_{X_s}. \end{aligned} \tag{O-2}$$

Using the former result, we have:

$$\mathcal{L}_\mathbf{a}[f] = \hbar \sum_{r=1}^{r_\infty-1} r \tilde{t}_{\infty, r} \partial_{\tilde{t}_{\infty, r}}[f] - \hbar \sum_{s=1}^n \sum_{r=1}^{r_s-1} r \tilde{t}_{s, r} \partial_{\tilde{t}_{s, r}}[f] - \hbar \sum_{s=1}^n X_s \partial_{X_s}[f]. \tag{O-3}$$

Similarly, we have

$$\begin{aligned} \mathcal{L}_\mathbf{b} &= \hbar \sum_{r=1}^{r_\infty-2} r(t_{\infty^{(1)}, r+1} \partial_{t_{\infty^{(1)}, r}} + t_{\infty^{(2)}, r+1} \partial_{t_{\infty^{(2)}, r}}) - \hbar \sum_{s=1}^n \partial_{X_s} \\ &= \hbar \sum_{r=1}^{r_\infty-2} r \tilde{t}_{\infty, r+1} \partial_{\tilde{t}_{\infty, r}} - \hbar \sum_{s=1}^n \partial_{X_s}. \end{aligned} \tag{O-4}$$

It is then obvious to verify that $(\tilde{X}_s)_{1 \leq s \leq n}$ and $(\tau_{X_s, k})_{1 \leq s \leq n, 1 \leq k \leq r_s-1}$ are solutions of the system (O-1). Therefore, only $(\tau_{\infty, k})_{1 \leq k \leq r_\infty-1}$ are non-trivial for $r_\infty \geq 3$.

Then, we have the following result

Proposition O.1. *The general solutions of the differential equation*

$$\hbar \sum_{r=1}^{r_\infty-1} r \tilde{t}_{\infty, r} \partial_{\tilde{t}_{\infty, r}}[f] - \hbar \sum_{s=1}^n \sum_{r=1}^{r_s-1} r \tilde{t}_{s, k} \partial_{\tilde{t}_{s, k}}[f] \tag{O-5}$$

are arbitrary functions of the variables

$$\begin{aligned} y_{\infty,k} &= \frac{\tilde{t}_{\infty,k}}{\tilde{t}_{\infty,r_{\infty}-1}^{\frac{k}{r_{\infty}-1}}}, \quad \forall k \in \llbracket 1, r_{\infty} - 2 \rrbracket \\ y_{s,k} &= \frac{\tilde{t}_{s,k}}{\tilde{t}_{s,r_s-1}^{\frac{k}{r_s-1}}}, \quad \forall s \in \llbracket 1, n \rrbracket, k \in \llbracket 1, r_s - 2 \rrbracket. \end{aligned} \quad (\text{O-6})$$

Proof. The proof is easy. We have for $k \in \llbracket 1, r_{\infty} - 2 \rrbracket$:

$$\begin{aligned} \mathcal{L}_{\mathbf{a}}[y_{\infty,k}] &= \hbar \sum_{r=1}^{r_{\infty}-1} r \tilde{t}_{\infty,r} \partial_{\tilde{t}_{\infty,r}} \left[\frac{\tilde{t}_{\infty,k}}{\tilde{t}_{\infty,r_{\infty}-1}^{\frac{k}{r_{\infty}-1}}} \right] \\ &= \hbar \left(k \frac{\tilde{t}_{\infty,k}}{\tilde{t}_{\infty,r_{\infty}-1}^{\frac{k}{r_{\infty}-1}}} - (r_{\infty} - 1) \frac{k}{r_{\infty} - 1} \tilde{t}_{\infty,r_{\infty}-1} \frac{\tilde{t}_{\infty,k}}{\tilde{t}_{\infty,r_{\infty}-1}^{\frac{k}{r_{\infty}-1} + 1}} \right) \\ &= 0 \end{aligned} \quad (\text{O-7})$$

and a similar computation for $(y_{s,k})_{1 \leq s \leq n, 1 \leq k \leq r_s - 2}$. $\square$

We now insert this result into $\mathcal{L}_{\mathbf{b}}$ that is given by

$$\mathcal{L}_{\mathbf{b}} = \hbar \sum_{r=1}^{r_{\infty}-2} r (t_{\infty(1),r+1} \partial_{t_{\infty(1),r}} + t_{\infty(2),r+1} \partial_{t_{\infty(2),r}}) - \hbar \sum_{s=1}^n \partial_{X_s}. \quad (\text{O-8})$$

We observe that

$$\begin{aligned} &\sum_{r=1}^{r_{\infty}-2} r (t_{\infty(1),r+1} \partial_{t_{\infty(1),r}} + t_{\infty(2),r+1} \partial_{t_{\infty(2),r}}) [f] = \sum_{r=1}^{r_{\infty}-2} r \tilde{t}_{\infty,r+1} \partial_{\tilde{t}_{\infty,r}} [f] \\ &= \sum_{r=1}^{r_{\infty}-2} r \tilde{t}_{\infty,r+1} \sum_{r'=1}^{r_{\infty}-2} \frac{\partial y_{\infty,r'}}{\partial \tilde{t}_{\infty,r}} \frac{\partial}{\partial y_{\infty,r'}} [f] = \sum_{r=1}^{r_{\infty}-2} r \tilde{t}_{\infty,r+1} \frac{1}{\tilde{t}_{\infty,r_{\infty}-1}^{\frac{r}{r_{\infty}-1}}} \frac{\partial}{\partial y_{\infty,r}} [f] \\ &= (r_{\infty} - 2) \tilde{t}_{\infty,r_{\infty}-1}^{\frac{r_{\infty}-2}{r_{\infty}-1}} \frac{\partial}{\partial y_{\infty,r_{\infty}-2}} [f] + \sum_{r=1}^{r_{\infty}-3} r \tilde{t}_{\infty,r+1} \frac{1}{\tilde{t}_{\infty,r_{\infty}-1}^{\frac{r}{r_{\infty}-1}}} \frac{\partial}{\partial y_{\infty,r}} [f] \\ &= \tilde{t}_{\infty,r_{\infty}-1}^{\frac{1}{r_{\infty}-1}} \left((r_{\infty} - 2) \frac{\partial}{\partial y_{\infty,r_{\infty}-2}} + \sum_{r=1}^{r_{\infty}-3} r y_{\infty,r+1} \frac{\partial}{\partial y_{\infty,r}} \right) [f] \end{aligned} \quad (\text{O-9})$$

so that we have to find general solutions of

$$\left((r_{\infty} - 2) \partial_{y_{\infty,r_{\infty}-2}} + \sum_{r=1}^{r_{\infty}-3} r y_{\infty,r+1} \partial_{y_{\infty,r}} \right) [f(y_{\infty,1}, \dots, y_{\infty,r_{\infty}-2})] = 0. \quad (\text{O-10})$$

Proposition O.2. *The general solutions of the differential equation*

$$(r_{\infty} - 2) \partial_{y_{r_{\infty}-2}} f(y_1, \dots, y_{r_{\infty}-2}) + \sum_{r=1}^{r_{\infty}-3} r y_{r+1} \partial_{y_r} f(y_1, \dots, y_{r_{\infty}-2}) = 0 \quad (\text{O-11})$$

are arbitrary functions of

$$\begin{aligned}
f_2(y_1, \dots, y_{r_\infty-2}) &= y_{r_\infty-3} - \frac{(r_\infty-3)}{2(r_\infty-2)} y_{r_\infty-2}^2 \\
f_3(y_1, \dots, y_{r_\infty-2}) &= y_{r_\infty-4} - \frac{r_\infty-4}{r_\infty-2} y_{r_\infty-2} y_{r_\infty-3} + \frac{(r_\infty-4)(r_\infty-3)}{3(r_\infty-2)^2} y_{r_\infty-2}^3 \\
f_k(y_1, \dots, y_{r_\infty-2}) &= y_{r_\infty-1-k} + \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^i y_{r_\infty-1-k+i}}{i! (r_\infty-2-k)! (r_\infty-2)^i} \\
&\quad + \frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^k}{k(k-2)! (r_\infty-2-k)! (r_\infty-2)^{k-1}}
\end{aligned} \tag{O-12}$$

where $k \in \llbracket 2, r_\infty-3 \rrbracket$.

Proof. Let $k \in \llbracket 2, r_\infty-3 \rrbracket$. We have:

$$\begin{aligned}
(r_\infty-2) \partial_{y_{r_\infty-2}} f_k &= \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^{i-1} y_{r_\infty-1-k+i}}{(i-1)! (r_\infty-2-k)! (r_\infty-2)^{i-1}} \\
&\quad + \frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^{k-1}}{(k-2)! (r_\infty-2-k)! (r_\infty-2)^{k-2}}.
\end{aligned} \tag{O-13}$$

Thus, we have by denoting $\Delta f_k = (r_\infty-2) \partial_{y_{r_\infty-2}} f_k(y_1, \dots, y_{r_\infty-2}) + \sum_{r=1}^{r_\infty-3} r y_{r+1} \partial_{y_r} f_k$:

$$\begin{aligned}
\Delta f_k &= (r_\infty-1-k) y_{r_\infty-k} + \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^{i-1} y_{r_\infty-1-k+i}}{(i-1)! (r_\infty-2-k)! (r_\infty-2)^{i-1}} + \frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^{k-1}}{(k-2)! (r_\infty-2-k)! (r_\infty-2)^{k-2}} \\
&\quad + \sum_{r=1}^{r_\infty-3} r y_{r+1} \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^i \partial_{y_r} [y_{r_\infty-1-k+i}]}{i! (r_\infty-2-k)! (r_\infty-2)^i} \\
&= (r_\infty-1-k) y_{r_\infty-k} + \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^{i-1} y_{r_\infty-1-k+i}}{(i-1)! (r_\infty-2-k)! (r_\infty-2)^{i-1}} + \frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^{k-1}}{(k-2)! (r_\infty-2-k)! (r_\infty-2)^{k-2}} \\
&\quad + \sum_{i=1}^{k-2} \frac{(r_\infty-1-k+i) (r_\infty-2-k+i)! y_{r_\infty-k+i} (-1)^i y_{r_\infty-2}^i}{i! (r_\infty-2-k)! (r_\infty-2)^i} \\
&= (r_\infty-1-k) y_{r_\infty-k} + \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^{i-1} y_{r_\infty-1-k+i}}{(i-1)! (r_\infty-2-k)! (r_\infty-2)^{i-1}} + \frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^{k-1}}{(k-2)! (r_\infty-2-k)! (r_\infty-2)^{k-2}} \\
&\quad + \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-1-k+i)! y_{r_\infty-2}^i y_{r_\infty-k+i}}{i! (r_\infty-2-k)! (r_\infty-2)^i} \\
&= (r_\infty-1-k) y_{r_\infty-k} + \sum_{i=1}^{k-2} \frac{(-1)^i (r_\infty-2-k+i)! y_{r_\infty-2}^{i-1} y_{r_\infty-1-k+i}}{(i-1)! (r_\infty-2-k)! (r_\infty-2)^{i-1}} + \frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^{k-1}}{(k-2)! (r_\infty-2-k)! (r_\infty-2)^{k-2}} \\
&\quad - \sum_{j=2}^{k-1} \frac{(-1)^j (r_\infty-2-k+j)! y_{r_\infty-2}^{j-1} y_{r_\infty-1-k+j}}{(j-1)! (r_\infty-2-k)! (r_\infty-2)^{j-1}} \\
&= 0
\end{aligned} \tag{O-14}$$

because the term $i=1$ in the first sum gives $-(r_\infty-1-k) y_{r_\infty-k}$ while the term $j=k-1$ in the last sum provides $-\frac{(-1)^{k-1} (r_\infty-3)! y_{r_\infty-2}^{k-1} y_{r_\infty-2-k+k}}{(k-2)! (r_\infty-k)! (r_\infty-2)^{k-2}}$ so that we end up with canceling terms. $\square$

From Proposition (O.2) and by replacing $y_{p,k}$ in terms of the original irregular times we get $(\tau_{\infty,k})_{1 \leq k \leq r_\infty-3}$. One can also verify by simple computations that the differential systems (O-1) is not satisfied for any trivial times. Since the set $\mathcal{T}$ of isomonodromic times and trivial times is in one-to-one correspondence with the set of irregular times, we conclude that functions of isomonodromic times are the only solutions of the differential systems (O-1).

P Proof of Theorem 6.3

In this section, we prove that the initial symplectic form

$$\begin{aligned}\Omega &= \hbar \sum_{j=1}^g dq_j \wedge dp_j - \sum_{s=1}^n \sum_{i=1}^2 \sum_{k=1}^{r_s-1} dt_{X_s^{(i)},k} \wedge d\text{Ham}^{(\mathbf{e}_{X_s^{(i)},k})} \\ &\quad - \sum_{i=1}^2 \sum_{k=1}^{r_\infty-1} dt_{\infty^{(i)},k} \wedge d\text{Ham}^{(\mathbf{e}_{\infty^{(i)},k})} - \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(\mathbf{e}_{X_s})}\end{aligned}\quad (\text{P-1})$$

is equal to

$$\tilde{\Omega} = \hbar \sum_{j=1}^g d\tilde{q}_j \wedge d\tilde{p}_j - \sum_{\tau \in \mathcal{T}_{\text{iso}}} d\tau \wedge d\text{Ham}^{(\alpha_\tau)} \quad (\text{P-2})$$

where the isomonodromic times $\mathcal{T}_{\text{iso}} = \{\tau_k\}_{1 \leq k \leq g}$ are given by Definitions 6.2, 6.3, 6.4 and 6.5 while the corresponding vectors $(\alpha_{\tau_k})_{1 \leq k \leq g}$ in the tangent space are given by Propositions 6.5, 6.7, 6.9 and 6.11 depending on the value of r_∞ and n .

P.1 First step: reduction to $\mathfrak{sl}_2(\mathbb{C})$

Let us first observe that we have from Theorem 5.1 that:

$$\begin{aligned}\text{Ham}^{(\mathbf{v}_{\infty,j})}(\mathbf{q}, \mathbf{p}) &= \frac{\hbar}{j} \sum_{i=1}^g q_i^j, \quad \forall j \in \llbracket 1, r_\infty - 1 \rrbracket, \\ \text{Ham}^{(\mathbf{v}_{X_s,j})}(\mathbf{q}, \mathbf{p}) &= \frac{\hbar}{j} \sum_{i=1}^g (q_i - X_s)^{-j}, \quad \forall (s, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket.\end{aligned}\quad (\text{P-3})$$

Let us now introduce an intermediate step and define some intermediate Darboux coordinates $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ by

$$\hat{q}_j := q_j, \quad \hat{p}_j := p_j - \frac{1}{2} P_1(q_j), \quad \forall j \in \llbracket 1, g \rrbracket. \quad (\text{P-4})$$

Note that they are related to $(\check{\mathbf{q}}, \check{\mathbf{p}})$ by the relation

$$\check{q}_j = T_2 \hat{q}_j + T_1, \quad \check{p}_j = T_2^{-1} \hat{p}_j, \quad \forall j \in \llbracket 1, g \rrbracket. \quad (\text{P-5})$$

The associated vectors in the tangent space are defined by:

$$\begin{aligned}\mathbf{w}_{\infty,k} &:= \mathbf{e}_{\infty^{(1)},k} - \mathbf{e}_{\infty^{(2)},k}, \quad \forall k \in \llbracket 0, r_\infty - 1 \rrbracket, \\ \mathbf{w}_{X_s,k} &:= \mathbf{e}_{X_s^{(1)},k} - \mathbf{e}_{X_s^{(2)},k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 0, r_s - 1 \rrbracket.\end{aligned}\quad (\text{P-6})$$

From the fact that $\partial_{\mathbf{v}_{X_s,j}} \hat{p}_i = -\partial_{q_i} \text{Ham}^{(\mathbf{v}_{X_s,j})} - \frac{1}{2} \partial_{\mathbf{v}_{X_s,j}} [P_1](q_i)$, it is straightforward to observe that

$$\begin{aligned}\text{Ham}^{(\mathbf{v}_{\infty,j})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 0, \quad \forall j \in \llbracket 1, r_\infty - 1 \rrbracket, \\ \text{Ham}^{(\mathbf{v}_{X_s,j})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 0, \quad \forall (s, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket\end{aligned}\quad (\text{P-7})$$

where we recall that

$$\begin{aligned}\mathbf{v}_{\infty,k} &:= \mathbf{e}_{\infty^{(1)},k} + \mathbf{e}_{\infty^{(2)},k}, \quad \forall k \in \llbracket 0, r_\infty - 1 \rrbracket, \\ \mathbf{v}_{X_s,k} &:= \mathbf{e}_{X_s^{(1)},k} + \mathbf{e}_{X_s^{(2)},k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 0, r_s - 1 \rrbracket.\end{aligned}\quad (\text{P-8})$$

Since we have

$$\begin{aligned}
\text{Ham}^{(\mathbf{v}_\infty, k)}(\mathbf{q}, \mathbf{p}) &= \text{Ham}^{(\mathbf{e}_{\infty(1), k})}(\mathbf{q}, \mathbf{p}) + \text{Ham}^{(\mathbf{e}_{\infty(2), k})}(\mathbf{q}, \mathbf{p}), \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\
\text{Ham}^{(\mathbf{w}_\infty, k)}(\mathbf{q}, \mathbf{p}) &= \text{Ham}^{(\mathbf{e}_{\infty(1), k})}(\mathbf{q}, \mathbf{p}) - \text{Ham}^{(\mathbf{e}_{\infty(2), k})}(\mathbf{q}, \mathbf{p}), \quad \forall k \in \llbracket 1, r_\infty - 1 \rrbracket, \\
\text{Ham}^{(\mathbf{v}_{X_s, k})}(\mathbf{q}, \mathbf{p}) &= \text{Ham}^{(\mathbf{e}_{X_s(1), k})}(\mathbf{q}, \mathbf{p}) + \text{Ham}^{(\mathbf{e}_{X_s(2), k})}(\mathbf{q}, \mathbf{p}), \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
\text{Ham}^{(\mathbf{w}_{X_s, k})}(\mathbf{q}, \mathbf{p}) &= \text{Ham}^{(\mathbf{e}_{X_s(1), k})}(\mathbf{q}, \mathbf{p}) - \text{Ham}^{(\mathbf{e}_{X_s(2), k})}(\mathbf{q}, \mathbf{p}), \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket,
\end{aligned} \tag{P-9}$$

it is a straightforward computation to get:

$$dq_j \wedge dp_j = d\hat{q}_j \wedge d\hat{p}_j + \frac{1}{2} \sum_{k=1}^{r_\infty-1} q_j^{k-1} dT_{\infty, k} \wedge d\hat{q}_j - \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} (q_j - X_s)^{-k-1} dT_{X_s, k} \wedge d\hat{q}_j. \tag{P-10}$$

At the level of coordinates, we shall introduce the corresponding times:

$$\begin{aligned}
T_{\infty, k} &= t_{\infty(1), k} + t_{\infty(2), k}, \quad \hat{T}_{\infty, k} := t_{\infty(1), k} - t_{\infty(2), k}, \quad \forall k \in \llbracket 0, r_\infty - 1 \rrbracket, \\
T_{X_s, k} &= t_{X_s(1), k} + t_{X_s(2), k}, \quad \hat{T}_{X_s, k} := t_{X_s(1), k} - t_{X_s(2), k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 0, r_s - 1 \rrbracket.
\end{aligned} \tag{P-11}$$

We get that

$$\begin{aligned}
&\sum_{s=1}^n \sum_{i=1}^2 \sum_{k=1}^{r_s-1} dt_{X_s(i), k} \wedge d\text{Ham}^{(\mathbf{e}_{X_s(i), k})}(\mathbf{q}, \mathbf{p}) + \sum_{i=1}^2 \sum_{k=1}^{r_\infty-1} dt_{\infty(i), k} \wedge d\text{Ham}^{(\mathbf{e}_{\infty(i), k})}(\mathbf{q}, \mathbf{p}) \\
&= \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} (dT_{X_s, k} \wedge d\text{Ham}^{(\mathbf{v}_{X_s, k})}(\mathbf{q}, \mathbf{p}) + d\hat{T}_{X_s, k} \wedge d\text{Ham}^{(\mathbf{w}_{X_s, k})}(\mathbf{q}, \mathbf{p})) \\
&+ \frac{1}{2} \sum_{k=1}^{r_\infty-1} (dT_{\infty, k} \wedge d\text{Ham}^{(\mathbf{v}_{\infty, k})}(\mathbf{q}, \mathbf{p}) + d\hat{T}_{\infty, k} \wedge d\text{Ham}^{(\mathbf{w}_{\infty, k})}(\mathbf{q}, \mathbf{p})) \\
&= \frac{1}{2} \sum_{p \in \mathcal{R}} d\hat{T}_{p, k} \wedge d\text{Ham}^{(\mathbf{w}_{p, k})}(\mathbf{q}, \mathbf{p}) + \frac{\hbar}{2} \sum_{j=1}^g \sum_{k=1}^{r_\infty-1} q_j^{k-1} dT_{\infty, k} \wedge dq_j \\
&- \frac{\hbar}{2} \sum_{j=1}^g \sum_{k=1}^{r_s-1} (q_j - X_s)^{-k-1} dT_{X_s, k} \wedge dq_j.
\end{aligned} \tag{P-12}$$

Combining this identity with (P-10) we obtain the intermediate identity

$$\Omega = \hbar \sum_{j=1}^g d\hat{q}_j \wedge d\hat{p}_j - \frac{1}{2} \sum_{p \in \mathcal{R}} \sum_{k=1}^{r_p-1} d\hat{T}_{p, k} \wedge d\text{Ham}^{(\mathbf{w}_{p, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \tag{P-13}$$

Let us also mention that Theorem 5.1 and Proposition 6.3 implies that

$$\begin{aligned}
\text{Ham}^{(\mathbf{a})}(\mathbf{q}, \mathbf{p}) &= -\hbar \sum_{j=1}^g q_j p_j - \delta_{r_\infty=1} \left(\sum_{s=1}^n t_{X_s(1), 0} t_{X_s(2), 0} \delta_{r_s=1} - t_{\infty(1), 0} (t_{\infty(2), 0} + \hbar) \right), \\
\text{Ham}^{(\mathbf{b})}(\mathbf{q}, \mathbf{p}) &= -\hbar \sum_{j=1}^g p_j + \delta_{r_\infty=2} \left(t_{\infty(1), 1} t_{\infty(2), 0} + t_{\infty(2), 1} t_{\infty(1), 0} + \hbar t_{\infty(1), 1} \right).
\end{aligned} \tag{P-14}$$

The quantities proportional to $\delta_{r_\infty=1}$ and $\delta_{r_\infty=2}$ do not depend on the Darboux coordinates and thus do not play any role in the Hamiltonian system. Therefore, one may discard them without

changing the corresponding Hamiltonian system and we get

$$\text{Ham}^{(a)}(\mathbf{q}, \mathbf{p}) = -\hbar \sum_{j=1}^g q_j p_j, \quad \text{Ham}^{(b)}(\mathbf{q}, \mathbf{p}) = -\hbar \sum_{j=1}^g p_j. \quad (\text{P-15})$$

However, one has to be extremely careful when changing $(\mathbf{q}, \mathbf{p}) \leftrightarrow (\hat{\mathbf{q}}, \hat{\mathbf{p}})$ because the change of coordinates is time dependent (because of the position of poles and times arising in P_1).

Using Proposition 6.12 and Theorem 6.1 we have for all $j \in \llbracket 1, g \rrbracket$:

$$\mathcal{L}_a[\hat{q}_j] = -\hbar \hat{q}_j, \quad \mathcal{L}_a[\hat{p}_j] = \hbar \hat{p}_j, \quad \mathcal{L}_b[\hat{q}_j] = -\hbar, \quad \mathcal{L}_b[\hat{p}_j] = 0. \quad (\text{P-16})$$

Thus, we end up with

$$\forall j \in \llbracket 1, g \rrbracket : \text{Ham}^{(a)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = -\hbar \sum_{j=1}^g \hat{q}_j \hat{p}_j, \quad \text{Ham}^{(b)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = -\hbar \sum_{j=1}^g \hat{p}_j. \quad (\text{P-17})$$

Note in particular that there is no $(P_1(q_j))_{1 \leq j \leq g}$ terms in the previous formulas as one would have obtained by simply replacing $(\mathbf{q}, \mathbf{p})$ in terms of $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ in (P-15).

The second step of the proof is to reduce (P-13) into (P-2). Since the definition of the trivial and isomonodromic times differs on the values of r_∞ and n , we shall study separately each case although the strategy is identical for all cases.

P.2 The case $r_\infty \geq 3$

For $r_\infty \geq 3$, Definition 6.2 and Proposition 6.4 are equivalent to

$$\begin{aligned} T_2 &= \left(\frac{\hat{T}_{\infty, r_\infty-1}}{2} \right)^{\frac{1}{r_\infty-1}} \\ T_1 &= \frac{\hat{T}_{\infty, r_\infty-2}}{(r_\infty-2) 2^{\frac{1}{r_\infty-1}} (\hat{T}_{\infty, r_\infty-1})^{\frac{r_\infty-2}{r_\infty-1}}} \\ \tau_{\infty, j} &= 2^{\frac{j}{r_\infty-1}} \left[\sum_{i=0}^{r_\infty-j-3} \frac{(-1)^i (j+i-1)!}{i! (j-1)! (r_\infty-2)^i} \frac{(\hat{T}_{\infty, r_\infty-2})^i \hat{T}_{\infty, j+i}}{(\hat{T}_{\infty, r_\infty-1})^{\frac{i(r_\infty-1)+j}{r_\infty-1}}} \right. \\ &\quad \left. + \frac{(-1)^{r_\infty-j-2} (r_\infty-3)!}{(r_\infty-1-j)(r_\infty-j-3)! (j-1)! (r_\infty-2)^{r_\infty-j-2}} \frac{(\hat{T}_{\infty, r_\infty-2})^{r_\infty-1-j}}{(\hat{T}_{\infty, r_\infty-1})^{\frac{(r_\infty-2)(r_\infty-1-j)}{r_\infty-1}}} \right], \quad \forall j \in \llbracket 1, r_\infty-3 \rrbracket \\ \tau_{X_s, k} &= \hat{T}_{X_s, k} \left(\frac{\hat{T}_{\infty, r_\infty-1}}{2} \right)^{\frac{k}{r_\infty-1}}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s-1 \rrbracket \\ \tilde{X}_s &= X_s \left(\frac{\hat{T}_{\infty, r_\infty-1}}{2} \right)^{\frac{1}{r_\infty-1}} + \frac{\hat{T}_{\infty, r_\infty-2}}{(r_\infty-2) 2^{\frac{1}{r_\infty-1}} (\hat{T}_{\infty, r_\infty-1})^{\frac{r_\infty-2}{r_\infty-1}}}, \quad \forall s \in \llbracket 1, n \rrbracket \end{aligned} \quad (\text{P-18})$$

and the inverse relations

$$\begin{aligned} \hat{T}_{\infty, r_\infty-1} &= 2T_2^{r_\infty-1} \\ \hat{T}_{\infty, r_\infty-2} &= 2(r_\infty-2)T_1T_2^{r_\infty-2} \\ \hat{T}_{\infty, k} &= T_2^k \left(2 \binom{r_\infty-2}{k-1} T_1^{r_\infty-1-k} + \sum_{j=2}^{r_\infty-1-k} \binom{r_\infty-2-j}{k-1} T_1^{r_\infty-1-j-k} \tau_{\infty, r_\infty-1-j} \right), \quad \forall k \in \llbracket 1, r_\infty-3 \rrbracket \\ \hat{T}_{X_s, k} &= T_2^{-k} \tau_{X_s, k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s-1 \rrbracket \\ X_s &= T_2^{-1} (\tilde{X}_s - T_1), \quad \forall s \in \llbracket 1, n \rrbracket. \end{aligned} \quad (\text{P-19})$$

We thus obtain at the level of differentials

$$d\hat{T}_{\infty, r_\infty-1} = 2(r_\infty-1)T_2^{r_\infty-2}dT_2$$

$$\begin{aligned}
d\hat{T}_{\infty, r_\infty-2} &= 2(r_\infty-2)T_2^{r_\infty-2}dT_1 + 2(r_\infty-2)^2T_1T_2^{r_\infty-3}dT_2 \\
d\hat{T}_{\infty, k} &= kT_2^{k-1} \left(2 \binom{r_\infty-2}{k-1} T_1^{r_\infty-1-k} + \sum_{m=2}^{r_\infty-1-k} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-1-m-k} \tau_{\infty, r_\infty-1-m} \right) dT_2 \\
&\quad + 2T_2^k (r_\infty-1-k) \binom{r_\infty-2}{k-1} T_1^{r_\infty-2-k} dT_1 \\
&\quad + T_2^k \sum_{m=2}^{r_\infty-2-k} (r_\infty-1-m-k) \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-2-m-k} \tau_{\infty, r_\infty-1-m} dT_1 \\
&\quad + T_2^k \sum_{m=2}^{r_\infty-1-k} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-1-m-k} d\tau_{\infty, r_\infty-1-m}, \quad \forall k \in \llbracket 1, r_\infty-3 \rrbracket \\
d\hat{T}_{X_s, k} &= -kT_2^{k-1} \tau_{X_s, k} dT_2 + T_2^{-k} d\tau_{X_s, k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s-1 \rrbracket \\
dX_s &= -T_2^{-2} (\tilde{X}_s - T_1) dT_2 + T_2^{-1} d\tilde{X}_s - T_2^{-1} dT_1, \quad \forall s \in \llbracket 1, n \rrbracket.
\end{aligned} \tag{P-20}$$

At the level of Hamiltonians, since $\text{Ham}^{(\mathbf{w}_{\infty, k})} = 2\text{Ham}^{(\alpha_{\hat{T}_{\infty, k}})}$ for all $k \in \llbracket 1, r_\infty-1 \rrbracket$ and $\text{Ham}^{(\mathbf{w}_{X_s, k})} = 2\text{Ham}^{(T_{X_s, k})}$ for all $(s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s-1 \rrbracket$, this is equivalent to

$$\begin{aligned}
\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= (r_\infty-2)T_2^{r_\infty-2}\text{Ham}^{(\mathbf{w}_{\infty, r_\infty-2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2^{-1} \sum_{s=1}^n \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_\infty-3} 2T_2^k (r_\infty-1-k) \binom{r_\infty-2}{k-1} T_1^{r_\infty-2-k} \text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_\infty-3} T_2^k \sum_{m=2}^{r_\infty-2-k} (r_\infty-1-m-k) \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-2-m-k} \tau_{\infty, r_\infty-1-m} \text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
\text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= (r_\infty-1)T_2^{r_\infty-2}\text{Ham}^{(\mathbf{w}_{\infty, r_\infty-1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + (r_\infty-2)^2T_1T_2^{r_\infty-3}\text{Ham}^{(\mathbf{w}_{\infty, r_\infty-2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_\infty-3} 2kT_2^{k-1} \binom{r_\infty-2}{k-1} T_1^{r_\infty-1-k} \text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_\infty-3} kT_2^{k-1} \sum_{m=2}^{r_\infty-1-k} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-1-m-k} \tau_{\infty, r_\infty-1-m} \text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&\quad - \sum_{s=1}^n \sum_{k=1}^{r_s-1} kT_2^{k-1} \tau_{X_s, k} \text{Ham}^{(\mathbf{w}_{X_s, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2^{-2} \sum_{s=1}^n (\tilde{X}_s - T_1) \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
\text{Ham}^{(\alpha_{\tau_{\infty, i}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{2} \sum_{k=1}^i T_2^k \binom{i-1}{k-1} T_1^{i-k} \text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall i \in \llbracket 1, r_\infty-3 \rrbracket \\
\text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{2} T_2^{-k} \text{Ham}^{(\mathbf{w}_{X_s, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s-1 \rrbracket \\
\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2^{-1} \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall s \in \llbracket 1, n \rrbracket.
\end{aligned} \tag{P-21}$$

Homogeneity in T_2 in (P-19) and Definition 6.1 implies that

$$\mathcal{L}_{\mathbf{a}} = \hbar T_2 \partial_{T_2} + \frac{1}{2} \sum_{k=1}^{r_\infty-1} T_{\infty, k} \mathcal{L}_{\mathbf{v}_{\infty, k}} - \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k T_{X_s, k} \mathcal{L}_{\mathbf{v}_{X_s, k}}. \tag{P-22}$$

From (P-19), we also get:

$$\begin{aligned}
\partial_{T_1} &= -T_2^{-1} \sum_{s=1}^n \partial_{X_s} + T_2^{-1} (r_\infty-2) \hat{T}_{\infty, r_\infty-1} \partial_{\hat{T}_{\infty, r_\infty-2}} + \sum_{k=1}^{r_\infty-3} T_2^k (r_\infty-1-k) \binom{r_\infty-2}{k-1} T_1^{r_\infty-2-k} \partial_{\hat{T}_{\infty, k}} \\
&\quad + \sum_{k=1}^{r_\infty-3} T_2^k \sum_{j=2}^{r_\infty-1-k} (r_\infty-1-j-k) \binom{r_\infty-2-j}{k-1} T_1^{r_\infty-2-j-k} \tau_{\infty, r_\infty-1-j} \partial_{\hat{T}_{\infty, k}} \\
&= -T_2^{-1} \sum_{s=1}^n \partial_{X_s} + T_2^{-1} (r_\infty-2) \hat{T}_{\infty, r_\infty-1} \partial_{\hat{T}_{\infty, r_\infty-2}} + T_2^{-1} \sum_{k=1}^{r_\infty-3} k \hat{T}_{\infty, k+1} \partial_{\hat{T}_{\infty, k}}.
\end{aligned} \tag{P-23}$$

Thus, we get from Definition 6.1 that

$$\mathcal{L}_{\mathbf{b}} = \hbar T_2 \partial_{T_1} + \frac{1}{2} \sum_{k=1}^{r_\infty-2} T_{\infty, k+1} \mathcal{L}_{\mathbf{v}_{\infty, k}}. \quad (\text{P-24})$$

Using (P-7) into (P-22) and (P-24) we finally obtain

$$\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2 \text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2 \text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \quad (\text{P-25})$$

Let now us observe that (P-17) implies that

$$\begin{aligned} & T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-1} dT_1 \wedge d\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2^{-2} \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) dT_1 \wedge dT_2 \\ &= - \left(T_2^{-1} \sum_{j=1}^g \hat{q}_j dT_2 \wedge d\hat{p}_j + T_2^{-1} \sum_{j=1}^g \hat{p}_j dT_2 \wedge d\hat{q}_j + T_2^{-1} \sum_{j=1}^g dT_1 \wedge d\hat{p}_j - T_2^{-2} \sum_{j=1}^g \hat{p}_j dT_1 \wedge dT_2 \right) \end{aligned} \quad (\text{P-26})$$

so that the shifted Darboux coordinates satisfy

$$\sum_{j=1}^g d\hat{q}_j \wedge d\hat{p}_j = \sum_{j=1}^g d\hat{q}_j \wedge d\hat{p}_j + T_2^{-1} \sum_{j=1}^g \hat{q}_j dT_2 \wedge d\hat{p}_j + T_2^{-1} \sum_{j=1}^g \hat{p}_j dT_2 \wedge d\hat{q}_j + T_2^{-1} \sum_{j=1}^g dT_1 \wedge d\hat{p}_j - T_2^{-2} \sum_{j=1}^g \hat{p}_j dT_1 \wedge dT_2 \quad (\text{P-27})$$

i.e.

$$\sum_{j=1}^g d\hat{q}_j \wedge d\hat{p}_j = \sum_{j=1}^g d\hat{q}_j \wedge d\hat{p}_j + T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-1} dT_1 \wedge d\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2^{-2} \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) dT_1 \wedge dT_2. \quad (\text{P-28})$$

We now need to invert the Hamiltonian relations (P.2). In order to do that, we observe that the following identities hold for any numbers $(h_j)_{j \geq 1}$ and $(\tau_k)_{k \geq 1}$:

$$\begin{aligned} & \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k (r_\infty - 1 - k) (-1)^{j+k} \binom{r_\infty-2}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-j} h_j = -(r_\infty - 2) \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} h_j \\ & \sum_{k=1}^{r_\infty-3} \sum_{m=2}^{r_\infty-2-k} \sum_{j=1}^k (-1)^{j+k} (r_\infty - 1 - m - k) \binom{r_\infty-2-m}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-m-j} \tau_{r_\infty-1-m} h_j = \sum_{k=1}^{r_\infty-4} k \tau_{k+1} h_k \\ & \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k k (-1)^{j+k} \binom{r_\infty-2}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-1-j} h_j \\ &= - \sum_{m=1}^{r_\infty-3} (r_\infty^2 - m r_\infty + 2m - 4r_\infty + 3) (-1)^{m+r_\infty} \binom{r_\infty-2}{m-1} T_1^{r_\infty-1-m} h_m \\ & \sum_{k=1}^{r_\infty-3} \sum_{m=2}^{r_\infty-1-k} \sum_{j=1}^k (-1)^{j+k} k \binom{r_\infty-2-m}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-1-m-j} \tau_{r_\infty-1-m} h_j = \sum_{k=1}^{r_\infty-3} k \tau_k h_k - T_1 \sum_{k=1}^{r_\infty-4} k \tau_{k+1} h_k. \end{aligned} \quad (\text{P-29})$$

Using these identities, we may invert the previous Hamiltonian relations P.2:

$$\begin{aligned} \text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^{-k} \sum_{j=1}^k (-1)^{j+k} \binom{k-1}{j-1} T_1^{k-j} \text{Ham}^{(\alpha_{\tau_{\infty, j}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \forall k \in \llbracket 1, r_\infty - 3 \rrbracket \\ \text{Ham}^{(\mathbf{w}_{X_s, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^k \text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket \\ \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2 \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \forall s \in \llbracket 1, n \rrbracket \\ \text{Ham}^{(\mathbf{w}_{\infty, r_\infty-2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{(r_\infty - 2) T_2^{r_\infty-2}} \text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \frac{1}{(r_\infty - 2) T_2^{r_\infty-2}} \sum_{s=1}^n \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\ &+ \frac{2}{T_2^{r_\infty-2}} \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_{\infty, j}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \frac{1}{(r_\infty - 2) T_2^{r_\infty-2}} \sum_{k=1}^{r_\infty-4} k \tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_{\infty, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\ \text{Ham}^{(\mathbf{w}_{\infty, r_\infty-1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{(r_\infty - 1) T_2^{r_\infty-2}} \text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \frac{1}{(r_\infty - 1) T_2^{r_\infty-1}} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \tau_{X_s, k} \text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{s=1}^n (\tilde{X}_s - T_1) \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{2}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{m=1}^{r_\infty - 3} (r_\infty^2 - mr_\infty + 2m - 4r_\infty + 3)(-1)^{m+r_\infty} \binom{r_\infty - 2}{m-1} T_1^{r_\infty - 1 - m} \text{Ham}^{(\alpha_{\tau_\infty, m})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \left(T_1 \sum_{k=1}^{r_\infty - 4} k\tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \sum_{k=1}^{r_\infty - 3} k\tau_{\infty, k} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \right) \\
& - \frac{(r_\infty - 2)}{(r_\infty - 1)T_2^{r_\infty - 1}} T_1 \text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \frac{(r_\infty - 2)}{(r_\infty - 1)T_2^{r_\infty - 1}} T_1 \sum_{s=1}^n \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& - 2 \frac{(r_\infty - 2)^2}{(r_\infty - 1)T_2^{r_\infty - 1}} T_1 \sum_{j=1}^{r_\infty - 3} (-1)^{j+r_\infty} \binom{r_\infty - 3}{j-1} T_1^{r_\infty - 2 - j} \text{Ham}^{(\alpha_{\tau_\infty, j})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{(r_\infty - 2)}{(r_\infty - 1)T_2^{r_\infty - 1}} T_1 \sum_{k=1}^{r_\infty - 4} k\tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}).
\end{aligned} \tag{P-30}$$

Note in particular that we have

$$\begin{aligned}
\text{Ham}^{(\mathbf{w}_{\infty, r_\infty - 1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{(r_\infty - 1)T_2^{r_\infty - 2}} \text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \frac{(r_\infty - 2)}{(r_\infty - 1)T_2^{r_\infty - 1}} T_1 \text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{s=1}^n \sum_{k=1}^{r_s - 1} k\tau_{X_s, k} \text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{s=1}^n (\tilde{X}_s - (r_\infty - 1)T_1) \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{1}{T_2^{r_\infty - 1}} T_1 \sum_{k=1}^{r_\infty - 4} k\tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{k=1}^{r_\infty - 3} k\tau_{\infty, k} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& - \frac{2}{T_2^{r_\infty - 1}} \sum_{m=1}^{r_\infty - 3} (-1)^{m+r_\infty} T_1^{r_\infty - 1 - m} \binom{r_\infty - 2}{m-1} \text{Ham}^{(\alpha_{\tau_\infty, m})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}).
\end{aligned} \tag{P-31}$$

Combining (P-30) with (P-30) and (P-31), we obtain:

$$\begin{aligned}
\text{Ham}^{(\mathbf{w}_{\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^{-k} \sum_{j=1}^k (-1)^{j+k} \binom{k-1}{j-1} T_1^{k-j} \text{Ham}^{(\alpha_{\tau_\infty, j})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall k \in \llbracket 1, r_\infty - 3 \rrbracket \\
\text{Ham}^{(\mathbf{w}_{X_s, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^k \text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket \\
\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2 \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall s \in \llbracket 1, n \rrbracket \\
\text{Ham}^{(\mathbf{w}_{\infty, r_\infty - 2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{(r_\infty - 2)T_2^{r_\infty - 1}} \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \frac{1}{(r_\infty - 2)T_2^{r_\infty - 2}} \sum_{s=1}^n \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{2}{T_2^{r_\infty - 2}} \sum_{j=1}^{r_\infty - 3} (-1)^{j+r_\infty} \binom{r_\infty - 3}{j-1} T_1^{r_\infty - 2 - j} \text{Ham}^{(\alpha_{\tau_\infty, j})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& - \frac{1}{(r_\infty - 2)T_2^{r_\infty - 2}} \sum_{k=1}^{r_\infty - 4} k\tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
\text{Ham}^{(\mathbf{w}_{\infty, r_\infty - 1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \frac{(r_\infty - 2)}{(r_\infty - 1)T_2^{r_\infty}} T_1 \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{s=1}^n \sum_{k=1}^{r_s - 1} k\tau_{X_s, k} \text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{s=1}^n (\tilde{X}_s - (r_\infty - 1)T_1) \text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& + \frac{1}{T_2^{r_\infty - 1}} T_1 \sum_{k=1}^{r_\infty - 4} k\tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& - \frac{1}{(r_\infty - 1)T_2^{r_\infty - 1}} \sum_{k=1}^{r_\infty - 3} k\tau_{\infty, k} \text{Ham}^{(\alpha_{\tau_\infty, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})
\end{aligned}$$

$$-\frac{2}{T_2^{r_\infty-1}} \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} T_1^{r_\infty-1-m} \binom{r_\infty-2}{m-1} \text{Ham}^{(\alpha_{\tau_\infty, m})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \quad (\text{P-32})$$

We may now use (P-22), (P-24) and (P-32) to compute (P-13). For compactness, we shall drop the notation of the variables $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ in the various Hamiltonians unless necessary. The different pieces arising in (P-13) are:

$$\begin{aligned} -\sum_{s=1}^n dX_s \wedge d\text{Ham}^{(X_s)} &= -\left(-T_2^{-2}(\tilde{X}_s - T_1)dT_2 + T_2^{-1}d\tilde{X}_s - T_2^{-1}dT_1\right) \wedge \left(T_2 d\text{Ham}^{(\alpha_{\tilde{X}_s})} + \text{Ham}^{(\alpha_{\tilde{X}_s})}dT_2\right) \\ &= -\sum_{s=1}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} + \sum_{s=1}^n T_2^{-1}(\tilde{X}_s - T_1)dT_2 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} + \sum_{s=1}^n T_2^{-1}\text{Ham}^{(\alpha_{\tilde{X}_s})}dT_2 \wedge d\tilde{X}_s \\ &\quad + \sum_{s=1}^n dT_1 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} + T_2^{-1} \sum_{s=1}^n \text{Ham}^{(\alpha_{\tilde{X}_s})}dT_1 \wedge dT_2. \end{aligned} \quad (\text{P-33})$$

$$\begin{aligned} &-\frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\hat{T}_{X_s, k} \wedge d\text{Ham}^{(\mathbf{w}_{X_s, k})} \\ &= -\frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} \left(-kT_2^{-k-1}\tau_{X_s, k}dT_2 + T_2^{-k}d\tau_{X_s, k}\right) \wedge \left(2kT_2^{k-1}\text{Ham}^{(\alpha_{\tau_{X_s, k}})}dT_2 + 2T_2^k d\text{Ham}^{(\alpha_{\tau_{X_s, k}})}\right) \\ &= -\sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} \\ &\quad + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s, k}dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k\text{Ham}^{(\alpha_{\tau_{X_s, k}})}dT_2 \wedge d\tau_{X_s, k}. \end{aligned} \quad (\text{P-34})$$

$$\begin{aligned} &-\frac{1}{2} d\hat{T}_{\infty, r_\infty-1} \wedge d\text{Ham}^{(\mathbf{w}_{\infty, r_\infty-1})} = -T_2^{-1}dT_2 \wedge d\text{Ham}^{(\mathbf{a})} + (r_\infty-2)T_2^{-2}dT_2 \wedge d(T_1\text{Ham}^{(\mathbf{b})}) \\ &\quad - T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} kdT_2 \wedge d(\tau_{X_s, k}\text{Ham}^{(\alpha_{\tau_{X_s, k}})}) - T_2^{-1} \sum_{s=1}^n dT_2 \wedge d(\tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})}) \\ &\quad + (r_\infty-1)T_2^{-1} \sum_{s=1}^n \text{Ham}^{(\alpha_{\tilde{X}_s})}dT_2 \wedge dT_1 + (r_\infty-1)T_2^{-1}T_1 \sum_{s=1}^n dT_2 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\ &\quad + (r_\infty-1)T_2^{-1} \sum_{k=1}^{r_\infty-4} k\tau_{\infty, k+1}\text{Ham}^{(\alpha_{\tau_{\infty, k}})}dT_1 \wedge dT_2 - (r_\infty-1)T_2^{-1}T_1 \sum_{k=1}^{r_\infty-4} kdT_2 \wedge d(\tau_{\infty, k+1}\text{Ham}^{(\alpha_{\tau_{\infty, k}})}) \\ &\quad + T_2^{-1}dT_2 \wedge \sum_{k=1}^{r_\infty-3} kd(\tau_{\infty, k}\text{Ham}^{(\alpha_{\tau_{\infty, k}})}) \\ &\quad + 2(r_\infty-1)T_2^{-1}dT_2 \wedge \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} \binom{r_\infty-2}{m-1} d\left(T_1^{r_\infty-1-m}\text{Ham}^{(\alpha_{\tau_\infty, m})}\right) \end{aligned} \quad (\text{P-35})$$

$$\begin{aligned} &-\frac{1}{2} d\hat{T}_{\infty, r_\infty-2} \wedge d\text{Ham}^{(\mathbf{w}_{\infty, r_\infty-2})} = -((r_\infty-2)T_2^{r_\infty-2}dT_1 + (r_\infty-2)^2T_1T_2^{r_\infty-3}dT_2) \wedge (d\text{Ham}^{(\alpha_{\mathbf{w}_{\infty, r_\infty-2}})}) \\ &= -T_2^{-1}dT_1 \wedge d\text{Ham}^{(\mathbf{b})} - \sum_{s=1}^n dT_1 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\ &\quad + 2T_2^{-1}(r_\infty-2)^2 \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j}\text{Ham}^{(\alpha_{\tau_{\infty, j}})}dT_1 \wedge dT_2 \\ &\quad - 2(r_\infty-2) \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j}dT_1 \wedge d\text{Ham}^{(\alpha_{\tau_{\infty, j}})} + \sum_{k=1}^{r_\infty-4} kdT_1 \wedge d(\tau_{\infty, k+1}\text{Ham}^{(\alpha_{\tau_{\infty, k}})}) \\ &\quad - (r_\infty-2)T_1T_2^{-2}dT_2 \wedge d\text{Ham}^{(\mathbf{b})} - (r_\infty-2)T_2^{-1}T_1 \sum_{s=1}^n dT_2 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\ &\quad - 2(r_\infty-2)^2T_1T_2^{-1} \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} dT_2 \wedge d(T_1^{r_\infty-2-j}\text{Ham}^{(\alpha_{\tau_{\infty, j}})}) \end{aligned}$$

$$\begin{aligned}
& +(r_\infty - 2)T_1T_2^{-1} \sum_{k=1}^{r_\infty-4} kdT_2 \wedge d(\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}) \\
& +(r_\infty - 1)T_2^{-2}\text{Ham}^{(\mathbf{b})}dT_1 \wedge dT_2 + (r_\infty - 2)T_2^{-1} \sum_{s=1}^n \text{Ham}^{(\alpha_{\tilde{x}_s})}dT_1 \wedge dT_2 \\
& -(r_\infty - 2)T_2^{-1} \sum_{k=1}^{r_\infty-4} k\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}dT_1 \wedge dT_2.
\end{aligned} \tag{P-36}$$

Let us observe that we have

$$\begin{aligned}
& -T_2^{-1}dT_2 \wedge d\text{Ham}^{(\mathbf{a})} + (r_\infty - 2)T_2^{-2}dT_2 \wedge d(T_1\text{Ham}^{(\mathbf{b})}) - T_2^{-1}dT_1 \wedge d\text{Ham}^{(\mathbf{b})} - (r_\infty - 2)T_1T_2^{-2}dT_2 \wedge d\text{Ham}^{(\mathbf{b})} \\
& = -T_2^{-1}dT_2 \wedge d\text{Ham}^{(\mathbf{a})} - T_2^{-1}dT_1 \wedge d\text{Ham}^{(\mathbf{b})} + (r_\infty - 2)T_2^{-2}\text{Ham}^{(\mathbf{b})}dT_2 \wedge dT_1.
\end{aligned} \tag{P-37}$$

Moreover, it is a trivial check to see that the terms in (P-33), (P-34), (P-35) and (P-36) simplify so that we obtain the intermediate identity:

$$\begin{aligned}
& -\sum_{s=1}^n dX_s \wedge d\text{Ham}^{(X_s)} - \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\hat{T}_{X_s,k} \wedge d\text{Ham}^{(\mathbf{w}_{X_s,k})} - \frac{1}{2} d\hat{T}_{\infty,r_\infty-1} \wedge d\text{Ham}^{(\mathbf{w}_{\infty,r_\infty-1})} \\
& -\frac{1}{2} d\hat{T}_{\infty,r_\infty-2} \wedge d\text{Ham}^{(\mathbf{w}_{\infty,r_\infty-2})} = -\sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} \\
& -T_2^{-1}dT_2 \wedge d\text{Ham}^{(\mathbf{a})} - T_2^{-1}dT_1 \wedge d\text{Ham}^{(\mathbf{b})} + (r_\infty - 2)T_2^{-2}\text{Ham}^{(\mathbf{b})}dT_2 \wedge dT_1 \\
& +(r_\infty - 1)T_2^{-1} \sum_{k=1}^{r_\infty-4} k\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}dT_1 \wedge dT_2 \\
& -(r_\infty - 1)T_2^{-1}T_1 \sum_{k=1}^{r_\infty-4} kdT_2 \wedge d(\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}) + T_2^{-1}dT_2 \wedge \sum_{k=1}^{r_\infty-3} kd(\tau_{\infty,k}\text{Ham}^{(\alpha_{\tau_\infty,k})}) \\
& +2(r_\infty - 1)T_2^{-1}dT_2 \wedge \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} d\left(T_1^{r_\infty-1-m} \binom{r_\infty-2}{m-1} \text{Ham}^{(\alpha_{\tau_\infty,m})}\right) \\
& +2T_2^{-1}(r_\infty - 2)^2 \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty,j})}dT_1 \wedge dT_2 \\
& -2(r_\infty - 2) \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} dT_1 \wedge d\text{Ham}^{(\alpha_{\tau_\infty,j})} + \sum_{k=1}^{r_\infty-4} kdT_1 \wedge d(\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}) \\
& -2(r_\infty - 2)T_1T_2^{-1} \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} dT_2 \wedge d(T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty,j})}) \\
& +(r_\infty - 2)T_1T_2^{-1} \sum_{k=1}^{r_\infty-4} kdT_2 \wedge d(\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}) \\
& +(r_\infty - 1)T_2^{-2}\text{Ham}^{(\mathbf{b})}dT_1 \wedge dT_2 - (r_\infty - 2)T_2^{-1} \sum_{k=1}^{r_\infty-4} k\tau_{\infty,k+1}\text{Ham}^{(\alpha_{\tau_\infty,k})}dT_1 \wedge dT_2.
\end{aligned} \tag{P-38}$$

In order to simplify the previous quantity, we need to compute the last missing terms of (P-13) using (P-32). They correspond to

$$\begin{aligned}
-\frac{1}{2} \sum_{k=1}^{r_\infty-3} d\hat{T}_{\infty,k} \wedge d\text{Ham}^{(\alpha_{\mathbf{w}_{\infty,k}})} & = \sum_{k=1}^{r_\infty-3} d\hat{T}_{\infty,k} \wedge \left(kT_2^{-k-1} \sum_{j=1}^k (-1)^{j+k} \binom{k-1}{j-1} T_1^{k-j} \text{Ham}^{(\alpha_{\tau_\infty,j})}dT_2 \right. \\
& \quad - T_2^{-k} \sum_{j=1}^k (-1)^{j+k} (k-j) \binom{k-1}{j-1} \text{Ham}^{(\alpha_{\tau_\infty,j})} T_1^{k-j-1} dT_1 \\
& \quad \left. - T_2^{-k} \sum_{j=1}^k (-1)^{j+k} \binom{k-1}{j-1} T_1^{k-j} d\text{Ham}^{(\alpha_{\tau_\infty,j})} \right).
\end{aligned} \tag{P-39}$$

To obtain compact formulas, we note that we have the following identities for any $(h_j)_{j \geq 1}$ and $(\tau_k)_{k \geq 1}$:

$$\begin{aligned}
& \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k k(r_\infty-1-k)(-1)^{j+k} \binom{r_\infty-2}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-j} h_j \\
&= -(r_\infty-2)^2 \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} h_m - (r_\infty-2)(r_\infty-3) T_1 h_{r_\infty-3} \\
&= \sum_{k=1}^{r_\infty-3} \sum_{m=2}^{r_\infty-2-k} \sum_{j=1}^k (-1)^{j+k} k(r_\infty-1-m-k) \binom{r_\infty-2-m}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-m-j} \tau_{r_\infty-1-m} h_j \\
&= \sum_{k=1}^{r_\infty-4} k^2 \tau_{k+1} h_k - T_1 \sum_{k=1}^{r_\infty-5} k(k+1) \tau_{k+2} h_k \\
&= \sum_{k=1}^{r_\infty-3} \sum_{m=2}^{r_\infty-1-k} \sum_{j=1}^k (-1)^{j+k} k \binom{r_\infty-2-m}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-1-m-j} \tau_{r_\infty-1-m} h_j \\
&= \sum_{k=1}^{r_\infty-3} k \tau_k h_k - T_1 \sum_{k=1}^{r_\infty-4} k \tau_{k+1} h_k \\
&= \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k k(k-j)(-1)^{j+k} \binom{r_\infty-2}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-j} h_j \\
&= - \sum_{m=1}^{r_\infty-4} (-1)^{m+r_\infty} (r_\infty-2)(r_\infty^2 - (m+5)r_\infty + 2m+5) \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} h_m \\
&= \sum_{k=1}^{r_\infty-3} \sum_{m=2}^{r_\infty-1-k} \sum_{j=1}^k (-1)^{j+k} k(k-j) \binom{r_\infty-2-m}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-m-j} \tau_{r_\infty-1-m} h_j \\
&= - \sum_{k=1}^{r_\infty-4} k(k+1) \tau_{k+1} h_k + T_1 \sum_{k=1}^{r_\infty-5} k(k+1) \tau_{k+2} h_k \\
&= \sum_{k=1}^{r_\infty-3} \sum_{m=2}^{r_\infty-1-k} \sum_{j=1}^k (-1)^{j+k} (k-j) \binom{r_\infty-2-m}{k-1} \binom{k-1}{j-1} T_1^{r_\infty-2-m-j} \tau_{r_\infty-1-m} h_j \\
&= - \sum_{k=1}^{r_\infty-4} k \tau_{k+1} h_k. \tag{P-40}
\end{aligned}$$

Let us now compute each of the three contributions of (P-39) using (P-20). The first contribution is:

$$\begin{aligned}
C_1 &= 2T_2^{-1} \sum_{k=1}^{r_\infty-2} \sum_{j=1}^k (-1)^{j+k} k(r_\infty-1-k) \binom{k-1}{j-1} \binom{r_\infty-3}{k-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} dT_1 \wedge dT_2 \\
&+ T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-2-k} (-1)^{j+k} k(r_\infty-1-m-k) \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-2-m-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} \tau_{\infty, r_\infty-1-m} dT_1 \wedge dT_2 \\
&+ T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-1-k} (-1)^{j+k} k \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-1-m-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} d\tau_{\infty, r_\infty-1-m} \wedge dT_2 \\
&= -2T_2^{-1} (r_\infty-2)^2 \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_1 \wedge dT_2 \\
&- 2(r_\infty-2)(r_\infty-3) T_2^{-1} T_1 \text{Ham}^{(\alpha_{\tau_\infty, r_\infty-3})} dT_1 \wedge dT_2 \\
&+ T_2^{-1} \sum_{k=1}^{r_\infty-4} k^2 \tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})} dT_1 \wedge dT_2 - T_2^{-1} T_1 \sum_{k=1}^{r_\infty-5} k(k+1) \tau_{\infty, k+2} \text{Ham}^{(\alpha_{\tau_\infty, k})} dT_1 \wedge dT_2 \\
&+ T_2^{-1} \sum_{k=1}^{r_\infty-3} k \text{Ham}^{(\alpha_{\tau_\infty, k})} d\tau_{\infty, k} \wedge dT_2 - T_2^{-1} T_1 \sum_{k=1}^{r_\infty-4} k \text{Ham}^{(\alpha_{\tau_\infty, k})} d\tau_{\infty, k+1} \wedge dT_2. \tag{P-41}
\end{aligned}$$

The second contribution is:

$$C_2 = -2T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k (-1)^{j+k} k(k-j) \binom{k-1}{j-1} \binom{r_\infty-2}{k-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} dT_2 \wedge dT_1$$

$$\begin{aligned}
& -T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-1-k} (-1)^{j+k} k(k-j) \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-2-m-j} \tau_{\infty, r_\infty-1-m} \text{Ham}^{(\alpha_{\tau_\infty, j})} dT_2 \wedge dT_1 \\
& -T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-1-k} (-1)^{j+k} k(k-j) \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-2-m-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} d\tau_{\infty, r_\infty-1-m} \wedge dT_1 \\
& = 2T_2^{-1} \sum_{m=1}^{r_\infty-4} (-1)^{m+r_\infty} (r_\infty-2)(r_\infty^2 - (m+5)r_\infty + 2m+5) \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_2 \wedge dT_1 \\
& + T_2^{-1} \sum_{k=1}^{r_\infty-4} k(k+1) \tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})} dT_2 \wedge dT_1 - T_2^{-1} T_1 \sum_{k=1}^{r_\infty-5} k(k+1) \text{Ham}^{(\alpha_{\tau_\infty, k})} \tau_{\infty, k+2} dT_2 \wedge dT_1 \\
& + T_2^{-1} \sum_{k=1}^{r_\infty-4} k \text{Ham}^{(\alpha_{\tau_\infty, k})} d\tau_{\infty, k+1} \wedge dT_1.
\end{aligned} \tag{P-42}$$

The third contribution is

$$\begin{aligned}
C_3 & = -2T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k (-1)^{j+k} k \binom{k-1}{j-1} \binom{r_\infty-2}{k-1} T_1^{r_\infty-1-j} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, j})} \\
& - T_2^{-1} \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-1-k} (-1)^{j+k} k \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-1-m-j} \tau_{\infty, r_\infty-1-m} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, j})} \\
& - \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k (-1)^{j+k} \binom{k-1}{j-1} (r_\infty-1-k) \binom{r_\infty-2}{k-1} T_1^{r_\infty-2-j} dT_1 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, j})} \\
& - 2 \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-2-k} (-1)^{j+k} (r_\infty-1-m-k) \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-2-m-j} \tau_{\infty, r_\infty-1-m} dT_1 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, j})} \\
& - \sum_{k=1}^{r_\infty-3} \sum_{j=1}^k \sum_{m=2}^{r_\infty-1-k} \binom{k-1}{j-1} \binom{r_\infty-2-m}{k-1} T_1^{r_\infty-1-m-j} (-1)^{j+k} d\tau_{\infty, r_\infty-1-m} \wedge d\text{Ham}^{(\alpha_{\tau_\infty, j})} \\
& = 2T_2^{-1} \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} (r_\infty^2 - (m+4)r_\infty + 2m+3) \binom{r_\infty-2}{m-1} T_1^{r_\infty-1-m} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, m})} \\
& - T_2^{-1} \sum_{k=1}^{r_\infty-3} k \tau_{\infty, k} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} + T_2^{-1} T_1 \sum_{k=1}^{r_\infty-4} k \tau_{\infty, k+1} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} \\
& + 2 \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} (r_\infty-2) \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} dT_1 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, m})} \\
& - \sum_{k=1}^{r_\infty-4} k \tau_{\infty, k+1} dT_1 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} - \sum_{k=1}^{r_\infty-3} d\tau_{\infty, k} \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})}.
\end{aligned} \tag{P-43}$$

Observe that some of the terms of (P-38), (P-41), (P-42) and (P-43) simplify so that we are left with

$$\begin{aligned}
& - \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(X_s)} - \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\hat{T}_{X_s, k} \wedge d\text{Ham}^{(\mathbf{w}_{X_s, k})} - \frac{1}{2} \sum_{k=1}^{r_\infty-1} d\hat{T}_{\infty, k} \wedge d\text{Ham}^{(\mathbf{w}_{\infty, k})} \\
& = - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} - \sum_{k=1}^{r_\infty-3} d\tau_{\infty, k} \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} \\
& - T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})} - T_2^{-1} dT_1 \wedge d\text{Ham}^{(\mathbf{b})} - (r_\infty-1) T_2^{-2} \text{Ham}^{(\mathbf{b})} dT_1 \wedge dT_2 + T_2^{-2} \text{Ham}^{(\mathbf{b})} dT_1 \wedge dT_2 \\
& + (r_\infty-1) T_2^{-1} \sum_{k=1}^{r_\infty-4} k \tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})} dT_1 \wedge dT_2 + T_2^{-1} dT_2 \wedge \sum_{k=1}^{r_\infty-3} k d(\tau_{\infty, k} \text{Ham}^{(\alpha_{\tau_\infty, k})}) \\
& + 2(r_\infty-1) T_2^{-1} dT_2 \wedge \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} d \left(T_1^{r_\infty-1-m} \binom{r_\infty-2}{m-1} \text{Ham}^{(\alpha_{\tau_\infty, m})} \right) \\
& + 2T_2^{-1} (r_\infty-2)^2 \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} dT_1 \wedge dT_2 \\
& - 2(r_\infty-2)^2 T_1 T_2^{-1} \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} dT_2 \wedge d(T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty, j})})
\end{aligned}$$

$$\begin{aligned}
& -2T_2^{-1}(r_\infty - 2)^2 \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_1 \wedge dT_2 \\
& -2T_2^{-1} T_1 (r_\infty - 2)(r_\infty - 3) \text{Ham}^{(\alpha_{\tau_\infty, r_\infty-3})} dT_1 \wedge dT_2 \\
& + 2T_2^{-1} \sum_{m=1}^{r_\infty-4} (-1)^{m+r_\infty} (r_\infty - 2)(r_\infty^2 - (m+5)r_\infty + 2m+5) \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_2 \wedge dT_1 \\
& + T_2^{-1} \sum_{k=1}^{r_\infty-3} k \text{Ham}^{(\alpha_{\tau_\infty, k})} d\tau_{\infty, k} \wedge dT_2 + T_2^{-1} \sum_{k=1}^{r_\infty-4} k \tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})} dT_2 \wedge dT_1 \\
& + 2T_2^{-1} \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} (r_\infty^2 - (m+4)r_\infty + 2m+3) \binom{r_\infty-2}{m-1} T_1^{r_\infty-1-m} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, m})} \\
& - T_2^{-1} \sum_{k=1}^{r_\infty-3} k \tau_{\infty, k} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} + (r_\infty - 1) T_2^{-2} \text{Ham}^{(\mathbf{b})} dT_1 \wedge dT_2 \\
& - (r_\infty - 2) T_2^{-1} \sum_{k=1}^{r_\infty-4} k \tau_{\infty, k+1} \text{Ham}^{(\alpha_{\tau_\infty, k})} dT_1 \wedge dT_2.
\end{aligned} \tag{P-44}$$

Note that many terms simplify so that the last identity is equivalent to:

$$\begin{aligned}
& - \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(X_s)} - \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\hat{T}_{X_s, k} \wedge d\text{Ham}^{(\mathbf{w}_{X_s, k})} - \frac{1}{2} \sum_{k=1}^{r_\infty-1} d\hat{T}_{\infty, k} \wedge d\text{Ham}^{(\mathbf{w}_{\infty, k})} \\
& = - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} - \sum_{k=1}^{r_\infty-3} d\tau_{\infty, k} \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} \\
& - T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})} - T_2^{-1} dT_1 \wedge d\text{Ham}^{(\mathbf{b})} + T_2^{-2} \text{Ham}^{(\mathbf{b})} dT_1 \wedge dT_2 \\
& - 2(r_\infty - 1) T_2^{-1} \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} (r_\infty - 1 - m) \binom{r_\infty-2}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_1 \wedge dT_2 \\
& + 2(r_\infty - 1) T_2^{-1} \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} \binom{r_\infty-2}{m-1} T_1^{r_\infty-1-m} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, m})} \\
& + 2T_2^{-1} (r_\infty - 2)^2 \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} dT_1 \wedge dT_2 \\
& + 2(r_\infty - 2)^2 T_2^{-1} \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} (r_\infty - 2 - j) \binom{r_\infty-3}{j-1} T_1^{r_\infty-2-j} \text{Ham}^{(\alpha_{\tau_\infty, j})} dT_1 \wedge dT_2 \\
& - 2(r_\infty - 2)^2 T_2^{-1} \sum_{j=1}^{r_\infty-3} (-1)^{j+r_\infty} \binom{r_\infty-3}{j-1} T_1^{r_\infty-1-j} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, j})} \\
& - 2T_2^{-1} (r_\infty - 2)^2 \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_1 \wedge dT_2 \\
& - 2(r_\infty - 2)(r_\infty - 3) T_2^{-1} T_1 \text{Ham}^{(\alpha_{\tau_\infty, r_\infty-3})} dT_1 \wedge dT_2 \\
& - 2T_2^{-1} \sum_{m=1}^{r_\infty-4} (-1)^{m+r_\infty} (r_\infty - 2)(r_\infty^2 - (m+5)r_\infty + 2m+5) \binom{r_\infty-3}{m-1} T_1^{r_\infty-2-m} \text{Ham}^{(\alpha_{\tau_\infty, m})} dT_1 \wedge dT_2 \\
& + 2T_2^{-1} \sum_{m=1}^{r_\infty-3} (-1)^{m+r_\infty} (r_\infty^2 - (m+4)r_\infty + 2m+3) \binom{r_\infty-2}{m-1} T_1^{r_\infty-1-m} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_\infty, m})} \\
& = - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})} - \sum_{k=1}^{r_\infty-3} d\tau_{\infty, k} \wedge d\text{Ham}^{(\alpha_{\tau_\infty, k})} \\
& - T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})} - T_2^{-1} dT_1 \wedge d\text{Ham}^{(\mathbf{b})} + T_2^{-2} \text{Ham}^{(\mathbf{b})} dT_1 \wedge dT_2
\end{aligned} \tag{P-45}$$

where terms proportional to $dT_2 \wedge d\text{Ham}^{(\tau_\infty, m)}$ easily cancel while most terms proportional to $dT_1 \wedge dT_2$ also cancel (note that the last line does not contribute for $m = r_\infty - 3$) because we

have for all $m \in \llbracket 1, r_\infty - 4 \rrbracket$ the identity:

$$(r_\infty - 1)(r_\infty - 1 - m) \binom{r_\infty - 2}{m - 1} - (r_\infty - 2)^2 (r_\infty - 2 - m) \binom{r_\infty - 3}{m - 1} + (r_\infty - 2)(r_\infty^2 - (m + 5)r_\infty + 2m + 5) \binom{r_\infty - 3}{m - 1} = 0. \quad (\text{P-46})$$

In the end, using (P-28), we are left with

$$\begin{aligned} \Omega &= \hbar \sum_{j=1}^g d\check{q}_j \wedge d\check{p}_j - \sum_{k=1}^{r_\infty - 3} d\tau_{\infty, k} \wedge d\text{Ham}^{(\alpha_{\tau_{\infty, k}})}(\check{\mathbf{q}}, \check{\mathbf{p}}) \\ &\quad - \sum_{s=1}^n \sum_{k=1}^{r_s - 1} d\tau_{X_s, k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s, k}})}(\check{\mathbf{q}}, \check{\mathbf{p}}) - \sum_{s=1}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})}(\check{\mathbf{q}}, \check{\mathbf{p}}). \end{aligned} \quad (\text{P-47})$$

P.3 The case $r_\infty = 2$

Let us study the case $r_\infty = 2$. In this case we have from Proposition 6.6:

$$\begin{aligned} T_2 &= \frac{1}{2} \hat{T}_{\infty, 1}, \\ T_1 &= -\frac{1}{2} \hat{T}_{\infty, 1} X_1, \\ \tau_{X_s, k} &= \left(\frac{\hat{T}_{\infty, 1}}{2} \right)^k \hat{T}_{X_s, k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ \tilde{X}_s &= \frac{1}{2} (X_s - X_1) \hat{T}_{\infty, 1}, \quad \forall s \in \llbracket 2, n \rrbracket, \end{aligned} \quad (\text{P-48})$$

or equivalently

$$\begin{aligned} \hat{T}_{\infty, 1} &= 2T_2, \\ X_1 &= -T_1 T_2^{-1}, \\ X_s &= (\tilde{X}_s - T_1) T_2^{-1}, \quad \forall s \in \llbracket 2, n \rrbracket, \\ \hat{T}_{X_s, k} &= T_2^{-k} \tau_{X_s, k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket. \end{aligned} \quad (\text{P-49})$$

At the level of differentials we obtain:

$$\begin{aligned} d\hat{T}_{\infty, 1} &= 2dT_2, \\ dX_1 &= -T_2^{-1} dT_1 + T_1 T_2^{-2} dT_2, \\ d\hat{T}_{X_s, k} &= -k T_2^{-k-1} \tau_{X_s, k} dT_2 + T_2^{-k} d\tau_{X_s, k}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\ dX_s &= -T_2^{-2} (\tilde{X}_s - T_1) dT_2 - T_2^{-1} dT_1 + T_2^{-1} d\tilde{X}_s, \quad \forall s \in \llbracket 2, n \rrbracket. \end{aligned} \quad (\text{P-50})$$

At the level of Hamiltonians, since $\text{Ham}^{(\mathbf{w}_{\infty, 1})} = 2\text{Ham}^{(\alpha_{\hat{T}_{\infty, 1}})}$ and $\text{Ham}^{(\mathbf{w}_{X_s, k})} = 2\text{Ham}^{(\alpha_{\hat{T}_{X_s, k}})}$ for all $(s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket$ this is equivalent to

$$\begin{aligned} \text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -T_2^{-1} \sum_{s=1}^n \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\ \text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \text{Ham}^{(\mathbf{w}_{\infty, 1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2^{-1} \sum_{s=1}^n X_s \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\ &\quad - \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s - 1} k T_2^{-k-1} \tau_{X_s, k} \text{Ham}^{(\mathbf{w}_{X_s, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \end{aligned}$$

$$\begin{aligned}
\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{2}T_2^{-k}\text{Ham}^{(\mathbf{w}_{X_s,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2^{-1}\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall s \in \llbracket 2, n \rrbracket.
\end{aligned} \tag{P-51}$$

A straightforward computation also provides

$$\begin{aligned}
\partial_{X_1} &= -T_2\partial_{T_1} - T_2\sum_{s=2}^n\partial_{\tilde{X}_s}, \\
\partial_{\hat{T}_{\infty,1}} &= \frac{1}{2}\partial_{T_2} + \frac{1}{2}T_1T_2^{-1}\partial_{T_1} + \frac{1}{2}T_2^{-1}\sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\partial_{\tau_{X_s,k}} + \frac{1}{2}T_2^{-1}\sum_{s=2}^n\tilde{X}_s\partial_{\tilde{X}_s}, \\
\partial_{\hat{T}_{X_s,k}} &= T_2^k\partial_{\tau_{X_s,k}}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
\partial_{X_s} &= T_2\partial_{\tilde{X}_s}, \quad \forall s \in \llbracket 2, n \rrbracket,
\end{aligned} \tag{P-52}$$

i.e.

$$\begin{aligned}
\text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -T_2\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2\sum_{s=2}^n\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\mathbf{w}_{\infty,1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \text{Ham}^{(T_2)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_1T_2^{-1}\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&\quad + T_2^{-1}\sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-1}\sum_{s=2}^n\tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\mathbf{w}_{X_s,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^k\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall s \in \llbracket 2, n \rrbracket.
\end{aligned} \tag{P-53}$$

Let us now observe that Definition 6.1 and the fact that $\mathcal{L}_{\mathbf{w}_{\infty,1}} = 2\hbar\partial_{\hat{T}_{\infty,1}}$ and $\mathcal{L}_{\mathbf{w}_{X_s,k}} = 2\hbar\partial_{\hat{T}_{X_s,k}}$ for all $(s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket$ provides

$$\mathcal{L}_{\mathbf{a}} = \hbar T_2\partial_{T_2} + \frac{1}{2}T_{\infty,1}\mathcal{L}_{\mathbf{v}_{\infty,1}} - \frac{1}{2}\sum_{s=1}^n\sum_{k=1}^{r_s-1}kT_{X_s,k}\mathcal{L}_{\mathbf{v}_{X_s,k}}, \quad \mathcal{L}_{\mathbf{b}} = \hbar T_2\partial_{T_1} \tag{P-54}$$

i.e. using (P-7)

$$\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2\text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \tag{P-55}$$

Combining (P-53) and (P-55), we get

$$\begin{aligned}
\text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - T_2\sum_{s=2}^n\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\mathbf{w}_{\infty,1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2^{-1}\text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_1T_2^{-2}\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
&\quad + T_2^{-1}\sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-1}\sum_{s=2}^n\tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\mathbf{w}_{X_s,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^k\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall s \in \llbracket 2, n \rrbracket.
\end{aligned} \tag{P-56}$$

Let us now insert this result and (P-50) into (P-13):

$$\begin{aligned}
& \frac{1}{2} \sum_{p \in \mathcal{R}} \sum_{k=1}^{r_p-1} d\hat{T}_{p,k} \wedge d\text{Ham}^{(\mathbf{w}_{p,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&= \frac{1}{2} (2dT_2) \wedge \left(T_2^{-1} d\text{Ham}^{(\mathbf{a})} + T_2^{-2} d(T_1 \text{Ham}^{(\mathbf{b})}) + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \text{Ham}^{(\alpha_{\tau_{X_s,k}})} d\tau_{X_s,k} \right. \\
&\quad + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \tau_{X_s,k} d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} + T_2^{-1} \sum_{s=2}^n \text{Ham}^{(\alpha_{\tilde{X}_s})} d\tilde{X}_s + T_2^{-1} \sum_{s=2}^n \tilde{X}_s d\text{Ham}^{(\alpha_{\tilde{X}_s})} \Big) \\
&\quad + \frac{1}{2} \sum_{s=1}^n \sum_{k=1}^{r_s-1} (-k T_2^{-k-1} \tau_{X_s,k} dT_2 + T_2^{-k} d\tau_{X_s,k}) \wedge \left(2k T_2^{k-1} \text{Ham}^{(\alpha_{\tau_{X_s,k}})} dT_2 + 2T_2^k d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} \right) \\
&\quad + (-T_2^{-1} dT_1 + T_1 T_2^{-2} dT_2) \wedge \left(-d\text{Ham}^{(\mathbf{b})} - \sum_{s=2}^n \text{Ham}^{(\alpha_{\tilde{X}_s})} dT_2 - T_2 \sum_{s=2}^n d\text{Ham}^{(\alpha_{\tilde{X}_s})} \right) \\
&\quad + \sum_{s=2}^n \left(-T_2^{-2} (\tilde{X}_s - T_1) dT_2 - T_2^{-1} dT_1 + T_2^{-1} d\tilde{X}_s \right) \wedge \left(\text{Ham}^{(\alpha_{\tilde{X}_s})} dT_2 + T_2 d\text{Ham}^{(\alpha_{\tilde{X}_s})} \right) \\
&= T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})} + T_2^{-2} dT_2 \wedge d(T_1 \text{Ham}^{(\mathbf{b})}) + (T_2^{-1} dT_1 - T_1 T_2^{-2} dT_2) \wedge d\text{Ham}^{(\mathbf{b})} \\
&\quad + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \text{Ham}^{(\alpha_{\tau_{X_s,k}})} dT_2 \wedge d\tau_{X_s,k} + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \tau_{X_s,k} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} \\
&\quad + T_2^{-1} \sum_{s=2}^n \text{Ham}^{(\alpha_{\tilde{X}_s})} dT_2 \wedge d\tilde{X}_s + T_2^{-1} \sum_{s=2}^n \tilde{X}_s dT_2 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} + \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} \\
&\quad - T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \tau_{X_s,k} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} + T_2^{-1} \sum_{s=1}^n \sum_{k=1}^{r_s-1} k \text{Ham}^{(\alpha_{\tau_{X_s,k}})} d\tau_{X_s,k} \wedge dT_2 \\
&\quad + T_2^{-1} \sum_{s=2}^n \text{Ham}^{(\alpha_{\tilde{X}_s})} dT_1 \wedge dT_2 + \sum_{s=2}^n dT_1 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} - T_1 T_2^{-1} \sum_{s=2}^n dT_2 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&\quad - T_2^{-1} \sum_{s=2}^n (\tilde{X}_s - T_1) dT_2 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} - T_2^{-1} \sum_{s=2}^n \text{Ham}^{(\alpha_{\tilde{X}_s})} dT_1 \wedge dT_2 \\
&\quad - \sum_{s=2}^n dT_1 \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} + T_2^{-1} \sum_{s=2}^n \text{Ham}^{(\alpha_{\tilde{X}_s})} d\tilde{X}_s \wedge dT_2 + \sum_{s=2}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&= T_2^{-1} dT_2 \wedge d\text{Ham}^{(\mathbf{a})} + T_2^{-2} dT_2 \wedge d(T_1 \text{Ham}^{(\mathbf{b})}) + (T_2^{-1} dT_1 - T_1 T_2^{-2} dT_2) \wedge d\text{Ham}^{(\mathbf{b})} \\
&\quad + \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} + \sum_{s=2}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&= - (T_2^{-2} T_1 dT_2 - T_2^{-1} dT_1) \wedge d\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-2} dT_2 \wedge \left(d(T_1 \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})) + d(T_2 \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})) \right) \\
&\quad + \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \sum_{s=2}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}).
\end{aligned} \tag{P-57}$$

Finally, using (P-17) and (P-28), equation (P-13) is rewritten into :

$$\Omega = \hbar \sum_{j=1}^g d\check{q}_j \wedge d\check{p}_j - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\check{\mathbf{q}}, \check{\mathbf{p}}) - \sum_{s=2}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})}(\check{\mathbf{q}}, \check{\mathbf{p}}). \tag{P-58}$$

P.4 The case $r_\infty = 1$ and $n \geq 2$

Let us study the case $r_\infty = 1$ and $n \geq 2$. In this case we have from Proposition 6.8:

$$\hat{T}_{X_s,k} = T_2^{-k} \tau_{X_s,k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket,$$

$$\begin{aligned}
X_1 &= -T_2^{-1}T_1, \\
X_2 &= T_2^{-1}(1-T_1), \\
X_s &= T_2^{-1}(\tilde{X}_s - T_1), \quad \forall s \in \llbracket 3, n \rrbracket.
\end{aligned} \tag{P-59}$$

Thus we immediately get

$$\begin{aligned}
d\hat{T}_{X_s,k} &= -kT_2^{-k-1}\tau_{X_s,k}dT_2 + T_2^{-k}d\tau_{X_s,k}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket, \\
dX_1 &= T_2^{-2}T_1dT_2 - T_2^{-1}dT_1, \\
dX_2 &= -T_2^{-2}(1-T_1)dT_2 - T_2^{-1}dT_1, \\
dX_s &= -T_2^{-2}(\tilde{X}_s - T_1)dT_2 + T_2^{-1}d\tilde{X}_s - T_2^{-1}dT_1, \quad \forall s \in \llbracket 3, n \rrbracket.
\end{aligned} \tag{P-60}$$

This provides

$$\begin{aligned}
\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -T_2^{-1} \sum_{s=1}^n \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -T_2^{-1} \sum_{s=1}^n X_s \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_2^{-1} \text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall s \in \llbracket 3, n \rrbracket, \\
\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{2} T_2^{-k} \text{Ham}^{(\mathbf{w}_{X_s,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket.
\end{aligned} \tag{P-61}$$

Let us now observe that

$$\begin{aligned}
\partial_{X_1} &= -T_2(1-T_1)\partial_{T_1} + T_2^2\partial_{T_2} + T_2 \sum_{s=3}^n (\tilde{X}_s - 1)\partial_{\tilde{X}_s} + T_2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s,k}\partial_{\hat{T}_{X_s,k}}, \\
\partial_{X_2} &= -T_1T_2\partial_{T_1} - T_2^2\partial_{T_2} - T_2 \sum_{s=3}^n \tilde{X}_s\partial_{\tilde{X}_s} - T_2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s,k}\partial_{\hat{T}_{X_s,k}}, \\
\partial_{X_s} &= T_2\partial_{\tilde{X}_s}, \quad \forall s \in \llbracket 3, n \rrbracket, \\
\partial_{\hat{T}_{X_s,k}} &= T_2^k\partial_{\tau_{X_s,k}}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket,
\end{aligned} \tag{P-62}$$

i.e.

$$\begin{aligned}
\text{Ham}^{(\mathbf{w}_1)} &= -T_2(1-T_1)\text{Ham}^{(\alpha_{T_1})} + T_2^2\text{Ham}^{(\alpha_{T_2})} + T_2 \sum_{s=3}^n (\tilde{X}_s - 1)\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&\quad + T_2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}, \\
\text{Ham}^{(\mathbf{w}_2)} &= -T_1T_2\text{Ham}^{(\alpha_{T_1})} - T_2^2\text{Ham}^{(\alpha_{T_2})} - T_2 \sum_{s=3}^n \tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&\quad - T_2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}, \\
\text{Ham}^{(\mathbf{w}_s)} &= T_2\text{Ham}^{(\alpha_{\tilde{X}_s})}, \quad \forall s \in \llbracket 3, n \rrbracket, \\
\text{Ham}^{(\mathbf{w}_{X_s,k})} &= 2T_2^k\text{Ham}^{(\alpha_{\tau_{X_s,k}})}, \quad \forall (s,k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket.
\end{aligned} \tag{P-63}$$

We have also from Definitions (6-5) and (6-6):

$$\begin{aligned}
\mathcal{L}_{\mathbf{b}} &= -\hbar\partial_{X_1} - \hbar\partial_{X_2} - \hbar\sum_{s=3}^n\partial_{X_s} = \hbar T_2\partial_{T_1}, \\
\mathcal{L}_{\mathbf{a}} &= -\frac{1}{2}\sum_{s=1}^n\sum_{r=1}^{r_s-1}rT_{X_s,r}\mathcal{L}_{\mathbf{v}_{X_s,r}} - \frac{1}{2}\sum_{s=1}^n\sum_{r=1}^{r_s-1}r\hat{T}_{X_s,r}\mathcal{L}_{\mathbf{w}_{X_s,r}} - \hbar X_1\partial_{X_1} - \hbar X_2\partial_{X_2} - \hbar\sum_{s=3}^nX_s\partial_{X_s} \\
&= \hbar T_2\partial_{T_2} - \frac{1}{2}\sum_{s=1}^n\sum_{r=1}^{r_s-1}rT_{X_s,r}\mathcal{L}_{\mathbf{v}_{X_s,r}},
\end{aligned} \tag{P-64}$$

i.e. using (P-7)

$$\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2\text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \tag{P-65}$$

Combining (P-63) and (P-65) implies

$$\begin{aligned}
\text{Ham}^{(\mathbf{w}_1)} &= -(1 - T_1)\text{Ham}^{(\mathbf{b})} + T_2\text{Ham}^{(\mathbf{a})} + T_2\sum_{s=3}^n(\tilde{X}_s - 1)\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&\quad + T_2\sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}, \\
\text{Ham}^{(\mathbf{w}_2)} &= -T_1\text{Ham}^{(\mathbf{b})} - T_2\text{Ham}^{(\mathbf{a})} - T_2\sum_{s=3}^n\tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
&\quad - T_2\sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}, \\
\text{Ham}^{(\mathbf{w}_s)} &= T_2\text{Ham}^{(\alpha_{\tilde{X}_s})}, \quad \forall s \in \llbracket 3, n \rrbracket, \\
\text{Ham}^{(\mathbf{w}_{X_s,k})} &= 2T_2^k\text{Ham}^{(\alpha_{\tau_{X_s,k}})}, \quad \forall (s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket.
\end{aligned} \tag{P-66}$$

Let us now insert these results into (P-13)

$$\begin{aligned}
&\frac{1}{2}\sum_{p \in \mathcal{R}}\sum_{k=1}^{r_p-1}d\hat{T}_{p,k} \wedge d\text{Ham}^{(\mathbf{w}_{p,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \sum_{s=1}^n dX_s \wedge d\text{Ham}^{(\mathbf{w}_s)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
&= \sum_{s=1}^n\sum_{k=1}^{r_s-1}\left(-kT_2^{-1}\tau_{X_s,k}dT_2 + \sum_{s=3}^nd\tau_{X_s,k}\right) \wedge \left(kT_2^{-1}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}dT_2 + d\text{Ham}^{(\alpha_{\tau_{X_s,k}})}\right) \\
&\quad + (T_2^{-2}T_1dT_2 - T_2^{-1}dT_1) \wedge \left(d(-(1 - T_1)\text{Ham}^{(\mathbf{b})}) + d(T_2\text{Ham}^{(\mathbf{a})}) + \sum_{s=3}^n(\tilde{X}_s - 1)\text{Ham}^{(\alpha_{\tilde{X}_s})}dT_2\right. \\
&\quad \left.+ T_2\sum_{s=3}^nd((\tilde{X}_s - 1)\text{Ham}^{(\alpha_{\tilde{X}_s})}) + \sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}dT_2 + T_2\sum_{s=1}^n\sum_{k=1}^{r_s-1}kd(\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})})\right) \\
&\quad - (T_2^{-2}(1 - T_1)dT_2 + T_2^{-1}dT_1) \wedge \left(-d(T_1\text{Ham}^{(\mathbf{b})}) - d(T_2\text{Ham}^{(\mathbf{a})}) - \sum_{s=3}^n\tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})}dT_2 - T_2\sum_{s=3}^nd(\tilde{X}_s\text{Ham}^{(\alpha_{\tilde{X}_s})})\right. \\
&\quad \left.- \sum_{s=1}^n\sum_{k=1}^{r_s-1}k\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})}dT_2 - T_2\sum_{s=1}^n\sum_{k=1}^{r_s-1}kd(\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})})\right) \\
&\quad + \sum_{s=3}^n\left(-T_2^{-1}(\tilde{X}_s - T_1)dT_2 + d\tilde{X}_s - dT_1\right) \wedge \left(d\text{Ham}^{(\alpha_{\tilde{X}_s})} + T_2^{-1}\text{Ham}^{(\alpha_{\tilde{X}_s})}dT_2\right) \\
&= \sum_{s=1}^n\sum_{k=1}^{r_s-1}d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} + \sum_{s=3}^nd\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} - \sum_{s=1}^n\sum_{k=1}^{r_s-1}kdT_2 \wedge d(\tau_{X_s,k}\text{Ham}^{(\alpha_{\tau_{X_s,k}})})
\end{aligned}$$

$$\begin{aligned}
& + (T_2^{-2}T_1dT_2 - T_2^{-1}dT_1) \wedge \left(\sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s,k} \text{Ham}^{(\alpha_{\tau_{X_s,k}})} dT_2 + T_2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} kd(\tau_{X_s,k} \text{Ham}^{(\alpha_{\tau_{X_s,k}})}) \right) \\
& + (T_2^{-2}(1-T_1)dT_2 + T_2^{-1}dT_1) \wedge \left(\sum_{s=1}^n \sum_{k=1}^{r_s-1} k\tau_{X_s,k} \text{Ham}^{(\alpha_{\tau_{X_s,k}})} dT_2 + T_2 \sum_{s=1}^n \sum_{k=1}^{r_s-1} kd(\tau_{X_s,k} \text{Ham}^{(\alpha_{\tau_{X_s,k}})}) \right) \\
& + (T_2^{-2}T_1dT_2 - T_2^{-1}dT_1) \wedge \left(\sum_{s=3}^n (\tilde{X}_s - 1) \text{Ham}^{(\alpha_{\tilde{X}_s})} dT_2 + T_2 \sum_{s=3}^n d((\tilde{X}_s - 1) \text{Ham}^{(\alpha_{\tilde{X}_s})}) \right) \\
& + (T_2^{-2}(1-T_1)dT_2 + T_2^{-1}dT_1) \left(\sum_{s=3}^n \tilde{X}_s \text{Ham}^{(\alpha_{\tilde{X}_s})} dT_2 + T_2 \sum_{s=3}^n d(\tilde{X}_s \text{Ham}^{(\alpha_{\tilde{X}_s})}) \right) \\
& + \sum_{s=3}^n \left(-T_2^{-1}(\tilde{X}_s - T_1)dT_2 - dT_1 \right) \wedge \left(d\text{Ham}^{(\alpha_{\tilde{X}_s})} + T_2^{-1}\text{Ham}^{(\alpha_{\tilde{X}_s})} dT_2 \right) + T_2^{-1}\text{Ham}^{(\alpha_{\tilde{X}_s})} d\tilde{X}_s \wedge dT_2 \\
& + (T_2^{-2}T_1dT_2 - T_2^{-1}dT_1) \wedge \left(-d((1-T_1)\text{Ham}^{(\mathbf{b})}) + d(T_2\text{Ham}^{(\mathbf{a})}) \right) \\
& - (T_2^{-2}(1-T_1)dT_2 + T_2^{-1}dT_1) \wedge \left(-d(T_1\text{Ham}^{(\mathbf{b})}) - d(T_2\text{Ham}^{(\mathbf{a})}) \right) \\
& = \sum_{s=1}^n d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})} + \sum_{s=3}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})} \\
& + (T_2^{-2}T_1dT_2 - T_2^{-1}dT_1) \wedge \left(-d((1-T_1)\text{Ham}^{(\mathbf{b})}) + d(T_2\text{Ham}^{(\mathbf{a})}) \right) \\
& - (T_2^{-2}(1-T_1)dT_2 + T_2^{-1}dT_1) \wedge \left(-d(T_1\text{Ham}^{(\mathbf{b})}) - d(T_2\text{Ham}^{(\mathbf{a})}) \right) \\
& = \sum_{s=1}^n d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \sum_{s=3}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& - (T_2^{-2}T_1dT_2 - T_2^{-1}dT_1) \wedge d\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-2}dT_2 \wedge \left(d(T_1\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})) + d(T_2\text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})) \right).
\end{aligned} \tag{P-67}$$

Using (P-28) we finally get

$$\Omega = \hbar \sum_{j=1}^g d\check{q}_j \wedge d\check{p}_j - \sum_{s=1}^n \sum_{k=1}^{r_s-1} d\tau_{X_s,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_s,k}})}(\check{\mathbf{q}}, \check{\mathbf{p}}) - \sum_{s=3}^n d\tilde{X}_s \wedge d\text{Ham}^{(\alpha_{\tilde{X}_s})}(\check{\mathbf{q}}, \check{\mathbf{p}}). \tag{P-68}$$

P.5 The case $r_\infty = 1$ and $n = 1$

Let us study the case $r_\infty = 1$ and $n = 1$. In this case we have from Proposition 6.10:

$$\begin{aligned}
\hat{T}_{X_1, r_1-1} &= 2T_2^{-(r_1-1)}, \\
\hat{T}_{X_1, k} &= T_2^{-k}\tau_{X_1, k}, \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket, \\
X_1 &= -T_2^{-1}T_1.
\end{aligned} \tag{P-69}$$

Thus we immediately get

$$\begin{aligned}
d\hat{T}_{X_1, r_1-1} &= -2(r_1 - 1)T_2^{-r_1}dT_2, \\
d\hat{T}_{X_1, k} &= -kT_2^{-k-1}\tau_{X_1, k}dT_2 + T_2^{-k}d\tau_{X_1, k}, \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket, \\
dX_1 &= -T_2^{-1}dT_1 + T_1T_2^{-2}dT_2,
\end{aligned} \tag{P-70}$$

i.e. using the fact that $\partial_{\hat{T}_{X_1, k}} = \frac{1}{2}\partial_{\mathbf{w}_{X_1, k}}$ (or using Proposition 6.11)

$$\begin{aligned}
\text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -T_2^{-1}\text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\
\text{Ham}^{(\alpha_{\tau_{X_1, k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= \frac{1}{2}T_2^{-k}\text{Ham}^{(\mathbf{w}_{X_1, k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket, \\
\text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= T_1T_2^{-2}\text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - (r_1 - 1)T_2^{-r_1}\text{Ham}^{(\mathbf{w}_{X_1, r_1-1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})
\end{aligned}$$

$$-\frac{1}{2} \sum_{k=1}^{r_1-2} k T_2^{-k-1} \tau_{X_1,k} \text{Ham}^{(\mathbf{w}_{X_1,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \quad (\text{P-71})$$

where we have denoted $\text{Ham}^{(\alpha_\tau)}(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ the Hamiltonian associated to the evolution $\hbar \partial_\tau$. Definitions (6-5) and (6-6) of $\mathcal{L}_{\mathbf{a}}$ and $\mathcal{L}_{\mathbf{b}}$ provide

$$\begin{aligned} \mathcal{L}_{\mathbf{a}} &= -\hbar \sum_{r=1}^{r_1-1} r \frac{t_{X_1^{(1)},r} + t_{X_1^{(2)},r}}{2} \left(\partial_{t_{X_1^{(1)},r}} + \partial_{t_{X_1^{(2)},r}} \right) + r \frac{t_{X_1^{(1)},r} - t_{X_1^{(2)},r}}{2} \left(\partial_{t_{X_1^{(1)},r}} - \partial_{t_{X_1^{(2)},r}} \right) - \hbar X_1 \partial_{X_1} \\ &= -\frac{1}{2} \sum_{r=1}^{r_1-1} r T_{X_1,r} \mathcal{L}_{\mathbf{v}_{X_1,r}} - \frac{1}{2} \sum_{r=1}^{r_1-1} r \hat{T}_{X_1,r} \mathcal{L}_{\mathbf{w}_{X_1,r}} - \hbar X_1 \partial_{X_1} \\ \mathcal{L}_{\mathbf{b}} &= -\hbar \partial_{X_1} \end{aligned} \quad (\text{P-72})$$

so that using (P-71)

$$\begin{aligned} \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -\text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = T_2 \text{Ham}^{(\alpha_{T_1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \\ \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -\frac{1}{2} \sum_{r=1}^{r_1-1} r T_{X_1,r} \text{Ham}^{(\mathbf{v}_{X_1,r})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2 \text{Ham}^{(\alpha_{T_2})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \end{aligned} \quad (\text{P-73})$$

Combining (P-71) and (P-73) and using (P-7) we obtain:

$$\begin{aligned} \text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = \hbar \sum_{j=1}^g \hat{p}_j, \\ \text{Ham}^{(\mathbf{w}_{X_1,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= 2T_2^k \text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}), \quad \forall k \in \llbracket 1, r_1 - 2 \rrbracket, \\ \text{Ham}^{(\mathbf{w}_{X_1,r_1-1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= -\frac{1}{r_1-1} T_2^{r_1-1} \text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) - \frac{1}{r_1-1} T_1 T_2^{r_1-2} \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\ &\quad - \frac{1}{r_1-1} T_2^{r_1-1} \sum_{k=1}^{r_1-2} k \tau_{X_1,k} \text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \end{aligned} \quad (\text{P-74})$$

A trivial computation from (P-17) gives:

$$-\frac{1}{r_1-1} T_2^{r_1-1} \left(\text{Ham}^{(\mathbf{a})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_1 T_2^{-1} \text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \right) = \frac{\hbar}{r_1-1} T_2^{r_1-1} \left(\sum_{j=1}^g \hat{q}_j \hat{p}_j + T_1 T_2^{-1} \sum_{j=1}^g \hat{p}_j \right) \quad (\text{P-75})$$

so that

$$\text{Ham}^{(\mathbf{w}_{X_1,r_1-1})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = \frac{\hbar}{r_1-1} T_2^{r_1-1} \left(\sum_{j=1}^g \hat{q}_j \hat{p}_j + T_1 T_2^{-1} \sum_{j=1}^g \hat{p}_j \right) - \frac{1}{r_1-1} T_2^{r_1-1} \sum_{k=1}^{r_1-2} k \tau_{X_1,k} \text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}). \quad (\text{P-76})$$

Finally we get from (P-74) and (P-76):

$$\begin{aligned} &\frac{1}{2} \sum_{k=1}^{r_1-1} d\hat{T}_{X_1,k} \wedge d\text{Ham}^{(\mathbf{w}_{X_1,k})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + dX_1 \wedge d\text{Ham}^{(\mathbf{w}_1)}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = \\ &-T_2^{-r_1} dT_2 \wedge \left(T_2^{r_1-1} d \left(\hbar \sum_{j=1}^g \hat{q}_j \hat{p}_j + \hbar T_1 T_2^{-1} \sum_{j=1}^g \hat{p}_j \right) - T_2^{r_1-1} \sum_{k=1}^{r_1-2} k d(\tau_{X_1,k} \text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})) \right) \\ &+ \frac{1}{2} \sum_{k=1}^{r_1-2} (-k T_2^{-k-1} \tau_{X_1,k} dT_2 + T_2^{-k} d\tau_{X_1,k}) \wedge \left(2k T_2^{k-1} \text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) dT_2 + 2T_2^k d\text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \right) \end{aligned}$$

$$\begin{aligned}
& -(-T_2^{-1}dT_1 + T_1T_2^{-2}dT_2) \wedge d\text{Ham}^{(\mathbf{b})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \\
& = \sum_{k=1}^g d\tau_{X_1,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{q}, \hat{p}) \\
& + T_2^{-1}dT_2 \wedge d\left(\hbar \sum_{j=1}^g \hat{q}_j \hat{p}_j + \hbar T_1 T_2^{-1} \sum_{j=1}^g \hat{p}_j\right) + T_2^{-1}dT_2 \wedge \sum_{k=1}^{r_1-2} kd(\tau_{X_1,k} \text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}})) \\
& - T_2^{-1} \sum_{k=1}^{r_1-2} k\tau_{X_1,k} dT_2 \wedge d\text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + T_2^{-1} \sum_{k=1}^{r_1-2} k\text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) d\tau_{X_1,k} \wedge dT_2 \\
& + \hbar \sum_{j=1}^g (-T_2^{-1}dT_1 + T_1T_2^{-2}dT_2) \wedge d\hat{p}_j \\
& = \sum_{k=1}^g d\tau_{X_1,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\hat{q}, \hat{p}) \\
& + \hbar T_2^{-1}dT_2 \wedge d\left(\sum_{j=1}^g \hat{q}_j \hat{p}_j + T_1 T_2^{-1} \sum_{j=1}^g \hat{p}_j\right) + \hbar \sum_{j=1}^g (-T_2^{-1}dT_1 + T_1T_2^{-2}dT_2) \wedge d\hat{p}_j.
\end{aligned} \tag{P-77}$$

From (P-27), equation (P-13) eventually reduces to

$$\Omega = \hbar \sum_{j=1}^g d\check{q}_j \wedge d\check{p}_j - \sum_{k=1}^g d\tau_{X_1,k} \wedge d\text{Ham}^{(\alpha_{\tau_{X_1,k}})}(\check{\mathbf{q}}, \check{\mathbf{p}}). \tag{P-78}$$

Q Proof of Theorem 6.4

Let $j \in \llbracket 1, g \rrbracket$. By definition, $(q_j)_{1 \leq j \leq g}$ are the zeros of the Wronskian W defined in (C-4):

$$W(\lambda) = \hbar(\Psi_1(\lambda)\Psi_2'(\lambda) - \Psi_2(\lambda)\Psi_1'(\lambda)) = \kappa \frac{\prod_{i=1}^g (\lambda - q_i)}{\prod_{s=1}^n (\lambda - X_s)^{r_s}} \exp\left(\frac{1}{\hbar} \int_0^\lambda P_1(\tilde{\lambda}) d\tilde{\lambda}\right). \tag{Q-1}$$

Asymptotics of the Wronskian at each pole are given by (C-2) and (C-3). It is then straightforward to observe that the asymptotics of $W(\lambda) \exp\left(-\frac{1}{\hbar} \int_0^\lambda P_1(\tilde{\lambda}) d\tilde{\lambda}\right)$ are invariant under the action of $\partial_{t_{X_s}^{(1)}} + \partial_{t_{X_s}^{(2)}}$ or $\partial_{t_\infty^{(1)}} + \partial_{t_\infty^{(2)}}$ so that

$$\begin{aligned}
0 &= (\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[q_j], \quad \forall s \in \llbracket 1, n \rrbracket, \\
0 &= (\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[q_j].
\end{aligned} \tag{Q-2}$$

Finally since it is obvious that $(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[T_1] = 0$ and $(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[T_2] = 0$ we get that:

$$\begin{aligned}
0 &= (\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[\check{q}_j], \quad \forall s \in \llbracket 1, n \rrbracket, \\
0 &= (\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[\check{q}_j].
\end{aligned} \tag{Q-3}$$

In fact, this observation (using Proposition 6.12 to deal with T_1 and T_2) is valid for any $(\partial_{t_{X_s(1),k}} + \partial_{t_{X_s(2),k}})_{1 \leq s \leq n, 0 \leq k \leq r_s-1}$ and $(\partial_{t_{\infty(1),k}} + \partial_{t_{\infty(2),k}})_{0 \leq k \leq r_\infty-1}$ recovering the fact that $(T_{\infty,k})_{0 \leq k \leq r_\infty-1}$ and $(T_{X_s,k})_{1 \leq s \leq n, 0 \leq k \leq r_s-1}$ are trivial times for $(\check{q}_j)_{1 \leq j \leq g}$. Finally note that this observation is

in agreement with the expression of $\mathcal{L}_\alpha[q_j]$ given in Theorem 5.1 where $p_j - \frac{1}{2}P_1(q_j) = T_2\check{p}_j$ only appears in the r.h.s.

Let us now look at $(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[p_j]$ for $s \in \llbracket 1, n \rrbracket$ or $(\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[p_j]$. By definition, we have

$$p_j = \text{Res}_{\lambda \rightarrow q_j} L_{2,1}(\lambda) = \hbar \text{Res}_{\lambda \rightarrow q_j} \frac{Y_2(\lambda)(\partial_\lambda Y_1)(\lambda) - Y_1(\lambda)(\partial_\lambda Y_2)(\lambda)}{Y_2(\lambda) - Y_1(\lambda)}. \quad (\text{Q-4})$$

We define $\hat{\Psi}_i(\lambda) = \Psi_i(\lambda) \exp\left(-\frac{1}{2\hbar} \int_0^\lambda P_1(\tilde{\lambda}) d\tilde{\lambda}\right)$ and $\hat{Y}_i(\lambda) = \hbar \frac{\partial_\lambda \hat{\Psi}_i(\lambda)}{\hat{\Psi}_i(\lambda)}$ for $i \in \{1, 2\}$. Observe in particular that $(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[\hat{\Psi}_i(\lambda)] = 0$ and $(\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[\hat{\Psi}(\lambda)] = 0$. Therefore

$$(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[\hat{Y}_i] = 0, (\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[\hat{Y}_i] = 0, \forall i \in \{1, 2\} \text{ and } s \in \llbracket 1, n \rrbracket. \quad (\text{Q-5})$$

Observe that

$$Y_i(\lambda) = \hat{Y}_i(\lambda) + \frac{1}{2}P_1(\lambda) \text{ and } (\partial_\lambda Y_i)(\lambda) = (\partial_\lambda \hat{Y}_i)(\lambda) + \frac{1}{2}P_1'(\lambda). \quad (\text{Q-6})$$

Thus we get:

$$\begin{aligned} p_j &= \hbar \text{Res}_{\lambda \rightarrow q_j} \frac{(\hat{Y}_2(\lambda) + \frac{1}{2}P_1(\lambda)) \left((\partial_\lambda \hat{Y}_1)(\lambda) + \frac{1}{2}P_1'(\lambda) \right) - (\hat{Y}_1(\lambda) + \frac{1}{2}P_1(\lambda)) \left((\partial_\lambda \hat{Y}_2)(\lambda) + \frac{1}{2}P_1'(\lambda) \right)}{\hat{Y}_2(\lambda) - \hat{Y}_1(\lambda)} \\ &= \text{Res}_{\lambda \rightarrow q_j} \frac{\hat{Y}_2(\lambda)(\partial_\lambda \hat{Y}_1)(\lambda) - \hat{Y}_1(\lambda)(\partial_\lambda \hat{Y}_2)(\lambda) + \frac{1}{2}P_1(\lambda) \left((\partial_\lambda \hat{Y}_1)(\lambda) - (\partial_\lambda \hat{Y}_2)(\lambda) \right) + \frac{1}{2}P_1'(\lambda) (\hat{Y}_2(\lambda) - \hat{Y}_1(\lambda))}{\hat{Y}_2(\lambda) - \hat{Y}_1(\lambda)} \\ &= -\frac{1}{2}P_1(q_j) + \text{Res}_{\lambda \rightarrow q_j} \frac{\hat{Y}_2(\lambda)(\partial_\lambda \hat{Y}_1)(\lambda) - \hat{Y}_1(\lambda)(\partial_\lambda \hat{Y}_2)(\lambda)}{\hat{Y}_2(\lambda) - \hat{Y}_1(\lambda)}. \end{aligned} \quad (\text{Q-7})$$

Thus, using (Q-2) and (Q-5) we get:

$$\begin{aligned} (\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[p_j] &= -\frac{1}{2}(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[P_1](q_j), \forall s \in \llbracket 1, n \rrbracket, \\ (\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[p_j] &= -\frac{1}{2}(\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[P_1](q_j). \end{aligned} \quad (\text{Q-8})$$

which is equivalent to say that

$$\begin{aligned} (\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[p_j - \frac{1}{2}P_1(q_j)] &= 0, \forall s \in \llbracket 1, n \rrbracket, \\ (\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[p_j - \frac{1}{2}P_1(q_j)] &= 0. \end{aligned} \quad (\text{Q-9})$$

Since we have $(\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[T_2] = 0$ for all $s \in \llbracket 1, n \rrbracket$ and $(\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[T_2] = 0$ we get that

$$\begin{aligned} (\partial_{t_{X_s(1),0}} + \partial_{t_{X_s(2),0}})[\check{p}_j] &= 0, \forall s \in \llbracket 1, n \rrbracket, \\ (\partial_{t_{\infty(1),0}} + \partial_{t_{\infty(2),0}})[\check{p}_j] &= 0. \end{aligned} \quad (\text{Q-10})$$

Note that the previous proof (using Proposition 6.12 to deal with T_2) is also valid for any $(\partial_{t_{X_s(1),k}} + \partial_{t_{X_s(2),k}})_{1 \leq s \leq n, 0 \leq k \leq r_s-1}$ and $(\partial_{t_{\infty(1),k}} + \partial_{t_{\infty(2),k}})_{0 \leq k \leq r_\infty-1}$ recovering the fact that $(T_{\infty,k})_{0 \leq k \leq r_\infty-1}$ and $(T_{X_s,k})_{1 \leq s \leq n, 0 \leq k \leq r_s-1}$ are trivial times for $(\check{p}_j)_{1 \leq j \leq g}$. Note also that the definitions of $(\hat{\Psi}_i)_{1 \leq i \leq 2}$ correspond to a diagonal gauge transformation for the matrix Ψ that symmetrizes both sheets.

R Proof of Theorem 7.1

Let us start with a trivial lemma

Lemma R.1. *Let us denote $S^{(d)}$ the $d \times d$ matrix with 1 on the antidiagonal and 0 everywhere else. In other words $S_{i,j}^{(d)} = \delta_{j=d+1-i}$. Then, for any $d \times d$ Toeplitz matrix T :*

$$S^{(d)} T^t S^{(d)} = T. \quad (\text{R-1})$$

The proof of the lemma is immediate from the fact that $T_{i,j} = \gamma_{i-j}$ for all $(i, j) \in \llbracket 1, d \rrbracket^2$ for some $(\gamma_{-(d-1)}, \dots, \gamma_{d-1})$. so that for any $(i, j) \in \llbracket 1, d \rrbracket^2$:

$$[S^{(d)} T^t S^{(d)}]_{i,j} = \sum_{k=1}^d \sum_{m=1}^d [S^{(d)}]_{i,k} T_{m,k} [S^{(d)}]_{m,j} \stackrel{k=d+1-i}{=} T_{d+1-j, d+1-i} = \gamma_{i-j} = T_{i,j} \quad (\text{R-2})$$

Note also that $(S^{(d)})^2 = I_d$ and that $S^{(d)}$ acts on vectors by reversing the order of the entries.

Let us then mention that (7-4) eliminates part of the Hamiltonian in Theorem 5.1. Then, for $r_\infty = 1$, all additional terms identically vanish. For $r_\infty = 2$, additional terms in Theorem 5.1 proportional to $\nu_{\infty,0}^{(\alpha)}$ identically vanish and terms for $r_\infty = 2$, since $t_{\infty(1), r_\infty-1}$ is fixed because of the definition of T_2 , Proposition 4.1 implies that $\nu_{\infty,-1}^{(\alpha_\tau)} = 0$ so that all additional terms vanish. Finally for $r_\infty \geq 3$, the choice of T_1 and T_2 implies that $t_{\infty(1), r_\infty-1}$ and $t_{\infty(2), r_\infty-1}$ are fixed so that $\nu_{\infty,-1}^{(\alpha_\tau)} = \nu_{\infty,0}^{(\alpha_\tau)} = 0$ from Proposition 4.1. Thus, (7-5) is proved.

Let us now consider a deformation relatively to $\tau_{\infty,j}$ with $j \in \llbracket 1, r_\infty - 3 \rrbracket$. In this case, the only non-vanishing term is $\alpha_{\infty(1),j} = \frac{1}{2}$. Consequently, $\nu_{X_s,k}^{(\alpha_{\infty,j})} = 0$ for all $(s, k) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s \rrbracket$. Hence we get

$$\tilde{M}_\infty \nu_j^{(\infty)} = \frac{1}{j} \mathbf{e}_{r_\infty-2-j} \text{ with } \nu_j^{(\infty)} = \left(\nu_{\infty,1}^{(\alpha_{\infty,j})}, \nu_{\infty,2}^{(\alpha_{\infty,j})}, \dots, \nu_{\infty, r_\infty-3}^{(\alpha_{\infty,j})} \right)^t. \quad (\text{R-3})$$

It gives

$$\text{Ham}^{(\alpha_{\infty,j})}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = \left(\nu_j^{(\infty)} \right)^t \mathbf{H}_\infty \text{ with } \mathbf{H}_\infty = (H_{\infty,0}, \dots, H_{\infty, r_\infty-4})^t. \quad (\text{R-4})$$

Thus we have,

$$\text{Ham}^{(\alpha_{\infty,j})}(\mathbf{q}, \mathbf{p}) = \left(\nu_j^{(\infty)} \right)^t \tilde{M}_\infty^t (\tilde{M}_\infty^t)^{-1} \mathbf{H}_\infty = \frac{1}{j} (\mathbf{e}_{r_\infty-2-j})^t (\tilde{M}_\infty^t)^{-1} \mathbf{H}_\infty. \quad (\text{R-5})$$

Since $\text{Ham}^{(\alpha_{\infty,j})}(\mathbf{q}, \mathbf{p})$ is scalar, it is equal to its transpose and we get for all $j \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$j \text{Ham}^{(\alpha_{\infty,j})}(\mathbf{q}, \mathbf{p}) = (\mathbf{H}_\infty)^t \tilde{M}_\infty^{-1} \mathbf{e}_{r_\infty-2-j}. \quad (\text{R-6})$$

Taking $i = r_\infty - 2 - j$, we get that for all $i \in \llbracket 1, r_\infty - 3 \rrbracket$:

$$(r_\infty - 2 - i) \text{Ham}^{(\alpha_{\infty, r_\infty-2-i})}(\mathbf{q}, \mathbf{p}) = (\mathbf{H}_\infty)^t \tilde{M}_\infty^{-1} \mathbf{e}_i \quad (\text{R-7})$$

$(\mathbf{H}_\infty)^t \tilde{M}_\infty^{-1}$ is a line vector and $(\mathbf{H}_\infty)^t \tilde{M}_\infty^{-1} \mathbf{e}_i$ extracts its i^{th} entry. Thus, the last identities are equivalent to

$$\left((r_\infty - 3) \text{Ham}^{(\alpha_{\infty, r_\infty-3})}, (r_\infty - 4) \text{Ham}^{(\alpha_{\infty, r_\infty-4})}, \dots, \text{Ham}^{(\alpha_{\infty, 1})} \right) = (\mathbf{H}_\infty)^t \tilde{M}_\infty^{-1} \quad (\text{R-8})$$

which is equivalent after transposition to

$$\begin{pmatrix} (r_\infty - 3)\text{Ham}^{(\alpha_{\tau_\infty, r_\infty-3})} \\ \vdots \\ \text{Ham}^{(\alpha_{\tau_\infty, 1})} \end{pmatrix} = \left(\tilde{M}_\infty^t\right)^{-1} \begin{pmatrix} H_{\infty, 0} \\ \vdots \\ H_{\infty, r_\infty-4} \end{pmatrix}. \quad (\text{R-9})$$

Thus, we get

$$(S^{(r_\infty-3)} \tilde{M}_\infty^t S^{(r_\infty-3)}) S^{(r_\infty-3)} \begin{pmatrix} (r_\infty - 3)\text{Ham}^{(\alpha_{\infty, r_\infty-3})} \\ \vdots \\ \text{Ham}^{(\alpha_{\infty, 1})} \end{pmatrix} = S^{(r_\infty-3)} \begin{pmatrix} H_{\infty, 0} \\ \vdots \\ H_{\infty, r_\infty-4} \end{pmatrix} \quad (\text{R-10})$$

i.e. using Lemma R.1:

$$\tilde{M}_\infty S^{(r_\infty-3)} \begin{pmatrix} (r_\infty - 3)\text{Ham}^{(\alpha_{\infty, r_\infty-3})} \\ \vdots \\ \text{Ham}^{(\alpha_{\infty, 1})} \end{pmatrix} = S^{(r_\infty-3)} \begin{pmatrix} H_{\infty, 0} \\ \vdots \\ H_{\infty, r_\infty-4} \end{pmatrix} \quad (\text{R-11})$$

which is equivalent to (7-6).

Similar computations can be carried out for deformations relatively to $\tau_{X_s, j}$ for $(s, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, r_s - 1 \rrbracket$. In this case, the only non-vanishing term is $\alpha_{X_s^{(1)}, j} = \frac{1}{2}$. Consequently, $\nu_{X_u, k}^{(\alpha_{X_s, j})} = 0$ for all $u \neq s$ and $\nu_{\infty, k}^{(\alpha_{X_s, j})} = 0$ for all $k \in \llbracket 1, r_\infty - 3 \rrbracket$ and

$$M_s \left(\nu_{X_s, 1}^{(\alpha_{X_s, j})} \dots, \nu_{X_s, r_s-1}^{(\alpha_{X_s, j})} \right)^t = -\frac{1}{j} \mathbf{e}_{r_s-j}. \quad (\text{R-12})$$

The corresponding Hamiltonian is given by:

$$\text{Ham}^{(\alpha_{X_s, j})}(\check{\mathbf{q}}, \check{\mathbf{p}}) = - \sum_{k=2}^{r_s} \nu_{X_s, k-1}^{(\alpha_{X_s, j})} H_{X_s, k} := - \left(\nu_j^{(X_s)} \right)^t \mathbf{H}_{X_s}. \quad (\text{R-13})$$

The rest of the computation is identical to the case at infinity presented above and provides (7-6).

Let us end with the case of deformations relatively to $\tilde{X}_s = X_s$ the position of the finite pole with $s \in \llbracket 1, n \rrbracket$. In this case, the only non-vanishing term is $\alpha_{X_s} = 1$. Thus, we get

$$\text{Ham}^{(\alpha_{\tilde{X}_s})}(\check{\mathbf{q}}, \check{\mathbf{p}}) = H_{X_s, 1}. \quad (\text{R-14})$$

Since it is valid for any $s \in \llbracket 1, n \rrbracket$, we end up with (7-6).

References

- [1] M. R. Adams, J. Harnad, and J. Hurtubise. Darboux coordinates and Liouville-Arnold integration in loop algebras. *Comm. Math. Phys.*, 155(2):385–413, 1993.
- [2] M. R. Adams, J. Harnad, and J. Hurtubise. Darboux coordinates on coadjoint orbits of lie algebras. *Lett. Math. Phys.*, 40:41–57, 1997.
- [3] M.R. Adams, J. Harnad, and J. Hurtubise. Dual moment maps into loop algebras. *Lett. Math. Phys.*, 20:299–308, 1990.
- [4] M. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Royal Soc. A*, 308(523-615), 1982.
- [5] M. Bergère, G. Borot, and B. Eynard. Rational differential systems, loop equations, and application to the q^{th} reductions of KP. *Ann. Henri Poincaré*, 16(12):2713–2782, 2015.
- [6] M. Bergère and B. Eynard. Determinantal formulae and loop equations, 2009. arXiv:0901.3273.
- [7] M. Bertola, J. Harnad, and J. Hurtubise. Hamiltonian structure of rational isomonodromic deformation systems. *J. Math. Phys.*, 64, 2023.
- [8] M. Bertola and O. Marchal. The partition function of the two-matrix model as an isomonodromic τ function. *J. Math. Phys.*, 50, 2008.
- [9] M. Bertola and M. Y. Mo. Isomonodromic deformation of resonant rational connections. *Int. Math. Res.*, 2005(11):565–635, 2005.
- [10] O. Biquard and P. Boalch. Wild non-abelian hodge theory on curves. *Compos. Math.*, 140(1):179–204, 2004.
- [11] G.D. Birkhoff. Singular points of ordinary linear differential equations. *Trans. Am. Math. Soc.*, 10(4):436–470, 1909.
- [12] P. Boalch. *Symplectic geometry and isomonodromic deformations*. PhD thesis, Oxford D.Phil., 1999.
- [13] P. Boalch. Symplectic manifolds and isomonodromic deformations. *Adv. Math.*, 163(2):137–205, 2001.
- [14] P. Boalch. Simply-laced isomonodromy systems. *Publ. Math. IHÉS*, 116:1–68, 2012.
- [15] P. Boalch. Geometry and braiding of stokes data; fission and wild character varieties. *Annals of Math.*, 179:301–365, 2014.
- [16] P. Boalch. Poisson varieties from Riemann surfaces. *Indag. Math.*, 25(5):872–900, 2014.
- [17] P. Boalch, J. Douçot, and G. Rembado. Twisted local wild mapping class groups: configuration spaces, fission trees and complex braids, 2022. arXiv:2209.12695.
- [18] P. Boalch and D. Yamakawa. Twisted wild character varieties, 2015. arXiv:1512.08091.
- [19] M. Cafasso, P. Gavrylenko, and O. Lisovyy. Tau functions as Widom constants. *Commun. Math. Phys.*, 365, 2019.

- [20] L. Chekhov and B. Eynard. Hermitian matrix model free energy: Feynman graph technique for all genera. *JHEP*, 3, 2006.
- [21] H. Chiba. Multi-poisson approach to the Painlevé equations: from the isospectral deformation to the isomonodromic deformation. *SIGMA*, 13, 2017.
- [22] O. Costin. Correlation between pole location and asymptotic behavior for Painlevé I solutions. *Comm. Pure Appl. Math.*, 52(4):461–478, 1999.
- [23] O. Costin and R. D. Costin. On the formation of singularities of solutions of nonlinear differential systems in antistokes directions. *Invent. Math.*, 145(3):425–485, 2001.
- [24] K. Diarra and F. Loray. Classification of algebraic solutions of irregular Garnier systems. *Compos. Math.*, 156(5):881–907, 2020.
- [25] B. Eynard, E. Garcia-Failde, O. Marchal, and N. Orantin. Quantization of classical spectral curves via topological recursion. *Commun. Math. Phys.*, 2024.
- [26] B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. *Commun. Number Theory Phys.*, 1(2), 2007.
- [27] B. Eynard and N. Orantin. Topological recursion in random matrices and enumerative geometry. *J. Phys. A: Mathematical and Theoretical*, 42(29), 2009.
- [28] V. V. Fock and A. A. Rosly. Poisson structure on moduli of flat connections on Riemann surfaces and r-matrix. *Am. Math. Soc. Transl.*, 191, 1999.
- [29] R. Fuchs. Sur quelques équations différentielles linéaires du second ordre. *Comptes Rendus*, 141:555–558, 1905.
- [30] I. Gaiur, M. Mazzocco, and V. Rubtsov. Isomonodromic deformations: Confluence, reduction & quantisation. *Commun. Math. Phys.*, 2023.
- [31] O. Gamayun, N. Iorgov, and O. Lisovyy. Conformal field theory of Painlevé VI. *JHEP*, 10, 2012.
- [32] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes. *Acta Math.*, 33:1–55, 1910.
- [33] R. Garnier. Sur des équations différentielles du troisième ordre dont l’intégrale générale est uniforme et sur une classe d’équations nouvelles d’ordre supérieur dont l’intégrale générale a ses points critiques fixes. *Ann. Sci. Éc. Norm. Supér.*, 29:1–126, 1912.
- [34] R. Garnier. Solution du problème de Riemann pour les systèmes différentiels linéaires du second ordre. *Ann. Sci. Éc. Norm. Supér.*, 43:177–307, 1927.
- [35] W. Goldman. The symplectic nature of the fundamental group of surfaces. *Adv. Math.*, 54:200–225, 1984.
- [36] J. Harnad. Hamiltonian dynamics, classical R-matrices and isomonodromic deformations. *Lect. Notes Phys.*, 502:137–205, 2007.
- [37] John P. Harnad. Dual isomonodromic deformations and moment maps to loop algebras. *Commun. Math. Phys.*, 166:337–366, 1994.

- [38] N. J. Hitchin. Geometrical aspects of Schlesinger’s equation. *J. Geom. Phys.*, 23(3-4), 1997.
- [39] J. Hurtubise. On the geometry of isomonodromic deformations. *J. Geom. Phys.*, 58(10):1394–1406, 2008.
- [40] K. Iwaki. 2-parameter τ -function for the first Painlevé equation: topological recursion and direct monodromy problem via exact WKB analysis. *Comm. Math. Phys.*, 377(2):1047–1098, 2020.
- [41] K. Iwaki, O. Marchal, and A. Saenz. Painlevé equations, topological type property and reconstruction by the topological recursion. *J. Geom. Phys.*, 124:16–54, 2018.
- [42] K. Iwaki and T. Nakanishi. Exact WKB analysis and cluster algebras. *J. Phys. A: Math. Theor.*, 47, 2014.
- [43] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. *From Gauss to Painlevé*, volume 16 of *Aspects of Mathematics*. Springer Vieweg-Teubner, 1991.
- [44] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D*, 2(3):407–448, 1981.
- [45] M. Jimbo, T. Miwa, and K. Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. general theory and τ -function. *Phys. D*, 2(2):306–352, 1981.
- [46] H. Kawamuko. On the holonomic deformation of linear differential equations. *Proc. Jpn. Acad. A: Math. Sci.*, 73(8):152–154, 1997.
- [47] H. Kawamuko. On the polynomial Hamiltonian structure associated with the $L(1, g + 2; g)$ type. *Proc. Jpn. Acad. A: Math. Sci.*, 73(8):155–157, 1997.
- [48] H. Kimura. The degeneration of the two dimensional Garnier system and the polynomial Hamiltonian structure. *Ann. di Mat. Pura ed Appl.*, 155:25–74, 1989.
- [49] A. Komyo. A nonclassical algebraic solution of a 3-variable irregular Garnier system, 2022. arXiv:2205.14979.
- [50] I. Krichever. Isomonodromy equations on algebraic curves, canonical transformations and whitham equations. *Mosc. Math. J.*, 2(4), 2002.
- [51] J. Malmquist. Sur les équations différentielles du second ordre dont l’intégrale général a ses points critiques fixes. *Ark. Mat. Astr. Fys.*, 17:1–89, 1922.
- [52] I. Manin. *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, volume 47 of *Colloquium publications*. American Mathematical Soc., 1999.
- [53] Yu.I. Manin. *Selected Papers of Yu I Manin*, volume 3. World Scientific Publishing, 1996.
- [54] Yu.I. Manin. Sixth Painlevé Equation, Universal Elliptic Curve, and Mirror of $\mathbb{P}^2$, 1996. arXiv:alg-geom/9605010.
- [55] O. Marchal and M. Alameddine. Hamiltonian representation of isomonodromic deformations of twisted rational connections: The Painlevé 1 hierarchy, 2023. arXiv:2302.13905.

- [56] O. Marchal and M. Alameddine. Isomonodromic and isospectral deformations of meromorphic connections: the $\mathfrak{sl}_2(\mathbb{C})$ case, 2023. arXiv:2306.07378.
- [57] O. Marchal and N. Orantin. Isomonodromic deformations of a rational differential system and reconstruction with the topological recursion: the $\mathfrak{sl}_2$ case. *J. Math. Phys.*, 61(6):061506, 33, 2020.
- [58] M. Mazzocco. Irregular Isomonodromic Deformations for Garnier Systems and Okamoto’s Canonical Transformations. *J. London Math. Soc.*, 70(2):405–419, 2004.
- [59] M. Mazzocco and M.Y. Mo. The Hamiltonian structure of the second Painlevé hierarchy. *Nonlinearity*, 20(12):2845, 2007.
- [60] M. Narasimhan and S. Seshadri. Stable and unitary vector bundles on compact Riemann surfaces. *Ann. Math.*, 82:540–567, 1965.
- [61] N. Nikolaev. Existence and uniqueness of exact WKB solutions for second-order singularly perturbed linear ODEs. *Commun. Math. Phys.*, 2022.
- [62] N. Nikolaev. Exact solutions for the singularly perturbed riccati equation and exact WKB analysis. *Nagoya Math. J.*, 250:434–469, 2023.
- [63] K. Okamoto. Polynomial Hamiltonians associated with Painlevé equations. *Proc. Japan Acad.*, 56:264–268, 1980.
- [64] K. Okamoto. Isomonodromic deformation and painlevé equations, and the Garnier system. *J. Fac. Sci. Univ. Tokyo*, 33:575–618, 1986.
- [65] K. Okamoto. Studies on the Painlevé equations: I.-Sixth Painlevé’ equation PVI. *Ann. di Mat. Pura ed Appl.*, 16:337–381, 1986.
- [66] Kazuo Okamoto. Déformation d’une équation différentielle linéaire avec une singularité irrégulière sur un tore. *J. Fac. Sci. Univ. Tokyo Sect. IA Math*, 26:501–518, 1979.
- [67] P. Painlevé. Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale générale est uniforme. *Acta Math.*, 25:1–85, 1902.
- [68] E. Picard. Mémoire sur la théorie des fonctions algébriques de deux variables. *J. Math. Pures Appl.*, 5:135–319, 1889.
- [69] L Schlesinger. Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten. *J. für Math.*, 141:96–145, 1912.
- [70] W. Wasow. Asymptotic Expansions for Ordinary Differential Equations. In *Asymptotic Expansions for Ordinary Differential Equations*. Dover Publications, 2018.
- [71] N.M.J. Woodhouse. Duality for the general isomonodromy problem. *J. Geom. Phys.*, 57(4):1147–1170, 2007.
- [72] D. Yamakawa. Tau functions and Hamiltonians of isomonodromic deformations. *Josai Mathematical Monographs*, 10, 2017.
- [73] D. Yamakawa. Fundamental two-forms for isomonodromic deformations. *Journal of Integrable Systems*, 4(1), 2019.

- [74] D. Yamakawa. Quantization of simply-laced isomonodromy systems by the quantum spectral curve method. *SUT J. Math.*, 58(1):23–50, 2022.