# Hamiltonian structure of rational isomonodromic deformation systems 

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#### Abstract

The Hamiltonian approach to isomonodromic deformation systems is extended to include generic rational covariant derivative operators on the Riemann sphere with irregular singularities of arbitrary Poincaré rank. The space of rational connections with given pole degrees carries a natural Poisson structure corresponding to the standard classical rational R-matrix structure on the dual space $L^{*} \mathfrak{g l}(r)$ of the loop algebra $L \mathfrak{g l}(r)$. Nonautonomous isomonodromic counterparts of the isospectral systems generated by spectral invariants are obtained by identifying the deformation parameters as Casimir elements on the phase space. These are shown to coincide with the higher Birkhoff invariants determining the local asymptotics near to irregular singular points, together with the pole loci. Pairs consisting of Birkhoff invariants, together with the corresponding dual spectral invariant Hamiltonians, appear as "mirror images" matching, at each pole, the negative power coefficients in the principal part of the Laurent expansion of the fundamental meromorphic differential on the associated spectral curve with the corresponding positive power terms in the analytic part. Infinitesimal isomonodromic deformations are shown to be generated by the sum of the Hamiltonian vector field and an explicit derivative vector field that is transversal to the symplectic foliation. The Casimir elements serve as coordinates complementing those along the symplectic leaves, defining a local symplectomorphism between them. The explicit derivative vector fields preserve the Poisson structure and define a flat transversal connection, spanning an integrable distribution whose leaves may be identified as the orbits of a free abelian local group action. The projection of the infinitesimal isomonodromic deformation vector fields to the quotient manifold under this action gives the commuting Hamiltonian vector fields corresponding to the spectral invariants dual to the Birkhoff invariants and the pole loci.


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## 1 Introduction and results

### 1.1 Isomonodromic systems and Hamiltonian structures

The study of isomonodromic deformations of linear differential equations with a finite number of isolated singular points dates back to the earliest works of Painlevé [56, 57, 58], Fuchs [16, 17], Garnier [20][22], Schlesinger [63] and others [59, 19]. A significant extension of what is meant by the generalized monodromy data for systems with irregular isolated singularities was made by Birkhoff [9], who also included the Stokes and connection matrices. A revival of interest in isomonodromic deformations was stimulated by the work of Flaschka and Newell [14, 15] and Jimbo, Miwa and Ueno [45, 46], inspired by developments in the theory of completely integrable systems [31].

Many subsequent studies were devoted to the six Painlevé transcendents [54, 46, 55, 41, 47, 13], the Schlesinger system [63, 45, 28, 37], governing Fuchsian first order matrix systems of arbitrary rank, and generalizations $[35,28,69,11,12,70]$ in which one or more irregular singular points are present, including polynomial systems [14, 67, 52], the Garnier system [20, 22], [32], Sec. 9.3, and many other particular cases. The general analysis of $[45,46]$ provides a uniform approach to rational first order systems and has led to a burgeoning literature on their symmetries [54, 55, 53], asymptotic properties [41, 47, 40, 48, 25], special classes of solutions, and a variety of applications [33, 13]. (See [44] for an elementary introduction, [32], Chapts. 9-11 and [35, 6, 7, 8, 33] for applications to the spectral statistics of random matrices and [49, 39, 26, 24, 42] for compendia of various classes of known solutions.)

One feature that was recognized since the very earliest studies $[50,54,45,46,55,28,10,11,43]$ is that many of these systems could be interpreted as having an underlying Hamiltonian structure of nonautonomous type. Closely linked to this property is the notion of isomonodromic $\tau$-functions [45, 46], which allow an alternative representation of these systems in a form that resembles the bilinear Hirota equations [62, 65, 36, 29] characterizing integrable hierarchies of autonomous systems, both finite and infinite dimensional. A remarkable feature was that for many classes of isomonodromic systems, including the Painlevé transcendents, the Schlesinger systems and some generalizations of the latter, the partial derivatives of the $\tau$-function with respect to the deformation parameters could be identified with the nonautonomous Hamiltonian functions generating the deformation dynamics, evaluated on the solution manifold.

In this work, previous results on the nonautonomous Hamiltonian structure of such systems are extended within a uniform framework that includes all the rational isomonodromic deformation systems introduced in [45, 46]. This is based on the well-known classical $R$-matrix structure [60, 64, 61] of rational, linear type, on (the dual space of) loop algebras. It is the same Poisson structure that plays a fundamental rôle in the analysis and solution of autonomous finite dimensional completely integrable Hamiltonian systems of isospectral type in terms of abelian functions [3, 1, 2, 27].

In $[35,28]$ it was noticed that certain classes of isomonodromic deformation equations could be derived using the same phase space and Hamiltonian structure, by simply treating the systems as nonautonomous deformations of corresponding isospectral ones. Rather than viewing the deformation parameters as independent time parameters, representing the evolution under a complete set of commuting flows, they
are identified with certain specific functions on the phase space which, as it turns out, are always Casimir elements of the underlying classical $R$-matrix structure. For the special case of the Schlesinger systems, this is exactly the converse of the procedure applied by Garnier [21] (which he called the "Painlevé simplification") in order to convert these into autonomous systems that are completely integrable.*

The main message of the present work is that if we do the reverse, namely deautonomize a certain subset of the integrable isospectral Hamiltonian systems with respect to the rational classical R-matrix structure, by identifying the deformation parameters with Casimir elements on the phase space, we obtain deformed Hamiltonian systems that agree exactly with the isomonodromic deformation dynamics introduced in [45, 46]. Strictly speaking, these are not really Hamiltonian systems since, added to the Hamiltonian vector fields, there is a "transversal component" which amounts to an "explicit" derivative of the corresponding rational Lax matrix $L(z)$ with respect to the Casimir parameters. This renders the system no longer isospectral, as a purely Hamiltonian flow would be, but rather of "zero-curvature" type, on the extended space consisting of the deformation parameters augmented by the spectral variable $z$ in the rational Lax matrix $L(z)$. The extended system therefore does not preserve the symplectic foliation, since it has a transversal component corresponding to the "explicit" dependence on the deformation parameters. However, the transverse foliation may be viewed as generated by a locally free abelian group action that preserves the Poisson structure, defining a local isomorphism between neighbouring leaves of the symplectic foliation. The quotient by this group action may be identified with any of the neighbouring symplectic leaves (augmented by some trivial Casimir elements, consisting of the exponents of formal monodromy $[4,45]$ ), and the projection of the infinitesimal isomonodromic deformation vector field to this quotient is the corresponding Hamiltonian vector field.

The key problem that needs to be resolved in the general case, with irregular singularities of arbitrary Poincaré rank, is: how do we define these "explicit derivative" vector fields, and how do we choose the spectral invariant Hamiltonians whose vector fields, when added to the "explicit derivative" ones, generate the corresponding 1-parameter family of isomonodromic deformations?

In the cases studied earlier [45, 46, 35, 28, 34], this was very straightforward. For the Schlesinger systems, the "explicit parameters" are simply the loci of the first order poles of the Lax matrix governing the Schlesinger equations. For systems with one further non-Fuchsian singularity at $\infty$, with Poincaré index 1 , the additional deformation parameters are the eigenvalues of the Lax matrix at $z=\infty$; i.e., the constant matrix term added to the Schlesinger Lax matrix [28]. Of the six Painlevé equations, two ( $P_{V I}$ and $P_{V}$ respectively) are special cases of these, and for the remaining four $\left(P_{I}-P_{I V}\right)$, the additional deformation parameter is one of the higher Birkhoff invariants identified in [46] and used in the canonical parametrization $[46,34]$ of their $2 \times 2$ Lax matrices.

In this work, the problem is solved for arbitrary (nonresonant) rational isomonodromic deformation equations of the type introduced in [45, 46], placing these in the Hamiltonian framework provided by the rational $R$-matrix structure. Some of the results presented here have appeared earlier [28, 30,5] in partial form. In [52], the $P_{I I}$ hierarchy, which consists of (reduced) isomonodromic deformation systems

[^1]involving $2 \times 2$ polynomial Lax matrices of any degree, plus a first order pole at $z=0$ was treated using the same rational $R$-matrix Poisson bracket structure. In [18], the Hamiltonian structure and quantization of rational isomonodromic deformations were studied, with irregular singularities obtained via confuence of poles. After this work was completed, we learned of ref. [71], in which some of the earlier results of $[28,30,5]$, which are included here in Section 3.3, were re-derived and extended in ways that overlap with parts of Sections 4 and 5.1. Another recent work [51] derived a Hamiltonian representation of rational $2 \times 2$ isomonodromic systems in a different way, making use of the spectral Darboux coordinates introduced in [2]. Our purpose here is to present a complete, self-contained account, which contains all results known to date.

The next three subsections recall the main implications of classical $R$-matrix theory, and summarize how the isospectral equations generated by a suitably chosen class of spectral invariant Hamiltonians, which are dual to the deformation parameters, in a natural sense, are converted into the isomonodromic deformation equations under consideration.

### 1.2 Rational $R$-matrix structure and isospectral systems

The phase space $\mathcal{L}_{r, \mathbf{d}}$ considered throughout this work consists of complex, traceless $r \times r$ rational matrixvalued functions of a spectral variable $z \in \mathbb{P}^{1}$

$$
\begin{equation*}
L(z)=-\sum_{j=0}^{d_{\infty}-1} L_{j+2}^{\infty} z^{j}+\sum_{\nu=1}^{N} \sum_{j=1}^{d_{\nu}+1} \frac{L_{j}^{\nu}}{\left(z-c_{\nu}\right)^{j}} \tag{1.1}
\end{equation*}
$$

henceforth referred to as the Lax matrix, where $\mathbf{d}:=\left(d_{1}, \ldots, d_{N}, d_{\infty}\right) \in \mathbb{N}_{+}^{N+1}$. The matrix differential $L(z) d z$ has pole divisor bounded by

$$
\begin{equation*}
\operatorname{div}_{p o l e}(L(z) d z) \geq-\left(d_{\infty}+1\right) \infty-\sum_{\nu=1}^{N}\left(d_{\nu}+1\right) c_{\nu} \tag{1.2}
\end{equation*}
$$

and the finite pole loci $\mathbf{c}=\left(c_{1}, \ldots, c_{N}\right)$ are distinct. The leading coefficients $\left\{L_{d_{\nu}+1}^{\nu}\right\}_{\nu=1, \ldots, N, \infty}$ of the polar parts, both at the finite $c_{\nu}$ 's and at $c_{\infty}=\infty$, are assumed diagonalizable, with distinct eigenvalues (the nonresonant condition [45, 5]). By conjugation with a constant matrix we may, without loss of generality, choose $L_{d_{\infty}+1}^{\infty}$ to be diagonal, with the entries all conserved quantities. The set $\mathcal{L}_{r, \mathrm{~d}}$ of such $L(z)^{\prime}$ 's may be interpreted as a Poisson submanifold of the dual space $L_{R}^{*} \mathfrak{g l}(r)$ of the modified loop algebra $L_{R} \mathfrak{g l}(r)$ (viewed as smooth maps $L: S^{1} \rightarrow \mathfrak{g l}(r)$ from the unit circle $S^{1}=\{z \in \mathbb{C} \| z \mid=1\}$ ) with respect to the modified Lie-Poisson bracket structure corresponding to the so-called split rational classical $R$-matrix [60, 64, 61].

Splitting the space $L_{R} \mathfrak{g l}(r) \sim L \mathfrak{g l}(r)$ as a direct sum

$$
\begin{equation*}
L_{R} \mathfrak{g l}(r)=L_{+} \mathfrak{g l}(r) \oplus L_{-} \mathfrak{g l}(r) \tag{1.3}
\end{equation*}
$$

of subspaces (and subalgebras) consisting of elements $X_{+} \in L_{+} \mathfrak{g l}(r)$ that admit analytic continuation inside the unit circle (or, simply, positive power Fourier series)

$$
\begin{equation*}
X_{+}=\sum_{i=0}^{\infty} X_{i} z^{i}, \quad X_{i} \in \mathfrak{g l}(r), z=e^{i \theta} \in S^{1} \tag{1.4}
\end{equation*}
$$

and those $X_{-} \in L_{-} \mathfrak{g l}(r)$ that admit analytic continuation outside, with $X_{-}(\infty)=0$ (or negative power Fourier series )

$$
\begin{equation*}
X_{-}=\sum_{i=1}^{\infty} X_{i} z^{-i}, \quad X_{i} \in \mathfrak{g l}(r), z=e^{i \theta} \in S^{1} \tag{1.5}
\end{equation*}
$$

the modified Lie bracket, denoted $[X, Y]_{R}$ is defined by

$$
\begin{equation*}
\left[X_{+}, Y_{+}\right]_{R}=\left[X_{+}, Y_{+}\right], \quad\left[X_{-}, Y_{-}\right]_{R}=-\left[X_{-}, Y_{-}\right], \quad\left[X_{+}, Y_{-}\right]_{R}=0 \tag{1.6}
\end{equation*}
$$

The dual space $L_{R}^{*} \mathfrak{g l}(r)$ (or $\left.L^{*} \mathfrak{g l}(r)\right)$ is identified with $L_{R} \mathfrak{g l}(r)$ (or $L \mathfrak{g l}(r)$ ) through the pairing

$$
\begin{equation*}
\mu(X)=\frac{1}{2 \pi i} \oint_{z \in \gamma} \operatorname{tr}(\mu(z) X(z)) d z, \quad \mu \in L^{*} \mathfrak{g l}(r), \quad X \in L \mathfrak{g l}(r) \tag{1.7}
\end{equation*}
$$

The annihilators $\left(L_{ \pm} \mathfrak{g l}(r)\right)^{0} \subset L_{R}^{*} \mathfrak{g l}(r)$ of the subalgebras $L_{+} \mathfrak{g l}(r)$ and $L_{+} \mathfrak{g l}(r)$ are therefore identified as

$$
\begin{equation*}
\left(L_{+} \mathfrak{g l}(r)\right)^{0}=L_{+} \mathfrak{g l}(r)=\left(L_{-} \mathfrak{g l}(r)\right)^{*}, \quad\left(L_{-} \mathfrak{g l}(r)\right)^{0}=L_{-} \mathfrak{g l}(r)=\left(L_{+} \mathfrak{g l}(r)\right)^{*} \tag{1.8}
\end{equation*}
$$

and these are mutually disjoint Poisson subspaces of $L_{R}^{*} \mathfrak{g l}(r)$. Here $\gamma$ can be chosen as the unit circle $S^{1}$ centered at the origin oriented counterclockwise, but it will be convenient, when considering finite dimensional Poisson subspaces of $L_{R}^{*} \mathfrak{g l}(r)$ consisting of rational elements $L \in L_{R}^{*} \mathfrak{g l}(r)$ of the form (1.1), to use the same notational conventions, but replace $S^{1}$ by a circle of any radius, chosen so that all the finite poles lie in the interior of the integration contour $\gamma$.

The canonical Lie-Poisson bracket with respect to the modified Lie algebra structure $L_{R} \mathfrak{g l}(r)$

$$
\begin{equation*}
\left.\{f, g\}_{R}\right|_{\mu \in L^{*} \mathfrak{g l}(r)}:=\mu\left([d f, d g]_{R}\right)=\mu\left(\left[(d f,)_{+},(d g)_{+}\right]-\left[(d f)_{-},(d g)_{-}\right]\right), \quad f, g \in C^{1}\left(L^{*} \mathfrak{g l}(r)\right) \tag{1.9}
\end{equation*}
$$

is dual to the Lie bracket on $L_{R} \mathfrak{g l}(r)$, and has a multitude of finite dimensional Poisson subspaces; in particular, the space $L_{(\mathbf{c}, \mathbf{d})}^{*} \mathfrak{g l}(r)$ of rational matrices of the form (1.1) for fixed pole loci

$$
\begin{equation*}
(\mathbf{c}, \infty), \quad \mathbf{c}:=\left\{c_{1}, \ldots, c_{N}\right\} \tag{1.10}
\end{equation*}
$$

and maximal degrees

$$
\begin{equation*}
\mathbf{d}:=\left\{d_{1}, \ldots, d_{N}, d_{\infty}\right\} \tag{1.11}
\end{equation*}
$$

This may equivalently be identified with the Lie-Poisson space consisting of the dual space to the finite dimensional Lie algebra

$$
\begin{equation*}
L_{\mathbf{d}} \mathfrak{g l}(r):=\oplus_{\nu=1}^{N} \mathfrak{g l}^{\left(d_{\nu}\right)}(r) \oplus \mathfrak{g l}^{\left(d_{\infty}+1\right)}(r) \tag{1.12}
\end{equation*}
$$

where $\mathfrak{g l}^{(j)}(r)$ denotes the $j$ th jet extension of $\mathfrak{g l}(r)$, and the leading polynomial coefficient matrix $L^{\left(d_{\infty}\right)}$, whose entries consists only of Casimirs elements, is chosen as diagonal. The corresponding Lie algebra $L_{(\mathbf{c}, \mathbf{d})} \mathfrak{g l}(r)$ may be viewed as the quotient of the full loop algebra $L \mathfrak{g l}(r)$ by the ideal

$$
\begin{equation*}
\mathfrak{I}_{\mathbf{c}, \mathbf{d}}:=z^{-d_{\infty}-1} \prod_{\nu=1}^{N}\left(z-c_{\nu}\right)^{d_{\nu}+1} L_{R} \mathfrak{g l}(r) \subset L_{R} \mathfrak{g l}(r) \tag{1.13}
\end{equation*}
$$

in which the pole loci $\left(c_{1}, \ldots, c_{n}\right)$ are "spectators", their values giving different equivalent identifications of $L_{(\mathbf{c}, \mathbf{d})} \mathfrak{g l}(r)$, for varying $\mathbf{c}$, as quotients of $L_{R} \mathfrak{g l}(r)$ by equivalent ideals. Taking a disjoint union over these, the pole locations $\left\{c_{1}, \ldots, c_{N}\right\}$, may be viewed as further coordinates, which are Casimir functions on the larger Poisson subspace, consisting of the product

$$
\begin{equation*}
\left(\mathbb{C}^{N}\right)^{\prime} \times L_{\mathbf{d}}^{*} \mathfrak{g l}(r), \quad\left(\mathbb{C}^{N}\right)^{\prime}:=\left\{\left(c_{1}, \ldots, c_{N}\right) \in \mathbb{C}^{N} \mid c_{i} \neq c_{j} \text { for } i \neq j\right\} \tag{1.14}
\end{equation*}
$$

that Poisson commute amongst themselves and with the elements of $L_{\mathbf{d} g l}(r)$.
The coadjoint action of the corresponding split loop group,

$$
\begin{equation*}
L_{+} \mathfrak{G} \mathfrak{l}(r) \times L_{-} \mathfrak{G} \mathfrak{l}(r)=\left\{\left(g_{+}(z), g_{-}(z)\right)\right\} \tag{1.15}
\end{equation*}
$$

where $g_{-}(\infty)=\mathbf{I}$, is given by dressing transformations:

$$
\begin{equation*}
A d_{R}^{*}\left(g_{+}, g_{-}\right):\left(X_{+}+X_{-}\right) \mapsto\left(g_{-} X_{+}\left(g_{-}\right)^{-1}\right)_{+}+\left(\left(g_{+}\right)^{-1} X_{-} g_{+}\right)_{-} \tag{1.16}
\end{equation*}
$$

where $(\ldots)_{ \pm}$denotes projection to the positive and negative parts of the Fourier series. The orbits under this action are the symplectic leaves of the $R$-matrix Poisson structure $\{,\}_{R}$, which we henceforth just denote as $\{$,$\} .$

Written in terms of matrix elements of $L$ evaluated at two values $(z, w)$ of the loop parameter, viewed as linear functionals on the loop algebra, the Lie-Poisson brackets corresponding to the split classical rational $R$-matrix structure (1.9) are given by

$$
\begin{equation*}
\left\{L_{a b}(z), L_{c d}(w)\right\}=\frac{1}{z-w}\left(\left(L_{a d}(z)-L_{a d}(w)\right) \delta_{c b}-\left(L_{c b}(z)-L_{c b}(w)\right) \delta_{a d}\right) \tag{1.17}
\end{equation*}
$$

Classical $R$-matrix theory $[60,64,61]$ then implies that:

1. All elements of the ring $\mathcal{I}^{\operatorname{Ad}^{*}}\left(L^{*} \mathfrak{g l}(r)\right)$ of (unmodified) $\mathrm{Ad}^{*}$ invariant functions of $L(z)$ (i.e., the ring of spectral invariants) Poisson commute amongst themselves.

$$
\begin{equation*}
\{f, g\}=0, \quad \forall f, g \in \mathcal{I}^{\operatorname{Ad}^{*}}\left(L^{*} \mathfrak{g l}(r)\right) \tag{1.18}
\end{equation*}
$$

This means that, on $L_{r, \mathbf{d}}$, all elements of the $\operatorname{ring} \mathcal{I}^{\operatorname{Ad}^{*}}\left(L^{*} \mathfrak{g l}(r)\right)$ generated by the coefficients of the characteristic polynomial

$$
\begin{equation*}
\operatorname{det}(L(z)-\lambda \mathbf{I})=0 \tag{1.19}
\end{equation*}
$$

defining the (planar) spectral curve $\mathcal{C}_{0}$ Poisson commute.
2. The Hamiltonian vector field $\mathbf{X}_{H}$ generated by any element $H \in \mathcal{I}^{\operatorname{Ad}^{*}}\left(L^{*} \mathfrak{g l}(r)\right)$ is given by a commutator

$$
\begin{align*}
\mathbf{X}_{H}(X) & =\{X, H\}=\left[R_{s}(d H), X\right]  \tag{1.20}\\
\forall H & \in \mathcal{I}^{\operatorname{Ad}^{*}}\left(L^{*} \mathfrak{g l}(r)\right), X \in L \mathfrak{g l}(r)
\end{align*}
$$

where $X \in L \mathfrak{g l}(r)$ is viewed as a linear functional on $L^{*} \mathfrak{g l}(r)$ under the pairing (1.7) and $R_{s}$ is the endomorphism of $L \mathfrak{g l}(r)$ defined by

$$
\begin{equation*}
R_{s}\left(Y_{+}+Y_{-}\right)=s Y_{+}+(s-1) Y_{-}, \quad Y \in L \mathfrak{g l}(r) \tag{1.21}
\end{equation*}
$$

for any $s \in \mathbb{C}$. In particular,

$$
\begin{equation*}
R_{1}\left(Y_{+}+Y_{-}\right)=Y_{+}, \quad \text { and } \quad R_{0}\left(Y_{+}+Y_{-}\right)=-Y_{-} \tag{1.22}
\end{equation*}
$$

Remark 1.1. For the isospectral systems generated by such Hamiltonians, the choice of $s \in \mathbb{C}$ is irrelevant, since changing it only adds a term proportional to $d H(L)$, which is in the commutant of L. But when the system is modified, as in Sections 1.3 and 1.4, by the addition of a transverse "explicit derivative" vector field, only one specific choice of $s$, usually $s=0$ or 1 , renders the system isomonodromic for any of the relevant spectral invariant Hamiltonians.
3. The Hamiltonian flow generated by any element $H \in \mathcal{I}^{\operatorname{Ad}^{*}}\left(L^{*} \mathfrak{g l}(r)\right)$ of the spectral ring leaves invariant the spectral curve (1.19), and hence the flow is isospectral. The time dependence of the corresponding integral curves $L(z, t)$ is determined by the Lax equation

$$
\begin{equation*}
\frac{d L}{d t}=\left[R_{s}(d H), L\right] \tag{1.23}
\end{equation*}
$$

and the Hamiltonian flow is given by

$$
\begin{equation*}
f_{H}(t): L(0) \rightarrow L(t)=\operatorname{Ad}_{R}^{*}\left(g_{+}(t), g_{-}(t)\right)(L(0)) \tag{1.24}
\end{equation*}
$$

where $\left(g_{+}(t), g_{-}(t)\right)$ is determined as the solution of the Riemann-Hilbert factorization problem

$$
\begin{equation*}
g_{+}(t) g_{-}(t)=e^{t d H(L(0))}, \quad\left(g_{+}(t), g_{-}(t)\right) \in L_{+} \mathfrak{G l l}(r) \times L_{-} \mathfrak{G l}(r) \tag{1.25}
\end{equation*}
$$

### 1.3 Examples of nonautonomous deformations as isomonodromic systems

The study of isospectral Hamiltonian systems generated by $A d^{*}$ invariant functions within the rational classical $R$-matrix Poisson bracket structure on (the dual space of) loop algebras was developed in [1, 2, 27] for autonomous systems and this was extended to nonautonomous isomonodromic ones in [35, 28, 34, 30]. The simplest case consists of the Schlesinger equations, which generate isomonodromic deformations of the Fuchsian system

$$
\begin{equation*}
\frac{\partial \Psi(z)}{\partial z}=L^{\mathrm{Sch}}(z) \Psi(z), \quad \Psi(z) \in \mathfrak{G} \mathfrak{G}(r) \tag{1.26}
\end{equation*}
$$

where

$$
\begin{equation*}
L^{S c h}(z):=\sum_{\nu=1}^{N} \frac{L^{\nu}}{z-c_{\nu}} \tag{1.27}
\end{equation*}
$$

The Casimir elements that serve as deformation parameters are the loci $\left(c_{1}, \ldots, c_{N}\right)$ of the poles. The corresponding spectral invariant Hamiltonians are

$$
\begin{equation*}
H_{\nu}:=\frac{1}{2} \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(L^{S c h}\right)^{2} d z, \quad \nu=1, \ldots, N \tag{1.28}
\end{equation*}
$$

and the Hamiltonian vector fields, acting on $L^{S c h}(z)$, are given by commutators with the matrices

$$
\begin{equation*}
R_{0}\left(d H_{\nu}\right)=-\left(d H_{\nu}\right)_{-}=-\frac{L^{\nu}}{z-c_{\nu}} \tag{1.29}
\end{equation*}
$$

The resulting nonautonomous equations, in which the explicit dependence of the Lax matrix $L(z)$ on the pole loci $\mathbf{c}$ is taken into account, are thus

$$
\begin{equation*}
\frac{\partial L^{\mathrm{Sch}}(z)}{\partial c_{\nu}}=\left[-\frac{L^{\nu}}{z-c_{\nu}}, L^{\mathrm{Sch}}(z)\right]+\frac{L^{\nu}}{\left(z-c_{\nu}\right)^{2}}, \quad \nu=1, \ldots, N \tag{1.30}
\end{equation*}
$$

The additional term $\frac{L^{\nu}}{\left(z-c_{\nu}\right)^{2}}$ in eq. (1.30) must be included because of the explicit dependence of $L(z)$ on the pole locations $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N}$. By equating the residues, these are equivalent to the Schlesinger equations

$$
\begin{align*}
& \frac{\partial L^{\mu}}{\partial c_{\nu}}=\frac{\left[L^{\mu}, L^{\nu}\right]}{c_{\mu}-c_{\nu}}, \quad \forall \nu \neq \mu  \tag{1.31a}\\
& \frac{\partial L^{\mu}}{\partial c_{\mu}}=-\sum_{\nu=1, \mu \neq \nu}^{N} \frac{\left[L^{\mu}, L^{\nu}\right]}{c_{\mu}-c_{\nu}} \tag{1.31b}
\end{align*}
$$

They are also the compatibility conditions for the overdetermined system consisting of (1.26), together with the infinitesimal deformation equations

$$
\begin{equation*}
\frac{\partial \Psi}{\partial c_{\nu}}=-\frac{L^{\nu}}{z-c_{\nu}} \Psi, \quad \nu=1, \ldots, N \tag{1.32}
\end{equation*}
$$

and hence imply invariance of the monodromy of the operator

$$
\begin{equation*}
\mathcal{D}^{L^{s c h}}:=\frac{\partial}{\partial z}-L^{S c h} \tag{1.33}
\end{equation*}
$$

under changes in the pole locations.
Viewed geometrically, eqs. (1.26), (1.32) represent parallel transport with respect to the connection form

$$
\begin{equation*}
\Omega:=-L^{\mathrm{Sch}} d z+\sum_{\nu=1}^{N} \frac{L^{\nu}}{z-c_{\nu}} d c_{\nu} \tag{1.34}
\end{equation*}
$$

over the product of the spectral parameter space $\left\{z \in \mathbb{P}^{1}\right\}$ ) and the space of distinct pole loci $\{\mathbf{c}\}$, with the loci of poles removed. Their compatibility conditions are equivalent to the commutativity of the covariant derivatives over the parameter space

$$
\begin{align*}
& {\left[\frac{\partial}{\partial z}-L^{S c h}(z), \frac{\partial}{\partial c_{\nu}}+\frac{L^{\nu}}{z-c_{\nu}}\right]=0}  \tag{1.35a}\\
& {\left[\frac{\partial}{\partial c_{\mu}}+\frac{L^{\mu}}{z-c_{\mu}}, \frac{\partial}{\partial c_{\nu}}+\frac{L^{\nu}}{z-c_{\nu}}\right]=0, \quad 1 \leq \mu, \nu \leq N} \tag{1.35b}
\end{align*}
$$

These are equivalent to the Schlesinger equations (1.31a), (1.31b), which therefore are interpretable as zero curvature equations for a flat connection. The additional term $\frac{L_{\nu}}{\left(z-c_{\nu}\right)^{2}}$ in eq. (1.30) turns the isospectral

Hamiltonian equations into the zero curvature equations (1.35) because of the identity ${ }^{\dagger}$

$$
\begin{equation*}
\frac{\partial}{\partial z}\left(-\frac{L^{\nu}}{z-c_{\nu}}\right)=\frac{\partial^{0} L^{S c h}}{\partial c_{\nu}^{0}}=\frac{L^{\nu}}{\left(z-c_{\nu}\right)^{2}} \tag{1.36}
\end{equation*}
$$

which we refer to as an isomonodromic identity, where $\frac{\partial^{0} L^{S c h}}{\partial c_{\nu}^{0}}$ denotes the explicit derivative with respect to the pole location parameter $c_{\nu}$, and the entries of the matrices $\left\{L^{\nu}\right\}_{\nu=1, \ldots, N}$ are considered as independent coordinates. This simple identity is the essential reason why the $R$-matrix approach which, in the autonomous case gives isospectral (Lax) equations, when applied to the nonautonomous system obtained by identifying the deformations parameters as pole loci, gives rise to the zero curvature equations (1.35).

Note that there are two ingredients leading to this result. The first is that the explicit parametric dependence in the nonautonomous system is just through the location of the poles, which are Casimir elements of the Poisson structure. The second is that the exact choice (1.28) of the corresponding Hamiltonians from amongst the various possible spectral invariants implies the isomonodromic identity (1.36). These must be matched in a special dual way with the deformation parameters in order that the identity (1.36), resulting in a zero curvature system, be satisfied. The notion of explicit dependence and dual pairing of the deformation parameters with the Hamiltonians, which is clear in this special case, is not obvious in the more general case of arbitrary rational Lax matrices. Making this precise will be one of the main points of the subsequent development. (See Theorems 4.1-4.6).

A slightly more general case was treated similarly in [28] where, in addition to the first order poles appearing in (1.27), a constant diagonal matrix $B=\operatorname{diag}\left(b_{1}, \ldots, b_{r}\right)$ was added to the Lax matrix, giving

$$
\begin{equation*}
L(z)=B+L^{S c h}(z) \tag{1.37}
\end{equation*}
$$

and hence adding a second order pole in the connection form $L(z) d z$ at $\infty$. The deformation parameters $\left(b_{1}, \ldots, b_{r}\right)$ are again Casimir elements in the $R$-matrix Lie-Poisson structure and there is a corresponding special set of $r$ spectral invariant Hamiltonians $\left\{K_{a}\right\}_{a=1, \ldots, r}$ (see [28]) generating the Hamiltonian vector fields as commutators. The deformation matrices $\left\{\left(d K_{a}\right)_{+}\right\}$again satisfy the necessary equality

$$
\begin{equation*}
\frac{\partial\left(d K_{a}\right)_{+}}{\partial z}=\frac{\partial^{0} L}{\partial b_{a}^{0}}=E_{a a}, \quad a=1, \ldots, r \tag{1.38}
\end{equation*}
$$

between their derivatives with respect to the spectral parameter $z$ and the "explicit" derivatives of the Lax matrix with respect to the parameters $\left(b_{1}, \ldots, b_{r}\right)$ (where $E_{a b} \in \mathfrak{g l}(r)$ is the elementary matrix whose only nonvanishing entry is a 1 in the $(a, b)$ position). Starting with the isospectral Hamiltonian Lax equations following from the $R$-matrix structure, adding the explicit derivatives with respect to the nonautonomous deformation parameters $\left(b_{1}, \ldots, b_{r}\right)$ in the Lax matrix and using the isomonodromic identity (1.38) again assures that the isospectral equations appearing in the autonomous case become zero curvature ones in the nonautonomous one. The compatibility conditions of the deformation equations again imply the invariance of the (generalized) monodromy.

In [34] the rational $R$-matrix approach was also applied to deriving the five Painlevé transcendent equations $P_{I}-P_{V}$, which are all reductions of $2 \times 2$ rational isomonodromic deformation equations, with

[^2]pole divisor having total degree -4 . (The $P_{V I}$ case is just a symmetry reduction of the $r=2$ Schlesinger system with 3 finite simple poles, plus one at $\infty$.) In [35], it was applied to rational systems in $L_{R}^{*} \mathfrak{g l}(2)$ with an arbitrary number of first order poles at the finite points $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N}$ plus an additional irregular singularity of Poincaré index 2 at $\infty$.

The question that naturally occurs is whether this approach can be extended to the general class of rational isomonodromic deformation systems introduced in [45, 46], in which the Lax matrix is of the form (1.28), and the connection has any number of irregular singularities of arbitrary Poincaré rank at finite points or at $z=\infty$. Completing the analysis for the general case is the main purpose of the present work. It follows along lines similar to the cases previously treated, but gives a clearer notion of what is meant by the explicit derivative of the Lax matrix with respect to the further deformation parameters which, as before, are viewed as functions on the phase space. As in the previous cases, it turns out that only Casimir elements can play this rôle and, in fact, (nearly) all of them do.

The full set of these provide a transversal, regular foliation complementary to the one given by the symplectic leaves, which are the "dressing transformation" orbits under the $R$-matrix Lie algebra structure. In addition to the pole loci $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N}$, the further Casimir elements that serve as deformation parameters turn out to coincide with the higher Birkhoff invariants $\left\{t_{j a}^{\nu}\right\}_{\nu=1, \ldots, N, \infty, j=1, \ldots, d_{\nu}, a=1, \ldots, r}$ appearing in $[4,45,46]$, which characterize the formal asymptotic behaviour of a fundamental system near the irregular singular points, but may also be expressed as spectral invariant functions of the Lax matrices, as in eqs. (1.41)

The second ingredient consists of finding the correct spectral invariant Hamiltonians that are paired "dually" with these Casimir elements. This turns out to have an elegant solution in terms of the structure of the spectral curve, and the local singularity structure of the naturally associated meromorphic differential over the poles in the Lax matrix. (See eqs. (1.43a), (1.43b), (1.44).) The "isomonodromic identities" (1.59) that allow the associated isospectral systems given by the $R$-matrix dynamics to be converted into isomonodromic ones are "reverse engineered", in the sense that eqs. (1.59) are used as the definition of what the "explicit derivatives" mean. It is then verified (Theorem 1.2) that these can, indeed, be interpreted as a set of commuting vector fields on the phase space. By Frobenius' theorem, they may be interpreted as commuting directional derivatives along transversal curves defined by fixing all but one of the Birkhoff invariants, viewed as coordinate functions on the phase space. They furthermore preserve the Poisson structure, and therefore provide an integrable distribution transverse to the symplectic foliation, allowing at least a local identification of the neighbouring symplectic leaves in a tubular neighbourhood as the orbit of an abelian group action on a single one of these. This gives a consistent Hamiltonian framework for the deautonomization of the isospectral equations in the general case, as summarized in the following subsection.

### 1.4 Rational isomonodromic systems and nonautonomous Hamiltonian deformations

In general, we consider isomonodromic deformations of rational covariant derivative operators:

$$
\begin{equation*}
\mathcal{D}_{z}^{L}:=\frac{\partial}{\partial z}-L(z) \tag{1.39}
\end{equation*}
$$

where $L(z)$ is a rational Lax matrix of the form (1.1), with fundamental systems $\Psi(z)$ of solutions of the equation

$$
\begin{equation*}
\frac{\partial \Psi(z)}{\partial z}=L(z) \Psi(z), \quad \Psi(z) \in \mathfrak{G l l}(r) \tag{1.40}
\end{equation*}
$$

In addition to the pole loci $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N}$, the independent variables parametrizing the deformations will be identified with a subset of the Casimir elements, defined in eqs. (1.41a), (1.41b), denoted $\left\{t_{j a}^{\nu}, t_{j a}^{\infty}\right\}$, which will be shown to coincide with the higher Birkhoff invariants [4] of Definition 3.1. Here, as in (1.1), the indices $\nu=1, \ldots, N$ denote the finite pole locations $\left\{z=c_{\nu}\right\}$, with corresponding negative powers $j=1, \ldots, d_{\nu}$ in the principal part of $L(z)$ at $z=c_{\nu}$ and $j=1, \ldots d_{\infty}$ at $z=\infty$, the positive powers in the polynomial part, The indices $a=1, \ldots, r$ correspond to the $r$ different local solutions $\left\{\lambda_{a}(z)\right\}_{a=1, \ldots, r}$ of the characteristic equation (1.19) near the poles $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$, given as Laurent series in a punctured neighbourhood of each pole location. Equivalently, the $a$ 's may be viewed as indexing the sheets of the spectral curve $\mathcal{C}$ obtained by compactifying the planar curve $\mathcal{C}_{0}$ defined by the characteristic equation (1.19). Because of the assumption that the leading coefficients around each singularity of $L$ have distinct eigenvalues, given by a meromorphic function $\lambda$ on $\mathcal{C}$ satisfying the characteristic equation (1.19), near each of the points $\left\{p_{\nu}^{(a)}, \infty^{(a)} \in \mathcal{C}\right\}_{a=1, \ldots, r,}$, over the points $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$, this has $r$ distinct local Laurent series' solutions $\left\{\lambda_{a}(z)\right\}_{a=1, \ldots, r}{ }^{\ddagger}$

In the autonomous case, the meromorphic differential $\lambda d z$ on the spectral curve $\mathcal{C}$ plays a fundamental rôle in explicitly integrating the Hamiltonian Lax equations corresponding to elements $H \in \mathcal{I}^{(\mathbf{c}, \mathbf{d})}$ of the spectral ring generated by the coefficients of the eigenvector equation (1.19) in terms of abelian functions [1, 2, 27]. A complete set of generators of the center of the Lie-Poisson algebra defined above (i.e. the Casimir elements) is given by the loci $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N}$ of the finite poles, together with the following further spectral invariants

$$
\begin{align*}
t_{j a}^{\nu} & :=-\underset{z=c_{\nu}}{\operatorname{res}}\left(z-c_{\nu}\right)^{j} \lambda_{a}(z) d z  \tag{1.41a}\\
\nu & =1, \ldots, N, \quad j=0, \ldots d_{\nu}, \quad a=1, \ldots, r \\
t_{j a}^{\infty} & :=-\underset{z=\infty}{\operatorname{res}} z^{-j} \lambda_{a}(z) d z  \tag{1.41b}\\
j & =1, \ldots d_{\infty}, \quad a=1, \ldots, r
\end{align*}
$$

which are just the coefficients of the principal parts of the Laurent expansion of $\lambda_{a}(z) d z$ at the pole locations. As shown in Section 3, these may be identified with the Birkhoff invariants (Definition 3.1) defining the formal local asymptotics of a fundamental system of solutions of (1.40) in a neighbourhood

[^3]of any finite irregular singular point $z=c_{\nu},\left(d_{\nu}>0\right)$ and at $z=\infty$. The parameters
\[

$$
\begin{equation*}
t_{0 a}^{\nu}:=-\operatorname{res}_{z=c_{\nu}} \lambda_{a}(z) d z, \quad \nu=1, \ldots, N, \quad a=1, \ldots, r \tag{1.42}
\end{equation*}
$$

\]

are known as the exponents of formal monodromy $[45,46])$.
Dual to these are the following non-Casimir spectral invariants

$$
\begin{align*}
H_{t_{j a}^{\nu}} & :=-\underset{z=c_{\nu}}{\operatorname{res}} \frac{1}{j\left(z-c_{\nu}\right)^{j}} \lambda_{a}(z) d z  \tag{1.43a}\\
\nu & =1, \ldots, N, \quad j=1, \ldots d_{\nu}, \quad a=1, \ldots, r \\
H_{t_{j a}^{\infty}} & :=-\operatorname{res}_{z=\infty} \frac{z^{j}}{j} \lambda_{a}(z) d z  \tag{1.43b}\\
& j=1, \ldots d_{\infty}, \quad a=1, \ldots r
\end{align*}
$$

which are the "mirror image" coefficients of the principal parts of the local Laurent expansion of $\lambda_{a}(z) d z$ near $z=c_{\nu}$, consisting of the coefficients of the first $d_{\nu}$ positive powers of $z-c_{\nu}$ in the analytic part, and the inverse powers at $z=\infty$, and

$$
\begin{equation*}
H_{c_{\nu}}:=\frac{1}{2} \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(L^{2}(z)\right) d z \tag{1.44}
\end{equation*}
$$

(Note that there are no non-Casimir spectral invariants dual to the exponents of formal monodromy $\left\{t_{0 a}^{\nu}\right\}_{\nu=1, \ldots, N, a=1, \ldots r .}$. In addition to these, we also have the exponents of formal monodromy at $z=\infty$ which we denote as

$$
\begin{equation*}
H_{a}^{\infty}:=t_{0 a}^{\infty}=-\underset{z=\infty}{\operatorname{res}} \lambda_{a}(z) d z, \quad a=1, \ldots, r \tag{1.45}
\end{equation*}
$$

Unlike those $\left\{t_{0 a}^{\nu}\right\}_{\nu=1, \ldots, N, a=1, \ldots, r}$ at the finite poles $z=c_{\nu}$, these are not Casimir elements, but dynamical spectral invariant Hamiltonians generating nontrivial Hamiltonian flows, whose Hamiltonian vector fields are given by the commutators

$$
\begin{equation*}
\mathbf{X}_{H_{a}^{\infty}} L(z)=\left\{L(z), H_{a}^{\infty}\right\}=\left[E_{a a}, L\right] \tag{1.46}
\end{equation*}
$$

The flows they generate, consisting of conjugation by invertible, $z$ independent diagonal matrices, are therefore both isospectral and isomonodromic, and should be understood as symmetries of the rational isomonodromic deformation systems of [45, 46].

It is important to note that, although $\left\{t_{j a}^{\nu}, t_{j a}^{\infty}, H_{t_{j a}^{\nu}}, H_{t_{j a}^{\infty}}, H_{c_{\nu}}, H_{a}^{\infty}\right\}$ are defined here by evaluation of residues of moments of $\left\{\lambda_{a}(z) d z\right\}_{a=1, \ldots, r}$ at the poles $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$, they may also be computed explicitly as polynomial expressions in the entries of the matrix terms $\left\{L_{j}^{\nu}\right\}$ appearing in eq. (1.1), and (see Corollary 4.4) depend rationally on the differences of the eigenvalues of the leading coefficients $L_{d_{\nu}+1}^{\nu}$.

To each of the Hamiltonians $\left\{H_{t_{j a}^{\nu}}, H_{c_{\nu}}\right\}_{\nu=1, \ldots, N, \infty, j=1, \ldots d_{\nu}, a=1, \ldots, r}$ defined above, there corresponds a Hamiltonian vector field $\left\{\mathbf{X}_{H_{t_{j a}}}, \mathbf{X}_{H_{c_{\nu}}}\right\}$ defined by the equations

$$
\begin{align*}
\mathbf{X}_{H_{t_{j a}^{\nu}}} L(z) & =\left\{L(z), H_{t_{j a}^{\nu}}\right\}=\left[U_{j a}^{\nu}, L(z)\right]  \tag{1.47a}\\
\mathbf{X}_{H_{c^{\nu}}} L(z) & =\left\{L(z), H_{c_{\nu}}\right\}=\left[V^{\nu}, L(z)\right] \tag{1.47b}
\end{align*}
$$

where the Poisson bracket (1.17) is applied to each of the entries of $L(z)$. Recall [45, 46] that the generalized isomonodromic deformations are determined by a system of PDEs of the form

$$
\begin{align*}
& \frac{\partial \Psi(z)}{\partial t_{j a}^{\nu}}=U_{j a}^{\nu}(z) \Psi(z), \quad \nu=1, \ldots, N, \infty, \quad j=1, \ldots, d_{\nu}, \quad a=1, \ldots, r  \tag{1.48a}\\
& \frac{\partial \Psi(z)}{\partial c_{\nu}}=V^{\nu}(z) \Psi(z), \quad \nu=1, \ldots, N \tag{1.48b}
\end{align*}
$$

where the matrices $\left\{U_{j a}^{\nu}(z), V^{\nu}(z)\right\}$ (whose definition is given in Section 3.2, eqs. (3.10), (3.11)) depend rationally on $z$ and are uniquely determined in terms of the entries of $L(z)$ (see Theorem 4.6). These equations generate deformations such that the extended monodromy data of the ODE (1.40) (monodromy matrices, connection matrices, Stokes' matrices) are constants in the deformation parameters. The compatibility of (1.48a), (1.48b) with (1.40) is equivalent to the set of equations

$$
\begin{align*}
\frac{\partial L}{\partial t_{j a}^{\nu}} & =\frac{\partial U_{j a}^{\nu}}{\partial z}+\left[U_{j a}^{\nu}, L\right]  \tag{1.49a}\\
\frac{\partial L}{\partial c_{\nu}} & =\frac{\partial V^{\nu}}{\partial z}+\left[V^{\nu}, L\right] \tag{1.49b}
\end{align*}
$$

known as zero curvature equations. (See Section 3.2.)
In addition to these, we also have the compatible set of (autonomous) isospectral deformation equations

$$
\begin{equation*}
\frac{\partial L}{\partial s_{a}}=\left[E_{a a}, L\right], \quad a=1, \ldots, r \tag{1.50}
\end{equation*}
$$

generated by the exponents of formal monodromy $\left\{H_{a}^{\infty}\right\}_{a=1, \ldots, r}$ at $\infty$ defined in (1.45), giving the $r$ dimensional abelian symmetry group consisting of conjugation by invertible diagonal matrices

$$
\begin{align*}
\mathcal{D}_{s}: L & \rightarrow \tilde{L}:=\mathcal{D}_{s} L \mathcal{D}_{s}^{-1}  \tag{1.51}\\
\mathcal{D}_{s} & :=\operatorname{Diag}\left(e^{s_{1}}, \ldots, e^{s_{r}}\right) \tag{1.52}
\end{align*}
$$

Corresponding to these flows, we may enhance the fundamental system of parallel transport equations (1.40), (1.48a), (1.48b) by defining

$$
\begin{equation*}
\tilde{\Psi}(z):=\mathcal{D}_{s} \tilde{\Psi}(z) \tag{1.53}
\end{equation*}
$$

which satisfies eqs. $(1.48 \mathrm{a}),(1.48 \mathrm{~b})$ and $(1.40)$ with $L$ replaced by $\tilde{L}$, as well as the further linear equations

$$
\begin{equation*}
\frac{\partial \tilde{\Psi}}{\partial s_{a}}=E_{a a} \tilde{\Psi}, \quad a=1, \ldots, r \tag{1.54}
\end{equation*}
$$

implying that the $s_{a}$ flows are both isospectral and isomonodromic. The parameters $\left\{s_{a}\right\}_{a=1, \ldots, r}$, however, are not functions on the phase space, but genuine independent flow variables.

Denote the set of isomonodromic deformation parameters

$$
\begin{equation*}
\mathbf{T}:=\left\{t_{j a}^{\nu}, c_{\nu}\right\}_{\nu=1, \ldots, N, \infty, j=1, \ldots, d_{\nu}, a=1, \ldots, r} \tag{1.55}
\end{equation*}
$$

As shown in Theorem 4.6, these consist entirely of Casimir elements for the Lie-Poisson bracket (1.17).

Remark 1.2. The only further Casimir functions needed to complete the center of the Poisson structure (restricted to rational Lax matrices (1.1)) are the exponents of formal monodromy $\left\{t_{0 a}^{\nu}\right\}_{1 \leq \nu \leq N, 1 \leq a \leq r}$ at the finite poles of the rational connection $L(z) d z$ defined in (1.42).

It follows from the $R$-matrix theory that the Hamiltonian vector fields generated by the spectral invariants (1.43a), (1.43b) and (1.44) can be interpreted as the commutator terms (1.20) in eq. (1.49). We will reprove this key fact explicitly for our Hamiltonians in Section 4.4, Theorem 4.6. Summarizing, we have the following theorem.

Theorem 1.1. The Hamiltonian vector fields corresponding to the Hamiltonians $H_{t}, t \in \mathbf{T}$ defined in (1.43a), (1.43b), (1.44), and $H_{a}^{\infty}$ defined in (1.46) are given by the following commutators

$$
\begin{align*}
\mathbf{X}_{H_{t_{j a}}} L(z) & =\left\{L(z), H_{t_{j a}^{\nu}}\right\}=\left[U_{j a}^{\nu}(z), L(z)\right]  \tag{1.56a}\\
\mathbf{X}_{H_{c_{\nu}}} L(z) & =\left\{L(z), H_{c_{\nu}}\right\}=\left[V^{\nu}(z), L(z)\right]  \tag{1.56b}\\
\mathbf{X}_{H_{a}^{\infty}} L(z) & =\left\{L(z), H_{a}^{\infty}\right\}=\left[E_{a a}, L\right], \tag{1.56c}
\end{align*}
$$

where the matrices $U_{j a}^{\nu}, V^{\nu}, E_{a a}$ are expressible as

$$
\begin{align*}
& U_{j a}^{\nu}(z)=-\left(d H_{t_{j a}^{\nu}}\right)_{-}, \quad V^{\nu}=-\left(d H_{c_{\nu}}\right)_{-}, \quad \nu=1, \ldots, N, j=1, \cdots d_{\nu}, a=1, \ldots, r  \tag{1.57a}\\
& U_{j a}^{\infty}(z)=\left(d H_{t_{j a}^{\infty}}\right)_{+}, \quad E_{a a}=\left(d H_{a}^{\infty}\right)_{+}, \quad j=1, \ldots, d_{\infty}, a=1, \ldots, r \tag{1.57b}
\end{align*}
$$

We also show (Theorem 3.3) that the isomonodromic $\tau$-function defined in [45, 46] (and eqs. (3.20), (3.21)) admits the following representation in terms of spectral invariants

$$
\begin{equation*}
d \ln \tau_{I M}=\sum_{\nu=1}^{N}\left(H_{c_{\nu}} d c_{\nu}+\sum_{j=1}^{d_{\nu}} \sum_{a=1}^{r} H_{t_{j a}^{\nu}} d t_{j a}^{\nu}\right)+\sum_{j=1}^{d_{\infty}} \sum_{a=1}^{r} H_{t_{j a}^{\infty}} d t_{j a}^{\infty} \tag{1.58}
\end{equation*}
$$

with the Hamiltonians defined as spectral invariants by formulae (1.43a), (1.43b), (1.44).
To each of the isomonodromic deformation parameters $t \in \mathbf{T}$ in (1.55) we associate a vector field $\nabla_{t}$ in $\Gamma\left(T \mathcal{L}_{r, \mathbf{d}}\right)$ determined by the formulae

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} L(z):=\frac{\partial U_{j a}^{\nu}(z)}{\partial z}, \quad \nabla_{t_{j a}^{\infty}} L(z):=\frac{\partial U_{j a}^{\infty}(z)}{\partial z}, \quad \nabla_{c_{\nu}} L(z):=\frac{\partial V^{\nu}(z)}{\partial z} \tag{1.59}
\end{equation*}
$$

These should be understood as defining the action of $\nabla_{t}$ on each of the coefficients of $L$ (viewed as linear coordinates on $\mathcal{L}_{r, \mathbf{d}}$ ) as well as the position of the poles, extended to arbitrary differentiable functions of $L$ by requiring it to be a derivation.

With these definitions, we can rewrite the isomonodromic equations (1.49) as

$$
\begin{equation*}
\partial_{t} L=\nabla_{t} L+\mathbf{X}_{H_{t}} L, \quad t \in \mathbf{T} \tag{1.60}
\end{equation*}
$$

In Theorems 4.1, 4.2 and Proposition 5.1, the vector fields $\left\{\nabla_{t}\right\}_{t \in \mathbf{T}}$ will be shown to act as coordinate curve directional derivatives, spanning an integrable distribution $\mathfrak{T} \subset T \mathcal{L}_{r, \mathbf{d}}$ within the tangent bundle of $\mathcal{L}_{r, \mathbf{d}}$, generating a maximal regular foliation transversal to the symplectic leaves. In Theorem 4.5 they
are shown to preserve the Poisson brackets, and we interpret them as "explicit derivatives" with respect to the various isomonodromic times $t=t_{j a}^{\nu}$ or $c_{\nu}$. This is clear, in particular, for the loci $\left(c_{1}, \ldots, c_{N}\right)$ of the poles, where it turns out that $V^{\nu}$ is precisely the (negative of the) singular part of $L(z)$ at $z=c_{\nu}$ and hence

$$
\begin{equation*}
\frac{\partial V^{\nu}(z)}{\partial z}=\frac{\partial^{0} L(z)}{\partial c_{\nu}^{0}} \tag{1.61}
\end{equation*}
$$

where the derivation $\nabla_{c_{\nu}}=\frac{\partial^{0}}{\partial c_{\nu}^{0}}$ is the same as "explicit derivative" in the obvious sense of differentiating the rational, matrix valued function $L(z)$ with respect to its pole locations while keeping all other parameters fixed.

For the higher Birkhoff invariants $\left\{t_{j a}^{\nu}\right\}_{\nu=1, \ldots, N, j=1, \ldots, d_{\nu}, a=1, \ldots, r}$, it is not at all obvious in which sense eq. (1.59) may be thought of as defining "explicit derivatives". The key point is that, in order to interpret the $\nabla_{t}$ 's as such, we must prove both that they mutually commute and that, when applied to the invariants $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$, they really act as directional derivatives along flow lines in which all the remaining transversal coordinates are kept fixed. These results are included in the following theorem, which combines those of Theorems 4.1, 4.2, and 4.5 of Section 4.1 and Proposition 5.1 of Section 5. Combined with Theorem 1.1, this shows how the Hamiltonian vectors fields of eqs. (1.56), when added to the "explicit" derivative vector fields, produce the zero curvature equations (1.49) that guarantee consistency of the equations (1.40), (1.48a), (1.48b), and ensure the invariance of the (generalized) monodromy of the system (1.40) under changes in the parameters $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$.

Theorem 1.2. 1. If $s, t$ denote any two isomonodromic times $s, t \in \mathbf{T}$ then

$$
\begin{equation*}
\nabla_{s} t=\delta_{s t} . \tag{1.62}
\end{equation*}
$$

2. The vector fields $\left\{\nabla_{t}\right\}_{t \in \mathbf{T}}$ in (1.59) commute amongst themselves and span an integrable distribution of constant rank in the space of rational Lax matrices of the form (1.1).
3. The $R$-matrix Poisson structure is invariant with respect to the local flows generated by the $\nabla_{t}$ 's. That is,

$$
\begin{equation*}
\nabla_{t}\{f, g\}=\left\{\nabla_{t} f, g\right\}+\left\{f, \nabla_{t} g\right\} \quad \forall t \in \mathbf{T}, f, g \in C^{\infty}\left(\mathcal{L}_{r, \mathbf{d}}\right) \tag{1.63}
\end{equation*}
$$

Remark 1.3. A generalization of the "explicit" derivative vector fields $\nabla_{t}$ for isomonodromic systems over an arbitrary Riemann surface is defined in [38], without further developing their properties.

## 2 An illustrative example: Hamiltonian theory of $P_{I I}$

Before proceeding to the general case, we recall the example of the second Painlevé transcendent $P_{I I}$ as the simplest illustration of an isomonodromic deformation system with $r=2$ and a polynomial Lax matrix of degree 2 .

The $P_{I I}$ equation is:

$$
\begin{equation*}
u^{\prime \prime}=2 u^{3}+t u+\alpha \tag{2.1}
\end{equation*}
$$

with arbitrary constant $\alpha$. To interpret (2.1) as an isomonodromic deformation equation [46, 34], we introduce a pair of $2 \times 2$ matrices $(L(z), U(z))$, which are second and first degree polynomials in $z$, parametrized as

$$
\begin{align*}
L(z) & :=z^{2}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)+z\left(\begin{array}{cc}
0 & -2 y_{1} \\
x_{2} & 0
\end{array}\right)+\left(\begin{array}{cc}
x_{2} y_{1}+\frac{t}{2} & -2 y_{2} \\
x_{1} & -x_{2} y_{1}-\frac{t}{2}
\end{array}\right)  \tag{2.2}\\
U(z) & :=\frac{z}{2}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)+\frac{1}{2}\left(\begin{array}{cc}
0 & -2 y_{1} \\
x_{2} & 0
\end{array}\right) . \tag{2.3}
\end{align*}
$$

The overdetermined pair of matrix equations

$$
\begin{align*}
& \frac{\partial \Psi(z)}{\partial z}=L(z) \Psi(z)  \tag{2.4}\\
& \frac{\partial \Psi(z)}{\partial t}=U(z) \Psi(z) \tag{2.5}
\end{align*}
$$

is compatible if and only if $L(z)$ satisfies the evolution equation

$$
\frac{\partial L}{\partial t}=[U(z), L(z)]+\frac{1}{2}\left(\begin{array}{cc}
1 & 0  \tag{2.6}\\
0 & -1
\end{array}\right)=[U(z), L(z)]+\frac{\partial U(z)}{\partial z}
$$

which guarantees the invariance of the (generalized) monodromy of the meromorphic covariant derivative operator

$$
\begin{equation*}
\frac{\partial}{\partial z}-L(z) \tag{2.7}
\end{equation*}
$$

under the $t$-deformations generated by (2.5).
Eq. (2.6) is of Hamiltonian type [46, 54, 55] with respect to the canonical 1-form

$$
\begin{equation*}
\theta:=y_{1} d x_{1}+y_{2} d x_{2} \tag{2.8}
\end{equation*}
$$

with $t$ viewed as a Casimir element. The map

$$
\begin{equation*}
\left(t, x_{1}, x_{2}, y_{1}, y_{2}\right) \rightarrow L(z) \tag{2.9}
\end{equation*}
$$

is then a Poisson map with respect to the Poisson bracket (1.17) on the dual $L^{*} \mathfrak{g l}{ }_{R}(2)$ of the loop algebra $L \mathfrak{g l} l_{R}(2)$ defined by the rational split $R$-matrix. The nonautonomous Hamiltonian $H_{I I}$ is the spectral invariant

$$
\begin{equation*}
H_{I I}=\frac{1}{4} \underset{z=0}{\operatorname{res}} z^{-1} \operatorname{tr}\left(L^{2}(z)\right)-\frac{t^{2}}{8}=\frac{1}{2}\left(x_{2}^{2} y_{1}^{2}+t x_{2} y_{1}-2 x_{1} y_{2}\right) \tag{2.10}
\end{equation*}
$$

and we have

$$
\begin{equation*}
-\left(d H_{I I}\right)_{-}=U(z) \tag{2.11}
\end{equation*}
$$

as required by the $R$-matrix theory. Taking into account the explicit dependence of $L(z)$ on the parameter $t$, viewed as a function on the phase space, note that $t$ is in fact a spectral invariant Casimir function, which can be computed as

$$
\begin{equation*}
t=\frac{1}{2} \underset{z=0}{\operatorname{res}} z^{-3} \operatorname{tr}\left(L^{2}(z)\right) d z=2 t_{11}^{\infty} \tag{2.12}
\end{equation*}
$$

and hence identified with the first Birkhoff invariant. That the zero curvature equations (2.6) are of the deformed Hamiltonian form (1.60) follows from the fact that the explicit $t$ derivative of $L(z)$ is

$$
\frac{\partial^{0} L}{\partial t^{0}}=\nabla_{t}(L)=\frac{1}{2}\left(\begin{array}{cc}
1 & 0  \tag{2.13}\\
0 & -1
\end{array}\right)=\frac{\partial U}{\partial z}
$$

To check that formulae (2.12), (2.10) coincide with the definitions (1.41b), (1.43b) of $t$ as twice the Birkhoff invariant $t_{11}^{\infty}$ and $H_{I I}$ as the "dual" Hamiltonian $H_{t_{11}^{\infty}}$, note that the meromorphic differential $\lambda(z) d z$ is defined, on its two branches, by the formulae

$$
\begin{equation*}
\lambda= \pm \sqrt{-\operatorname{det}(L)}= \pm\left(z^{2}+\frac{t}{2}-\frac{x_{1} y_{1}+x_{2} y_{2}}{z}+\frac{H_{I I}}{z^{2}}+\ldots\right) \tag{2.14}
\end{equation*}
$$

Choosing new canonical coordinates

$$
\begin{equation*}
u:=\frac{x_{1}}{x_{2}}, \quad v:=x_{2} y_{1}, \quad w:=\ln x_{2}, \quad a:=x_{1} y_{1}+x_{2} y_{2} \tag{2.15}
\end{equation*}
$$

the canonical 1-form (2.8) becomes

$$
\begin{equation*}
\theta=v d u+a d w \tag{2.16}
\end{equation*}
$$

and the Hamiltonian is

$$
\begin{equation*}
H_{I I}=\frac{1}{2} v^{2}+\frac{1}{2}\left(t+2 u^{2}\right) v-a u \tag{2.17}
\end{equation*}
$$

Note that $a$ is an autonomous spectral invariant of the Lax matrix, and hence a conserved quantity. It can be written equivalently as

$$
\begin{equation*}
a=-\frac{1}{4} \operatorname{res}_{z=0} z^{-2} \operatorname{tr}(L(z))^{2}=H_{1}^{\infty}=-H_{2}^{\infty} \tag{2.18}
\end{equation*}
$$

which is equal to the exponent of formal monodromy $t_{01}^{\infty}$ at $\infty$. The Hamiltonian flow it generates is the group of scaling symmetries

$$
\begin{gather*}
f_{s}:\left(x_{1}, y_{1}, x_{2}, y_{2}, t\right) \rightarrow\left\{e^{s} x_{1}, e^{-s} y_{1}, e^{s} x_{2}, e^{-s} y_{2}, t\right\}  \tag{2.19}\\
f_{s}: L(z) \rightarrow e^{s \sigma_{3}} L(z) e^{-s \sigma_{3}} \tag{2.20}
\end{gather*}
$$

The corresponding canonically conjugate position variable $w$ is thus an ignorable coordinate, and Hamilton's equations reduce to the two-dimensional system

$$
\begin{equation*}
\frac{d u}{d t}=v+u^{2}+\frac{t}{2}, \quad \frac{d v}{d t}=-2 u v+a \tag{2.21}
\end{equation*}
$$

Eliminating the $v$ variable to obtain an equivalent second degree equation for $u$ gives the $P_{I I}$ equation (2.1) with $\alpha=a-\frac{1}{2}$. The associated isomonodromic $\tau$-function is related to the Hamiltonian $H_{I I}$ by

$$
\begin{equation*}
\mathrm{d}(\ln \tau)=H_{I I} d t \tag{2.22}
\end{equation*}
$$

## 3 Isomonodromic deformations: general rational Lax matrices

### 3.1 Rational Lax matrices

We start our analysis of the general case by defining notation and reviewing the underlying notions. Consider the space of matrices of the form (1.1):

$$
\begin{align*}
\mathcal{L}_{r, \mathbf{d}}:=\{L(z)=- & \sum_{j=0}^{d_{\infty}-1} L_{j+2}^{\infty} z^{j}+\sum_{\nu=1}^{N} \sum_{j=1}^{d_{\nu}+1} \frac{L_{j}^{\nu}}{\left(z-c_{\nu}\right)^{j}}, \\
& \left.L_{d_{\infty}+1}^{\infty} \in \mathfrak{h}_{r e g} \subset \mathfrak{g l}(r), L_{d_{\nu}+1}^{\nu} \in \mathfrak{g}_{r e g} \subset \mathfrak{g l}(r), \quad c_{\nu} \neq c_{\mu}, \quad \nu \neq \mu\right\} \tag{3.1}
\end{align*}
$$

Here $\mathfrak{h}_{\text {reg }}$ denotes the set of ad-regular diagonal matrices (i.e. with distinct eigenvalues) in $\mathfrak{g l}(r)$, while $\mathfrak{g}_{\text {reg }}$ is the set of all $\mathfrak{g l}(r)$ matrices with distinct eigenvalues. The rationale behind the indexing of the matrices $\left\{L_{j}^{\nu}, L_{j+2}^{\infty}\right\}$ is that the subscript should coincide with the order of the pole of the matrix-valued differential form $L(z) d z$ near the corresponding pole. In particular, if we define local coordinates near its poles by

$$
\zeta_{\nu}=\left\{\begin{align*}
\left(z-c_{\nu}\right), & & \nu & =1, \ldots, N  \tag{3.2}\\
\frac{1}{z}, & & \nu & =\infty
\end{align*}\right.
$$

the singular part of $L(z) d z$ near each pole can be written in a uniform manner as

$$
\begin{equation*}
(L(z))_{\text {sing }} d z=\sum_{j=1}^{d_{\nu}+1} \frac{L_{j}^{\nu}}{\zeta_{\nu}^{j}} d \zeta_{\nu}, \quad \nu=1, \ldots, N, \infty \tag{3.3}
\end{equation*}
$$

We now recall the classical description $[4,45,46,68,66]$ of formal solutions near an isolated singular point of a linear first order system of ordinary differential equations of the form (1.40).

Proposition 3.1. Consider the linear system of first order ODE's in the complex plane

$$
\begin{equation*}
\frac{d \Psi(z)}{d z}=L(z) \Psi(z), \quad L(z) \in \mathcal{L}_{r, \mathbf{d}}, \quad \Psi(z) \in \mathfrak{G} \mathfrak{l}(r) \tag{3.4}
\end{equation*}
$$

with $\mathcal{L}_{r, \mathbf{d}}$ defined in (3.1), and $L_{d_{\infty}+1}^{\infty}$ chosen to be diagonal. In terms of the local parameters $\zeta_{\nu}$ defined in (3.2), there exist local formal series solutions of the form

$$
\begin{equation*}
\Psi_{\text {form }}^{\nu}(z)=Y^{\nu}\left(\zeta_{\nu}\right) \mathrm{e}^{T^{\nu}\left(\zeta_{\nu}\right)}, \quad Y^{\nu}\left(\zeta_{\nu}\right):=G^{\nu}\left(\mathbf{I}+\sum_{j \geq 1} Y_{j}^{\nu} \zeta_{\nu}{ }^{j}\right) \tag{3.5}
\end{equation*}
$$

in a punctured neighbourhood of each of the singular points $\left\{z=c_{\nu}\right\}$ (including $\infty$ ), where $T^{\nu}\left(\zeta_{\nu}\right) \in \mathfrak{h}_{\text {reg }}$ is a diagonal matrix of the form

$$
\begin{equation*}
T^{\nu}\left(\zeta_{\nu}\right)=\sum_{j=1}^{d_{\nu}} \frac{T_{j}^{\nu}}{j \zeta_{\nu}{ }^{j}}+T_{0}^{\nu} \ln \zeta_{\nu}, \quad T_{d_{\nu}}^{\nu}=-\left(G^{\nu}\right)^{-1} L_{d_{\nu}+1}^{\nu} G^{\nu} \tag{3.6}
\end{equation*}
$$

for $\nu=1, \ldots, N, \infty$. The columns of the invertible matrices $G^{\nu} \in G L(r, \mathbb{C})$ are the independent eigenvectors of $L_{d_{\nu}+1}^{\nu}$ and $G^{\infty}=\mathbf{I}$.

We use the following notation for the diagonal values

$$
\begin{equation*}
T_{j}^{\nu}=\operatorname{diag}\left(t_{j 1}^{\nu}, \ldots, t_{j r}^{\nu}\right), \quad j=0, \ldots, d_{\nu} \tag{3.7}
\end{equation*}
$$

so

$$
\begin{equation*}
T^{\nu}\left(\zeta_{\nu}\right)=\sum_{j=1}^{d_{\nu}} \sum_{a=1}^{r} t_{j a}^{\nu} E_{a a} \frac{1}{j \zeta_{\nu}^{j}}+\sum_{a=1}^{r} t_{0 a}^{\nu} E_{a a} \ln \zeta_{\nu} \tag{3.8}
\end{equation*}
$$

where $E_{a b}$ is the elementary matrix whose only nonzero entry is 1 in the $a b$ position and

$$
\begin{equation*}
t_{j a}^{\nu} \neq t_{j b}^{\nu} \quad \text { for } a \neq b, \quad j=1, \ldots, d_{\nu}, \quad \nu=1, \ldots, N, \infty \tag{3.9}
\end{equation*}
$$

Definition 3.1. The entries $\left(t_{j 1}^{\nu}, \ldots, t_{j r}^{\nu}\right)$ of the diagonal matrices $\left\{T_{j}^{\nu}\right\}_{j=0, \ldots, d_{\nu}}$ are called the Birkhoff invariants. In particular, the entries $\left(t_{01}^{\nu}, \ldots, t_{0 r}^{\nu}\right)$ in the matrices $T_{0}^{\nu}$ are called the exponents of formal monodromy [4, 45] and the entries $\left(t_{j a}^{\nu}, \ldots, t_{j r}^{\nu}\right)$ for $1 \leq j \leq d_{\nu}$ the higher Birkhoff invariants. The integers $\mathbf{d}=\left(d_{1}, \ldots d_{N}, d_{\infty}\right)$ are the Poincaré ranks of the various singular points $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$.

### 3.2 Isomonodromic deformations and zero curvature equations

The higher Birkhoff invariants $\left\{t_{j a}^{\nu}\right\}_{\nu=1, \ldots, N, \infty, j=1, \ldots d_{\nu}, a=1, \ldots r}$, together with the loci $\left\{c_{\nu}\right\}_{\nu=1, \ldots, N}$ of the finite poles will serve as isomonodromic deformation parameters. To each, there corresponds an infinitesimal deformation equation which we now recall, following [45, 46]. For $\nu=1 \ldots, N, \infty$ and $j=1, \ldots, d_{\nu}$, define the following matrices in terms of the Birkhoff invariants in $T^{\nu}\left(\zeta_{\nu}\right)$ and the formal series $Y^{\nu}\left(\zeta_{\nu}\right)$ in (3.5) near $z=\zeta_{\nu}$.

$$
\begin{align*}
U_{j a}^{\nu}(z ; L) & :=\left(Y^{\nu}\left(\zeta_{\nu}\right) \frac{\partial T^{\nu}\left(\zeta_{\nu}\right)}{\partial t_{j a}^{\nu}}\left(Y^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\text {sing }}=\left(Y^{\nu}\left(\zeta_{\nu}\right) \frac{E_{a a}}{j \zeta_{\nu}^{j}}\left(Y^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\text {sing }}  \tag{3.10}\\
V^{\nu}(z ; L) & :=\left(Y^{\nu}\left(\zeta_{\nu}\right) \frac{\partial T^{\nu}\left(\zeta_{\nu}\right)}{\partial c_{\nu}}\left(Y^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\text {sing }}=-\left(Y^{\nu}\left(\zeta_{\nu}\right) \frac{d T^{\nu}\left(\zeta_{\nu}\right)}{d z}\left(Y^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\operatorname{sing}}=-\sum_{j=1}^{d_{\nu}+1} \frac{L_{j}^{\nu}}{\left(z-c_{\nu}\right)^{j}} . \tag{3.11}
\end{align*}
$$

where $(\cdot)_{\operatorname{sing}}$ denotes the principal part ${ }^{\S}$ of the Laurent series $(\cdot)$ near $z=c_{\nu}$. For a Laurent series

$$
\begin{equation*}
f(z)=\sum_{n \in \mathbb{Z}} a_{n}(z-c)^{n} \tag{3.12}
\end{equation*}
$$

centered at $c \in \mathbb{C}^{1}$ the principal part may be expressed as

$$
\begin{equation*}
(f(z))_{\text {sing }}:=\sum_{n \geq 1} \frac{a_{-n}}{(z-c)^{n}}=\operatorname{res}_{w=c} \frac{f(w) d w}{z-w} \tag{3.13}
\end{equation*}
$$

[^4]At $c_{\infty}=\infty$, the principal part is simply the polynomial part of the expression (including the constant term). The main result of [45, 46] is that if the following system of equations is satisfied,

$$
\begin{align*}
& \frac{d \Psi(z)}{d z}=L(z) \Psi(z),  \tag{3.14a}\\
& \frac{\partial \Psi(z)}{\partial t_{j a}^{\nu}}=U_{j a}^{\nu}(z) \Psi(z), \quad j=1, \ldots, d_{\nu}  \tag{3.14b}\\
& \frac{\partial \Psi(z)}{\partial c_{\nu}}=V^{\nu}(z) \Psi(z), \quad \nu=1, \ldots, N \tag{3.14c}
\end{align*}
$$

(where, for brevity, we write $U_{j a}^{\nu}(z), V^{\nu}(z)$ for $U_{j a}^{\nu}(z ; L), V^{\nu}(z, ; L)$ ), the generalized monodromy (including the values of the Stokes matrices defined in a neighbourhood of each irregular singular point) is independent of the deformation parameters $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$.

This means that the matrices $\left\{L, U_{j a}^{\nu}, V^{\nu}\right\}$, restricted to the isomonodromic solution manifold, satisfy the zero curvature equations (1.49). Equivalently, defining the connection form

$$
\begin{equation*}
\Omega(z ; \boldsymbol{t}, \boldsymbol{c}):=-L(z) d z-\sum_{\nu=1}^{N, \infty}\left(\sum_{j=1}^{d_{\nu}} \sum_{a=1}^{r} U_{j a}^{\nu} d t_{j a}^{\nu}+V^{\nu} d c_{\nu}\right) \tag{3.15}
\end{equation*}
$$

the corresponding curvature vanishes

$$
\begin{equation*}
\delta \Omega+[\Omega, \Omega]=0 \tag{3.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta:=d z \frac{\partial}{\partial z}+\sum_{\nu=1}^{N} d c_{\nu} \frac{\partial}{\partial c_{\nu}}+\sum_{\nu=1}^{N} \sum_{j=1}^{d_{\nu}} \sum_{a=1}^{r} d t_{j a}^{\nu} \frac{\partial}{\partial t_{j a}^{\nu}}+\sum_{j=1}^{d_{\infty}} \sum_{a=1}^{r} d t_{j a}^{\infty} \frac{\partial}{\partial t_{j a}^{\infty}} \tag{3.17}
\end{equation*}
$$

is the total differential operator acting on scalar functions of $\left\{z, t_{j a}^{\nu}, c_{\nu}\right\}$ extended, as usual, to the exterior differential on the space of differential forms.

We denote differentials with respect to the parameters at each pole location as

$$
\begin{equation*}
\mathrm{d}_{\nu}:=d c_{\nu} \frac{\partial}{\partial c_{\nu}}+\sum_{j=1}^{d_{\nu}} \sum_{a=1}^{r} d t_{j a}^{\nu} \frac{\partial}{\partial t_{j a}^{\nu}}, \quad \mathrm{d}_{\infty}:=\sum_{j=1}^{d_{\infty}} \sum_{a=1}^{r} d t_{j a}^{\infty} \frac{\partial}{\partial t_{j a}^{\infty}} \tag{3.18}
\end{equation*}
$$

so

$$
\begin{equation*}
\mathrm{d}=\sum_{\nu=1}^{N} \mathrm{~d}_{\nu}+\mathrm{d}_{\infty} \tag{3.19}
\end{equation*}
$$

is the total differential on the space of deformation parameters. It is proved in $[45,46]$ that the differential

$$
\begin{equation*}
\omega_{I M}:=-\sum_{\nu=1, \ldots, N, \infty} \operatorname{res}_{z=c_{\nu}}\left(\operatorname{tr}\left(\left(Y^{\nu}\left(\zeta_{\nu}\right)\right)^{-1} \partial_{z} Y^{\nu}\left(\zeta_{\nu}\right) \mathrm{d}_{\nu} T^{\nu}\left(\zeta_{\nu}\right)\right) d z\right) \tag{3.20}
\end{equation*}
$$

is closed when restricted to the solution manifold of the isomonodromic equations and hence locally exact. The isomonodromic $\tau$-function $\tau_{I M}$ is locally defined, up to a parameter independent normalization, by

$$
\begin{equation*}
\mathrm{d} \ln \tau_{I M}:=\sum_{\nu=1, \ldots, N, \infty} \mathrm{~d}_{\nu} \ln \tau_{I M}=\omega_{I M} \tag{3.21}
\end{equation*}
$$

Note that this actually defines a section of a line bundle over the solution space of the isomonodromic equations, and that the "function" $\tau_{I M}$ is not, in general, single valued. Moreover, it is only defined up to a multiplicative factor that does not depend on the deformation parameters $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$.

### 3.3 Expression in terms of spectral invariants

Our first result consists of a representation of formula (3.21) determining the isomonodromic $\tau$-function in terms of spectral invariants. This was announced previously [30] in the form expressed in Theorems $3.2,3.3$ below, appeared in slightly different contexts in [5, 38] and more recently was re-derived in [71]. The proofs given here are complete and self-contained.

We define the spectral curve $\mathcal{C}$ as the compactification of the affine curve $\mathcal{C}_{0}$ cut out by the characteristic equation (1.19) of $L(z)$ obtained by adding $r$ points over each finite $z=c_{\nu}$, and over $z=\infty$. This allows us to view $\lambda d z$ as a globally defined meromorphic differential on the compactified spectral curve $\mathcal{C}$, with the locally defined Laurent series for the functions $\lambda_{1}(z), \ldots, \lambda_{N}(z)$ in punctured neighbourhoods of the points $\left\{z=c_{\mu}\right\}_{\nu=1, \ldots, N, \infty}$ representing the values of $\lambda$ near the $r$ distinct points of $\mathcal{C}$ projecting to the punctured neighborhood of $z=c_{\nu}$. By our requirement that the highest degree terms $L_{d_{\nu}+1}^{\nu}$ 's all have distinct eigenvalues, the eigenvalues $\left\{\lambda_{a}^{\nu}\left(\zeta_{\nu}\right)\right\}_{a=1, \ldots r}$ of $L(z)$, in a neighbourhood of each of the poles $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$, will be expressed by $r$ distinct Laurent series expansions, with poles of order $d_{\nu}+1$ for the poles at the finite points $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N}$ and $d_{\infty}-1$ for the pole at infinity. (As indicated above, the difference in indexing at finite poles and at $\infty$ is due to the fact that the objects of interest are the matrix valued 1-form $L(z) d z$ and the meromorphic differential $\lambda d z$, which have poles of order $d_{\nu}+1$ at the $z=c_{\nu}$ 's, including $c_{\infty}=\infty$, since $d z$ has a double pole at $\infty$.)

The first step is to prove formulae (1.41a), (1.41b), (1.43a), (1.43b) and (1.44); i.e., to express the Birkhoff invariants $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$ and their "dual" partners, the Hamiltonians $\left\{H_{t_{j a}^{\nu}}, H_{c_{\nu}}\right\}$, in terms of the spectral curve $\mathcal{C}$ defined in (1.19). In order to treat the local behaviour near the finite poles $\{z=$ $\left.c_{\nu}\right\}_{\nu=1, \ldots, N}$ and at $z=c_{\infty}$ uniformly in what follows, we will always consider the matrix-valued 1-form $L(z) d z$, with $\zeta_{\nu}$ chosen as one of the local coordinates (3.2). Note that

$$
\begin{equation*}
\frac{d z}{d \zeta_{\nu}}=1, \quad \nu=1, \ldots, N, \quad \frac{d z}{d \zeta_{\infty}}=-\frac{1}{\zeta_{\infty}^{2}} \tag{3.22}
\end{equation*}
$$

Choose a suitably normalized invertible matrix $P(z)$, whose columns are the linearly independent eigenvectors of $L(z)$, and which has local Taylor expansions near to each $z=c_{\nu}$

$$
\begin{equation*}
\left.P(z)\right|_{\text {near } z=c_{\nu}}=: P^{\nu}\left(\zeta_{\nu}\right)=G^{\nu}\left(\mathbf{I}+F_{1}^{\nu} \zeta_{\nu}+F_{2}^{\nu} \zeta_{\nu}^{2}+\ldots\right) \tag{3.23}
\end{equation*}
$$

where $G^{\nu}$ is the same invertible matrix as in (3.5). These satisfy

$$
\begin{align*}
L(z) P^{\nu}\left(\zeta_{\nu}\right) & =P^{\nu}\left(\zeta_{\nu}\right) \Lambda^{\nu}\left(\zeta_{\nu}\right), \quad \nu=1, \ldots, N  \tag{3.24a}\\
L(z) P^{\infty}\left(\zeta_{\infty}\right) & =-\zeta_{\infty}^{2} P^{\infty}\left(\zeta_{\infty}\right) \Lambda^{\nu}\left(\zeta_{\infty}\right) \tag{3.24b}
\end{align*}
$$

in punctured neighbourhoods of each pole $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$, where

$$
\begin{align*}
\Lambda^{\nu}\left(\zeta_{\nu}\right) & =\operatorname{diag}\left(\lambda_{1}(z), \ldots, \lambda_{r}(z)\right)=\operatorname{diag}\left(\lambda_{1}^{\nu}\left(\zeta_{\nu}\right), \ldots, \lambda_{r}^{\nu}\left(\zeta_{\nu}\right)\right), \quad \nu=1, \ldots, N,  \tag{3.25a}\\
\Lambda^{\nu}\left(\zeta_{\infty}\right) & =-z^{2} \operatorname{diag}\left(\lambda_{1}(z), \ldots, \lambda_{r}(z)\right)=-\zeta_{\infty}^{-2} \operatorname{diag}\left(\lambda_{1}^{\infty}\left(\zeta_{\infty}\right), \ldots, \lambda_{r}^{\infty}\left(\zeta_{\infty}\right)\right) \tag{3.25b}
\end{align*}
$$

is the diagonal matrix of eigenvalues, which are the $r$ (distinct) solutions of the characteristic equation (1.19), as local Laurent series near each pole $z=c_{\nu}$. We then have the following key result, which implies formulae (1.41a), (1.41b)

Theorem 3.2. (Residue formulae for the Birkhoff invariants.)

1. For a suitable choice of normalization of the eigenvectors forming the columns of $P^{\nu}\left(\zeta_{\nu}\right)$, the matrix $Y^{\nu}\left(\zeta_{\nu}\right)$, with formal expansion (3.5), coincides with the analytic series (3.23) for $P^{\nu}\left(\zeta_{\nu}\right)$ up to terms of order $\mathcal{O}\left(\zeta_{\nu}^{d_{\nu}+1}\right)$.
2. The matrix $\frac{d T^{\nu}\left(\zeta_{\nu}\right)}{d \zeta_{\nu}}$ equals the principal part of the local Laurent series of the matrix $\Lambda^{\nu}\left(\zeta_{\nu}\right)$ of eigenvalues near $z=c_{\nu}$

$$
\begin{equation*}
\frac{d T^{\nu}}{d \zeta_{\nu}}\left(\zeta_{\nu}\right)=\left(\Lambda^{\nu}\left(\zeta_{\nu}\right)\right)_{\text {sing }} \tag{3.26}
\end{equation*}
$$

where

$$
\begin{align*}
\lambda_{a}^{\nu}\left(\zeta_{\nu}\right) & =-\sum_{j=0}^{d_{\nu}} \frac{t_{j a}^{\nu}}{\zeta_{\nu}^{j+1}}+\mathcal{O}(1), \quad \nu=1, \ldots, N  \tag{3.27a}\\
\lambda_{a}^{\infty}\left(\zeta_{\infty}\right) & =\sum_{j=0}^{d_{\infty}} \frac{t_{j a}^{\infty}}{\zeta_{\infty}^{j-1}}+\mathcal{O}\left(\zeta_{\infty}^{2}\right) \tag{3.27b}
\end{align*}
$$

and hence

$$
\begin{align*}
& t_{j a}^{\nu}=-\underset{z=c_{\nu}}{\operatorname{res}}\left(z-c_{\nu}\right)^{j} \lambda_{a}(z) d z, \quad \nu=0, \ldots, N, \quad j=1, \ldots, d_{\nu}  \tag{3.28a}\\
& t_{j a}^{\infty}=-\underset{z=\infty}{\operatorname{res}} z^{-j} \lambda_{a}(z) d z, \quad j=0, \ldots d_{\infty} . \tag{3.28b}
\end{align*}
$$

The results of Theorem 3.2 should be compared with Remark 1 and eqs. (2.31)-(2.32) in [45] and eq. (4.5) in [40].

Proof. Near each pole $z=c_{\nu}$, the locally analytic matrix $P^{\nu}\left(\zeta_{\nu}\right)$ of eigenvectors of $L(z)$ is given by the series (3.23). In a punctured neighbourhood of the pole $z=c_{\nu}$, where the formal series expansions (3.5), (3.6) hold, we can express (3.4) as

$$
\begin{equation*}
\frac{d Y^{\nu}}{d \zeta_{\nu}}+Y^{\nu} \frac{d T^{\nu}}{d \zeta_{\nu}}=L(z) \frac{d z}{d \zeta_{\nu}} Y^{\nu} \tag{3.29}
\end{equation*}
$$

Expanding $L(z) \frac{d z}{d \zeta_{\nu}}$ in the local coordinate $\zeta_{\nu}$ (where $\frac{d z}{d \zeta_{\nu}}$ is given by (3.22)), we have

$$
\begin{equation*}
L(z) \frac{d z}{d \zeta_{\nu}}=\sum_{j=1}^{d_{\nu}+1} \frac{L_{j}^{\nu}}{\zeta_{\nu}^{j}}+\mathcal{O}(1) \tag{3.30}
\end{equation*}
$$

Near each $z=c_{\nu}$, the eigenvector equation can be written in matrix form as (3.24a), (3.24b), where

$$
\begin{equation*}
\Lambda^{\nu}\left(\zeta_{\nu}\right)=\operatorname{diag}\left(\lambda_{1}^{\nu}\left(\zeta_{\nu}\right), \ldots, \lambda_{r}^{\nu}\left(\zeta_{\nu}\right)\right) \tag{3.31}
\end{equation*}
$$

is the diagonal matrix with entries the distinct solutions of the characteristic equation (1.19), whose local Laurent series' have a pole of order $d_{\nu}+1$ at $\zeta_{\nu}=0$. Any (formally analytic) series $P^{\nu}\left(\zeta_{\nu}\right)$ that satisfies (3.24a), (3.24b) with diagonal $\Lambda^{\nu}\left(\zeta_{\nu}\right)$ or $\Lambda^{\infty}\left(\zeta_{\infty}\right)$ is an eigenvector matrix. Note that the singular part of (3.29) is identical to the singular part of $(3.24 \mathrm{a})$ or $(3.24 \mathrm{~b})$ and contains the expansion up to order $d_{\nu}$. Thus the first $d_{\nu}$ terms in the expansion of $Y^{\nu}\left(\zeta_{\nu}\right)$ coincide with the eigenvector matrix. Eqs. (3.24a), (3.24b) then imply eq. (3.26), concluding the proof.

As a consequence of Theorem 3.2, it also follows that formulae (1.43a), (1.43b), (1.44) are equivalent to the coefficients in $(3.20),(3.21)$, expressed as spectral invariant residues, viewed as Hamiltonian functions on the phase space $\mathcal{L}_{r, \mathbf{d}}$, evaluated on the isomonodromic solution manifold.

Theorem 3.3. The components of formulae (3.20), (3.21)

$$
\begin{align*}
& \partial_{t_{j a}^{\nu}} \ln \tau_{I M}=-\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d z} \frac{\partial T^{\nu}}{\partial t_{j a}^{\nu}}\right) d z  \tag{3.32a}\\
& \nu=1, \ldots, N, \infty, \quad j=1, \ldots, d_{\nu}, \quad a=1, \ldots, r \\
& \partial_{c_{\nu}} \ln \tau_{I M}=-\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d z} \frac{\partial T^{\nu}}{\partial c_{\nu}}\right) d z \\
& \nu=1, \ldots, N, \tag{3.32b}
\end{align*}
$$

are equivalent to the following expressions in terms of residues at the singular points of the meromorphic differential $\lambda d z$

$$
\begin{align*}
& \partial_{t_{j a}^{\nu}} \ln \tau_{I M}=-\frac{1}{j} \underset{z=c_{\nu}}{\operatorname{res}} \frac{1}{\zeta_{\nu}^{j}} \lambda_{a}(z) d z=H_{t_{j a}^{\nu}}  \tag{3.33a}\\
& \nu=1, \ldots, N, \infty, \quad j=1, \ldots, d_{\nu}, \quad a=1, \ldots, r \\
& \partial_{c_{\nu}} \ln \tau_{I M}=\frac{1}{2} \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(L^{2}(z)\right) d z=H_{c_{\nu}}  \tag{3.33b}\\
& \nu=1, \ldots, N
\end{align*}
$$

where the second equalities in (3.33a) (3.33b) are the definitions (1.43a), (1.43b), (1.44) of the Hamiltonians $\left\{H_{t_{j a}^{\nu}}, H_{c^{\nu}}\right\}$ as spectral invariants, and hence

$$
\begin{align*}
\lambda_{a}^{\nu}\left(\zeta_{\nu}\right) & =-\sum_{j=1}^{d_{\nu}} \frac{t_{j a}^{\nu}}{\zeta_{\nu}^{j+1}}-\frac{t_{0 a}^{\nu}}{\zeta_{\nu}}-\sum_{j=1}^{d_{\nu}} j H_{t_{j a}^{\nu}} \zeta_{\nu}^{j-1}+\mathcal{O}\left(\zeta_{\nu}^{d_{\nu}}\right), \quad \nu=1, \ldots, N,  \tag{3.34a}\\
\lambda_{a}^{\infty}\left(\zeta_{\infty}\right) & =\sum_{j=1}^{d_{\nu}} \frac{t_{j a}^{\infty}}{\zeta_{\infty}^{j-1}}+t_{0 a}^{\infty} \zeta_{\infty}+\sum_{j=1}^{d_{\infty}} j H_{t_{j a}^{\infty}} \zeta_{\infty}^{j+1}+\mathcal{O}\left(\zeta_{\infty}^{d_{\infty}+2}\right) \tag{3.34b}
\end{align*}
$$

The results of Theorem 3.3 should be compared with Remark 5.2 and eqs. (5.1), (5.1)' in [45] and Lemma 4.1 in [40].

Proof. In terms of the Laurent expansion in the local parameter $\zeta_{\nu}$ near to $z=c_{\nu}$, eq. (3.4) implies

$$
\begin{equation*}
\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d \zeta_{\nu}}=\frac{d T^{\nu}}{d \zeta_{\nu}}+\frac{d z}{d \zeta_{\nu}}\left(Y^{\nu}\right)^{-1} L Y^{\nu} \tag{3.35}
\end{equation*}
$$

Substituting this expression into the RHS of (3.32a), we obtain

$$
\begin{align*}
& -\quad \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d z} \frac{\partial T^{\nu}}{\partial t_{j a}^{\nu}}\right) d z=-\underset{\zeta_{\nu}=0}{\operatorname{res}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d \zeta_{\nu}} \frac{\partial T^{\nu}}{\partial t_{j a}^{\nu}}\right) d \zeta_{\nu} \\
& \quad=-\underset{\zeta_{\nu}=0}{\text { res }} \operatorname{tr}\left(\frac{d T^{\nu}}{d \zeta_{\nu}} \frac{\partial T^{\nu}}{\partial t_{j a}^{\nu}}\right) d \zeta_{\nu}-\underset{\zeta_{\nu}=0}{\operatorname{res}} \operatorname{tr}\left(\frac{d z}{d \zeta_{\nu}}\left(Y^{\nu}\right)^{-1} L Y^{\nu} \frac{\partial T^{\nu}}{\partial t_{j a}^{\nu}}\right) d \zeta_{\nu} \tag{3.36}
\end{align*}
$$

The first residue vanishes because $\frac{\partial T^{\nu}}{\partial t_{j a}^{\nu}}$ is a Laurent polynomial containing only negative powers of $\zeta_{\nu}$. The second residue written explicitly is

$$
\begin{equation*}
-\operatorname{res}_{\zeta_{\nu}=0}^{\operatorname{tr}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} L Y^{\nu} \frac{E_{a a}}{j \zeta_{\nu}^{j}}\right) \frac{d z}{d \zeta_{\nu}} d \zeta_{\nu} \tag{3.37}
\end{equation*}
$$

The proof will be complete if we can show that the diagonal part of $\left(Y^{\nu}\right)^{-1} L Y^{\nu} \frac{d z}{d \zeta_{\nu}}$ coincides with the Taylor expansion of the eigenvalue matrix $\Lambda^{\nu}$ up terms of order $\mathcal{O}\left(\zeta_{\nu}^{d_{\nu}+1}\right)$. For simplicity, consider a pole at a finite $z=c_{\nu}$, so that $d z / d \zeta_{\nu}=1$ and let $d=d_{\nu}$. We then have

$$
\begin{equation*}
\left(Y^{\nu}\right)^{-1} L Y^{\nu}=\overbrace{\frac{d T^{\nu}}{d \zeta_{\nu}}}^{\Lambda_{-}^{\nu}\left(\zeta_{\nu}\right)}+\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d \zeta_{\nu}} . \tag{3.38}
\end{equation*}
$$

Let $\eta_{+}\left(\zeta_{\nu}\right)$ denote the diagonal part of $\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d \zeta_{\nu}}$ and $F\left(\zeta_{\nu}\right)$ the off-diagonal part (both as formal analytic series). The goal is to show that $\eta_{+}$coincides with the Taylor expansion of the eigenvalue matrix $\Lambda^{\nu}$ up to order $\zeta_{\nu}^{d+1}$.

To see this, note that the spectrum of $\left(Y^{\nu}\right)^{-1} L Y^{\nu}$ coincides with that of $L$ and hence with the spectrum of

$$
\begin{equation*}
\Lambda_{-}^{\nu}\left(\zeta_{\nu}\right)+\eta_{+}\left(\zeta_{\nu}\right)+F\left(\zeta_{\nu}\right) \tag{3.39}
\end{equation*}
$$

Define

$$
\begin{equation*}
D:=\operatorname{diag}\left(D_{1}, \ldots, D_{r}\right), \quad D_{a}\left(\lambda, \zeta_{\nu}\right):=\lambda-\left(\Lambda_{-}^{\nu}\left(\zeta_{\nu}\right)+\eta_{+}\left(\zeta_{\nu}\right)\right)_{a a} \tag{3.40}
\end{equation*}
$$

and consider the characteristic polynomial

$$
\begin{equation*}
\operatorname{det}\left(\lambda \mathbf{I}-\Lambda_{-}^{\nu}\left(\zeta_{\nu}\right)-\eta_{+}\left(\zeta_{\nu}\right)-F\left(\zeta_{\nu}\right)\right)=\left(\prod_{a=1}^{r} D_{a}\right) \operatorname{det}\left[\mathbf{I}-D^{-1} F\right] \tag{3.41}
\end{equation*}
$$

Recalling that the diagonal elements of $F$ vanish, the expansion of the determinant has the form

$$
\begin{equation*}
\operatorname{det}\left[\mathbf{I}-D^{-1} F\right]=1+\mathcal{O}\left(\zeta_{\nu}^{2 d_{\nu}+2}\right) \tag{3.42}
\end{equation*}
$$

near to $z-c_{\nu}$. The characteristic equation (1.19) implies that one of the $D_{a}\left(\lambda_{a}, \zeta_{\nu}\right)$ 's vanishes to order $\zeta_{\nu}^{d+1}$,

$$
\begin{equation*}
\lambda_{a}=\left(\Lambda_{-}^{\nu}+\eta_{+}\right)_{a a}+\mathcal{O}\left(\zeta_{\nu}^{d_{\nu}+1}\right) \tag{3.43}
\end{equation*}
$$

This concludes the proof of equality between (3.32a) and (3.33a).
For (3.32b), start by observing that

$$
\begin{equation*}
\partial_{c_{\nu}} T^{\nu}=-\partial_{z} T^{\nu} \tag{3.44}
\end{equation*}
$$

These equations hold for $\nu=1, \ldots, N$, so $d z / d \zeta_{\nu}=1$, and the RHS of (3.32b) is

$$
\operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} \frac{d Y^{\nu}}{d z} \frac{d T^{\nu}}{d z}\right) d z
$$

$$
\begin{equation*}
=\operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(\frac{d T^{\nu}}{d z}\right)^{2} d z+\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(\left(Y^{\nu}\right)^{-1} L Y^{\nu} \frac{d T^{\nu}}{d z}\right) d z \tag{3.45}
\end{equation*}
$$

The first residue is again zero because the integrand is a negative power Laurent polynomial starting with $\left(z-c_{\nu}\right)^{-2}$. We have already shown that the diagonal part of $\left(Y^{\nu}\right)^{-1} L Y^{\nu}$ coincides with the diagonal matrix of eigenvalues $\Lambda^{\nu}$, up to order $\mathcal{O}\left(\left(z-c_{\nu}\right)^{d_{\nu}+1}\right)$. Therefore, in the second residue in (3.45), we can substitute $\Lambda^{\nu}$ for $\left(Y^{\nu}\right)^{-1} L Y^{\nu}$. Since $d T^{\nu} / d z$ is the singular part of $\Lambda^{\nu}$ at $z=c_{\nu}$, the second residue equals

$$
\begin{equation*}
\operatorname{res}_{z=c_{\nu}} \operatorname{Tr}\left(\Lambda^{\nu}(z) \Lambda_{\text {sing }}(z)\right) d z=\frac{1}{2} \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{Tr}\left(\Lambda^{\nu}(z) \Lambda^{\nu}(z)\right) d z=\frac{1}{2} \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{Tr}\left(L^{2}(z)\right) d z \tag{3.46}
\end{equation*}
$$

We thus have shown the equivalence of (3.32a) with (3.33a) and (3.32b) with (3.33b).

## 4 Birkhoff connection, commutativity and Poisson property

### 4.1 The Birkhoff connection

Consider the isomonodromic deformation equations (3.14b), (3.14c) whose compatibility conditions express the vanishing curvature (3.16) of the connection $\Omega$ in (3.15). In particular the Lax matrix $L(z)$ satisfies the following zero curvature equations [45]

$$
\begin{equation*}
\frac{\partial L(z)}{\partial t_{j a}^{\nu}}=\frac{d U_{j a}^{\nu}(z ; L)}{d z}-\left[L(z), U_{j a}^{\nu}(z ; L)\right], \quad \frac{\partial L(z)}{\partial c_{\nu}}=\frac{d V^{\nu}(z ; L)}{d z}-\left[L(z), V^{\nu}(z ; L)\right] \tag{4.1}
\end{equation*}
$$

where the matrices $U_{j a}^{\nu}, V^{\nu}$ are defined in (3.10), (3.11). These equations should be viewed as defining commuting vector fields on the manifold $\mathcal{L}_{r, \mathbf{d}}$ of rational matrices of the form (3.1), which is why we have explicitly indicated the dependence of the connection component matrices $\left\{U_{j a}^{\nu}(z ; L), V^{\nu}(z ; L)\right\}$ on the Lax matrix $L$.

They can equivalently be written in terms of spectral data. The following is equivalent to eqs. (3.10), (3.11):

$$
\begin{align*}
U_{j a}^{\nu}(z ; L) & :=\left(P^{\nu}\left(\zeta_{\nu}\right) \frac{\partial T^{\nu}\left(\zeta_{\nu}\right)}{\partial t_{j a}^{\nu}}\left(P^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\text {sing }}=\left(P^{\nu}\left(\zeta_{\nu}\right) \frac{E_{a a}}{j \zeta_{\nu}^{j}}\left(P^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\text {sing }}  \tag{4.2}\\
\nu & =1, \ldots, N, \infty \\
V^{\nu}(z ; L) & :=\left(P^{\nu}\left(\zeta_{\nu}\right) \frac{\partial T^{\nu}(z)}{\partial c_{\nu}}\left(P^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\operatorname{sing}}=-\left(P^{\nu}\left(\zeta_{\nu}\right) \frac{d T^{\nu}(z)}{d \zeta_{\nu}}\left(P^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right)_{\text {sing }}  \tag{4.3}\\
\nu & =1 \ldots, N
\end{align*}
$$

The only difference is that we have replaced the formal series $Y^{\nu}\left(\zeta_{\nu}\right)$ in (3.5) with a local analytic series of eigenvectors. Note that right multiplication of the matrix of eigenvectors $P(z)$ by an invertible diagonal matrix changes the normalization of the eigenvector, but does not affect formulae (3.10), (3.11). The equivalence is due to the fact that

$$
\begin{equation*}
Y^{\nu}\left(\zeta_{\nu}\right)=P^{\nu}\left(\zeta_{\nu}\right)+\mathcal{O}\left(\zeta_{\nu}^{d_{\nu}+1}\right) \tag{4.4}
\end{equation*}
$$

Now we come to the crux of the matter. The vector fields defined by (4.1) contain the sum of two terms: a commutator term and the $z$-derivative of the deformation matrices appearing in eqs. (3.14b). (3.14c). The former has a clear interpretation. As will be proved explicitly in Section 4.4, the equations

$$
\begin{equation*}
\mathbf{X}_{H_{t}{ }_{j a}^{\nu}} L:=\left[U_{j a}^{\nu}, L\right], \quad \mathbf{X}_{H_{c_{\nu}}} L:=\left[V^{\nu}, L\right], \quad \mathbf{X}_{H_{a}^{\infty}} L:=\left[E_{a a}, L\right] \tag{4.5}
\end{equation*}
$$

give the infinitesimal isospectral deformations generated by the spectral invariant Hamiltonians $\left\{H_{t_{j a}^{\nu}}, H_{c_{\nu}}, H_{a}^{\infty}\right\}$, defined by the action of their Hamiltonian vector fields $\mathbf{X}_{H_{t_{j a}^{\nu}}}, \mathbf{X}_{H_{c_{\nu}}}, \mathbf{X}_{H_{a}^{\infty}}$. Our focus however is on the other terms; namely, the vector fields $\nabla_{t_{j a}^{\nu}}$ and $\nabla_{c^{\nu}}$ that act as follows on the matrix $L$ (i.e. on the linear functions of $L$ )

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} L(z):=\frac{d}{d z} U_{j a}^{\nu}(z ; L), \quad \nabla_{c^{\nu}} L(z):=\frac{d}{d z} V^{\nu}(z ; L) \tag{4.6}
\end{equation*}
$$

This is understood as defining the action of $\nabla_{t_{j a}^{\nu}}$ and $\nabla_{c^{\nu}}$ on each of the coefficients of $L$ (viewed as linear coordinates on $\mathcal{L}_{r, \mathbf{d}}$ ) as well as the position of the poles, extended to arbitrary differentiable functions of $L$ by requiring it to be a derivation. We would like to interpret equations (4.6) as "explicit derivatives" with respect to the parameters $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$. But for this to make sense, we need to verify that:

1. The vector fields $\nabla_{t_{j a}^{\nu}}, \nabla_{c^{\nu}}$ commute for all $\nu, j$ and $a$.
2. They act on the Casimir functions $\left\{t_{j a}^{\nu}, c_{\nu}\right\}$ as directional derivatives along coordinate curves.

Definition 4.1. We call the map that associates the vector field $\nabla_{t}$ to $t \in \mathbf{T}$ as in (4.6) the Birkhoff connection.

The justification of the term "connection" will be given in Section 5, where we interpret $\nabla$ as a flat Ehresmann connection. In addition we will verify that these vector fields preserve the Poisson structure.

We then have
Theorem 4.1. The vector fields $\nabla_{t_{j a}^{\nu}}, \nabla_{c^{\nu}}$ given by (4.6) act as follows on the Casimir functions $t_{k b}^{\mu}, c_{\nu}$ :

$$
\begin{align*}
\nabla_{t_{j a}^{\nu}} c_{\mu} & =0  \tag{4.7a}\\
\nabla_{c_{\nu}} c_{\mu} & =\delta_{\mu \nu}  \tag{4.7b}\\
\nabla_{t_{j a}^{\nu}} t_{k b}^{\mu} & =\delta_{\nu \mu} \delta_{a b} \delta_{j k}  \tag{4.7c}\\
\nabla_{c^{\nu}} t_{j a}^{\mu} & =0 \tag{4.7~d}
\end{align*}
$$

That is, for any $t, s \in \mathbf{T}$ we have

$$
\begin{equation*}
\nabla_{s} t=\delta_{t, s} \tag{4.8}
\end{equation*}
$$

Proof. To prove (4.7a), (4.7b) we first need to express $c_{\nu}$ as a function on $\mathcal{L}_{r, \mathbf{d}}$. Take any of the entries of $L(z)$ that actually has a pole of order $d_{\nu}+1$ at $c_{\nu}$. We can then write

$$
\begin{equation*}
c_{\nu}=-\frac{1}{1+d_{\nu}} \underset{z=c_{\nu}}{\operatorname{res}} z \frac{d}{d z} \ln L_{k \ell}(z) d z . \tag{4.9}
\end{equation*}
$$

The residue may be understood as a contour integral on a small circle around $z=c_{\nu}$, which remains constant under the deformations $\nabla_{t_{j a}^{\nu}}, \nabla_{c_{\mu}}$.

To prove (4.7a), apply $\nabla_{t_{j a}^{\nu}}$ to (4.9) and use the definition (4.6) of its action on $L_{k \ell}$, together with (4.2) to obtain

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} c_{\mu}=-\frac{1}{1+d_{\mu}} \operatorname{res}_{z=c_{\mu}} z \frac{d}{d z}\left(\frac{\frac{d}{d z}\left(U_{j a}^{\nu}\right)_{k \ell}(z)}{L_{k \ell}(z)}\right) d z=\frac{1}{1+d_{\mu}} \underset{z=c_{\mu}}{\operatorname{res}}\left(\frac{\frac{d}{d z}\left(U_{j a}^{\nu}\right)_{k \ell}(z)}{L_{k \ell}(z)}\right) d z \tag{4.10}
\end{equation*}
$$

where in the second equality we have used integration by parts. The numerator is either analytic at $z=c_{\mu}$ if $\nu \neq \mu$, or has a pole of order at most $d_{\nu}+1$. In either case the ratio is analytic at $c_{\mu}$ and the residue is zero.

To prove (4.7b) we similarly use the fact that

$$
\begin{equation*}
\nabla_{c_{\mu}} L_{i j}(z)=\frac{d}{d z} V_{i j}^{\mu}(z) \tag{4.11}
\end{equation*}
$$

is analytic at $z=c_{\nu}$. We then have

$$
\begin{equation*}
\nabla_{c_{\mu}} c_{\nu}=0 \tag{4.12}
\end{equation*}
$$

for $\nu \neq \mu$. If $\mu=\nu$, note that $V_{i j}^{\nu}(z)$ has a pole of order $d_{\nu}+2$ at $c_{\nu}$ and hence

$$
\begin{equation*}
\nabla_{c_{\nu}} c_{\nu}=-\frac{1}{1+d_{\nu}} \operatorname{res}_{z=c_{\nu}} z \frac{d}{d z}\left(\frac{\frac{d}{d z} V_{i j}^{\nu}(z)}{L_{i j}(z)}\right) d z=\frac{1}{1+d_{\nu}} \operatorname{res}_{z=c_{\nu}}\left(\frac{\frac{d}{d z} V_{i j}^{\nu}(z)}{L_{i j}(z)}\right) d z \tag{4.13}
\end{equation*}
$$

Recalling that $V^{\nu}$ is just the negative of the singular part of $L$ at $c_{\nu}$, in the above residue we can add the regular part without changing the value of the integral, to get

$$
\begin{equation*}
\nabla_{c_{\nu}} c_{\nu}=-\frac{1}{1+d_{\nu}} \operatorname{res}_{z=c_{\nu}}\left(\frac{\frac{d}{d z} L_{i j}(z)}{L_{i j}(z)}\right) d z=1 \tag{4.14}
\end{equation*}
$$

We next prove (4.7c). For all $\mu=1, \ldots, N, \infty$, the diagonalization (3.24a), (3.24b) can be written uniformly as

$$
\begin{equation*}
\frac{d z}{d \zeta_{\mu}} L(z)=P^{\mu}\left(\zeta_{\mu}\right) \Lambda^{\nu}\left(\zeta_{\mu}\right) P^{\mu}\left(\zeta_{\mu}\right)^{-1} \tag{4.15}
\end{equation*}
$$

where $\Lambda^{\mu}$ is the diagonal matrix whose entries consist of the Laurent series in $\lambda_{a}\left(\zeta_{\mu}\right)$ with a pole of order $d_{\nu}+1$. Formulae (3.28a), (3.28b) are equivalent to the matrix identities:

$$
\begin{equation*}
T_{k}^{\mu}=-\operatorname{res}_{z=c_{\mu}} \zeta_{\mu}^{k} P^{\mu}\left(\zeta_{\mu}\right)^{-1} L(z) P^{\mu}\left(\zeta_{\mu}\right) d z \tag{4.16}
\end{equation*}
$$

Applying the derivation $\nabla_{t_{j a}^{\nu}}$ to all the matrix components on both sides, using (4.7a), we obtain:

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} T_{k}^{\mu}=-\operatorname{res}_{z=c_{\mu}}\left(P^{\mu}\left(\zeta_{\mu}\right)^{-1} \frac{d U_{j a}^{\nu}(z)}{d z} P^{\mu}\left(\zeta_{\mu}\right)-\left[\left(P^{\mu}\right)^{-1} \nabla_{t_{j a}^{\nu}} P^{\mu}, \Lambda^{\mu}\right]\right) \zeta_{\mu}^{k} d z \tag{4.17}
\end{equation*}
$$

The commutator in (4.17) is diagonal free and hence we can discard it because the left side must be a diagonal matrix. If $\mu \neq \nu$, the residue vanishes because the matrix $U_{j a}^{\nu}$ is analytic at $z=c_{\mu}$. If $\mu=\nu$ we
substitute the definition (3.10) of $U_{j a}^{\nu}$ and simplify to obtain

$$
\begin{align*}
\nabla_{t_{j a}^{\nu}} T_{k}^{\nu} & =-\operatorname{res}_{z=c_{\nu}}\left(\left(P^{\nu}\right)^{-1} \frac{d}{d z}\left(P^{\nu} \frac{E_{a a}}{j \zeta_{\nu}^{j}}\left(P^{\nu}\right)^{-1}\right)_{\text {sing }} P^{\nu}\right)_{D} \zeta_{\nu}^{k} d z \\
& =-\operatorname{res}_{\zeta_{\nu}=0}\left(\left(P^{\nu}\right)^{-1} \frac{d}{d \zeta_{\nu}}\left(P^{\nu} \frac{E_{a a}}{j \zeta_{\nu}^{j}}\left(P^{\nu}\right)^{-1}\right) P^{\nu}\right)_{D} \zeta_{\nu}^{k} d \zeta_{\nu} \tag{4.18}
\end{align*}
$$

where $(\cdot)_{D}$ denotes the diagonal part of the matrix $(\cdot)$, and we have added the regular part since it does not contribute to the residue. Finally, using the Leibniz rule we obtain

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} T_{k}^{\nu}=-\operatorname{res}_{\zeta_{\nu}=0}\left(\left[\left(P^{\nu}\right)^{-1} \frac{d P^{\nu}}{d \zeta_{\nu}}, \frac{E_{a a}}{j \zeta_{\nu}^{j}}\right]-\frac{E_{a a}}{\zeta_{\nu}^{j+1}}\right)_{D} \zeta_{\nu}^{k} d \zeta_{\nu}=E_{a a} \delta_{j, k} \tag{4.19}
\end{equation*}
$$

where, again, we have used the fact that the commutator is diagonal free. This proves eq. (4.7c).
To prove $(4.7 \mathrm{~d})$, we repeat the argument used in the first part of the proof, with $U_{j a}^{\nu}$. replaced by $V^{\nu}$. (for $\nu=1, \ldots, N$.) We have

$$
\begin{equation*}
\frac{d T^{\mu}\left(\zeta_{\mu}\right)}{d \zeta_{\mu}}=-\operatorname{res}_{\xi_{\mu}=0} P^{\mu}\left(\xi_{\mu}\right)^{-1} L(w) P^{\mu}\left(\xi_{\mu}\right) \frac{d \xi_{\mu}}{\zeta_{\mu}-\xi_{\mu}} \tag{4.20}
\end{equation*}
$$

where

$$
\begin{equation*}
\xi_{\mu}:=w-c_{\mu}, \mu=1, \ldots, N, \quad \text { or } \quad \xi_{\infty}=1 / w \tag{4.21}
\end{equation*}
$$

Acting with $\nabla_{c^{\nu}}$ and using (4.14) for $\mu=\nu$, we get

$$
\begin{equation*}
\nabla_{c^{\nu}}\left(\frac{d T^{\mu}\left(\zeta_{\mu}\right)}{d \zeta_{\mu}}\right)=-\delta_{\mu \nu} \frac{d^{2} T^{\nu}}{d \zeta_{\mu}^{2}}\left(\zeta_{\mu}\right)-\operatorname{res}_{\xi_{\mu}=0}\left(\left(P^{\mu}\right)^{-1} V^{\nu} P^{\mu}-\left[\left(P^{\mu}\right)^{-1} \nabla_{c^{\nu}} P^{\mu}, L\right]\right) \frac{d \xi_{\mu}}{\zeta_{\mu}-\xi_{\mu}} \tag{4.22}
\end{equation*}
$$

The commutator is again diagonal free and the first term in the residue is analytic if $\nu \neq \mu$, so this vanishes. For $\mu=\nu=1, \ldots, N$ we get

$$
\begin{gather*}
\nabla_{c^{\nu}} \frac{d T^{\nu}}{d \zeta_{\nu}}=-\frac{d^{2} T^{\nu}}{d \zeta_{\nu}^{2}}-\operatorname{res}_{\xi_{\nu}=0}\left(\left(P^{\nu}\right)^{-1} \frac{d V^{\nu}}{d \zeta^{2}} P^{\nu}\right)_{D} \frac{d \xi_{\nu}}{\xi_{\nu}-\zeta_{\nu}} \\
(3.10),(3.11)  \tag{4.23}\\
= \\
d \zeta_{\nu}^{2} \\
\operatorname{d}_{\xi_{\nu}=0}^{\operatorname{res}}\left(\left(P^{\nu}\right)^{-1} \frac{d}{d \zeta_{\nu}}\left(P^{\nu}\left(\xi_{\nu}\right) \frac{d T^{\nu}\left(\xi_{\nu}\right)}{d \xi_{\nu}} P^{\nu}\left(\xi_{\nu}\right)^{-1}\right) P^{\nu}\right)_{D} \frac{d \xi_{\nu}}{\zeta_{\nu}-\xi_{\nu}}
\end{gather*}
$$

Integrating by parts and discarding the commutator term (which is diagonal free) we obtain

$$
\begin{equation*}
\nabla_{c^{\nu}} \frac{d T^{\nu}}{d \zeta_{\nu}}=-\frac{d^{2} T^{\nu}}{d \zeta_{\nu}^{2}}-\operatorname{res}_{\xi_{\nu}=0}\left(\frac{d T^{\nu}\left(\xi_{\nu}\right)}{d \xi_{\nu}}\right) \frac{d \xi_{\nu}}{\left(\xi_{\nu}-\zeta_{\nu}\right)^{2}}=-\frac{d^{2} T^{\nu}}{d \zeta_{\nu}^{2}}+\frac{d^{2} T^{\nu}}{d \zeta_{\nu}^{2}}=0 \tag{4.24}
\end{equation*}
$$

showing that $\nabla_{c^{\nu}}$ annihilates all the Birkhoff invariants.

### 4.2 Commutativity of the vector fields $\nabla$

Theorem 4.2. For all $\mu, \nu=1, \ldots, N$, and $\nu=\infty$, the vector fields $\left\{\nabla_{c^{\mu}}, \nabla_{t_{j a}^{\nu}}\right\}_{j=1, \ldots d_{\nu}, a=1, \ldots, r}$ commute amongst themselves.

We give the proof in two parts. First we show that the vector fields $\nabla_{t_{j a}^{\mu}}, \nabla_{t_{k b}}$ commute for $\mu \neq \nu$. This follows from the following Lemma 4.3, which actually contains a stronger statement. We then localize the proof for a single $\nu$ and show that $\left\{\nabla_{c^{\nu}}, \nabla_{t_{j a}^{\nu}}\right\}$ commute for all pairs $(j, a)$.

Lemma 4.3. For $\mu \neq \nu$ the vector fields satisfy

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} L_{k}^{\mu}=0, \quad \nabla_{c_{\nu}} L_{k}^{\mu}=0 \tag{4.25}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} \nabla_{t_{k b}^{\mu}} L \equiv 0, \quad \nabla_{c^{\mu}} \nabla_{t_{j a}^{\nu}} L \equiv 0, \quad \nabla_{c^{\nu}} \nabla_{c^{\mu}} L \equiv 0 \tag{4.26}
\end{equation*}
$$

In particular, this implies that the vector fields $\nabla_{t_{j a}^{\nu}}, \nabla_{t_{k b}^{\mu}}, \nabla_{c_{\nu}}$ all commute for $\mu \neq \nu$.
Proof. Formula (4.25) follows from the definition (4.6) of $\nabla$. For example the matrix $\nabla_{t_{j a}^{\nu}} L(z)=U_{j a}^{\nu}(z)$ (see (4.2)) is analytic at $z=c_{\mu}(\nu \neq \mu)$, which means that $\nabla_{t_{j a}^{\nu}}$ annihilates all entries of the coefficient matrices $L_{k}^{\mu}$ in the polar expansion of $L$ near $z=c_{\mu}$, and similarly for $\nabla_{c_{\nu}}$.

To prove (4.26), note that

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} \nabla_{t_{k b}^{\mu}} L(z)=\frac{d}{d z} \nabla_{t_{j a}^{\nu}} U_{k b}^{\mu}(z) \tag{4.27}
\end{equation*}
$$

so it suffices to show that $\nabla_{t_{j a}^{\nu}} U_{k b}^{\mu}(z)$ is independent of $z$, with a similar statement for $V^{\nu}$. To prove this, observe that $U_{k b}^{\mu}$ depends only on the Taylor expansion of $P^{\mu}\left(\zeta_{\mu}\right)$ up to order $d_{\mu}$. As will be shown below, these coefficients, in turn, depend only on the coefficients $L_{\left\{d_{\mu}+1, \ldots, 1\right\}}^{\mu}$, and therefore are annihilated by $\nabla_{t_{j a}^{\nu}}$. This implies that

$$
\begin{equation*}
\nabla_{t_{j a}^{\nu}} U_{k b}^{\mu}(z) \equiv 0 \tag{4.28}
\end{equation*}
$$

To see this, we need to consider the equations defining the Taylor expansion of $P^{\mu}\left(\zeta_{\mu}\right)$. By comparing the series expansions of both sides of the eigenvector equation

$$
\begin{equation*}
\frac{d z}{d \zeta} L(z) P^{\mu}\left(\zeta_{\mu}\right)=P^{\mu}\left(\zeta_{\mu}\right) \Lambda^{\mu}\left(\zeta_{\mu}\right) \tag{4.29}
\end{equation*}
$$

it follows that we can always express the series $P^{\mu}\left(\zeta_{\mu}\right), \Lambda^{\mu}\left(\zeta_{\mu}\right)$ as

$$
\begin{align*}
P^{\mu}\left(\zeta_{\mu}\right)=P_{0}^{\mu}+P_{1}^{\mu} \zeta_{\mu}+P_{2}^{\mu} \zeta_{\mu}^{2}+\ldots & =G^{\mu}\left(\mathbf{I}+F_{1}^{\mu} \zeta_{\mu}+F_{2}^{\mu} \zeta_{\mu}^{2}+\ldots\right)  \tag{4.30}\\
\Lambda^{\mu}\left(\zeta_{\mu}\right) & =\sum_{j=-d_{\mu}-1}^{\infty} \Lambda_{j}^{\mu} \zeta_{\mu}^{j} \tag{4.31}
\end{align*}
$$

where the coefficients $F_{j}^{\mu}$ are diagonal free matrices and the $\Lambda_{j}^{\mu}$ 's are purely diagonal. Setting $F_{0}^{\nu}:=\mathbf{I}$ and defining the coefficient matrices $\widehat{L}_{j}^{\nu}$ by the identity

$$
\begin{equation*}
\left(G^{\nu}\right)^{-1} L(z) G^{\nu} \frac{d z}{d \zeta_{\mu}}=\sum_{j=-d_{\nu}-1}^{\infty} \widehat{L}_{-j}^{\nu} \zeta_{\mu}^{j} \tag{4.32}
\end{equation*}
$$

the matrices $P^{\nu}, \Lambda^{\nu}$ are determined by the recurrence relations

$$
\begin{aligned}
\Lambda_{-d_{\nu}-1}^{\nu} & :=-T_{d_{\nu}}^{\nu}=\widehat{L}_{d_{\nu}+1}^{\nu} \in \mathfrak{h} \\
F_{\ell}^{\nu} & :=-\operatorname{ad}_{T_{d_{\nu}}}^{-1}\left(\sum_{j=1}^{\ell}\left(\widehat{L}_{d_{\nu}+1-j}^{\nu} F_{\ell-j}^{\nu}-F_{\ell-j}^{\nu} \Lambda_{d_{\nu}+1-j}^{\nu}\right)\right)
\end{aligned}
$$

$$
\begin{equation*}
\Lambda_{-d_{\nu}-1+\ell}^{\nu}:=\left(\sum_{j \geq 1}\left(\widehat{L}_{d_{\nu}+1-j}^{\nu} F_{\ell-j}^{\nu}-F_{\ell-j}^{\nu} \Lambda_{-d_{\nu}-1+j}^{\nu}\right)\right)_{D}, \quad \ell \geq 1 . \tag{4.33}
\end{equation*}
$$

From these recursive formulae it follows that the matrix $P_{\ell}^{\mu}$ depends only on $L_{d_{\nu}+1}^{\nu}, \ldots, L_{d_{\nu}+1-\ell}^{\nu}$. This implies that $U_{j a}^{\nu}$ (and similarly $V^{\nu}$ ) only depend only on the singular part of $L \frac{d z}{d \zeta_{\nu}}$ at the point $z=c_{\nu}$, proving the claim that the coefficients in the Taylor expansion of $P^{\mu}$ up to order $d_{\mu}$ depend only on the coefficients $L_{\left\{d_{\mu}+1, \ldots, 1\right\}}^{\mu}$.

Formulae (4.33) show that the coefficients of $P^{\nu}\left(\zeta_{\nu}\right)$, and hence also of the matrices $U_{j a}^{\nu}, V^{\nu}$, depend polynomially on the coefficient matrices $L_{j}^{\nu}$ and are Laurent polynomials in the differences of the eigenvalues of the leading coefficient matrices $L_{d_{\nu}+1}^{\nu}$, and hence in the differences of the diagonal entries of $T_{d_{\nu}}^{\nu}$. We state this in the following.

Corollary 4.4. The matrices $U_{j a}^{\nu}(z ; L), V^{\nu}(z ; L), U_{j a}^{\infty}(z ; L)$ depend polynomially on the entries of the coefficient matrices of $L$ and as Laurent polynomials in the differences of the eigenvalues $\left\{-t_{d_{\nu}, 1}^{\nu}, \ldots,-t_{d_{\nu}, r}^{\nu}\right\}$ of the leading order singularity matrices $L_{d_{\nu}+1}^{\nu}$.

Proof. (Of Theorem 4.2.) We first prove the commutativity of the various vector fields $\nabla$ attached to a single pole. It is clear that the nature of the proof is entirely local, so we will write $P$ instead of $P^{\nu}\left(\zeta_{\nu}\right)$ for the local analytic series of eigenvectors, and $\Lambda$ for the local Laurent series $\Lambda^{\nu}\left(\zeta_{\nu}\right)$ in a punctured neighbourhood of $z=c_{\nu}$. To reduce the number of indices, we denote the two vector fields

$$
\begin{equation*}
\nabla_{1}:=\nabla_{t_{k a}^{\nu}}, \quad \nabla_{2}:=\nabla_{t_{\ell b}^{\nu}} \tag{4.34}
\end{equation*}
$$

and define correspondingly

$$
\begin{equation*}
Q_{1}:=P \frac{\partial T^{\nu}}{\partial t_{k a}^{\nu}} P^{-1}, \quad Q_{2}:=P \frac{\partial T^{\nu}}{\partial t_{\ell b}^{\nu}} P^{-1}, \quad U_{1}:=\left(Q_{1}\right)_{\operatorname{sing}} \quad U_{2}:=\left(Q_{2}\right)_{\operatorname{sing}} \tag{4.35}
\end{equation*}
$$

which, more explicitly, are:

$$
\begin{equation*}
Q_{1}=\frac{1}{k \zeta_{\nu}^{k}} P E_{a a} P^{-1}, \quad Q_{2}=\frac{1}{\ell \zeta_{\nu}^{\ell}} P E_{b b} P^{-1} \tag{4.36}
\end{equation*}
$$

The theorem will follow if we can show that

$$
\begin{equation*}
\nabla_{1} \nabla_{2} L(z)=\nabla_{2} \nabla_{1} L(z) \tag{4.37}
\end{equation*}
$$

where, by (4.6),

$$
\begin{equation*}
\nabla_{j} L(z):=U_{j}(z)^{\prime} \tag{4.38}
\end{equation*}
$$

with $U_{1}, U_{2}$ defined in (4.35) and ' denotes $\frac{d}{d z}$. Note that

$$
\begin{equation*}
\nabla_{2} U_{1}^{\prime}=\left(\left[\nabla_{2} P P^{-1}, Q_{1}\right]+P \nabla_{2} \nabla_{1} \Lambda P^{-1}\right)_{s i n g}^{\prime}=\left(\left[\nabla_{2} P P^{-1}, Q_{1}\right]\right)_{s i n g}^{\prime} \tag{4.39}
\end{equation*}
$$

where in the second equality we have used

$$
\nabla_{2} \nabla_{1} \Lambda=\nabla_{2}\left(-\frac{1}{\zeta_{\nu}^{k+1-2 \epsilon}} E_{a a}+\mathcal{O}\left(\zeta^{\epsilon}\right)\right)=\mathcal{O}\left(\zeta^{\epsilon}\right)
$$

with $\epsilon=1$ if $\nu=\infty$ and 0 otherwise. In either case, the term is locally analytic and hence drops out of the formula, so it is sufficient to prove that

$$
\begin{equation*}
\frac{d}{d z}\left[\nabla_{2} P P^{-1}, Q_{1}\right]_{s i n g}=\frac{d}{d z}\left[\nabla_{1} P P^{-1}, Q_{2}\right]_{s i n g} \tag{4.40}
\end{equation*}
$$

The proof will be complete if we can show that:

$$
\begin{equation*}
\left[\nabla_{2} P P^{-1}, Q_{1}\right]_{s i n g}=\left[\nabla_{1} P P^{-1}, Q_{2}\right]_{\text {sing }}+\text { const. } \tag{4.41}
\end{equation*}
$$

To do this, we introduce the following notation: given any matrix $M(z)$, define $\widehat{M}(z)$ to be

$$
\begin{equation*}
\widehat{M}(z):=P\left(P^{-1} M P\right)_{O D} P^{-1} \tag{4.42}
\end{equation*}
$$

where $(\cdot)_{O D}$ denotes the off-diagonal part the matrix $(\cdot)$. Thus, we consider $M$ modulo the commutant of $L$. Also define the inverse of the adjoint map

$$
\begin{equation*}
\operatorname{ad}_{L}^{-1} M(z):=P\left(\operatorname{ad}_{\Lambda}^{-1}\left(P^{-1} M P\right)_{O D}\right) P^{-1} \tag{4.43}
\end{equation*}
$$

where all terms are viewed as formal Laurent series near $z=c_{\nu}$. Note that for any ad-regular diagonal matrix $D=\operatorname{diag}\left(d_{1}, \ldots, d_{r}\right)$ and any off-diagonal matrix $M$, the inverse of $\operatorname{ad}_{D}$ is well defined:

$$
\begin{equation*}
\left(\operatorname{ad}_{D}^{-1} M\right)_{i j}=\frac{M_{i j}}{d_{i}-d_{j}} \tag{4.44}
\end{equation*}
$$

Definitions (4.42), (4.43) imply

$$
\begin{equation*}
\left[L(z), \operatorname{ad}_{L}^{-1}(M)\right]=\widehat{M}(z) \tag{4.45}
\end{equation*}
$$

By the definition (4.6) of $\nabla$,

$$
\begin{equation*}
\nabla_{j} L(z)=\frac{d U_{j}}{d z} \tag{4.46}
\end{equation*}
$$

On the other hand, by the Leibniz rule, we have

$$
\begin{equation*}
\nabla_{j} L(z)=\left[\nabla_{j} P P^{-1}, L\right]+P \nabla_{j} \Lambda P^{-1} \tag{4.47}
\end{equation*}
$$

Equating the two and using (4.42), (4.43), we deduce that

$$
\begin{equation*}
\widehat{\nabla_{j} P P^{-1}}=-\operatorname{ad}_{L}^{-1}\left(U_{j}^{\prime}\right) \tag{4.48}
\end{equation*}
$$

We now prove the identity (4.41). First, we have

$$
\begin{equation*}
\left[\nabla_{2} P P^{-1}, Q_{1}\right]_{s i n g}=-\left[\widehat{\nabla_{2} P P^{-1}}, Q_{1}\right]_{s i n g} \stackrel{(4.48)}{=}\left[\operatorname{ad}_{L}^{-1}\left(U_{2}^{\prime}\right), Q_{1}\right]_{s i n g} \tag{4.49}
\end{equation*}
$$

The first equality is due to the fact that $Q_{1}$ commutes with $L$. Since $Q_{1,2}$ contains at most the power $\zeta_{\nu}^{-d_{\nu}}$ and $L \frac{d z}{d \zeta_{\nu}}$ has a pole of order $d_{\nu}+1$ at $\zeta_{\nu}$, it follows that $\mathrm{ad}_{L}^{-1}$ decreases the power by $d_{\nu}+1$ (or $d_{\infty}-1$ if $\nu=\infty$ ). Therefore we obtain (recalling $U_{2}=\left(Q_{2}\right)_{\text {sing }}$ )

$$
\begin{equation*}
\operatorname{ad}_{L}^{-1}\left(U_{2}^{\prime}\right)=\operatorname{ad}_{L}^{-1}\left(Q_{2}^{\prime}\right)+\mathcal{O}\left(\zeta_{\nu}^{d_{\nu}+\delta}\right) \tag{4.50}
\end{equation*}
$$

where $\delta=-1$ for $\nu=1, \ldots, N$ and +1 for $\nu=\infty$.
We can thus replace $U_{2}^{\prime}$ by $Q_{2}^{\prime}$ in (4.49) (within, at most, an additive constant, if $\nu=\infty, k=d_{\infty}$ ), to obtain

$$
\begin{equation*}
\left[\nabla_{2} P P^{-1}, Q_{1}\right]_{s i n g}=\left[\operatorname{ad}_{L}^{-1}\left(Q_{2}^{\prime}\right), Q_{1}\right]_{\text {sing }}+\text { const } \tag{4.51}
\end{equation*}
$$

where

$$
\begin{align*}
{\left[\operatorname{ad}_{L}^{-1}\left(Q_{2}^{\prime}\right), Q_{1}\right]_{\operatorname{sing}} } & =\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{2}\right]+P \partial_{2} T^{\prime} P^{-1}\right), Q_{1}\right]_{\operatorname{sing}} \\
& =\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{2}\right]\right), Q_{1}\right]_{\text {sing }} \tag{4.52}
\end{align*}
$$

In the last step, we have used the fact that the second term in the argument of $\operatorname{ad}_{L}^{-1}$ commutes with $L$ and hence drops out.

Consider now the quantity being projected on the principal part and observe that it belongs to the range of $\operatorname{ad}_{L}$ since $Q_{1}$ commutes with $L$. (Recall that $\operatorname{ad}_{L}$ is an invertible map on matrices of this form.) Using

$$
\begin{equation*}
\operatorname{ad}_{L} Q_{j}=0 \tag{4.53}
\end{equation*}
$$

we get

$$
\begin{equation*}
\operatorname{ad}_{L}\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{2}\right]\right), Q_{1}\right] \stackrel{\left.\left(\mathrm{Jacobi}+\stackrel{[L}{ }, Q_{1}\right]=0\right)}{=}\left[\left[P^{\prime} P^{-1}, Q_{2}\right], Q_{1}\right] \stackrel{\left(\left[Q_{1}, Q_{2}\right]=0\right)}{=}\left[Q_{2},\left[P^{\prime} P^{-1}, Q_{1}\right]\right] \tag{4.54}
\end{equation*}
$$

which shows that

$$
\begin{equation*}
\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{2}\right]\right), Q_{1}\right]=\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{1}\right]\right), Q_{2}\right] \tag{4.55}
\end{equation*}
$$

Combining these results, we can write the following chain of equalities, where above each the relevant identity is indicated.

$$
\begin{align*}
{\left[\nabla_{2} P P^{-1}, Q_{1}\right]_{\text {sing }} } & \stackrel{(4.51)}{=}\left[\operatorname{ad}_{L}^{-1}\left(Q_{2}^{\prime}\right), Q_{1}\right]_{\text {sing }}+\text { const } \stackrel{(4.52)}{=}\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{2}\right]\right), Q_{1}\right]_{\text {sing }}+\text { const } \\
& \stackrel{(4.55)}{=}\left[\operatorname{ad}_{L}^{-1}\left(\left[P^{\prime} P^{-1}, Q_{1}\right]\right), Q_{2}\right]+\text { const } \stackrel{(4.52)}{=}\left[\operatorname{ad}_{L}^{-1}\left(Q_{1}^{\prime}\right), Q_{2}\right]_{\text {sing }}+\text { const } \\
& \stackrel{(4.51)}{=\stackrel{(4)}{=}}\left[\nabla_{1} P P^{-1}, Q_{2}\right]_{\text {sing }}+\text { const. } \tag{4.56}
\end{align*}
$$

This proves (4.41) and hence completes the proof for this case. The case $\nabla_{1}=\nabla_{c^{\nu}}$ is proved similarly.

### 4.3 Poisson preserving property of $\nabla$

Denote by $\mathcal{P}$ the bivector field defining the Poisson bracket (1.17)

$$
\begin{equation*}
\{f, g\}=\mathcal{P}(d f, d g) \tag{4.57}
\end{equation*}
$$

Formulae (4.6) define the vector fields $\left\{\nabla_{t_{j a}^{\nu}}, \nabla_{c^{\nu}}\right\}$ on the finite-dimensional manifold $\mathcal{L}_{r, \mathbf{d}}$ of rational matrices (3.1), and we have proved that they commute and act on the Casimir functions $\left\{t_{j a}^{\nu}, c^{\nu}\right\}$ as
we would expect from an "explicit derivative". It remains to show that these fields are infinitesimal generators of Poisson morphisms; i.e., that

$$
\begin{equation*}
\mathcal{L}_{\nabla_{t}} \mathcal{P}=0, \tag{4.58}
\end{equation*}
$$

where $\mathcal{L}_{\nabla_{t}}$ denotes the Lie derivative with respect to $\nabla_{t}$ for $t \in \mathbf{T}$. This follows from the fact that the $\nabla_{t_{j a}^{\prime}}, \nabla_{c^{\nu}}$ 's are differences between Hamiltonian vector fields and "isomonodromic" vector fields. The first automatically preserve the Poisson brackets, and the second do also (cf. e.g., Hitchin [37] and Boalch [10]). However, it is quite straightforward to show this directly, which we now do.

The Poisson invariance condition (4.58) is equivalent to the following.
Theorem 4.5. Let $t$ denote any of the isomonodromic times $t \in \mathbf{T}$ and $\nabla_{t}$ be the corresponding vector field. Then

$$
\begin{equation*}
\nabla_{t}\{f, g\}=\left\{\nabla_{t} f, g\right\}+\left\{f, \nabla_{t} g\right\} \tag{4.59}
\end{equation*}
$$

In particular, if $f, g$ are in the joint kernel of all the $\nabla_{t}$ 's, their Poisson bracket $\{f, g\}$ is also.
Proof. It is sufficient to verify (4.59) for any two linear functionals of $L$. We take linear functionals $\mathcal{X}, \mathcal{Y}$ of the form

$$
\begin{align*}
\mathcal{X}(L):=\sum_{\mu} \operatorname{res}_{c_{\mu}} \operatorname{tr}\left(X_{\mu}(z) L(z)\right) d z, & X_{\mu}(z):=\sum_{j=\epsilon_{\mu}}^{d_{\mu}+\epsilon_{\mu}} X_{\mu j} \zeta_{\mu}^{j}  \tag{4.60}\\
\mathcal{Y}(L) & :=\sum_{\mu} r_{c_{\mu}} \operatorname{rest} \operatorname{tr}\left(Y_{\mu}(z) L(z)\right) d z, \quad Y_{\mu}(z):=\sum_{j=\epsilon_{\mu}}^{d_{\mu}+\epsilon_{\mu}} Y_{\mu j} \zeta_{\mu}^{j}, \tag{4.61}
\end{align*}
$$

where $\epsilon_{\mu}=0$ for $\mu=1, \ldots, N$ and $\epsilon_{\infty}=1$. Note that $\mathcal{X}$ and $\mathcal{Y}$ do not depend on the positions of the poles, only on the matrix elements of the $L_{j}^{\mu}$ 's. For example

$$
\begin{equation*}
\mathcal{X}=\sum_{\mu} \sum_{j} \operatorname{tr}\left(X_{\mu j} L_{j}^{\mu}\right) . \tag{4.62}
\end{equation*}
$$

Furthermore we have

$$
\begin{equation*}
\{\mathcal{X}, \mathcal{Y}\}=\sum_{\mu} \underset{z=c_{\mu}}{\text { res }} \operatorname{tr}\left(\left[X_{\mu}(z), Y_{\mu}(z)\right] L(z)\right) d z \tag{4.63}
\end{equation*}
$$

For similar reasons, the bracket here is independent of the pole loci. The identity is therefore trivially satisfied by the $\nabla_{c_{\nu}}$ 's and we focus only on the action of $\nabla_{t}$ where $t$ is one of the higher Birkhoff invariants $t_{j a}^{\nu}$. Writing $L=P^{\mu} \Lambda^{\mu}\left(P^{\mu}\right)^{-1}$ we have

$$
\begin{align*}
d \mathcal{X} & =\sum_{\mu} \underset{z=c_{\mu}}{\operatorname{res}} \operatorname{tr}\left(X_{\mu}\left[d P^{\mu}\left(P^{\mu}\right)^{-1}, L\right]+X_{\mu} P^{\mu} d \Lambda^{\mu}\left(P^{\mu}\right)^{-1}\right) d z \\
& =\sum_{\mu} \underset{z=c_{\mu}}{\operatorname{res}} \operatorname{tr}\left(\left[L, X_{\mu}\right] d P^{\mu}\left(P^{\mu}\right)^{-1}+\left(P^{\mu}\right)^{-1} X_{\mu} P^{\mu} d \Lambda^{\mu}\right) d z \tag{4.64}
\end{align*}
$$

We now need to compute $d\left(\nabla_{t} \mathcal{X}\right)$. For $t$ equal to any of the parameters $\left\{t_{j a}^{\nu}\right\}$ denote, for brevity,

$$
\begin{equation*}
U_{t}:=U_{j a}^{\nu} \tag{4.65}
\end{equation*}
$$

Then

$$
\begin{equation*}
\nabla_{t}(\mathcal{X})=\sum_{\mu} \underset{z=c_{\mu}}{\text { res }} \operatorname{tr}\left(\nabla_{t}\left(L X_{\mu}\right)\right) d z=\sum_{\mu} \underset{z=c_{\mu}}{\text { res }} \operatorname{tr}\left(U_{t}^{\prime} X_{\mu}\right) d z=-\sum_{\mu} \underset{z=c_{\mu}}{\text { res }} \operatorname{tr}\left(U_{t} X_{\mu}^{\prime}\right) d z . \tag{4.66}
\end{equation*}
$$

Observe that, for $\mu \neq \nu, U_{t}$ has a pole only at $c_{\nu}$ and is analytic at $c_{\mu}$ so the only residue that contributes is at $\nu=\mu$. In this case we have

$$
\begin{equation*}
\nabla_{t}(\mathcal{X})=-\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(U_{t} X_{\nu}^{\prime}\right) d z=-\underset{z=c_{\nu}}{\text { res }} \operatorname{tr}\left(P^{\nu} \nabla_{t} T^{\nu}\left(P^{\nu}\right)^{-1} X_{\nu}^{\prime}\right) d z, \tag{4.67}
\end{equation*}
$$

where in the last step we have added the negative part of

$$
\begin{equation*}
Q_{t}:=P^{\nu} \nabla_{t}\left(T^{\nu}\right)\left(P^{\nu}\right)^{-1} \tag{4.68}
\end{equation*}
$$

which does not contribute to the residue.
Now take the differential of this function. Consider first the case when $t=t_{j a}^{\nu}$ and observe that

$$
\begin{equation*}
\nabla_{t} T^{\nu}=\frac{E_{a a}}{j \zeta_{\nu}^{j}} \tag{4.69}
\end{equation*}
$$

so that

$$
\begin{equation*}
d \nabla_{t} T^{\nu}=\frac{E_{a a}}{\zeta_{\nu}^{j+1}} d c_{\nu} . \tag{4.70}
\end{equation*}
$$

Then

$$
\begin{align*}
d\left(\nabla_{t} \mathcal{X}\right)(L)=- & \operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(\left[d P^{\nu}\left(P^{\nu}\right)^{-1}, Q_{t}\right] X_{\nu}^{\prime}+P^{\nu} \frac{E_{a a}}{\zeta_{\nu}^{j+1}} d c_{\nu}\left(P^{\nu}\right)^{-1} X_{\mu}^{\prime}\right) d z= \\
& =\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(d P^{\nu}\left(P^{\nu}\right)^{-1}\left[Q_{t}, X_{\nu}^{\prime}\right]+P^{\nu} \frac{E_{a a}}{\zeta_{\nu}^{j+1}} d c_{\nu}\left(P^{\nu}\right)^{-1} X_{\mu}^{\prime}\right) d z \tag{4.71}
\end{align*}
$$

By inspection and comparison with (4.64), we conclude that $d \nabla_{t} \mathcal{X}$ is given by

$$
\begin{equation*}
d \nabla_{t} \mathcal{X}=\operatorname{ad}_{L}^{-1}\left(\left[Q_{t}, X_{\nu}^{\prime}\right]\right)+\left(\left(P^{\nu}\right)^{-1} X_{\nu}^{\prime} P_{\nu}\right)_{a a} \frac{\left(P^{\nu}\right)^{-1} E_{a a} P^{\nu}}{\zeta_{\nu}^{j+1}} \mathrm{~d} c_{\nu} . \tag{4.72}
\end{equation*}
$$

Note that $\mathrm{ad}_{L}^{-1}$ is well defined because $\left[Q_{t}, X^{\prime}\right]$ annihilates the commutant part of $X^{\prime}$, given that $Q_{t}=$ $P \partial_{t} T P^{-1}$ belongs to the commutant subalgebra of $L$. We can now complete the computation

$$
\begin{equation*}
\left\{\nabla_{t} \mathcal{X}, \mathcal{Y}\right\}(L)+\left\{\mathcal{X}, \nabla_{t} \mathcal{Y}\right\}(L)=\underset{z=c_{\nu}}{\text { res }} \operatorname{tr}\left(L\left[d \nabla_{t} \mathcal{X}, d \mathcal{Y}\right]\right) d z+\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(L\left[d \mathcal{X}, d \nabla_{t} \mathcal{Y}\right]\right) d z . \tag{4.73}
\end{equation*}
$$

Consider the first term and observe that in the first equality below we can drop the second term in (4.72) from the trace, since it commutes with $L$ and (using the cyclicity of the trace) yields a vanishing contribution. Therefore

$$
\begin{gather*}
\left.\operatorname{res}_{z=c_{\nu}}^{\operatorname{tr}}\left(L\left[d \nabla_{t} \mathcal{X}, d \mathcal{Y}\right]\right) d z \stackrel{(4.72)}{=} \underset{z=c_{\nu}}{\operatorname{res} \operatorname{tr}} \operatorname{tL}\left[\operatorname{ad}_{L}^{-1}\left(\left[Q_{t}, X_{\nu}^{\prime}\right]\right), Y_{\nu}\right]\right) d z \\
=\operatorname{res}_{z=c_{\nu}}^{\operatorname{tr}}\left(\left[L, \operatorname{ad}_{L}^{-1}\left(\left[Q_{t}, X_{\nu}^{\prime}\right]\right)\right] Y_{\nu}\right) d z=\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(\left[Q_{t}, X_{\nu}^{\prime}\right] Y_{\nu}\right) d z=\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(Q_{t}\left[Y_{\nu}, X_{\nu}^{\prime}\right]\right) d z . \tag{4.74}
\end{gather*}
$$

Repeating the computation for the second term we have

$$
\begin{align*}
\left\{\nabla_{t} \mathcal{X}, \mathcal{Y}\right\}(L)+\left\{\mathcal{X}, \nabla_{t} \mathcal{Y}\right\}(L) & ={\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(Q_{t}\left[Y_{\nu}, X_{\nu}^{\prime}\right]\right) d z+\operatorname{res}_{z=c_{\nu}}^{\operatorname{er}} \operatorname{tr}\left(Q_{t}\left[Y_{\nu}^{\prime}, X_{\nu}\right]\right) d z} \\
=\operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(Q_{t} \frac{d}{d z}\left[Y_{\nu}, X_{\nu}\right]\right) d z={\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(Q_{t}^{\prime}\left[X_{\nu}, Y_{\nu}\right]\right) d z} & =\operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(U_{t}^{\prime}\left[X_{\nu}, Y_{\nu}\right]\right) d z=\operatorname{res}_{z=c_{\nu}}^{\operatorname{tr}}\left(\nabla_{t} L\left[X_{\nu}, Y_{\nu}\right]\right) d z \\
& =\nabla_{t}\{\mathcal{X}, \mathcal{Y}\}(L),
\end{align*}
$$

which completes the proof.

### 4.4 Isospectral flows and their Hamiltonian formulation

In this section we reprove, by direct computation, that the Hamiltonian vector fields generated by the Hamiltonian functions $H_{j a}^{\nu}, H_{c_{\nu}}, H_{a}^{\infty}$, defined in (1.43a), (1.43b), (1.44), (1.45) coincide with the vector fields $\mathbf{X}_{H_{t} j_{j a}}, \mathbf{X}_{H_{c_{\nu}}}, \mathbf{X}_{H_{a}^{\infty}}$ defined in (4.5). For example this means that

$$
\begin{equation*}
\left\{L, H_{t_{k j}^{\nu}}(L)\right\}=\left[U_{j a}^{\nu}, L\right] \tag{4.76}
\end{equation*}
$$

where the Poisson bracket is as expressed in (1.17). We recall that a compact way of expressing the Poisson bracket, using tensor product notation, is

$$
\begin{equation*}
\{\stackrel{1}{L}(z), \stackrel{2}{L}(w)\}=\left[\Pi, \frac{\stackrel{1}{L}(z)-\stackrel{1}{L}(w)}{z-w}\right] \tag{4.77}
\end{equation*}
$$

where

$$
\begin{gather*}
\Pi: \mathbb{C}^{r} \otimes \mathbb{C}^{r} \rightarrow \mathbb{C}^{r} \otimes \mathbb{C}^{r} \\
\Pi(\underline{v} \otimes \underline{w})=\underline{w} \otimes \underline{v} \tag{4.78}
\end{gather*}
$$

is the order-reversing operator.
Theorem 4.6. The Hamiltonian vector fields on $\mathcal{L}_{r, \mathbf{d}}$ corresponding to the spectral invariant Hamiltonians $H_{t_{j a}^{\nu}}, H_{t_{j a}^{\infty}}, H_{c_{\nu}}$, and $H_{a}^{\infty}$ defined in (1.43a), (1.43b), 1.44) and (1.45) take the form:

$$
\begin{align*}
\mathbf{X}_{H_{t_{j a}^{\nu}}} L(z)=\left\{L(z), H_{t_{j a}^{\nu}}\right\} & =\left[U_{j a}^{\nu}(z ; L), L(z)\right]  \tag{4.79a}\\
\mathbf{X}_{H_{c_{\nu}}} L(z)=\left\{L(z), H_{c_{\nu}}\right\} & =\left[V^{\nu}(z ; L), L(z)\right]  \tag{4.79~b}\\
\mathbf{X}_{H_{a}^{\infty}} L(z)=\left\{L(z), H_{a}^{\infty}\right\} & =\left[E_{a a}, L(z)\right] \tag{4.79c}
\end{align*}
$$

where the matrices $\left\{U_{j a}^{\nu}, V^{\nu}\right\}$ are defined either by eqs. (3.10), (3.11) in terms of the formal asymptotic expansions or, equivalently, by eqs. (4.2), (4.3). In the notation (1.20), (1.22), viewed as elements of the loop algebra $L \mathfrak{g l}(r)$ they are equal to

$$
\begin{align*}
U_{j a}^{\nu} & =-\left(d H_{t_{j a}^{\nu}}\right)_{-}, \quad V^{\nu}=-\left(d H_{c_{\nu}}\right)_{-}, \quad \nu=1, \ldots, N,  \tag{4.80a}\\
U_{j a}^{\infty} & =\left(d H_{t_{j a}^{\infty}}\right)_{+}, \quad E_{a a}=\left(d H_{a}^{\infty}\right)_{+} \tag{4.80b}
\end{align*}
$$

The Birkhoff invariants $\left\{t_{j a}^{\nu}\right\}_{\nu=1, \ldots, N, j=0,1, \ldots, d_{\nu}, a=1, \ldots, r}$ and $\left\{t_{j a}^{\infty}\right\}_{j=1, \ldots, d_{\infty}, a=1, \ldots, r}$ (i.e., including the exponents of formal monodromy at the finite poles but not those at $\infty$ ), are all Casimir elements of the Poisson bracket.

Proof. From the equality between expressions (3.32a) and (3.33a) of Theorem 3.3 for $H_{j a}^{\nu}$ and the definition of the matrix of eigenvectors

$$
\begin{equation*}
L(z)=P^{\nu}\left(\zeta_{\nu}\right) \Lambda^{\nu}\left(\zeta_{\nu}\right) P^{\nu}\left(\zeta_{\nu}\right)^{-1} \text { near to } z=c_{\nu} \tag{4.81}
\end{equation*}
$$

we have

$$
\begin{equation*}
H_{t_{j a}^{\nu}}=-\underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(L P^{\nu}\left(\zeta_{\nu}\right) \frac{1}{j \zeta_{\nu}^{j}} E_{a a}\left(P^{\nu}\left(\zeta_{\nu}\right)\right)^{-1}\right) d z \tag{4.82}
\end{equation*}
$$

We first compute the differential of $H_{j a}^{\nu}$ on the manifold $\mathcal{L}_{r, \mathbf{d}}$. Denote by

$$
\begin{equation*}
M(z, \lambda)=\widehat{\lambda \mathbf{I}-L(z)} \tag{4.83}
\end{equation*}
$$

the classical adjoint of $\lambda \mathbf{I}-L(z)$. For $(\lambda, z)$ on the spectral curve in a neighbourhood of a point $\left(\lambda_{a}\left(c_{\nu}\right), c_{\nu}\right)$ over $z=c_{\nu}$,

$$
\begin{equation*}
d \lambda_{a}=\operatorname{tr}\left(\frac{M\left(z, \lambda_{a}\right) d L(z)}{\operatorname{tr} M\left(z, \lambda_{a}\right)}\right)=\operatorname{tr}\left(P^{\nu}\left(\zeta_{\nu}\right) E_{a a} P^{\nu}\left(\zeta_{\nu}\right)^{-1} d L(z)\right) \tag{4.84}
\end{equation*}
$$

where we have used the fact that for a matrix $L$ with simple spectrum, the matrix

$$
\begin{equation*}
\frac{\widetilde{\lambda_{a} \mathbf{I}-L}}{\operatorname{tr}\left(\widetilde{\lambda_{a} \mathbf{I}-L}\right)} \tag{4.85}
\end{equation*}
$$

is the spectral projector onto the 1-dimensional eigenspace with the eigenvalue $\lambda_{a}$. Defining

$$
\begin{equation*}
Q_{j a}^{\nu}\left(\zeta_{\nu}\right):=P^{\nu}\left(\zeta_{\nu}\right) \frac{E_{a a}}{j \zeta_{\nu}^{j}} P^{\nu}\left(\zeta_{\nu}\right)^{-1} \tag{4.86}
\end{equation*}
$$

we have

$$
\begin{equation*}
d H_{t_{j a}^{\nu}}=-\operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(d L Q_{j a}^{\nu}+L\left[d P^{\nu}\left(P^{\nu}\right)^{-1}, Q_{j a}^{\nu}\right]\right) d z=-\operatorname{res}_{z=c_{\nu}} \operatorname{tr}\left(d L Q_{j a}^{\nu}\right) d z \tag{4.87}
\end{equation*}
$$

where in the last equality we have used the cyclicity of the trace and the fact that $\left[L, Q_{j a}^{\nu}\right]=0$. Therefore

$$
\begin{equation*}
d H_{t_{j a}^{\nu}}=-\operatorname{res}_{z=c_{\nu}} \frac{1}{j \zeta_{\nu}{ }^{j}} \operatorname{tr}\left(P^{\nu} E_{a a}\left(P^{\nu}\right)^{-1} d L(z)\right) d z=-\operatorname{res}_{z=c_{\nu}}^{\operatorname{tr}}\left(Q_{j a}^{\nu}\left(\zeta_{\nu}\right) d L(z)\right) d z \tag{4.88}
\end{equation*}
$$

Now, for brevity, set $H=H_{t_{j a}^{\nu}}$ and $Q(z)=Q_{j a}^{\nu}\left(\zeta_{\nu}\right)$. (We write $Q(z)$ for simplicity, but keep in mind that this is a Laurent series centered at $z=c_{\nu}$, and for $\nu=\infty$ we set $c_{\infty}=\infty$.) We then have

$$
\begin{align*}
\{L(z), H\} & =-\underset{w=c_{\nu}}{\operatorname{res}} \operatorname{tr}_{2}\{\stackrel{1}{L}(z), \stackrel{2}{L}(w)\} \stackrel{2}{Q}(w) d w=-\underset{w=c_{\nu}}{\operatorname{res}} \operatorname{tr}_{2}\left[\Pi, \frac{\stackrel{1}{L}(z)-\stackrel{1}{L}(w)}{z-w}\right] \stackrel{2}{Q}(w) d w= \\
& =\operatorname{res}_{w=c_{\nu}} \frac{[Q(w), L(z)-L(w)]}{z-w} d w=\left[\operatorname{res}_{w=c_{\nu}} \frac{Q(w) d w}{z-w}, L(z)\right] \tag{4.89}
\end{align*}
$$

where we have again used the fact that $[Q(w), L(w)]=0$ and that, for any pair of matrices $A, B$

$$
\begin{equation*}
\operatorname{tr}_{2}([\Pi, \stackrel{1}{A}] \stackrel{2}{B})=[A, B] \tag{4.90}
\end{equation*}
$$

The residue in (4.89) produces precisely the singular part of $Q(w)$ at $z=c_{\nu}$, which completes the proof of the first equation in (4.80a).

For $H_{c_{\nu}}=\frac{1}{2} \underset{z=c_{\nu}}{\operatorname{res}} \operatorname{tr}\left(L^{2}(z)\right) d z$ we have similarly

$$
\begin{equation*}
d H_{c_{\nu}}=\operatorname{res}_{z=c_{\nu}} \operatorname{tr}(L(z) d L(z)) d z \tag{4.91}
\end{equation*}
$$

so that (with $H=H_{c_{\nu}}$ )

$$
\begin{align*}
\{L(z), H\} & =\underset{w=c_{\nu}}{\operatorname{res}} \operatorname{tr}_{2}\{\stackrel{1}{L}(z), \stackrel{2}{L}(w)\} \stackrel{2}{L}(w)=\underset{w=c_{\nu}}{\operatorname{res}} \operatorname{tr}_{2}\left[\Pi, \frac{\stackrel{1}{L}(z)-\stackrel{1}{L}(w)}{z-w}\right] \stackrel{2}{L}(w)= \\
& =\underset{w=c_{\nu}}{\operatorname{res}} \frac{[L(z)-L(w), L(w)]}{z-w}=\left[-\underset{w=c_{\nu}}{\operatorname{res}} \frac{L(w)}{z-w}, L(z)\right] \tag{4.92}
\end{align*}
$$

The residue produces minus the singular part of $L(w)$ at $w=c_{\nu}$ (in the variable $z$ ) which is precisely the matrix $V^{\nu}(z)$, see (3.10), (3.11).

To prove (4.80a) and (4.80b), we need to identify $d H_{t}$ with an element of the co-tangent space to the submanifold $\mathcal{L}_{r, \mathbf{d}}$, viewed as a Poisson submanifold of $L^{*} \mathfrak{g l}(r)$. Under the pairing (1.7), $T^{*} \mathcal{L}_{r, \mathbf{d}}$ is identified with the quotient of $L \mathfrak{g l}(r)$ by the ideal $\mathfrak{I}_{\mathbf{c}, \mathbf{d}}$. We do that only for one case, leaving the remainder to the reader. Consider a finite pole $c_{\nu}, \nu \in\{1, \ldots, N\}$ and one of the Hamiltonians $H_{t_{j a}^{\nu}}$.

We pick up the computation from (4.88) and observe that the residue remains unchanged if we truncate $Q_{j a}^{\nu}$ modulo $\mathcal{O}\left(z-c_{\nu}\right)^{d_{\nu}+2}$. Provisionally denote the resulting rational function, with only one pole at $z=c_{\nu}$ and a polynomial part at $\infty$, as $\widehat{Q}_{j a}^{\nu}$, We would like to use Cauchy's theorem to deform the contour of integration from a small circle $\left|z-c_{\nu}\right|=\epsilon$ to a large circle $|z|=R$. However, in doing so, we would pick up the residues at all the other poles of $d L(z)$. To prevent this, multiply $\widehat{Q}_{j a}^{\nu}$ by a scalar polynomial $h(z)$ such that

$$
h(z)=\left\{\begin{array}{cc}
1+\mathcal{O}\left(z-c_{\nu}\right)^{2 d_{\nu}+2}, & z \rightarrow c_{\nu}  \tag{4.93}\\
\mathcal{O}\left(z-c_{\mu}\right)^{d_{\mu}+2}, & z \rightarrow c_{\mu}, \quad \mu \neq \nu .
\end{array}\right.
$$

We can then replace $Q_{j a}^{\nu}$ in the residue (4.88) with $h(z) \widehat{Q}_{j a}^{\nu}(z)$ without affecting the value of the residue. But now we can use Cauchy's theorem, since $d L(z) h(z) \widehat{Q}_{j a}^{\nu}(z)$ has only one finite pole at $z=c_{\nu}$. Thus we have shown that $d H_{j a}^{\nu}$ can be identified with $-h(z) \widehat{Q}_{j a}^{\nu}(z)$ in the cotangent space, where the minus sign is due to the sign in front of the residue (4.88). Finally, from the properties of $h$, it follows that the projection of $h(z) \widehat{Q}_{j a}^{\nu}(z)$ in $L_{-} \mathfrak{g l}(r)$ is exactly the same as the principal part of $Q_{j a}^{\nu}$ at $z=c_{\nu}$. That is,

$$
\begin{equation*}
R_{0}\left(d H_{j a}^{\nu}\right)=-\left(-h(z) \widehat{Q}_{j a}^{\nu}\right)_{-}=\left(Q_{j a}^{\nu}\right)_{s i n g}=U_{j a}^{\nu} \tag{4.94}
\end{equation*}
$$

For the case $\nu=\infty$, since the residue formula for $H_{t_{j a}^{\infty}}$ involves minus the residue at $\infty$ (which is a positively oriented contour integral along a circle), we find that $d H_{t_{j a}^{\infty}}$ is identified simply with the Laurent expansion of $Q_{j a}^{\infty}(z)$ ad then

$$
\begin{equation*}
R_{1}\left(d H_{t_{j a}^{\infty}}\right)=\left(Q_{j a}^{\infty}\right)_{+}=U_{j a}^{\infty} \tag{4.95}
\end{equation*}
$$

To prove that the Birkhoff invariants are Casimir elements we proceed similarly. Consider $t_{j a}^{\nu}$ as in (1.41), (1.42). (This computation includes the exponents of formal monodromy at the finite poles.) Then

$$
\begin{equation*}
\mathrm{d} t_{j a}^{\nu}=-\operatorname{res}_{z=c_{\nu}}^{\operatorname{es}} \operatorname{tr}(W(z) d L(z)) d z, \quad W:=\zeta_{\nu}^{j} P^{\nu}\left(\zeta_{\nu}\right) E_{a a} P^{\nu}\left(\zeta_{\nu}\right)^{-1} \tag{4.96}
\end{equation*}
$$

We now proceed similarly to the proof of (4.89):

$$
\begin{align*}
\left\{L(z), t_{j a}^{\nu}\right\} & =-\underset{w=c_{\nu}}{\operatorname{res}} \operatorname{tr}_{2}\{\stackrel{1}{L}(z), \stackrel{2}{L}(w)\} \stackrel{2}{W}(w) d w=-\underset{w=c_{\nu}}{\operatorname{res}} \operatorname{tr}_{2}\left[\Pi, \frac{\stackrel{1}{L}(z)-\stackrel{1}{L}(w)}{z-w}\right] \stackrel{2}{W}(w) d w= \\
& =\operatorname{res}_{w=c_{\nu}} \frac{[W(w), L(z)-L(w)]}{z-w} d w=\left[\operatorname{res}_{w=c_{\nu}} \frac{W(w) d w}{z-w}, L(z)\right] \tag{4.97}
\end{align*}
$$

Since $W(w)$ is locally analytic at $z=c_{\nu}$, the residue in (4.97) is the null matrix and the Poisson bracket vanishes for all $t_{j a}^{\nu}$, which are therefore all Casimir elements.

Note that for $\nu=\infty$ and $j=0$ we have $t_{0 a}^{\infty}=H_{a}^{\infty}$, and these are not Casimir elements. However the same computation shows that in this case the residue equals the constant matrix $E_{a a}$, proving (4.79c) and showing that the exponents of formal monodromy at $\infty$ are indeed the Hamiltonian generators of the group of invertible diagonal constant matrices, acting by conjugation on $L$.

## 5 Further discussion and open problems

### 5.1 Birkhoff fibration

To further clarify the meaning of the Birkhoff connection, denote the space of loci of the finite, distinct poles of the rational Lax matrices in $\mathcal{L}_{r, \mathbf{d}}$ by

$$
\begin{equation*}
\mathbb{C}_{\Delta}^{N}:=\left\{\mathbf{c}:=\left(c_{1}, \ldots, c_{N}\right),\left\{c_{\nu} \neq c_{\mu}, \text { for } \nu \neq \mu\right\}\right. \tag{5.1}
\end{equation*}
$$

the space of $\left(\sum_{\nu=1}^{N} d_{\nu}+d_{\infty}\right)$-tuples of diagonal $r \times r$ matrices by

$$
\begin{equation*}
\prod_{\nu=1 . . N, \infty}(\mathfrak{h})^{d_{\nu}-1} \times \mathfrak{h}_{r e g}:=\left\{T_{j}^{\nu} \in \mathfrak{h}\right\}_{j=1 \ldots d_{\nu}, \nu=1, \ldots, N, \infty, \text { with } T_{d_{\nu}}^{\nu} \in \mathfrak{h}_{r e g}, ~}^{\text {, }} \tag{5.2}
\end{equation*}
$$

where $T_{d_{\nu}}^{\nu} \in \mathfrak{h}_{\text {reg }}$ has distinct eigenvalues, and the Cartesian product of these by

$$
\begin{equation*}
\mathcal{T}:=\mathbb{C}_{\Delta}^{N} \times \prod_{\nu=1 . . N, \infty}(\mathfrak{h})^{d_{\nu}-1} \times \mathfrak{h}_{\text {reg }} \tag{5.3}
\end{equation*}
$$

The map $\Phi: \mathcal{L}_{r, \mathbf{d}} \rightarrow \mathcal{T}$ that associates to each $L \in \mathcal{L}_{r, \mathbf{d}}$ the loci of its poles and the (higher) Birkhoff invariants is surjective, and the fibers are the union of the symplectic leaves over all values of the residual Casimir invariants $\left(T_{0}^{1}, \ldots, T_{0}^{N}\right) \in(\mathfrak{h})^{N}$. If all $d_{\nu}$ 's are assumed $>1$, this realizes $\mathcal{L}_{r, \mathbf{d}}$ within the open dense stratum of generic symplectic leaves as a fiber bundle over $\mathcal{T}$, with fibers isomorphic to the space given by

$$
\begin{align*}
& \Phi^{-1}(\{\mathbf{c}, \mathbf{t}\})=\left\{P^{\infty}(\zeta)=\left(\mathbf{I}+\sum_{j=1}^{d_{\infty}-1} F_{j}^{\infty} \zeta^{j}\right) \in \mathfrak{G} \mathfrak{l}(r)[[\zeta]] / \operatorname{Diag}(r)[[\zeta]] \bmod \zeta^{d_{\infty}}\right\} \\
& \times \prod_{j=1}^{N}\left\{P^{\nu}(\zeta)=G^{\nu}\left(\mathbf{I}+\sum_{j=1}^{d_{\nu}} F_{j}^{\nu} \zeta^{j}\right) \in \mathfrak{G} \mathfrak{l}(r)[[\zeta]] / \operatorname{Diag}(r)[[\zeta]] \bmod \zeta^{d_{\nu}+1}\right\} \times \mathfrak{h}^{N}, \tag{5.4}
\end{align*}
$$

where $\operatorname{Diag}(r)[[\zeta]]$ denotes the subgroup of formal series of $r \times r$ diagonal matrices

$$
\begin{equation*}
D(\zeta)=D_{0}+D_{1} \zeta+\ldots, \quad \operatorname{det} D_{0} \neq 0 \tag{5.5}
\end{equation*}
$$

acting by right multiplication. (If some of the $d_{\nu}=0$, then as many terms in the last factor $\mathfrak{h}^{N}$ should be replaced by $\mathfrak{h}_{\text {reg }}$ ).

Given $(\mathbf{c}, \mathbf{t}) \in \mathcal{T}$ and an element $\left(P^{\infty}, P^{1}, \ldots, P^{N}\right) \times\left(T_{0}^{1}, \ldots, T_{0}^{N}\right)$, we recover the matrix $L(z) \in \mathcal{L}_{r, \mathbf{d}}$ from the formula

$$
\begin{align*}
L(z)= & \left(P^{\infty}\left(\frac{1}{z}\right)\left(\sum_{j=1}^{d_{\infty}} T_{j}^{\infty} z^{j-1}\right) P^{\infty}\left(\frac{1}{z}\right)^{-1}\right)_{+}^{+} \\
& +\sum_{\nu=1}^{N} \sum_{j=1}^{N}\left(P^{\nu}\left(\zeta_{\nu}\right)\left(-\sum_{j=1}^{d_{\nu}} \frac{T_{j}^{\nu}}{\left(z-c_{\nu}\right)^{j+1}}+\frac{T_{0}^{\nu}}{z-c_{\nu}}\right) P^{\nu}\left(\zeta_{\nu}\right)^{-1}\right)_{\operatorname{sing}} \tag{5.6}
\end{align*}
$$

Thus $\frac{\partial}{\partial t} \rightarrow \nabla_{t}$ lifts the tangent vectors from $T \mathcal{T}$ to $T \mathcal{L}_{r, \mathbf{d}}$ as a flat connection preserving the Poisson structure.

### 5.2 Quotient manifold and deautonomization

Let

$$
\begin{equation*}
\mathfrak{T}:=\operatorname{Span}\left\{\nabla_{t}, t \in \mathbf{T}\right\} \tag{5.8}
\end{equation*}
$$

Proposition 5.1. $\mathfrak{T}$ is an integrable distribution of constant, maximal rank

$$
\begin{equation*}
N+r \sum_{\nu=1}^{N} d_{\nu}+r d_{\infty} \tag{5.9}
\end{equation*}
$$

and the canonical projection $\pi: \mathcal{L}_{r, \mathbf{d}} \rightarrow \mathcal{W}=\mathcal{L}_{r, \mathbf{d}} / \mathfrak{T}$ is Poisson.
Proof. The integrability follows from Theorem 4.2. The rank condition is almost obvious, but we provide a proof nevertheless. Suppose, on the contrary, that there exists $L_{0}(z) \in \mathcal{L}_{r, \mathbf{d}}$ such that that the vector fields $\nabla_{t_{j a}^{\nu}}$ are linearly dependent in $T_{L_{0}} \mathcal{L}_{r, \mathbf{d}}$. This implies that there are Laurent polynomial diagonal matrices $D_{\nu}\left(\zeta_{\nu}\right)$ with poles of degrees $\leq d_{\nu}$ such that

$$
\begin{equation*}
\sum_{\nu=1, \ldots, N, \infty} \frac{d}{d z}\left(P^{\nu}\left(\zeta_{\nu}\right) D_{\nu}\left(\zeta_{\nu}\right)\left(P^{\nu}\right)^{-1}\left(\zeta_{\nu}\right)\right)_{\operatorname{sing}} \equiv 0 \tag{5.10}
\end{equation*}
$$

Since each term in the sum is rational, with a single pole at $z=c_{\nu}$, they must separately vanish. By definition of the projection $(\cdot)_{\operatorname{sing}}$ at the finite poles $c_{\nu}, \nu=1, \ldots, N$, the terms are Laurent polynomials without constant term, but then $D_{\nu}\left(\zeta_{\nu}\right) \equiv 0$ (since the projection cannot result in a constant). For $\nu=\infty$, eq. (5.10) implies that $\left(P^{\infty} D_{\infty}(z)\left(P^{\infty}\right)^{-1}\right)_{\operatorname{sing}}$ should be a constant, which is possible only if $D_{\infty}$ is a constant diagonal matrix. However this is not possible because $\nabla T^{\infty}$ is either a polynomial without constant coefficient or zero.

Now consider the second statement. This is equivalent to saying that the algebra of $\nabla_{t}$ invariant functions is a Poisson subalgebra, which follows from Theorem 4.5.

The dimension of $\mathfrak{T}$ is

$$
\begin{equation*}
\operatorname{dim} \mathfrak{T}=N+r d_{\infty}+r \sum_{\nu=1}^{N} d_{\nu} \tag{5.11}
\end{equation*}
$$

while the dimension of $\mathcal{L}_{r, \mathbf{d}}$ is

$$
\begin{equation*}
\operatorname{dim} \mathcal{L}_{r, \mathbf{d}}=N+r+r^{2}\left(d_{\infty}-1\right)+r^{2} \sum_{\nu=1}^{N}\left(d_{\nu}+1\right) \tag{5.12}
\end{equation*}
$$

Recall that the fibers of the projection $\mathcal{L}_{r, \mathbf{d}} \rightarrow \mathcal{T}$ are unions of symplectic leaves over the $r N$ values of the Casimir functions $\left\{t_{0 a}^{\nu}\right\}_{\substack{\nu=1, \ldots, N \\ a=1, \ldots r}}^{\substack{\text {. }}}$ (the exponents of formal monodromy). There are an additional

$$
\begin{align*}
2 K+r N & :=\operatorname{dim} \mathcal{L}_{r, \mathbf{d}}-\operatorname{dim} \mathfrak{T}=r+r^{2}\left(d_{\infty}-1\right)+r^{2} \sum_{\nu=1}^{N} d_{\nu}-r d_{\infty}-r \sum_{\nu=1}^{N}\left(d_{\nu}+1\right) \\
& =r(r-1)\left(d_{\infty}+\sum_{\nu=1}^{N} d_{\nu}+N-1\right)+r N \tag{5.13}
\end{align*}
$$

functionally independent $\nabla$ invariants. These include the $r N$ exponents of formal monodromy $\left\{t_{0 a}^{\nu}\right\}_{\nu=1, \ldots, N, \infty, a=1, \ldots, r}$ at the finite poles, which we denote as

$$
\begin{equation*}
\mathcal{T}_{0}:=\left\{t_{0 a}^{\nu}, t_{0 a}^{\nu} \neq t_{0 b}^{\nu} \text { if } d_{\nu} \neq 0 \text { and } a \neq b\right\}_{\nu=1, \ldots N, a, b=1, \ldots r} \tag{5.14}
\end{equation*}
$$

Setting both these and all elements of $\mathcal{T}$ equal to constants, we obtain the symplectic leaves, which are of dimension $2 K$. We may therefore choose a complementary set of coordinates consisting of $\nabla$-invariant coordinates on the symplectic leaves, which we denote

$$
\begin{equation*}
\mathbf{W}:=\left(w_{1}, \ldots, w_{2 K}\right) \tag{5.15}
\end{equation*}
$$

Using the Riemann-Hurwitz formula, we can verify that the dimension $2 K$ of the symplectic leaf is related to the genus $g$ of generic spectral curves $\mathcal{C}$ by

$$
\begin{equation*}
K=g+r-1 \tag{5.16}
\end{equation*}
$$

We thus have, at least locally, a full set of complementary (holomorphic) functions $\left\{w_{\alpha}, t_{0 a}^{\nu}\right\}_{\substack{\alpha=1, \ldots, 2 K \\ \nu=1, \ldots, N \\ a=1, \ldots r}}^{\substack{1 \\ \hline}}$ which, together with the isomonodromic deformation parameters $\left\{t_{j a}^{\nu}, c_{\nu}\right\}_{\substack{\nu=1, \ldots, N, \infty \\ j=1, \ldots, r \\ j=1, \ldots d_{\nu}}}$, form a (local) coordinate system on $\mathcal{L}_{r, \mathbf{d}}$. Note that the first $r$ of these $\left(w_{1}, \ldots w_{r}\right)$ may be chosen to be the exponents of formal monodromy $\left\{t_{0 a}^{\infty}=H_{a}^{\infty}\right\}_{a=1, \ldots, r}$ at $\infty$, which all Poisson commute amongst themselves, and with all the spectral invariants $\left\{H_{j a}^{\nu}\right\}$, and these can be completed to form Darboux coordinate systems, at least locally, on the symplectic leaves.

### 5.3 Isomonodromic deformations as deautonomization of isospectral flows.

Suppose that we have found a full set of $2 K+r N$ independent $\nabla$-invariants $\left(\mathbf{W}, \mathcal{T}_{0}\right)$, including the exponents of formal monodromy $\mathcal{T}_{0}$.

If $w$ is a $\nabla$-invariant function then the isomonodromic equations read

$$
\begin{equation*}
\frac{\partial}{\partial t} w=\underbrace{\nabla_{t} w}_{=0}+\left\{w, H_{t}\right\}, \quad \forall t \in \mathbf{T} \tag{5.17}
\end{equation*}
$$

with the Hamiltonians $H_{t}$ defined in (1.43a), (1.43b), (1.44). This means that the isomonodromic equations would then take standard non-autonomous Hamiltonian form (since the Hamiltonians also depend explicitly on the isomonodromic times).

The question that remains is how to determine such invariants explicitly and use them to parametrize the Lax matrix $L(z)$ canonically. This means completing the transverse Birkhoff invariants and pole loci with $\nabla$ invariants so as to form a (local) coordinate system on the phase space, which is canonical on the symplectic leaves, thereby simultaneously implementing the transversal foliation and the symplectic one as a local product.

This has been resolved in special cases in the literature; e.g., in [34] for the six Painlevé equations. In [35] it was done for extensions of the $P_{I I}$ system to systems of isomonodromic deformation equations in which the $\mathfrak{s l}(2)$-valued Lax matrix consists of the sum of a first degree polynomial part plus any number first order poles. In [52] it was done for the $P_{I I}$ hierarchy, represented as zero curvature equations involving $\mathfrak{s l}(2)$-valued Lax matrices $L(z)$ that are polynomials in $z$ plus a fixed first order pole at $z=0$, that satisfy the involutive symmetry

$$
\begin{equation*}
L(-z)=\sigma_{1} L(z) \sigma_{1} \tag{5.18}
\end{equation*}
$$

The system was shown to consist of nonautonomous deformations of Hamiltonian equations with respect to the rational classical $R$-matrix structure on loop algebras, with the Lax matrix parametrized by a combination of the Casimir invariants of higher Birkhoff type, and $\nabla$-invariants that are chosen to form a Darboux coordinate system.

It is not evident how to generalize these constructions even to arbitrary polynomial $\mathfrak{s l}(2)$-valued $L(z)$. A further example, however, given in Appendix A, does this for the case of cubic polynomials, suggesting that it may be possible for all $\mathfrak{s l}(2)$-valued polynomial $L(z)$ 's, and perhaps more generally, for all rational ones.

## A Example: Traceless polynomial $L(z)$ with $r=2, d_{\infty}=4$

Consider the case of a polynomial traceless Lax matrix $L(z) \in \mathfrak{s l}_{2}$ with $d_{\infty}=4$. Since the traces are all Casimir elements, the above proofs hold when $L(z)$ is traceless; only the dimension count is slightly different. An explicit parametrization in terms of canonical coordinates is given as follows:

$$
L=\left(t_{4} z^{3}+t_{3} z^{2}+\left(t_{2}-\sqrt{t_{4}} y_{2} x_{1}\right) z-\sqrt[4]{t_{4}}\left(x_{1} y_{3}+x_{3} y_{2}\right)+t_{1}-\frac{t_{3} x_{1} y_{2}}{3 \sqrt{t_{4}}}\right) \sigma_{3}
$$

$$
\begin{align*}
& +\sqrt{2}\left(-t_{4}{ }^{3 / 4} x_{1} z^{2}-\left(\sqrt{t_{4}} x_{3}+\frac{2 t_{3} x_{1}}{3 \sqrt[4]{t_{4}}}\right) z+\frac{1}{4} \sqrt[4]{t_{4}} x_{1}{ }^{2} y_{2}-\frac{t_{3} x_{3}}{3 \sqrt{t_{4}}}-\sqrt[4]{t_{4}} x_{2}-\frac{t_{2} x_{1}}{2 \sqrt[4]{t_{4}}}+\frac{t_{3}{ }^{2} x_{1}}{18 t_{4}{ }^{5 / 4}}\right) \sigma_{+} \\
& +\sqrt{2}\left(-t_{4}{ }^{3 / 4} y_{2} z^{2}-\left(\sqrt{t_{4}} y_{3}+\frac{2 t_{3} y_{2}}{3 \sqrt[4]{t_{4}}}\right) z+\frac{1}{4} \sqrt[4]{t_{4}} y_{2}{ }^{2} x_{1}-\frac{t_{3} y_{3}}{3 \sqrt{t_{4}}}-\sqrt[4]{t_{4}} y_{1}-\frac{t_{2} y_{2}}{2 \sqrt[4]{t_{4}}}+\frac{t_{3}{ }^{2} y_{2}}{18 t_{4}{ }^{5 / 4}}\right) \sigma_{-} \tag{A.1}
\end{align*}
$$

where

$$
\begin{equation*}
t_{j}=t_{j 1}^{\infty}=-t_{j 2}^{\infty}, \quad j=1, \cdots, 4 \tag{A.2}
\end{equation*}
$$

are the Casimir elements defined in (1.41b), and $\left(x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, y_{3}\right)$ are Darboux coordinates on the symplectic leaves.

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=\left\{y_{i}, y_{j}\right\}=0, \quad\left\{x_{i}, y_{j}\right\}=\delta_{i j}, \quad i, j,=1,2,3 \tag{A.3}
\end{equation*}
$$

(These coordinates were found by an explicit computation of the $\nabla_{t_{i}}$-invariants using a computer algebra system.)

The exponent of formal monodromy is

$$
\begin{equation*}
t_{0}^{\infty}:=-\operatorname{res}_{z=\infty} \sqrt{-\operatorname{det} L(z)} d z=a:=x_{1} y_{1}+x_{2} y_{2}+x_{3} y_{3} \tag{A.4}
\end{equation*}
$$

which is a constant of motion that Poisson commutes with all other spectral invariants, but is not a Casimir invariant. The Hamiltonian vector field corresponding to $a$ is given by the commutator

$$
\begin{equation*}
\mathbf{X}_{a}(L)=\{L, a\}=\left[\sigma_{3}, L\right] \tag{A.5}
\end{equation*}
$$

and the flow it generates is the group of scaling transformations, given by conjugation with invertible diagonal matrices

$$
\begin{align*}
f_{s}:\left\{x_{i}, y_{i}, t_{a}\right\} & \rightarrow\left\{e^{s} x_{i}, e^{-s} y_{i}, t_{a}\right\}, \quad i=1,2,3, \quad a=1,2,3,4,  \tag{A.6}\\
f_{s}: L(z) & \rightarrow e^{s \sigma_{3}} L(z) e^{-s \sigma_{3}} . \tag{A.7}
\end{align*}
$$

It follows by direct computation that

$$
\begin{equation*}
\partial_{t_{j}} L(z)=U_{j}^{\prime}(z)=\frac{\mathrm{d}}{d z}\left(\frac{z^{j}}{j} P \sigma_{3} P^{-1}\right)_{+}, \quad j=1, \ldots, 4 \tag{A.8}
\end{equation*}
$$

The affine symmetry group of dilations and translations in $z$ acts on the Lax matrix $L(z)$ as follows

$$
\begin{align*}
& c \widetilde{L}(c z+b, \widetilde{\mathbf{t}})=L(z, \mathbf{t})  \tag{A.9}\\
& \left(\widetilde{t}_{1}, \widetilde{t}_{2}, \widetilde{t}_{3}, \widetilde{t}_{4}\right):=\left(c\left(t_{1}+b t_{2}+b^{2} t_{3}+b^{3} t_{4}\right), c^{2}\left(t_{2}+2 b t_{3}+3 b^{2} t_{4}\right), c^{3}\left(t_{3}+3 t_{4} b\right), c^{4} t_{4}\right) \tag{A.10}
\end{align*}
$$

Choosing $c=t_{4}^{-\frac{1}{4}}$ and $b=-\frac{t_{3}}{3 t_{4}}$, this has the same effect as setting $t_{4}=1, t_{3}=0$, reducing $L(z)$ to

$$
\begin{align*}
L(z) & =\left(z^{3}+\left(t_{2}-x_{1} y_{2}\right) z-x_{1} y_{3}-x_{3} y_{2}+t_{1}\right) \sigma_{3} \\
& -\sqrt{2}\left(x_{1}\left(z^{2}+\frac{t_{2}}{2}\right)+x_{3} z+x_{2}-\frac{1}{4} y_{2} x_{1}^{2}\right) \sigma_{+} \\
& -\sqrt{2}\left(y_{2}\left(z^{2}+\frac{t_{2}}{2}\right)+y_{3} z+y_{1}-\frac{1}{4} x_{1} y_{2}^{2}\right) \sigma_{-} . \tag{A.11}
\end{align*}
$$

There are therefore only two relevant isomonodromic deformation parameters, $\left(t_{1}, t_{2}\right)$. Expanding the eigenvalue $\lambda=\sqrt{-\operatorname{det} L}$ (on one of the sheets)

$$
\begin{equation*}
\lambda=z^{3}+t_{2} z+t_{1}+\frac{a}{z}+\frac{H_{1}}{2 z^{2}}+\frac{H_{2}}{2 z^{3}}+\mathcal{O}\left(z^{-4}\right) \tag{A.12}
\end{equation*}
$$

where

$$
\begin{align*}
H_{1}= & \frac{3}{2} x_{1}^{2} y_{2} y_{3}+\left(\frac{3}{2} x_{3} y_{2}^{2}-2 t_{1} y_{2}-t_{2} y_{3}\right) x_{1}+2 x_{2} y_{3}+\left(2 y_{1}-y_{2} t_{2}\right) x_{3}  \tag{A.13}\\
H_{2}= & \frac{1}{16} x_{1}^{3} y_{2}^{3}+\left(\frac{1}{2} y_{3}^{2}-\frac{1}{4} t_{2} y_{2}^{2}-\frac{5}{4} y_{1} y_{2}\right) x_{1}^{2}+\left(\frac{1}{2} t_{2} y_{1}+\left(\frac{1}{4} t_{2}^{2}+t_{0}\right) y_{2}-\frac{5}{4} x_{2} y_{2}^{2}-t_{1} y_{3}\right) x_{1} \\
& +\left(\frac{1}{2} y_{2} t_{2}+y_{1}\right) x_{2}+\frac{1}{2} x_{3}^{2} y_{2}^{2}-t_{1} x_{3} y_{2}-t_{2} a . \tag{A.14}
\end{align*}
$$

are the nonautonomous Hamiltonians $H_{1}:=H_{t_{11}^{\infty}}, H_{2}:=H_{t_{21}^{\infty}}$ obtained from formulae (1.43b), and

$$
\begin{equation*}
a=H_{1}^{\infty}=-H_{2}^{\infty} \tag{A.15}
\end{equation*}
$$

is the exponent of formal monodromy at $z=\infty$. The deformation matrices $U_{1}, U_{2}$ then become

$$
U_{1}=\left[\begin{array}{cc}
z & -\sqrt{2} x_{1}  \tag{A.16}\\
-\sqrt{2} y_{2} & -z
\end{array}\right], \quad U_{2}=\frac{1}{2}\left[\begin{array}{cc}
-x_{1} y_{2}+z^{2} & -\sqrt{2}\left(x_{1} z+x_{3}\right) \\
-\sqrt{2}\left(y_{2} z+y_{3}\right) & x_{1} y_{2}-z^{2}
\end{array}\right] .
$$

In these coordinates, the isomonodromic deformation equations are simply Hamiltonian's equations for the time-dependent Hamiltonians $H_{1}$ and $H_{2}$, modified by the "explicit" dependence of $L(z)$ on the deformation parameters $\left(t_{1}, t_{2}\right)$.

Defining

$$
\begin{align*}
& x_{1}:=u_{1} \mathrm{e}^{w}, \quad x_{2}:=u_{2} \mathrm{e}^{w}, \quad x_{3}:=\mathrm{e}^{w}, \\
& y_{1}:=v_{1} \mathrm{e}^{-w}, \quad y_{2}:=v_{2} \mathrm{e}^{-w}, \quad y_{3}:=\left(a-u_{1} v_{1}-u_{2} v_{2}\right) \mathrm{e}^{-w}, \tag{A.17}
\end{align*}
$$

the canonical 1-form becomes

$$
\begin{equation*}
\theta=\sum_{i=1}^{3} y_{i} d x_{i}=v_{1} d u_{1}+v_{2} d u_{2}+a d w \tag{A.18}
\end{equation*}
$$

so the coordinate change from $\left(x_{1}, y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right)$ to $\left(u_{1}, v_{1}, u_{2}, v_{2}, w, a\right)$ is canonical. The conserved quantity $a$, defined in (A.15), that generates the scaling symmetry (A.6), (A.7) may also be expressed as

$$
\begin{equation*}
a=\frac{1}{4} \operatorname{res}_{z=0} z^{-3} \operatorname{tr}\left(L^{2}(z)\right)-\frac{1}{2} t_{2}^{2} \tag{A.19}
\end{equation*}
$$

and can be set equal to any constant value. The scaling flow (A.6), (A.7) that it generates is just translation of the canonically conjugate position variable $w$ :

$$
\begin{equation*}
f_{s}:\left(u_{1}, v_{1}, u_{2}, v_{2}, w, a\right) \rightarrow\left(u_{1}, v_{1}, u_{2}, v_{2}, w+s, a\right) \tag{A.20}
\end{equation*}
$$

which is an ignorable canonical coordinate for all Hamiltonians in the ring of spectral invariants of $L(z)$ :
The reduced Hamiltonians are

$$
H_{1}=\left(\frac{3}{2} v_{2} u_{1}^{2}-t_{2} u_{1}+2 u_{2}\right) a-2 t_{1} u_{1} v_{2}+\left(u_{1}^{2} v_{1}+u_{1} u_{2} v_{2}-v_{2}\right) t_{2}
$$

$$
\begin{align*}
& -\frac{3}{2} u_{1}^{3} v_{1} v_{2}-\frac{3}{2} u_{1}^{2} u_{2} v_{2}^{2}-2 u_{1} u_{2} v_{1}+\frac{3}{2} u_{1} v_{2}^{2}-2 u_{2}^{2} v_{2}+2 v_{1}  \tag{A.21}\\
H_{2}= & \frac{1}{2} a^{2} u_{1}^{2}+\left(-u_{1} t_{1}-t_{2}-u_{1}\left(u_{1}^{2} v_{1}+u_{1} u_{2} v_{2}-v_{2}\right)\right) a+\left(u_{1}^{2} v_{1}+u_{1} u_{2} v_{2}-v_{2}\right) t_{1}+ \\
& +\frac{1}{4} t_{2}^{2} u_{1} v_{2}+\left(-\frac{1}{4} v_{2}^{2} u_{1}^{2}+\frac{1}{2} u_{1} v_{1}+\frac{1}{2} u_{2} v_{2}\right) t_{2}+\frac{1}{2} u_{1}^{4} v_{1}^{2}+u_{1}^{3} u_{2} v_{1} v_{2}+\frac{1}{16} v_{2}^{3} u_{1}^{3}+ \\
& +\frac{1}{2} u_{1}^{2} u_{2}^{2} v_{2}^{2}-\frac{5}{4} u_{1}^{2} v_{1} v_{2}-\frac{5}{4} u_{1} u_{2} v_{2}^{2}+\frac{1}{2} v_{2}^{2}+u_{2} v_{1} . \tag{A.22}
\end{align*}
$$

The isomonodromic deformation equations are then Hamiltonian's equations for the time-dependent Hamiltonians $H_{1}$ and $H_{2}$.

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[^1]:    *We now recognize these as the classical limit of the general rank-r Gaudin systems [23], which were studied much later as models for quantum integrable spin chains.

[^2]:    ${ }^{\dagger}$ The notation $\frac{\partial^{0}}{\partial c_{\nu}^{0}}$ for the "explicit derivatives" will be changed to $\nabla_{c_{\nu}}$ in what follows and, more generally, $\nabla_{t}:=\frac{\partial^{0}}{\partial t^{0}}$ for the further deformation parameters $\left\{t=t_{j a}^{\nu}\right\}$ to be introduced in (1.41a), (1.41b) below.

[^3]:    ${ }^{\ddagger}$ The various local Laurent series $\left\{\lambda_{a}(z)\right\}$ near $\left\{z=c_{\nu}\right\}_{\nu=1, \ldots, N, \infty}$ depend, of course, on $\nu$ as well, but we omit indicating this explicitly to avoid a plethora of indices.

[^4]:    ${ }^{\S}$ The notation $(\cdot)_{\operatorname{sing}}$ means the principal part at a particular point $c_{\nu} \in \mathbb{P}^{1}$, which should be clear from the context or, if not, will be specified.

