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Frölicher structures, diffieties, and a formal KP hierarchy

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This paper is dedicated to Alexandre M. Vinogradov, whose breakthrough insights are at the core of the concepts utilized herein

Abstract. We propose a definition of a diffiety based on the theory of Frölicher structures. As a consequence, we obtain a natural Vinogradov sequence and, under the assumption of the existence of a suitable derivation, we can form on it a Kadomtsev-Petviashvili hierarchy which is well-posed.

1. Introduction

The Erlangen Program of F. Klein and works of S. Lie became a base of the "Geometric Revolution" in the classical XIX century approach to differential equations (DEs). The next stage of this revolution was manifested by applications to DEs of powerful algebro-geometric and algebro-topological methods developed by the French School (E. and H. Cartan, Ch. Ehresmann, J. Leray, J.-P. Serre and of course A. Grothendieck). Other very similar or close (differential-geometric) toolbox was developed in Japan by K. Kodaira and M. Kuranishi and in US by D. Spencer, H. Goldschmidt, D. Quillen, V. Guillemin and S. Sternberg.

A. M. Vinogradov (whose scientific start and first twenty years of work coincided with the so-called "Golden Age" of the Moscow Mathematical School) was probably the first person who unified successfully two sides of algebro-geometric instrumentaria in his approach to the study of (non-)linear Partial Differential Equations ((N)PDEs). He developed and combined ideas of Grothendieck and the differential-geometric viewpoint of Ehresmann, and introduced a category whose objects he called *diffieties (differential varieties)* to study *infinitely prolonged differential equations*. He also proposed a theory for the study of diffieties, what has became known as the *secondary calculus*, see [**45**]. From a technical point of view, the main ingredients of secondary calculus are an appropriate mixture of commutative and homological algebra with differential geometry. The concept of a diffiety plays the same role in the theory of PDEs that affine algebraic varieties do in the theory of algebraic equations. Various natural characteristics of a diffiety and,

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consequently, of the corresponding system of PDEs, are expressed in terms of this calculus and vice versa.

A.M. Vinogradov himself says the following about the appearance of the notion of a diffiety (see [**46**]):

> ... it was understood that various natural differential operators and constructions that are necessary for the study of a system of PDEs of order k do not live necessarily on the k–th order jet space, but involve jet spaces of any order. This is equivalent to say that a conceptually complete theory of PDEs is possible only on *infinite order* jet spaces. A logical consequence of this fact is that objects of the *category of partial differential equations* are *diffieties*, which duly formalize the vague idea of the "space of all solutions" of a PDE. Diffieties are a kind of infinite dimensional manifolds, and the specific differential calculus on them, called *secondary calculus*, is a native language to deal with PDEs and especially with NPDEs.

Roughly speaking, for each differential equation $\mathcal{E}$, the basic geometric object $\mathcal{E}^\infty$, called "diffiety", is introduced. This is the infinitely prolonged equation of the PDE $\mathcal{E}$ or, the space of (pointwise!) formal solutions of $\mathcal{E}$, and it is endowed with the Cartan distribution $\mathcal{C}$, which is integrable. The dimension of this distribution is equal to the number of independent variables of the given PDE, while local solutions of the system correspond to "integral submanifolds" of $\mathcal{C}$. This structure is induced by infinite prolongations of PDE, and it is based on them. The vector fields on $\mathcal{E}^\infty$ which preserve $\mathcal{C}$ are called $\mathcal{C}-$fields, and the infinitesimal symmetries of $\mathcal{E}$ are defined to be the equivalence classes of $\mathcal{C}-$fields modulo vector fields that are tangent to $\mathcal{C}$.

From an algebraic point of view, diffieties are spectra of scalar differential invariants (see examples in [**45**]). One can think about them as about infinite-dimensional varieties enabled with a finite dimensional involutive Cartan distribution. By means of this distribution, the de Rham algebra on a diffiety can be used to define a spectral sequence (so-called *Vinogradov's $\mathcal{C}-$ spectral sequence*) that encodes information on the equation at hand, see [**1, 25, 45**]. The $\mathcal{C}-$ spectral sequence corresponding to the "empty" equation, (that is, $\mathcal{E}^\infty = \mathcal{J}^\infty$) is the so-called variational bicomplex associated to the bundle E, see [**1**].

Let us give a more precise and computable definition of a diffiety. Let $\mathcal{E}^\infty$ be the infinite prolongation of a differential equation $\mathcal{E}$ (*i.e.*, $\mathcal{E}$ is a submanifold of the $k-$th jet fibration $\mathcal{J}^k(\pi)$ of a fibration $\pi : E \mapsto M$).

A diffiety $\mathcal{O} = (\mathcal{O}, C^\infty(\mathcal{O}), \mathcal{C}(\mathcal{O}))$ is a triple consisting of a manifold, a function field on it, and a finite-dimensional distribution, that is locally of the form $(\mathcal{E}^\infty, C^\infty(\mathcal{E}^\infty), \mathcal{C}(\mathcal{E}))$, where $\mathcal{C}(\mathcal{E}))$ is the restriction of the Cartan distribution on $J^\infty E$ to $\mathcal{E}^\infty$. The dimension of $\mathcal{C}(\mathcal{E})$ is called the dimension of $\mathcal{O}$. A smooth map $F : \mathcal{O}_1 \mapsto \mathcal{O}_2$ is called a morphism of diffieties if

$$dF_x(\mathcal{C}_x(\mathcal{O}_1)) \subset \mathcal{C}_{F(x)}(\mathcal{O}_2), \quad x \in \mathcal{O}_1.$$

We note at this point that the notion of a diffiety appears to be very useful also in conceptual definitions of non-local symmetries for PDEs, see [**24, 25**].

In spite of its "universality", "naturality" and "conceptuality" (the three most popular adjectives used by A. M. Vinogradov) these important and useful (and, usually) infinite-dimensional (in a standard sense) objects, may look "rather esoteric" at a first glance (to quote Toru Tsujishita in his MR reference to the paper [**44**], but in the op. cit. he immediately stresses that, after the paper [**44**], this notion "now becomes open to everybody."). We believe that esoteric they are not: there are some not sufficiently known curious links between the notion of a diffiety and other not-so-well-known concepts like Souriau diffeology and Frölicher space structure. In fact, we hope to show in this paper that *Frölicher spaces* are a very natural arena for the development of the theory of diffieties. We can rephrase the definition given above fully rigorously, and we can rather straightforwardly set up the Vinogradov sequence.

Frölicher spaces were first described by A. Frölicher in a serie of works in the 1980's (see e.g. [**18, 19**]) and most results of this period are gathered in his book with A. Kriegl [**20**]. The terminology "Frölicher space" can be found for the first time, to our knowledge, in papers by Cherenack (see e.g. [**10**]), and Kriegl and Michor confirmed that these spaces were due to A. Frölicher in [**26**]. This setting is designed to obtain a safe differential calculus and a safe differential geometry, and the presence of charts is not assumed. Smoothness depends directly on tests on a space of functions and on a space of paths, that are both assumed to be spaces of smooth maps, under mild conditions for coherence. Refined in [**29, 47**] by means of the notion of *diffeology* due to J.-M. Souriau (see e.g. [**32**]), the technical possibilities in this setting have increased drastically these last few years. Considerations on Frölicher spaces can be found in various contexts, see e.g. [**2, 3, 4, 5, 7, 8, 14, 29, 31, 32, 33, 40**].

In this work, written for a proceedings volume in memory of the scientific work of Alexandre Vinogradov, we intend to propose a formal description of the interplay of Frölicher spaces and diffieties, as stated in the penultimate paragraph. We are aware that our formulations may not fully fill all the refinements on diffieties that have been produced till now (for example the theory of non-local symmetries and coverings, see [**24**]), but this note intends to be a primary base for adaptations and discussions. We organize our paper as follows: Section 2 is an introduction to the theory of Frölicher spaces and some basic geometric structures on them (tangent vectors, differential forms, etc.); Section 3 is our proposal for a definition of a diffiety *via* Frölicher spaces; Section 4 is both an application of the theory and a prelude to further work (we provide some details in the following paragraph); section 5 contains a final remark.

On Section 4: we have explored the initial value problem of the Kadomtsev-Petviashvili (KP) hierarchy and related non-linear KP equations in several papers, see [**16, 17, 30, 32, 34**]. In particular, in [**34**] we have presented several curious equations that correspond to "KP flows" posed on particular algebras equipped with derivations, and in [**35**] we have applied our constructions to algebras arising from the *elementary diffiety* $(\mathcal{E}^\infty, C^\infty(\mathcal{E}^\infty), \mathcal{C}(\mathcal{E}))$. Can we apply them to general diffieties as defined in Section 3? In Section 4 we show that the answer is affirmative, provided that the diffiety is equipped with a derivation.

2. Preliminaries on Frölicher spaces

2.1. Frölicher spaces and their diffeology.

Definition 2.1. • A **Frölicher** space is a triple $(X, \mathcal{F}, \mathcal{P})$ such that
- $\mathcal{P}$ is a set of paths $\mathbb{R} \to X$,

- A function $f : X \to \mathbb{R}$ is in $\mathcal{F}$ if and only if for any $c \in \mathcal{P}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;
- A path $c : \mathbb{R} \to X$ is in $\mathcal{P}$ (i.e. is a **contour**) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

• Let $(X, \mathcal{F}, \mathcal{P})$ et $(X', \mathcal{F}', \mathcal{P}')$ be two Frölicher spaces, a map $f : X \to X'$ is **differentiable** (=smooth) if and only if one of the following equivalent conditions is fulfilled:

- $\mathcal{F}' \circ f \circ \mathcal{P} \subset C^\infty(\mathbb{R}, \mathbb{R})$
- $f \circ \mathcal{P} \subset \mathcal{P}'$
- $\mathcal{F}' \circ f \subset \mathcal{F}$

Any family of maps $\mathcal{F}_g$ from X to $\mathbb{R}$ generates a Frölicher structure $(X, \mathcal{F}, \mathcal{P})$, setting [**26**]:
- $\mathcal{P} = \{c : \mathbb{R} \to X$ such that $\mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \to \mathbb{R}$ such that $f \circ \mathcal{P} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$.

We easily see that $\mathcal{F}_g \subset \mathcal{F}$. This notion will be useful in the sequel to describe in a simple way a Frölicher structure. A Frölicher space carries a natural topology, the pull-back topology of $\mathbb{R}$ via $\mathcal{F}$. In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ, in general. We have to mention here that the space of smooth functions $\mathcal{F}$ is a vector space, which has a natural Frölicher structure called functional Frölicher structure.

Associated to the notion of Frölicher structure, we find the notion of **(reflexive) diffeology**.

Definition 2.2. Let X be a set.

• A **p-parametrization** of dimension p on X is a map from an open subset O of $\mathbb{R}^p$ to X.

• A **diffeology** on X is a set $\mathcal{D}$ of parametrizations on X such that:

- For each $p \in \mathbb{N}$, any constant map $\mathbb{R}^p \to X$ is in $\mathcal{D}$;
- For each arbitrary set of indexes I and family $\{f_i : O_i \to X\}_{i\in I}$ of compatible maps that extend to a map $f : \bigcup_{i\in I} O_i \to X$, if $\{f_i : O_i \to X\}_{i\in I} \subset \mathcal{D}$, then $f \in \mathcal{P}$.
- For each $f \in \mathcal{D}$, $f : O \subset \mathbb{R}^p \to X$, and $g : O' \subset \mathbb{R}^q \to O$, in which g is a smooth map (in the usual sense) from an open set $O' \subset \mathbb{R}^q$ to O, we have $f \circ g \in \mathcal{P}$.

If $\mathcal{D}$ is a diffeology on X, then $(X, \mathcal{D})$ is called a **diffeological space** and, if $(X, \mathcal{D})$ and $(X', \mathcal{D}')$ are two diffeological spaces, a map $f : X \to X'$ is **smooth** if and only if $f \circ \mathcal{D} \subset \mathcal{D}'$.

The notion of a diffeological space is due to J.M. Souriau, see [**41**]; a previous but not fully equivalent notion is due to Chen, see [**9**] and [**42**] for a very careful comparison. A comprehensive exposition of basic concepts can be found in [**23**]. The category of diffeological spaces is very large, and it contains many different pathological examples even if it enables a very easy-to-use framework for infinite dimensional objects. Therefore, the category of Frölicher spaces as a subcategory of this category (see [**29, 47, 42, 33**]) with less technical problems may be useful. The first steps of the comparison were published in [**29**]; the reader can also see [**30, 32, 47**] for extended expositions. In particular, it is explained in [**32**] that *Diffeological, Frölicher and Gateaux smoothness are the same notion if we restrict* *ourselves to a Fréchet context,* in a sense that we explain here. For this, we first need to analyze how we generate a Frölicher or a diffeological space, that is, how we implement a Frölicher or a diffeological structure on a given set X.

If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we define a natural diffeology on X by using the following family of maps f defined on open domains $D(f)$ of Euclidean spaces, see [**29**]:

$$\mathcal{D}_\infty(\mathcal{F}) = \coprod_{p \in \mathbb{N}^*} \{ f : D(f) \to X \mid D(f) \text{ is open in } \mathbb{R}^p \text{ and } \mathcal{F} \circ f \in C^\infty(D(f), \mathbb{R}) \} \ .$$

Now we can easily show the following:

Proposition 2.1. [**29**] *Let$(X, \mathcal{F}, \mathcal{P})$ and $(X', \mathcal{F}', \mathcal{P}')$ be two Frölicher spaces. A map $f : X \to X'$ is smooth in the sense of Frölicher if and only if it is smooth for the underlying diffeologies $\mathcal{D}_\infty(\mathcal{F})$ and $\mathcal{D}_\infty(\mathcal{F}')$.*

Thus, Proposition 2.1 and the foregoing remarks imply that the following implications hold:

smooth manifold $\Rightarrow$ Frölicher space $\Rightarrow$ diffeological space

The reader is referred to the Ph.D. thesis [**47**] for a deeper analysis of these implications. A more complete and up-to-date exposition, based on infinite dimensional examples, is actually under preparation in [**22**], but let us make some short precisions for the reader. We follow [**18**, Section 3] closely: as a consequence of Boman's theorem, see [**26**], the triple $(\mathbb{R}^n, C^\infty(\mathbb{R}^n, \mathbb{R}), C^\infty(\mathbb{R}, \mathbb{R}^n))$ is a Frölicher structure on $\mathbb{R}^n$. More generally, if V is a finite-dimensional paracompact smooth manifold, $(V, C^\infty(V, \mathbb{R}), C^\infty(\mathbb{R}, V))$ is a Frölicher structure on V and the smooth (in the standard sense) maps between two such manifolds are precisely the smooth maps in the Frölicher sense. We can replace $\mathbb{R}^n$ for a Fréchet space E in our first example, and we can replace V for a Fréchet manifold V' in our second example, *provided that* V' is a paracompact topological space and it is modelled on a Fréchet space E' that satisfies the following property: for each neighbourhood U of $0 \in E'$ there exists a smooth function $f : U \to \mathbb{R}$ such that $f(0) = 1$ and $f(x) = 0$ for $x \notin U$.

2.2. Tangent spaces, differential forms, diffeomorphisms and all that. On Frölicher spaces, we can define internal tangent cones ${}^iT_xX$, see e.g. [**30**]. We mention that, diffeologically, internal tangent spaces are defined in a different way, see e.g. [**11**], but the approach that we present here is

- coherent with the definition of the kinematic tangent space of a manifold in the c^∞ category studied in [**26**]
- and chosen as an interesting definition for tangent space in an applied setting [**21, 30, 32**].

For each $x \in X$, we consider

$$C_x = \{c \in C^\infty(\mathbb{R}, X) | c(0) = x\}$$

and take the equivalence relation $\mathcal{R}$ given by

$$c\mathcal{R}c' \Leftrightarrow \forall f \in C^\infty(X, \mathbb{R}), \quad \partial_t(f \circ c)|_{t=0} = \partial_t(f \circ c')|_{t=0} \ .$$

The internal tangent cone at x is the quotient

$$^iT_xX = C_x/\mathcal{R}.$$

If $X = \partial_t c(t)|_{t=0} \in {}^iT_X$, we define the simplified notation

$$Df(X) = \partial_t (f \circ c)|_{t=0} \, .$$

Under these constructions, we can define the total space of internal tangent cones

$$^iTX = \coprod_{x \in X} {}^iT_xX$$

with canonical projection $\pi : u \in {}^iT_xX \mapsto x$. A Frölicher structure can be defined on it. We note that in general, the fibration π on ^iTX may have non-isomorphic fibers that are all cones but not vector spaces (an example appears in [**33**]).

Differential forms on X and a de Rham differential also exist in this setting. It is easier to understand their construction under the diffeology viewpoint:

Definition 2.3. [**41**] Let $(X, \mathcal{D})$ be a diffeological space and let V be a vector space equipped with a differentiable structure. A $V-$valued $n-$differential form α on X (noted $\alpha \in \Omega^n(X, V)$) is a map

$$\alpha : \{p : O_p \to X\} \in \mathcal{D} \mapsto \alpha_p \in \Omega^n(O_p; V)$$

such that

- Let $x \in X$. $\forall p, p' \in \mathcal{D}$ such that $x \in Im(p) \cap Im(p')$, the forms α_p and $\alpha_{p'}$ are of the same order n.
- Moreover, let $y \in O_p$ and $y' \in O_{p'}$. If $(X_1, ..., X_n)$ are n germs of paths in $Im(p) \cap Im(p')$, if there exists two systems of $n-$vectors $(Y_1, ..., Y_n) \in (T_yO_p)^n$ and $(Y_1', ..., Y_n') \in (T_{y'}O_{p'})^n$, if $p_*(Y_1, ..., Y_n) = p'_*(Y_1', ..., Y_n') = (X_1, ..., X_n)$,

$$\alpha_p(Y_1, ..., Y_n) = \alpha_{p'}(Y_1', ..., Y_n').$$

We note by

$$\Omega(X; V) = \oplus_{n \in \mathbb{N}} \Omega^n(X, V)$$

the set of $V-$valued differential forms.

We feel we need to make two remarks:

- If do not exist n linearly independent vectors $(Y_1, ..., Y_n)$ defined as in the last point of the definition, $\alpha_p = 0$ at y.
- Let $(\alpha, p, p') \in \Omega(X, V) \times \mathcal{D}^2$. If there exists $g \in C^\infty(D(p); D(p'))$ (in the usual sense) such that $p' \circ g = p$, then $\alpha_p = g^* \alpha_{p'}$.

Proposition 2.2. *The set $\mathcal{P}(\Omega^n(X, V))$ made of maps $q : x \mapsto \alpha(x)$ from an open subset O_q of a finite dimensional vector space to $\Omega^n(X, V)$ such that for each $p \in \mathcal{P}$,*

$$\{x \mapsto \alpha_p(x)\} \in C^\infty(O_q, \Omega^n(O_p, V)),$$

is a diffeology on $\Omega^n(X, V)$.

Working with plots of the diffeology, we can define the wedge product and the exterior differential of differential forms, which are (diffeologically) smooth and have the same properties as in the standard case.

We need to introduce one last construction. Again following [**33**], we let $Diff(X)$ be the group of diffeomorphisms of the Frölicher space X. This is a well-defined object, it can be equipped with a diffeology, and its internal tangent space at the identity is a diffeological vector space that defines (see [**33**, Section

2]) a restricted tangent space on X called diff-tangent space and noted by ${}^dTX$. The space ${}^dTX$ is a Frölicher subspace of ${}^iTX$ and it is also a Frölicher vector bundle. Hereafter we assume that $Diff(X)$ is a **Frölicher Lie group**, that is, the tangent space at the identity has a natural diffeology for which the Lie bracket is smooth. This condition is not automatically fulfilled, and the study [**28**], completed in the paper [**27**], shows that this happens only on a "restricted" class of examples. However it is only through this setting that one can, to our knowledge, define a Lie bracket on vector fields that are identified as elements of the Lie algebra of $Diff(X)$.

3. Diffieties and their Frölicher structure

We recall from [**45**, p. 7, 191] and Section 1, that a *diffiety* is, classically, a (finite or infinite dimensional) manifold $\mathcal{O}$ that is locally of the form $\mathcal{E}^\infty$, where $\mathcal{E}^\infty$ denotes an infinitely prolonged equation that fibers over an n-dimensional manifold of "independent variables". We refer the reader to [**25**] for information on equation manifolds. This definition means, in particular, that $\mathcal{O}$ must be equipped with an n-dimensional involutive distribution that coincides (locally) with the Cartan distribution of $\mathcal{E}^\infty$.

We now propose a very general definition of a diffiety that does not use topology explicitly. Let $\mathcal{O}$ be a set, and let $(\mathcal{O}, \mathcal{F}(\mathcal{O}), \mathcal{P}(\mathcal{O}))$ be a Frölicher structure. Since $\mathcal{P}(\mathcal{O})$ is generated by $\mathcal{F}(\mathcal{O})$, we remark that it is redundant to mention it.

Definition 3.1. We consider a Frölicher space $(\mathcal{O}, \mathcal{F}(\mathcal{O}), \mathcal{P}(\mathcal{O}))$. A **Cartan distribution** $\mathcal{C}(\mathcal{O})$ on $\mathcal{O}$ is a finite-dimensional vector subbundle of ${}^dTX$ that is involutive, that is, there is a Lie subalgebra $\mathcal{H}$ of the Lie algebra of $Diff(\mathcal{O})$ such that for all $p \in \mathcal{O}$,

$$\mathcal{C}(\mathcal{O})_p = \{X(p) : X \in \mathcal{H}\} \; .$$

A **diffiety** is a triple $(\mathcal{O}, \mathcal{F}(\mathcal{O}), \mathcal{P}(\mathcal{O}))$.

This definition includes the particular case of $\mathcal{O} = \mathcal{E}^\infty$ if all prolongations of $\mathcal{E}$ satisfy natural conditions as in the standard case of [**1, 25**]. Indeed, let us assume that $\mathcal{E}^\infty$ is contained in the infinite jet bundle $J^\infty(E)$ for some fiber bundle $E \to M$. The bundle $J^\infty(E)$ is the inverse limit of the sequence of finite-dimensional jet bundles $J^k(E)$, $k \geq 0$, and we have observed that these manifolds have natural Frölicher structures. Now, the category of Frölicher spaces and smooth maps is closed under inverse limits (see e.g., [**20**]), and therefore we obtain that $J^\infty(E)$ is a Frölicher space. Since $\mathcal{E}^\infty \subseteq J^\infty(E)$, we conclude that $\mathcal{E}^\infty$ has a natural Frölicher structure. It follows from the standard geometry reviewed for example in [**1, 25**], that functions over $\mathcal{E}^\infty$ that are Frölicher-smooth are precisely the standard smooth functions on $\mathcal{E}^\infty$. Thus, the classical example of a diffiety (what Vinogradov calls an *elementary diffiety* in [**45**, P. 191]) *is* a diffiety in our sense.

We now show that within this framework, we can build the **Vinogradov sequence** along the lines of [**43, 45**]: the algebra of differential forms

$$\Omega^*(\mathcal{O}) = \sum_{k \in \mathbb{N}} \Omega^*(\mathcal{O})$$

is a graded differential algebra for the de Rham differential operator. Let us define

$$C\Omega^k(\mathcal{O}) = \left\{ \alpha \in \Omega^k \mid \alpha|_{\mathcal{C}(\mathcal{O})} = 0 \right\} ,$$

$$\mathcal{C}\Omega(\mathcal{O}) = \sum_{k\in\mathbb{N}} \mathcal{C}\Omega^k(\mathcal{O})$$

and

$$\mathcal{C}^l\Omega(\mathcal{O}) = \mathcal{C}\Omega(\mathcal{O})^{\wedge l}.$$

Since $\mathcal{C}(\mathcal{O})$ is involutive, $\mathcal{C}(\mathcal{O})$ is a differential ideal, and so is $\mathcal{C}^l(\mathcal{O})$ for $l \in \mathbb{N}^*$. Therefore we can define

$$E_0^{p,q} = \frac{\mathcal{C}^p\Omega^{p+q}(\mathcal{O})}{\mathcal{C}^{p+1}\Omega^{p+q}(\mathcal{O})},$$
$$E_{r+1}^{p,q} = H^*(E_r^{p,q})$$

and

$$\mathcal{C}E(\mathcal{O}) = \{E_r^{p,q}\}$$

equipped with the restriction/extension of the de Rham differential to the corresponding spaces. This is the spectral sequence we sought.

4. KP hierarchy on a diffiety

In this section we consider the KP hierarchy. As explained in Section 1, we present this section as an advance of [**35**], where we study differential equations posed on *elementary* diffieties. Here, for the sake of brevity we simply assume that there is a non-trivial derivation D on $\mathcal{F}(\mathcal{O})$ that is smooth (in [**35**] we use explicit derivations). We denote by A any Frölicher subalgebra of $\mathcal{F}(\mathcal{O})$ such that $D\,A \subset A$.

Let ξ be a formal variable not in A. The *algebra of symbols* over A is the vector space

$$\Psi_\xi(A) = \left\{ P_\xi = \sum_{\nu\in\mathbf{Z}} a_\nu\, \xi^\nu : a_\nu \in A\,,\ a_\nu = 0 \text{ for } \nu \gg 0 \right\}$$

equipped with the associative multiplication $\circ$, where

$$P_\xi \circ Q_\xi = \sum_{k\geq 0} \frac{1}{k!}\, \frac{\partial^k P_\xi}{\partial \xi^k}\, D^k Q_\xi\ , \tag{4.1}$$

with the prescription that the multiplication on the right hand side is the standard multiplication of Laurent series in ξ with coefficients in A. The algebra A is included in $\Psi_\xi(A)$.

The *algebra of formal pseudo-differential operators* over A is the vector space

$$\Psi(A) = \left\{ P = \sum_{\nu\in\mathbb{Z}} a_\nu\, D^\nu : a_\nu \in A\,,\ a_\nu = 0 \text{ for } \nu \gg 0 \right\}\ ,$$

in which a multiplication $\circ$ is defined so that the map

$$\sum_{\nu\in\mathbb{Z}} a_\nu\, \xi^\nu \ \mapsto\ \sum_{\nu\in\mathbb{Z}} a_\nu\, D^\nu$$

from $\Psi_\xi(A)$ to $\Psi(A)$ is an algebra homomorphism. The algebra $\Psi(A)$ is associative; it becomes a Lie algebra over K if we define, as usual,

$$[P, Q] = P \circ Q - Q \circ P\ . \tag{4.2}$$

The *order* of $P \neq 0 \in \Psi(A)$ is N if $a_N \neq 0$ and $a_\nu = 0$ for all $\nu > N$, and $\mathrm{order}(P) = -\infty$ if $P = 0$. If P is of order N, the coefficient a_N is called the *leading term* of P.

We collect some properties of $\Psi(A)$ for completeness.

LEMMA 4.1. (1) *If P and Q are formal pseudo–differential operators over A, and the leading terms of P, Q are not divisors of zero in A, then* $\mathrm{order}(P \circ Q) = \mathrm{order}(P) + \mathrm{order}(Q)$, *and* $\mathrm{order}([P, Q]) \leq \mathrm{order}(P) + \mathrm{order}(Q) - 1$. *In particular, if A is an integral domain, the ring $\Psi(A)$ does not have zero divisors.*

(2) *Every non–zero formal pseudo–differential operator P for which its leading term is invertible has an inverse in $\Psi(A)$.*

The following observation is at the basis of all our analysis: the Lie algebra $\Psi(A)$ admits a left A–module direct sum decomposition

$$\Psi(A) = \mathcal{I}_A \oplus \mathcal{D}_A \ , \tag{4.3}$$

in which $\mathcal{D}_A$ is the Lie subalgebra of all *differential* operators of order greater or equal to zero, and $\mathcal{I}_A$ is the Lie subalgebra of *integral* operators, that is, the set of all formal pseudo-differential operators in $\Psi(A)$ of order at most -1. It is known since [**37, 38, 39**] that there exist formal Lie groups $G(\Psi(A))$, $G_+(\mathcal{D}_A)$, and $G_-(\mathcal{I}_A)$ with Lie algebras $\Psi(A)$, $\mathcal{D}_A$, and $\mathcal{I}_A$ respectively, such that

$$G(\Psi(A)) = G_-(\mathcal{I}_A) \cdot G_+(\mathcal{D}_A) \, . \tag{4.4}$$

In order to explain what these groups are, we need to specialize the algebra A considered in the previous subsection. Let R be a fixed commutative algebra over a field K equipped with a derivation D. We take as A the ring of formal power series over R in an infinite number of variables $\tau_1, \tau_2, \cdots$. Certainly, it is not necessary to make this particular choice in the classical algebraic theory of KP (see for instance [**15, 36**]) but, unless we equip A with a further structure as in Demidov [**12, 13**] and specially [**17**], taking A as a ring of power series is crucial for the explicit construction of the groups $G(\Psi(A))$, $G_+(\mathcal{D}_A)$ and $G_-(\mathcal{I}_A)$.

REMARK 4.2. For the benefit of the reader, we recall the construction of A following Bourbaki [**6**, p. 454-457]. We consider a countable set of indices I, and we take T as the additive monoid of all sequences of natural numbers $t = (n_i)_{i \in I}$ such that $n_i = 0$ except for a finite number of indices. A *formal power series* is a function u from T to R, $u = (u_t)_{t \in T}$. Consider an infinite number of formal variables τ_i, $i \in I$ and set $\tau = (\tau_1, \tau_2, \cdots)$. Then, the formal power series u is also written as $u = \sum_{t \in T} u_t \tau^t$ in which $\tau^t = \tau_1^{n_1} \tau_2^{n_2} \cdots$. We say that $u_t \in R$ is a coefficient and that $u_t \tau^t$ is a term.
Operations on the ring A are defined in an usual manner: If $u = (u_t)_{t \in T}$, $v = (v_t)_{t \in T}$, then we set

$$u + v = (u_t + v_t)_{t \in T} \quad \text{and} \quad uv = w,$$

in which $w = (w_t)_{t \in T}$ and

$$w_t = \sum_{\substack{p,q \in T \\ p+q=t}} u_p v_q .$$

It is shown in [**6**, p. 455] that this multiplication is well defined, and that A equipped with these two operations is a commutative algebra with unit. The derivation D on R extends to a derivation on A via

$$Du = \sum_{t \in T} (Du_t) \tau^t .$$

We also define the *order* of a power series. Let $u \in A$, $u \neq 0$. Let us write $u = \sum_{t\in T} u_t \tau^t$, and if $t = (n_i)_{i\in I}$, let us set $\mid t \mid = \sum n_i$. The terms $u_t\tau^t$ such that $\mid t \mid = p$ are called *terms of total degree* p. The formal power series u_p whose terms of total degree p are those of u, and whose other terms are zero, is called *the homogeneous part of* u *of degree* p. The series u_0, the homogeneous part of u of degree 0, is identified with an element of R called the constant term of u. For a formal series $u \neq 0$, the least integer $p \geq 0$ such that $u_p \neq 0$ is called the *order* of u, and it is denoted by $ord_t(u)$, and we extend this definition to the case $u = 0$ setting $ord_t(0) = \infty$ (see [**6**, p. 457]). The following properties hold: if u, v are formal power series different from zero, then

$$\begin{aligned} ord_t(u+v) &\geq inf(ord_t(u), ord_t(v)) \ , \quad \text{if } u+v \neq 0, \\ ord_t(u+v) &= inf(ord_t(u), ord_t(v)) \ , \quad \text{if } ord_t(u) \neq ord_t(v), \\ ord_t(uv) &\geq ord_t(u) + ord_t(v) \ , \quad \text{if } uv \neq 0. \end{aligned}$$

Following Mulase, [**39**], we now define the space

$$\widehat{\Psi}(A) = \left\{ P = \sum_{\alpha\in\mathbb{Z}} a_\alpha D^\alpha : a_\alpha \in A \text{ and } \exists C \in \mathbb{R}^+, N \in \mathbb{Z}^+ \text{ so that } ord_t(a_\alpha) > C\,\alpha - N \;\forall\; \alpha \gg 0 \right\} \tag{4.5}$$

and the subspace

$$\widehat{\mathcal{D}}_A = \left\{ P = \sum_{\alpha\in\mathbb{Z}} a_\alpha D^\alpha : P \in \widehat{\Psi}(A) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\} . \tag{4.6}$$

The definition of order explained in the last remark implies that A and $\Psi(A)$ are contained in $\widehat{\Psi}(A)$. The operations on $\widehat{\Psi}(A)$ are natural extensions of the operations on $\Psi(A)$. As mentioned by Mulase in [**39**] and proved explicitly in [**16, 17**], the following holds:

Lemma 4.3. *The space* $\widehat{\Psi}(A)$ *has a natural algebra structure, and* $\widehat{\mathcal{D}}_A$ *is a subalgebra of* $\widehat{\Psi}(A)$.

Definition 4.4. Let $\mathcal{K}$ be the ideal of A generated by $t_1, t_2, \cdots$. If $P \in \widehat{\Psi}(A)$, we denote by $P|_{t=0}$ the equivalence class $P\, mod\ \mathcal{K}$, and we identify it with an element of $\Psi(A)$. We also set $G_A = 1 + \mathcal{I}_A$, and we define the spaces

$$\widehat{\Psi}(A)^\times = \{P \in \widehat{\Psi}(A) : P|_{t=0} \in G_A\}$$

and

$$\widehat{\mathcal{D}}_A^\times = \{P \in \widehat{\mathcal{D}}_A : P|_{t=0} = 1\} .$$

The most important fact about these two spaces is the following result proven by Mulase in [**39**], and reviewed in full detail in [**16**] and [**17**].

Proposition 4.1. *The spaces* $\widehat{\Psi}(A)^\times$ *and* $\widehat{\mathcal{D}}_A^\times$ *are formal Lie groups: each element* P *in* $\widehat{\Psi}(A)^\times$ *and* $\widehat{\mathcal{D}}_A^\times$ *has an inverse of the form*

$$P^{-1} = \sum_{n\geq 0} (1-P)^n .$$

Moreover, we have: $G(\Psi(A)) = \widehat{\Psi}(A)^\times$, $G_+(\mathcal{D}(A)) = \widehat{\mathcal{D}}_A^\times$, *and* $G_-(\mathcal{I}_A) = 1 + \mathcal{I}_A$.

THEOREM 4.5. *For any $U \in G(\Psi(A))$ there exist unique $W \in G_-(\mathcal{I}_A)$ and $Y \in G_+(\mathcal{D}_A)$ such that*

$$U = W^{-1}\, Y \ .$$

In other words, there exists a unique global factorization of the formal Lie group $G(\Psi(A))$ as a product group,

$$G(\Psi(A)) = G_-(\mathcal{I}_A)\, G_+(\mathcal{D}_A) \ .$$

This theorem is proven in [**39**], see also the later papers [**16, 17**] for full proofs.

The *Kadomtsev-Petviashvili (KP) hierarchy* on $\mathcal{O}$ reads

$$\frac{dL}{dt_k} = \left[(L^k)_+, L\right] \ , \qquad k \geq 1 \ , \tag{4.7}$$

with initial condition $L(0) = L_0 \in \Psi^1(R)$. The dependent variable L is chosen to be of the form

$$L = D + \sum_{\alpha \leq -1} u_\alpha\, D^\alpha \in \Psi^1(A_t) \ .$$

Standard references on (4.7) are [**15, 36, 37, 38, 39**]. The following result gives a solution to the Cauchy problem for the KP hierarchy (4.7).

THEOREM 4.6. *Consider the KP hierarchy with initial condition $L(0) = L_0$.*

(1) *There exists a pair $(S, Y) \in G_-(\mathcal{I}_A) \times \widehat{\mathcal{D}}_A^\times$ such that the unique solution to Equation (4.7) with $L(0) = L_0$ is*

$$L(t) = Y\, L_0\, Y^{-1} = S L_0 S^{-1} \ .$$

(2) *The pair (S, Y) is uniquely determined by the decomposition problem*

$$exp\left(\sum_{k\in\mathbb{N}} t_k L_0^k\right) = S^{-1}Y.$$

(3) *The solution operator L is smoothly dependent on the variable t and on the initial value L_0. This means that the map*

$$(L_0, s) \in (D + \Psi^{-1}(A)) \times T \mapsto \sum_{n\in\mathbb{N}} \left(\sum_{|t|=n} [L(s)]_t\right) \in (D + \Psi^{-1}(R))^{\mathbb{N}}$$

is smooth, in which $s \in \cup_{n\in\mathbb{N}} \mathbb{K}^n$ and this set is equipped with the structure of locally convex topological space given by the inductive limit.

The existence of an algebraic decomposition as in Parts (1) and (2) of Theorem 4.6 appears already in Mulase's seminal papers [**37, 39**], and a formal solution to (4.7) as in Part (1) is in [**16**]. The richness of Theorem 4.6 steams from the fact that we pose the KP equations (4.7) in the Frölicher algebra $\Psi(A)$ using an analytically rigorous factorization in Frölicher groups of the infinite-dimensional Frölicher group $G(\Psi(A))$. For all details, we refer to [**17, 32**].

5. Outlook

As anticipated, we have a more concrete example of Frölicher structure on a classical elementary diffiety in the work in progress [**35**]; in [**35**] we also investigate analogues to the KP hierarchy and equations very explicitly. The definition of a diffiety given here, plus Section 5, compels us to study non-linear equations "of KP type" in full detail in this global setting. We hope to do so in the near future.

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