Multi-point correlation functions in the boundary XXZ chain at finite temperature

Karol K. Kozlowski [1]

Univ Lyon, ENS de Lyon, Univ Claude Bernard Lyon 1, CNRS, Laboratoire de Physique, F-69342 Lyon, France

Véronique Terras [2]

Université Paris-Saclay, CNRS, LPTMS, 91405, Orsay, France

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Abstract

We consider multi-point correlation functions in the open XXZ chain with longitudinal boundary fields and in a uniform external magnetic field. We show that, at finite temperature, these correlation functions can be written in the quantum transfer matrix framework as sums over thermal form factors. More precisely, and quite remarkably, each term of the sum is given by a simple product of usual matrix elements of the quantum transfer matrix multiplied by a unique factor containing the whole information about the boundary fields. As an example, we provide a detailed expression for the longitudinal spin one-point functions at distance m from the boundary. This work thus solves the long-standing problem of setting up form factor expansions in integrable models subject to open boundary conditions.

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[1]e-mail: karol.kozlowski@ens-lyon.fr

[2]e-mail: veronique.terras@universite-paris-saclay.fr

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1 Introduction

The calculation of the correlation functions in interacting integrable models is a long-standing problem. The first attempts in this direction goes back to the works of Takahashi [40]: the latter developed ingenious roundabout arguments leading to closed expressions for some of the next-to-neighbouring correlation functions in the XXX chain, the one of the nearest neighbouring spin operator following trivially from Hulten's result [12] for the model's ground state energy. It however turned out to be very difficult to go beyond these special cases in the framework of the coordinate Bethe Ansatz. In fact, genuine progress could only be made after the formulation of the algebraic Bethe Ansatz [7]. Then, two structurally different approaches to the problem emerged. The first approach goes back to the works of the Kyoto school [16, 17, 18] for the XXZ chain in the infinite volume limit: there, the $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry of the infinite XXZ chain was used so as to set up a vertex operator approach and systems of q-KZ equations, leading to the computation of the form factor densities of local operators in the massive regime of the XXZ chain [17] along with the zero temperature reduced density matrix [16, 18] in all regimes of the XXZ chain. The *per se* algebraic Bethe Ansatz approach to the calculation of correlation functions in integrable models in finite volume was pioneered in the works [13, 14], see also [1]. However, in this early stage, the obtained representations for the correlators suffered from an important combinatorial intricacy. This problem was later overcome thanks to the obtention of a convenient determinant representation for the scalar product of off-shell/on-shell Bethe vectors [39] along with the resolution of the so-called quantum inverse scattering problem [30]. Eventually, the algebraic Bethe Ansatz approach gave rise to various types of series of multiple integral representations for the correlation functions, both in finite volume and in the thermodynamic limit [21, 27, 28, 29]. In particular, it reproduced the integral representations for the zero temperature reduced density matrix of the XXZ infinite chain previously obtained in the framework of the aforementioned first approach, and extended them to the case of finite overall magnetic fields [31]. >From these representations it was eventually possible to extract, without any *ad hoc* hypothesis or handling, the long-distance asymptotic behaviour of two-point functions in the massless regime of the chain [22]. Later on, it turned out that the determinant representations for the form factors of local operators in the XXZ chain obtained in [30] could be analysed in the large-volume, thermodynamic, limit [23, 25], which then made it possible to efficiently describe, by means of form factor expansions, the dynamical two-point correlation functions, as well as to grasp their critical behaviours [24, 26, 34, 35, 36]. Moreover, by using the quantum transfer matrix formalism, it was also possible to generalise these approaches to the case of the correlation functions at finite temperature [9]. In particular, the thermal form factor expansions [5, 6] turned out to be most efficient for studying the critical regime.

All the results mentioned so far pertained to the XXZ spin $1/2$ chain or to the one-dimensional Bose gaz subject to periodic boundary conditions. The two mentioned approaches to the computation of correlation functions could also be adapted, at least to some extent, to deal with other kinds of integrable boundary conditions. It particular, it has been possible to provide closed expressions for the zero temperature reduced density matrix [15, 19] and the two-point correlation functions [20] of the XXZ chain subject to diagonal boundary fields. Unfortunately, these representations did not turn out to be efficient enough for the analysis of the long-distance asymptotic behaviour of the correlation functions. Also, the lack of translational invariance hindered the implementation of the form factor approach to criticality, an approach which bare its fruits in the periodic case. The sole case that could have been treated so far by the form factor method corresponds to the edge spin-spin dynamical autocorrelation function [10]. Finally, some progress was achieved in describing the XXZ chain subject to open boundary conditions at finite temperature: the surface free energy was characterised in the works [2, 37]. However, so far, no closed expressions for the thermal correlation functions exist in the case of open boundary conditions.

This paper aims at filling this gap. More precisely, we establish a setting allowing one to obtain thermal form factor expansions for multi-point correlation functions in the XXZ chain subject to diagonal boundary fields. As in the periodic case, these expansions fully factorise the distance dependence of the correlation function and bare, in fact, several structural similarities with those arising in the case of the periodic chain. While, as already mentioned, the lack of translation invariance renders the zero-temperature form factor expansions ineffective in the case of open boundary conditions, the key observation of this work is that the presence of finite temperature allows one to bypass these limitations. Indeed, as observed in [2], the quantum transfer matrix – *viz.* the auxiliary object naturally describing the thermodynamics of the *periodic* XXZ chain – appears as a key ingredient for computing the surface free energy of the XXZ chain subject to diagonal boundary fields. The surface free energy is expressed as the Trotter limit of the average of a specific projector calculated in respect to the dominant state of the quantum transfer matrix. The Trotter limit of that representation can be taken upon observing that the latter may be recast [37] in terms of the partition functions of the six-vertex model with reflecting ends which admits a determinant representation [41]. In the present work, we push further this construction by conforming it to the computation of multi-point correlation functions. The connection between the periodic and open boundary conditions for finite temperature quantum integrable models unravelled in [2] allows us to build on the concept of thermal form factor expansions [5, 6] and the projector representation of [2] so as to set up thermal boundary form factor expansions in XXZ chain subject to diagonal boundary fields. This constitutes the main result of this work.

The paper is organised as follows. In Section 2, the description of the open XXZ chain at finite temperature T within the Trotter approximation method is briefly recalled, and this setting is applied to the derivation of a finite Trotter approximant for the thermal multi-point correlation functions in the open XXZ chain. In Section 3, we reorganise the result in the form of a Trotter limit of a finite Trotter number thermal form factor expansion. In Section 4 is developed a scheme, following [5, 37], for taking the Trotter limit in these expressions. We more particularly focus there on the case of the longitudinal spin one point function, the calculation of the Trotter limit for the multi-point correlation functions being a trivial generalisation thereof. The result of this process for the one-point function at infinite Trotter limit, *viz.* a closed expression for this one-point function at finite temperature, is finally presented in Section 5.

2 Multi-point correlation functions at finite temperature: setting of the problem

We consider in this paper the XXZ spin-1/2 open chain with longitudinal boundary fields h_- and h_+, and in a uniform external magnetic field h. The corresponding Hamiltonian is

$$\mathtt{H}_h \;=\; \mathtt{H} - \frac{h}{2}\sum_{k=1}^{L}\sigma_k^z\,, \tag{2.1}$$

with

$$\mathtt{H} = J\sum_{m=1}^{L-1}\Big\{\sigma_m^x\,\sigma_{m+1}^x + \sigma_m^y\,\sigma_{m+1}^y + \Delta\,(\sigma_m^z\,\sigma_{m+1}^z + \mathrm{id})\Big\} + h_-\,\sigma_1^z + h_+\,\sigma_L^z + c\,. \tag{2.2}$$

This Hamiltonian acts on the quantum space of states $\mathfrak{h}_{XXZ} = \otimes_{p=1}^{L}\mathfrak{h}_p$, where the local quantum space $\mathfrak{h}_a$ associated with the a^{th} site is isomorphic to $\mathbb{C}^2$. Here J is a global coupling constant, Δ is some anisotropy parameter along the z-direction, $\sigma^{x,y,z}$ are the Pauli matrices and given an operator $\mathtt{O} \in \mathrm{End}(\mathbb{C}^2)$, $\mathtt{O}_m \in \mathrm{End}(\mathfrak{h}_{XXZ})$ acts as $\mathtt{O}$ on the tensor component $\mathfrak{h}_m$ and as the identity on all the other tensor components. In the following, we shall parameterise the anisotropy parameters and boundary fields in terms of three parameters ζ and $\xi_\pm$ as

$$\Delta = \cos(\zeta), \qquad h_\pm^z = J\,\sinh(-\mathrm{i}\zeta)\,\coth\xi_\pm, \tag{2.3}$$

and fix for convenience the constant c in (2.2) to be equal to

$$c = J\,\frac{\cos(2\zeta)}{\cos(\zeta)}\,. \tag{2.4}$$

Although the parameterisation (2.3) is most suited for the study of the massless regime $-1 < \cos\zeta < 1$, we stress that the adaptation of our handlings to the other regimes of the couplings is straightforward.

Given r local operators $\mathtt{O}^{(1)}_{m_1+1},\dots,\mathtt{O}^{(r)}_{m_r+1}$, our aim is to compute the thermal average at temperature T,

$$\mathbb{E}_{L;T}\Big[\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\Big] \;=\; \mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big\{\mathtt{O}^{(1)}_{m_1+1}\dots\mathtt{O}^{(r)}_{m_r+1}\mathrm{e}^{-\frac{\mathtt{H}_h}{T}}\Big\}\cdot Z_L^{-1}, \tag{2.5}$$

in which Z_L is the partition function of the model:

$$Z_L \;=\; \mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big[\mathrm{e}^{-\frac{\mathtt{H}_h}{T}}\Big]. \tag{2.6}$$

More precisely, we aim at computing the thermodynamic limit of (2.5):

$$\langle\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\rangle_T \;=\; \lim_{L\to+\infty}\mathbb{E}_{L;T}\Big[\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\Big]. \tag{2.7}$$

2.1 A Trotter approximant of the partition function

The Hamiltonian (2.1) can be diagonalised by means of the boundary version of the algebraic Bethe Ansatz approach [38]. For this, one needs to introduce two monodromy matrices,

$$T_a(\lambda) \;=\; R_{aL}(\lambda-\xi_L)\dots R_{a1}(\lambda-\xi_1) \qquad \text{and} \qquad \widehat{T}_a(\lambda) \;=\; R_{1a}(\lambda+\xi_1)\dots R_{La}(\lambda+\xi_L)\,. \tag{2.8}$$

There ξ_k represent inhomogeneity parameters, the roman index a refers to an auxiliary two-dimensional space whereas the indices $1,\ldots,L$ refer to the various quantum spaces $\mathfrak{h}_1,\ldots,\mathfrak{h}_L$ arising in the tensor product decomposition of the model's Hilbert space $\mathfrak{h}_{XXZ}=\otimes_{p=1}^{L}\mathfrak{h}_p$. The above monodromy matrices are built from the six-vertex R-matrix taken in the polynomial normalisation:

$$\mathrm{R}(\lambda)=\frac{1}{\sinh(-\mathrm{i}\zeta)}\begin{pmatrix}\sinh(\lambda-\mathrm{i}\zeta) & 0 & 0 & 0\\ 0 & \sinh(\lambda) & \sinh(-\mathrm{i}\zeta) & 0\\ 0 & \sinh(-\mathrm{i}\zeta) & \sinh(\lambda) & 0\\ 0 & 0 & 0 & \sinh(\lambda-\mathrm{i}\zeta)\end{pmatrix}. \tag{2.9}$$

Further, one also needs to introduce the diagonal solutions of the reflection equations that have been first found by Cherednik [3],

$$K_a^{\pm}(\lambda)=K_a\Big(\lambda-\mathrm{i}\tfrac{\zeta}{2}\mp\mathrm{i}\tfrac{\zeta}{2};\xi_{\pm}\Big)\qquad\text{with}\quad K_a(\lambda;\xi)=\begin{pmatrix}\sinh(\lambda+\xi) & 0\\ 0 & \sinh(\xi-\lambda)\end{pmatrix}_{[a]}. \tag{2.10}$$

These K-matrices can be checked to satisfy

$$\mathrm{tr}_a[K_a^{+}(0)]=2\sinh(\xi_+)\cos(\zeta)\qquad\text{and}\qquad \mathrm{tr}_a[K_a^{-}(0)]=2\sinh(\xi_-)\,. \tag{2.11}$$

Then, the model's transfer matrix takes the form

$$\boldsymbol{\tau}(\lambda)\;=\;\mathrm{tr}_{\mathfrak{h}_a}\Big[K_a^{+}(\lambda)T_a(\lambda)K_a^{-}(\lambda)\widehat{T}_a(\lambda)\Big]\,. \tag{2.12}$$

One can show [3] that, in the homogeneous limit ($\xi_k=0$, $k=1,\ldots,L$), $\boldsymbol{\tau}(\lambda)$ enjoys the properties

$$\boldsymbol{\tau}(0)_{|\xi_a=0}=\frac{\mathrm{tr}_a[K_a^{+}(0)]\mathrm{tr}_a[K_a^{-}(0)]}{2}\mathtt{id}\qquad\text{and}\qquad \mathrm{H}=\frac{J\sinh(-\mathrm{i}\zeta)}{\boldsymbol{\tau}(0)_{|\xi_a=0}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\boldsymbol{\tau}(\lambda)_{|\lambda=0,\xi_a=0}\,. \tag{2.13}$$

There $\mathtt{id}$ stands for the identity operator on $\mathfrak{h}_{XXZ}$ and H is given by (2.2). As a consequence,

$$\Big(\Big[\boldsymbol{\tau}(-\tfrac{\beta}{N})\cdot\boldsymbol{\tau}^{-1}(0)\Big]_{|\xi_a=0}\Big)^{N}=\mathrm{e}^{-\frac{\mathrm{H}}{T}}\cdot\Big(1+\mathrm{O}(N^{-1})\Big)\qquad\text{with}\qquad\beta=\frac{J\sinh(-\mathrm{i}\zeta)}{T}. \tag{2.14}$$

This leads to a Trotter limit-based representation for the partition function (2.6) of the open XXZ spin-1/2 chain in a uniform external magnetic field:

$$Z_L=\lim_{N\to+\infty}\mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big\{\Big[\boldsymbol{\tau}(-\tfrac{\beta}{N})\cdot\boldsymbol{\tau}^{-1}(0)\Big]^{N}\prod_{a=1}^{L}\mathrm{e}^{\frac{h}{2T}\sigma_a^z}\Big\}_{|\xi_a=0}\,. \tag{2.15}$$

2.2 A Trotter approximant for multi-point functions

The above approach also allows one to obtain a Trotter approximant of the thermal average (2.5):

$$\begin{aligned}\mathbb{E}_{L;T}\Big[\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\Big]\cdot Z_L\;&=\;\mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big\{\mathtt{O}^{(1)}_{m_1+1}\ldots\mathtt{O}^{(r)}_{m_r+1}\mathrm{e}^{-\frac{\mathrm{H}_h}{T}}\Big\}\\ &=\;\lim_{N\to+\infty}\mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big\{\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\cdot\boldsymbol{\tau}^{N}(-\tfrac{\beta}{N})\cdot\boldsymbol{\tau}^{-N}(0)\cdot\prod_{a=1}^{L}\mathrm{e}^{\frac{h}{2T}\sigma_a^z}\Big\}_{|\xi_a=0}\,.\end{aligned} \tag{2.16}$$

This may be recast in terms of quantities naturally associated with the quantum transfer matrix. For doing so, one needs to rely on the representation, first observed in [2],

$$\boldsymbol{\tau}(\lambda) \;=\; \mathrm{tr}_{\mathfrak{h}_a\otimes\mathfrak{h}_b}\Big[\mathtt{P}_{ab}(\lambda)T_b^{\mathrm{t}_b}(\lambda)\widehat{T}_a(\lambda)\Big] \qquad \text{with} \qquad \mathtt{P}_{ab}(\lambda) = K_a^{+}(\lambda)\mathcal{P}_{ab}^{\mathrm{t}_a}K_a^{-}(\lambda)\;, \tag{2.17}$$

in which $\mathcal{P}_{ab}$ is the permutation operator on $\mathfrak{h}_a\otimes\mathfrak{h}_b$. Then direct algebra leads to

$$\begin{aligned}\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\cdot\boldsymbol{\tau}^N\big(-\tfrac{\beta}{N}\big)\prod_{a=1}^{L}\mathrm{e}^{\frac{h}{2T}\sigma_a^z} \;=\; \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\Pi_{\mathfrak{q}}\big(-\tfrac{\beta}{N}\big)\,\mathtt{T}_{\mathfrak{q};1}(\xi_1)\cdots\mathtt{T}_{\mathfrak{q};m_1}(\xi_{m_1})\,\mathtt{O}^{(1)}_{m_1+1}\,\mathtt{T}_{\mathfrak{q};m_1+1}(\xi_{m_1+1})\cdots \\ \cdots\mathtt{O}^{(r)}_{m_r+1}\,\mathtt{T}_{\mathfrak{q};m_r+1}(\xi_{m_r+1})\,\mathtt{T}_{\mathfrak{q};m_r+2}(\xi_{m_r+2})\cdots\mathtt{T}_{\mathfrak{q};L}(\xi_L)\Big]\end{aligned} \tag{2.18}$$

in which $\mathfrak{h}_{\mathfrak{q}} = \otimes_{k=1}^{2N}\mathfrak{h}_{a_k}$, $\mathfrak{h}_{a_k}\simeq\mathbb{C}^2$, and $\mathtt{T}_{\mathfrak{q};k}$ is the quantum monodromy matrix with auxiliary space $\mathfrak{h}_k$:

$$\mathtt{T}_{\mathfrak{q};k}(\xi) \;=\; R_{a_{2N}k}^{\mathrm{t}_{a_{2N}}}\big(-\xi-\tfrac{\beta}{N}\big)\,R_{ka_{2N-1}}\big(\xi-\tfrac{\beta}{N}\big)\cdots R_{a_2k}^{\mathrm{t}_{a_2}}\big(-\xi-\tfrac{\beta}{N}\big)\,R_{ka_1}\big(\xi-\tfrac{\beta}{N}\big)\,\mathrm{e}^{\frac{h}{2T}\sigma_k^z} \;=\; \begin{pmatrix}\mathtt{A}(\lambda) & \mathtt{B}(\lambda)\\ \mathtt{C}(\lambda) & \mathtt{D}(\lambda)\end{pmatrix}_{[k]}\;. \tag{2.19}$$

Finally, $\Pi_{\mathfrak{q}}(\lambda)$ is a rank one matrix defined as

$$\Pi_{\mathfrak{q}}(\lambda) \;=\; \mathtt{P}_{a_1a_2}(\lambda)\dots\mathtt{P}_{a_{2N-1}a_{2N}}(\lambda)\;, \tag{2.20}$$

and which can be factorised as

$$\Pi_{\mathfrak{q}}(\lambda) \;=\; K_{a_1}^{+}(\lambda)\cdots K_{a_{2N-1}}^{+}(\lambda)\,\mathtt{v}\cdot\mathtt{v}^{\mathrm{t}}K_{a_{2N-1}}^{-}(\lambda)\cdots K_{a_1}^{-}(\lambda)\,, \tag{2.21}$$

where

$$\mathtt{v} \;=\; \Big(\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\boldsymbol{e}_2\otimes\boldsymbol{e}_2\Big)\otimes\cdots\otimes\Big(\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\boldsymbol{e}_2\otimes\boldsymbol{e}_2\Big)\,, \tag{2.22}$$

$\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ being the elements of the canonical basis of $\mathbb{C}^2\simeq\mathfrak{h}_{a_k}$. Note that, in the case where there are no operators in (2.18), one simply obtains the representation obtained in [2]:

$$\boldsymbol{\tau}^N\big(-\tfrac{\beta}{N}\big)\prod_{a=1}^{L}\mathrm{e}^{\frac{h}{2T}\sigma_a^z} \;=\; \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\Pi_{\mathfrak{q}}\big(-\tfrac{\beta}{N}\big)\,\mathtt{T}_{\mathfrak{q};1}(\xi_1)\cdots\mathtt{T}_{\mathfrak{q};L}(\xi_L)\Big]\,. \tag{2.23}$$

Thus, upon taking the traces over the local quantum spaces $\mathfrak{h}_a$ building up the model's Hilbert space $\mathfrak{h}_{XXZ}$, one arrives to the below representation for the finite volume L thermal average of a multi-point function:

$$\begin{aligned}\mathbb{E}_{L;T}\Big[\mathtt{O}^{(1)}_{m_1+1}\cdots\mathtt{O}^{(r)}_{m_r+1}\Big] \;=\; \lim_{N\to+\infty}\bigg\{\Big\{\mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\Pi_{\mathfrak{q}}\big(-\tfrac{\beta}{N}\big)\Big(\mathtt{t}_{\mathfrak{q}}(0)\Big)^{L}\Big]\Big\}^{-1} \\ \times\,\mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\Pi_{\mathfrak{q}}\big(-\tfrac{\beta}{N}\big)\cdot\Big(\mathtt{t}_{\mathfrak{q}}(0)\Big)^{m_1}\cdot\Xi^{(1)}\cdot\Big(\mathtt{t}_{\mathfrak{q}}(0)\Big)^{m_2-m_1-1}\cdot\Xi^{(2)}\cdots\Xi^{(r-1)}\cdot\Big(\mathtt{t}_{\mathfrak{q}}(0)\Big)^{m_r-m_{r-1}-1}\cdot\Xi^{(r)}\cdot\Big(\mathtt{t}_{\mathfrak{q}}(0)\Big)^{L-m_r-1}\Big]\bigg\}\,.\end{aligned} \tag{2.24}$$

Here $\mathtt{t}_{\mathfrak{q}}(0)$ stands for the quantum transfer matrix

$$\mathtt{t}_{\mathfrak{q}}(0) \;=\; \mathrm{tr}_{\mathfrak{h}_0}\Big[\mathtt{T}_{\mathfrak{q};0}(0)\Big]\,, \tag{2.25}$$

and we have also defined

$$\Xi^{(k)} \;=\; \mathrm{tr}_{\mathfrak{h}_0}\Big[\mathtt{T}_{\mathfrak{q};0}(0)\,\mathtt{O}^{(k)}_0\Big]\,. \tag{2.26}$$

To proceed further, *viz.* take the thermodynamic limit of the thermal expectation (2.24), one needs to assume that one may exchange the Trotter and the thermodynamic limits. This has been rigorously established, for T sufficiently large, in the case of the quantum transfer matrix approach to the periodic chain in [11]. Further, that work also established that, at least for T large enough, $\mathtt{t}_{\mathrm{q}}(0)$ admits a maximal in modulus, real Eigenvalue $\widehat{\Lambda}_{0;0}$. The associate dominant Eigenvector is denoted by $\boldsymbol{\Psi}_0^{(0)}$. Thus, under the assumption of commutativity of the Trotter and thermodynamic limits, the thermal average of an r point function projects onto the Trotter limit of an average of a string of operators:

$$\begin{aligned}&\langle \mathtt{O}^{(1)}_{m_1+1}\cdots \mathtt{O}^{(r)}_{m_r+1}\rangle_T \\ &= \lim_{N\to+\infty}\left\{\frac{\Big(\boldsymbol{\Psi}_0^{(0)},\Pi_{\mathrm{q}}\big(-\frac{\beta}{N}\big)\big(\mathtt{t}_{\mathrm{q}}(0)\big)^{m_1}\,\Xi^{(1)}\big(\mathtt{t}_{\mathrm{q}}(0)\big)^{m_2-m_1-1}\,\Xi^{(2)}\cdots\Xi^{(r-1)}\big(\mathtt{t}_{\mathrm{q}}(0)\big)^{m_r-m_{r-1}-1}\,\Xi^{(r)}\,\boldsymbol{\Psi}_0^{(0)}\Big)}{\Big(\boldsymbol{\Psi}_0^{(0)},\Pi_{\mathrm{q}}\big(-\frac{\beta}{N}\big)\boldsymbol{\Psi}_0^{(0)}\Big)\cdot\widehat{\Lambda}_{0;0}^{m_r+1}}\right\}. \qquad (2.27)\end{aligned}$$

The above representation constitutes the starting point of our derivation of the boundary thermal form factor expansion of the multi-point functions.

3 The finite Trotter thermal form factor series expansion

In this section, we explicitly express the general multi-point correlation function (2.27) as the Trotter limit of a sum over a product of finite Trotter thermal form factors — the bulk quantum transfer matrix elements — multiplied by some boundary factor which contains the whole information on the boundary fields. So as to write this form factor expansion, we need to make a short detour relatively to the spectrum of the quantum transfer matrix (2.25).

3.1 The algebraic Bethe Ansatz approach to the quantum transfer matrix

Recall that Eigenstates of the quantum transfer matrix (2.25) can be constructed from the knowledge of the solutions to the following Bethe Ansatz equations,

$$-1 = (-1)^s\,\mathrm{e}^{-\frac{h}{T}}\prod_{k=1}^{M}\left\{\frac{\sinh(\mathrm{i}\zeta+\mu_k-\mu_p)}{\sinh(\mathrm{i}\zeta+\mu_p-\mu_k)}\right\}\cdot\left[\frac{\sinh(\mu_p+\beta/N+\mathrm{i}\zeta)\sinh(\mu_p-\beta/N)}{\sinh(\mathrm{i}\zeta-\mu_p+\beta/N)\sinh(\mu_p+\beta/N)}\right]^N, \qquad (3.1)$$

under the hypothesis that the roots are pairwise distinct, see [11]. Here $s = N - M$ is called the spin. The Eigenstates associated with a solution $\{\mu_a\}_1^M$ will be written as $\boldsymbol{\Psi}\big(\{\mu_a\}_1^M\big)$. These satisfy

$$\mathtt{t}_{\mathrm{q}}(0)\cdot\boldsymbol{\Psi}\big(\{\mu_a\}_1^M\big) \;=\; \tau(0\mid\{\mu_a\}_1^M)\cdot\boldsymbol{\Psi}\big(\{\mu_a\}_1^M\big) \qquad (3.2)$$

in which

$$\begin{aligned}\tau(\xi\mid\{\mu_k\}_1^M) \;=\;& (-1)^N\mathrm{e}^{\frac{h}{2T}}\prod_{k=1}^{M}\left\{\frac{\sinh(\xi-\mu_k+\mathrm{i}\zeta)}{\sinh(\xi-\mu_k)}\right\}\cdot\left(\frac{\sinh(\xi+\beta/N)\sinh(\xi-\beta/N-\mathrm{i}\zeta)}{\sinh^2(-\mathrm{i}\zeta)}\right)^N \\ &+\;(-1)^N\mathrm{e}^{-\frac{h}{2T}}\prod_{k=1}^{M}\left\{\frac{\sinh(\xi-\mu_k-\mathrm{i}\zeta)}{\sinh(\xi-\mu_k)}\right\}\cdot\left(\frac{\sinh(\xi+\beta/N+\mathrm{i}\zeta)\sinh(\xi-\beta/N)}{\sinh^2(-\mathrm{i}\zeta)}\right)^N. \qquad (3.3)\end{aligned}$$

For convenience, we shall sometimes insist on the dependence on the magnetic field h

i) of the matrix entries of the quantum monodromy matrix $\mathtt{A}_h, \mathtt{D}_h, ...,$

ii) of the quantum transfer matrix $\mathtt{t}_{\mathfrak{q};h}$,

iii) of the Bethe roots $\{\mu_a(h)\}_1^M$,

iii) of the Eigenvalues $\tau_h(0 \mid \{\mu_a\}_1^M)$.

It has been rigorously shown in [11] that, for T large enough, uniformly in N, $\mathtt{t}_{\mathfrak{q}}(0)$ has a dominant Eigenvalue $\widehat{\Lambda}_{0;0}$ which can be built from a solution $\{\lambda_a\}_1^N$ to (3.1) with $M = N$. As already discussed, the associated Eigenvector $\mathbf{\Psi}\big(\{\lambda_a\}_1^M\big)$ is denoted $\mathbf{\Psi}_0^{(0)}$ and called the dominant state of the quantum transfer matrix. >From now on, $\{\lambda_a\}_1^N$ will always refer to the set of Bethe roots describing the dominant state.

In a given spin sector s above the dominant state, *viz.* corresponding to Eigenvectors parameterised by

$$M = N - s \tag{3.4}$$

Bethe roots, one may denote, for short, the Eigenvectors of the quantum transfer matrix as $\mathbf{\Psi}_k^{(s)}$. In this notation, $\mathbf{\Psi}_k^{(s)}$ is assumed to be a Bethe vector parameterised by the Bethe roots $\{\mu_a^{(k;s)}\}_1^{N-s}$. Then, one also introduces the notation $\widehat{\Lambda}_{k;s}$ for the associated Eigenvalues, *viz.*

$$\mathtt{t}_{\mathfrak{q}}(0) \cdot \mathbf{\Psi}_k^{(s)} \; = \; \widehat{\Lambda}_{k;s} \cdot \mathbf{\Psi}_k^{(s)} \; . \tag{3.5}$$

Clearly, in order for the above to be well defined, one needs that the quantum transfer matrix is diagonalisable and that the Bethe Ansatz construction provides one with its complete set of Eigenvectors. We shall not dwell on the diagonalisability/completeness issues here and take them for granted or simply irrelevant for the limits considered.

3.2 The general case of multi-point functions

Upon inserting a sum over a complete set of Eigenstates $\mathbf{\Psi}_{k_a}^{(s_a)}$ in front of each of the operators $\Xi^{(k)}$ in the expression (2.27), one obtains a form factor representation for the multi-point correlation functions:

$$\begin{aligned} &\big\langle \mathsf{o}_{m_1+1}^{(1)} \cdots \mathsf{o}_{m_r+1}^{(r)} \big\rangle_T \\ &= \lim_{N\to+\infty} \left\{ \sum_{\substack{k_a \in \mathfrak{D}_a \\ a=1,\dots,r}} \widehat{\Lambda}_{k_1;s_1}^{m_1} \prod_{a=2}^{r} \Big\{ \widehat{\Lambda}_{k_a;s_a}^{m_a - m_{a-1} - 1} \Big\} \widehat{\Lambda}_{0;0}^{-m_r - 1} \cdot \frac{\Big(\mathbf{\Psi}_0^{(0)}, \Pi_{\mathfrak{q}}\big(-\frac{\beta}{N}\big)\mathbf{\Psi}_{k_1}^{(s_1)}\Big)}{\Big(\mathbf{\Psi}_0^{(0)}, \Pi_{\mathfrak{q}}\big(-\frac{\beta}{N}\big)\mathbf{\Psi}_0^{(0)}\Big)} \cdot \prod_{a=1}^{r} \frac{\Big(\mathbf{\Psi}_{k_a}^{(s_a)}, \Xi^{(a)} \mathbf{\Psi}_{k_{a+1}}^{(s_{a+1})}\Big)}{\Big(\mathbf{\Psi}_{k_a}^{(s_a)}, \mathbf{\Psi}_{k_a}^{(s_a)}\Big)} \right\} . \end{aligned} \tag{3.6}$$

Here we have used the notations (3.5), and we agree upon $k_{r+1} = s_{r+1} = 0$. Also, the spins s_a of the inserted Eigenstates are to be taken such that

$$\Big(\mathbf{\Psi}_{k_a}^{(s_a)}, \Xi^{(a)} \mathbf{\Psi}_{k_{a+1}}^{(s_{a+1})}\Big) \; \neq \; 0. \tag{3.7}$$

In (3.13), the summations run over appropriate subsets $\mathfrak{D}_a$ for the k_a's which are compatible with the spin constraints. Also, note that the constraint (3.7) means that we focus on a class of local operators $\mathsf{o}^{(a)} \in \mathrm{End}(\mathbb{C}^2)$ having a definite spin (*i.e.* $\mathrm{id}, \sigma^z, \sigma^{\pm}$). More general operators are of course allowed but then one needs to sum in (3.6) also over the spins s_a on the intermediate states.

We would like to underline that the matrix elements (3.7) are the bulk quantum transfer matrix elements, for which there exist a convenient determinant representation at finite Trotter number N, see [39]. Their Trotter limit can be studied similarly as in [5]. Quite remarkably, the whole information on the boundary field is contained, for each term of the sum, in a single factor, the ratio of matrix elements

$$\frac{\Big(\mathbf{\Psi}_0^{(0)}, \Pi_{\mathfrak{q}}\big(-\frac{\beta}{N}\big)\mathbf{\Psi}_{k_1}^{(s_1)}\Big)}{\Big(\mathbf{\Psi}_0^{(0)}, \Pi_{\mathfrak{q}}\big(-\frac{\beta}{N}\big)\mathbf{\Psi}_0^{(0)}\Big)} . \tag{3.8}$$

The latter can also be represented, at finite Trotter number N in terms of a determinant of size N, as recalled here below.

Given two Bethe vectors $\mathbf{\Psi}_0^{(0)}$ and $\mathbf{\Psi}_k^{(0)}$ parameterised by the roots $\{\lambda_a\}_1^N$, resp. $\{\mu_a^{(k;0)}\}_1^N$, it was indeed established in [37] that one has

$$\Big(\mathbf{\Psi}_0^{(0)}, \Pi_{\mathfrak{q}}\big(-\tfrac{\beta}{N}\big)\mathbf{\Psi}_k^{(0)}\Big) \;=\; \mathcal{F}^{(+)}(\{\lambda_a\}_1^N)\cdot\mathcal{F}^{(-)}(\{\mu_a^{(k;0)}\}_1^N) \tag{3.9}$$

in which

$$\mathcal{F}^{(-)}(\{\mu_a\}_1^N) \;=\; \mathrm{e}^{-\frac{Nh}{2T}}\cdot\mathcal{Z}_N\Big(\{-\tfrac{\beta}{N}\}_1^N;\{\mu_a\}_1^N;\xi_-\Big) \tag{3.10}$$

is expressed in terms of the partition function of the six-vertex model with reflecting ends [41]:

$$\mathcal{Z}_N\Big(\{\xi_a\}_1^N;\{\mu_a\}_1^N;\xi_-\Big) \;=\; \frac{\prod\limits_{a,b=1}^{N}\prod\limits_{\epsilon=\pm}\Big\{\sinh(\xi_a+\epsilon\mu_b)\sinh(\xi_a-\mathrm{i}\zeta+\epsilon\mu_b)\Big\}}{\prod\limits_{a<b}^{N}\Big\{\sinh(\xi_a-\xi_b)\sinh(\xi_a+\xi_b-\mathrm{i}\zeta)\prod\limits_{\epsilon=\pm}\sinh(\mu_b+\epsilon\mu_a)\Big\}} \times \det_N\left[\frac{\sinh(-\mathrm{i}\zeta)\sinh(\xi_-+\mu_b)\sinh(2\xi_a)}{\prod\limits_{\epsilon=\pm}\sinh(\xi_a-\mathrm{i}\zeta+\epsilon\mu_b)\sinh(\xi_a+\epsilon\mu_b)}\right]. \tag{3.11}$$

A similar expression can be written for $\mathcal{F}^{(+)}(\{\lambda_a\}_1^N)$ (see [37]), but it is clear that this factor disappears when considering the ratio (3.8) and therefore will not play any role in the following, so that we omit to recall it here. Also, due to symmetry reasons, it holds that

$$\Big(\mathbf{\Psi}_0^{(0)}, \Pi_{\mathfrak{q}}\big(-\tfrac{\beta}{N}\big)\mathbf{\Psi}_k^{(s)}\Big) \;=\; 0 \tag{3.12}$$

whenever $s\neq 0$.

Hence the form factor representation (3.6) can be rewritten as

$$\begin{aligned}&\langle \mathsf{o}_{m_1+1}^{(1)}\cdots\mathsf{o}_{m_r+1}^{(r)}\rangle_T \\ &\quad= \lim_{N\to+\infty}\left\{\sum_{\substack{k_a\in\mathfrak{D}_a\\ a=1,\dots,r}} \widehat{\Lambda}_{k_1;0}^{m_1}\prod_{a=2}^{r}\Big\{\widehat{\Lambda}_{k_a;s_a}^{m_a-m_{a-1}-1}\Big\}\widehat{\Lambda}_{0;0}^{-m_r-1}\cdot\frac{\mathcal{F}^{(-)}(\{\mu_a^{(k_1;0)}\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a\}_1^N)}\cdot\prod_{a=1}^{r}\frac{\Big(\mathbf{\Psi}_{k_a}^{(s_a)},\Xi^{(a)}\mathbf{\Psi}_{k_{a+1}}^{(s_{a+1})}\Big)}{\Big(\mathbf{\Psi}_{k_a}^{(s_a)},\mathbf{\Psi}_{k_a}^{(s_a)}\Big)}\right\}, \end{aligned}\tag{3.13}$$

with here $s_1 = 0$ due to (3.12). We remind that $\{\lambda_a\}_1^N$ stands for the set of Bethe roots describing the dominant Eigenstate of the quantum transfer matrix. Also, we used explicitly that the number of Bethe roots describing the first inserted excited state $\mathbf{\Psi}_{k_1}^{(0)}$ is exactly N.

3.3 The case of one-point functions

We now specialise the above framework to the interesting particular case of the one-point functions, *viz.* the thermal average of some local operator at distance m from the boundary. By symmetry, the sole non-trivial one-point functions that are non-zero are the thermal expectation of the operator σ_{m+1}^z. For the sake of simplicity, from now on, we denote by $\{\mu_a(h')\}_1^N$ a set of solutions to the Bethe Ansatz equations (3.1) at external magnetic field h'. With this being settled, one may specialise the previous results as follows.

Lemma 3.1. *It holds*

$$\langle \sigma^z_{m+1} \rangle_T \;=\; \lim_{N\to+\infty} \Big\{ 2T \partial_{h'} \mathfrak{D}_m Q_N(h',m) \Big\}_{|h'=h} \tag{3.14}$$

where $\mathfrak{D}_m u = u_{m+1} - u_m$ *and*

$$Q_N(h',m) \;=\; \sum_{\{\mu_a(h')\}_1^N} \mathrm{e}^{\frac{N(h'-h)}{2T}} \, \frac{\mathcal{F}^{(-)}(\{\mu_a(h')\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)} \cdot \frac{\Big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\lambda_a(h)\}_1^N)\Big)}{\Big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\mu_a(h')\}_1^N)\Big)} \cdot \bigg(\frac{\tau_{h'}(0 \mid \{\mu_a(h')\}_1^N)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N)} \bigg)^m . \tag{3.15}$$

Above, the sum runs over all solutions to the Bethe Ansatz equations (3.1)

Proof —

Making the expansion (3.13) explicit, one gets

$$\langle \sigma^z_{m+1} \rangle_T = \lim_{N\to+\infty} \left\{ \sum_{\{\mu_a(h)\}_1^N} \frac{\mathcal{F}^{(-)}(\{\mu_a(h)\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)} \cdot \frac{\Big(\mathbf{\Psi}(\{\mu_a(h)\}_1^N), (\mathtt{A}-\mathtt{D})(0)\cdot \mathbf{\Psi}(\{\lambda_a(h)\}_1^N)\Big)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N) \cdot \Big(\mathbf{\Psi}(\{\mu_a(h)\}_1^N), \mathbf{\Psi}(\{\mu_a(h)\}_1^N)\Big)} \cdot \bigg(\frac{\tau_h(0 \mid \{\mu_a(h)\}_1^N)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N)} \bigg)^m \right\}.$$

Observe that it holds

$$\partial_{h'} \mathtt{t}_{\mathfrak{q};h'}(0)_{|h'=h} \;=\; \frac{1}{2T} \Big(\mathtt{A}_h(0) - \mathtt{D}_h(0) \Big) . \tag{3.16}$$

From there, it readily follows that

$$\frac{\Big(\mathbf{\Psi}(\{\lambda_a(h)\}_1^N), (\mathtt{A}_h - \mathtt{D}_h)(0) \mathbf{\Psi}(\{\mu_a(h)\}_1^N)\Big)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N)} = 2T \frac{\partial}{\partial h'} \left\{ \bigg(\frac{\tau_{h'}(0 \mid \{\mu_a(h')\}_1^N)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N)} - 1 \bigg) \cdot \Big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\lambda_a(h)\}_1^N) \Big) \right\}_{|h=h'} .$$

Then, by using the discrete lattice derivative $\mathfrak{D}_m u = u_{m+1} - u_m$, one gets that

$$\begin{aligned} & \frac{\mathcal{F}^{(-)}(\{\mu_a(h)\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)} \cdot \frac{\Big(\mathbf{\Psi}(\{\mu_a(h)\}_1^N), (\mathtt{A}_h-\mathtt{D}_h)(0)\cdot \mathbf{\Psi}(\{\lambda_a(h)\}_1^N)\Big)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N) \cdot \Big(\mathbf{\Psi}(\{\mu_a(h)\}_1^N), \mathbf{\Psi}(\{\mu_a(h)\}_1^N)\Big)} \cdot \bigg(\frac{\tau_h(0 \mid \{\mu_a(h)\}_1^N)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N)} \bigg)^m \\ & \quad = 2T \partial_{h'} \mathfrak{D}_m \left\{ \mathrm{e}^{\frac{N(h'-h)}{2T}} \, \frac{\mathcal{F}^{(-)}(\{\mu_a(h')\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)} \cdot \frac{\Big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\lambda_a(h)\}_1^N)\Big)}{\Big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\mu_a(h')\}_1^N)\Big)} \cdot \bigg(\frac{\tau_{h'}(0 \mid \{\mu_a(h')\}_1^N)}{\tau_h(0 \mid \{\lambda_a(h)\}_1^N)} \bigg)^m \right\} . \end{aligned} \tag{3.17}$$

The claim follows upon putting the formulae together. ■

4 Taking the Trotter limit

In order to obtain a closed expression for the multi-point correlation functions at finite T one should still show that the Trotter limit can be taken on the level of (3.13). This demands to extract the large-N behaviour of the matrix elements ratios

$$\frac{\Big(\mathbf{\Psi}^{(s_a)}_{k_a}, \Xi^{(a)} \mathbf{\Psi}^{(s_{a+1})}_{k_{a+1}}\Big)}{\Big(\mathbf{\Psi}^{(s_a)}_{k_a}, \mathbf{\Psi}^{(s_a)}_{k_a}\Big)} , \tag{4.1}$$

as well as the one of the boundary factor given by the ratios of partition functions of the six-vertex model with reflecting ends

$$\mathcal{F}_{\mathcal{B}}(\{\mu_a(h')\}_1^N;\{\lambda_a(h)\}_1^N;\xi_-) = \mathrm{e}^{\frac{N(h'-h)}{2T}}\,\frac{\mathcal{F}^{(-)}(\{\mu_a(h')\}_1^N)}{\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)} \;=\; \frac{\mathcal{Z}_N\Big(\{-\frac{\beta}{N}\}_1^N;\{\mu_a(h')\}_1^N;\xi_-\Big)}{\mathcal{Z}_N\Big(\{-\frac{\beta}{N}\}_1^N;\{\lambda_a(h)\}_1^N;\xi_-\Big)}\;. \tag{4.2}$$

As already mentioned, the ratios (4.1) can be expressed in terms of ratios of determinants whose size grows linearly with N by use of the Gaudin and Slavnov representations [33, 39]. When $k_{a+1} = s_{a+1} = 0$, their Trotter limit was considered in [5] and, although cumbersome, the generalisation of these considerations to generic k_{a+1} and s_{a+1} is straightforward. The large-N behaviour of $\mathcal{F}^{(-)}(\{\lambda_a(h)\}_1^N)$, *viz.* for the distribution of Bethe roots describing the dominant state of the quantum transfer matrix, was also considered in [37]. Note that, for the consideration of the one-point function (3.14) or of the more general correlation function (3.13), we need here to consider the more general case involving the set of Bethe roots for an arbitrary excited state of the quantum transfer matrix.

The purpose of this section is to explain how one can reformulate the matrix elements (4.1) and the boundary factor (4.2) in a convenient way for the consideration of the Trotter limit. As usual in the QTM approach, we use for that the reformulation of the QTM spectrum in terms of some non-linear integral equation for the associated counting function. We shall see that both quantities (4.1) and (4.2) can be expressed as appropriate contour integrals involving this counting function.

In the following, we shall not consider the most general case and focus on the ratios arising in the study of one-point functions (3.14)-(3.15) since the latter already highlights all the technicalities arising in the computation of the Trotter limit. Also, for definiteness, we shall focus on the regime $\zeta \in]0;\pi[$ $(-1 < \Delta < 1)$.

4.1 The non-linear integral description of the spectrum of the quantum transfer matrix

To study the solutions to the Bethe equations and provide an exact characterisation of certain spectral properties of the quantum transfer matrix - see [11] for a precise statement-, following [4, 32], it is useful to introduce the counting function associated with a solution $\{\mu_a\}_1^M$ to the Bethe Ansatz equations

$$\widehat{\mathfrak{a}}(\xi \mid \{\mu_a\}_1^M) \;=\; \mathrm{e}^{-\frac{h}{T}}(-1)^s \prod_{k=1}^{M}\left\{\frac{\sinh(\mathrm{i}\zeta-\xi+\mu_k)}{\sinh(\mathrm{i}\zeta+\xi-\mu_k)}\right\}\cdot\left\{\frac{\sinh(\xi-\beta/N)\sinh(\mathrm{i}\zeta+\xi+\beta/N)}{\sinh(\xi+\beta/N)\sinh(\mathrm{i}\zeta-\xi+\beta/N)}\right\}^N\;. \tag{4.3}$$

4.1.1 The non-linear equation at finite Trotter number

The main point is that the counting function can be expressed in an alternative way which allows one to study the Trotter limit and, subsequently, the low-T limit efficiently. For that purpose, one first focuses on the solution describing the dominant state of the quantum transfer matrix. We remind that these roots are denoted by $\{\lambda_a\}_1^N$. Those roots were thoroughly characterised, on rigorous grounds, in [11] for temperatures large enough. Basically, one chooses a base contour $\mathscr{C}$ that encircles all the roots $\{\lambda_a\}_1^N$ as well as a neighbourhood of the origin, but not any other roots of $1+\widehat{\mathfrak{a}}(\xi \mid \{\lambda_a\}_1^N)$. By construction, the contour $\mathscr{C}$ is such that $1+\widehat{\mathfrak{a}}(\xi \mid \{\lambda_a\}_1^N)$ enjoys the zero monodromy condition relatively to it:

$$0 = \oint_{\mathscr{C}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\widehat{\mathfrak{a}}'(\xi \mid \{\lambda_a\}_1^N)}{1+\widehat{\mathfrak{a}}(\xi \mid \{\lambda_a\}_1^N)}\;. \tag{4.4}$$

We shall denote by $\mathscr{D}$ the compact domain such that $\partial\mathscr{D} = \mathscr{C}$.

One may also construct an associated contour $\mathscr{C}_s$ relatively to the dominant Eigenstate of the quantum transfer matrix in the spin s sector. From now on, for practical reasons, we shall restrict our analysis to the case $s = 0$ which is directly relevant to the one-point function setting.

Then a sub-dominant Eigenstate of the quantum transfer matrix in the $s = 0$ sector is characterised by the data:

- a set $\widehat{\mathfrak{X}}$ gathering the positions of hole roots contained inside of $\mathscr{C}$;
- a set $\widehat{\mathcal{Y}}$ gathering the positions of the particle roots contained outside of $\mathscr{C}$ within any $\mathrm{i}\pi$-periodic strip;
- the set of singular roots $\widehat{\mathcal{Y}}_{\mathrm{sg}}$ built out of a subset of $\widehat{\mathcal{Y}}$: $\widehat{\mathcal{Y}}_{\mathrm{sg}} = \{y - \mathrm{i}\zeta_{\mathrm{m}}\mathfrak{s}_2 \, : \, y \in \widehat{\mathcal{Y}} \text{ and } y - \mathrm{i}\zeta_{\mathrm{m}}\mathfrak{s}_2 \in \mathrm{Int}(\mathscr{C})\}$, where $\zeta_{\mathrm{m}} = \min(\zeta, \pi - \zeta)$ and $\mathfrak{s}_2 = \mathrm{sign}(\pi - 2\zeta)$.

While this is not essential for our calculations, we shall henceforth make the simplifying assumption that the roots building $\widehat{\mathcal{Y}}$, $\widehat{\mathcal{Y}}_{\mathrm{sg}}$ and $\widehat{\mathfrak{X}}$ are all pairwise distinct and that we deal solely with excited states having no singular roots, *viz* $\widehat{\mathcal{Y}}_{\mathrm{sg}} = \emptyset$. Treating the general case will not pose any problems but will definitely lead to cumbersome calculations.

Following the notations of [11], it is convenient to gather all the roots parameterising a sub-dominant state in terms of the formal difference of sets

$$\widehat{\mathbb{Y}} = \widehat{\mathcal{Y}} \ominus \widehat{\mathfrak{X}} \, . \tag{4.5}$$

The main statement of the non-linear integral equation based approach to the spectrum of the quantum transfer matrix approach is that one may equivalently parameterise the counting function associated with an excited state $\{\mu_a\}_1^M$ in terms of the set $\widehat{\mathbb{Y}}$ and of s. Henceforth, we shall thus denote the associated counting function as $\widehat{\mathfrak{a}}_{\mathbb{Y}}$ for short. In particular, he counting function associated with the dominant state is denoted $\widehat{\mathfrak{a}}_{\emptyset}$. Within this convention, the zero monodromy condition on the contour $\mathscr{C}$ translates itself into a constraint on the cardinalities of the particle and hole roots' sets:

$$0 = \oint_{\mathscr{C}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\widehat{\mathfrak{a}}'_{\mathbb{Y}}(\mu)}{1 + \widehat{\mathfrak{a}}_{\mathbb{Y}}(\mu)} = |\widehat{\mathfrak{X}}| - |\widehat{\mathcal{Y}}| - s \, . \tag{4.6}$$

In particular, in the $s = 0$ sector, $|\widehat{\mathfrak{X}}| = |\widehat{\mathcal{Y}}|$. Also, the introduction of $\widehat{\mathbb{Y}}$ allows us to make use of the below conventions:

$$\sum_{\lambda \in \widehat{\mathbb{Y}}} f(\lambda) = \sum_{y \in \widehat{\mathcal{Y}}} f(y) - \sum_{x \in \widehat{\mathfrak{X}}} f(x) \qquad \text{and} \qquad \prod_{\lambda \in \widehat{\mathbb{Y}}} f(\lambda) = \frac{\prod\limits_{y \in \widehat{\mathcal{Y}}} f(y)}{\prod\limits_{x \in \widehat{\mathfrak{X}}} f(x)} \, . \tag{4.7}$$

Following the standard procedure, one gets the non-linear integral equation satisfied by $\widehat{\mathfrak{a}}_{\mathbb{Y}}(\xi) = \mathrm{e}^{\widehat{\mathfrak{A}}_{\mathbb{Y}}(\xi)}$:

$$\widehat{\mathfrak{A}}_{\mathbb{Y}}(\xi) = -\frac{h}{T} + \mathfrak{w}_N(\xi) - \mathrm{i}\pi s + \mathrm{i} \sum_{y \in \widehat{\mathbb{Y}}} \theta(\xi - y) + \oint_{\mathscr{C}} K(\xi - u) \cdot \mathscr{L}\mathrm{n}\Big[1 + \mathrm{e}^{\widehat{\mathfrak{A}}_{\mathbb{Y}}}\Big](u) \cdot \mathrm{d}u \tag{4.8}$$

with $\xi \in \mathcal{S}_{\zeta_{\mathrm{m}}/2}$, the strip of width $\zeta_{\mathrm{m}} = \min(\zeta, \pi - \zeta)$ around $\mathbb{R}$. Above, for $v \in \mathscr{C}$, one has

$$\mathscr{L}\mathrm{n}\Big[1 + \mathrm{e}^{\widehat{\mathfrak{A}}_{\mathbb{Y}}}\Big](v) = \int_{\kappa}^{v} \frac{\widehat{\mathfrak{A}}'_{\mathbb{Y}}(u)}{1 + \mathrm{e}^{-\widehat{\mathfrak{A}}_{\mathbb{Y}}(u)}} \cdot \mathrm{d}u + \ln\Big[1 + \mathrm{e}^{\widehat{\mathfrak{A}}_{\mathbb{Y}}(\kappa)}\Big] \, . \tag{4.9}$$

Here κ is some point on $\mathscr{C}$ and the integral is taken, in the positive direction along $\mathscr{C}$, from κ to ν. The function "ln" appearing above corresponds to the principal branch of the logarithm extended to $\mathbb{R}^-$ with the convention $\arg(z) \in [-\pi\,;\pi[$. The functions θ and K are respectively defined as

$$\theta(\lambda) = \begin{cases} \mathrm{i}\ln\Big(\dfrac{\sinh(\mathrm{i}\zeta+\lambda)}{\sinh(\mathrm{i}\zeta-\lambda)}\Big) & \text{for} \quad |\Im(\lambda)| < \zeta_m, \\ -\pi\,\mathfrak{s}_2 + \mathrm{i}\ln\Big(\dfrac{\sinh(\mathrm{i}\zeta+\lambda)}{\sinh(\lambda-\mathrm{i}\zeta)}\Big) & \text{for} \quad \zeta_m < |\Im(\lambda)| < \pi/2, \end{cases} \tag{4.10}$$

$$K(\lambda) = \frac{1}{2\pi}\theta'(\lambda) = \frac{\mathfrak{s}_2}{2\pi\mathrm{i}}\big[\coth(\lambda-\mathrm{i}\zeta_m) - \coth(\lambda+\mathrm{i}\zeta_m)\big] = \frac{\sin(2\zeta)}{2\pi\,\sinh(\lambda-\mathrm{i}\zeta)\,\sinh(\lambda+\mathrm{i}\zeta)}\;, \tag{4.11}$$

whereas

$$\mathfrak{w}_N(\xi) \;=\; N\ln\Big(\frac{\sinh(\xi-\beta/N)\sinh(\xi+\beta/N-\mathrm{i}\zeta)}{\sinh(\xi+\beta/N)\sinh(\xi-\beta/N-\mathrm{i}\zeta)}\Big)\,. \tag{4.12}$$

One should note that the particle and hole roots forming $\widehat{\mathbb{Y}}$ are subject to the subsidiary conditions

$$\widehat{\mathfrak{a}}_{\mathbb{Y}}(x) \;=\; -1 \quad \forall x \in \widehat{\mathfrak{X}} \qquad \text{and} \qquad \widehat{\mathfrak{a}}_{\mathbb{Y}}(y) \;=\; -1 \quad \forall y \in \widehat{\mathcal{Y}}\,. \tag{4.13}$$

In the remaining part of the paper, we shall use the following convenient shorthand notation for an appropriate determination of the logarithm of $1+\widehat{\mathfrak{a}}_{\mathbb{Y}}$:

$$\widehat{\mathscr{L}_{\mathbb{Y}}}(s) \;=\; \frac{1}{2\mathrm{i}\pi}\mathscr{L}n\Big(1+\widehat{\mathfrak{a}}_{\mathbb{Y}}\Big)(s)\,. \tag{4.14}$$

4.1.2 The non-linear equation in the infinite Trotter number limit

To compute the infinite Trotter number limit, one *assumes* that $\widehat{\mathfrak{A}}_{\mathbb{Y}} \underset{N\to+\infty}{\longrightarrow} \mathfrak{A}_{\mathbb{Y}}$ pointwise on $\mathscr{C}$, and that all properties of the non-linear problem are preserved under this limit. Then, one may readily deduce a non-linear integral equation satisfied by $\mathfrak{A}_{\mathbb{Y}}$. This procedure has been set into a rigorous setting, for T large enough, in [11]. Here we stick with a formal exposure and refer the interested reader to the mentioned work for more analytic details.

In the infinite Trotter number limit, one has

$$\mathfrak{w}_N(\xi) \underset{N\to+\infty}{\longrightarrow} -2\beta\Big(\coth(\xi)-\coth(\xi-\mathrm{i}\zeta)\Big) = \frac{2J\sin^2(\zeta)}{T\sinh(\xi)\sinh(\xi-\mathrm{i}\zeta)}\,. \tag{4.15}$$

In addition one *assumes* the existence of the limit of the particle and hole roots in the below sense. First, one parameterises $\widehat{\mathfrak{X}} = \{\widehat{x}_a\}_1^{|\widehat{\mathfrak{X}}|}$ and $\widehat{\mathcal{Y}} = \{\widehat{y}_a\}_1^{|\widehat{\mathcal{Y}}|}$, so that, when $N\to+\infty$

$$\widehat{x}_a \to x_a + \mathrm{i}\frac{\zeta}{2}\;,\;\; a=1,\dots,|\mathfrak{X}| \qquad \text{and} \qquad \widehat{y}_a \to y_a + \mathrm{i}\frac{\zeta}{2}\;,\;\; a=1,\dots,|\mathcal{Y}|\,. \tag{4.16}$$

All these assumptions result in the non-linear integral equation satisfied by the limit function on $\mathcal{S}_{\zeta_{\mathrm{m}}/2}$, the strip of width ζ_{m} centered around $\mathbb{R}$:

$$\mathfrak{A}_{\mathbb{Y}}(\xi) \;=\; -\frac{1}{T}e_0(\xi) \;-\; \mathrm{i}\pi s \;+\; \mathrm{i}\sum_{y\in\mathbb{Y}}\theta(\xi-y) \;+\; \oint_{\mathscr{C}} K(\xi-u)\cdot\mathscr{L}\mathrm{n}\Big[1+\mathrm{e}^{\mathfrak{A}_{\mathbb{Y}}}\Big](u)\cdot\mathrm{d}u\,. \tag{4.17}$$

Here,

$$e_0(\xi) \;=\; h \,-\, \frac{2J\sin^2(\zeta)}{\sinh(\xi)\sinh(\xi - \mathrm{i}\zeta)} \;. \tag{4.18}$$

Furthermore,

$$\mathbb{Y} \;=\; \Big\{\mathcal{Y} \,+\, \mathrm{i}\tfrac{\zeta}{2}\Big\} \ominus \Big\{\mathfrak{X} \,+\, \mathrm{i}\tfrac{\zeta}{2}\Big\} \,, \tag{4.19}$$

where

$$\mathfrak{X} \;=\; \{x_a\}_{a=1}^{|\mathfrak{X}|} \qquad \text{and} \qquad \mathcal{Y} \;=\; \{y_a\}_{a=1}^{|\mathcal{Y}|} \;. \tag{4.20}$$

The non-linear integral equation at infinite Trotter number is to be supplemented with the constraints

$$0 \;=\; \oint\limits_{\mathscr{C}} \frac{\mathfrak{A}'_{\mathbb{Y}}(u)}{1+\mathrm{e}^{-\mathfrak{A}_{\mathbb{Y}}(u)}} \cdot \frac{\mathrm{d}u}{2\mathrm{i}\pi} \;=\; |\mathfrak{X}| - |\mathcal{Y}| - s \quad \text{and} \quad \left\{ \begin{array}{rcl} \mathfrak{a}_{\mathbb{Y}}(x+\mathrm{i}\frac{\zeta}{2}) & = & -1 \quad \forall x \in \mathfrak{X} \\ \mathfrak{a}_{\mathbb{Y}}(y+\mathrm{i}\frac{\zeta}{2}) & = & -1 \quad \forall y \in \mathcal{Y} \end{array} \right. \;, \tag{4.21}$$

with $\mathfrak{a}_{\mathbb{Y}}(\xi) = \mathrm{e}^{\mathfrak{A}_{\mathbb{Y}}(\xi)}$. Similarly as in (4.14), we shall also denote

$$\mathscr{L}_{\mathbb{Y}}(s) \;=\; \frac{1}{2\mathrm{i}\pi}\mathscr{L}n\Big(1+\mathfrak{a}_{\mathbb{Y}}\Big)(s) \;. \tag{4.22}$$

4.1.3 The Eigenvalues of the quantum transfer matrix

One readily obtains the below representation for the Eigenvalues of the transfer matrix, labelled by the particle-hole roots' parameters $\widehat{\mathbb{Y}}$:

$$\widehat{\tau}_{\mathbb{Y}}(0) \;=\; \prod_{y\in\widehat{\mathbb{Y}}} \frac{\sinh(y-\mathrm{i}\zeta)}{\sinh(y)} \cdot \Big(\frac{\sinh(\beta/N+\mathrm{i}\zeta)}{\sinh(\mathrm{i}\zeta)}\Big)^{2N} \cdot \exp\Bigg\{ \frac{h}{2T} \,-\, \oint\limits_{\mathscr{C}} \frac{\mathrm{i}\sin(\zeta)\mathscr{L}\mathrm{n}\big[1+\mathrm{e}^{\widehat{\mathfrak{A}}_{\mathbb{Y}}}\big](u)}{\sinh(u-\mathrm{i}\zeta)\sinh(u)} \frac{\mathrm{d}u}{2\mathrm{i}\pi} \Bigg\} \,. \tag{4.23}$$

The Trotter limit can easily be taken on the level of this representation, leading to $\widehat{\tau}_{\mathbb{Y}}(0) \to \tau_{\mathbb{Y}}(0)$ where

$$\tau_{\mathbb{Y}}(0) \;=\; \prod_{y\in\mathcal{Y}\ominus\mathfrak{X}} \frac{\sinh(y-\mathrm{i}\zeta/2)}{\sinh(y+\mathrm{i}\zeta/2)} \cdot \exp\Bigg\{ \frac{h}{2T} \,-\, \frac{2J}{T}\cos(\zeta) \,-\, \oint\limits_{\mathscr{C}-\mathrm{i}\frac{\zeta}{2}} \frac{\mathrm{i}\sin(\zeta)\mathscr{L}\mathrm{n}\big[1+\mathrm{e}^{\mathfrak{A}_{\mathbb{Y}}}\big](u+\mathrm{i}\zeta/2)}{\sinh(u-\mathrm{i}\zeta/2)\sinh(u+\mathrm{i}\zeta/2)} \frac{\mathrm{d}u}{2\mathrm{i}\pi} \Bigg\} \,. \tag{4.24}$$

4.2 The boundary factor

We now explain how to represent the boundary factor (4.2) in terms of appropriate contour integrals involving the corresponding counting function $\widehat{\mathfrak{a}}_{\mathbb{Y}}$ solution of (4.4).

It was established in [37] that, for any set of Bethe roots $\{\mu_a\}_1^N$ at magnetic field h' solving (3.1) and labelled by the particle-hole roots' parameters $\widehat{\mathbb{Y}}$, the partition function (3.11) involved in the expression of the boundary factor (4.2) can be represented in the following form:

$$\mathcal{Z}_N\Big(\{-\tfrac{\beta}{N}\}_1^N;\{\mu_a\}_1^N;\xi_-\Big) \;=\; \mathcal{P}(\{\mu_a\}_1^N)\cdot \mathrm{e}^{\widehat{\mathscr{F}}_{\mathbb{Y}}} \cdot \Big(\frac{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)}{1-\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)}\Big)^{\frac{1}{4}} \;. \tag{4.25}$$

The latter involves the counting function $\widehat{\mathfrak{a}}_{\mathbb{Y}}$ that has been defined in (4.3), a pure product function $\mathcal{P}$ which reads

$$\mathcal{P}(\{\mu_a\}_1^N) \;=\; \prod_{a=1}^{N} G(\mu_a)\cdot\prod_{a<b}^{N} f(\mu_a,\mu_b) \tag{4.26}$$

with

$$f(\lambda,\mu) \;=\; \frac{\sinh(\lambda+\mu-\mathrm{i}\zeta)}{\sinh(\lambda+\mu)}, \tag{4.27}$$

$$G(\mu) \;=\; \frac{\sinh(2\beta/N)\sinh(\mu+\xi_-)}{\sinh(2\mu)}\Big\{\sinh(\mu-\tfrac{\beta}{N})\sinh(\mathrm{i}\zeta+\mu+\tfrac{\beta}{N})\Big\}^N\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\mu)\Big\}^{\frac{1}{2}}\;, \tag{4.28}$$

and a contribution $\widehat{\mathscr{F}}_{\mathbb{Y}}$ which is given by a series of multiple contour integrals:

$$\widehat{\mathscr{F}}_{\mathbb{Y}} \;=\; \sum_{k=0}^{+\infty}\oint_{\mathscr{C}^{(1)}\supset\cdots\supset\mathscr{C}^{(2k+1)}}\sum_{n=k}^{+\infty}\frac{\Big[\widehat{\mathfrak{r}}_{\mathbb{Y}}(\omega_{2k+1})\Big]^{n-k}}{2n+1}\;\cdot\;\prod_{p=1}^{2k+1}\widehat{U}_{\mathbb{Y}}(\omega_p,\omega_{p+1})\;\frac{\mathrm{d}^{2k+1}\omega}{(2\mathrm{i}\pi)^{2k+1}}$$
$$-\sum_{k=1}^{+\infty}\oint_{\mathscr{C}^{(1)}\supset\cdots\supset\mathscr{C}^{(2k)}}\sum_{n=k}^{+\infty}\frac{\Big[\widehat{\mathfrak{r}}_{\mathbb{Y}}(\omega_{2k})\Big]^{n-k}}{2n}\;\cdot\;\prod_{p=1}^{2k}\widehat{U}_{\mathbb{Y}}(\omega_p,\omega_{p+1})\;\frac{\mathrm{d}^{2k}\omega}{(2\mathrm{i}\pi)^{2k}}\;. \tag{4.29}$$

In (4.29), the integration contours are encased contours such that, for any p, $\mathscr{C}^{(1)}\supset\cdots\supset\mathscr{C}^{(p)}$, and that $\mathscr{C}^{(k)}$, $k=1,\ldots,p$, enlaces the roots $\mu_1,\ldots,\mu_N$ but not the ones that are shifted by $\pm\mathrm{i}\zeta$. The integration variables in (4.29) satisfy to the convention $\omega_{2k+2}\equiv\omega_1$ while the integrands are built up from the function

$$\widehat{\mathfrak{r}}_{\mathbb{Y}}(\omega) = \mathrm{e}^{-\frac{2h'}{T}}\prod_{a=1}^{N}\frac{\sinh(\mu_a+\omega+\mathrm{i}\zeta)\sinh(\mu_a-\omega+\mathrm{i}\zeta)}{\sinh(\mu_a+\omega-\mathrm{i}\zeta)\sinh(\mu_a-\omega-\mathrm{i}\zeta)}\;, \tag{4.30}$$

as well as the kernel

$$\widehat{U}_{\mathbb{Y}}(\omega,\omega') = \frac{-\mathrm{e}^{-\frac{h'}{T}}\sinh(2\omega'-\mathrm{i}\zeta)}{\sinh(\omega+\omega')\sinh(\omega-\omega'-\mathrm{i}\zeta)}\prod_{a=1}^{N}\frac{\sinh(\omega'+\mu_a)\sinh(\omega'-\mu_a-\mathrm{i}\zeta)}{\sinh(\omega'-\mu_a)\sinh(\omega'+\mu_a-\mathrm{i}\zeta)}\;. \tag{4.31}$$

In [37], the representation (4.25), for the particular case of the Bethe roots $\{\lambda_a\}_1^N$ describing the dominant Eigenstate of the quantum transfer matrix, was reformulated in a smooth way for the consideration of the Trotter limit, in terms of contour integrals involving the function $\widehat{\mathfrak{a}}_{\emptyset}$. We now explain how to similarly rewrite, in terms of the function $\widehat{\mathfrak{a}}_{\mathbb{Y}}$, the partition function (4.25) for the more general case of a set of Bethe roots $\{\mu_a\}_1^N$ describing some sub-dominant state of the quantum transfer matrix labelled by the particle-hole roots' parameters $\widehat{\mathbb{Y}}$. For the purpose of the calculations to come, it is useful to introduce an auxiliary set of roots $\{\nu_a\}_1^N$ which corresponds to the collection of all the roots of $1+\widehat{\mathfrak{a}}_{\mathbb{Y}}$ located inside of $\mathscr{C}$:

$$\{\mu_a\}_1^N \;=\; \{\nu_a\}_1^{N-n}\cup\{\widehat{y}_a\}_1^n \qquad\text{and}\qquad \{\nu_a\}_1^N \;=\; \{\nu_a\}_1^{N-n}\cup\{\widehat{x}_a\}_1^n \tag{4.32}$$

where we chose the parameterisation of the sets $\widehat{\mathcal{Y}}=\{\widehat{y}_a\}_1^n$ and $\widehat{\mathfrak{X}}=\{\widehat{x}_a\}_1^n$.

For a sub-dominant state of the quantum transfer matrix as described above, one can factorise the product function $\mathcal{P}$ (4.26) in the form

$$\mathcal{P}(\{\mu_a\}_1^N) \;=\; \mathcal{P}(\{\nu_a\}_1^N)\cdot\mathcal{P}_{\mathrm{bk}}\Big(\{\nu_a\}_1^N;\widehat{\mathbb{Y}}\Big)\cdot\mathcal{P}_{\mathrm{loc}}\Big(\widehat{\mathbb{Y}}\Big)\,, \tag{4.33}$$

in which, using the product convention (4.7), we have defined

$$\mathcal{P}_{\text{bk}}\Big(\{\nu_a\}_1^N;\widehat{\mathbb{Y}}\Big) \;=\; \prod_{a=1}^{N}\prod_{z\in\widehat{\mathbb{Y}}} f(\nu_a,z)\,, \tag{4.34}$$

$$\mathcal{P}_{\text{loc}}\Big(\widehat{\mathbb{Y}}\Big) \;=\; \frac{\prod\limits_{a<b}^{n}\Big\{f(\widehat{y}_a,\widehat{y}_b)\,f(\widehat{x}_a,\widehat{x}_b)\Big\}}{\prod\limits_{a,b=1}^{n} f(\widehat{x}_a,\widehat{y}_b)}\prod_{a=1}^{n}\Big\{\frac{G(\widehat{y}_a)}{G(\widehat{x}_a)}f(\widehat{x}_a,\widehat{x}_a)\Big\}\,. \tag{4.35}$$

This decomposition follows readily from the product identity

$$\prod_{a<b}^{N} f(\mu_a,\mu_b) \;=\; \prod_{a<b}^{N} f(\nu_a,\nu_b)\cdot\prod_{a=1}^{N}\prod_{b=1}^{n}\frac{f(\nu_a,\widehat{y}_b)}{f(\nu_a,\widehat{x}_b)}\cdot\frac{\prod\limits_{a<b}^{n}\Big\{f(\widehat{y}_a,\widehat{y}_b)f(\widehat{x}_a,\widehat{x}_b)\Big\}}{\prod\limits_{a,b=1}^{n} f(\widehat{x}_a,\widehat{y}_b)}\cdot\prod_{a=1}^{n} f(\widehat{x}_a,\widehat{x}_a)\,. \tag{4.36}$$

The rewriting of $\mathcal{P}$ in terms of contour integrals is based on the following lemma concerning the rewriting of products over the Bethe roots ν_a, $1\le a\le N$:

Lemma 4.1. *Let* $\{\mu_a\}_1^N$ *be a solution of the Bethe Ansatz equations* (3.1)*,* $\widehat{\mathfrak{a}}_{\mathbb{Y}}$ *its associated counting function and let the roots* $\{\nu_a\}_1^N$ *be defined as in* (4.32)*. Then, for* z *an arbitrary parameter, we have*

$$\prod_{a=1}^{N}\sinh(\nu_a+z) \;=\; \Big\{\sinh(z-\tfrac{\beta}{N})\Big\}^{N}\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-z)\Big\}^{\mathbf{1}_{\mathscr{D}}(-z)}\cdot\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\coth(s+z)\Big\}\,, \tag{4.37}$$

$$\prod_{a=1}^{N}\sinh(2\nu_a+z) \;=\; \Big\{\sinh(z-2\tfrac{\beta}{N})\Big\}^{N}\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\frac{z}{2})\Big\}^{\mathbf{1}_{\mathscr{D}}(-\frac{z}{2})}\cdot\exp\Big\{-2\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\coth(2s+z)\Big\}\,, \tag{4.38}$$

as well as

$$\prod_{a,b=1}^{N}\sinh(\nu_a+\nu_b+z) \;=\; \Big\{\sinh(z-\tfrac{2\beta}{N})\Big\}^{N^2}\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-z+\tfrac{\beta}{N})\Big\}^{N\cdot\mathbf{1}_{\mathscr{D}}(-z+\frac{\beta}{N})}\cdot\prod_{a=1}^{N}\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-z-\nu_a)\Big\}^{\mathbf{1}_{\mathscr{D}}(-z-\nu_a)}$$
$$\times\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\,\coth'(s+s'+z)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s')\;-\;2N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\coth(s-\tfrac{\beta}{N}+z)\Big\}\,. \tag{4.39}$$

Here $\mathbf{1}_{\mathscr{D}}(\omega)$ *is equal to* 1 *when* $\omega\in\mathscr{D}$ *mod* $\mathrm{i}\pi\mathbb{Z}$ *and to* 0 *otherwise. In* (4.39) *the contour* $\mathscr{C}'\subset\mathscr{C}$ *has to be chosen so that it encircles the points* ν_a*,* $1\le a\le N$*, as well as the neighbourhood of the origin, but not the poles at* $s'=-s-z$ *for* $s\in\mathscr{C}$*.*

Proof —

So as to estimate the products (4.37), one introduces the function

$$\mathcal{S}_N^{(1)}(\omega) \;=\; \sum_{a=1}^{N}\ln\sinh(\nu_a+\omega+z)\,. \tag{4.40}$$

Its derivative can be written as

$$\big(\mathcal{S}_N^{(1)}\big)'(\omega) \;=\; \oint\limits_{\mathscr{C}} \frac{\mathrm{d}s}{2\mathrm{i}\pi} \coth(s+\omega+z) \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}'(s)}{1 \,+\, \widehat{\mathfrak{a}}_{\mathbb{Y}}(s)} \;+\; N \coth(\omega - \tfrac{\beta}{N} + z) \;-\; \mathbf{1}_{\mathscr{D}}(-\omega - z) \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}'(-\omega-z)}{1 \,+\, \widehat{\mathfrak{a}}_{\mathbb{Y}}(-\omega-z)} \,. \tag{4.41}$$

Upon taking the ante-derivative and then the exponent, this yields

$$\mathrm{e}^{\mathcal{S}_N^{(1)}(\omega)} \;=\; \Big\{ \sinh(\omega - \tfrac{\beta}{N} + z) \Big\}^{N} \cdot \Big[1 \,+\, \widehat{\mathfrak{a}}_{\mathbb{Y}}(-\omega-z) \Big]^{\mathbf{1}_{\mathscr{D}}(-\omega-z)} \mathrm{e}^{C^{(1)}(\omega)} \cdot \exp\Big\{ - \oint\limits_{\mathscr{C}} \mathrm{d}s \coth(s+\omega+z) \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \Big\} \,. \tag{4.42}$$

There, $C^{(1)}(\omega)$ is $\mathrm{i}\pi$-periodic and constant on $-\mathscr{D} - z$ and on $\{\lambda \;:\; |\Im(\lambda)| \leq \pi/2\} \setminus \{ -\overline{\mathscr{D}} - z\}$. However, observe that, by construction, the function $\omega \mapsto \mathrm{e}^{\mathcal{S}_N^{(1)}(\omega)}$ is continuous across $-\mathscr{C} - z$. Also, it is easy to see by using the Sokhotsky-Plemejl formulae that the function

$$\omega \mapsto \Big[1 \,+\, \widehat{\mathfrak{a}}_{\mathbb{Y}}(-\omega-z) \Big]^{\mathbf{1}_{\mathscr{D}}(-\omega-z)} \cdot \exp\Big\{ - \oint\limits_{\mathscr{C}} \mathrm{d}s \coth(s+\omega+z)\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \Big\} \tag{4.43}$$

is also continuous across $-\mathscr{C} - z$. The two formulae can thus match only if $C^{(1)}(\omega)$ is continuous across $-\mathscr{C} - z$. Thus $C^{(1)}(\omega) = C^{(1)}$ does not depend on ω. One can then fix its value by comparing the $\omega \to +\infty$ behaviour of the two sides of (4.42). On the one hand, it is direct to infer from the finite product representation issued from (4.40) that

$$\mathrm{e}^{\mathcal{S}_N^{(1)}(\omega)} \underset{\omega \to +\infty}{\sim} \frac{1}{2^N} \, \mathrm{e}^{N(\omega+z)} \cdot \prod_{a=1}^{N} \mathrm{e}^{\nu_a} \,. \tag{4.44}$$

On the other hand, one has that

$$- \oint\limits_{\mathscr{C}} \mathrm{d}s \, \coth(s+\omega+z)\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \underset{\omega \to +\infty}{\sim} - \oint\limits_{\mathscr{C}} \mathrm{d}s \, \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \;=\; \oint\limits_{\mathscr{C}} \frac{\mathrm{d}s}{2\mathrm{i}\pi} \, s \, \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}'(s)}{1 \,+\, \widehat{\mathfrak{a}}_{\mathbb{Y}}(s)} \;=\; \sum_{a=1}^{N} \nu_a \;+\; \beta \,, \tag{4.45}$$

which leads to

$$\begin{aligned} &\{ \sinh(\omega - \tfrac{\beta}{N} + z) \}^{N} \cdot \Big[1 \,+\, \widehat{\mathfrak{a}}_{\mathbb{Y}}(-\omega-z) \Big]^{\mathbf{1}_{\mathscr{D}}(-\omega-z)} \mathrm{e}^{C^{(1)}} \cdot \exp\Big\{ - \oint\limits_{\mathscr{C}} \mathrm{d}s \coth(s+\omega+z)\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \Big\} \\ &\qquad\qquad \underset{\omega \to +\infty}{\sim} \frac{1}{2^N} \, \mathrm{e}^{N(\omega+z) + C^{(1)}} \cdot \prod_{a=1}^{N} \mathrm{e}^{\nu_a} \,. \end{aligned} \tag{4.46}$$

By comparing the form of these asymptotics, one gets that $C^{(1)} = 0$. The identity (4.37) hence follows by setting $\omega = 0$.

The product (4.38) can be computed similarly.

Let us now establish (4.39). Using (4.37) we can write

$$\begin{aligned} \prod_{a,b=1}^{N} \sinh(\nu_a + \nu_b + z) \;=\; \prod_{a=1}^{N} \bigg\{ & \Big[\sinh(\nu_a + z - \tfrac{\beta}{N}) \Big]^{N} \Big[1 + \widehat{\mathfrak{a}}_{\mathbb{Y}}(-\nu_a - z) \Big]^{\mathbf{1}_{\mathscr{D}}(-\nu_a - z)} \\ & \times \exp\Big[- \oint\limits_{\mathscr{C}} \mathrm{d}s \, \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \coth\big(s + \nu_a + z\big) \Big] \bigg\} \,. \end{aligned} \tag{4.47}$$

Using again (4.37) to compute the first product in (4.47),

$$\prod_{a=1}^{N}\Big[\sinh(\nu_a+z-\tfrac{\beta}{N})\Big]^{N} = \Big\{\sinh(z-2\tfrac{\beta}{N})\Big\}^{N^2}\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-z+\tfrac{\beta}{N})\Big\}^{N\cdot\mathbf{1}_{\mathscr{D}}(-z+\frac{\beta}{N})}$$
$$\times\exp\Big\{-N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\coth\big(s+z-\tfrac{\beta}{N}\big)\Big\},\quad(4.48)$$

and observing that

$$-\sum_{a=1}^{N}\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\coth(s+\nu_a+z)$$
$$=-\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\frac{\mathrm{d}s'}{2\mathrm{i}\pi}\coth(s+s'+z)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\frac{\widehat{\mathfrak{a}}'_{\mathbb{Y}}(s')}{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(s')}-N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\coth\Big(s+z-\tfrac{\beta}{N}\Big)$$
$$=\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\coth'(s+s'+z)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s')-N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\coth\Big(s+z-\tfrac{\beta}{N}\Big),\quad(4.49)$$

we obtain (4.39). ■

This lemma allows us to formulate the following proposition:

Proposition 4.2. *The partition function* (3.11) *involved in the expression of the boundary factor* (4.2) *can be represented as in* (4.25).

In this expression, the product function $\mathcal{P}$ can be rewritten as

$$\mathcal{P}(\{\mu_a\}_1^N) = \mathfrak{p}_N\cdot\mathcal{K}_N^{(1)}(\widehat{\mathbb{Y}})\cdot\frac{[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\xi_-)]^{\mathbf{1}_{\mathscr{D}}(-\xi_-)}}{[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)]^{\frac{1}{2}}}\cdot\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\,\phi_2^{(+)}(s,s')\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s')$$
$$+\oint_{\mathscr{C}}\mathrm{d}s\,\phi_1(s)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\partial_s\ln\sinh(s,\mathrm{i}\zeta+\tfrac{\beta}{N})\Big\},\quad(4.50)$$

in which

$$\mathfrak{p}_N = \Big\{\frac{\sinh(\mathrm{i}\zeta+2\beta/N)}{\sinh(2\beta/N)}\Big\}^{\frac{N(N-1)}{2}}\cdot\Big\{\sinh(\mathrm{i}\zeta)\,\sinh\Big(2\tfrac{\beta}{N}\Big)\Big\}^{N^2}\cdot\Big\{\sinh\Big(\xi_--\tfrac{\beta}{N}\Big)\Big\}^{N}.\quad(4.51)$$

and

$$\mathcal{K}_N^{(1)}(\widehat{\mathbb{Y}}) = \frac{\prod_{a<b}^{n}f(\widehat{y}_a,\widehat{y}_b)f(\widehat{x}_a,\widehat{x}_b)}{\prod_{a,b=1}^{n}f(\widehat{x}_a,\widehat{y}_b)}\cdot\prod_{a=1}^{n}\Big\{\frac{\sinh(\widehat{y}_a+\xi_-)}{\sinh(\widehat{x}_a+\xi_-)}\cdot\frac{\sinh(2\,\widehat{x}_a-\mathrm{i}\zeta)}{\sinh(2\,\widehat{y}_a)}\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\widehat{x}_a)\Big\}^{\frac{1}{2}}\cdot\Big\{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\widehat{y}_a)\Big\}^{\frac{1}{2}}\Big\}$$
$$\times\prod_{a=1}^{n}\Big\{\frac{\sinh(\mathrm{i}\zeta+\beta/N,\widehat{y}_a)}{\sinh(\mathrm{i}\zeta+\beta/N,\widehat{x}_a)}\Big\}^{N}\cdot\prod_{z\in\widehat{\mathbb{Y}}}\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\partial_s\ln f(s,z)\Big\}.\quad(4.52)$$

Note that $\mathscr{C}$ is taken here such that it also satisfies the property that if $x \in \widehat{\mathfrak{X}}$ then also $-x \in \mathrm{Int}(\mathscr{C})$. Also, we have introduced the functions

$$\phi_2^{(+)}(s,s') \;=\; \frac{1}{2}\Big[\coth'(s+s'-\mathrm{i}\zeta) \;-\; \coth'(s+s')\Big]\,, \tag{4.53}$$

$$\phi_1(s) \;=\; \coth(2s-\mathrm{i}\zeta) \;+\; \coth(2s) \;-\; \coth(s+\xi_-)\,, \tag{4.54}$$

and the shortcut notation

$$\sinh(s,s') \;=\; \sinh(s+s')\sinh(s-s')\,. \tag{4.55}$$

The contours are chosen in such a way that the poles of the integral involving ϕ_1 at $\mathrm{i}\zeta/2$ and $\mathrm{i}(\zeta-\pi)/2$ are located outside of $\mathscr{C}$. Likewise, in the double integral involving $\phi_2^{(+)}$, the contours are chosen in such a way that the poles at $s+s'-\mathrm{i}\zeta$ are always located outside from the contours $\mathscr{C}$ or $\mathscr{C}'$.

The contribution $\widehat{\mathscr{F}_{\mathbb{Y}}}$ is given as in (4.29), in which the kernel $\widehat{U}_{\mathbb{Y}}$ is given by

$$\widehat{U}_{\mathbb{Y}}(\omega,\omega') = \frac{-\mathrm{e}^{-\frac{h'}{T}}\sinh(2\omega'-\mathrm{i}\zeta)}{\sinh(\omega+\omega')\sinh(\omega-\omega'-\mathrm{i}\zeta)}\cdot\left[\frac{\sinh(\omega'-\beta/N)\sinh(\omega'+\beta/N-\mathrm{i}\zeta)}{\sinh(\omega'+\beta/N)\sinh(\omega'-\beta/N-\mathrm{i}\zeta)}\right]^N$$
$$\times\exp\bigg\{-\oint\limits_{\mathscr{C}_U}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\Big[\coth(\omega'+s)\;+\;\coth(\omega'-s)\;-\;\coth(\omega'+s-\mathrm{i}\zeta)\;-\;\coth(\omega'-s-\mathrm{i}\zeta)\Big]\bigg\}\,, \tag{4.56}$$

and the function $\widehat{\mathfrak{r}}_{\mathbb{Y}}$ by

$$\widehat{\mathfrak{r}}_{\mathbb{Y}}(\omega) = \mathrm{e}^{-\frac{2h'}{T}}\left[\frac{\sinh(\omega-\beta/N+\mathrm{i}\zeta)\sinh(\omega+\beta/N-\mathrm{i}\zeta)}{\sinh(\omega+\beta/N+\mathrm{i}\zeta)\sinh(\omega-\beta/N-\mathrm{i}\zeta)}\right]^N$$
$$\times\exp\bigg\{-\oint\limits_{\mathscr{C}_U}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\Big[\coth(s+\omega+\mathrm{i}\zeta)\;+\;\coth(\omega-s+\mathrm{i}\zeta)\;-\;\coth(\omega+s-\mathrm{i}\zeta)\;-\;\coth(\omega-s-\mathrm{i}\zeta)\Big]\bigg\}\,. \tag{4.57}$$

In the above expressions, the contour $\mathscr{C}_U$ encircles the points $\{\mu_a\}_1^N$ as well as the origin, and is such that, given any $\omega\in\mathscr{C}^{(p)}$, where $\mathscr{C}^{(p)}$ refers to any of the encasted contours introduced in (4.29), the points $\pm\omega$, $\pm(\omega+\mathrm{i}\zeta)$, $\pm(\omega-\mathrm{i}\zeta)$ are not surrounded by $\mathscr{C}_U$.

Proof —

The formulas (4.56) and (4.57) are direct rewritings of (4.31) and (4.30). Let us therefore prove (4.50).

Using (4.39) with $z=0$ and $z=-\mathrm{i}\zeta$, together with the decomposition

$$\prod_{a<b}^{N}\sinh(\nu_a+\nu_b+z) \;=\; \frac{\Big\{\prod\limits_{a,b=1}^{N}\sinh(\nu_a+\nu_b+z)\Big\}^{\frac{1}{2}}}{\Big\{\prod\limits_{a=1}^{N}\sinh(2\nu_a+z)\Big\}^{\frac{1}{2}}}\,, \tag{4.58}$$

and observing that $\widehat{\mathfrak{a}}_{\mathbb{Y}}(\frac{\beta}{N})=0$, we obtain that

$$\prod_{a<b}^{N}\sinh(\nu_a+\nu_b) \;=\; \Big\{\sinh(-2\tfrac{\beta}{N})\Big\}^{\frac{N^2}{2}}\cdot\prod_{a=1}^{N}\bigg\{\frac{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\nu_a)}{\sinh(2\nu_a)}\bigg\}^{\frac{1}{2}}$$
$$\times\exp\bigg\{\frac{1}{2}\oint\limits_{\mathscr{C}}\mathrm{d}\omega\oint\limits_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}\omega'\coth'(\omega+\omega')\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega')\;-\;N\oint\limits_{\mathscr{C}}\mathrm{d}\omega\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega)\coth(\omega-\tfrac{\beta}{N})\bigg\}\,, \tag{4.59}$$

and

$$\prod_{a<b}^{N} \sinh(\nu_a+\nu_b-\mathrm{i}\zeta) \;=\; \Big\{\sinh(-\mathrm{i}\zeta-2\tfrac{\beta}{N})\Big\}^{\frac{N^2}{2}}\cdot\prod_{a=1}^{N}\Big\{\frac{1}{\sinh(2\nu_a-\mathrm{i}\zeta)}\Big\}^{\frac{1}{2}}$$

$$\times\exp\Big\{\frac{1}{2}\oint_{\mathscr{C}}\mathrm{d}\omega\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}\omega'\coth'(\omega+\omega'-\mathrm{i}\zeta)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega')\;-\;N\oint_{\mathscr{C}}\mathrm{d}\omega\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega)\coth(\omega-\tfrac{\beta}{N}-\mathrm{i}\zeta)\Big\}. \quad (4.60)$$

Here the integration contour $\mathscr{C}'\subset\mathscr{C}$ is chosen such that the poles at $\omega'=-\omega$ and at $\omega'=-\omega+\mathrm{i}\zeta$ all lie outside of $\mathscr{C}'$ whereas those at $\omega=-\omega'+\mathrm{i}\zeta$ lie outside of $\mathscr{C}$. Using also (4.38), we get

$$\prod_{a=1}^{N}\Big\{\frac{\sinh(2\nu_a)}{\sinh(2\nu_a-\mathrm{i}\zeta)}\Big\}^{\frac{1}{2}} \;=\; \Big\{\frac{\sinh(2\frac{\beta}{N})}{\sinh(\mathrm{i}\zeta+2\frac{\beta}{N})}\Big\}^{\frac{N}{2}}[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)]^{\frac{1}{2}}\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\Big[\coth(2s-\mathrm{i}\zeta)\;-\;\coth(2s)\Big]\Big\}. \quad (4.61)$$

It follows that

$$\prod_{a<b}^{N} f(\nu_a,\nu_b) \;=\; \Big\{\frac{\sinh(\mathrm{i}\zeta+2\frac{\beta}{N})}{\sinh(2\frac{\beta}{N})}\Big\}^{\frac{N(N-1)}{2}}\cdot\frac{[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)]^{\frac{1}{2}}}{\prod_{a=1}^{N}[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\nu_a)]^{\frac{1}{2}}}\cdot\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\Big[\coth(2s-\mathrm{i}\zeta)\;-\;\coth(2s)\Big]\Big\}$$

$$\times\exp\Big\{N\oint_{\mathscr{C}}\mathrm{d}\omega\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega)\Big[\coth(\omega-\tfrac{\beta}{N})-\coth(\omega-\tfrac{\beta}{N}-\mathrm{i}\zeta)\Big]\Big\}$$

$$\times\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}\omega\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}\omega'\,\phi_2^{(+)}(\omega,\omega')\,\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega)\widehat{\mathscr{L}_{\mathbb{Y}}}(\omega')\Big\}, \quad (4.62)$$

in which $\phi_2^{(+)}$ is given by (4.53).

Moreover, it also follows from (4.37) that

$$\prod_{a=1}^{N}\sinh(\nu_a+\xi_-) \;=\; \sinh^N(\xi_- -\tfrac{\beta}{N})\cdot[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\xi_-)]^{\mathbf{1}_{\mathscr{D}}(-\xi_-)}\cdot\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\coth(s+\xi_-)\Big\}, \quad (4.63)$$

$$\prod_{a=1}^{N}\sinh^N(\nu_a-\tfrac{\beta}{N}) \;=\; \sinh^{N^2}(-2\tfrac{\beta}{N})\cdot\exp\Big\{-N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\coth(s-\tfrac{\beta}{N})\Big\}, \quad (4.64)$$

$$\prod_{a=1}^{N}\sinh^N(\mathrm{i}\zeta+\nu_a+\tfrac{\beta}{N}) \;=\; \sinh^{N^2}(\mathrm{i}\zeta)\cdot\exp\Big\{-N\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\coth(s+\mathrm{i}\zeta+\tfrac{\beta}{N})\Big\}, \quad (4.65)$$

and from (4.38) that

$$\prod_{a=1}^{N}\sinh(2\nu_a) \;=\; \sinh^N(-2\tfrac{\beta}{N})\cdot[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)]\cdot\exp\Big\{-2\oint_{\mathscr{C}}\mathrm{d}s\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\coth(2s)\Big\}, \quad (4.66)$$

so that

$$\prod_{a=1}^{N} G(\nu_a) \;=\; \frac{\prod\limits_{a=1}^{N} [1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\nu_a)]^{\frac{1}{2}} \cdot [1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\xi_-)]^{\mathbf{1}_{\mathscr{D}}(-\xi_-)}}{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)} \cdot \Big\{ \sinh(\mathrm{i}\zeta)\, \sinh\big(2\tfrac{\beta}{N}\big) \Big\}^{N^2} \sinh^N(\xi_- - \tfrac{\beta}{N})$$

$$\times \exp\Big\{ \oint\limits_{\mathscr{C}} \mathrm{d}\omega\, \widehat{\mathscr{L}_{\mathbb{Y}}}(\omega) \Big[2\coth(2\omega) - \coth(\omega+\xi_-) - N\coth(\omega-\tfrac{\beta}{N}) - N\coth(\omega+\tfrac{\beta}{N}+\mathrm{i}\zeta) \Big] \Big\} \,. \qquad (4.67)$$

Finally,

$$\begin{aligned} \mathcal{P}_{\mathrm{bk}}\big(\{\nu_a\}_1^N; \widehat{\mathbb{Y}}\big) \;&=\; \prod_{a=1}^{N} \left\{ \prod_{y\in\widehat{\mathcal{Y}}} \frac{\sinh(\nu_a+y-\mathrm{i}\zeta)}{\sinh(\nu_a+y)} \prod_{x\in\widehat{\mathfrak{X}}} \frac{\sinh(\nu_a+x)}{\sinh(\nu_a+x-\mathrm{i}\zeta)} \right\} \\ &=\; \prod_{z\in\widehat{\mathbb{Y}}} \left\{ \big[f(-\tfrac{\beta}{N}, z) \big]^N \exp\Big[-\oint\limits_{\mathscr{C}} \mathrm{d}s\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s)\, \partial_s \ln f(s,z) \Big] \right\} \cdot \prod_{x\in\widehat{\mathfrak{X}}} [1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-x)] \,, \qquad (4.68) \end{aligned}$$

in which we have used (4.37) with $z = y, y-\mathrm{i}\zeta$ for $y\in\widehat{\mathcal{Y}}$, $-y$ and $-y+\mathrm{i}\zeta$ being outside of $\mathscr{D}$ mod $\mathrm{i}\pi\mathbb{Z}$, and with $z = x, x-\mathrm{i}\zeta$ for $x\in\widehat{\mathfrak{X}}$, $-x\in\mathscr{D}$ and $-x+\mathrm{i}\zeta\notin\mathscr{D}$ mod $\mathrm{i}\pi\mathbb{Z}$.

It then remains to put all these partial results together to obtain (4.50). ■

As a result, we can formulate the following rewriting of the boundary factor (4.2) at finite Trotter number:

Proposition 4.3. *Let $\{\lambda_a(h)\}_1^N$ stand for the Bethe roots describing the dominant state of the quantum transfer matrix $\mathtt{t}_{\mathfrak{q};h}$ at magnetic field h, and $\{\mu_a(h')\}_1^N$ be a set of Bethe roots for a sub-dominant state of the quantum transfer matrix $\mathtt{t}_{\mathfrak{q};h'}$ at magnetic field h', characterised by a set* (4.5) $\widehat{\mathbb{Y}}$.

Then, the boundary factor (4.2) *admits the following representation:*

$$\mathcal{F}_{\mathscr{B}}(\{\mu_a(h')\}_1^N; \{\lambda_a(h)\}_1^N; \xi_-) \;=\; \mathrm{e}^{\widehat{\mathscr{F}}_{\mathbb{Y}} - \widehat{\mathscr{F}}_{\emptyset}} \cdot \left[\frac{1-\widehat{\mathfrak{a}}_{\emptyset}(0)^2}{1-\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)^2} \right]^{\frac{1}{4}} \cdot \left[\frac{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\xi_-)}{1+\widehat{\mathfrak{a}}_{\emptyset}(-\xi_-)} \right]^{\mathbf{1}_{\mathscr{D}}(-\xi_-)} \cdot \mathcal{K}_N^{(1)}(\widehat{\mathbb{Y}})$$

$$\times \exp\Big\{ \oint\limits_{\mathscr{C}} \mathrm{d}s \oint\limits_{\mathscr{C}'\subset\mathscr{C}} \mathrm{d}s'\, \phi_2^{(+)}(s,s')\, \Big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s') \;-\; \widehat{\mathscr{L}_{\emptyset}}(s)\, \widehat{\mathscr{L}_{\emptyset}}(s') \Big]$$

$$+ \oint\limits_{\mathscr{C}} \mathrm{d}s\, \phi_1(s)\, \Big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s) \;-\; \widehat{\mathscr{L}_{\emptyset}}(s) \Big] - N \oint\limits_{\mathscr{C}} \mathrm{d}s\, \Big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s) \;-\; \widehat{\mathscr{L}_{\emptyset}}(s) \Big]\, \partial_s \ln \sinh(s, \mathrm{i}\zeta + \tfrac{\beta}{N}) \Big\} \,. \qquad (4.69)$$

In this expression, $\widehat{\mathfrak{a}}_{\mathbb{Y}} \equiv \widehat{\mathfrak{a}}_{\mathbb{Y};h'}$ and $\widehat{\mathfrak{a}}_{\emptyset} \equiv \widehat{\mathfrak{a}}_{\emptyset;h}$ stand for the respective counting functions of $\{\mu_a(h')\}_1^N$ and $\{\lambda_a(h)\}_1^N$, whereas $\widehat{\mathscr{L}_{\mathbb{Y}}}$ and $\widehat{\mathscr{L}_{\emptyset}}$ are defined from these counting functions as in (4.14)*. The contribution $\widehat{\mathscr{F}}_{\mathbb{Y}}$ is given as in* (4.29)*, in which the kernel $\widehat{U}_{\mathbb{Y}} \equiv \widehat{U}_{\mathbb{Y};h'}$ and the function $\widehat{\mathfrak{r}}_{\mathbb{Y}} \equiv \widehat{\mathfrak{r}}_{\mathbb{Y};h'}$ are given respectively by* (4.56) *and* (4.57) *with h replaced by h', the contribution $\widehat{\mathscr{F}}_{\emptyset}$ being defined similarly in terms of h and $\widehat{\mathscr{L}_{\emptyset}}$. The functions $\phi_2^{(+)}$ and ϕ_1 are defined in* (4.53) *and* (4.54)*. The factor $\mathcal{K}_N^{(1)}(\widehat{\mathbb{Y}})$ is expressed in terms of the particle-hole configuration of the set of Bethe roots $\{\mu_a(h')\}_1^N$ as in* (4.52)*. Finally, $\mathscr{C}$ has the property that if $x\in\widehat{\mathfrak{X}}$ then $-x\in\mathrm{Int}(\mathscr{C})$.*

4.3 The matrix element

We now repeat the analysis relatively to the relevant matrix element to the problem. For that purpose, we need to recall a few know facts.

The scalar product between two on-shell Bethe vectors at different magnetic fields and parameterised by roots $\{\lambda_a(h)\}_1^N$ and $\{\mu_a(h')\}_1^N$ have been shown [22] to admit the determinant representation

$$\Big(\boldsymbol{\Psi}(\{\lambda_a\}_1^N),\boldsymbol{\Psi}(\{\mu_a\}_1^N)\Big) \; = \; \Big(\boldsymbol{\Psi}(\{\mu_a\}_1^N),\boldsymbol{\Psi}(\{\lambda_a\}_1^N)\Big) \; = \; \prod_{a=1}^{N}\Big\{a_{h'}(\lambda_a)d_{h'}(\mu_a)\prod_{b=1}^{N}\frac{\sinh(\mu_b-\lambda_a-\mathrm{i}\zeta)}{\sinh(\lambda_a-\mu_b)}\Big\}$$
$$\times\prod_{b=1}^{N}\Big\{\varkappa\frac{V_+(\lambda_b)}{V_-(\lambda_b)}\;-\;1\Big\}\cdot\frac{1\;-\;\varkappa}{V_+^{-1}(\theta)\;-\;\varkappa V_-^{-1}(\theta)}\cdot\operatorname{det}_N\Big[\delta_{jk}\;+\;U_{jk}^{(\lambda)}(\theta)\Big]\,, \quad (4.70)$$

for $\varkappa=\mathrm{e}^{\frac{h-h'}{T}}$ and θ an arbitrary complex number. Here we have used the following notations:

$$a_h(\lambda) \;=\; \mathrm{e}^{\frac{h}{2T}}\Big\{\frac{\sinh(\lambda-\frac{\beta}{N}-\mathrm{i}\zeta)\sinh(\lambda+\frac{\beta}{N})}{\sinh^2(-\mathrm{i}\zeta)}\Big\}^N\,, \quad (4.71)$$

$$d_h(\lambda) \;=\; \mathrm{e}^{-\frac{h}{2T}}\Big\{\frac{\sinh(\lambda+\frac{\beta}{N}+\mathrm{i}\zeta)\sinh(\lambda-\frac{\beta}{N})}{\sinh^2(-\mathrm{i}\zeta)}\Big\}^N\,, \quad (4.72)$$

$$V_{\pm}(\omega) \;=\; \prod_{b=1}^{N}\frac{\sinh\left(\lambda_b-\omega\mp\mathrm{i}\zeta\right)}{\sinh\left(\mu_b-\omega\mp\mathrm{i}\zeta\right)}\,, \quad (4.73)$$

and

$$U_{jk}^{(\lambda)}(\theta) \;=\; \frac{\prod\limits_{a=1}^{N}\sinh\left(\mu_a-\lambda_j\right)}{\prod\limits_{\substack{a=1\\ \neq j}}^{N}\sinh(\lambda_a-\lambda_j)}\cdot\frac{K_\varkappa(\lambda_j-\lambda_k)\;-\;K_\varkappa(\theta-\lambda_k)}{V_+^{-1}(\lambda_j)\;-\;\varkappa V_-^{-1}(\lambda_j)}, \quad (4.74)$$

in which

$$K_\varkappa(\omega) \;=\; \coth(\lambda+\mathrm{i}\zeta)\;-\;\varkappa\coth(\lambda-\mathrm{i}\zeta)\,. \quad (4.75)$$

Note that, if $\{\lambda_a(h)\}_1^N$ stand for the Bethe roots describing the dominant state of the quantum transfer matrix $\mathtt{t}_{\mathrm{q};h}$ and $\{\mu_a(h')\}_1^N$ a sub-dominant state of the quantum transfer matrix $\mathtt{t}_{\mathrm{q};h'}$ characterised by a set (4.5) $\widehat{\mathbb{Y}}$, we can write

$$\varkappa\frac{V_+(\omega)}{V_-(\omega)} \;=\; \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\omega)}{\widehat{\mathfrak{a}}_{\emptyset}(\omega)}\,, \quad (4.76)$$

in terms of the corresponding counting functions $\widehat{\mathfrak{a}}_{\mathbb{Y}}\equiv\widehat{\mathfrak{a}}_{\mathbb{Y};h'}$ and $\widehat{\mathfrak{a}}_{\emptyset}\equiv\widehat{\mathfrak{a}}_{\emptyset;h}$. Similarly as in [22], we can also rewrite the finite size determinant in (4.70) as a Fredholm determinant of an integral operator acting on a the contour surrounding the points $\{\lambda_a(h)\}_1^N$. For latter convenience, we choose this contour $\Gamma(\mathscr{C})$ to surround also the contour $\mathscr{C}$. Hence (4.70) can be rewritten as

$$\Big(\boldsymbol{\Psi}(\{\lambda_a\}_1^N),\boldsymbol{\Psi}(\{\mu_a\}_1^N)\Big) \;=\; (-1)^N\prod_{k=1}^{N}\Big\{a_{h'}(\lambda_k)d_{h'}(\mu_k)\prod_{b=1}^{N}\frac{\sinh(\mu_b-\lambda_k-\mathrm{i}\zeta)}{\sinh(\lambda_k-\mu_b)}\Big\}$$
$$\times\prod_{b=1}^{N}\Big(\widehat{\mathfrak{a}}_{\mathbb{Y}}(\lambda_b)\;+\;1\Big)\cdot\frac{(1\;-\;\varkappa)V_+(\theta)}{1\;-\;\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\theta)}{\widehat{\mathfrak{a}}_{\emptyset}(\theta)}}\cdot\operatorname{det}_{\Gamma(\mathscr{C})}\Big[\mathtt{id}\;+\;\widehat{\mathtt{U}}_{\theta}^{(\lambda)}\Big]\,. \quad (4.77)$$

There, we have set

$$
\begin{aligned}
\widehat{\mathrm{U}}_{\theta}^{(\lambda)}(\omega,\omega') \; &= \; -\frac{1}{2\pi\mathrm{i}} \prod_{a=1}^{N} \frac{\sinh\left(\mu_a-\omega\right)}{\sinh(\lambda_a-\omega)} \cdot \frac{K_{\varkappa}(\omega-\omega') \; - \; K_{\varkappa}(\theta-\omega')}{V_{+}^{-1}(\omega) \; - \; \kappa V_{-}^{-1}(\omega)} \\
&= \; -\frac{V_{+}(\omega)}{2\pi\mathrm{i}} \prod_{a=1}^{N} \frac{\sinh\left(\mu_a-\omega\right)}{\sinh(\lambda_a-\omega)} \cdot \frac{K_{\varkappa}(\omega-\omega') \; - \; K_{\varkappa}(\theta-\omega')}{1 \; - \; \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\omega)}{\widehat{\mathfrak{a}}_{\emptyset}(\omega)}} \,.
\end{aligned} \tag{4.78}
$$

Finally, one recalls the "norm" formula for Bethe vectors. Given any solution $\{\mu_a\}_1^N$ to the Bethe equation at non-zero magnetic field h', and a contour $\mathscr{C}_{\mathbb{Y}}$ which surrounds the points $\{\mu_a(h')\}_1^N$ as well as a neighbourhood of the origin, one has that

$$
\Big(\mathbf{\Psi}(\{\mu_a\}_1^N),\mathbf{\Psi}(\{\mu_a\}_1^N)\Big) \; = \; \prod_{s=1}^{N} \bigg\{ \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}'(\mu_s)}{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\mu_s)} \cdot a_{h'}(\mu_s) d_{h'}(\mu_s) \bigg\} \cdot \frac{\prod\limits_{a,b=1}^{N} \sinh(\mu_a-\mu_b-\mathrm{i}\zeta)}{\prod\limits_{a\neq b}^{N} \sinh(\mu_a-\mu_b)} \cdot \operatorname{det}_{\mathscr{C}_{\mathbb{Y}}}\Big[\mathtt{id} \; + \; \widehat{\overline{K}}_{\mathbb{Y}}\Big] \,. \tag{4.79}
$$

In the above, $\widehat{\overline{K}}_{\mathbb{Y}}$ is an integral operator on $L^2(\mathscr{C}_{\mathbb{Y}})$ with integral kernel

$$
\widehat{\overline{K}}_{\mathbb{Y}}(\omega,\omega') \; = \; -\frac{K(\omega-\omega')}{1 \; + \; \widehat{\mathfrak{a}}_{\mathbb{Y}}^{-1}(\omega')} \tag{4.80}
$$

Although this is not directly necessary here for our purpose of computing the infinite Trotter number limit, it is also possible to re-express the Fredholm determinant in (4.79) in terms of an integral operator acting on the contour $\mathscr{C}$ instead of $\mathscr{C}_{\mathbb{Y}}$. We refer to Appendix C of [8] for details about such a reformulation.

Before stating the main result of the section, we first need to establish a technical lemma.

Lemma 4.4. *It holds*

$$
\begin{aligned}
&\frac{\prod\limits_{a\neq b}^{N} \sinh(\nu_a-\nu_b)}{\prod\limits_{a,b=1}^{N} \sinh(\nu_a-\lambda_b)} \cdot \prod_{b=1}^{N} \bigg\{ \frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\lambda_b) \; + \; 1}{\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(\nu_b)} \bigg\} \prod_{a=1}^{N} \bigg\{ \frac{\sinh(\lambda_a+\beta/N)}{\sinh(\nu_a+\beta/N)} \bigg\}^{N} \\
&\qquad = \; \exp\bigg\{ \oint_{\mathscr{C}} \mathrm{d}s \oint_{\mathscr{C}'\subset\mathscr{C}} \mathrm{d}s' \coth'(s-s') \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \Big[\widehat{\mathscr{L}_{\emptyset}}(s') \; - \; \widehat{\mathscr{L}_{\mathbb{Y}}}(s') \Big] \bigg\} \,.
\end{aligned} \tag{4.81}
$$

Also, for $|\epsilon|$ small enough and generic, it holds

$$
\begin{aligned}
&\prod_{z\in\widehat{\mathbb{Y}}} \prod_{a=1}^{N} \bigg\{ \frac{\sinh(\lambda_a-z-\epsilon)}{\sinh(\nu_a-z+\epsilon)\sinh(\nu_a-z-\epsilon)} \bigg\} \; = \; \prod_{z\in\widehat{\mathbb{Y}}} \bigg\{ \frac{1}{\sinh(\epsilon-z-\frac{\beta}{N})} \bigg\}^{N} \prod_{x\in\widehat{\mathfrak{X}}} \bigg\{ \frac{[1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(x-\epsilon)][1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(x+\epsilon)]}{[1+\widehat{\mathfrak{a}}_{\emptyset}(x+\epsilon)]} \bigg\} \\
&\qquad \times \prod_{z\in\widehat{\mathbb{Y}}} \exp\bigg\{ \oint_{\mathscr{C}} \mathrm{d}s \coth(s-z+\epsilon) \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \; - \; \coth(s-z-\epsilon) \Big[\widehat{\mathscr{L}_{\emptyset}}(s) \; - \; \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \Big] \bigg\} \,.
\end{aligned} \tag{4.82}
$$

Proof —

By virtue of (4.37), it is easy to see that, for ϵ small enough,

$$\prod_{a,b=1}^{N} \frac{\sinh(\nu_a-\lambda_b+\epsilon)}{\sinh(\nu_a-\nu_b+\epsilon)} = \prod_{a=1}^{N}\Big\{\frac{\sinh(\lambda_a-\epsilon+\beta/N)}{\sinh(\nu_a-\epsilon+\beta/N)}\Big\}^{N}\cdot\prod_{b=1}^{N}\Big\{\frac{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(\lambda_b-\epsilon)}{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(\nu_b-\epsilon)}\Big\} \tag{4.83}$$

$$\times\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\coth'(s-s'+\epsilon)\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big[\widehat{\mathscr{L}_{\emptyset}}(s')-\widehat{\mathscr{L}_{\mathbb{Y}}}(s')\big]\Big\}. \tag{4.84}$$

Then, the $\epsilon\to0^{+}$ limit yields (4.81). The representation (4.82) follows from (4.37) through even more direct handlings. ■

This lemma enables us to formulate the following rewriting of the ratio of matrix elements (4.1) that is used for the expression (3.15) of the one-point function (3.14):

Proposition 4.5. *Let $\{\lambda_a(h)\}_1^N$ stand for the Bethe roots describing the dominant state of the quantum transfer matrix $\mathtt{t}_{\mathrm{q};h}$ at magnetic field h, and $\{\mu_a(h')\}_1^N$ be a set of Bethe roots for a sub-dominant state of the quantum transfer matrix $\mathtt{t}_{\mathrm{q};h'}$ at magnetic field h', characterised by a set* (4.5) $\widehat{\mathbb{Y}}$.

Then, the ratio of matrix elements (4.70) *and* (4.79) *admits the following representation:*

$$\frac{\big(\boldsymbol{\Psi}(\{\lambda_a\}_1^N),\boldsymbol{\Psi}(\{\mu_a(h')\}_1^N)\big)}{\big(\boldsymbol{\Psi}(\{\mu_a(h')\}_1^N),\boldsymbol{\Psi}(\{\mu_a(h')\}_1^N)\big)} = \mathcal{K}_N^{(2)}(\widehat{\mathbb{Y}})\cdot\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\,\phi_2^{(-)}(s,s')\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s')-\widehat{\mathscr{L}_{\emptyset}}(s')\big]$$

$$+\;N\oint_{\mathscr{C}}\mathrm{d}s\big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-\widehat{\mathscr{L}_{\emptyset}}(s)\big]\partial_s\ln\sinh(s,\mathrm{i}\zeta+\beta/N)\Big\}\cdot\frac{(1-\kappa)\,V_+(\theta)}{1-\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\theta)}{\widehat{\mathfrak{a}}_{\emptyset}(\theta)}}\cdot\frac{\det_{\Gamma(\mathscr{C})}\big[id+\widehat{U}_{\theta}^{(\lambda)}\big]}{\det_{\mathscr{C}_{\mathbb{Y}}}\big[id+\widehat{\overline{K}}_{\mathbb{Y}}\big]}, \tag{4.85}$$

where the factor $\mathcal{K}_N^{(2)}(\widehat{\mathbb{Y}})$ is expressed in terms of the particle-hole configuration of the set of Bethe roots $\{\mu_a(h')\}_1^N$ as

$$\mathcal{K}_N^{(2)}(\widehat{\mathbb{Y}}) = \prod_{z\in\widehat{\mathbb{Y}}}\Big\{\sinh\big(z,\mathrm{i}\zeta+\tfrac{\beta}{N}\big)\Big\}^{-N}\cdot\frac{\prod\limits_{\substack{z\neq z'\\ z,z'\in\widehat{\mathbb{Y}}}}\sinh(z-z')}{\prod\limits_{z,z'\in\widehat{\mathbb{Y}}}\sinh(z-z'-\mathrm{i}\zeta)}\cdot\prod_{x\in\widehat{\mathfrak{X}}}\Big\{\frac{\widehat{\mathfrak{a}}_{\emptyset}(x)+1}{-\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(x)}\Big\}\cdot\prod_{y\in\widehat{\mathcal{Y}}}\Big\{\frac{1}{\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(y)}\Big\}$$

$$\times\prod_{z\in\widehat{\mathbb{Y}}}\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\Big[\widetilde{\psi}_1(s,z)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)+\widetilde{\psi}_2(s,z)\big[\widehat{\mathscr{L}_{\emptyset}}(s)-\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big]\Big]\Big\}. \tag{4.86}$$

In these expressions, $\widehat{\mathfrak{a}}_{\mathbb{Y}}\equiv\widehat{\mathfrak{a}}_{\mathbb{Y};h'}$ and $\widehat{\mathfrak{a}}_{\emptyset}\equiv\widehat{\mathfrak{a}}_{\emptyset;h}$ stand for the respective counting functions of $\{\mu_a(h')\}_1^N$ and $\{\lambda_a(h)\}_1^N$, whereas $\widehat{\mathscr{L}_{\mathbb{Y}}}$ and $\widehat{\mathscr{L}_{\emptyset}}$ are defined from these counting functions as in (4.14). *We have also defined $\kappa=\mathrm{e}^{\frac{h-h'}{T}}$, used the shortcut notation* (4.55) *and introduced the functions*

$$\phi_2^{(-)}(s,s') = \coth'(s-s'-\mathrm{i}\zeta)-\coth'(s-s'), \tag{4.87}$$

$$\widetilde{\psi}_1(s,z) = \coth(s-z-\mathrm{i}\zeta)-\coth(s-z), \tag{4.88}$$

$$\widetilde{\psi}_2(s,z) = \coth(s-z)-\coth(s-z+\mathrm{i}\zeta)\,. \tag{4.89}$$

θ is here an arbitrary complex number such that $\theta+\mathrm{i}\zeta\notin\mathscr{D}$, so that the coefficient $V_+(\theta)$ can be rewritten as

$$V_+(\theta) = \prod_{z\in\widehat{\mathbb{Y}}}\Big\{\frac{1}{\sinh(z-\theta-\mathrm{i}\zeta)}\Big\}\cdot\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\,\coth(s-\theta-\mathrm{i}\zeta)\big[\widehat{\mathscr{L}_{\emptyset}}(s)-\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big]\Big\}. \tag{4.90}$$

Finally, $\widehat{\widehat{K}}_{\mathbb{Y}}$ is an integral operator on $L^2(\mathscr{C}_{\mathbb{Y}})$, where $\mathscr{C}_{\mathbb{Y}}$ surrounds the points $\{\mu_a(h')\}_1^N$ and the origin, with integral kernel given by (4.80), *whereas $\widehat{\mathtt{U}}_{\theta}^{(\lambda)}$ is an integral operator on $L^2\big(\Gamma(\mathscr{C})\big)$, where $\Gamma(\mathscr{C})$ surrounds the contour $\mathscr{C}$, with integral kernel given as*

$$\widehat{U}_{\theta}^{(\lambda)}(\omega,\omega') \;=\; -\frac{1}{2\pi\mathrm{i}}\prod_{z\in\widehat{\mathbb{Y}}}\Big\{\frac{\sinh(z-\omega)}{\sinh(z-\omega-\mathrm{i}\zeta)}\Big\}\cdot\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\Big[\coth(s-\omega-\mathrm{i}\zeta)-\coth(s-\omega)\Big]\cdot\Big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-\widehat{\mathscr{L}_{\emptyset}}(s)\Big]\Big\}$$
$$\times\frac{K_{\kappa}(\omega-\omega')\;-\;K_{\kappa}(\theta-\omega')}{1\;-\;\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\omega)}{\widehat{\mathfrak{a}}_{\emptyset}(\omega)}}\;. \qquad (4.91)$$

Proof —

It follows from the norm and scalar product formulae (4.79) and (4.77) that the normalised twisted scalar product can be written as

$$\frac{\Big(\mathbf{\Psi}(\{\lambda_a\}_1^N),\mathbf{\Psi}(\{\mu_a(h')\}_1^N)\Big)}{\Big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N),\mathbf{\Psi}(\{\mu_a(h')\}_1^N)\Big)}\;=\;\prod_{b=1}^{N}\Big\{\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\lambda_b)\;+\;1}{\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(\mu_b)}\cdot\frac{a_{h'}(\lambda_b)}{a_{h'}(\mu_b)}\cdot\prod_{a=1}^{N}\frac{\sinh(\mu_a-\lambda_b-\mathrm{i}\zeta)}{\sinh(\mu_a-\mu_b-\mathrm{i}\zeta)}\Big\}$$
$$\times\frac{\prod_{a\neq b}^{N}\sinh(\mu_a-\mu_b)}{\prod_{a,b=1}^{N}\sinh(\mu_a-\lambda_b)}\cdot\frac{(1\;-\;\varkappa)V_{+}(\theta)}{1\;-\;\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\theta)}{\widehat{\mathfrak{a}}_{\emptyset}(\theta)}}\cdot\frac{\det_{\Gamma(\mathscr{C})}\Big[\mathrm{id}\;+\;\widehat{\mathtt{U}}_{\theta}^{(\lambda)}\Big]}{\det_{\mathscr{C}_{\mathbb{Y}}}\Big[\mathrm{id}\;+\;\widehat{\widehat{K}}_{\mathbb{Y}}\Big]}\;. \qquad (4.92)$$

Further, for any arbitrary ϵ, one readily infers the product decomposition

$$\prod_{a,b=1}^{N}\frac{\sinh(\mu_a-\lambda_b+\epsilon)}{\sinh(\mu_a-\mu_b+\epsilon)}\;=\;\prod_{b=1}^{N}\prod_{a=1}^{N}\Big\{\frac{\sinh(\nu_a-\lambda_b+\epsilon)}{\sinh(\nu_a-\nu_b+\epsilon)}\Big\}$$
$$\times\prod_{z\in\widehat{\mathbb{Y}}}\prod_{a=1}^{N}\Big\{\frac{\sinh(z-\lambda_a+\epsilon)}{\sinh(\nu_a-z+\epsilon)\sinh(z-\nu_a+\epsilon)}\Big\}\cdot\prod_{z,z'\in\widehat{\mathbb{Y}}}\Big\{\frac{1}{\sinh(z-z'+\epsilon)}\Big\}\;. \qquad (4.93)$$

Then, it remains to invoke Lemma 4.4 and take the $\epsilon\to 0$ limit so as to obtain

$$\frac{\prod_{a\neq b}^{N}\sinh(\mu_a-\mu_b)}{\prod_{a,b=1}^{N}\sinh(\mu_a-\lambda_b)}\cdot\prod_{b=1}^{N}\Big\{\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\lambda_b)\;+\;1}{\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(\mu_b)}\Big\}\prod_{a=1}^{N}\Big\{\frac{\sinh(\lambda_a+\beta/N)}{\sinh(\mu_a+\beta/N)}\Big\}^{N}$$
$$=\prod_{x\in\widehat{\mathfrak{X}}}\Big\{\frac{1+\widehat{\mathfrak{a}}_{\emptyset}(x)}{-\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(x)}\Big\}\cdot\prod_{y\in\widehat{\mathcal{Y}}}\Big\{\frac{1}{\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(y)}\Big\}\cdot\prod_{z\neq z'\in\widehat{\mathbb{Y}}}\Big\{\sinh(z-z')\Big\}\cdot\prod_{z\in\widehat{\mathbb{Y}}}\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\,\coth(s-z)\Big[\widehat{\mathscr{L}_{\emptyset}}(s)\;-\;2\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\Big]\Big\}$$
$$\times\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\;\coth'(s-s')\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\Big[\widehat{\mathscr{L}_{\emptyset}}(s')\;-\;\widehat{\mathscr{L}_{\mathbb{Y}}}(s')\Big]\Big\}\;. \qquad (4.94)$$

Quite similarly, one gets

$$\prod_{a,b=1}^{N}\frac{\sinh(\mu_a-\lambda_b-\mathrm{i}\zeta)}{\sinh(\mu_a-\mu_b-\mathrm{i}\zeta)} = \prod_{a=1}^{N}\Big\{\frac{\sinh(\lambda_a+\frac{\beta}{N}+\mathrm{i}\zeta)}{\sinh(\nu_a+\frac{\beta}{N}+\mathrm{i}\zeta)}\Big\}^N\cdot\prod_{z,z'\in\widehat{\mathbb{Y}}}\Big\{\frac{1}{\sinh(z-z'-\mathrm{i}\zeta)}\Big\}$$
$$\times\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\,\mathrm{d}s'\ \coth'(s-s'-\mathrm{i}\zeta)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big[\widehat{\mathscr{L}_{\emptyset}}(s')-\widehat{\mathscr{L}_{\mathbb{Y}}}(s')\big]\Big\}$$
$$\times\prod_{z\in\widehat{\mathbb{Y}}}\Big[\frac{1}{\sinh^N(-z-\mathrm{i}\zeta-\frac{\beta}{N})}\cdot\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\ \coth(s-z-\mathrm{i}\zeta)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-\coth(s-z+\mathrm{i}\zeta)\big[\widehat{\mathscr{L}_{\emptyset}}(s)-\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big]\Big\}\Big]. \tag{4.95}$$

Using again (4.37), one also obtains

$$V_{\pm}(\omega) = \prod_{z\in\widehat{\mathbb{Y}}}\Big\{\frac{1}{\sinh(z-\omega\mp\mathrm{i}\zeta)}\Big\}\cdot\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\coth(s-\omega\mp\mathrm{i}\zeta)\big[\widehat{\mathscr{L}_{\emptyset}}(s)-\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big]\Big\} \tag{4.96}$$

whenever $\omega\pm\mathrm{i}\zeta\notin\mathscr{D}$,

$$\prod_{a=1}^{N}\frac{\sinh(\mu_a-\omega)}{\sinh(\lambda_a-\omega)} = \prod_{z\in\widehat{\mathbb{Y}}}\big\{\sinh(z-\omega)\big\}\cdot\exp\Big\{-\oint_{\mathscr{C}}\mathrm{d}s\coth(s-\omega)\big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-\widehat{\mathscr{L}_{\emptyset}}(s)\big]\Big\} \tag{4.97}$$

whenever $\omega\notin\mathscr{D}$, and

$$\prod_{a=1}^{N}\Big\{\frac{\sinh(\lambda_a,\mathrm{i}\zeta+\frac{\beta}{N})}{\sinh(\nu_a,\mathrm{i}\zeta+\frac{\beta}{N})}\Big\}^N = \exp\Big\{N\oint_{\mathscr{C}}\mathrm{d}s\big[\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-\widehat{\mathscr{L}_{\emptyset}}(s)\big]\partial_s\ln\big\{\sinh(s,\mathrm{i}\zeta+\tfrac{\beta}{N})\big\}\Big\}. \tag{4.98}$$

By gathering all these results, the claim follows. ■

4.4 Taking the infinite Trotter number limit in the complete representation

We gather here the results of Proposition 4.3 and Proposition 4.5:

Theorem 4.6. *With the hypothesis and notations of Proposition 4.3 and Proposition 4.5,*

$$\mathcal{F}_{\mathcal{B}}(\{\mu_a(h')\}_1^N;\{\lambda_a(h)\}_1^N;\xi_-)\cdot\frac{\big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N),\mathbf{\Psi}(\{\lambda_a(h)\}_1^N)\big)}{\big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N),\mathbf{\Psi}(\{\mu_a(h')\}_1^N)\big)} = \mathrm{e}^{\widehat{\mathscr{F}}_{\mathbb{Y}}-\widehat{\mathscr{F}}_{\emptyset}}\cdot\Big[\frac{1-\widehat{\mathfrak{a}}_{\emptyset}(0)^2}{1-\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)^2}\Big]^{\frac{1}{4}}\cdot\Big[\frac{1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\xi_-)}{1+\widehat{\mathfrak{a}}_{\emptyset}(-\xi_-)}\Big]^{\mathbf{1}_{\mathscr{D}}(-\xi_-)}\cdot$$
$$\times\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\oint_{\mathscr{C}'\subset\mathscr{C}}\mathrm{d}s'\Big[\phi_2^{(+)}(s,s')\big(\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s')-\widehat{\mathscr{L}_{\emptyset}}(s)\,\widehat{\mathscr{L}_{\emptyset}}(s')\big)+\phi_2^{(-)}(s,s')\,\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big(\widehat{\mathscr{L}_{\mathbb{Y}}}(s')-\widehat{\mathscr{L}_{\emptyset}}(s')\big)\Big]\Big\}$$
$$\times\exp\Big\{\oint_{\mathscr{C}}\mathrm{d}s\,\phi_1(s)\big(\widehat{\mathscr{L}_{\mathbb{Y}}}(s)-\widehat{\mathscr{L}_{\emptyset}}(s)\big)\Big\}\cdot\mathcal{K}_N^{(\mathrm{tot})}(\widehat{\mathbb{Y}})\cdot\frac{(1-\varkappa)V_+(\theta)}{1-\frac{\widehat{\mathfrak{a}}_{\mathbb{Y}}(\theta)}{\widehat{\mathfrak{a}}_{\emptyset}(\theta)}}\cdot\frac{\det_{\Gamma(\mathscr{C})}\big[\mathit{id}+\widehat{U}_{\theta}^{(\lambda)}\big]}{\det_{\mathscr{C}_{\mathbb{Y}}}\big[\mathit{id}+\widehat{\overline{K}}_{\mathbb{Y}}\big]}, \tag{4.99}$$

in which $\mathcal{K}_N^{(\mathrm{tot})}(\widehat{\mathbb{Y}})$ *is expressed in terms of the particle-hole configuration of the set of Bethe roots* $\{\mu_a(h')\}_1^N$ *as*

$$\begin{aligned}\mathcal{K}_N^{(\mathrm{tot})}(\widehat{\mathbb{Y}}) \;&=\; \mathcal{K}_N^{(1)}(\widehat{\mathbb{Y}})\,\mathcal{K}_N^{(2)}(\widehat{\mathbb{Y}}) \\ &=\; \frac{\prod\limits_{a<b}^{n} f(\widehat{y}_a,\widehat{y}_b) f(\widehat{x}_a,\widehat{x}_b)}{\prod\limits_{a,b=1}^{n} f(\widehat{x}_a,\widehat{y}_b)} \cdot \frac{\prod\limits_{\substack{z\neq z' \\ z,z'\in\widehat{\mathbb{Y}}}} \sinh(z-z')}{\prod\limits_{z,z'\in\widehat{\mathbb{Y}}} \sinh(z-z'-\mathrm{i}\zeta)} \cdot \prod_{x\in\widehat{\mathfrak{X}}} \Big\{ \frac{\widehat{\mathfrak{a}}_{\emptyset}(x)\;+\;1}{-\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(x)} \Big\} \cdot \prod_{y\in\widehat{\mathcal{Y}}} \Big\{ \frac{1}{\ln'\widehat{\mathfrak{a}}_{\mathbb{Y}}(y)} \Big\} \\ &\times \prod_{a=1}^{n} \Big\{ \frac{\sinh(\widehat{y}_a+\xi_-)}{\sinh(\widehat{x}_a+\xi_-)} \cdot \frac{\sinh(2\widehat{x}_a-\mathrm{i}\zeta)}{\sinh(2\widehat{y}_a)} \cdot [1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\widehat{x}_a)]^{\frac{1}{2}} \cdot [1+\widehat{\mathfrak{a}}_{\mathbb{Y}}(-\widehat{y}_a)]^{\frac{1}{2}} \Big\} \\ &\times \prod_{z\in\widehat{\mathbb{Y}}} \exp\Big\{ -\oint_{\mathscr{C}} \mathrm{d}s\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s)\, \partial_s \ln f(s,z) \;+\; \oint_{\mathscr{C}} \mathrm{d}s\, \widetilde{\psi}_1(s,z)\, \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \;+\; \widetilde{\psi}_2(s,z) \Big[\widehat{\mathscr{L}_{\emptyset}}(s) \;-\; \widehat{\mathscr{L}_{\mathbb{Y}}}(s) \Big] \Big\}. \end{aligned} \tag{4.100}$$

It is now straightforward to take the infinite Trotter number limit of this expression, along the lines settled in Section 4.1.2. The limiting expression

$$\mathscr{A}_{h,h'}^{(z)}(\mathbb{Y}) \;=\; \lim_{N\to+\infty} \Big\{ \mathcal{F}_{\mathscr{B}}(\{\mu_a(h')\}_1^N; \{\lambda_a(h)\}_1^N; \xi_-) \cdot \frac{\big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\lambda_a(h)\}_1^N)\big)}{\big(\mathbf{\Psi}(\{\mu_a(h')\}_1^N), \mathbf{\Psi}(\{\mu_a(h')\}_1^N)\big)} \Big\} \tag{4.101}$$

is then simply given by a mere replacement $\widehat{\mathfrak{a}}_{\mathbb{Y}} \hookrightarrow \mathfrak{a}_{\mathbb{Y}}$ and $\widehat{\mathscr{L}_{\mathbb{Y}}} \hookrightarrow \mathscr{L}_{\mathbb{Y}}$ in the above formulas, and by taking the limiting values of the particle and hole roots as in (4.16). Also (see [37]), the limiting expression $\mathscr{F}_{\mathbb{Y}}$ of $\widehat{\mathscr{F}_{\mathbb{Y}}}$ is obtained by replacing in (4.29) the kernel $\widehat{U}_{\mathbb{Y}}$ (4.56) and the function $\widehat{\mathfrak{r}}_{\mathbb{Y}}$ (4.57) respectively by

$$\begin{aligned} U_{\mathbb{Y}}(\omega,\omega') &= \frac{-\mathrm{e}^{-\frac{h'}{T}} \sinh(2\omega'-\mathrm{i}\zeta)}{\sinh(\omega+\omega')\sinh(\omega-\omega'-\mathrm{i}\zeta)} \cdot \exp\{-2\beta\,[\coth(\omega')-\coth(\omega'-\mathrm{i}\zeta)]\} \\ &\times \exp\Big\{ -\oint_{\mathscr{C}_U} \mathrm{d}s\, \mathscr{L}_{\mathbb{Y}}(s) \Big[\coth(\omega'+s) \;+\; \coth(\omega'-s) \;-\; \coth(\omega'+s-\mathrm{i}\zeta) \;-\; \coth(\omega'-s-\mathrm{i}\zeta) \Big] \Big\}, \end{aligned} \tag{4.102}$$

and

$$\begin{aligned} \mathfrak{r}_{\mathbb{Y}}(\omega) &= \mathrm{e}^{-\frac{2h'}{T}} \cdot \exp\{-2\beta\,[\coth(\omega'+\mathrm{i}\zeta)-\coth(\omega'-\mathrm{i}\zeta)]\} \\ &\times \exp\Big\{ -\oint_{\mathscr{C}_U} \mathrm{d}s\, \mathscr{L}_{\mathbb{Y}}(s) \Big[\coth(s+\omega+\mathrm{i}\zeta) \;+\; \coth(\omega-s+\mathrm{i}\zeta) \;-\; \coth(\omega+s-\mathrm{i}\zeta) \;-\; \coth(\omega-s-\mathrm{i}\zeta) \Big] \Big\}. \end{aligned} \tag{4.103}$$

The only point of attention in taking the limit in (4.99) comes from the fact that, strictly speaking, the limiting value of $\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)$ is not well defined. However, the limiting value of its square $\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)^2$ can be taken without problem from the integral equation

$$\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)^2 \;=\; \Big(\frac{\sinh(\beta/N-\mathrm{i}\zeta)}{\sinh(-\beta/N-\mathrm{i}\zeta)}\Big)^{2N} \mathrm{e}^{-2\frac{h'}{T}} \prod_{y\in\widehat{\mathbb{Y}}} \Big\{\mathrm{e}^{2\mathrm{i}\theta(-y)}\Big\} \cdot \exp\Big\{4\mathrm{i}\pi \oint_{\mathscr{C}} K(u)\, \widehat{\mathscr{L}_{\mathbb{Y}}}(u)\, \mathrm{d}u\Big\}, \tag{4.104}$$

from which it follows that

$$\lim_{N\to+\infty} \Big\{\widehat{\mathfrak{a}}_{\mathbb{Y}}(0)^2\Big\} \;=\; \mathcal{A}(\mathbb{Y}) \tag{4.105}$$

is well defined and finite.

5 The thermal form factor expansion for the one point function

From the previous results, one therefore obtains the below thermal form factor series expansion for the generating function $Q(h', m) = \lim_{N\to+\infty} Q_N(h', m)$ of the one-point function at distance m from the boundary. One has

$$\langle \sigma^z_{m+1} \rangle_T = 2T\partial_{h'} \mathfrak{D}_m Q(h', m)_{|h'=h} \tag{5.1}$$

where

$$Q(h', m) = \sum_{\mathbb{Y}} \Big(\frac{\tau_{\mathbb{Y}}(0)}{\tau_{\emptyset}(0)} \Big)^m \mathscr{A}^{(z)}_{h,h'}(\mathbb{Y}) \, . \tag{5.2}$$

Above, the summation runs through all possible particle-hole excitations, *viz.* though all the solutions to the non-linear integral equation (4.17) subject to the conditions which fix a given particle-hole excitations (4.21), and $\mathscr{A}^{(z)}_{h,h'}(\mathbb{Y})$ denotes the infinite Trotter limit (4.101) of the product (4.99). Defining $\mathscr{C} = C + \mathrm{i}\frac{\zeta}{2}$, $\mathcal{D} = \mathrm{Int}(C)$, and setting

$$\mathcal{L}_{\mathbb{Y}}(\lambda) = \frac{1}{2\mathrm{i}\pi} \mathscr{L}n\Big[1 + \mathrm{e}^{-\frac{1}{T}u_{\mathbb{Y}}(\lambda)}\Big] \, , \quad \text{with} \quad u_{\mathbb{Y}}(\lambda) = -T\mathfrak{A}_{\mathbb{Y}}(u + \mathrm{i}\tfrac{\zeta}{2}) \, , \tag{5.3}$$

this quantity $\mathscr{A}^{(z)}_{h,h'}(\mathbb{Y})$ can explicitly be rewritten as

$$\mathscr{A}^{(z)}_{h,h'}(\mathbb{Y}) = \mathrm{e}^{\mathscr{F}_{\mathbb{Y}} - \mathscr{F}_{\emptyset}} \cdot \Big(\mathcal{E}^{(r)} \cdot \mathcal{K}^{(r)} \Big)(\mathbb{Y}) \cdot \Big(\mathcal{E}^{(s)} \cdot \mathcal{K}^{(s)} \Big)(\mathbb{Y}) \, . \tag{5.4}$$

Here we have introduced

$$\mathcal{E}^{(r)}(\mathbb{Y}) = \exp\bigg\{ -\oint_{C} \mathrm{d}\lambda \oint_{C' \subset C} \mathrm{d}\mu \Big[\frac{1}{2} \coth'(\lambda + \mu + \mathrm{i}\zeta) \cdot \Big(\mathcal{L}_{\mathbb{Y}}(\lambda) \mathcal{L}_{\mathbb{Y}}(\mu) - \mathcal{L}_{\emptyset}(\lambda) \mathcal{L}_{\emptyset}(\mu) \Big)$$
$$+ \coth'(\lambda - \mu - \mathrm{i}\zeta) \cdot \mathcal{L}_{\mathbb{Y}}(\lambda) \Big(\mathcal{L}_{\emptyset}(\mu) - \mathcal{L}_{\mathbb{Y}}(\mu) \Big) \Big] \bigg\} \cdot \exp\bigg\{ \lim_{\epsilon \to 0^+} \oint_{C} \mathrm{d}\lambda \widetilde{\phi}_1(\lambda + \mathrm{i}\epsilon) \Big(\mathcal{L}_{\mathbb{Y}}(\lambda) - \mathcal{L}_{\emptyset}(\lambda) \Big) \bigg\} \, , \tag{5.5}$$

in which we agree upon

$$\widetilde{\phi}_1(\lambda) = \coth(2\lambda) + \coth(2\lambda + \mathrm{i}\zeta) - \coth(\lambda + \xi_- + \mathrm{i}\tfrac{\zeta}{2}) \, , \tag{5.6}$$

and

$$\mathcal{E}^{(s)}(\mathbb{Y}) = \exp\bigg\{ \oint_{C} \mathrm{d}\lambda \oint_{C' \subset C} \mathrm{d}\mu \Big[\frac{1}{2} \coth'(\lambda + \mu) \cdot \Big(\mathcal{L}_{\mathbb{Y}}(\lambda) \mathcal{L}_{\mathbb{Y}}(\mu) - \mathcal{L}_{\emptyset}(\lambda) \mathcal{L}_{\emptyset}(\mu) \Big)$$
$$+ \coth'(\lambda - \mu) \cdot \mathcal{L}_{\mathbb{Y}}(\lambda) \Big(\mathcal{L}_{\emptyset}(\mu) - \mathcal{L}_{\mathbb{Y}}(\mu) \Big) \Big] \bigg\} \, . \tag{5.7}$$

We have also defined

$$\mathcal{K}^{(r)}(\mathbb{Y}) \;=\; \Big(\frac{1-\mathcal{A}(\emptyset)}{1-\mathcal{A}(\mathbb{Y})}\Big)^{\frac{1}{4}} \cdot \frac{\det_{\Gamma(\mathscr{C})}\big[\mathtt{id} \,+\, \mathtt{U}^{(\lambda)}_{\theta}\big]}{\det_{\mathscr{C}_{\mathbb{Y}}}\big[\mathtt{id} \,+\, \overline{K}_{\mathbb{Y}}\big]} \cdot \frac{(1\,-\,\varkappa)V_{+}(\theta)}{1\,-\,\frac{\mathfrak{a}_{\mathbb{Y}}(\theta)}{\mathfrak{a}_{\emptyset}(\theta)}} \cdot \prod_{a=1}^{n}\Big\{\frac{-1\,-\,\mathrm{e}^{-\frac{1}{T}u_{\emptyset}(x_a)}}{\widehat{u}'_{\mathbb{Y}}(x_a)\cdot\widehat{u}'_{\mathbb{Y}}(y_a)}\Big\} \cdot \prod_{z\in\mathcal{Y}\ominus\mathfrak{X}}\big\{\mathcal{B}^{(r)}(z)\big\}$$
$$\times\prod_{a=1}^{n}\Big\{\frac{\sinh(y_a+\xi_-+\mathrm{i}\zeta/2)}{\sinh(x_a+\xi_-+\mathrm{i}\zeta/2)}\cdot\frac{\sinh(2x_a)}{\sinh(2\,y_a+\mathrm{i}\zeta)}\cdot\Big\{1+\mathrm{e}^{-\frac{1}{T}u_{\mathbb{Y}}(-x_a-\mathrm{i}\zeta)}\Big\}^{\frac{1}{2}}\cdot\Big\{1+\mathrm{e}^{-\frac{1}{T}u_{\mathbb{Y}}(-y_a-\mathrm{i}\zeta)}\Big\}^{\frac{1}{2}}\Big\}$$
$$\times\prod_{a,b=1}^{n}\Big\{\frac{\sinh(x_a+y_b+\mathrm{i}\zeta)\sinh(x_a-y_b,\mathrm{i}\zeta)}{\sinh(x_a-x_b-\mathrm{i}\zeta)\sinh(y_a-y_b-\mathrm{i}\zeta)}\Big\}\cdot\prod_{a<b}^{n}\Big\{\frac{1}{\sinh(x_a+x_b+\mathrm{i}\zeta)\sinh(y_a+y_b+\mathrm{i}\zeta)}\Big\}$$
$$\times\Bigg(\frac{1+\mathrm{e}^{-\frac{1}{T}u_{\mathbb{Y}}(-\xi_--\mathrm{i}\zeta/2)}}{1+\mathrm{e}^{-\frac{1}{T}u_{\emptyset}(-\xi_--\mathrm{i}\zeta/2)}}\Bigg)^{\mathbf{1}_{\mathcal{D}}(-\xi_--\mathrm{i}\zeta/2)} \;. \tag{5.8}$$

Here the integral kernels $\mathtt{U}^{(\lambda)}_{\theta}$ and $\overline{K}_{\mathbb{Y}}$ are obtained upon the replacement $\widehat{\mathfrak{a}}_{\mathbb{Y}} \hookrightarrow \mathfrak{a}_{\mathbb{Y}}$ in the associated expressions at finite Trotter number (4.91) and (4.80), the quantity $\mathcal{A}(\mathbb{Y})$ is given by (see (4.105))

$$\mathcal{A}(\mathbb{Y}) \;=\; \prod_{y\in\mathcal{Y}\ominus\mathfrak{X}}\big\{\mathrm{e}^{2\mathrm{i}\theta(-y-\mathrm{i}\frac{\zeta}{2})}\big\}\cdot\exp\Big\{-2\frac{h}{T}+4\beta\coth(\mathrm{i}\zeta)\;+\;4\mathrm{i}\pi\oint_{C}K(\lambda+\mathrm{i}\tfrac{\zeta}{2})\cdot\mathcal{L}_{\mathbb{Y}}(\lambda)\cdot\mathrm{d}\lambda\Big\}, \tag{5.9}$$

and we have introduced

$$\mathcal{B}^{(r)}(z) \;=\; \exp\Big\{\oint_{C}\mathrm{d}\lambda\Big[\mathcal{L}_{\mathbb{Y}}(\lambda)\big[\coth(\lambda-z-\mathrm{i}\zeta)+\coth(\lambda+z+\mathrm{i}\zeta)\big]\;-\;\coth(\lambda-z+\mathrm{i}\zeta)\big(\mathcal{L}_{\emptyset}(\lambda)\;-\;\mathcal{L}_{\mathbb{Y}}(\lambda)\big)\Big]\Big\}. \tag{5.10}$$

Further, one has

$$V_{+}(\theta) \;=\; \prod_{z\in\mathcal{Y}\ominus\mathfrak{X}}\Big\{\frac{1}{\sinh(z-\theta-\mathrm{i}\zeta/2)}\Big\}\cdot\exp\Big\{-\oint_{C}\mathrm{d}s\,\coth\big(s-\theta-\frac{\mathrm{i}\zeta}{2}\big)\big[\widehat{\mathscr{L}_{\emptyset}}(s)\;-\;\widehat{\mathscr{L}_{\mathbb{Y}}}(s)\big]\Big\}. \tag{5.11}$$

Finally, the last factor appearing in (5.4) takes the form

$$\mathcal{K}^{(s)}(\mathbb{Y}) \;=\; T^{2n}\cdot\prod_{z\in\mathcal{Y}\ominus\mathfrak{X}}\big\{\mathcal{B}^{(s)}(z)\big\}\cdot\frac{\prod\limits_{a<b}^{n}\big\{\sinh^2(x_a-x_b)\sinh^2(y_a-y_b)\sinh(x_a+x_b)\sinh(y_a+y_b)\big\}}{\prod\limits_{a,b=1}^{n}\big\{\sinh^2(x_a-y_b)\sinh(x_a+y_b)\big\}}\;, \tag{5.12}$$

which involves the function

$$\mathcal{B}^{(s)}(z) \;=\; \exp\Big\{\oint_{C}\mathrm{d}\lambda\Big[\coth(\lambda-z)\big(\mathcal{L}_{\emptyset}(\lambda)\;-\;\mathcal{L}_{\mathbb{Y}}(\lambda)\big)\;-\;\mathcal{L}_{\mathbb{Y}}(\lambda)\big[\coth(\lambda+z)+\coth(\lambda-z)\big]\Big]\Big\}\,. \tag{5.13}$$

We would also like to recall that this representation holds when the parameter θ is such that $\theta+\mathrm{i}\zeta/2\notin\mathrm{Int}(C)$ and that $\varkappa=\mathrm{e}^{\frac{h-h'}{T}}$.

6 Conclusion

This work develops a setting allowing one to produce thermal form factor expansions for multi-point correlation functions in open quantum integrable models. Such kinds of thermal form factor expansions, which had been so far only developed in the periodic case, have proven to be very efficient for the computation of correlation functions, and notably for the determination of their long-distance asymptotic behaviour. It is worth recalling that, for integrable models with open boundary conditions, the only exact representations that could have been obtained so far for the correlation functions turned out to be too intricate for allowing any asymptotic analysis. We therefore hope that the new approach that we propose here will open the way to solve this long-standing problem.

Our method relies on the ideas developed in [5, 37]. We have here explicitly worked out the expansion describing the one-point function at distance m from the boundary in the XXZ spin-1/2 chain with diagonal boundary fields. It is however clear that the thermal form factor expansion for multi-point correlation functions can be worked out completely similarly. We also stress that, by conforming the techniques of [8], it is possible to generalise our setting to dynamical correlation functions as well. Let us finally mention that our method is *a priori* applicable to all quantum integrable models whose thermodynamics can be addressed within the quantum transfer matrix method.

The next problem that we would like to consider is the analysis of the low-temperature limit of our final result. Indeed, similarly as what happens in the periodic case [5, 6], we expect to be able to grasp, from this form factor expansion, the asymptotic behaviour of the one-point functions (5.1) in terms of the distance m from the boundary. To this aim, we would like to mention that our final result has already been explicitly decomposed into two parts: one which has a finite limit when $T \to 0^+$, the factors $\mathcal{E}^{(r)}(\mathbb{Y})$, $\mathcal{K}^{(r)}(\mathbb{Y})$, and one which we expect to produce a power-law behaviour in T, the factors $\mathcal{E}^{(s)}(\mathbb{Y})$, $\mathcal{K}^{(s)}(\mathbb{Y})$. The latter should be responsible for the emergence of a conformal field theoretic description of the long-distance asymptotic behaviour of the one point function. We plan to consider this interesting problem in a next publication.

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