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Two-species chemotaxis-competition system with singular sensitivity: Global existence, boundedness, and persistence

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Abstract

This paper is concerned with the following parabolic-parabolic-elliptic chemotaxis system with singular sensitivity and Lotka-Volterra competitive kinetics,

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w} \nabla w) + u(a_1 - b_1 u - c_1 v), & x \in \Omega \\ v_t = \Delta v - \chi_2 \nabla \cdot (\frac{v}{w} \nabla w) + v(a_2 - b_2 v - c_2 u), & x \in \Omega \\ 0 = \Delta w - \mu w + \nu u + \lambda v, & x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and χ_i, a_i, b_i, c_i $(i = 1, 2)$ and μ, ν, λ are positive constants. This is the first work on two-species chemotaxis-competition system with singular sensitivity and Lotka-Volterra competitive kinetics. Among others, we prove that for any given nonnegative initial data $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 + v_0 \not\equiv 0$, (0.1) has a unique globally defined classical solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0), w(t, x; u_0, v_0))$ with $u(0, x; u_0, v_0) = u_0(x)$ and $v(0, x; u_0, v_0) = v_0(x)$ provided that $\min\{a_1, a_2\}$ is large relative to χ_1, χ_2 and $u_0 + v_0$ is not small. Moreover, under the same condition, we prove that

$$\limsup_{t\to\infty} \|u(t, \cdot; u_0, v_0) + v(t, \cdot; u_0, v_0)\|_\infty \le M^*,$$

and

$$\liminf_{t\to\infty} \inf_{x\in\Omega} (u(t, x, u_0, v_0) + v(t, x; u_0, v_0)) \ge m^*,$$

for some positive constants M^*, m^* independent of u_0, v_0, the latter is referred to as combined pointwise persistence.

Key words. Parabolic-parabolic-elliptic chemotaxis system, singular sensitivity, Lotka-Volterra competitive kinetics, global existence, global boundedness, combined mass persistence, combined pointwise persistence.

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1 Introduction and Main Results

Chemotaxis refers to the movement of cells or organisms in response to chemicals in their environments, and plays a crucial role in many biological processes such as immune system response,

tumor growth, population dynamics, gravitational collapse, the governing of immune cell migration. Since the pioneering works by Keller and Segel ([23], [24]) on chemotaxis models, a lot of works have been carried out on the qualitative properties of various chemotaxis models such as the analysis of global existence, boundedness, blow-up in finite time, and asymptotic behavior of globally defined solutions, etc. The reader is referred to [4, 16, 17] and the references therein for some detailed introduction into the mathematics of chemotaxis models.

There are a large number of works on various two competing species chemotaxis models. For example, consider the following two-species chemotaxis system,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + u(a_1 - b_1 u - c_1 v), & x \in \Omega \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + v(a_2 - b_2 v - c_2 u), & x \in \Omega \\ \tau w_t = \Delta w - \mu w + \nu u + \lambda v, & x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $a_1, b_1, c_1, a_2, b_2, c_2, \nu, \lambda$ are positive numbers, and $\tau \geq 0$. Biologically, (1.1) models the evolution of two competitive species subject to a chemical substance, which is produced by the two species themselves. Here the unknown functions $u(t,x)$ and $v(t,x)$ represent the population densities of two competitive biological species and $w(t,x)$ represents the concentration of the chemical substance. The terms $u(a_1 - b_1 u - c_1 v)$ and $v(a_2 - b_2 v - c_2 u)$ are referred to as Lotka-Volterra competitive terms. The parameter μ is the degradation rate of the chemical substance and ν and λ are the production rates of the chemical substance by the species u and v, respectively. $\tau \geq 0$ is related to the diffusion rate of the chemical substance. The functions $\chi_1(w)$ and $\chi_2(w)$ reflect the strength of the chemical substance on the movements of two species, and are referred to as chemotaxis sensitivity functions or coefficients.

When $\chi_1(w) \equiv \chi_1 > 0$, $\chi_2(w) \equiv \chi_2 > 0$, and $\tau = 0$, it is known that if $N \leq 2$, or $N \geq 3$ and χ_i is small relative to b_i and c_i $(i = 1, 2)$, then for any nonnegative initial data $u_0, v_0 \in C^0(\bar{\Omega})$, system (1.1) possesses a unique globally defined classical solution $(u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$ with $u(0,x;u_0,v_0) = u_0(x)$ and $v(0,x;u_0,v_0) = v_0(x)$ (see [20, 22, 28, 32, 34] and references therein). Moreover, the large time behaviors of globally defined classical solutions of (1.1) such as competitive exclusion, coexistence, stabilization, etc., are investigated in [7, 20, 21, 22, 29, 30, 32, 34], etc.

When $\chi_1(\omega) \equiv \chi_1 > 0$, $\chi_2(w) \equiv \chi_2 > 0$ and $\tau = 1$, it is proved that, if $N \leq 2$, or $N \geq 3$ and χ_1 and χ_2 are small relative to other parameters in (1.1), then for any nonnegative initial data $u_0, v_0 \in C(\bar{\Omega})$, $w_0 \in W^{1,\infty}(\Omega)$, system (1.1) possesses a unique globally defined classical solution $(u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$ with $u(0,x;u_0,v_0) = u_0(x)$, $v(0,x;u_0,v_0) = v_0(x)$, and $w(0,x;u_0,v_0) = w_0(x)$ (see [3, 28, 36], etc.). Moreover, the large-time behaviors of globally defined classical solutions are investigated in [3, 36], etc. We refer to the readers to the articles [18, 36] for the further details.

The aim of current paper is to investigate the global existence, boundedness, and combined persistence of classical solutions of (1.1) with $\chi_1(w) = \frac{\chi_1}{w}$ and $\chi_2(w) = \frac{\chi_2}{w}$ for some positive constants χ_1 and χ_2, and $\tau = 0$, that is, the following parabolic-parabolic-elliptic chemotaxis

system with singular sensitivity and Lotka-Volterra competitive kinetics,

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w} \nabla w) + u(a_1 - b_1 u - c_1 v), & x \in \Omega \\ v_t = \Delta v - \chi_2 \nabla \cdot (\frac{v}{w} \nabla w) + v(a_2 - b_2 v - c_2 u), & x \in \Omega \\ 0 = \Delta w - \mu w + \nu u + \lambda v, & x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

It is seen that the chemotaxis sensitivities $\frac{\chi_i}{w}$ $(i = 1, 2)$ are singular near $w = 0$, reflecting an inhibition of chemotactic migration at high signal concentrations. Such a sensitivity describing the living organisms' response to the chemical signal was derived by the Weber-Fechner law (see [24]).

We consider classical solutions of (1.2) with initial functions $u_0, v_0 \in C^0(\bar{\Omega})$ with $u_0 \geq 0$, $v_0 \geq 0$, and $\int_\Omega (u_0 + v_0) > 0$. Note that for such initial functions, if $v_0 = 0$ (resp. $u_0 = 0$), then $v(t, x) \equiv 0$ (resp. $u(t, x) \equiv 0$) on the existence interval. Note also that if $v(t, x) \equiv 0$, then (1.2) becomes

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w} \nabla w) + u(a_1 - b_1 u), & x \in \Omega \\ 0 = \Delta w - \mu w + \nu u, & x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

and if $u(t, x) \equiv 0$, then (1.2) becomes

$$\begin{cases} v_t = \Delta v - \chi_2 \nabla \cdot (\frac{v}{w} \nabla w) + v(a_2 - b_2 v), & x \in \Omega \\ 0 = \Delta w - \mu w + \lambda v, & x \in \Omega \\ \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

Systems (1.3) and (1.4) are essentially the same, and are referred to as one species chemotaxis models with logistic source and singular sensitivity. They have been studied in many works (see [5, 6, 8, 13, 14, 25, 26, 31], etc.). Let us briefly review some known results for one species chemotaxis models with logistic source and singular sensitivity.

Consider (1.3) with $a_1 = b_1 \equiv 0$ and $\mu = \nu = 1$. Fujie, Winkler, and Yokota in [14] proved the global existence and boundedness of positive classical solutions when $\chi_1 < \frac{2}{N}$ and $N \geq 2$. More recently, Fujie and Senba in [12] proved the global existence and boundedness of classical positive solutions for the case of $N = 2$ for any $\chi_1 > 0$. The existence of finite-time blow-up is then completely ruled out for any $\chi_1 > 0$ in the case $N = 2$. When $N \geq 3$, finite-time blow-up may occur (see [31]).

Consider (1.3) with $a_1, b_1 > 0$. Central questions include whether the logistic source prevents the occurrence of finite-time blow-up in (1.3) (i.e. any positive solution exists globally); if so, whether any globally defined positive solution is bounded, and what is the long time behavior of globally defined bounded positive solutions, etc. When $N = 2$ and a_1, b_1 are positive constants, it is proved in [13] that finite-time blow-up does not occur (see [13, Theorem 1.1]), and moreover, if

$$a_1 > \begin{cases} \frac{\mu \chi_1^2}{4}, & \text{if } 0 < \chi_1 \leq 2 \\ \mu(\chi_1 - 1), & \text{if } \chi_1 > 2, \end{cases} \tag{1.5}$$

then any globally defined positive solution is bounded. Under some additional assumption, it is proved in [8] that the constant solution $(\frac{a}{b}, \frac{\nu}{\mu}\frac{a}{b})$ is exponentially stable (see [8, Theorem 1]). Very recently, among others, we proved that, in any space dimensional setting, a positive classical solution $(u(t,x), w(t,x))$ of (1.3) exists globally and stays bounded provided that a_1 is large relative to χ_1 and the initial function $u(0,x)$ is not too small (see [25, Theorem 1.2(3)]. Recall that, with $\chi_1 \nabla \cdot (\frac{u}{w}\nabla w)$ in (1.3) being replaced by $\chi_1 \cdot \nabla(u\nabla w)$, then a positive classical solution $(u(t,x), w(t,x))$ exists globally provided that b is large relative to χ_1 (see [33, Theorem 2.5]). Hence [25, Theorem 1.2(3)] reveals some possible interesting difference between singular and regular chemotaxis sensitivities. We also proved that the globally defined positive solutions of (1.3) are away from 0 provided that a_1 is large relative to χ_1 and the initial functions are not too small (see [26, Theorem 1.3]).

However, as far as we know, there is little study on the two-species chemotaxis system (1.2). It is the aim of this paper to investigate the global existence, boundedness, and combined persistence of classical solutions of (1.2).

Definition 1.1. *For given $u_0(\cdot) \in C^0(\bar{\Omega})$ and $v_0(\cdot) \in C^0(\bar{\Omega})$ satisfying that $u_0 \geq 0$, $v_0 \geq 0$, and $\int_\Omega (u_0(x) + v_0(x))dx > 0$, we say $(u(t,x), v(t,x), w(t,x))$ is a* positive classical solution *of* (1.2) *on $(0,T)$ for some $T \in (0,\infty]$ with initial condition $(u(0,x), v(0,x)) = (u_0(x), v_0(x))$ if*

$$u(t,x;u_0,v_0) + v(t,x;u_0,v_0) > 0,\ w(t,x;u_0,v_0) > 0 \quad \forall t \in (0,T),\ x \in \bar{\Omega}, \tag{1.6}$$

$$u(\cdot,\cdot), v(\cdot,\cdot) \in C([0,T) \times \bar{\Omega}) \cap C^{1,2}((0,T) \times \bar{\Omega}), \quad w(\cdot,\cdot) \in C^{0,2}((0,T) \times \bar{\Omega}),$$

$$\lim_{t \to 0+} \|u(t,\cdot) - u_0(\cdot)\|_{C^0(\bar{\Omega})} = 0, \quad \lim_{t \to 0+} \|v(t,\cdot) - v_0(\cdot)\|_{C^0(\bar{\Omega})} = 0,$$

and $(u(t,x), v(t,x), w(t,x))$ satisfies (1.2) *for all $(t,x) \in (0,T) \times \Omega$.*

The following proposition on the local existence of classical solutions of (1.2) can be proved by the similar arguments as those in [13, Lemma 2.2].

Proposition 1.1 (Local existence)**.** *For given $u_0(\cdot) \in C^0(\bar{\Omega})$ and $v_0(\cdot) \in C^0(\bar{\Omega})$ satisfying that $u_0 \geq 0$, $v_0 \geq 0$, and $\int_\Omega (u_0(x) + v_0(x))dx > 0$, there exists $T_{\max}(u_0,v_0) \in (0,\infty]$ such that* (1.2) *has a unique positive classical solution, denoted by $(u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$, on $(0, T_{\max}(u_0,v_0))$ with initial condition $(u(0,x;u_0,v_0), v(0,x;u_0,v_0)) = (u_0(x), v_0(x))$. Moreover, if $\int_\Omega u_0(x)dx > 0$ and $\int_\Omega v_0(x)dx > 0$, then*

$$u(t,x;u_0,v_0) > 0 \quad \text{and} \quad v(t,x;u_0,v_0) > 0 \quad \forall\, t \in (0, T_{\max}(u_0,v_0)),\ x \in \bar{\Omega}.$$

If $T_{\max}(u_0,v_0) < \infty$, then either

$$\limsup_{t \nearrow T_{\max}(u_0,v_0)} \Big(\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^0(\bar{\Omega})} \Big) = \infty,$$

or

$$\liminf_{t \nearrow T_{\max}(u_0,v_0)} \inf_{x \in \Omega} w(t,x;u_0,v_0) = 0. \tag{1.7}$$

We will focus on the following problems in this paper: whether $(u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$ exists globally, i.e., $T_{\max}(u_0,v_0)=\infty$, for any $u_0(\cdot)\in C^0(\bar{\Omega})$ and $v_0(\cdot)\in C^0(\bar{\Omega})$ satisfying that $u_0\geq 0$, $v_0\geq 0$, and $\int_\Omega(u_0(x)+v_0(x))dx>0$; If $T_{\max}(u_0,v_0)=\infty$, whether $(u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$ is bounded above and stays away from 0.

We point out that in the study of global existence and boundedness of classical solutions of one species chemotaxis model (1.3) with singular sensitivity, it is crucial to prove the boundedness of $(\int_\Omega u(t,x)dx)^{-1}$ and the boundedness of $\int_\Omega u^p(t,x)dx$ for some $p\gg 1$. The novel idea discovered in this paper for the study of global existence and boundedness of classical solutions of (1.2) is to prove the boundedness of $(\int_\Omega(u(t,x)+v(t,x))dx)^{-1}$ and the boundedness of $\int_\Omega(u(t,x)+v(t,x))^p dx$ for $p\gg 1$. Note that for (1.2), $(\int_\Omega u(t,x)dx)^{-1}$ (resp. $(\int_\Omega v(t,x)dx)^{-1}$) may not be bounded. Note also that the boundedness of $(\int_\Omega(u(t,x)+v(t,x))dx)^{-1}$ is strongly related to the boundedness of $\int_\Omega(u(t,x)+v(t,x))^{-q}dx)$ for some $q>0$.

In the rest of the introduction, we introduce some standing notations and assumptions in subsection 1.1, and state the main results of the paper and provide some remarks on the main results in subsection 1.2.

1.1 Notations and assumptions

In this subsection, we introduce some standing notations and assumptions to be used throughout the paper.

Observe that for given $u_0(\cdot)\in C^0(\bar{\Omega})$ and $v_0(\cdot)\in C^0(\bar{\Omega})$ satisfying that $u_0\geq 0$, $v_0\geq 0$, and $\int_\Omega(u_0(x)+v_0(x))dx>0$, if $v_0\equiv 0$ (resp. $u_0\equiv 0$), then $v(t,x;u_0,v_0)\equiv 0$ (resp. $u(t,x;u_0,v_0)\equiv 0$) for $t\in(0,T_{\max}(u_0,v_0))$ and $(u(t,x;u_0), w(t,x;u_0)) := (u(t,x;u_0,0), w(t,x;u_0,0))$ (resp. $v(t,x;v_0)$, $w(t,x;v_0)) := (v(t,x;0,v_0),\ w(t,x;0,v_0)))$ is the solution of (1.3) (resp. (1.4)) with initial condition $u(0,x;u_0)=u_0(x)$ (resp. $v(0,x;v_0)=v_0(x)$). Hence, throughout the rest of this paper, we consider classical solutions of (1.2) with the initial function $u_0(x), v_0(x)$ satisfying

$$u_0, v_0\in C^0(\bar{\Omega}),\quad u_0,v_0\geq 0,\quad \text{and}\quad \int_\Omega u_0>0,\ \int_\Omega v_0>0. \tag{1.8}$$

Let

$$\begin{cases} a_{\min}=\min\{a_1,a_2\}, & a_{\max}=\max\{a_1,a_2\}\\ b_{\min}=\min\{b_1,b_2\}, & b_{\max}=\max\{b_1,b_2\}\\ c_{\min}=\min\{c_1,c_2\}, & c_{\max}=\max\{c_1,c_2\}.\end{cases}$$

For given $B>0$ and $\beta\neq\chi_2-B$, let

$$f(\mu,\chi_1,\chi_2,\beta,B)=\mu(B+\beta)\Big(1+\frac{B(\chi_2-B-\beta)^2+(\chi_1-\chi_2)^2\beta}{4B\beta}\Big). \tag{1.9}$$

For fixed $\mu>0$, $\chi_1>0$ and $\chi_2>0$, let

$$\chi_1^*(\mu,\chi_1,\chi_2)=\inf\Big\{f(\mu,\chi_1,\chi_2,\beta,B)\,|\,B>0,\beta>0,\beta\neq\chi_2-B\Big\}.$$

Similarly, let

$$\chi_2^*(\mu,\chi_1,\chi_2)=\inf\Big\{f(\mu,\chi_2,\chi_1,\beta,B)\,|\,B>0,\beta>0,\beta\neq\chi_1-B\Big\}.$$

Let

$$\chi^*(\mu,\chi_1,\chi_2)=\min\{\chi_1^*(\mu,\chi_1,\chi_2),\chi_2^*(\mu,\chi_1,\chi_2)\}, \tag{1.10}$$

and

$$\chi^{**}(\mu,\chi_1,\chi_2)=\begin{cases}2\mu\ \max\{\chi_1,\chi_2\} & \text{if } (\chi_1-\chi_2)^2\le\max\{4\chi_1,4\chi_2\}\\ \mu\max\{\chi_1,\chi_2\}+\frac{\mu(\chi_1-\chi_2)^2}{4} & \text{if } (\chi_1-\chi_2)^2>\max\{4\chi_1,4\chi_2\}.\end{cases}$$

Note that

$$\chi^*(\mu,\chi_1,\chi_2)\le\chi^{**}(\mu,\chi_1,\chi_2)$$

(see (2.1)). The numbers $\chi^*(\mu,\chi_1,\chi_2)$ and $\chi^{**}(\mu,\chi_1,\chi_2)$ will be used to get lower bounds for $\int(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)$, which are essential in the proofs of global existence of $(u(t,x;u_0,v_0)$, $v(t,x;u_0,v_0),w(t,x;u_0,v_0))$. Let

$$q^*=q_{\chi_1,\chi_2}=\begin{cases}1 & \text{if } (\chi_1-\chi_2)^2\le\max\{4\chi_1,4\chi_2\},\\ \max\Big\{\frac{4\chi_1}{(\chi_1-\chi_2)^2},\frac{4\chi_2}{(\chi_1-\chi_2)^2}\Big\} & \text{if } (\chi_1-\chi_2)^2>\max\{4\chi_1,4\chi_2\},\end{cases}$$

and

$$\chi_{1,2}=\min\Big\{\chi_1^2+(\chi_1-\chi_2)^2,\chi_2^2+(\chi_1-\chi_2)^2\Big\},$$

as well as

$$C_{\chi_1,\chi_2}=\begin{cases}1 & \text{if } \chi_{1,2}\ge 1\\ \sqrt{\chi_{1,2}} & \text{if } \chi_{1,2}<1.\end{cases}$$

Note that there is $\delta_0>0$ such that for any (u_0,v_0) satisfying (1.8),

$$w(t,x;u_0,v_0)\ge\delta_0\int_\Omega(u(t,x;u_0,v_0)+v(t,x;u_0,v_0))>0\quad\forall\, t\in[0,T_{\max}(u_0,v_0)),\ x\in\Omega \tag{1.11}$$

(see Lemma 2.4). Note also that, for any $p\ge 3$ and $p-\sqrt{2p-3}<k<p+\sqrt{2p-3}$, there are $M(p,k)>0$ and $\tilde M(p,k)>0$ such that

$$\begin{aligned}\int_\Omega\frac{|\nabla w(t,x;u_0,v_0)|^{2p}}{w^k(t,x;u_0,v_0)}\le& M(p,k)\int_\Omega\frac{(\nu u(t,x;u_0,v_0)+\lambda v(t,x;u_0,v_0))^p}{w^{k-p}(t,x;u_0,v_0)}\\ &+\tilde M(p,k)\int_\Omega w^{2p-k}(t,x;u_0,v_0)\quad\forall\ t\in(0,T_{\max}(u_0,v_0))\end{aligned} \tag{1.12}$$

(see Proposition 4.1).

At some places, we impose the following conditions on the parameters in (1.2) and on (u_0,v_0).

(H1) $a_{\min}>\chi^{**}(\mu,\chi_1,\chi_2)+\dfrac{\big(\chi_{1,2}\cdot\max\{\nu,\lambda\}\cdot N\big)^{q^*}\big(M(N+1,N+1)\big)^{\frac{q^*}{N+1}}\big(b_{\max}+c_{\max}\big)a_{\max}^{1-q^*}}{(2\delta_0|\Omega|C_{\chi_1,\chi_2})^{q^*}\cdot\min\{b_{\min},c_{\min}\}}$.

(H2) (u_0,v_0) satisfies (1.8) and there is $\tau_0\in[0,T_{\max}(u_0,v_0))$ such that

$$\int_\Omega\big(u+v\big)^{-1}(\tau_0,x;u_0,v_0)\le\frac{(b_{\max}+c_{\max})|\Omega|}{\big(a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)\big)\cdot C_{\chi_1,\chi_2}} \tag{1.13}$$

in the case that $(\chi_1-\chi_2)^2 \le \max\{4\chi_1, 4\chi_2\}$, and

$$\begin{cases} \int_\Omega \big(u+v\big)(\tau_0, x; u_0, v_0) \le \frac{a_{\max}|\Omega|}{\min\{b_{\min}, c_{\min}\}\cdot C_{\chi_1,\chi_2}^{\frac{q^*}{1-q^*}}}, \\ \\ \int_\Omega \big(u+v\big)^{-q^*}(\tau_0, x; u_0, v_0) \le \frac{(b_{\max}+c_{\max})|\Omega|\big(a_{\max}|\big)^{1-q^*}}{\big(a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)\big)\cdot\big(\min\{b_{\min}, c_{\min}\}\big)^{1-q^*}\cdot C_{\chi_1,\chi_2}^{q^*}} \end{cases} \tag{1.14}$$

in the case that $(\chi_1-\chi_2)^2 > \max\{4\chi_1, 4\chi_2\}$.

Note that the assumption **(H1)** indicates that a_1, a_2 are large relative to χ_1, χ_2, which would prevent w becomes too small as time evolutes and is a biologically meaningful condition. We point out that, when $\chi_1, \chi_2 \to 0$, **(H1)** becomes

$$a_{\min} > 0,$$

which always holds. The assumption **(H2)** includes implicitly the condition $a_{\min} > \chi^{**}(\mu, \chi_1, \chi_2)$. The condition (1.13) indicates that u_0+v_0 is not small in the case that χ_1 and χ_2 are close in the sense that $(\chi_1-\chi_2)^2 \le \max\{4\chi_1, 4\chi_2\}$. The condition (1.14) indicates that u_0+v_0 is neither big nor small in the case that χ_1 and χ_2 are not close in the sense that $(\chi_1-\chi_2)^2 > \max\{4\chi_1, 4\chi_2\}$. Both (1.13) and (1.14) would also prevent w becomes too small as time evolutes and are biologically meaningful conditions. We point out that when $(\chi_1-\chi_2)^2 \le \max\{4\chi_1, 4\chi_2\}$, it is not required that u_0+v_0 is not big. We also point out that, when $0<\chi_1<1$ and $0<\chi_2<1$, we always have

$$(\chi_1-\chi_2)^2 \le \max\{4\chi_1, \chi_2\}.$$

Hence, when $0<\chi_1<1$ and $0<\chi_2<1$, **(H2)** becomes (1.13), and when $\chi_1, \chi_2 \to 0$, (1.13) becomes

$$\int_\Omega (u+v)^{-1}(\tau_0, u_0, v_0) < \infty$$

for some $\tau_0 > 0$, which always holds.

1.2 Main results and remarks

In this subsection, we state the main results of the paper and provide some remarks on the main results.

Observe that, for any given u_0, v_0 satisfying (1.8), by the third equation in (1.2),

$$\mu \int_\Omega w(t,x;u_0,v_0)dx = \int_\Omega (\nu u(t,x;u_0,v_0) + \lambda v(t,x;u_0,v_0))dx$$

and

$$\begin{aligned} \min\{\nu,\lambda\}\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))dx &\le \int_\Omega (\nu u(t,x;u_0,v_0)+\lambda v(t,x;u_0,v_0))dx \\ &\le \max\{\nu,\lambda\}\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))dx \end{aligned}$$

for any $t \in (0, T_{\max}(u_0, v_0))$. Hence $\liminf_{t \nearrow T_{\max}(u_0,v_0)} \int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx = 0$ implies that (1.7) holds. On the other hand, by (1.11),

$$\liminf_{t \nearrow T_{\max}(u_0,v_0)} \inf_{x \in \Omega} w(t,x;u_0,v_0) = 0 \text{ iff } \liminf_{t \nearrow T_{\max}(u_0,v_0)} \int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx = 0.$$

The first main theorem of the current paper is on the lower bounds of the combined mass $\int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx$ on any bounded subinterval of $(0, T_{\max}(u_0,v_0))$, which would provides the upper bounds of $\big(\int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx\big)^{-1}$ on bounded subintervals of $(0, T_{\max}(u_0,v_0))$.

Theorem 1.1 (Local lower bound of the combined mass)**.** *For any $T \in (0,\infty)$ and any u_0, v_0 satisfying* (1.8)*,*

$$\inf_{0 \le t < \min\{T, T_{\max}(u_0,v_0)\}} \int_\Omega \big(u(t,x;u_0,v_0) + v(t,x;u_0,v_0)\big)dx > 0. \tag{1.15}$$

Remark 1.1. (1) By Proposition 1.1, (1.11) and (1.15), if $T_{\max}(u_0,v_0) < \infty$, then we must have

$$\limsup_{t \nearrow T_{\max}(u_0,v_0)} \Big(\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^0(\bar\Omega)}\Big) = \infty.$$

(2) If

$$\inf_{0 \le t < \min\{T, T_{\max}(u_0,v_0)\}} \int_\Omega u(t,x;u_0,v_0)dx > 0$$

and

$$\inf_{0 \le t < \min\{T, T_{\max}(u_0,v_0)\}} \int_\Omega v(t,x;u_0,v_0)dx > 0,$$

then (1.15) holds. But the converse may not be true because competitive exclusion may occur in (1.2), which will be studied somewhere else.

(3) It will be proved that $T_{\max}(u_0,v_0) = \infty$ provided that **(H1)** and **(H2)** hold (see Theorem 1.4).

The second main theorem of the current paper is on the local L^p- and C^θ-boundedness of positive classical solutions of (1.2).

Theorem 1.2 (Local L^p- and C^θ-boundedness)**.**

(1) (Local L^p-boundedness) Assume that **(H1)** *holds. There are $p^* > \max\{2, N\}$, $\tilde p^* > 1$, and $M_1 > 0$ such that for any (u_0, v_0) satisfies* **(H2)***, there holds*

$$\begin{aligned}\int_\Omega (u+v)^p(t,x;u_0,v_0)dx &\le e^{-(t-\tau_0)} \int_\Omega (u+v)^p(\tau_0,x;u_0,v_0)dx \\ &\quad + M_1 e^{-(t-\tau_0)}(t-\tau_0)\Big(\int_\Omega (u+v)^{p^*}(\tau_0,x;u_0,v_0)\Big)^{\tilde p^*} + M_1, \ \forall\, \tau_0 < t < T_{\max}(u_0,v_0),\end{aligned}$$

where $p = p^*$ or $2p^*$.

(2) (Local C^θ-boundedness) For any $p > 2N$ and $0 < \theta < 1 - \frac{2N}{p}$, there are $M_2 > 0$, $\beta > 0$, and $\gamma > 0$ such that for any (u_0, v_0) satisfies (1.8) and $T \in (0, \infty)$, there holds

$$\begin{aligned}
&\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^\theta(\bar\Omega)} \\
&\le M_2\Big[(t-\tau)^{-\beta}e^{-\gamma(t-\tau)}\|u(\tau,\cdot;u_0,v_0) + v(\tau,\cdot;u_0,v_0)\|_{L^p} \\
&\quad + \frac{\sup\limits_{\tau\le t<\hat T}\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{L^p}^2}{\inf\limits_{\tau\le t<\hat T, x\in\Omega} w(t,x;u_0,v_0)} \\
&\quad + \sup_{\tau\le t<\hat T}\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{L^p} + 1\Big]
\end{aligned} \tag{1.16}$$

for any $0 < \tau < t < \hat T = \min\{T, T_{\max}(u_0, v_0)\}$.

Remark 1.2.

(1) By Theorem 1.2(2), if $\inf_{x\in\Omega} w(t,x;u_0,v_0)$ is bounded away from zero on $(0, T_{\max}(u_0,v_0))$ and $\int_\Omega (u+v)^p(t,x;u_0,v_0)dx$ is bounded on $(0, T_{\max}(u_0,v_0))$ for some $p > 2N$, then $\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^\theta(\bar\Omega)}$ is bounded on $[\tau, T_{\max}(u_0,v_0))$ for some $0 < \theta < 1$ and any $0 < \tau < T_{\max}(u_0,v_0)$.

(2) By (1.11), $w(t,x;u_0,v_0)$ is bounded below by $\int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx$. By Theorems 1.1 and 1.2(2), $\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^\theta(\bar\Omega)}$ is bounded on $[\tau, \hat T)$ for some $\theta \in (0,1)$, any $T \in (0,\infty)$, and any $0 < \tau < \hat T := \min\{T, T_{\max}(u_0,v_0)\}$, which plays an important role in the study of global existence of classical solutions of (1.2).

The third main theorem of the current paper is on global existence of classical solutions of (1.2).

Theorem 1.3 (Global existence)**.** *Assume that* **(H1)** *holds. For any u_0, v_0 satisfying* **(H2)***, the solution $(u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$ exists globally, that is,*

$$T_{\max}(u_0,v_0) = \infty.$$

Remark 1.3. The proof of Theorem 1.3 relies on Theorems 1.1 and 1.2. The proof of Theorem 1.2(1) replies on estimates on the lower bounds of $w(t,x;u_0,v_0)$, which relies on estimates on the upper bounds of $\int_\Omega (u+v)^{-q}$ for some $q > 0$.

The fourth main theorem of the current paper is on the boundedness of $\int_\Omega (u+v)^{-q}$ for some $q > 0$. For given u_0, v_0 satisfying (1.8) and $\tau \in [0, T_{\max}(u_0,v_0))$, let

$$m^*(\tau,u_0,v_0) := \max\Big\{\int_\Omega (u(\tau,x;u_0,v_0) + v(\tau,x;u_0,v_0))dx, \frac{a_{\max}|\Omega|}{\min\{b_{\min}, c_{\min}\}}\Big\}.$$

Theorem 1.4. *(1) (Boundedness of $\int_\Omega (u+v)^{-q}$) Assume that $a_{\min} > \chi^*(\mu,\chi_1,\chi_2)$. Then there is $q>0$ such that for any u_0, v_0 satisfying (1.8) and for every $0<\tau<T_{\max}(u_0,v_0)$,*

$$\begin{aligned}&\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))^{-q}dx\\ &\le e^{-\frac{\epsilon_0 q}{2}(t-\tau)}\int_\Omega (u(\tau,x;u_0,v_0)+v(\tau,x;u_0,v_0))^{-q}dx + 2C_{\epsilon_0,\tau,u_0,v_0}\epsilon_0^{-1}\end{aligned}$$

for all $\tau<t<T_{\max}(u_0,v_0)$, where $\epsilon_0 = a_{\min}-\chi^*(\mu,\chi_1,\chi_2)$ and

$$C_{\epsilon_0,\tau,u_0,v_0} = \begin{cases} (b_{\max}+c_{\max})|\Omega| & \text{if } q=1,\\ \frac{(b_{\max}+c_{\max})^q}{q}\left(\frac{4}{\epsilon_0}\right)^q\left(\frac{q-1}{q}\right)^{q-1}|\Omega| & \text{if } q>1,\\ (b_{\max}+c_{\max})|\Omega|^q(m^*(\tau,u_0,v_0))^{1-q} & \text{if } q<1.\end{cases}$$

(2) (Boundedness of $\int_\Omega (u+v)^{-q}$) Assume that $a_{\min} > \chi^{**}(\mu,\chi_1,\chi_2)$. Then for any u_0, v_0 satisfying (1.8) and for every $0<\tau<T_{\max}(u_0,v_0)$, the following holds,

$$\begin{aligned}&\int_\Omega (u(x,t;u_0,v_0)+v(x,t;u_0,v_0))^{-q}dx\\ &\le e^{-(a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2))q(t-\tau)}\int_\Omega (u(x,\tau;u_0,v_0)+v(x,\tau;u_0,v_0))^{-q}dx + \tilde{C}_{\tau,u_0,v_0}\end{aligned}$$

for all $\tau<t<T_{\max}(u_0,v_0)$, where $q=q_{\chi_1,\chi_2}$ and

$$\tilde{C}_{\tau,u_0,v_0} = \begin{cases} \frac{(b_{\max}+c_{\max})|\Omega|}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} & \text{if } q_{\chi_1,\chi_2}=1,\\ \frac{(b_{\max}+c_{\max})|\Omega|^{q_{\chi_1,\chi_2}}\left(m^*(\tau,u_0,v_0)\right)^{1-q_{\chi_1,\chi_2}}}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} & \text{if } q_{\chi_1,\chi_2}<1.\end{cases}$$

Remark 1.4. *(1) The condition $a_{\min} > \chi^{**}(\mu,\chi_1,\chi_2)$ in Theorem 1.4(2) implies the condition $a_{\min} > \chi^*(\mu,\chi_1,\chi_2)$ in Theorem 1.4(1), but the q in Theorem 1.4(2) is explicit.*

(2) Assume that $a_{\min} > \chi^*(\mu,\chi_1,\chi_2)$ or $\chi^{**}(\mu,\chi_1,\chi_2)$. By Hölder's inequality, for any $q>0$,

$$\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0)) \ge \frac{|\Omega|^{\frac{q+1}{q}}}{\left(\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))^{-q}\right)^{\frac{1}{q}}} \tag{1.17}$$

for all $0<t<T_{\max}(u_0,v_0)$. By (1.17), and Theorem 1.4(1) or Theorem 1.4(2),

$$\inf_{0\le t<T_{\max}(u_0,v_0)}\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))dx > 0,$$

which improves (1.15).

(3) Assume that **(H1)** *and* **(H2)** *hold. By Theorem 1.4, $\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))^{-q}dx$ stays bounded for some $q>0$. This plays important roles in the study of L^p-boundedness and global existence of classical solutions of (1.2). We will therefore prove Theorem 1.4 before proving Theorems 1.2 and 1.3.*

The last main theorem of the current paper is about initial independent ultimate upper and lower bounds of classical solutions of (1.2), some of which follows from Theorems 1.2-1.4 quite directly. We state them in a theorem since they have important impacts on the understanding of the asymptotic behavior of globally defined positive solutions of (1.2).

Theorem 1.5. *Assume that* **(H1)** *holds. Then the following hold.*

(1) (Uniform boundedness) Let $q > 0$ be as in Theorem 1.4(2) and $p = p^* > \max\{2, N\}$, where p^* is as in Theorem 1.2(1). Let $0 < \theta < 1 - \frac{N}{p}$. There are $M_1^* > 0$, $M_2^* > 0$ and $M_3^* > 0$ such that for any u_0, v_0 satisfying **(H2)**,

$$\limsup_{t\to\infty} \int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))^{-q} dx \le M_1^*, \tag{1.18}$$

$$\limsup_{t\to\infty} \int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))^{p} dx \le M_2^*, \tag{1.19}$$

and

$$\limsup_{t\to\infty} \|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^\theta(\bar\Omega)} \le M_3^*. \tag{1.20}$$

(2) (Combined pointwise persistence) There is $M_0^* > 0$ such that for any u_0, v_0 satisfying **(H2)**,

$$\liminf_{t\to\infty} \inf_{x\in\Omega} (u(t,x;u_0,v_0) + v(t,x;u_0,v_0)) \ge M_0^*. \tag{1.21}$$

Remark 1.5. *(1) Let p, q be as in Theorem 1.5. Let*

$$\mathcal{E} = \Big\{u, v \in C^0(\bar\Omega) \,|\, u \ge 0,\ v \ge 0,\ \int_\Omega (u(x) + v(x))dx > 0,$$
$$\int_\Omega (u(x) + v(x))^{-q} \le M_1^*,\ \int_\Omega (u(x) + v(x))^p dx \le M_2^*\Big\}.$$

If **(H1)** *holds, Theorem 1.5 shows that $\mathcal{E}$ eventually attracts all globally defined positive solutions of* (1.2) *with initial conditions satisfying* **(H2)**.

(2) (1.21) *implies that, if* **(H1)** *holds, then there is $m_0^* > 0$ such that for any u_0, v_0 satisfying* **(H2)**,

$$\liminf_{t\to\infty} \int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx \ge m_0^*, \tag{1.22}$$

which is refereed to as combined mass persistence. It remains open whether $T_{\max}(u_0, v_0) = \infty$ and (1.22) *holds without the assumption* **(H2)**.

(3) (1.20) *implies that, if* **(H1)** *holds, then for any u_0, v_0 satisfying* **(H2)**,

$$\limsup_{t\to\infty} \|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{L^\infty(\Omega)} \le M_3^*. \tag{1.23}$$

It also remains open whether $T_{\max}(u_0, v_0) = \infty$ and (1.23) *holds without the assumption* **(H2)**.

(4) We say both species persistent in mass *if for any* u_0, v_0 *satisfying* (1.8),

$$\liminf_{t\to\infty}\int_\Omega u(t,x;u_0,v_0)dx>0 \text{ and } \liminf_{t\to\infty}\int_\Omega v(t,x;u_0,v_0)dx>0,$$

and say competitive exclusion in mass *occurs if for any* u_0, v_0 *satisfying* (1.8),

$$\limsup_{t\to\infty}\int_\Omega u(t,x;u_0,v_0)dx=0 \text{ and } \liminf_{t\to\infty}\int_\Omega v(t,x;u_0,v_0)dx>0$$

or

$$\liminf_{t\to\infty}\int_\Omega u(t,x;u_0,v_0)dx>0 \text{ and } \limsup_{t\to\infty}\int_\Omega v(t,x;u_0,v_0)dx=0.$$

We say both species persistent pointwise *if for any* u_0, v_0 *satisfying* (1.8),

$$\liminf_{t\to\infty}\inf_{x\in\Omega} u(t,x;u_0,v_0)>0 \text{ and } \liminf_{t\to\infty}\inf_{x\in\Omega} v(t,x;u_0,v_0)>0,$$

and say pointwise competitive exclusion *occurs if for any* u_0, v_0 *satisfying* (1.8),

$$\limsup_{t\to\infty}\sup_{x\in\Omega} u(t,x;u_0,v_0)=0 \text{ and } \liminf_{t\to\infty}\inf_{x\in\Omega} v(t,x;u_0,v_0)>0$$

or

$$\liminf_{t\to\infty}\inf_{x\in\Omega} u(t,x;u_0,v_0)>0 \text{ and } \limsup_{t\to\infty}\sup_{x\in\Omega} v(t,x;u_0,v_0)=0.$$

We will study the persistence of both species and competitive exclusion somewhere else.

We conclude the introduction with some remarks on the following full parabolic counterpart of (1.2),

$$\begin{cases} u_t=\Delta u-\chi_1\nabla\cdot(\frac{u}{w}\nabla w)+u(a_1-b_1u-c_1v), & x\in\Omega\\ v_t=\Delta v-\chi_2\nabla\cdot(\frac{v}{w}\nabla w)+v(a_2-b_2v-c_2u), & x\in\Omega\\ w_t=\Delta w-\mu w+\nu u+\lambda v, & x\in\Omega\\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=\frac{\partial w}{\partial n}=0, & x\in\partial\Omega. \end{cases} \tag{1.24}$$

For given u_0, v_0, w_0 satisfying

$$u_0, v_0(\cdot)\in C^0(\bar\Omega),\ w_0\in W^{1,\infty}(\Omega),\ u_0\geq 0,\ v_0\geq 0,\ \int_\Omega(u_0(x)+v_0(x))dx>0,\ w_0(x)>0, \tag{1.25}$$

by standard contraction arguments (see [4, 13, 19]), there exists $T_{\max}(u_0,v_0,w_0)\in(0,\infty]$ such that (1.24) possesses a unique positive classical solution, denoted by $(u(t,x;u_0,v_0,w_0), v(t,x;u_0,v_0,w_0), w(t,x;u_0,v_0,w_0))$, on $(0,T_{\max}(u_0,v_0,w_0))$ with initial condition $(u(0,x;u_0,v_0,w_0)\ , v(0,x;u_0,v_0,w_0), w(0,x;u_0,v_0,w_0))=(u_0(x),v_0(x),w_0(x))$. Moreover if $T_{\max}(u_0,v_0,w_0)<\infty$, then either

$$\limsup_{t\nearrow T_{\max}(u_0,v_0,w_0)}\Big(\|u(t,\cdot;u_0,v_0,w_0)+v(t,\cdot;u_0,v_0,w_0)\|_{C^0(\bar\Omega)}+\|w(t,\cdot;u_0,v_0,w_0)\|_{W^{1,\infty}(\Omega)}\Big)=\infty,$$

or

$$\liminf_{t\nearrow T_{\max}(u_0,v_0,w_0)}\inf_{x\in\Omega} w(t,x;u_0,v_0,w_0)=0.$$

Observe that, for given u_0, v_0, w_0 satisfying (1.25), if $v_0(x) \equiv 0$, then $v(t,x;u_0,v_0,w_0) \equiv 0$ for $t \in (0, T_{\max}(u_0,v_0,w_0))$ and $(u(t,x;u_0,w_0), w(t,x;u_0,w_0)) := (u(t,x;u_0,v_0,w_0), w(t,x;u_0,v_0,w_0))$ is the classical solution of the following full parabolic counterpart of (1.3)

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w} \nabla w) + u(a_1 - b_1 u), & x \in \Omega \\ w_t = \Delta w - \mu w + \nu u, & x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.26}$$

with initial condition $(u(0,x;u_0,w_0), w(0,x;u_0,w_0)) = (u_0(x), w_0(x))$. Similarly, if $u_0 \equiv 0$, then $u(t,x;u_0,v_0,w_0) \equiv 0$ on $(0, T_{\max}(u_0,v_0,w_0))$ and $(v(t,x;v_0,w_0), w(t,x;v_0,w_0)) := (v(t,x;u_0,v_0,w_0),$ $w(t,x;u_0,v_0,w_0))$ is the classical solution of

$$\begin{cases} v_t = \Delta v - \chi_2 \nabla \cdot (\frac{v}{w} \nabla w) + v(a_2 - b_2 v), & x \in \Omega \\ w_t = \Delta w - \mu w + \lambda v, & x \in \Omega \\ \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.27}$$

with initial condition $(v(0,x;v_0,w_0), w(0,x;v_0,w_0)) = (v_0(x), w_0(x))$.

Systems (1.26) and (1.27) are essentially the same. There are several works on the global existence and asymptotic behavior of classical or weak solutions of (1.26). For example, in the case $N = 2$, the authors of [1] proved the global existence of solutions of (1.26) with initial functions $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, and $w_0 \in H^{1+\theta_0}(\Omega)$ for some $\theta_0 \in (0, 1/2)$, $\inf_{x\in\Omega} w_0(x) > 0$ (see [1, Theorem 2.1]); the authors of [37] proved the global existence and boundedness of classical solutions of (1.26) with initial functions $u_0 \in C^0(\bar{\Omega})$, $u_0(x) \geq 0$, $u_0 \not\equiv 0$, and $w_0 \in W^{1,q}(\Omega)$ for some $q > 2$, $w_0(x) > 0$ provided that χ_1 is relatively small with respect to a_1 (see [37, Theorem 1]); the authors of [39] showed the global stability of the positive constant solution of (1.26) provided that χ_1 is relatively small with respect to a_1 (see [39, Theorem 1.1]). For general $N \geq 1$, global existence of weak solutions of (1.26) is studied in [10, 38]. The authors of [11, 27, 35] studied the global existence and boundedness of classical solutions of (1.26) with $a_1 = b_1 = 0$.

Up to our knowledge, it remains open whether (1.26) has a global classical solution for any given initial functions $u_0 \in C^0(\bar{\Omega})$, $u_0(x) \geq 0$, $u_0 \not\equiv 0$, and $w_0 \in W^{1,\infty}(\Omega)$, $w_0(x) > 0$ in any space dimensional setting. There is little study on the global existence and boundedness of classical solutions of (1.24) for any given initial functions u_0, v_0, w_0 satisfying (1.25). We remark that finite upper bounds of $\int_\Omega (u(t,x) + v(t,x))^p dx$ for some $p \gg 1$ and positive lower bounds of $w(t,x)$, or equivalently, finite upper bounds of $\int_\Omega (u(t,x) + v(t,x))^{-q}$ for some $q > 0$, are among the key ingredients in the proofs of global existence and boundedness of the classical solution $u(t,x), v(t,x), w(t,x))$ of (1.2) with initial functions u_0, v_0 satisfying (1.8). We expect that such bounds for the solutions of (1.24) if they can be obtained will also ensure the global existence and boundedness of classical solutions of (1.24) with initial conditions u_0, v_0, w_0 satisfying (1.25). However, the techniques in the current paper to obtain the upper bounds of $\int_\Omega (u+v)^p$ for $p \gg 1$ and $\int_\Omega (u+v)^{-q}$ for some $q > 0$ for the solutions of (1.2) rely on the fact that the equation for w in (1.2) is elliptic (see Remarks 4.1 and 3.1). New techniques/methods need to be developed to get such bounds for the solutions of (1.24). We wish to carry out some study on the global existence and various properties of classical solutions of (1.24) in the near future.

The rest of the paper is organized as follows. In section 2, we present some preliminary lemmas. In section 3, we prove Theorems 1.1 and and 1.4. Section 4 is devoted to the proof of Theorem 1.2. Theorems 1.3 and 1.5 are proved in section 5. We prove one important technical proposition in the Appendix.

2 Preliminary lemmas

In this section, we present some lemmas to be used in later sections.

First, let $\chi^*(\mu,\chi_1,\chi_2)$ be defined as in (1.10). We present the following lemma on the continuity and upper bounds of $\chi^*(\mu,\chi_1,\chi_2)$.

Lemma 2.1. *(1) $\chi^*(\mu,\chi_1,\chi_2)$ is upper semicontinuous in $\mu>0$, $\chi_1>0$ and $\chi_2>0$, that is,*

$$\limsup_{(\mu,\chi_1,\chi_2)\to(\mu^0,\chi_1^0,\chi_2^0)} \chi^*(\mu,\chi_1,\chi_2) \le \chi^*(\mu^0,\chi_1^0,\chi_2^0) \quad \forall\, \mu^0>0,\ \chi_1^0>0,\ \chi_2^0>0.$$

(2) For any $\mu>0$, $\chi_1>0$ and $\chi_2>0$,

$$\chi^*(\mu,\chi_1,\chi_2) \le \min\Big\{\mu\chi_2+\frac{\mu(\chi_1-\chi_2)^2}{4}, \mu\chi_1+\frac{\mu(\chi_2-\chi_1)^2}{4}\Big\}, \tag{2.1}$$

and when $\chi_1=\chi_2=\chi$,

$$\chi^*(\mu,\chi,\chi) \le \begin{cases} \frac{\mu\chi^2}{4} & \text{if } 0<\chi<2 \\ \mu(\chi-1) & \text{if } \chi\ge 2. \end{cases} \tag{2.2}$$

Remark 2.1. *We point out that the boundedness of $\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))^{-q}dx$ for some $q>0$ plays a crucial role in the study of the boundedness of $u(t,x;u_0,v_0)+v(t,x;u_0,v_0)$. The assumption that $a_{\min}>\chi^*(\mu,\chi_1,\chi_2)$ indicates that the chemotaxis sensitivities χ_1,χ_2 and the degradation rate μ of the chemical substance are small relatively with respect to $a_{\min}$ and ensures the boundedness of $\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))^{-q}dx$ for some $q>0$ (see Lemma 3.2 and the arguments of Theorem 1.2). In the following, we provide some discussions on some specific upper bounds of $\chi^*(\mu,\chi_1,\chi_2)$ for the reader to get some comprehensive feeling about the impact of χ_1,χ_2 and μ on the boundedness of $\int_\Omega (u+v)^{-q}$. First, note that*

$$\begin{aligned}\frac{\partial}{\partial t}(u+v) =& \Delta(u+v)-\nabla\cdot\left(\frac{\chi_1 u+\chi_2 v}{w}\nabla w\right)\\ &+(a_1u+a_2v)-(b_1u^2+b_2v^2)-(c_1+c_2)uv.\end{aligned} \tag{2.3}$$

Next, for any $q>0$, multiplying (2.3) by $(u+v)^{-q-1}$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} =& -(q+1)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2\\ &+(q+1)\int_\Omega (u+v)^{-q-2}\frac{\chi_1u+\chi_2v}{w}\nabla(u+v)\cdot\nabla w\\ &-\int_\Omega (u+v)^{-q-1}(a_1u+a_2v)+\int_\Omega (u+v)^{-q-1}(b_1u^2+b_2v^2)\\ &+\int_\Omega (u+v)^{-q-1}(c_1+c_2)uv\end{aligned}$$

$$
\begin{aligned}
&\le -(q+1)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2\\
&\quad + (q+1)\chi_2\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w\\
&\quad + (q+1)(\chi_1-\chi_2)\int_\Omega (u+v)^{-q-2}\frac{u}{w}\nabla(u+v)\cdot\nabla w\\
&\quad - a_{\min}\int_\Omega (u+v)^{-q} + b_{\max}\int_\Omega (u+v)^{-q+1} + c_{\max}\int_\Omega (u+v)^{-q+1}\\
&\le -(q+1)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2\\
&\quad + (q+1)\chi_2\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w\\
&\quad + \frac{(\chi_1-\chi_2)^2 q(q+1)}{4B}\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2 + \frac{B(q+1)}{q}\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2}\\
&\quad - a_{\min}\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1}, \quad \forall\, B>0. \qquad (2.4)
\end{aligned}
$$

Now, by the third equation in (1.2)*, we have*

$$
\frac{B}{q}\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2} \le \frac{B\mu}{q}\int_\Omega (u+v)^{-q} - B\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w
$$

(see the arguments of Lemma 3.2 for detail). Hence

$$
\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le &-(q+1)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2\\
&+ (q+1)\chi_2\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w\\
&+ \frac{(\chi_1-\chi_2)^2 q(q+1)}{4B}\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2\\
&+ \frac{B(q+1)\mu}{q}\int_\Omega (u+v)^{-q} - B(q+1)\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w\\
&- a_{\min}\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1}
\end{aligned}
$$

for any $q>0, B>0$. Let $B=\chi_2$ and $q=\frac{4\chi_2}{(\chi_1-\chi_2)^2}$ in the case $\chi_1\neq\chi_2$. We get

$$
\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le -\Big(a_{\min} - \mu\big(\chi_2 + \frac{(\chi_1-\chi_2)^2}{4}\big)\Big)\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1}.
$$

Therefore, $a_{\min} > \mu\big(\chi_2 + \frac{(\chi_1-\chi_2)^2}{4}\big)$ ensures the boundedness of $\int_\Omega (u+v)^{-q}$ (see the arguments of Theorem 1.2) and $\mu\big(\chi_2+\frac{(\chi_1-\chi_2)^2}{4}\big)$ is an upper bound of $\chi^*(\mu,\chi_1,\chi_2)$. Similarly, $\mu\big(\chi_1+\frac{(\chi_1-\chi_2)^2}{4}\big)$ is an upper bound of $\chi^*(\mu,\chi_1,\chi_2)$.

Proof of Lemma 2.1. (1) We prove that $\chi^*(\mu,\chi_1,\chi_2)$ is upper semicontinuous in $\mu>0$, $\chi_1>0$ and $\chi_2>0$.

Fix $(\mu^0,\chi_1^0,\chi_2^0)$ with $\mu^0,\chi_1^0,\chi_2^0>0$. Without loss of generality, we may assume that $\chi^*(\mu^0,\chi_1^0,\chi_2^0)=\chi_1^*(\mu^0,\chi_1^0,\chi_2^0)$. Then, for any $\epsilon>0$, there are $\tilde\beta>0$ and $\tilde B>0$ satisfying $\tilde\beta\neq\chi_2^0-\tilde B$ such that

$$\chi_1^*(\mu^0,\chi_1^0,\chi_2^0)\geq f(\mu^0,\chi_1^0,\chi_2^0,\tilde\beta,\tilde B)-\epsilon,$$

and there is $\delta>0$ such that

$$\tilde\beta\neq\chi_2-\tilde B$$

and

$$f(\mu,\chi_1,\chi_2,\tilde\beta,\tilde B)\leq f(\mu^0,\chi_1^0,\chi_2^0,\tilde\beta,\tilde B)+\epsilon$$

for all $\mu>0,\chi_1>0,\chi_2>0$ satisfying $|\mu-\mu^0|<\delta$, $|\chi_1-\chi_1^0|<\delta$, and $|\chi_2-\chi_2^0|<\delta$. This implies that

$$f(\mu,\chi_1,\chi_2,\tilde\beta,\tilde B)\leq f(\mu^0,\chi_1^0,\chi_2^0,\tilde\beta,\tilde B)+\epsilon\leq\chi_1^*(\mu^0,\chi_1^0,\chi_2^0)+2\epsilon,$$

and then

$$\chi_1^*(\mu,\chi_1,\chi_2)\leq\chi_1^*(\mu^0,\chi_1^0,\chi_2^0)+2\epsilon.$$

Therefore,

$$\limsup_{(\mu,\chi_1,\chi_2)\to(\mu^0,\chi_1^0,\chi_2^0)}\chi^*(\mu,\chi_1,\chi_2)\leq\chi^*(\mu^0,\chi_1^0,\chi_2^0)\quad\forall\,\mu^0,\chi_1^0,\chi_2^0>0.$$

(2) We first prove (2.2) when $\chi_1=\chi_2$. In this case, we have

$$f(\mu,\chi_1,\chi_2,\beta,B)=\mu(B+\beta)\big(1+\frac{(\chi_2-B-\beta)^2}{4\beta}\big).$$

This implies that

$$\begin{aligned}\chi_1^*(\mu,\chi_1,\chi_2)&=\inf\Big\{\mu(B+\beta)\big(1+\frac{(\chi_2-B-\beta)^2}{4\beta}\big)\,|\,B>0,\,\beta>0,\beta\neq\chi_2-B\Big\}\\&\leq\inf\Big\{\mu\beta\Big(1+\frac{(\chi_2-\beta)^2}{4\beta}\Big)\,|\,0<\beta<\chi_2\Big\}\\&=\begin{cases}\frac{\mu\chi_2^2}{4} & \text{if } 0<\chi_2<2\\ \mu(\chi_2-1) & \text{if } \chi_2>2.\end{cases}\end{aligned}$$

Similarly,

$$\chi_2^*(\mu,\chi_1,\chi_2)\leq\begin{cases}\frac{\mu\chi_1^2}{4} & \text{if } 0<\chi_1<2\\ \mu(\chi_1-1) & \text{if } \chi_1>2.\end{cases}$$

Hence (2.2) holds.

Next we prove (2.1) for any $\mu>0$, $\chi_1>0$ and $\chi_2>0$. Recall that

$$f(\mu,\chi_1,\chi_2,\beta,B)=\mu(B+\beta)\Big(1+\frac{B(\chi_2-B-\beta)^2+(\chi_1-\chi_2)^2\beta}{4B\beta}\Big).$$

Then

$$f(\mu,\chi_1,\chi_2,\beta,\chi_2)=\mu(\chi_2+\beta)\Big(1+\frac{\beta\chi_2+(x_1-x_2)^2}{4\chi_2}\Big)$$

and

$$\chi_1^*(\mu,\chi_1,\chi_2)\le\inf\Big\{f(\mu,\chi_1,\chi_2,\beta,\chi_2)\,|\,\beta>0\Big\}=\mu\chi_2+\frac{\mu(\chi_1-\chi_2)^2}{4}.$$

Similarly,

$$\chi_2^*(\mu,\chi_1,\chi_2)\le\inf\Big\{f(\mu,\chi_2,\chi_1,\beta,\chi_1)\,|\,\beta>0\Big\}=\mu\chi_1+\frac{\mu(\chi_2-\chi_1)^2}{4}.$$

Hence (2.1) holds. □

Next, we present some properties of the semigroup generated by $-\Delta+\mu I$ complemented with Neumann boundary condition on $L^p(\Omega)$. For given $1<p<\infty$, let $X_p=L^p(\Omega)$ and

$$A_p=-\Delta+\mu I:D(A_p)\subset L^p(\Omega)\to L^p(\Omega) \tag{2.5}$$

with

$$D(A_p)=\Big\{u\in W^{2,p}(\Omega)\,|\,\frac{\partial u}{\partial n}=0\quad\text{on }\partial\Omega\Big\}.$$

Then $-A_p$ generates an analytic semigroup on $L^p(\Omega)$. We denote it by e^{-tA_p}. Note that $\mathrm{Re}\sigma(A_p)>0$. Let A_p^β be the fractional power operator of A_p (see [15, Definition 1.4.1]). Let $X_p^\beta=\mathcal{D}(A_p^\beta)$ with graph norm $\|u\|_{X_p^\beta}=\|A_p^\beta u\|_{L^p(\Omega)}$ for $\beta\ge0$ and $u\in X_p^\beta$ (see [15, Definition 1.4.7]).

Lemma 2.2. *(i) For each $p\in(1,\infty)$ and $\beta\ge0$, there is $C_{p,\beta}>0$ such that for some $\gamma>0$,*

$$\|A_p^\beta e^{-A_pt}\|_{L^p(\Omega)}\le C_{p,\beta}t^{-\beta}e^{-\gamma t}\quad for\ t>0.$$

(ii) If $m\in\{0,1\}$ and $q\in[p,\infty]$ are such that $m-\frac{N}{q}<2\beta-\frac{N}{p}$, then $X_p^\beta\hookrightarrow W^{m,q}(\Omega)$.

(iii) If $2\beta-\frac{N}{p}>\theta\ge0$, then $X_p^\beta\hookrightarrow C^\theta(\Omega)$.

Proof. (i) It follows from [15, Theorem 1.4.3].

(ii) It follows from [15, Theorem 1.6.1].

(iii) It also follows from [15, Theorem 1.6.1]. □

Lemma 2.3. *Let $\beta\ge0$, $p\in(1,\infty)$. Then for any $\epsilon>0$ there exists $C_{p,\beta,\epsilon}>0$ such that for any $w\in C_0^\infty(\Omega)$ we have*

$$\|A_p^\beta e^{-tA_p}\nabla\cdot w\|_{L^p(\Omega)}\le C_{p,\beta,\epsilon}t^{-\beta-\frac12-\epsilon}e^{-\gamma t}\|w\|_{L^p(\Omega)}\quad for\ all\ t>0\ and\ some\ \gamma>0. \tag{2.6}$$

Consequently, for all $t>0$ the operator $A_p^\beta e^{-tA_p}\nabla\cdot$ admits a unique extension to all of $L^p(\Omega)$ which is again denoted by $A_p^\beta e^{-tA_p}\nabla\cdot$ and satisfies (2.6) for all $\mathbb{R}^N$-valued $w\in L^p(\Omega)$.

Proof. It follows from [19, Lemma 2.1]. □

Remark 2.2. *We remark that the fractional powers of A_p are used to prove the boundedness of the $C^\theta(\bar{\Omega})$-norm of the solutions of* (1.2) *for some $\theta > 0$. Such techniques are also used in [19] (see [19, Theorem 4.1]). It should be pointed out that we can prove the boundedness of the $L^p(\Omega)$-norm of the solutions of* (1.2) *for any $p \geq 1$ via the standard estimates for the Neumann heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ in Ω such as*

$$\|e^{t\Delta}u\|_{L^q(\Omega)} \leq C\Big(1 + t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big)\|u\|_{L^p(\Omega)},$$

$$\|\nabla e^{t\Delta}u\|_{L^q(\Omega)} \leq C\Big(1 + t^{-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big)e^{-\lambda_1 t}\|u\|_{L^p(\Omega)},$$

for some $C > 0$, all $t > 0$, $u \in L^p(\Omega)$, and $q > p$ (see [35, Lemma 1.3] for some other estimates for the Neumann heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$). Due to the presence of the chemotaxis in (1.2)*, some estimates for the fractional powers of A_p or some other arguments such as bootstrap arguments are needed to get the $C^\theta(\bar{\Omega})$-boundedness of the solutions of* (1.2) *for some $\theta > 0$.*

We then present some lemmas on the lower and upper bounds of the solutions of

$$\begin{cases} -\Delta w + \mu w = \nu u + \lambda v, & x \in \Omega \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

where μ, ν, and λ are positive constants. For given $u, v \in L^p(\Omega)$, let $w(\cdot; u, v)$ be the solution of (2.7).

Lemma 2.4. *Let $u, v \in C^0(\bar{\Omega})$ be nonnegative function such that $\int_\Omega u > 0$ and $\int_\Omega v > 0$. Then*

$$w(x; u, v) \geq \delta_0 \int_\Omega (u + v) > 0 \quad \text{in} \quad \Omega,$$

where δ_0 is some positive constant independent of u, v.

Proof. It follows from the arguments of [14, Lemma 2.1] and the Gaussian lower bound for the heat kernel of the laplacian with Neumann boundary condition on smooth domain (see [9, Theorem 4]). □

Lemma 2.5. *For any $p \geq 1$, there exists $C_p > 0$ such that*

$$\max\Big\{\|w(\cdot; u, v)\|_{L^p(\Omega)}, \|\nabla w(\cdot; u, v)\|_{L^p(\Omega)}\Big\} \leq C_p\|u(\cdot) + v(\cdot)\|_{L^p(\Omega)} \quad \forall u, v \in L^p(\Omega).$$

Proof. It follows from L^p-estimates for elliptic equations (see [2, Theorem 12.1]). □

Lemma 2.6. *For any nonnegative $u, v \in C(\bar{\Omega})$,*

$$\int_\Omega \frac{|\nabla w(\cdot; u, v)|^2}{w(\cdot; u, v)^2} \leq \mu|\Omega|.$$

Proof. Multiplying (2.7) by $\frac{1}{w}$ and integrating it over Ω yields that

$$0 = \int_\Omega \frac{1}{w} \cdot \Big(\Delta w - \mu w + \nu u + \lambda v\Big) = \int_\Omega \frac{|\nabla w|^2}{w^2} - \mu|\Omega| + \nu \int_\Omega \frac{u}{w} + \lambda \int_\Omega \frac{v}{w}$$

for all $t \in (0, T_{\max})$. The lemma thus follows. □

Throughout the rest of this section, we assume that $u_0(x)$ and $v_0(x)$ satisfies (1.8) and $(u(t,x), v(t,x), w(t,x)) := (u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0))$ is the unique classical solution of (1.2) on the maximal interval $(0, T_{\max}) := (0, T_{\max}(u_0,v_0))$ with the initial condition $(u(0,x), v(0,x)) = (u_0(x), v_0(x))$. Note that

$$u(t,x), v(t,x), w(t,x) > 0 \quad \forall\, x \in \Omega,\ t \in (0, T_{\max}).$$

We may drop (t,x) in $u(t,x)$, $v(t,x)$, and $w(t,x)$ if no confusion occurs.

We now present some upper bound for $\int_\Omega u(t,x;u_0,v_0)dx$ and $\int_\Omega v(t,x;u_0,v_0)dx$.

Lemma 2.7. *For any $\tau \in [0, T_{\max})$, the followings hold.*

$$\int_\Omega u(t,x;u_0,v_0)dx \le m_1^*(\tau,u_0,v_0) := \max\Big\{\int_\Omega u(\tau,x;u_0,v_0)dx, \frac{a_1|\Omega|}{b_1}\Big\}, \tag{2.8}$$

and

$$\int_\Omega v(t,x;u_0,v_0)dx \le m_2^*(\tau,u_0,v_0) := \max\Big\{\int_\Omega v(\tau,x;u_0,v_0)dx, \frac{a_2|\Omega|}{b_2}\Big\}, \tag{2.9}$$

as well as

$$\begin{aligned}&\int_\Omega \big(u(t,x;u_0,v_0) + v(t,x;u_0,v_0)\big)dx\\ &\le m^*(\tau,u_0,v_0) := \max\Big\{\int_\Omega (u(\tau,x;u_0,v_0) + v(\tau,x;u_0,v_0))dx, \frac{a_{\max}|\Omega|}{\min\{b_{\min}, c_{\min}\}}\Big\}\end{aligned} \tag{2.10}$$

for any $t \in [\tau, T_{\max})$, where $|\Omega|$ is the Lebesgue measure of Ω. Moreover, if $T_{\max}(u_0,v_0) = \infty$, then

$$\limsup_{t\to\infty}\int_\Omega u(t,x;u_0,v_0)dx \le \frac{a_1|\Omega|}{b_1}, \quad \limsup_{t\to\infty}\int_\Omega v(t,x;u_0,v_0)dx \le \frac{a_2|\Omega|}{b_2}, \tag{2.11}$$

and

$$\limsup_{t\to\infty}\int_\Omega (u(t,x;u_0,v_0) + v(t,x;u_0,v_0))dx \le \frac{a_{\max}|\Omega|}{\min\{b_{\min}, c_{\min}\}}. \tag{2.12}$$

Proof. By integrating the first equation in (1.2) with respect to x, we get that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u &= \int_\Omega \Delta u - \chi\int_\Omega \nabla\cdot\Big(\frac{u}{w}\nabla w\Big) + a_1\int_\Omega u - b_1\int_\Omega u^2 - c_1\int_\Omega vu\\ &\le a_1\int_\Omega u - \frac{b_1}{|\Omega|}\Big(\int_\Omega u\Big)^2 \quad \forall\, t \in (0, T_{\max}).\end{aligned}$$

This together with comparison principle for scalar ODEs implies (2.8) and the first inequality in (2.11) when $T_{\max} = \infty$.

Similarly, we can prove (2.9) and the second inequality in (2.11) when $T_{\max} = \infty$.

In the following, we prove (2.10). By (2.3),

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+v) &= \int_\Omega (a_1u + a_2v) - \int_\Omega (b_1u^2 + b_2v^2) - \int_\Omega (c_1+c_2)uv\\ &\le a_{\max}\int_\Omega (u+v) - b_{\min}\int_\Omega (u^2+v^2) - 2c_{\min}\int_\Omega uv\\ &= a_{\min}\int_\Omega (u+v) - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^2\\ &\le a_{\max}\int_\Omega (u+v) - \frac{\min\{b_{\min}, c_{\min}\}}{|\Omega|}\Big(\int_\Omega (u+v)\Big)^2,\end{aligned}$$

for all $t \in (0, T_{\max})$. This together with comparison principle for scalar ODEs implies (2.10) and (2.12) when $T_{\max}(u_0, v_0) = \infty$. $\square$

Lemma 2.8. *For any $\varepsilon > 0$ and $p > 0$, there is $C(\varepsilon, p) > 0$ such that*

$$\int_\Omega w^{p+1} \le \varepsilon \int_\Omega (u+v)^{p+1} + C(\varepsilon, p)\Big(\int_\Omega (u+v)\Big)^{p+1} \quad \textit{for all } \ t \in (0, T_{\max}).$$

Proof. First, multiplying the third equation in (1.2) by w^p and integrating over Ω, along with applying Young's inequality, we have

$$\begin{aligned} p\int_\Omega w^{p-1}|\nabla w|^2 + \mu \int_\Omega w^{p+1} &\le \max\{\nu, \lambda\} \int_\Omega (u+v) w^p \\ &\le \left(\frac{\max\{\nu,\lambda\}}{p+1}\right)^{p+1} \left(\frac{p}{\mu}\right)^p \int_\Omega (u+v)^{p+1} + \mu \int_\Omega w^{p+1}, \end{aligned}$$

which implies

$$\int_\Omega w^{p-1}|\nabla w|^2 \le \frac{1}{p}\left(\frac{\max\{\nu,\lambda\}}{p+1}\right)^{p+1} \left(\frac{p}{\mu}\right)^p \int_\Omega (u+v)^{p+1} \quad \text{for all } \ t \in (0, T_{\max}). \tag{2.13}$$

Then, for any $\tilde{\varepsilon} > 0$, there are $C_1, \tilde{C}_1 > 0$ such that

$$\begin{aligned} &\int_\Omega w^{p+1} = \int_\Omega (w^{\frac{p+1}{2}})^2 \\ &\le \tilde{\varepsilon} \int_\Omega |\nabla w^{\frac{p+1}{2}}|^2 + C_1 \left(\int_\Omega w^{\frac{p+1}{2}}\right)^2 && \text{(by Ehrling's lemma)} \\ &\le \frac{\tilde{\varepsilon}(p+1)^2}{4} \int_\Omega w^{p-1}|\nabla w|^2 + C_1\Big(\int_\Omega w^p\Big)\Big(\int_\Omega w\Big) && \text{(by Hölder's ineq.)} \\ &\le \frac{\tilde{\varepsilon}(p+1)^2}{4p} \left(\frac{\max\{\nu,\lambda\}}{p+1}\right)^{p+1} \left(\frac{p}{\mu}\right)^p \int_\Omega (u+v)^{p+1} + C_1\Big(\int_\Omega w^p\Big)\Big(\int_\Omega w\Big) && \text{(by (2.13))} \\ &\le \frac{\tilde{\varepsilon}(p+1)^2}{4p} \left(\frac{\max\{\nu,\lambda\}}{p+1}\right)^{p+1} \left(\frac{p}{\mu}\right)^p \int_\Omega (u+v)^{p+1} + \frac{1}{2}\int_\Omega w^{p+1} + \tilde{C}_1\Big(\int_\Omega w\Big)^{p+1} && \text{(by Young's ineq.)} \end{aligned}$$

for all $t \in (0, T_{\max})$. The lemma then follows with the fact $\int_\Omega w \le \frac{\max\{\nu,\lambda\}}{\mu} \int_\Omega (u+v)$ and taking $C = \tilde{C}_1 (\frac{\max\{\nu,\lambda\}}{\mu})^{p+1}$ and $\tilde{\varepsilon} = \frac{2p}{(p+1)^2} (\frac{p+1}{\max\{\nu,\lambda\}})^{p+1} (\frac{\mu}{p})^p \varepsilon$. $\square$

3 Proofs of Theorems 1.1, and 1.4

In this section, we investigate lower bound of the combined mass, and L^{-q}-boundedness of classical solutions of (1.2), and prove Theorems 1.1, and 1.4.

Throughout this section, for given u_0, v_0 satisfying (1.8), we put

$$(u(t,x), v(t,x), w(t,x)) := (u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0)),$$

and if no confusion occurs, we may drop (t,x) in $u(t,x)$ (resp. $v(t,x)$, $w(t,x)$).

3.1 Lower bound of the combined mass on bounded intervals and proof of Theorem 1.1

In this subsection, we study the lower bound of the combined mass on bounded intervals and proof of Theorem 1.1.

We first present a lemma on $\int_\Omega \ln(u+v)dx$. Note that, by (1.6), $u(t,x;u_0,v_0)+v(t,x;u_0,v_0)>0$ for all $t\in(0,T_{\max}(u_0,v_0))$ and $x\in\bar{\Omega}$,. Hence $\int_\Omega \ln(u(t,x;u_0,v_0)+v(t,x;u_0,v_0))dx$ is well defined for $t\in(0,T_{\max}(u_0,v_0))$.

Lemma 3.1. *For any $u_0(x)$ and $v_0(x)$ satisfying* (1.8)*, there exists $K=K(u_0,v_0)>0$ such that*

$$\frac{d}{dt}\int_\Omega \ln\big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx \geq -K \quad \text{for all } t\in(0,T_{\max}(u_0,v_0)).$$

Proof. First, multiplying (2.3) by $\frac{1}{u+v}$ and then integrating on Ω yields that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \ln(u+v) \geq \int_\Omega \frac{|\nabla(u+v)|^2}{(u+v)^2} - \underbrace{\int_\Omega \frac{\chi_1 u+\chi_2 v}{(u+v)^2}\nabla(u+v)\cdot\frac{\nabla w}{w}}_{I}\\ + \int_\Omega \frac{a_1u+a_2v}{u+v} - \int_\Omega \frac{b_1u^2+b_2v^2}{u+v} - \int_\Omega \frac{(c_1+c_2)uv}{u+v}\end{aligned}$$

for all $t\in(0,T_{\max}(u_0,v_0))$. By Young's inequality and Lemma 2.6, we have

$$\begin{aligned}I=\int_\Omega \frac{\chi_1u+\chi_2v}{(u+v)^2}\nabla(u+v)\cdot\frac{\nabla w}{w} &\leq \int_\Omega \frac{|\nabla(u+v)|^2}{(u+v)^2} + \frac{\max\{\chi_1^2,\chi_2^2\}}{4}\int_\Omega \frac{|\nabla w|^2}{w^2}\\ &\leq \int_\Omega \frac{|\nabla(u+v)|^2}{(u+v)^2} + \frac{\mu|\Omega|\max\{\chi_1^2,\chi_2^2\}}{4}.\end{aligned}$$

Then by Lemma 2.7 we have

$$\frac{d}{dt}\int_\Omega \ln(u+v) \geq -\frac{\mu|\Omega|\max\{\chi_1^2,\chi_2^2\}}{4} + a_{\min}|\Omega| - (b_{\max}+c_{\max})\left(m_1^*(0,u_0,v_0)+m_2^*(0,u_0,v_0)\right)$$

for all $t\in(0,T_{\max})$. The lemma is thus proved. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Fix a $\tau\in(0,T_{\max}(u_0,v_0))$. It is clear that

$$\inf_{0\leq t\leq\tau}\int_\Omega \big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx>0.$$

It then suffices to prove that for any $T>0$, there exist $C=C(T)>0$ such that

$$\int_\Omega \big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx \geq C(T) \quad \text{for all } t\in(\tau,\hat{T}), \tag{3.1}$$

where $\hat{T}=\min\{T,T_{\max}(u_0,v_0)\}$.

Note that $L := \int_\Omega \ln\big(u(\tau,x;u_0,v_0)+v(\tau,x;u_0,v_0)\big)dx$ is finite. By Lemma 3.1, there exists $K=K(u_0,v_0)>0$ such that

$$\frac{d}{dt}\int_\Omega \ln\big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx \geq -K$$

for all $t\in(0,T_{\max}(u_0,v_0))$. We thus have that

$$\begin{aligned}\int_\Omega \ln\big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx &\geq \int_\Omega \ln\left(u(\tau,x;u_0,v_0)+v(\tau,x;u_0,v_0)\right)dx - K\cdot(t-\tau)\\ &\geq L-K\cdot(\hat{T}-\tau)=C(K,L,\tau)\quad \text{for all } t\in(\tau,\hat{T}).\end{aligned}$$

Therefore Jensen's inequality asserts that

$$\begin{aligned}\int_\Omega \ln\big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx &= |\Omega|\cdot\int_\Omega \ln\big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)\frac{dx}{|\Omega|}\\ &\leq |\Omega|\cdot\ln\Big(\int_\Omega \big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)\frac{dx}{|\Omega|}\Big)\end{aligned}$$

for all $t\in(0,T_{\max}(u_0,v_0))$, which implies

$$\begin{aligned}\int_\Omega \big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx &\geq |\Omega|\cdot\exp\Big(\frac{1}{|\Omega|}\cdot\int_\Omega \ln\big(u(t,x;u_0,v_0)+v(t,x;u_0,v_0)\big)dx\Big)\\ &\geq |\Omega|\cdot e^{\frac{1}{|\Omega|}\cdot C(K,L,\tau)}\quad \text{for all } t\in(\tau,\hat{T}),\end{aligned}$$

which implies (3.1). The theorem is thus proved. □

3.2 L^{-q}-boundedness and proof of Theorem 1.4

In this subsection, we study the boundedness of $\int_\Omega (u+v)^{-q}$ and prove Theorem 1.4.

We first prove Theorem 1.4(1). To this end. we first prove a lemma.

Lemma 3.2. *Let $\mu>0$, $\chi_1>0$ and $\chi_2>0$ be given.*

(1) For any $B>0$ and $0<\beta\neq\chi_2-B$,

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \leq &\Big(f(\mu,\chi_1,\chi_2,\beta,B)-a_{\min}\Big)\int_\Omega (u+v)^{-q}\\ &+(b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1},\quad \forall\, t\in(0,T_{\max}(u_0,v_0)),\end{aligned}\tag{3.2}$$

where $f(\mu,\chi_1,\chi_2,\beta,B)$ is as in (1.9) and

$$q=\frac{4B\beta}{B(\chi_2-B-\beta)^2+(\chi_1-\chi_2)^2\beta}.\tag{3.3}$$

(2) For any $B>0$ and $0<\beta\neq\chi_1-B$,

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \leq &\Big(f(\mu,\chi_2,\chi_1,\beta,B)-a_{\min}\Big)\int_\Omega (u+v)^{-q}\\ &+(b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1},\quad \forall\, t\in(0,T_{\max}(u_0,v_0)),\end{aligned}\tag{3.4}$$

where

$$q=\frac{4B\beta}{B(\chi_1-B-\beta)^2+(\chi_1-\chi_2)^2\beta}.\tag{3.5}$$

Remark 3.1. *Lemma 3.2 plays a crucial role in the proof of the global existence and boundedness of the solutions of* (1.2)*. The proof of Lemma 3.2 strongly relies on the fact that the third equation in* (1.2) *is elliptic. Therefore new techniques need to be developed to investigate the global existence and boundedness of solutions of* (1.24)*.*

Proof of Lemma 3.2. In this proof, we always assume that $t \in (0, T_{\max}(u_0, v_0))$.

(1) First, by (2.4), we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \leq &-(q+1)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2 \\ &+(q+1)\chi_2\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w \\ &+\frac{(\chi_1-\chi_2)^2 q(q+1)}{4B}\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2 + \frac{B(q+1)}{q}\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2} \\ &-a_{\min}\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1} \end{aligned} \tag{3.6}$$

for any $q > 0$ and $B > 0$.

Next, multiplying the third equation in (1.2) by $\frac{(u+v)^{-q}}{w}$ and then integrating over Ω with respect to x, we obtain that

$$\begin{aligned} 0 &= \int_\Omega \frac{(u+v)^{-q}}{w}\cdot\big(\Delta w - \mu w + \nu u + \lambda v\big) \\ &= -\int_\Omega \frac{(-q)(u+v)^{-q-1}w\nabla(u+v) - (u+v)^{-q}\nabla w}{w^2}\cdot\nabla w \\ &\quad -\mu\int_\Omega (u+v)^{-q} + \nu\int_\Omega \frac{u(u+v)^{-q}}{w} + \lambda\int_\Omega \frac{v(u+v)^{-q}}{w} \end{aligned}$$

for all $t \in (0, T_{\max})$. Thus we have,

$$q\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w + \int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2} \leq \mu\int_\Omega (u+v)^{-q} \tag{3.7}$$

for all $t \in (0, T_{\max})$. It then follows that

$$\frac{B}{q}\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2} \leq \frac{B\mu}{q}\int_\Omega (u+v)^{-q} - B\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w.$$

This together with (3.6) implies that

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \leq &-(q+1)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2 \\ &+(q+1)(\chi_2-B)\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w \\ &+\frac{(\chi_1-\chi_2)^2 q(q+1)}{4B}\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2 \\ &+\frac{B(q+1)\mu}{q}\int_\Omega (u+v)^{-q} \\ &-a_{\min}\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1} \end{aligned} \tag{3.8}$$

for any $q > 0$ and $B > 0$.

Now, let $\tilde{\chi} = \chi_2 - B$. Note that, when $B > 0$ and $q > 0$ are such that $D := \frac{(\chi_1-\chi_2)^2}{4B}q < 1$, for any $\beta > 0$, by Young's inequality and (3.7), we have

$$\begin{aligned}\tilde{\chi}\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w = (\tilde{\chi}-\beta)\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w \\ + \beta\int_\Omega \frac{(u+v)^{-q-1}}{w}\nabla(u+v)\cdot\nabla w \\ \le (1-D)\int_\Omega (u+v)^{-q-2}|\nabla(u+v)|^2 + \frac{|\tilde{\chi}-\beta|^2}{4(1-D)}\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2} \\ + \frac{\beta\mu}{q}\int_\Omega (u+v)^{-q} - \frac{\beta}{q}\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2}.\end{aligned}$$

This together with (3.8) implies that

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le (q+1)\Big(\frac{B|\tilde{\chi}-\beta|^2}{4B-(\chi_1-\chi_2)^2 q} - \frac{\beta}{q}\Big)\int_\Omega (u+v)^{-q}\frac{|\nabla w|^2}{w^2} \\ + (q+1)\Big(\frac{B\mu}{q} + \frac{\beta\mu}{q}\Big)\int_\Omega (u+v)^{-q} \\ - a_{\min}\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1}\end{aligned} \tag{3.9}$$

for any $\beta > 0$ and $B > 0$, $q > 0$ with $\frac{(\chi_1-\chi_2)^2}{4B}q < 1$.

Finally, for any $B > 0$ and $\beta > 0$, $\beta \neq \tilde{\chi}$, let q be as in (3.3). Then

$$\frac{(\chi_1-\chi_2)^2}{4B}q < 1,$$

and

$$\frac{B|\tilde{\chi}-\beta|^2}{4B-(\chi_1-\chi_2)^2 q} - \frac{\beta}{q} = 0.$$

Then by (3.9), we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le \Big((B\mu+\beta\mu)\Big(1+\frac{1}{q}\Big) - a_{\min}\Big)\int_\Omega (u+v)^{-q} \\ + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1}.\end{aligned}$$

Note that

$$(B\mu+\beta\mu)\Big(1+\frac{1}{q}\Big) = f(\mu,\chi_1,\chi_2,\beta,B).$$

This implies that (3.2) holds.

(2) It can be proved by the similar arguments as in (1). □

Remark 3.2. *(1) If $(\chi_1-\chi_2)^2 \le 4\chi_2$, letting $B = \chi_2$ and $q = 1$, by (3.8), we have*

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-1} \le \Big(2\mu\chi_2 - a_{\min}\Big)\int_\Omega (u+v)^{-1} \\ + (b_{\max}+c_{\max})|\Omega|, \quad \forall\, t \in (0, T_{\max}(u_0,v_0)).\end{aligned}$$

Similarly, if $(\chi_1-\chi_2)^2 \le 4\chi_1$, *letting* $B=\chi_1$ *and* $q=1$, *we have*

$$\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-1} \le \Big(2\mu\chi_1 - a_{\min}\Big)\int_\Omega (u+v)^{-1} + (b_{\max}+c_{\max})|\Omega|, \quad \forall\, t\in(0,T_{\max}(u_0,v_0)).$$

(2) Assume that $(\chi_1-\chi_2)^2 > \max\{4\chi_1, 4\chi_2\}$. *Let* $B=\chi_2$ *and* $q=\frac{4\chi_2}{(\chi_1-\chi_2)^2}$. *By* (3.8) *again, we have*

$$\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le \Big(\mu\chi_2 + \frac{\mu(\chi_1-\chi_2)^2}{4} - a_{\min}\Big)\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega u^{-q+1}, \quad \forall\, t\in(0,T_{\max}(u_0,v_0)).$$

Similarly, let $B=\chi_1$ *and* $q=\frac{4\chi_1}{(\chi_1-\chi_2)^2}$. *We have*

$$\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le \Big(\mu\chi_1 + \frac{\mu(\chi_1-\chi_2)^2}{4} - a_{\min}\Big)\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega u^{-q+1}, \quad \forall\, t\in(0,T_{\max}(u_0,v_0)).$$

Proof of Theorem 1.4(1). In this proof, we always assume that $t\in(0,T_{\max}(u_0,v_0))$ unless specified otherwise.

Without loss of generality, we assume that

$$\chi^*(\mu,\chi_1,\chi_2) = \min\{\chi_1^*(\mu,\chi_1,\chi_2), \chi_2^*(\mu,\chi_1,\chi_2)\} = \chi_1^*(\mu,\chi_1,\chi_2).$$

Let $\epsilon_0 = a_{\min} - \chi^*(\mu,\chi_1,\chi_2)$. Then by the definition of $\chi_1^*(\mu,\chi_1,\chi_2)$, there are $B>0$ and $0<\beta\ne\chi_2-B$ such that

$$a_{\min} - f(\mu,\chi_1,\chi_2,\beta,B) > \frac{3\epsilon_0}{4}.$$

By (3.2), we then have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le -\frac{3\epsilon_0}{4}\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1} \tag{3.10}$$

for all $t>0$, where q is as in (3.3).

Note that if $q\ge 1$, by Young's inequality,

$$(b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1} \le \frac{\epsilon_0}{4}\int_\Omega (u+v)^{-q} + C_{\epsilon_0},$$

where $C_{\epsilon_0} = \frac{(b_{\max}+c_{\max})^q}{q}\big(\frac{4}{\epsilon_0}\big)^q\big(\frac{q-1}{q}\big)^{q-1}|\Omega|$ if $q>1$ and $C_{\epsilon_0}=(b_{\max}+c_{\max})|\Omega|$ if $q=1$. If $q<1$, then by the Hölder inequality

$$\begin{aligned}(b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1} &\le (b_{\max}+c_{\max})|\Omega|^q\cdot\left(\int_\Omega u+v\right)^{1-q}\\ &\le (b_{\max}+c_{\max})|\Omega|^q (m^*(\tau,u_0,v_0))^{1-q}\end{aligned} \tag{3.11}$$

for any $0 < \tau < t < T_{\max}(u_0, v_0)$, where $m^*(\tau, u_0, v_0)$ is as in (2.10). By (3.10)-(3.11), there is $\tilde{C}_{\epsilon_0,\tau,u_0,v_0} > 0$ such that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le -\frac{\epsilon_0}{2}\int_\Omega (u+v)^{-q} + C_{\epsilon_0,\tau,u_0,v_0} \quad \forall\, 0 < \tau < t < T_{\max}(u_0, v_0),$$

where

$$C_{\epsilon_0,\tau,u_0,v_0} = \begin{cases} \frac{(b_{\max}+c_{\max})^q}{q}\left(\frac{4}{\epsilon_0}\right)^q\left(\frac{q-1}{q}\right)^{q-1}|\Omega| & \text{if } q > 1, \\ (b_{\max}+c_{\max})|\Omega| & \text{if } q = 1, \\ (b_{\max}+c_{\max})|\Omega|^q (m^*(\tau, u_0, v_0))^{1-q} & \text{if } q < 1. \end{cases} \tag{3.12}$$

This implies that

$$\begin{aligned} &\int_\Omega (u(x,t;u_0,v_0) + v(x,t;u_0,v_0))^{-q}dx \\ &\le e^{-\frac{\epsilon_0 q}{2}(t-\tau)}\int_\Omega (u(x,\tau;u_0,v_0) + v(x,\tau;u_0,v_0))^{-q}dx + 2C_{\epsilon_0,\tau,u_0,v_0}\epsilon_0^{-1} \end{aligned} \tag{3.13}$$

for any $\tau > 0$ and $\tau < t < T_{\max}(u_0, v_0)$. Theorem 1.3(1) thus follows in the case that $a_{\min} > \chi^*(\mu, \chi_1, \chi_2)$. □

Next, we prove Theorem 1.4(2).

Proof of Theorem 1.4(2). In this proof, we always assume that $t \in (0, T_{\max}(u_0, v_0))$ unless specified otherwise.

Assume that $a_{\min} > \chi^{**}(\mu, \chi_1, \chi_2)$. Let $q = q_{\chi_1,\chi_2}$. By Remark 3.2, we have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega (u+v)^{-q} \le -\big(a_{\min} - \chi^{**}(\mu,\chi_1,\chi_2)\big)\int_\Omega (u+v)^{-q} + (b_{\max}+c_{\max})\int_\Omega (u+v)^{-q+1}. \tag{3.14}$$

Note that $q_{\chi_1,\chi_2} \le 1$. By the arguments of (3.13), we have

$$\begin{aligned} &\int_\Omega (u(x,t;u_0,v_0) + v(x,t;u_0,v_0))^{-q}dx \\ &\le e^{-(a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2))q(t-\tau)}\int_\Omega (u(x,\tau;u_0,v_0) + v(x,\tau;u_0,v_0))^{-q}dx + \tilde{C}_{\tau,u_0,v_0}, \end{aligned}$$

where

$$\tilde{C}_{\tau,u_0,v_0} = \begin{cases} \frac{(b_{\max}+c_{\max})|\Omega|}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} & \text{if } q = 1, \\ \frac{(b_{\max}+c_{\max})|\Omega|^q\left(m^*(\tau,u_0,v_0)\right)^{1-q}}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} & \text{if } q < 1. \end{cases}$$

Theorem 1.3(2) follows in the case that $a_{\min} > \chi^{**}(\mu, \chi_1, \chi_2)$. □

Remark 3.3. *Assume that* $a_{\min} > \chi^{**}(\mu, \chi_1, \chi_2)$. *By* (3.14), *we also have*

$$\int_\Omega (u+v)^{-q_{\chi_1,\chi_2}}(t,x)dx \le \begin{cases} \max\left\{ \int_\Omega (u+v)^{-q_{\chi_1,\chi_2}}(\tau,x)dx, \frac{(b_{\max}+c_{\max})|\Omega|}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} \right\} \\ \qquad\qquad \text{if } (\chi_1-\chi_2)^2 \le \max\{4\chi_1, 4\chi_2\} \\ \\ \max\left\{ \int_\Omega (u+v)^{-q_{\chi_1,\chi_2}}(\tau,x)dx, \frac{(b_{\max}+c_{\max})|\Omega|^{q_{\chi_1,\chi_2}}\left(m^*(\tau,u_0,v_0)\right)^{1-q_{\chi_1,\chi_2}}}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} \right\} \\ \qquad\qquad \text{if } (\chi_1-\chi_2)^2 > \max\{4\chi_1, 4\chi_2\} \end{cases}$$

for all $0 < \tau \le t < T_{\max}(u_0, v_0)$.

4 L^p- and C^θ-boundedness and proof of Theorem 1.2

In this section, we study the L^p- and C^θ-boundedness of classical solutions of (1.2) and prove Theorem 1.2.

4.1 Proof of Theorem 1.2(1)

In this subsection, we prove Theorem 1.2(1). To this end, we first make some observations. In the following, again we always assume that $t \in (0, T_{\max}(u_0, v_0))$ unless specified otherwise.

Fix $p > 2$. First, observe that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega (u+v)^p =& -(p-1)\int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2 \\ &+ (p-1)\int_\Omega (u+v)^{p-2}\frac{\chi_1 u+\chi_2 v}{w}\nabla(u+v)\cdot\nabla w \\ &+ \int_\Omega (u+v)^{p-1}(a_1u+a_2v) - \int_\Omega (u+v)^{p-1}(b_1u^2+b_2v^2) - \int_\Omega (u+v)^{p-1}(c_1+c_2)uv \\ \le& -(p-1)\int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2 \\ &+ (p-1)\int_\Omega (u+v)^{p-2}\frac{\chi_1 u+\chi_2 v}{w}\nabla(u+v)\cdot\nabla w \\ &+ a_{\max}\int_\Omega (u+v)^p - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{p+1}. \end{aligned} \tag{4.1}$$

To prove Theorem 1.2(1), it is then essential to provide proper estimates for the integral $\int_\Omega (u+v)^{p-2}\frac{\chi_1 u+\chi_2 v}{w}\nabla(u+v)\cdot\nabla w$.

Next, observe that

$$\begin{aligned} \int_\Omega \frac{(u+v)^{p-2}(\chi_1 u+\chi_2 v)}{w}\nabla(u+v)\cdot\nabla w =& \chi_2\int_\Omega \frac{(u+v)^{p-1}}{w}\nabla(u+v)\cdot\nabla w \\ &+ (\chi_1-\chi_2)\int_\Omega \frac{(u+v)^{p-2}}{w}u\nabla(u+v)\cdot\nabla w. \end{aligned}$$

Since $u^2 \le (u+v)^2$, by Young's inequality, we have that

$$\begin{aligned}&\big|(\chi_1-\chi_2)\int_\Omega \frac{(u+v)^{p-2}}{w}u\nabla(u+v)\cdot\nabla w\big|\\ &\le \frac{1}{2}\int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2+\frac{(\chi_1-\chi_2)^2}{2}\int_\Omega (u+v)^p\frac{|\nabla w|^2}{w^2}\end{aligned}$$

and

$$|\chi_2\int_\Omega \frac{(u+v)^{p-1}}{w}\nabla(u+v)\cdot\nabla w| \le \frac{1}{2}\int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2+\frac{\chi_2^2}{2}\int_\Omega (u+v)^p\frac{|\nabla w|^2}{w^2}.$$

Therefore,

$$\begin{aligned}&\int_\Omega \frac{(u+v)^{p-2}(\chi_1 u+\chi_2 v)}{w}\nabla(u+v)\cdot\nabla w\\ &\le \int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2+\frac{\chi_2^2+(\chi_1-\chi_2)^2}{2}\int_\Omega (u+v)^p\frac{|\nabla w|^2}{w^2}\\ &\le \int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2+\frac{\chi_2^2+(\chi_1-\chi_2)^2}{2\inf\limits_{x\in\Omega} w(t,x)}\int_\Omega (u+v)^p\frac{|\nabla w|^2}{w}\\ &\le \int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2+\frac{\chi_2^2+(\chi_1-\chi_2)^2}{2\inf\limits_{x\in\Omega} w(t,x)}\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{p+1}}\Big)^{\frac{1}{p+1}}. \qquad (4.2)\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_\Omega \frac{(u+v)^{p-2}(\chi_1 u+\chi_2 v)}{w}\nabla(u+v)\cdot\nabla w\\ &\le \int_\Omega (u+v)^{p-2}|\nabla(u+v)|^2+\frac{\chi_1^2+(\chi_1-\chi_2)^2}{2\inf\limits_{x\in\Omega} w(t,x)}\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{p+1}}\Big)^{\frac{1}{p+1}}. \qquad (4.3)\end{aligned}$$

To provide proper estimates for $\int_\Omega (u+v)^{p-2}\frac{\chi_1 u+\chi_2 v}{w}\nabla(u+v)\cdot\nabla w$, it is then essential to provide proper estimates for $\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{p+1}}$. We have the following proposition on estimates for $\int_\Omega \frac{|\nabla w|^{2p}}{w^k}$.

Proposition 4.1. *Let $p \ge 3$ and $p-\sqrt{2p-3} < k < p+\sqrt{2p-3}$. There are $M(p,k) > 0$ and $\tilde{M}(p,k) > 0$ such that*

$$\int_\Omega \frac{|\nabla w|^{2p}}{w^k} \le M(p,k)\int_\Omega \frac{(\nu u+\lambda v)^p}{w^{k-p}}+\tilde{M}(p,k)\int_\Omega w^{2p-k} \quad \forall\, t\in(0,T_{\max}(u_0,v_0)).$$

Remark 4.1. *We remark that the estimate in Proposition 4.1 plays a crucial role in the proof of the boundedness of the solutions of* (1.2)*. The proof of Proposition 4.1 strongly relies on the fact that the third equation in* (1.2) *is elliptic. It therefore requires new techniques to investigate the global existence and boundedness of solutions of* (1.24)*.*

In the rest of this subsection, we prove Theorem 1.2(1) by applying Proposition 4.1. Proposition 4.1 can be proved by the similar arguments as those in [25, Proposition 1.3]. For the reader's convenience, we will provide the outline of the proof of Proposition 4.1 in the appendix.

Proof of Theorem 1.2(1). First, for any $p > \max\{2, N\}$, by (4.1), (4.2), and (4.3), we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+v)^p \le \frac{(p-1)\big(\chi_i^2 + (\chi_1-\chi_2)^2\big)}{2\inf\limits_{x\in\Omega} w(t,x)}\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega \frac{|\nabla w|^{2p+2}}{w^{p+1}}\Big)^{\frac{1}{p+1}}$$
$$+ a_{\max}\int_\Omega (u+v)^p - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{p+1} \tag{4.4}$$

for all $t \in (0, T_{\max}(u_0, v_0))$ and $i = 1, 2$. Let

$$C^*(p) = M(p+1, p+1), \quad C^{**}(p) = \tilde{M}(p+1, p+1),$$

and

$$\tilde{\chi} = \min\Big\{\chi_1^2 + (\chi_1-\chi_2)^2, \chi_2^2 + (\chi_1-\chi_2)^2\Big\}.$$

By (4.4) and Proposition 4.1, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+v)^p &\le \frac{\tilde{\chi}(p-1)}{2\inf\limits_{x\in\Omega} w(t,x)}\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(C^*(p)\int_\Omega (\nu u+\lambda v)^{p+1} + C^{**}(p)\int_\Omega w^{p+1}\Big)^{\frac{1}{p+1}} \\
&\quad + a_{\max}\int_\Omega (u+v)^p - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{p+1} \\
&\le \frac{\tilde{\chi}(p-1)\big(C^*(p)\big)^{\frac{1}{p+1}}\max\{\nu,\lambda\}}{2\inf\limits_{x\in\Omega} w(t,x)}\int_\Omega (u+v)^{p+1} \\
&\quad + \frac{\tilde{\chi}(p-1)\big(C^{**}(p)\big)^{\frac{1}{p+1}}}{2\inf\limits_{x\in\Omega} w(t,x)}\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega w^{p+1}\Big)^{\frac{1}{p+1}} \\
&\quad + a_{\max}\int_\Omega (u+v)^p - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{p+1}.
\end{aligned} \tag{4.5}$$

Next, by **(H2)** and Remark 3.3,

$$\int_\Omega (u+v)^{-q^*} \le \frac{(b_{\max}+c_{\max})|\Omega|^{q^*}\big(m^*\big)^{1-q^*}}{\big(a_{\min} - \chi^{**}(\mu,\chi_1,\chi_2)\big)C^{q^*}_{\chi_1,\chi_2}}, \ \forall\ \tau_0 \le t < T_{\max}(u_0, v_0).$$

where $q^* = q_{\chi_1,\chi_2}$, $\tau_0 \in [0, T_{\max}(u_0, v_0))$ is as in **(H2)**, and

$$m^* = \frac{a_{\max}|\Omega|}{\min\{b_{\min}, c_{\min}\}}.$$

Let

$$\delta^* = \frac{\delta_0|\Omega|(a_{\min} - \chi^{**}(\mu,\chi_1,\chi_2))^{\frac{1}{q^*}}\big(\min\{b_{\min}, c_{\min}\}\big)^{\frac{1}{q^*}-1}}{(b_{\max}+c_{\max})^{\frac{1}{q^*}}\big(a_{\max}\big)^{\frac{1}{q^*}-1}}.$$

Then by Lemma 2.4 and (1.17), we have

$$\inf_{x\in\Omega} w(t,x) \ge \delta_0\int_\Omega (u+v) \ge \frac{\delta_0|\Omega|^{\frac{q^*+1}{q^*}}}{\Big(\int_\Omega (u+v)^{-q^*}\Big)^{\frac{1}{q^*}}} \ge \delta^* C_{\chi_1,\chi_2} \quad \forall\, \tau_0 \le t < T_{\max}(u_0, v_0).$$

By Lemma 2.8, for any $\varepsilon_1 > 0$, there is $C(\varepsilon_1, p) > 0$ such that

$$\int_\Omega w^{p+1} \le \varepsilon_1 \int_\Omega (u+v)^{p+1} + C(\varepsilon_1, p)\Big(\int_\Omega (u+v)\Big)^{p+1} \quad \text{for all } \ 0 < t < T_{\max}(u_0, v_0).$$

By Young's inequality, any $\varepsilon_2 > 0$, there is $\tilde{C}(\varepsilon_2, p) > 0$ such that

$$\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega (u+v)\Big) \le \varepsilon_2 \int_\Omega (u+v)^{p+1} + \tilde{C}(\varepsilon_2, p)\Big(\int_\Omega (u+v)\Big)^{p+1}.$$

Then, by (4.5), we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+v)^p \le{}& \frac{\tilde{\chi}(p-1)\big(C^*\big)^{\frac{1}{p+1}}\max\{\nu,\lambda\}}{2\delta^* C_{\chi_1,\chi_2}}\int_\Omega (u+v)^{p+1} + \frac{\tilde{\chi}(p-1)\big(C^{**}\big)^{\frac{1}{p+1}}\varepsilon_1^{\frac{1}{p+1}}}{2\delta^* C_{\chi_1,\chi_2}}\int_\Omega (u+v)^{p+1} \\
&+ \frac{\tilde{\chi}(p-1)\big(C^{**}\big)^{\frac{1}{p+1}}\big(C(\varepsilon_1,p)\big)^{\frac{1}{p+1}}}{2\delta^* C_{\chi_1,\chi_2}}\Big(\int_\Omega (u+v)^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega (u+v)\Big) \\
&+ a_{\max}\int_\Omega (u+v)^p - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{p+1} \\
\le{}& \frac{\tilde{\chi}(p-1)\big(C^*\big)^{\frac{1}{p+1}}\max\{\nu,\lambda\}}{2\delta^* C_{\chi_1,\chi_2}}\int_\Omega (u+v)^{p+1} \\
&+ \frac{\tilde{\chi}(p-1)\big(C^{**}\big)^{\frac{1}{p+1}}\big(\varepsilon_1^{\frac{1}{p+1}} + \big(C(\varepsilon_1,p)\big)^{\frac{1}{p+1}}\varepsilon_2\big)}{2\delta^* C_{\chi_1,\chi_2}}\int_\Omega (u+v)^{p+1} \\
&+ \frac{\tilde{\chi}(p-1)\big(C^{**}\big)^{\frac{1}{p+1}}\big(C(\varepsilon_1,p)\big)^{\frac{1}{p+1}}\tilde{C}(\varepsilon_2,p)}{2\delta^* C_{\chi_1,\chi_2}}\Big(\int_\Omega (u+v)\Big)^{p+1} \\
&+ a_{\max}\int_\Omega (u+v)^p - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{p+1}.
\end{aligned}$$

Now, by **(H1)**, there are $p^* > \max\{2, N\}$, $\varepsilon_1^* > 0$ and $\varepsilon_2^* > 0$ such that

$$\min\big\{b_{\min}, c_{\min}\big\} > \frac{\tilde{\chi}(p-1)\big(C^*\big)^{\frac{1}{p^*+1}}\max\{\nu,\lambda\}}{2\delta^* C_{\chi_1,\chi_2}} + \frac{\tilde{\chi}(p-1)\big(C^{**}\big)^{\frac{1}{p^*+1}}\big((\varepsilon_1^*)^{\frac{1}{p^*+1}} + \big(C(\varepsilon_1^*,p^*)\big)^{\frac{1}{p^*+1}}\varepsilon_2^*\big)}{2\delta^* C_{\chi_1,\chi_2}}.$$

Therefore, there is $b^* > 0$ such that

$$\begin{aligned}
\frac{1}{p^*}\frac{d}{dt}\int_\Omega (u+v)^{p^*} \le{}& \frac{\tilde{\chi}(p-1)\big(C^{**}\big)^{\frac{1}{p+1}}\big(C(\varepsilon_1^*,p^*)\big)^{\frac{1}{p^*+1}}\tilde{C}(\varepsilon_2^*,p^*)}{2\delta^* C_{\chi_1,\chi_2}}\Big(\int_\Omega (u+v)\Big)^{p^*+1} \\
&+ a_{\max}\int_\Omega (u+v)^{p^*} - b^*\int_\Omega (u+v)^{p^*+1} \quad \forall\, \tau_0 \le t < T_{\max}(u_0, v_0).
\end{aligned}$$

This implies that there is $M_1' > 0$ independent of (u_0, v_0) such that

$$\frac{d}{dt}\int_\Omega (u+v)^{p^*} \le -\int_\Omega (u+v)^{p^*} + M_1' \quad \forall\ \tau_0 < t < T_{\max}(u_0, v_0).$$

Hence

$$\int_\Omega (u+v)^p(t,x;u_0,v_0) \le e^{-(t-\tau_0)}\int_\Omega (u+v)(\tau_0,x;u_0,v_0) + M_1' \quad \forall\, \tau_0 \le t < T_{\max}(u_0, v_0). \tag{4.6}$$

Therefore Theorem 1.2(1) holds with $p = p^*$, $M_1 = M_1'$, and any $\tilde{p}^* > 1$.

It remains to prove that Theorem 1.2(1) holds with $p = 2p^*$. To this end, let $\tilde{p} = 2p^*$. By (4.1), we have

$$\begin{aligned}\frac{1}{\tilde{p}}\frac{d}{dt}\int_\Omega (u+v)^{\tilde{p}} \le &-(\tilde{p}-1)\int_\Omega (u+v)^{\tilde{p}-2}|\nabla(u+v)|^2 - \int_\Omega (u+v)^{\tilde{p}} \\ &+(\tilde{p}-1)\int_\Omega (u+v)^{\tilde{p}-2}\frac{\chi_1 u+\chi_2 v}{w}\nabla(u+v)\cdot\nabla w \\ &+\big(a_{\max}+1\big)\int_\Omega (u+v)^{\tilde{p}} - \min\{b_{\min}, c_{\min}\}\int_\Omega (u+v)^{\tilde{p}+1}. \end{aligned} \tag{4.7}$$

Note that

$$\int_\Omega (u+v)^{\tilde{p}} \le \frac{\tilde{p}}{\tilde{p}+1}\int_\Omega (u+v)^{\tilde{p}+1} + \frac{|\Omega|}{\tilde{p}+1}. \tag{4.8}$$

Note also that

$$\begin{aligned}&\int_\Omega (u+v)^{\tilde{p}-2}\frac{\chi_1 u+\chi_2 v}{w}\nabla(u+v)\cdot\nabla w \\ &\le (\chi_1+\chi_2)\int_\Omega \frac{(u+v)^{\tilde{p}-1}}{w}|\nabla(u+v)|\,|\nabla w| \\ &\le \frac{1}{2}\int_\Omega (u+v)^{\tilde{p}-2}|\nabla(u+v)|^2 + \frac{(\chi_1+\chi_2)^2}{2}\int_\Omega (u+v)^{\tilde{p}}\frac{|\nabla w|^2}{w^2} \\ &\le \frac{1}{2}\int_\Omega (u+v)^{\tilde{p}-2}|\nabla(u+v)|^2 + \frac{(\chi_1+\chi_2)^2}{2}\frac{\tilde{p}+1}{\tilde{p}}\int_\Omega (u+v)^{\tilde{p}+1} \\ &\quad + \frac{(\chi_1+\chi_2)^2}{2}\frac{1}{\tilde{p}+1}\int_\Omega \frac{|\nabla w|^{2\tilde{p}+2}}{w^{2\tilde{p}+2}}. \end{aligned} \tag{4.9}$$

By Proposition 4.1, we have

$$\int_\Omega \frac{|\nabla w|^{2\tilde{p}+2}}{w^{2\tilde{p}+2}} \le \frac{1}{\Big(\inf_{x\in\Omega} w(t,x)\Big)^{\tilde{p}+1}}\Big(M(\tilde{p}+1,\tilde{p}+1)\int_\Omega (u+v)^{\tilde{p}+1} + \tilde{M}(\tilde{p}+1,\tilde{p}+1)\int_\Omega w^{\tilde{p}+1}\Big) \tag{4.10}$$

By **(H2)** and Theorem 1.4(2),

$$\int_\Omega (u+v)^{-q} \le e^{-(a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2))q(t-\tau)}\int_\Omega (u(x,\tau_0;u_0,v_0)+v(x,\tau_0;u_0,v_0))^{-q}dx + \tilde{C}_1$$

for all $\tau_0 < t < T_{\max}(u_0,v_0)$, where $q = q_{\chi_1,\chi_2}$ and

$$\tilde{C}_1 = \frac{(b_{\max}+c_{\max})|\Omega|}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)}\Big(\frac{a_{\max}}{\min\{b_{\min},c_{\min}\}}\Big)^{1-q}.$$

This together with (1.17) implies that

$$\begin{aligned}\frac{1}{\inf_{x\in\Omega} w(t,x)} &\le \frac{1}{\delta_0}\frac{1}{\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))} \\ &\le \frac{1}{\delta_0|\Omega|^{\frac{q+1}{q}}}\Big(\int_\Omega (u(t,x;u_0,v_0)+v(t,x;u_0,v_0))^{-q}\Big)^{\frac{1}{q}} \\ &\le \frac{1}{\delta_0|\Omega|^{q+1}q}\Big(e^{-(a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2))q(t-\tau)}\int_\Omega (u(x,\tau_0;u_0,v_0)+v(x,\tau_0;u_0,v_0))^{-q}dx + \tilde{C}_1\Big) \end{aligned} \tag{4.11}$$

for $\tau_0 < t < T_{\max}(u_0, v_0)$. By Lemma 2.5, there is $\tilde{C}_2 > 0$ such that

$$\int_\Omega w^{\tilde{p}+1} \le \tilde{C}_2 \int_\Omega (u+v)^{\tilde{p}+1}. \tag{4.12}$$

By **(H2)** and (4.7)-(4.12), there is $\tilde{C}_2 > 0$ independent of (u_0, v_0) such that

$$\frac{1}{\tilde{p}} \cdot \frac{d}{dt} \int_\Omega (u+v)^{\tilde{p}} \le -\frac{\tilde{p}-1}{2} \int_\Omega (u+v)^{\tilde{p}-2} |\nabla(u+v)|^2 - \frac{1}{p} \int_\Omega (u+v)^{\tilde{p}} + \tilde{C}_2 \int_\Omega (u+v)^{\tilde{p}+1} + \tilde{C}_2$$

for all $\tau_0 < t < T_{|max}(u_0, v_0)$. By Gagliardo-Nirenberg inequality, there are $\tilde{C}_3 > 0$, $\tilde{C}_4 > 0$, and $\tilde{C}_5 > 0$ such that

$$\begin{aligned}
\int_\Omega (u+v)^{\tilde{p}+1} &= \int_\Omega \Big((u+v)^{\frac{\tilde{p}}{2}}\Big)^{\frac{2(\tilde{p}+1)}{\tilde{p}}} = \|(u+v)^{\frac{\tilde{p}}{2}}\|_{L^{\frac{2(\tilde{p}+1)}{\tilde{p}}}}^{\frac{2(\tilde{p}+1)}{\tilde{p}}} \\
&\le \Big(\tilde{C}_3 \|\nabla(u+v)^{\frac{\tilde{p}}{2}}\|_{L^2}^{\theta} \|(u+v)^{\frac{\tilde{p}}{2}}\|_{L^1}^{1-\theta} + \tilde{C}_3 \|(u+v)^{\frac{\tilde{p}}{2}}\|_{L^1}\Big)^{\frac{2(\tilde{p}+1)}{\tilde{p}}} \\
&\le \tilde{C}_4 \|\nabla(u+v)^{\frac{\tilde{p}}{2}}\|_{L^2}^{\theta \frac{2(\tilde{p}+1)}{\tilde{p}}} \|(u+v)^{\frac{\tilde{p}}{2}}\|_{L^1}^{(1-\theta)\frac{2(\tilde{p}+1)}{\tilde{p}}} + \tilde{C}_4 \|(u+v)^{\frac{\tilde{p}}{2}}\|_{L^1}^{\frac{2(\tilde{p}+1)}{\tilde{p}}} \\
&\le \tilde{C}_5 \Big(\int_\Omega u^{\tilde{p}-2} |\nabla(u+v)|^2\Big)^{\theta \frac{\tilde{p}+1}{\tilde{p}}} \Big(\int_\Omega (u+v)^{\frac{\tilde{p}}{2}}\Big)^{(1-\theta)\frac{2(\tilde{p}+1)}{\tilde{p}}} + \tilde{C}_4 \Big(\int_\Omega (u+v)^{\frac{\tilde{p}}{2}}\Big)^{\frac{2(\tilde{p}+1)}{\tilde{p}}},
\end{aligned}$$

where

$$\theta = \frac{1 - \frac{\tilde{p}}{2(\tilde{p}+1)}}{\frac{1}{2} + \frac{1}{N}} = \frac{\frac{1}{2} + \frac{1}{2(\tilde{p}+1)}}{\frac{1}{2} + \frac{1}{N}} = \frac{\tilde{p}+2}{\tilde{p}+1} \cdot \frac{N}{N+2} \in (0,1),$$

and

$$\frac{\theta(\tilde{p}+1)}{\tilde{p}} = \frac{\tilde{p}+2}{\tilde{p}+1} \cdot \frac{N}{N+2} \cdot \frac{\tilde{p}+1}{\tilde{p}} = \frac{\tilde{p}N + 2N}{\tilde{p}N + 2\tilde{p}} < 1 \quad \text{and} \quad \tilde{p} = 2p^*.$$

Then by Young's inequality, for any $0 < \epsilon < \frac{\tilde{p}-1}{2}$, there is $\tilde{C}_\epsilon > 0$ such that

$$\int_\Omega (u+v)^{\tilde{p}+1} \le \epsilon \int_\Omega (u+v)^{\tilde{p}-2} |\nabla(u+v)|^2 + \tilde{C}_\epsilon \Big(1 + \int_\Omega (u+v)^{p^*}\Big)^{\tilde{p}^*},$$

where $\tilde{p}^* = \frac{2(1-\theta)(\tilde{p}+1)}{p - \theta(p+1)}$. Hence there is $\tilde{C}_6 > 0$ independent of (u_0, v_0) such that

$$\frac{d}{dt} \int_\Omega (u+v)^{\tilde{p}} \le -\int_\Omega (u+v)^{\tilde{p}} + \tilde{C}_\epsilon \tilde{C}_2 \Big(\int_\Omega (u+v)^{p^*}\Big)^{\tilde{p}^*} + \tilde{C}_6 \quad \forall\, \tau_0 < t < T_{\max}(u_0, v_0).$$

This together with (4.6) implies that there is $M_1'' > 0$ independent of (u_0, v_0) such that

$$\begin{aligned}
&\int_\Omega (u+v)^p(t, x; u_0, v_0) dx \\
&\le e^{-(t-\tau_0)} \int_\Omega (u+v)^p(\tau_0, x; u_0, v_0) dx \\
&\quad + M_1'' e^{-(t-\tau_0)} (t - \tau_0) \Big(\int_\Omega (u+v)^{p^*}(\tau_0, x; u_0, v_0)\Big)^{\tilde{p}^*} + M_1'',\ \forall\, \tau_0 < t < T_{\max}(u_0, v_0).
\end{aligned} \tag{4.13}$$

Theorem 1.2(1) then follows with $M_1 = \max\{M_1', M_1''\}$. □

4.2 Proof of Theorem 1.2(2)

In this subsection, we prove Theorem 1.2(2).

Proof of Theorem 1.2(2). For given $p > 2N$ and $0 < \theta < 1 - \frac{2N}{p}$, choose $\beta \in (0, \frac{1}{2})$ such that $2\beta - \frac{2N}{p} > \theta$. By Lemma 2.2, there is $C_{p,\beta} > 0$ such that for any u_0, v_0 satisfying (1.8),

$$\|u(\cdot,\cdot) + v(\cdot,\cdot)\|_{C^\theta(\bar{\Omega})} \le C_{p,\beta}\|u(t,\cdot) + v(t,\cdot)\|_{X^\beta_{\frac{p}{2}}} \quad \forall\, 0 < t < T_{\max}(u_0, v_0). \tag{4.14}$$

Note that in view of (2.3), we have that

$$\begin{aligned}\|u(t,\cdot) + v(t,\cdot)\|_{X^\beta_{\frac{p}{2}}} \le{}& \|A^\beta_{\frac{p}{2}} e^{-A_{\frac{p}{2}}(t-\tau)}(u(\tau,\cdot;u_0,v_0) + v(\tau,\cdot;u_0,v_0))\|_{L^{p/2}(\Omega)} \\ &+ \int_\tau^t \left\| A^\beta_{\frac{p}{2}} e^{-A_{\frac{p}{2}}(t-s)} \nabla\cdot\left(\frac{\chi_1 u(s,\cdot) + \chi_2 v(s,\cdot)}{w(s,\cdot)}\nabla w(s,\cdot)\right)\right\|_{L^{p/2}(\Omega)} ds \\ &+ \int_\tau^t \|A^\beta_{\frac{p}{2}} e^{-A_{\frac{p}{2}}(t-s)} g(u(s,\cdot), v(s,\cdot))\|_{L^{p/2}(\Omega)} ds \end{aligned} \tag{4.15}$$

for any $0 \le \tau < T_{\max}(u_0, v_0)$ and $\tau < t < T_{\max}(u_0, v_0)$, where $A_{\frac{p}{2}} = -\Delta + \mu I$ is defined as in (2.5) with p being replaced by $\frac{p}{2}$, and

$$g(u,v) = (\mu + a_1)u + (\mu + a_2)v - (b_1 u^2 + b_2 v^2) - (c_1 + c_2)uv.$$

By Lemma 2.2 again, there is $\gamma > 0$ such that

$$\begin{aligned}&\|A^\beta_{\frac{p}{2}} e^{-A_{\frac{p}{2}}(t-\tau)}(u(\tau,\cdot;u_0,v_0) + v(\tau,\cdot;u_0,v_0))\|_{L^{p/2}(\Omega)} \\ &\le C_{p,\beta}(t-\tau)^{-\beta} e^{-\gamma(t-\tau)}\|u(\tau,\cdot;u_0,v_0) + v(\tau,\cdot;u_0,v_0)\|_{L^{p/2}} \quad \forall\, \tau < t < T_{\max}(u_0,v_0). \end{aligned} \tag{4.16}$$

By Lemma 2.3 and Lemma 2.5, for any $\epsilon > 0$, there are $\gamma > 0$, $C_p > 0$ and $C_{p,\beta,\epsilon} > 0$ such that

$$\begin{aligned}&\int_\tau^t \left\| A^\beta_{\frac{p}{2}} e^{-A+\frac{p}{2}(t-s)} \nabla\cdot\left(\frac{\chi_1 u(s,\cdot) + \chi_2 v(s,\cdot)}{w(s,\cdot)}\nabla w(s,\cdot)\right)\right\|_{L^{p/2}(\Omega)} ds \\ &\le C_{p,\beta,\epsilon}\int_\tau^t (1 + (t-s)^{-\beta-\frac{1}{2}-\epsilon}) e^{-\gamma(t-s)} \left\| \frac{\chi_1 u(s,\cdot) + \chi_2 v(s,\cdot)}{w(s,\cdot)} \cdot \nabla w(s,\cdot)\right\|_{L^{p/2}(\Omega)} ds \\ &\le C_{p,\beta,\epsilon}(\chi_1 + \chi_2)\int_\tau^t \frac{(1 + (t-s)^{-\beta-\frac{1}{2}-\epsilon}) e^{-\gamma(t-s)}\, \|u(s,\cdot) + v(s,\cdot)\|_{L^p(\Omega)}\, \|\nabla w(s,\cdot)\|_{L^p(\Omega)}}{\inf\limits_{\tau \le t < \min\{T, T_{\max}\}, x\in\Omega} w(t,x)} ds \\ &\le C_p C_{p,\beta,\epsilon}(\chi_1 + \chi_2)\int_\tau^t \frac{(1 + (t-s)^{-\beta-\frac{1}{2}-\epsilon}) e^{-\gamma(t-s)}\, \|u(s,\cdot) + v(s,\cdot)\|^2_{L^p(\Omega)}}{\inf\limits_{\tau \le t < \min\{T, T_{\max}\}, x\in\Omega} w(t,x)} ds. \end{aligned} \tag{4.17}$$

By Lemma 2.2 again,

$$\begin{aligned}&\int_\tau^t \|A^\beta_{\frac{p}{2}} e^{-A_{\frac{p}{2}}(t-s)} g(u(s,\cdot), v(s,\cdot))\|_{L^{p/2}(\Omega)} ds \\ &\le C_{p,\beta}\int_\tau^t (t-s)^{-\beta} e^{-\gamma(t-s)}\Big\{\mu|\Omega|^{2/p} + a_{\max}\|u(s,\cdot) + v(s,\cdot)\|_{L^{p/2}} + b_{\max}\|(u(s,\cdot) + v(s,\cdot))^2\|_{L^{p/2}} \\ &\qquad + \frac{c_{\max}}{2}\|(u(s,\cdot) + v(s,\cdot))^2\|_{L^{p/2}}\Big\} ds. \end{aligned} \tag{4.18}$$

(1.16) then follows from (4.14) to (4.18). □

5 Proofs of Theorems 1.3 and 1.5

In this section, we study the global existence and uniform boundedness and uniform pointwise persistence of classical solutions of (1.2), and prove Theorems 1.3 and 1.5.

Throughout this section, for given u_0, v_0 satisfying (1.8), we put

$$(u(t,x), v(t,x), w(t,x)) := (u(t,x;u_0,v_0), v(t,x;u_0,v_0), w(t,x;u_0,v_0)),$$

and if no confusion occurs, we may drop (t,x) in $u(t,x)$ (resp. $v(t,x)$, $w(t,x)$).

5.1 Global existence and proof of Theorem 1.3

In this subsection, we study the global existence of classical solutions of (1.2) and prove Theorem 1.3.

Proof of Theorem 1.3. We prove that $T_{\max} = \infty$ by contradiction. Assume that $T_{\max} < \infty$. Then by Lemma 2.4 and Theorem 1.1, there is $\delta > 0$ such that

$$w(t,x) \geq \delta \qquad \text{for all } x \in \Omega \quad \text{and } t \in (0, T_{\max}), \tag{5.1}$$

and then by Proposition 1.1, we have

$$\limsup_{t \nearrow T_{\max}} \|u(t,\cdot) + v(t,\cdot)\|_{C^0(\bar{\Omega})} = \infty.$$

But, by (4.13), (5.1), and Theorem 1.2(2), we have

$$\lim_{t \nearrow T_{\max}} \|u(t,\cdot) + v(t,\cdot)\|_{C^0(\bar{\Omega})} < \infty,$$

which is a contradiction. Therefore $T_{\max} = \infty$.

□

5.2 Proof of Theorem 1.5

In this subsection, we investigate the ultimate upper bounds of $\int_\Omega (u+v)^{-q}$, $\int_\Omega (u+v)^p$, and $\|u+v\|_\infty$ for globally defined classical solutions of (1.2), ultimate lower bound of $\inf_{x\in\Omega}(u+v)$, and prove Theorem 1.5.

Proof of Theorem 1.5. First, we prove (1.18). By Theorem 1.3, $T_{\max}(u_0,v_0) = \infty$. By the assumption **(H1)**, we have $a_{\min} > \chi^{**}(\mu,\chi_1,\chi_2)$. By **(H2)** and Theorem 1.4(2),

$$\begin{aligned}
&\int_\Omega (u(x,t;u_0,v_0) + v(x,t;u_0,v_0))^{-q} dx \\
&\leq e^{-(a_{\min} - \chi^{**}(\mu,\chi_1,\chi_2))(t-\tau_0)} \int_\Omega (u(\tau_0,x;u_0,v_0) + v(\tau_0,x;u_0,v_0))^{-q} dx + \tilde{C}_{\tau_0,u_0,v_0},
\end{aligned} \tag{5.2}$$

for all $\tau_0 < t < \infty$, where $q = q_{\chi_1,\chi_2} (\leq 1)$, and

$$\tilde{C}_{\tau,u_0,v_0} = \frac{(b_{\max} + c_{\max})|\Omega|}{a_{\min} - \chi^{**}(\mu,\chi_1,\chi_2)} \times \Big(\frac{a_{\max}}{\min\{b_{\min}, c_{\min}\}}\Big)^{1-q}.$$

This together with (5.2) implies that (1.18) holds with $M_1^* = \frac{(b_{\max}+c_{\max})|\Omega|}{a_{\min}-\chi^{**}(\mu,\chi_1,\chi_2)} \times \Big(\frac{a_{\max}}{\min\{b_{\min},c_{\min}\}}\Big)^{1-q}$.

Next, it is clear that (1.19) with $M_2^* = M_1$ follows from Theorem 1.2(1).

Now, we prove (1.20). By (1.11) and (1.17), we have

$$w(t,x;u_0,v_0) \geq \delta_0 \int_\Omega (u+v)(t,x;u_0,v_0) \geq \frac{\delta_0|\Omega|^{\frac{q+1}{q}}}{\Big(\int_\Omega (u+v)^{-q}(t,x;u_0,v_0)\Big)^{\frac{1}{q}}} \tag{5.3}$$

for all $0 < t < \infty$. Note that $0 < \theta < \frac{N}{p^*} = \frac{2N}{2p^*}$. By Theorem 1.2(2), we have

$$\begin{aligned}
&\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{C^\theta(\bar\Omega)} \\
&\leq M_2\Big[(t-\tau)^{-\beta}e^{-\gamma(t-\tau)}\|u(\tau,\cdot;u_0,v_0) + v(\tau,\cdot;u_0,v_0)\|_{L^{2p^*}} \\
&\quad + \frac{\sup\limits_{\tau\leq t<\hat T}\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|^2_{L^{2p^*}}}{\inf\limits_{\tau\leq t<\hat T, x\in\Omega} w(t,x;u_0,v_0)} \\
&\quad + \sup_{\tau\leq t<\hat T}\|u(t,\cdot;u_0,v_0) + v(t,\cdot;u_0,v_0)\|_{L^{2p^*}} + 1\Big]
\end{aligned}$$

for any $0 < \tau < t < \hat T < \infty$. By Theorem 1.2(1) and (5.3), and letting $\hat T - \tau \to \infty$ and $\tau \to \infty$, we obtain (1.20) with $M_3^* = M_1\Big[\frac{M_1^{\frac{1}{p^*}}(M_1^*)^{\frac{1}{q}}}{\delta_0|\Omega|^{\frac{q+1}{q}}} + M_2^{\frac{1}{2p^*}} + 1\Big]$.

(2) We prove it by contradiction. Assume that there is no $M_0^* > 0$ such that (1.21) holds. Then for any $n \in \mathbb{N}$, there are u_n, v_n satisfying **(H2)** such that

$$\liminf_{t\to\infty}\inf_{x\in\Omega}(u(t,x;0,u_n,v_n) + v(t,x;0,u_n,v_n)) \leq m_n := \frac{1}{n}. \tag{5.4}$$

By (1.18), (1.20), and (5.3), there are $T_n > 0$ and $\tilde M_1^*$ such that

$$\int_\Omega u(t,x;u_n,v_n) + v(t,x;u_n,v_n)dx \geq \tilde M_1^* \quad \forall\, t \geq T_n \tag{5.5}$$

and

$$\|u(t,x;u_n,v_n) + v(t,x;u_n,v_n)\|_{C^\theta(\bar\Omega)} \leq 2M_3^* \quad \forall\, t \geq T_n. \tag{5.6}$$

By (5.4), there are $t_n \in \mathbb{R}$ with $t_n \geq T_n + 1$ and $x_n \in \bar\Omega$ such that

$$u(t_n,x_n;u_n,v_n) + v(t_n,x_n;u_n,v_n) \leq \frac{2}{n}. \tag{5.7}$$

Note that

$$u(t_n,x;u_n,v_n) = u(1,x;u(t_n-1,\cdot;u_n,v_n),v(t_n-1,u_n,v_n)) \tag{5.8}$$

and

$$v(t_n,x;0,u_n,v_n) = v(1,x;u(t_n-1,\cdot;u_n,v_n),v(t_n-1,\cdot;u_n,v_n)). \tag{5.9}$$

By (5.6), without loss of generality, we may assume that there are $u_0^*, v_0^*, u_1^*, v_1^* \in C(\bar{\Omega})$ such that

$$\lim_{n\to\infty} u(t_n - 1, x; u_n, v_n) = u_0^*(x), \quad \lim_{n\to\infty} v(t_n - 1, x; u_n, v_n)) = v_0^*(x),$$

and

$$\lim_{n\to\infty} u(t_n, x; u_n, v_n) = u_1^*(x), \quad \lim_{n\to\infty} v(t_n, x; u_n, v_n)) = v_1^*(x).$$

uniformly in $x \in \bar{\Omega}$. Then, by (5.8) and (5.9), we have that

$$u_1^*(x) + v_1^*(x) = u(1, x; u_0^*, v_0^*) + v(1, x; u_0^*, v_0^*). \tag{5.10}$$

By (5.5), (5.6), and the Dominated Convergence Theorem,

$$\int_\Omega u_0^*(x) + v_0^*(x) = \lim_{n\to\infty} \int_\Omega (u(t_n - 1, x; 0, u_n, v_n) + v(t_n - 1, x; 0, u_n, v_n))dx \geq \tilde{M}_1^*.$$

Then by (5.10) and comparison principles for parabolic equations,

$$\inf_{x\in\bar{\Omega}} (u_1^*(x) + v_1^*(x)) > 0.$$

This together with (5.7) implies that

$$\frac{2}{n} \geq \inf_{x\in\Omega} (u(t_n, x; 0, u_n, v_n) + v(t_n, x; 0, u_n, v_n)) \geq \frac{1}{2} \inf_{x\in\Omega} (u_1^*(x) + v_1^*(x)) \quad \forall\, n \gg 1,$$

which is a contradiction. Therefore (2) holds. □

6 Appendix: Proof of Proposition 4.1

Proposition 4.1 can be proved by the similar arguments as those in [25, Proposition 4.2]. For the reader's convenience, we provide a proof of Proposition 4.1 in this section.

Throughout this section, for given u_0, v_0 satisfying (1.8), we put

$$(u(t,x), v(t,x), w(t,x)) := (u(t,x; u_0, v_0), v(t,x; u_0, v_0), w(t,x; u_0, v_0)),$$

and if no confusion occurs, we may drop (t,x) in $u(t,x)$ (resp. $v(t,x)$, $w(t,x)$).

We first present some lemmas.

Lemma 6.1. *Let $p \geq 3$ and $k \geq 2$. Then*

$$(k-1)\int_\Omega \frac{|\nabla w|^{2p}}{w^k} \leq (p-1)\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 + \mu \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}} \tag{6.1}$$

for all $t \in (0, T_{\max}(u_0, v_0))$.

Proof. Multiplying the third equation in (1.2) by $\frac{|\nabla w|^{2p-2}}{w^{k-1}}$ and integrating it over Ω yields that

$$\begin{aligned}(k-1)\int_\Omega \frac{|\nabla w|^{2p}}{w^k} + \nu \int_\Omega \frac{u|\nabla w|^{2p-2}}{w^{k-1}} + \lambda \int_\Omega \frac{v|\nabla w|^{2p-2}}{w^{k-1}} \\ = (p-1)\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 + \mu \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}}\end{aligned}$$

for all $t \in (0, T_{\max}(u_0, v_0))$. This implies (6.1). □

To prove Proposition 4.1, it is then essential to provide proper estimate for the integral $\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}}\nabla w\cdot\nabla|\nabla w|^2$. By Young's inequality, for any $0<\varepsilon<p-\frac{3}{2}$, we have

$$\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}}\nabla w\cdot\nabla|\nabla w|^2 \le \frac{p-\frac{3}{2}-\varepsilon}{p+k-3}\int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}}\big|\nabla|\nabla w|^2\big|^2 + \frac{p+k-3}{4\left(p-\frac{3}{2}-\varepsilon\right)}\int_\Omega \frac{|\nabla w|^{2p}}{w^k}. \tag{6.2}$$

By a direct calculation, we have

$$(p-2)\int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}}\big|\nabla|\nabla w|^2\big|^2 = \int_{\partial\Omega}\frac{|\nabla w|^{2p-4}}{w^{k-2}}\frac{\partial|\nabla w|^2}{\partial\nu} + (k-2)\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}}\nabla w\cdot\nabla|\nabla w|^2 - \int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}}\Delta|\nabla w|^2. \tag{6.3}$$

Note that

$$\Delta|\nabla w|^2 = 2\nabla w\cdot\nabla(\Delta w) + 2|D^2 w|^2 = 2\nabla w\cdot\nabla(\mu w-\nu u-\lambda v) + 2|D^2 w|^2,$$

and

$$\big|\nabla|\nabla w|^2\big|^2 = 2\sum_{i=1}^n |\nabla w\cdot\nabla w_{x_i}|^2 \le 4|\nabla w|^2\sum_{i=1}^n|\nabla w_{x_i}|^2 = 4|\nabla w|^2|D^2 w|^2.$$

It then follows that

$$-\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}}\Delta|\nabla w|^2 \le 2\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}}\nabla(\nu u+\lambda v)\cdot\nabla w - 2\mu\int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}} - \frac{1}{2}\int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}}\big|\nabla|\nabla w|^2\big|^2. \tag{6.4}$$

By (6.3) and (6.4), we have

$$\left(p-\frac{3}{2}\right)\int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}}\big|\nabla|\nabla w|^2\big|^2 \le \int_{\partial\Omega}\frac{|\nabla w|^{2p-4}}{w^{k-2}}\frac{\partial|\nabla w|^2}{\partial\nu} + (k-2)\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}}\nabla w\cdot\nabla|\nabla w|^2 + 2\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}}\nabla(\nu u+\lambda v)\cdot\nabla w - 2\mu\int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}}. \tag{6.5}$$

To get proper estimate for $\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}}\nabla w\cdot\nabla|\nabla w|^2$, it is then essential to provide proper estimates for the integrals $\int_{\partial\Omega}\frac{|\nabla w|^{2p-4}}{w^{k-2}}\frac{\partial|\nabla w|^2}{\partial\nu}$ and $\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}}\nabla(\nu u+\lambda v)\cdot\nabla w$.

Lemma 6.2. *For every $\epsilon>0$, there is $C>0$ such that*

$$\int_{\partial\Omega}\frac{|\nabla w|^{2p-4}}{w^{k-2}}\frac{\partial|\nabla w|^2}{\partial\nu} \le \epsilon\int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}}\big|\nabla|\nabla w|^2\big|^2 + C\int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}},$$

for all $t\in(0,T_{\max}(u_0,v_0))$.

Proof. It follows from the similar arguments in [25, Lemma 4.4]. □

Lemma 6.3. *Let $p \geq 3$ and $k \geq p$. For every given $\varepsilon > 0$, there is $M > 0$ such that*

$$\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}} \nabla(\nu u + \lambda v) \cdot \nabla w \leq M \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + \varepsilon \int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}} \big|\nabla |\nabla w|^2\big|^2 + \varepsilon \int_\Omega \frac{|\nabla w|^{2p}}{w^k}$$

for all $t \in (0, T_{\max}(u_0, v_0))$.

Proof. First, we have that

$$\int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-2}} \nabla(\nu u + \lambda v) \cdot \nabla w = \underbrace{(k-2) \int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-2}}{w^{k-1}}}_{I_1} - \underbrace{\int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-4}}{w^{k-2}} \Delta w}_{I_2}$$
$$\underbrace{- (p-2) \int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-6}}{w^{k-2}} \nabla w \cdot \nabla |\nabla w|^2}_{I_3}.$$

Next, by Young's inequality, for every $B_1 > 0$, there exists a positive constant $A_1 = A_1(k, p, B_1) > 0$ such that

$$\begin{aligned} I_1 = (k-2) \int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-2}}{w^{k-1}} &= (k-2) \int_\Omega \frac{\nu u + \lambda v}{w^{\frac{k-p}{p}}} \cdot \frac{|\nabla w|^{2p-2}}{w^{\frac{k(p-1)}{p}}} \\ &\leq A_1 \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + B_1 \int_\Omega \frac{|\nabla w|^{2p}}{w^k}. \end{aligned} \tag{6.6}$$

Now, by the fact that $\Delta w = \mu w - \nu u - \lambda v$, and Young's inequality, for every $B_2 > 0$, there exists a positive constant $A_2 = A_2(k, p, \nu, \lambda, B_2) > 0$ such that

$$\begin{aligned} -I_2 &= - \int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-4}}{w^{k-2}} \Delta w \\ &= -\mu \int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-4}}{w^{k-3}} + \int_\Omega \frac{(\nu u + \lambda v)^2 |\nabla w|^{2p-4}}{w^{k-2}} \\ &\leq \int_\Omega \frac{(\nu u + \lambda v)^2}{w^{\frac{2(k-p)}{p}}} \cdot \frac{|\nabla w|^{2p-4}}{w^{\frac{k(p-2)}{p}}} \\ &\leq A_2 \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + B_2 \int_\Omega \frac{|\nabla w|^{2p}}{w^k}. \end{aligned} \tag{6.7}$$

Finally, for every $\varepsilon > 0$ and $B_3 > 0$, there are positive constants $A_3 = A_3(\varepsilon, k, p, \nu, B_3) > 0$ and $A_4 = A_4(A_4, B_3)$ such that

$$\begin{aligned} -I_3 &= -(p-2) \int_\Omega \frac{(\nu u + \lambda v)|\nabla w|^{2p-6}}{w^{k-2}} \nabla w \cdot \nabla |\nabla w|^2 \\ &\leq A_3 \int_\Omega \frac{(\nu u + \lambda v)^2 |\nabla w|^{2p-4}}{w^{k-2}} + \varepsilon \int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}} \big|\nabla |\nabla w|^2\big|^2 \\ &\leq A_4 \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + B_3 \int_\Omega \frac{|\nabla w|^{2p}}{w^k} + \varepsilon \int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}} \big|\nabla |\nabla w|^2\big|^2. \end{aligned} \tag{6.8}$$

Combining (6.6), (6.7) and (6.8) with $B_1 = B_2 = B_3 = \frac{1}{3}\varepsilon$ and $M = M(\varepsilon, k, p, \nu) := A_1 + A_2 + A_4$ completes the proof. □

We now prove Proposition 4.1.

Proof of Proposition 4.1. First, by (6.5) and Lemmas 6.2 and 6.3, for any $0 < \varepsilon < p - \frac{3}{2}$, there are $M, C > 0$ such that

$$\begin{aligned}\left(p - \frac{3}{2}\right) \int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}} \left|\nabla |\nabla w|^2\right|^2 \leq \varepsilon \int_\Omega \frac{|\nabla w|^{2p-6}}{w^{k-2}} \left|\nabla |\nabla w|^2\right|^2 + \varepsilon \int_\Omega \frac{|\nabla w|^{2p}}{w^k} \\ + (k-2) \int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 \\ + M \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + C \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}}. \end{aligned} \tag{6.9}$$

Next, by (6.2) and (6.9), we have

$$\begin{aligned} &\frac{p-1}{p+k-3} \int_\Omega \frac{|\nabla w|^{2p-4}}{w^{k-1}} \nabla w \cdot \nabla |\nabla w|^2 \\ &\leq \frac{\varepsilon}{p+k-3} \int_\Omega \frac{|\nabla w|^{2p}}{w^k} + \frac{M}{p+k-3} \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} \\ &+ \frac{C}{p+k-3} \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}} + \frac{p+k-3}{4\left(p - \frac{3}{2} - \varepsilon\right)} \int_\Omega \frac{|\nabla w|^{2p}}{w^k}. \end{aligned} \tag{6.10}$$

Now, by (6.1), and (6.10), for any $0 < \varepsilon < p - \frac{3}{2}$, there holds

$$\begin{aligned} &\left(k - 1 - \frac{(p+k-3)^2}{4(p - \frac{3}{2} - \varepsilon)} - \varepsilon\right) \int_\Omega \frac{|\nabla w|^{2p}}{w^k} \\ &\leq M \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + (C + \mu) \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}}. \end{aligned} \tag{6.11}$$

By Young's inequality, for any $0 < \varepsilon < p - \frac{3}{2}$, there is $\tilde{M}(\varepsilon, p)$ such that

$$(C + \mu) \int_\Omega \frac{|\nabla w|^{2p-2}}{w^{k-2}} \leq \varepsilon \int_\Omega \frac{|\nabla w|^{2p}}{w^k} + \tilde{M} \int_\Omega w^{2p-k}. \tag{6.12}$$

By (6.11) and (6.12),

$$\begin{aligned} &\left(k - 1 - \frac{(p+k-3)^2}{4(p - \frac{3}{2} - \varepsilon)} - 2\varepsilon\right) \int_\Omega \frac{|\nabla w|^{2p}}{w^k} \\ &\leq M \int_\Omega \frac{(\nu u + \lambda v)^p}{w^{k-p}} + \tilde{M} \int_\Omega w^{2p-k}. \end{aligned} \tag{6.13}$$

Finally, since $p - \sqrt{2p-3} \leq k < p + \sqrt{2p-3}$, one can find $\varepsilon > 0$ such that

$$k - 1 - 2\varepsilon - \frac{(p+k-3)^2}{4\left(p - \frac{3}{2} - \varepsilon\right)} > 0.$$

The proposition then follows from (6.13). □

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