# ON THE HILBERT SPACE DERIVED FROM THE WEIL DISTRIBUTION 

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#### Abstract

We study the Hilbert space obtained by completing the space of all smooth and compactly supported functions on the real line with respect to the hermitian form arising from the Weil distribution under the Riemann hypothesis. It turns out that this Hilbert space is isomorphic to a de Branges space by a composition of the Fourier transform and a simple map. This result is applied to state a new equivalence condition for the Riemann hypothesis in a series of equalities.


## 1. Introduction

The Weil distribution is a distribution associated with the Riemann zeta-function $\zeta(s)$. Let

$$
\xi(s)=\frac{1}{2} s(s-1) \pi^{-s / 2} \Gamma\left(\frac{s}{2}\right) \zeta(s)
$$

be the Riemann xi-function, where $\Gamma(s)$ is the gamma-function. Let $\Gamma$ be the set of all zeros of $\xi(1 / 2-i z)$ without multiplicity and let $m_{\gamma}$ denote the multiplicity of $\gamma \in \Gamma$. The Riemann hypothesis ( RH , for short) claims that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s)=1 / 2$. It is equivalent to the assertion that all $\gamma \in \Gamma$ are real.

The Weil distribution is the linear functional $W: C_{c}^{\infty}(\mathbb{R}) \rightarrow \mathbb{C}$ defined by

$$
C_{c}^{\infty}(\mathbb{R}) \ni \phi \mapsto W(\phi):=\sum_{\gamma \in \Gamma} m_{\gamma} \widehat{\phi}(-\gamma)
$$

where $C_{c}^{\infty}(\mathbb{R})$ is the space of all smooth and compactly supported functions on $\mathbb{R}$ and

$$
\begin{equation*}
\widehat{f}(z):=(\mathrm{F} f)(z):=\int_{-\infty}^{\infty} f(x) e^{i z x} d x \tag{1.1}
\end{equation*}
$$

is the Fourier transform. We omit the description of the topology of $C_{c}^{\infty}(\mathbb{R})$, since we do not need it later. A. Weil [14 (see also the note in [11, Section 3.2]) discovered that the RH is true if and only if the Weil distribution $W$ is non-negative definite, that is,

$$
W(\psi * \widetilde{\psi}) \geq 0 \quad \text { for every } \psi \in C_{c}^{\infty}(\mathbb{R})
$$

where

$$
(\phi * \psi)(x):=\int_{-\infty}^{\infty} \phi(y) \psi(x-y) d y \quad \text { and } \quad \widetilde{\psi}(x):=\overline{\psi(-x)}
$$

Further, if the RH is true, the Weil distribution is positive definite, that is, $W(\psi * \widetilde{\psi})>0$ for every nonzero $\psi \in C_{c}^{\infty}(\mathbb{R})$.

Using the Weil distribution, we define the hermitian form $\langle\cdot, \cdot\rangle_{W}$ on $C_{c}^{\infty}(\mathbb{R})$ by

$$
\begin{equation*}
\left\langle\psi_{1}, \psi_{2}\right\rangle_{W}=W\left(\psi_{1} * \widetilde{\psi_{2}}\right)=\sum_{\gamma \in \Gamma} m_{\gamma} \widehat{\psi_{1}}(-\gamma)\left(\widehat{\psi_{2}}\right)^{\sharp}(-\gamma), \quad \psi_{1}, \psi_{2} \in C_{c}^{\infty}(\mathbb{R}), \tag{1.2}
\end{equation*}
$$

where

$$
F^{\sharp}(z):=\overline{F(\bar{z})}
$$

[^0]for complex-valued functions of a complex variable. We often use this $\sharp$ notation. We call this the Weil hermitian form. H. Yoshida [16] has studied the Weil hermitian form in detail by restricting it to a function space on a finite interval $[-a, a](a>0)$. The subject of the present paper is the behavior of the Weil hermitian form on the whole line $\mathbb{R}$. Yoshida proposed a method to complete a function space on a finite interval with respect to the Weil hermitian form without assuming the RH , but it does not work on the whole line.

Suppose that the RH is true. Then the Weil hermitian form $\langle\cdot, \cdot\rangle_{W}$ is positive definite on $C_{c}^{\infty}(\mathbb{R})$. Therefore, the completion $\mathcal{H}_{W}$ of the pre-Hilbert space $C_{c}^{\infty}(\mathbb{R})$ with respect to $\langle\cdot, \cdot\rangle_{W}$ is defined. The first main result is an explicit description of the Hilbert space $\mathcal{H}_{W}$. The elements of $\mathcal{H}_{W}$ are equivalence classes of Cauchy sequences with respect to $\langle\cdot, \cdot\rangle_{W}$, where two Cauchy sequences are equivalent if their difference converges to zero with respect to $\langle\cdot, \cdot\rangle_{W}$. The representative of each class can be chosen from $L^{2}(\mathbb{R})$ (Theorem 5.1 below). Therefore, we denote the class represented by $\psi \in L^{2}(\mathbb{R})$ as $[\psi]$ and often identify $\psi$ with $[\psi]$.

For the concrete description of $\mathcal{H}_{W}$, we use a de Branges space and a model space. The entire function $E_{\xi}$ defined by

$$
\begin{equation*}
E_{\xi}(z):=\xi(1 / 2-i z)+\xi^{\prime}(1 / 2-i z) \tag{1.3}
\end{equation*}
$$

belongs to the Hermite-Biehler class under the RH ([5, Theorem 1]) and hence it defines the de Branges space $\mathcal{H}\left(E_{\xi}\right)$, where the dash on the right-hand side of (1.3) means differentiation of $\xi(s)$ with respect to $s$. Furthermore, the meromorphic function

$$
\begin{equation*}
\Theta_{\xi}(z):=E_{\xi}^{\sharp}(z) / E_{\xi}(z) \tag{1.4}
\end{equation*}
$$

in $\mathbb{C}$ is a meromorphic inner function in the upper-half plane $\mathbb{C}_{+}=\{z \mid \Im(z)>0\}$ under the RH, and therefore it defines the model space $\mathcal{K}\left(\Theta_{\xi}\right)$. These two Hilbert spaces $\mathcal{H}\left(E_{\xi}\right)$ and $\mathcal{K}\left(\Theta_{\xi}\right)$ are isomorphic with $\left\|E_{\xi} F\right\|_{\mathcal{H}\left(E_{\xi}\right)}=\|F\|_{\mathcal{K}\left(\Theta_{\xi}\right)}$ for every $F \in \mathcal{K}(\Theta)$ (see Section 2 for details on the Hermite-Biehler class, de Branges spaces, and model spaces). Then the first result is stated as follows.

Theorem 1.1. Assume that the $R H$ is true. Let $\mathcal{H}_{W}, \mathcal{H}\left(E_{\xi}\right)$, and $\mathcal{K}\left(\Theta_{\xi}\right)$ be Hilbert spaces as above. Then, the map $\mathcal{K}\left(\Theta_{\xi}\right) \rightarrow \mathcal{H}_{W}$ defined by

$$
F \mapsto\left[\psi_{F}\right], \quad \psi_{F}:=\mathrm{F}^{-1}(F)
$$

is an isomorphism between Hilbert spaces and satisfies

$$
\left\|E_{\xi} F\right\|_{\mathcal{H}\left(E_{\xi}\right)}^{2}=\|F\|_{\mathcal{K}\left(\Theta_{\xi}\right)}^{2}=\pi\left\langle\psi_{F}, \psi_{F}\right\rangle_{W}=\pi\left\langle\left[\psi_{F}\right],\left[\psi_{F}\right]\right\rangle_{W}
$$

for $F \in \mathcal{K}(\Theta)$, where $\mathrm{F}^{-1}$ is the Fourier inversion on $L^{2}(\mathbb{R})$.
This result is proved in Section 5. J. C. Lagarias suggested after Theorem 1 of 5 ] that the norm of the de Branges space $\mathcal{H}\left(E_{\xi}\right)$ and the Weil hermitian form (the spectral side of the "explicit formula" of prime number theory) are similar. Theorem 1.1 shows that they are naturally coincident. Hence, $\mathcal{H}_{W}$ and $\mathcal{H}\left(E_{\xi}\right)$ must have an "arithmetic structure" through the Weil explicit formula (3.3) below, but we will not discuss this further.

One of the remarkable properties of de Branges spaces is the structure of subspaces. The set of all de Branges subspaces of a given de Branges space is totally ordered by set-theoretical inclusion (see [15, pp. 500-506] for details). Such a structure also comes to $\mathcal{H}_{W}$ through the isomorphism of Theorem 1.1 as stated in Theorem 5.3 below.

Another notable property of de Branges spaces is the explicit description of the family of self-adjoint extensions of the multiplication operator by an independent variable $F(z) \mapsto z F(z)$. It enables us to interpret the set of zeros $\Gamma$ as the set of eigenvalues of a self-adjoint operator on $\mathcal{H}_{W}$. This means that one of the Hilbert-Pólya spaces is the

Hilbert space $\mathcal{H}_{W}$ naturally obtained from the Weil distribution. See Sections 2.3 and 6 for details.

As stated in Theorem 1.1, the Hilbert space $\mathcal{H}_{W}$ is isomorphic to a de Branges space under the RH . Moreover, representatives of classes in $\mathcal{H}_{W}$ can be chosen from the concrete subspace $V(0)$ of $L^{2}(\mathbb{R})$. It is surprising that such an explicit description of $\mathcal{H}_{W}$ is possible, and interesting in itself. However, it is a matter of concern that it is not even possible to define $\mathcal{H}_{W}, \mathcal{H}\left(E_{\xi}\right)$, and $\mathcal{K}\left(\Theta_{\xi}\right)$ without assuming the RH . However, by considering a screw line of the screw function attached to $\zeta(s)$, which will be explained in Sections 2.1 and 4.2, we can unconditionally construct two special Hilbert spaces $\mathcal{H}_{0}$ and $\mathcal{K}_{0}$ isomorphic to $\mathcal{H}_{W}$ and $\mathcal{K}\left(\Theta_{\xi}\right)$, respectively, under the RH (Theorem 5.2 below). The construction of such spaces leads to an equivalence condition for the RH stated below. That is the second main result.

Let $L^{2}(\mathbb{R})$ be the usual $L^{2}$-space on the real line with respect to the Lebesgue measure. We define

$$
\begin{equation*}
\mathfrak{S}_{t}(z):=\frac{i\left(1+\Theta_{\xi}(z)\right)}{2} \mathfrak{P}_{t}(z) \tag{1.5}
\end{equation*}
$$

with

$$
\begin{align*}
\mathfrak{P}_{t}(z): & \frac{4\left(e^{t / 2}-1\right)}{1+2 i z}+\frac{4\left(e^{-t / 2}-1\right)}{1-2 i z} \\
& +\frac{e^{-i z t}-1}{i z} \frac{\zeta^{\prime}}{\zeta}\left(\frac{1}{2}-i z\right)+\sum_{n \leq e^{t}} \frac{\Lambda(n)}{\sqrt{n}} \frac{e^{-i z(t-\log n)}-1}{i z}  \tag{1.6}\\
& -\frac{1}{2 i z}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}-\frac{i z}{2}\right)-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)\right] \\
& -\frac{1}{2 i z} e^{-t / 2}\left[\Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right)-\Phi\left(e^{-2 t}, 1, \frac{1}{4}\right)\right]
\end{align*}
$$

for a non-negative real number $t$ and a complex number $z$, where $\Lambda(n)$ is the von Mangoldt function defined by $\Lambda(n)=\log p$ if $n=p^{k}$ with $k \in \mathbb{Z}_{>0}$ and $\Lambda(n)=0$ otherwise, and

$$
\Phi(z, s, a)=\sum_{n=0}^{\infty} \frac{z^{n}}{(n+a)^{s}}
$$

is the Hurwitz-Lerch zeta-function. For negative $t$, we set $\mathfrak{S}_{t}(z):=\mathfrak{S}_{-t}(z)$. For this $\mathfrak{S}_{t}$, we first obtain the following.

Proposition 1.1. For any fixed $t \in \mathbb{R}, \mathfrak{S}_{t}(z)$ belongs to $L^{2}(\mathbb{R})$ as a function of $z$.
Proof. See Section 3.2.
From this result, the mapping $t \mapsto \mathfrak{S}_{t}(z)$ from $\mathbb{R}$ to $L^{2}(\mathbb{R})$ is defined. By the uniformity of the $L^{2}$-norm of $\mathfrak{S}_{t}(z)$ on a compact set of $t$ obtained in the proof of Proposition 1.1 and Minkowski's integral inequality, the following holds.

Proposition 1.2. For $\phi \in C_{c}^{\infty}(\mathbb{R})$, we define

$$
\begin{equation*}
\widehat{\mathcal{P}_{\phi}}(z):=\int_{-\infty}^{\infty} \mathfrak{S}_{t}^{\sharp}(z) \phi(t) d t \quad\left(=\int_{-\infty}^{\infty} \overline{\mathfrak{S}_{t}(\bar{z})} \phi(t) d t\right) \tag{1.7}
\end{equation*}
$$

using (1.5). Then $\widehat{\mathcal{P}_{\phi}}(z)$ belongs to $L^{2}(\mathbb{R})$.
Using the image of the composition $\widehat{\mathcal{P}_{D}}:=\widehat{\mathcal{P}} \circ D$ of the integral operator $\widehat{\mathcal{P}}$ and the differential operator

$$
\begin{equation*}
(D \psi)(t):=i \psi^{\prime}(t) \tag{1.8}
\end{equation*}
$$

we obtain the following equivalence condition for the RH .

Theorem 1.2. The $R H$ is true if and only if the equality

$$
\begin{equation*}
\left\|\widehat{\mathcal{P}_{D \psi}}\right\|_{L^{2}(\mathbb{R})}^{2}=\pi\langle\psi, \psi\rangle_{W} \tag{1.9}
\end{equation*}
$$

holds for all $\psi \in C_{c}^{\infty}(\mathbb{R})$.
Proof. See Section 4.3,
Equation (1.9) is reformulated to the following simpler form.
Corollary 1.1. Define the subspace $V^{\circ}(0)$ of $L^{2}(\mathbb{R})$ by

$$
V^{\circ}(0):=\left\{\mathrm{F}^{-1} \widehat{\mathcal{P}_{D \psi}} \mid \psi \in C_{c}^{\infty}(\mathbb{R})\right\} .
$$

Then the RH is true if and only if the equality

$$
\begin{equation*}
2\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\langle\psi, \psi\rangle_{W} \tag{1.10}
\end{equation*}
$$

holds for all $\psi \in V^{\circ}(0)$.
Proof. See Section 4.3 and Theorem 5.2 ,
The advantage of Theorem 1.2 and Corollary 1.1 is that it has turned the criterion of the RH from a set of inequalities like Weil's criterion into a set of equalities. It should also be noted that equations (1.9) and (1.10) can be expressed without zeros of $\xi(1 / 2-i z)$ by (1.5) and (1.6). Furthermore, equations (1.9) and (1.10) claim that the non-negativity of Weil's hermitian form is explained by the non-negativity of the $L^{2}$-norm.

In the following sections, first, in Section 2, we briefly review necessary notions such as screw functions, screw lines, the Hermite-Biehler class, de Branges spaces, and model spaces. Then, in Section 3, we state and prove unconditional results that we need to prove the main results. Moreover, we unconditionally define two Hilbert spaces $\mathcal{H}_{0}$ and $\mathcal{K}_{0}$ that agree with the Hilbert spaces $\mathcal{H}_{W}$ and $\mathcal{K}(\Theta)$, respectively, under the RH .

In Section 4, we show that $\mathfrak{S}_{t}(z)$ in (1.5) gives a screw line of the screw function corresponding to the Riemann zeta-function under the RH (Theorem4.1). Furthermore, we prove Theorem 1.2 and Corollary [1.1. The strategy of the proof of Theorem 4.1 is similar to [12], however the computational details change. In [12], the analytic or geometric meaning of the functions that give the norms is unknown, but in this paper the functions that give the norms have the meaning as a screw line. Furthermore, as an advantage of using the screw line $\mathfrak{S}_{t}$, we obtain Theorem 1.2, whose analogue was not obtained in [12].

In Section 5, we prove Theorem 1.1 in a more detailed form. In addition, we prove that $\mathcal{H}_{0}=\mathcal{H}_{W}$ and $\mathcal{K}_{0}=\mathcal{K}(\Theta)$ under the RH . Afterwards, we explain that the Hilbert space $\mathcal{H}_{W}$ is one of the Hilbert-Pólya spaces in Section 6. Finally, we mention two special values of $\mathfrak{S}_{t}(z)$ in Section 7 as an appendix.

## 2. REVIEW ON NECESSARY NOTIONS

2.1. Screw functions and screw lines. In this and the next part, we refer to 4, Sections 5 and 12]. See also its references for details. Following M. G. Kreĭn, we denote by $\mathcal{G}_{\infty}$ the space of all continuous functions $g(t)$ on $\mathbb{R}$ such that $g(-t)=\overline{g(t)}$ and the kernel

$$
\begin{equation*}
G_{g}(t, u):=g(t-u)-g(t)-g(-u)+g(0) \tag{2.1}
\end{equation*}
$$

is non-negative definite on $\mathbb{R}$, that is, $\sum_{i, j=1}^{n} G_{g}\left(t_{i}, t_{j}\right) \xi_{i} \overline{\xi_{j}} \geq 0$ for all $n \in \mathbb{N}, t_{i} \in \mathbb{R}$, and $\xi_{i} \in \mathbb{C}(i=1,2, \ldots, n)$. Functions belonging to $\mathcal{G}_{\infty}$ are called screw functions on $\mathbb{R}$.

If an (even) real-valued function $g(t)$ is a screw function, there exists a Hilbert space $\mathcal{H}$ and a continuous mapping $t \mapsto x(t)$ from $\mathbb{R}$ into $\mathcal{H}$ such that

$$
\langle x(t+v)-x(v), x(u+v)-x(v)\rangle_{\mathcal{H}}
$$

is independent of $v \in \mathbb{R}$ for all $t, u \in \mathbb{R}$ and the equality $\langle x(t)-x(0), x(u)-x(0)\rangle_{\mathcal{H}}=$ $G_{g}(t, u)$ holds. Therefore, $\|x(t)-x(0)\|_{\mathcal{H}}^{2}=-2 g(t)$ if $g(0)=0$. A mapping $x: \mathbb{R} \rightarrow \mathcal{H}$ endowed with the translation-invariance described above is called a screw line for $g(t)$.
2.2. Hilbert spaces associated with screw functions. Each $g \in \mathcal{G}_{\infty}$ defines a nonnegative definite hermitian form on $\mathbb{R}$ by

$$
\begin{equation*}
\left\langle\phi_{1}, \phi_{2}\right\rangle_{G_{g}}:=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{g}(t, u) \phi_{1}(u) \overline{\phi_{2}(t)} d u d t \tag{2.2}
\end{equation*}
$$

According to [4, Section 5], we denote by $\mathcal{L}\left(G_{g}\right)$ the space $C_{0}(\mathbb{R})$ of all continuous and compactly supported functions $\phi$ on $\mathbb{R}$ such that $\widehat{\phi}(0)=0$ equipped with the hermitian inner product $\langle\cdot, \cdot\rangle_{G_{g}}$. We also denote by $\mathcal{H}\left(G_{g}\right)$ the completion of the factor space $\mathcal{L}\left(G_{g}\right) / \mathcal{L}^{\circ}\left(G_{g}\right)$, where $\mathcal{L}^{\circ}\left(G_{g}\right)=\left\{\phi \in \mathcal{L}\left(G_{g}\right) \mid\langle\phi, \phi\rangle_{G_{g}}=0\right\}$. Note that even if $\langle\cdot, \cdot\rangle_{G_{g}}$ is positive definite on $\mathcal{L}\left(G_{g}\right)$, that is, $\mathcal{L}^{\circ}\left(G_{g}\right)=\{0\}$, there possibly exists a sequence $\left(\phi_{n}\right)_{n}$ of $\mathcal{L}\left(G_{g}\right)$ such that $\phi_{n} \rightarrow 0$ as $n \rightarrow \infty$ with respect to $\langle\cdot, \cdot\rangle_{G_{g}}$. The completion $\mathcal{H}\left(G_{g}\right)$ is a space of equivalence classes of Cauchy sequences with respect to $\langle\cdot, \cdot\rangle_{G_{g}}$. Two Cauchy sequences are equivalent if their difference converges to zero with respect to $\langle\cdot, \cdot\rangle_{G_{g}}$. We denote by $[\phi] \in \mathcal{H}\left(G_{g}\right)$ the equivalence class represented by $\phi$. In general, elements of $\mathcal{H}\left(G_{g}\right)$ are not necessarily represented by functions unlike $\mathcal{H}_{W}$ (cf. [4, Section 4.3]).

Every $g \in \mathcal{G}_{\infty}$ admits a representation

$$
\begin{equation*}
g(t)=g(0)+i b t+\int_{-\infty}^{\infty}\left(e^{i \lambda t}-1-\frac{i \lambda t}{1+\lambda^{2}}\right) \frac{d \tau(\lambda)}{\lambda^{2}} \tag{2.3}
\end{equation*}
$$

with $b \in \mathbb{R}$ and a measure $\tau$ on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} d \tau(\lambda) /\left(1+\lambda^{2}\right)<\infty$ and vice versa. If $g(t)$ is real-valued, $b=0$. Without loss of generality, we suppose that $g(0)=0$.

We define

$$
\Phi_{1}(\phi, \lambda):=\int_{-\infty}^{\infty} \frac{e^{i \lambda x}-1}{\lambda} \phi(x) d x=\frac{\widehat{\phi}(\lambda)-\widehat{\phi}(0)}{\lambda}=\frac{\widehat{\phi}(\lambda)}{\lambda}
$$

for $\phi \in \mathcal{L}\left(G_{g}\right)$. Then, $\left\langle\phi_{1}, \phi_{2}\right\rangle_{G_{g}}=\left\langle\Phi_{1}\left(\phi_{1}\right), \Phi_{1}\left(\phi_{2}\right)\right\rangle_{L^{2}(\tau)}$ for $\phi_{1}, \phi_{2} \in \mathcal{L}\left(G_{g}\right)$ and $\Phi_{1}$ establishes an isomorphism between $\mathcal{H}\left(G_{g}\right)$ and $L^{2}(\tau)$.
2.3. De Branges spaces. In this part, we refer to 9, 15. See also those references for details. Let $H^{2}:=H^{2}\left(\mathbb{C}_{+}\right)=\mathrm{F}\left(L^{2}(0, \infty)\right)$ be the Hardy space in the upper half-plane. As usual, we identify $H^{2}$ with a closed subspace of $L^{2}(\mathbb{R})$ via boundary values. Then, the inner product of $H^{2}$ coincides with the standard inner product of $L^{2}(\mathbb{R})$.

The Hermite-Biehler class consists of entire functions $E$ satisfying $\left|E^{\sharp}(z)\right|<|E(z)|$ for all $z \in \mathbb{C}_{+}$. For each entire function $E$ belonging to the Hermite-Biehler class, the de Branges space $\mathcal{H}(E)$ is defined as a Hilbert space consisting of entire functions $F(z)$ such that both $F(z) / E(z)$ and $F^{\sharp}(z) / E(z)$ belong to $H^{2}$ and have the norm

$$
\begin{equation*}
\|F\|_{\mathcal{H}(E)}:=\|F / E\|_{L^{2}(\mathbb{R})} \tag{2.4}
\end{equation*}
$$

The multiplication operator M by an independent variable is defined by $\mathfrak{D}(\mathrm{M})=$ $\{F(z) \in \mathcal{H}(E) \mid z F(z) \in \mathcal{H}(E)\}$ and $(\mathrm{M} F)(z)=z F(z)$ for $F \in \mathfrak{D}(\mathrm{M})$. The domain $\mathfrak{D}(\mathrm{M})$ is dense in $\mathcal{H}(E)$ if and only if

$$
S_{\theta}(z):=\frac{i}{2}\left(e^{i \theta} E(z)-e^{-i \theta} E^{\sharp}(z)\right)
$$

does not belong to $\mathcal{H}(E)$ for all $\theta \in[0, \pi)([9$, Theorem 11]). The particular two $\theta$ cases are often written as $A(z):=-S_{\pi / 2}(z)$ and $B(z):=S_{0}(z)$.

If $\mathfrak{D}(\mathrm{M})$ is dense in $\mathcal{H}(E)$, all self-adjoint extensions of M are parametrized by $\theta \in[0, \pi)$ and are described as follows. The domain of $M_{\theta}$ is

$$
\mathfrak{D}\left(\mathrm{M}_{\theta}\right)=\left\{\left.G(z)=\frac{S_{\theta}\left(w_{0}\right) F(z)-S_{\theta}(z) F\left(w_{0}\right)}{z-w_{0}} \right\rvert\, F(z) \in \mathcal{H}(E)\right\},
$$

and the operation is defined by

$$
\mathrm{M}_{\theta} G(z)=z G(z)+F\left(w_{0}\right) S_{\theta}(z),
$$

where $w_{0}$ is a fixed complex number with $S_{\theta}\left(w_{0}\right) \neq 0$ ( 9 , Theorem 12]). The domain $\mathfrak{D}\left(\mathrm{M}_{\theta}\right)$ is independent of the choice of the number $w_{0}$. For a fixed $\theta \in[0, \pi)$, we confirm that $G(z)=S_{\theta}(z) /(z-\gamma)$ belongs to $\mathfrak{D}\left(\mathrm{M}_{\theta}\right)$ by taking

$$
F(z)=\frac{S_{\theta}(z)}{S_{\theta}\left(w_{0}\right)} \frac{\gamma-w_{0}}{z-\gamma}
$$

for every zero $\gamma$ of $S_{\theta}(z)$ and is an eigenfunction of $\mathrm{M}_{\theta}$ with the eigenvalue $\gamma$. Further, $\left\{S_{\theta}(z) /(z-\gamma) \mid S_{\theta}(\gamma)=0\right\}$ forms an orthogonal basis of $\mathcal{H}(E)$ ([9, Theorem 8]).
2.4. Model subspaces. In this part, we refer to [6, Section 2], [10, Section 3.5] and [12, Section 3.1]. See also those references for details.
Let $H^{\infty}=H^{\infty}\left(\mathbb{C}_{+}\right)$be the space of all bounded analytic functions in $\mathbb{C}_{+}$. A function $\Theta \in H^{\infty}$ is called an inner function in $\mathbb{C}_{+}$if $\lim _{y \rightarrow 0+}|\Theta(x+i y)|=1$ for almost all $x \in \mathbb{R}$. For an inner function $\Theta$, a model space $\mathcal{K}(\Theta)$ is defined as the orthogonal complement $\mathcal{K}(\Theta)=H^{2} \ominus \Theta H^{2}$ and has the alternative representation

$$
\begin{equation*}
\mathcal{K}(\Theta)=H^{2} \cap \Theta \bar{H}^{2} \tag{2.5}
\end{equation*}
$$

where $\Theta H^{2}=\left\{\Theta(z) F(z) \mid F \in H^{2}\right\}$ and $\bar{H}^{2}=H^{2}\left(\mathbb{C}_{-}\right)$is the Hardy space in the lower half-plane. The model space $\mathcal{K}(\Theta)$ is a subspace of $L^{2}(\mathbb{R})$ as a Hilbert space. In particular, the inner product of $\mathcal{K}(\Theta)$ matches that of $L^{2}(\mathbb{R})$ on the real line.

If an inner function $\Theta$ in $\mathbb{C}_{+}$extends to a meromorphic function in $\mathbb{C}$, it is called a meromorphic inner function in $\mathbb{C}_{+}$. For any meromorphic inner function $\Theta$, there exists $E$ of the Hermite-Biehler class such that $\Theta=E^{\sharp} / E$. The de Branges space $\mathcal{H}(E)$ is isometrically isomorphic to $\mathcal{K}(\Theta)$ by $F(z) \mapsto E(z) F(z)$. In particular, $\mathcal{H}(E)=$ $E H^{2} \cap E^{\sharp} \bar{H}^{2}$

For a meromorphic inner function $\Theta$, let $\mu_{\Theta}$ be the positive discrete measure on $\mathbb{R}$ supported on $\sigma(\Theta)=\{x \in \mathbb{R} \mid \Theta(x)=-1\}$ and

$$
\begin{equation*}
\mu_{\Theta}(x)=\frac{2 \pi}{\left|\Theta^{\prime}(x)\right|} \tag{2.6}
\end{equation*}
$$

Then the restriction map $\left.F \mapsto F\right|_{\sigma(\Theta)}$ is an isometric operator from $\mathcal{K}(\Theta)$ to $L^{2}\left(\mu_{\Theta}\right)$ ( 6 , Theorem 2.1]). The isometric property of the map implies that the family of functions

$$
\begin{equation*}
f_{\gamma}(z)=\sqrt{\frac{2}{\pi\left|\Theta^{\prime}(\gamma)\right|}} \frac{1+\Theta(z)}{2(z-\gamma)}=\sqrt{\frac{2}{\pi\left|\Theta^{\prime}(\gamma)\right|}} \frac{A(z)}{(z-\gamma) E(z)} \tag{2.7}
\end{equation*}
$$

parametrized by all zeros $\gamma$ of $A(z)=-S_{\pi / 2}(z)$ forms an orthonormal basis of $\mathcal{K}(\Theta)$ if $\mathfrak{D}(\mathrm{M})$ is dense in $\mathcal{H}(E)$.

## 3. Unconditional Results

Throughout this and later sections, we denote $E=E_{\xi}$ and $\Theta=\Theta_{\xi}=E_{\xi}^{\sharp} / E_{\xi}$ for functions defined in (1.3) and (1.4), respectively. Otherwise, it is mentioned.
3.1. Expansion of $\mathfrak{P}_{t}(z)$ over the zeros. For the basic properties of the Riemann zeta-function, we refer to [13]. By two functional equations $\xi(s)=\xi(1-s)$ and $\xi(s)=$ $\xi^{\sharp}(s)$, if $\gamma$ belongs to the set of zeros $\Gamma$, then both $-\gamma$ and $\bar{\gamma}$ also belong to $\Gamma$ with the same multiplicity. On the other hand, $|\Im(\gamma)|<1 / 2$ for every $\gamma \in \Gamma$, since all zeros of $\xi(s)$ lie in the strip $0<\Re(s)<1$. For $E(z)$ of (1.3), we define

$$
\begin{equation*}
A(z):=\left(E(z)+E^{\sharp}(z)\right) / 2 \tag{3.1}
\end{equation*}
$$

as in Section [2.3. Then $A(z)=\xi(1 / 2-i z)$, because $E^{\sharp}(z)=\overline{E(\bar{z})}=\xi(1 / 2-i z)-$ $\xi^{\prime}(1 / 2-i z)$ by functional equations of $\xi(s)$. Therefore, the set $\Gamma$ coincides with the set of all zeros of both $A(z)$ and $1+\Theta(z)$. We define

$$
\begin{equation*}
P_{t}(z):=\sum_{\gamma \in \Gamma} m_{\gamma} \frac{e^{-i \gamma t}-1}{\gamma} \cdot \frac{1}{z-\gamma} \tag{3.2}
\end{equation*}
$$

for non-negative $t$. For negative $t$, we set $P_{t}(z):=P_{-t}(z)$. The series on the right-hand side of (3.2) converges absolutely and uniformly on every compact subset of $\mathbb{C} \backslash \Gamma$, since $\sum_{\gamma \in \Gamma} m_{\gamma}|\gamma|^{-1-\delta}<\infty$ for any $\delta>0$, because $A(z)$ is an entire function of order one. Therefore, $P_{t}(z)$ is a meromorphic function on $\mathbb{C}$ with $\Gamma$ as the set of all poles.

Proposition 3.1. Let $\mathfrak{P}_{t}(z)$ and $P_{t}(z)$ be meromorphic functions defined by (1.6) and (3.2), respectively. Then, both coincide.

Proof. For $t \geq 0$ and $z \in \mathbb{C}_{+}$, we define

$$
\phi_{z, t}(x)= \begin{cases}(i z)^{-1} e^{i z x}\left(e^{-i z t}-1\right), & t<x \\ (i z)^{-1} e^{i z x}\left(e^{-i z x}-1\right), & 0 \leq x \leq t\end{cases}
$$

The main tool for the proof is the Weil explicit formula

$$
\begin{align*}
\lim _{X \rightarrow \infty} & \sum_{\substack{\gamma \in \Gamma \\
|\gamma| \leq X}} m_{\gamma} \int_{-\infty}^{\infty} \phi(x) e^{-i \gamma x} d x \\
= & \int_{-\infty}^{\infty} \phi(x)\left(e^{x / 2}+e^{-x / 2}\right) d x-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(\log n)-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(-\log n)  \tag{3.3}\\
& -\left(\log 4 \pi+\gamma_{0}\right) \phi(0)-\int_{0}^{\infty}\left\{\phi(x)+\phi(-x)-2 e^{-x / 2} \phi(0)\right\} \frac{e^{x / 2} d x}{e^{x}-e^{-x}}
\end{align*}
$$

which is obtained from the explicit formula in [1, p. 186] by taking $\phi(x)=e^{x / 2} f\left(e^{x}\right)$ for test functions $f(t)$ in that formula with the conditions for $f(t)$ in [2, Section 3], where $\gamma_{0}$ is the Euler-Mascheroni constant. (Note that the formula in [2] has two typographical errors in the second line of the right-hand side.)

As is easily seen, Weil's explicit formula can be applied to $\phi(x)=\phi_{z, t}(x)$. We have

$$
\int_{-\infty}^{\infty} \phi_{z, t}(x) e^{-i \gamma x} d x=\frac{e^{-i \gamma t}-1}{\gamma} \cdot \frac{1}{z-\gamma} \quad \text { when } \Im(z)>\Im(\gamma) \text {. }
$$

Therefore, the left-hand side of Weil's explicit formula for $\phi_{z, t}(x)$ gives $P_{t}(z)$ of (3.2) when $\Im(z)>1 / 2$. Hence, if it is shown that the right-hand side is equal to $\mathfrak{P}_{t}(z)$ for $\Im(z)>1 / 2$, then the conclusion of the proposition follows by analytic continuation.

It is easy to verify

$$
\int_{-\infty}^{\infty} \phi_{z, t}(x)\left(e^{x / 2}+e^{-x / 2}\right) d x=\frac{4\left(e^{t / 2}-1\right)}{1+2 i z}+\frac{4\left(e^{-t / 2}-1\right)}{1-2 i z}
$$

and

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi_{z, t}(\log n) & =\frac{1}{i z} \sum_{n \leq e^{t}} \frac{\Lambda(n)}{\sqrt{n}}\left(1-n^{i z}\right)+\frac{e^{-i z t}-1}{i z} \sum_{t<\log n} \frac{\Lambda(n)}{n^{1 / 2-i z}} \\
& =-\sum_{n \leq e^{t}} \frac{\Lambda(n)}{\sqrt{n}} \frac{e^{-i z(t-\log n)}-1}{i z}-\frac{e^{-i z t}-1}{i z} \frac{\zeta^{\prime}}{\zeta}\left(\frac{1}{2}-i z\right), \\
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi_{z, t}(-\log n) & =0, \quad \phi_{z, t}(0)=0
\end{aligned}
$$

for $\Im(z)>1 / 2$ by direct calculation.

Therefore, the remaining task is to calculate the fifth term on the right-hand side. We split it into $\int_{t}^{\infty}$ and $\int_{0}^{t}$. For the first integral,

$$
\begin{aligned}
\int_{t}^{\infty} & \left\{\phi_{z, t}(x)+\phi_{z, t}(-x)-2 e^{-x / 2} \phi_{z, t}(0)\right\} \frac{e^{x / 2} d x}{e^{x}-e^{-x}} \\
& =\frac{e^{-i z t}-1}{i z} \int_{t}^{\infty} e^{i z x} \frac{e^{x / 2} d x}{e^{x}-e^{-x}}=\frac{e^{-i z t}-1}{i z} \int_{t}^{\infty} e^{i z x} e^{-x / 2} \sum_{n=0}^{\infty} e^{-2 n x} d x \\
& =\frac{e^{-i z t}-1}{2 i z} e^{-t\left(\frac{1}{2}-i z\right)} \sum_{n=0}^{\infty} \frac{e^{-2 n t}}{n+\frac{1}{2}\left(\frac{1}{2}-i z\right)}=\frac{e^{-i z t}-1}{2 i z} e^{-t\left(\frac{1}{2}-i z\right)} \Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right) .
\end{aligned}
$$

For the second integral,

$$
\begin{aligned}
& \int_{0}^{t}\left\{\phi_{z, t}(x)+\phi_{z, t}(-x)-2 e^{-x / 2} \phi_{z, t}(0)\right\} \frac{e^{x / 2} d x}{e^{x}-e^{-x}} \\
& \quad=-\frac{1}{i z} \int_{0}^{t}\left(e^{i z x}-1\right) \frac{e^{x / 2} d x}{e^{x}-e^{-x}}=-\frac{1}{i z} \int_{0}^{t}\left(e^{i z x}-1\right) e^{-x / 2} \sum_{n=0}^{\infty} e^{-2 n x} d x .
\end{aligned}
$$

To handle the right-hand side, we calculate as

$$
\begin{aligned}
& \int_{0}^{t}\left(e^{i z x}-1\right) e^{-x / 2} \sum_{n=0}^{N} e^{-2 n x} d x \\
&= \frac{1}{2} \sum_{n=0}^{N}\left[\frac{1-e^{-2 t\left(n+\frac{1}{2}\left(\frac{1}{2}-i z\right)\right)}}{n+\frac{1}{2}\left(\frac{1}{2}-i z\right)}-\frac{1-e^{-2 t\left(n+\frac{1}{4}\right)}}{n+\frac{1}{4}}\right] \\
&=-\frac{1}{2} e^{-t\left(\frac{1}{2}-i z\right)} \sum_{n=0}^{N} \frac{e^{-2 t n}}{n+\frac{1}{2}\left(\frac{1}{2}-i z\right)}+\frac{1}{2} e^{-t / 2} \sum_{n=0}^{N} \frac{e^{-2 t n}}{n+\frac{1}{4}} \\
&+\frac{1}{2} \sum_{n=0}^{N}\left[\frac{1}{n+\frac{1}{2}\left(\frac{1}{2}-i z\right)}-\frac{1}{n+\frac{1}{4}}\right] \\
&=-\frac{1}{2} e^{-t\left(\frac{1}{2}-i z\right)} \Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right)+\frac{1}{2} e^{-t / 2} \Phi\left(e^{-2 t}, 1, \frac{1}{4}\right) \\
&-\frac{1}{2}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}-\frac{i z}{2}\right)-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)\right]+O\left(e^{-2 N t}\right)+O\left(N^{-1}\right)
\end{aligned}
$$

using the well-known series expansion

$$
\begin{equation*}
\frac{\Gamma^{\prime}}{\Gamma}(w)=-\gamma_{0}-\sum_{n=0}^{\infty}\left(\frac{1}{w+n}-\frac{1}{n+1}\right) \tag{3.4}
\end{equation*}
$$

where the implied constant depends on $t$ and $z$. Therefore, we obtain

$$
\begin{aligned}
& \int_{0}^{t}\left\{\phi_{z, t}(x)+\phi_{z, t}(-x)-2 e^{-x / 2} \phi_{z, t}(0)\right\} \frac{e^{x / 2} d x}{e^{x}-e^{-x}} \\
& =\frac{1}{2 i z} e^{-t\left(\frac{1}{2}-i z\right)} \Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right)-\frac{1}{2 i z} e^{-t / 2} \Phi\left(e^{-2 t}, 1, \frac{1}{4}\right) \\
& \quad+\frac{1}{2 i z}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}-\frac{i z}{2}\right)-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)\right] .
\end{aligned}
$$

Combining the results for $\int_{t}^{\infty}$ and $\int_{0}^{t}$,

$$
\begin{aligned}
& \int_{0}^{\infty}\left\{\phi_{z, t}(x)+\phi_{z, t}(-x)-2 e^{-x / 2} \phi_{z, t}(0)\right\} \frac{e^{x / 2} d x}{e^{x}-e^{-x}} \\
&= \frac{1}{2 i z} e^{-t / 2}\left[\Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right)-\Phi\left(e^{-2 t}, 1, \frac{1}{4}\right)\right] \\
& \quad+\frac{1}{2 i z}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}-\frac{i z}{2}\right)-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)\right]
\end{aligned}
$$

From the calculation of the five terms on the right-hand side above, we conclude that the right-hand side of Weil's explicit formula for $\phi_{z, t}(x)$ is equal to (1.6).
3.2. Proof of Proposition 1.1, We have $|\Theta(z)|=1$ for every $z \in \mathbb{R}$ by definition. In fact, zeros of $E(z)$ in the denominator cancel out in the numerator $E^{\sharp}(z)$, even if they exist. Further, $\mathfrak{P}_{t}(z)$ has poles of order one at $\gamma \in \Gamma$, but $\mathfrak{S}_{t}(z)$ is holomorphic there, since $(1+\Theta(z)) / 2=A(z) / E(z)=A(z) /\left(A(z)+i A^{\prime}(z)\right)=(z-\gamma)\left(-i / m_{\gamma}+o(1)\right)$ near $z=\gamma$ by direct calculation. Hence, $\mathfrak{S}_{t}(z)$ is bounded and holomorphic on the real line by (1.5), (3.2), and Proposition 3.1. On the other hand, in the horizontal strip $|\Im(z)| \leq 1 / 2$, we have the well-known estimate $\left(\Gamma^{\prime} / \Gamma\right)(1 / 4+i z / 2) \ll \log |z|$ and

$$
\frac{\zeta^{\prime}}{\zeta}\left(\frac{1}{2}-i z\right)=\sum_{|\Re(z)-\gamma| \leq 1} \frac{i}{z-\gamma}+O(\log |z|)
$$

by [13, Theorem 9.6 (A)]. In both estimates, implied constants are uniform in $|\Im(z)| \leq$ $1 / 2$. The number of zeros $\gamma \in \Gamma$ satisfying $|\Re(z)-\gamma| \leq 1$ is $O(\log |z|)$ counting with multiplicity by [13, Theorem 9.2]. Therefore, $\mathfrak{S}_{t}(z) \ll|z|^{-1} \log |z|$ as $|z| \rightarrow \infty$ with an implied constant depending on a compact set of $t$ by (1.6). Hence $\mathfrak{S}_{t}(z)$ belongs to $L^{2}(\mathbb{R})$ and the norm is uniformly bounded on a compact set of $t$.
3.3. Two special Hilbert spaces. We first introduce the set of meromorphic functions

$$
\begin{equation*}
F_{\gamma}(z):=\sqrt{\frac{m_{\gamma}}{\pi}} \frac{i(1+\Theta(z))}{2(z-\gamma)}, \quad \gamma \in \Gamma . \tag{3.5}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\mathfrak{S}_{t}(z)=\sum_{\gamma \in \Gamma} \sqrt{\pi m_{\gamma}} \frac{e^{-i \gamma t}-1}{\gamma} F_{\gamma}(z) \tag{3.6}
\end{equation*}
$$

by Proposition 3.1. Therefore,

$$
\begin{equation*}
\widehat{\mathcal{P}_{\phi}}(z)=\sum_{\gamma \in \Gamma} \sqrt{\pi m_{\gamma}} \frac{\widehat{\phi}(\gamma)-\widehat{\phi}(0)}{\gamma} F_{\gamma}(z) \tag{3.7}
\end{equation*}
$$

for any $\phi \in C_{c}^{\infty}(\mathbb{R})$ by definition (1.7) and the symmetry $\gamma \mapsto \bar{\gamma}$ of $\Gamma$ with $m_{\gamma}=m_{\bar{\gamma}}$. This implies

$$
\begin{equation*}
\widehat{\mathcal{P}_{D \psi}}(z)=\sum_{\gamma \in \Gamma} \sqrt{\pi m_{\gamma}} \widehat{\psi}(\gamma) F_{\gamma}(z) \tag{3.8}
\end{equation*}
$$

for any $\psi \in C_{c}^{\infty}(\mathbb{R})$, since $(\widehat{D \psi}(z)-\widehat{D \psi}(0)) / z=\widehat{D \psi}(z) / z=\widehat{\psi}(z)$ for $D$ in (1.8).
On the other hand, we define the norm $\left\|\|_{0}\right.$ on $C_{c}^{\infty}(\mathbb{R})$ by

$$
\begin{equation*}
\|\psi\|_{0}:=\frac{1}{\sqrt{\pi}}\left\|\widehat{\mathcal{P}_{D \psi}}\right\|_{L^{2}(\mathbb{R})}, \quad \psi \in C_{c}^{\infty}(\mathbb{R}) \tag{3.9}
\end{equation*}
$$

based on Proposition 1.2. Then, we have:
Lemma 3.1. Equation (3.9) defines a norm on $C_{c}^{\infty}(\mathbb{R})$.

Proof. We obtain $\left\|\psi_{1}+\psi_{2}\right\|_{0} \leq\left\|\psi_{1}\right\|_{0}+\left\|\psi_{2}\right\|_{0}$ and $\|k \psi\|_{0}=|k|\|\psi\|_{0}$ for $\psi_{1}, \psi_{2}, \psi \in$ $C_{c}^{\infty}(\mathbb{R})$ and $k \in \mathbb{C}$ by the obvious linearity of $\widehat{\mathcal{P}_{D}}$. Therefore, the proof is completed if it is shown that $\|\psi\|_{0}=0$ implies $\psi=0$. If $\|\psi\|_{0}=0$, the image $\widehat{\mathcal{P}_{D \psi}}(z)$ is identically zero. The latter means that $\widehat{\psi}(\gamma)=0$ for all $\gamma \in \Gamma$, because, if not, there must exist a sequence $\left(c_{\gamma}\right)_{\gamma \in \Gamma}$ such that $\sum_{\gamma \in \Gamma} c_{\gamma}(z-\gamma)^{-1}$ is identically zero on $\mathbb{C}$ by (3.5) and (3.8), but it is impossible. If $\widehat{\psi}(\gamma)=0$ for all $\gamma \in \Gamma$, it implies that $\psi$ is identically zero by [11, Lemma 2.1].

By Lemma 3.1, we can complete the space $C_{c}^{\infty}(\mathbb{R})$ with respect to $\left\|\|_{0}\right.$. We denote the completion by $\mathcal{H}_{0}$. On the other hand, we denote the $L^{2}$-closure of the image $\widehat{\mathcal{P}_{D}}\left(C_{c}^{\infty}(\mathbb{R})\right)$ in $L^{2}(\mathbb{R})$ by $\mathcal{K}_{0}$. Then, two Hilbert spaces $\mathcal{H}_{0}$ and $\mathcal{K}_{0}$ are isometrically isomorphic up to a constant multiple. The map $\widehat{\mathcal{P}_{D}}$ from $C_{c}^{\infty}(\mathbb{R})$ to $\widehat{\mathcal{P}_{D}}\left(C_{c}^{\infty}(\mathbb{R})\right) \subset L^{2}(\mathbb{R})$ extends to the map from $\mathcal{H}_{0}$ to $\mathcal{K}_{0}$ by (3.9). As proved in Theorem 5.2 below, $\mathcal{H}_{0}=\mathcal{H}_{W}$ and $\mathcal{K}_{0}=\mathcal{K}(\Theta)$ under the RH.

## 4. A screw line of the Riemann zeta-function

4.1. A special orthonormal basis. Assuming the RH is true, $E=E_{\xi}$ belongs to the Hermite-Biehler class ([5, Theorem 1]), and thus $\Theta=\Theta_{\xi}$ is a meromorphic innber function. Therefore, they define the de Branges space $\mathcal{H}(E)$ and the model space $\mathcal{K}(\Theta)$, respectively. We need the following result for the later discussion.

Proposition 4.1. Assume that the RH is true. Then, the family (3.5) forms an orthonormal basis of the Hilbert space $\mathcal{K}(\Theta)$. Furthermore,

$$
\begin{equation*}
\frac{\Theta^{\prime}(\gamma)}{2}=-\frac{i}{m_{\gamma}} \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{\gamma}(\gamma)=\frac{1}{\sqrt{m_{\gamma} \pi}}, \quad F_{\gamma}\left(\gamma^{\prime}\right)=0 \quad \text { for every } \quad \gamma \in \Gamma, \gamma^{\prime} \in \Gamma \backslash\{\gamma\} . \tag{4.2}
\end{equation*}
$$

Proof. See [12, Proposition 3.2] and its proof.
4.2. Screw line of the Riemann zeta-function. We define the even real-valued function $g_{\xi}(t)$ on the real line by

$$
\begin{align*}
g_{\xi}(t):= & -4\left(e^{t / 2}+e^{-t / 2}-2\right)+\sum_{n \leq e^{t}} \frac{\Lambda(n)}{\sqrt{n}}(t-\log n)  \tag{4.3}\\
& -\frac{t}{2}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)-\log \pi\right]-\frac{1}{4}\left(\Phi(1,2,1 / 4)-e^{-t / 2} \Phi\left(e^{-2 t}, 2,1 / 4\right)\right)
\end{align*}
$$

for non-negative $t$. We easily obtain $g_{\xi}(0)=0$. Then, $g_{\xi}(t)$ is a screw function on $\mathbb{R}$ under the RH as stated in [11, Theorem 1.2]. One of the screw lines corresponding to $g_{\xi}(t)$ can be constructed as follows.

Let $\tau_{\xi}$ be the non-negative measure representing $g_{\xi}(t)$ as in (2.3) under the RH. Then the Hilbert space $\mathcal{H}=L^{2}\left(\tau_{\xi}\right)$ and the mapping $t \mapsto x(t):=\left(e^{i t \gamma}-1\right) / \gamma$ provide a screw line satisfying $\|x(t)-x(0)\|_{\mathcal{H}}^{2}=-2 g_{\xi}(t)$ ([4, Section 12]). This spectral construction for a screw line is important and useful in analysis, but it is not very useful for studying the nontrivial zeros of $\zeta(s)$ without assuming the RH. In the following, we show that $\mathfrak{S}_{t}$ gives a screw line of $g_{\xi}(t)$. In contrast to the above spectral screw line, this screw line will be used later to study $\mathcal{H}_{W}$.

Theorem 4.1. Assume the RH is true and let $g(t)=g_{\xi}(t)$. Then, the mapping $t \mapsto$ $\pi^{-1 / 2} \mathfrak{S}_{\mathfrak{t}}(z)$ from $\mathbb{R}$ to $L^{2}(\mathbb{R})$ is a screw line of $g(t)$. That is,

$$
\begin{equation*}
\frac{1}{\pi}\left\langle\mathfrak{S}_{t}, \mathfrak{S}_{u}\right\rangle_{L^{2}(\mathbb{R})}=G_{g}(t, u) \tag{4.4}
\end{equation*}
$$

holds for $t, u \in \mathbb{R}$.
Proof. The sum of coefficients on the right-hand side of (3.6) is convergent in $L^{2}$-sense:

$$
\sum_{\gamma \in \Gamma}\left|\sqrt{\pi m_{\gamma}} \frac{e^{-i \gamma t}-1}{\gamma}\right|^{2} \leq \pi \sum_{\gamma \in \Gamma} \frac{m_{\gamma}}{|\gamma|^{2}}<\infty
$$

Therefore, applying Proposition 4.1 to $\mathfrak{S}_{t}(z)$ via formula (3.6), we find that it belongs to the subspace $\mathcal{K}(\Theta)$ of $L^{2}(\mathbb{R})$ and

$$
\begin{equation*}
\frac{1}{\pi}\left\langle\mathfrak{S}_{t+v}-\mathfrak{S}_{v}, \mathfrak{S}_{u+v}-\mathfrak{S}_{v}\right\rangle_{L^{2}(\mathbb{R})}=\sum_{\gamma \in \Gamma} m_{\gamma} \frac{e^{-i \gamma t}-1}{\gamma} \cdot \frac{e^{i \gamma u}-1}{\gamma} \tag{4.5}
\end{equation*}
$$

holds. The right-hand side is equal to $G_{g}(t, u)$ by

$$
\begin{equation*}
G_{g}(t, u)=\sum_{\gamma \in \Gamma} \frac{\left(e^{i \gamma t}-1\right)\left(e^{-i \gamma u}-1\right)}{\gamma^{2}} \tag{4.6}
\end{equation*}
$$

in [11, (1.9)] and the symmetry $\gamma \mapsto-\gamma$ of $\Gamma$ with $m_{\gamma}=m_{-\gamma}$. Hence, $\pi^{-1 / 2} \mathfrak{S}_{t}: \mathbb{R} \rightarrow$ $L^{2}(\mathbb{R})$ is a screw line of $g(t)$ under the RH.

We find that $\mathfrak{S}_{0}(z)$ is identically zero by (1.5) and (1.6), since

$$
\lim _{t \rightarrow 0}\left(\Phi\left(e^{-2 t}, 1, \frac{1}{4}\right)-\Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right)\right)=-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)+\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{2}\left(\frac{1}{2}-i z\right)\right)
$$

by (2.6). Therefore, by taking $v=0$ in (4.5), we obtain (4.4).
The following immediately follows from Theorem 4.1.
Corollary 4.1. The $R H$ is true if and only if the equality

$$
\begin{equation*}
\frac{1}{2 \pi}\left\|\mathfrak{S}_{t}\right\|_{L^{2}(\mathbb{R})}^{2}=-g(t) \tag{4.7}
\end{equation*}
$$

holds for all $t \geq t_{0}$ for some $t_{0} \geq 0$.
Proof. Assuming the RH, we obtain (4.7) by taking $u=t$ in (4.4), since $G_{g}(t, t)=$ $-2 g(t)$ by (2.1) and $g(0)=0$. Conversely, we suppose that equality (4.7) holds for all $t \geq t_{0}$. Then $-g(t)$ is non-negative on $\left[t_{0}, \infty\right)$, which implies that the RH is true by [11, Theorems 1.7 and 11.1].
4.3. Proof of Theorem $\mathbf{1 . 2}$, Theorem 1.2 is a corollary of the following result.

Theorem 4.2. Let $g(t)=g_{\xi}(t)$. The $R H$ is true if and only if the equality

$$
\begin{equation*}
\left\|\widehat{\mathcal{P}_{\phi}}\right\|_{L^{2}(\mathbb{R})}^{2}=\pi\langle\phi, \phi\rangle_{G_{g}} \tag{4.8}
\end{equation*}
$$

holds for all $\phi \in C_{c}^{\infty}(\mathbb{R})$ satisfying $\widehat{\phi}(0)=0$. If the $R H$ is true, equality (4.8) holds for all $\phi \in C_{c}^{\infty}(\mathbb{R})$.

Proof. First, we prove equation (4.8) assuming that the RH is true. We have

$$
\begin{equation*}
\left\|\widehat{\mathcal{P}_{\phi}}\right\|_{L^{2}(\mathbb{R})}^{2}=\pi \sum_{\gamma \in \Gamma} m_{\gamma}\left|\frac{\widehat{\phi}(\gamma)-\widehat{\phi}(0)}{\gamma}\right|^{2} \tag{4.9}
\end{equation*}
$$

by (3.7) and Proposition 4.1. Applying (4.6) to (2.2) and noting the symmetry $\gamma \mapsto-\gamma$ of $\Gamma$ with $m_{\gamma}=m_{-\gamma}$, we find that the right-hand side of (4.9) is equal to $\pi\langle\phi, \phi\rangle_{G_{g}}$.

Conversely, we prove that the RH is true assuming equality (4.8). We show that a contradiction arises if the RH is false. We take a non-real $\gamma_{0} \in \Gamma$. For any $\epsilon>0$, there exists $\psi_{1}, \psi_{2} \in C_{c}^{\infty}(\mathbb{R})$ such that $\widehat{\psi_{1}}\left(-\gamma_{0}\right)=i, \widehat{\psi_{2}}\left(-\overline{\gamma_{0}}\right)=-i,\left|\widehat{\psi_{1}}(-\gamma)\right| \leq \epsilon\left|\gamma_{0}-\gamma\right|^{-1-\delta}$ for every $\gamma \in \Gamma \backslash\left\{\gamma_{0}\right\}$, and $\left|\widehat{\psi_{2}}(-\gamma)\right| \leq \epsilon\left|\overline{\gamma_{0}}-\gamma\right|^{-1-\delta}$ for every $\gamma \in \Gamma \backslash\left\{\overline{\gamma_{0}}\right\}$ by [16, Lemma

1]. We define $\psi:=\psi_{1}+\psi_{2}(\neq 0)$ and set $\phi:=D \psi$. Then, $\widehat{\phi}(0)=0$ by definition, and $\langle\phi, \phi\rangle_{G_{g}}=\langle\psi, \psi\rangle_{W}$ holds by the relation

$$
\begin{equation*}
\left\langle D \psi_{1}, D \psi_{2}\right\rangle_{G_{g}}=\left\langle\psi_{1}, \psi_{2}\right\rangle_{W} \tag{4.10}
\end{equation*}
$$

in [11, Proposition 3.1]. The right-hand side is equal to $\sum_{\gamma \in \Gamma} m_{\gamma} \widehat{\psi}(-\gamma)(\widehat{\psi})^{\sharp}(-\gamma)=$ $-m_{\gamma_{0}}+O(\epsilon)$, since $\sum_{\gamma \in \Gamma} m_{\gamma}|\gamma|^{-1-\delta}<\infty$. Therefore, $\langle\phi, \phi\rangle_{G_{g}}$ is negative for a sufficiently small $\epsilon>0$, but it contradicts the non-negativity that follows from (4.8).

Proof of Theorem 1.2. The conclusion follows from Theorem 4.2 and the relation (4.10) of hermitian forms, since the differential operator $D$ in (1.8) gives a bijection from $C_{c}^{\infty}(\mathbb{R})$ to the subspace of $C_{c}^{\infty}(\mathbb{R})$ consisting of $\phi$ with $\widehat{\phi}(0)=0$.

Proof of Corollary 1.1. The RH is true if (1.10) holds by the same argument as the second half of the proof of Theorem 4.2, Therefore, we prove (1.10) assuming the RH.

Let $\psi \in V^{\circ}(0)$. Then $\widehat{\psi}(z)=\widehat{\mathcal{P}_{D \psi_{0}}}(z)$ for some $\psi_{0} \in C_{c}^{\infty}(\mathbb{R})$ by definition. Therefore, $\widehat{\psi}(z)=\sum_{\gamma \in \Gamma} \sqrt{\pi m_{\gamma}} \widehat{\psi_{0}}(\gamma) F_{\gamma}(z)$ by (3.8). The equality shows that $\widehat{\psi}(z)$ is a continuous function of $z \in \mathbb{R}$ by the uniform convergence of the right-hand side on a compact set of $z$. Taking $z=\gamma$ in this equality, we have $\widehat{\psi}(\gamma)=\widehat{\psi_{0}}(\gamma)$ by (4.2). Therefore, $\langle\psi, \psi\rangle_{W}$ is defined and satisfies $\langle\psi, \psi\rangle_{W}=\left\langle\psi_{0}, \psi_{0}\right\rangle_{W}$. The right-hand side is equal to $\left\|\widehat{\psi_{0}}\right\|_{L^{2}(\mathbb{R})}^{2}=2 \pi\left\|\psi_{0}\right\|_{L^{2}(\mathbb{R})}^{2}$ by (1.9) and Plancherel's identity. The same argument works if we start with $\psi_{0} \in C_{c}^{\infty}(\mathbb{R})$. Hence, we obtain (1.10).

## 5. Proof of Theorem 1.1 and its refinement

Throughout this section, we assume that the RH is true and denote $E=E_{\xi}, \Theta=$ $\Theta_{\xi}=E_{\xi}^{\sharp} / E_{\xi}$ as before, and denote $g=g_{\xi}$. Therefore, $E$ belongs to the Hermite-Biehler class, $\Theta$ is a meromorphic inner function in $\mathbb{C}_{+}$, and $g$ belongs to $\mathcal{G}_{\infty}$.

For use in the proof of Theorem 1.1 and its refinement, we introduce the operator K acting on $L^{2}(\mathbb{R})$ by

$$
\begin{equation*}
\mathrm{K}:=\mathrm{F}^{-1} \mathrm{M}_{\Theta} \mathrm{JF} \tag{5.1}
\end{equation*}
$$

with

$$
\left(\mathrm{M}_{\Theta} F\right)(z):=\Theta(z) F(z) \quad \text { and } \quad(\mathrm{J} F)(z):=F^{\sharp}(z)
$$

The Fourier transform $F$, the multiplication operator $M_{\Theta}$, and the involution $J$ are defined for functions of a complex variable and all isometries on $L^{2}(\mathbb{R})$. Therefore, K is isometric on $L^{2}(\mathbb{R})$. Further, K is invertible by $\mathrm{K}^{2}=\mathrm{id}$. By definition, K is not $\mathbb{C}$-linear but $\mathbb{R}$-linear and conjugate linear. Using the isometric operator $K$, we define

$$
\begin{equation*}
V(t):=L^{2}(t, \infty) \cap \mathrm{K} L^{2}(t, \infty) \tag{5.2}
\end{equation*}
$$

and

$$
\mathcal{H}_{W}(t):=\{[\psi] \mid \psi \in V(t)\}
$$

for $t \geq 0$. The set of subspaces $V(t)$ of $L^{2}(\mathbb{R})$ are clearly totally ordered by the settheoretical inclusion.

First, Theorem 1.1 is shown using $V(t)$ for $t=0$, and it is refined using general $t \geq 0$.
Lemma 5.1. Let $V(0)=L^{2}(0, \infty) \cap \mathrm{K} L^{2}(0, \infty)$. Then, we have $\mathcal{K}(\Theta)=\mathrm{F}(V(0))$, and therefore $\mathcal{H}(E)=E F(V(0))=\{E(z) \widehat{\psi}(z) \mid \psi \in V(0)\}$.

Proof. It is sufficient to show that $\mathcal{K}(\Theta)=\mathrm{F}(V(0))$, since $\mathcal{H}(E)=E \mathcal{K}(\Theta)$. The proof below is essentially the same as the proof in [10, Lemma 4.1].

If $\psi \in V(0)$, both $\mathrm{F} \psi$ and $\mathrm{FK} \psi$ belong to the Hardy space $H^{2}$ by definition (5.1) and $H^{2}=\mathrm{F}\left(L^{2}(0, \infty)\right)$. On the other hand, we have $(\mathrm{FK} \psi)(z)=\Theta(z)(\mathrm{F} \psi)^{\sharp}(z)$ by definition (5.1) again. This implies $(\mathrm{F} \psi)(z)=\Theta(z)(\mathrm{FK} \psi)^{\sharp}(z)$, since $\Theta(z) \Theta^{\sharp}(z)=1$ by definition (1.4). Therefore, $\mathrm{F} \psi$ belongs to $\mathcal{K}(\Theta)$ by (2.5).

Conversely, if $F \in \mathcal{K}(\Theta)$, there exists $f \in L^{2}(0, \infty)$ and $g \in L^{2}(-\infty, 0)$ such that

$$
F(z)=(\mathrm{F} f)(z)=\Theta(z)(\mathrm{Fg})(z) .
$$

We have $(\mathrm{Fg})^{\sharp}(z)=\Theta(z)(\mathrm{F} f)^{\sharp}(z)$ by using $\Theta(z) \Theta^{\sharp}(z)=1$ again. Here $(\mathrm{Fg})^{\sharp}(z)=(\mathrm{Fg})(z)$ for $\tilde{g}(x)=\overline{g(-x)} \in L^{2}(0, \infty)$, and $\Theta(z)(\mathrm{F} f)^{\sharp}(z)=(\mathrm{FK} f)(z)$ as above. Hence $\mathrm{K} f$ belongs to $L^{2}(0, \infty)$, and thus $f \in V(0)$.

By Lemma 5.1, it is concluded that the RH is false if $V(0)=\{0\}$ is shown, since $A(z) /(z-\gamma)=\xi(1 / 2-i z) /(z-\gamma)$ belongs to $\mathcal{H}(E)$ for all $\gamma \in \Gamma$ if the RH is true. Therefore, it is interesting to prove or disprove $V(0) \neq\{0\}$ unconditionally, but we do not discuss that issue further in this paper.

Let $\tau=\tau_{\xi}$ be the measure on $\mathbb{R}$ determined from the screw function $g=g_{\xi}$ by (2.3). Then, we have $g(0)=0, b=0$, and

$$
\begin{equation*}
d \tau(\lambda)=\sum_{\gamma \in \Gamma} m_{\gamma} \delta(\lambda-\gamma) d \lambda, \quad \lambda \in \mathbb{R}, \tag{5.3}
\end{equation*}
$$

since

$$
g(t)=\sum_{\gamma \in \Gamma} m_{\gamma} \frac{e^{i \gamma t}-1}{\gamma^{2}}
$$

by [11, Theorem $1.1(2)]$, where $\delta$ is the Dirac mass at $\lambda=0$, We understand that the Hilbert space $L^{2}(\tau)$ is the space of sequences $S=(S(\gamma))_{\gamma \in \Gamma}$ with

$$
\begin{equation*}
\|S\|_{L^{2}(\tau)}^{2}=\sum_{\gamma \in \Gamma} m_{\gamma}|S(\gamma)|^{2} . \tag{5.4}
\end{equation*}
$$

Then, we prove two isomorphisms for $L^{2}(\tau)$ necessary for the proof of Theorem 1.1.
Lemma 5.2. Hilbert spaces $V(0)$ and $L^{2}(\tau)$ are isomorphic by the linear map

$$
V(0) \ni \psi \mapsto S_{\psi}:=(\widehat{\psi}(\gamma))_{\gamma \in \Gamma} \in L^{2}(\tau)
$$

with

$$
\begin{equation*}
2\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\left\|S_{\psi}\right\|_{L^{2}(\tau)}^{2} . \tag{5.5}
\end{equation*}
$$

Proof. Let $\mu_{\Theta}$ be the measure on $\mathbb{R}$ determined from $\Theta=\Theta_{\xi}$ by (2.6). Then, the linear map $\mathcal{K}(\Theta) \rightarrow L^{2}\left(\mu_{\Theta}\right)$ given by $\widehat{\psi} \mapsto S_{\psi}$ is an isometric isomorphism as reviewed in Section [2.4. On the other hand, $L^{2}\left(\mu_{\Theta}\right)=L^{2}(\tau)$ with $\|S\|_{L^{2}\left(\mu_{\Theta}\right)}^{2}=\pi\|S\|_{L^{2}(\tau)}^{2}$ by (2.6), (4.1), and (5.3). Therefore, by composing the maps $V(0) \rightarrow \mathcal{K}(\Theta)=\mathcal{F}(V(0))$ and $\mathcal{K}(\Theta) \rightarrow L^{2}\left(\mu_{\Theta}\right)$, we get the conclusion of the lemma, since $2 \pi\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\|\widehat{\psi}\|_{L^{2}(\mathbb{R})}^{2}$.
Lemma 5.3. For $\psi=\lim _{n \rightarrow \infty} \psi_{n} \in \mathcal{H}_{W}$ with $\left\{\psi_{n}\right\}_{n \geq 1} \subset C_{c}^{\infty}(\mathbb{R})$, we define $S_{\psi} \in L^{2}(\tau)$ by

$$
S_{\psi}:=\lim _{n \rightarrow \infty}\left(\widehat{\psi_{n}}(\gamma)\right)_{\gamma \in \Gamma} \quad \text { in } \quad L^{2}(\tau) .
$$

Then, it is well-defined and provides an isomorphism between $\mathcal{H}_{W}$ and $L^{2}(\tau)$ through the mapping

$$
\mathcal{H}_{W} \ni \psi \mapsto S_{\psi} \in L^{2}(\tau)
$$

with

$$
\begin{equation*}
\langle\psi, \psi\rangle_{W}=\left\|S_{\psi}\right\|_{L^{2}(\tau)}^{2} . \tag{5.6}
\end{equation*}
$$

Proof. We consider $C_{0}^{\infty}(\mathbb{R})=\left\{\phi \in C_{c}^{\infty}(\mathbb{R}) \mid \widehat{\phi}(0)=0\right\}$, since we obtain the same completion $\mathcal{H}\left(G_{g}\right)$ even if starting from this space instead of $C_{0}(\mathbb{R})$. Then differentiation $\psi \mapsto \psi^{\prime}$ gives a bijection from $C_{c}^{\infty}(\mathbb{R})$ to $C_{0}^{\infty}(\mathbb{R})$. The inverse map is $\phi \mapsto \int_{-\infty}^{x} \phi(y) d y$.

The Weil hermitian form and the hermitian form $\langle\cdot, \cdot\rangle_{G_{g}}$ defined by (2.2) for the screw function $g$ are related as in (4.10), which is written as

$$
\begin{equation*}
\langle\phi, \phi\rangle_{G_{g}}=\langle\psi, \psi\rangle_{W}, \quad \psi(x)=\int_{-\infty}^{x} \phi(y) d y, \quad \psi \in C_{c}^{\infty}(\mathbb{R}) \tag{5.7}
\end{equation*}
$$

(Although not necessary for the proof, $\langle\phi, \phi\rangle_{G_{g}}$ and $\langle\psi, \psi\rangle_{W}$ are positive definite on $C_{0}^{\infty}(\mathbb{R})$ and $C_{c}^{\infty}(\mathbb{R})$, respectively, by [11, Lemma 2.1].) Relation (5.7) extends to the completed Hilbert spaces. Therefore, $\mathcal{H}_{W}$ is isometrically isomorphic to the Hilbert space $\mathcal{H}\left(G_{g}\right)$ by $\mathcal{H}\left(G_{g}\right) \rightarrow \mathcal{H}_{W}:[\phi] \mapsto[\psi]$ with $\psi=\lim _{n \rightarrow \infty} \psi_{n}$ and $\psi_{n}(x)=\int_{-\infty}^{x} \phi_{n}(y) d y$ for $\phi=\lim _{n \rightarrow \infty} \phi_{n}\left(\phi_{n} \in C_{c}^{\infty}(\mathbb{R})\right)$.

We define $\mathcal{H}\left(G_{g}\right) \rightarrow L^{2}(\tau)$ as follows. For $[\phi] \in \mathcal{H}\left(G_{g}\right)$, we define $S_{\phi}=\left(S_{\phi}(\gamma)\right)_{\gamma \in \Gamma}$ $\in L^{2}(\tau)$ by

$$
\lim _{n \rightarrow \infty}\left(\widehat{\phi_{n}}(\gamma) / \gamma\right)_{\gamma \in \Gamma} \quad \text { in } \quad L^{2}(\tau)
$$

using a sequence $\left(\phi_{n}\right)_{n}$ in $C_{0}^{\infty}(\mathbb{R})$ satisfying $\phi=\lim _{n \rightarrow \infty} \phi_{n}$. Then, the map is welldefined and $\langle[\phi],[\phi]\rangle_{G_{g}}=\langle\phi, \phi\rangle_{G_{g}}=\left\|S_{\phi}\right\|_{L^{2}(\tau)}$ by (2.2), (4.6), and (5.4). Therefore, it establishes the isomorphic isomorphism $\mathcal{H}\left(G_{g}\right) \rightarrow L^{2}(\tau):[\phi] \mapsto S_{\phi}$ ([4), Sections 5.3 and 12.5]). Using $\mathcal{H}\left(G_{g}\right) \rightarrow \mathcal{H}_{W}$ and noting $\widehat{\phi}(\lambda) / \lambda=i \widehat{\psi}(\lambda)$ for $\phi \in C_{0}^{\infty}(\mathbb{R})$, we define $\mathcal{H}_{W} \rightarrow L^{2}(\tau)$ by $[\psi] \mapsto S_{\psi}$ with

$$
S_{\psi}=\left(S_{\psi}(\gamma)\right)_{\gamma \in \Gamma}=\lim _{n \rightarrow \infty}\left(\widehat{\psi_{n}}(\gamma)\right)_{\gamma \in \Gamma}=\lim _{n \rightarrow \infty}\left(-i \widehat{\phi_{n}}(\gamma) / \gamma\right)_{\gamma \in \Gamma} \quad \text { in } \quad L^{2}(\tau)
$$

where $\left(\phi_{n}\right)_{n}$ is a sequence in $C_{0}^{\infty}(\mathbb{R})$ such that $\psi=\lim _{n \rightarrow \infty} \psi_{n}$ with $\phi_{n}=\psi_{n}^{\prime}$. Then, the map is well-defined and

$$
\langle[\psi],[\psi]\rangle_{W}=\langle\psi, \psi\rangle_{W}=\left\|S_{\psi}\right\|_{L^{2}(\tau)}=\left\|S_{\phi}\right\|_{L^{2}(\tau)}=\langle\phi, \phi\rangle_{G_{g}}=\langle[\phi],[\phi]\rangle_{G_{g}}
$$

holds, where $\phi=\lim _{n \rightarrow \infty} \phi_{n}$ and the second equality follows from (1.2) and (5.4). Hence, it establishes an isometric isomorphism $\mathcal{H}_{W} \rightarrow L^{2}(\tau)$ by $[\psi] \mapsto S_{\psi}$. As a result, the mapping $\mathcal{H}_{W} \rightarrow L^{2}(\tau)$ is directly defined by $S_{\psi}=\lim _{n \rightarrow \infty}\left(\widehat{\psi_{n}}(\gamma)\right)_{\gamma \in \Gamma}$ and $[\psi] \mapsto S_{\psi}$ for $\psi=\lim _{n \rightarrow \infty} \psi_{n}$ with the desired equality for norms.
Theorem 5.1. Assume that the $R H$ is true. Let $\mathcal{H}_{W}, \mathcal{H}(E)$, and $\mathcal{K}(\Theta)$ be as above. Let $V(t)$ be the spaces defined in (5.2). Then the following hold:
(1) $\|E \widehat{\psi}\|_{\mathcal{H}(E)}^{2}=\|\widehat{\psi}\|_{L^{2}(\mathbb{R})}^{2}=2 \pi\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\pi\langle\psi, \psi\rangle_{W}$ for $\psi \in V(0)$.
(2) The map from $\mathcal{K}(\Theta)$ to $\mathcal{H}_{W}$ obtained by the composition of the inverse of

$$
\begin{equation*}
V(0) \rightarrow \mathcal{K}(\Theta): \psi \mapsto \widehat{\psi}(z), \quad 2 \pi\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\|\widehat{\psi}\|_{L^{2}(\mathbb{R})}^{2} \tag{5.8}
\end{equation*}
$$

and

$$
\begin{equation*}
V(0) \rightarrow \mathcal{H}_{W}: \psi \mapsto[\psi], \quad 2\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\langle[\psi],[\psi]\rangle_{W}=\langle\psi, \psi\rangle_{W} \tag{5.9}
\end{equation*}
$$

agrees with the isomorphism $F \mapsto \psi_{F}$ in Theorem 1.1. In particular, (5.9) is an isometric isomorphism up to a constant multiple.

Proof. (1) It suffices to show that the equality

$$
\begin{equation*}
\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\frac{1}{2}\langle\psi, \psi\rangle_{W} \tag{5.10}
\end{equation*}
$$

holds, since $\|E \widehat{\psi}\|_{\mathcal{H}(E)}=\|\widehat{\psi}\|_{L^{2}(\mathbb{R})}$ by (2.4) and $\|\widehat{\psi}\|_{L^{2}(\mathbb{R})}^{2}=2 \pi\|\psi\|_{L^{2}(\mathbb{R})}^{2}$ by (1.1). For each $\gamma \in \Gamma$, we define $\psi_{\gamma} \in L^{2}(\mathbb{R})$ by

$$
\begin{equation*}
F_{\gamma}=\widehat{\psi_{\gamma}} \tag{5.11}
\end{equation*}
$$

Then, each $\psi_{\gamma}$ belongs to $V(0)$ and $\left\{\psi_{\gamma}\right\}_{\gamma \in \Gamma}$ forms an orthogonal basis satisfying $2 \pi\left\|\psi_{\gamma}\right\|_{L^{2}(\mathbb{R})}^{2}=$ $\left\|\widehat{\psi_{\gamma}}\right\|_{\mathcal{K}(\Theta)}^{2}=\left\|F_{\gamma}\right\|_{\mathcal{K}(\Theta)}^{2}=1$ by Proposition 4.1 and Lemma 5.1, since the orthogonality of $F_{\gamma}$ 's is inherited via the Fourier transform. For $\psi=\sum_{\gamma} c_{\gamma} \psi_{\gamma} \in V(0)$, we have

$$
\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\frac{1}{2 \pi} \sum_{\gamma \in \Gamma}\left|c_{\gamma}\right|^{2}
$$

by the orthogonality and

$$
\widehat{\psi}(\gamma)=\frac{1}{\sqrt{m_{\gamma} \pi}} c_{\gamma}
$$

by applying (4.2) to $\widehat{\psi}=\sum_{\gamma} c_{\gamma} F_{\gamma}$. From these two and (1.2), we get (5.10).
(2) It is clear that the composition of the inverse of (5.8) and (5.9) agrees with the map $F \mapsto \psi_{F}$ of Theorem 1.1 including the equality for norms, and we observed in the proof of Lemma 5.2 that the map (5.8) is an isometric isomorphism up to the multiple $\sqrt{2 \pi}$. Therefore, it suffices to show that the map (5.9) gives an isometric isomorphism up to the multiple $\sqrt{2}$.

For $\psi \in V(0), S_{\psi} \in L^{2}(\tau)$ is defined and satisfy $2\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\left\|S_{\psi}\right\|_{L^{2}(\tau)}^{2}$ by Lemma 5.2. Then, there exists a sequence $\left(\psi_{n}^{*}\right)_{n}$ in $C_{c}^{\infty}(\mathbb{R})$ such that converges to $\psi^{*}$ with respect to $\langle\cdot, \cdot\rangle_{W}$ and $S_{\psi}=S_{\psi^{*}}$ by Lemma [5.3. The later implies $\left\langle\psi-\psi_{n}^{*}, \psi-\psi_{n}^{*}\right\rangle_{W}=$ $\left\langle\psi^{*}-\psi_{n}^{*}, \psi^{*}-\psi_{n}^{*}\right\rangle_{W} \rightarrow 0(n \rightarrow \infty)$. Therefore, $\psi=\psi^{*}$ and hence $V(0) \rightarrow \mathcal{H}_{W}$ is directly defined by $\psi \mapsto[\psi]$. Furthermore, we obtain $2\|\psi\|_{L^{2}(\mathbb{R})}^{2}=\langle\psi, \psi\rangle_{W}$ from (5.5) and (5.6). Hence, this map is nothing but (5.9).

The equality $\|\psi\|_{L^{2}(\mathbb{R})}=2^{-1}\langle\psi, \psi\rangle_{W}$ in Theorem 5.1(1) shows that the $L^{2}$-structure induced from $L^{2}(\mathbb{R})$ and an "arithmetic structure" (or a "local structure") coming from the geometric side of the Weil explicit formula (3.3) are coincident on a dense subspace of $V(0)$ consisting of functions such that the Weil explicit formula holds.

Theorem 5.2. Assume that the RH is true. Then, $\mathcal{H}_{0}=\mathcal{H}_{W}$ and $\mathcal{K}_{0}=\mathcal{K}(\Theta)$ and the extended map $\widehat{\mathcal{P}_{D}}: \mathcal{H}_{W} \rightarrow \mathcal{K}(\Theta)$ provides the inverse of the map in Theorem 5.1 (2). In particular, $V(0)$ is the $L^{2}$-closure of $V^{\circ}(0)$ in Corollary 1.1.

Proof. For $\psi \in C_{c}^{\infty}(\mathbb{R})$, we have

$$
\left\|\widehat{\mathcal{P}_{D \psi}}\right\|_{L^{2}(\mathbb{R})}^{2}=\pi \sum_{\gamma \in \Gamma} m_{\gamma}|\widehat{\psi}(\gamma)|^{2}=\pi\langle\psi, \psi\rangle_{W}
$$

by (1.2), (3.8), and Proposition 4.1. Hence, $\mathcal{H}_{0}$ coincides with $\mathcal{H}_{W}$ by definition (3.9). Formula (3.8) shows that the image $\widehat{\mathcal{P}_{D \psi}}$ is defined independent of the representatives of $\mathcal{H}_{W}$. On the other hand, $\mathcal{K}_{0}$ is a subspace of $\mathcal{K}(\Theta)$ by Proposition 4.1 again.

We denote $F=\widehat{\mathcal{P}_{D \psi}}$ for $[\psi] \in \mathcal{H}_{W}$ and set $\psi_{F}=\mathrm{F}^{-1}(F)$ as in Theorem 1.1. Then, $F(\gamma)=\widehat{\psi}(\gamma)$ by (3.8) and (4.2). Therefore, $\widehat{\psi_{F}}(\gamma)=\psi(\gamma)$ for all $\gamma \in \Gamma$, and hence $[\psi]=\left[\psi_{F}\right]$ in $\mathcal{H}_{W}$. On the other hand, $\widehat{\mathcal{P}_{D \psi_{F}}}(z)=F$ by (3.8), since $\widehat{\psi_{F}}=F$ by definition and $F(\gamma)=\widehat{\psi}(\gamma)$. Hence, we obtain the desired conclusion.

The totally ordered structure of the subspaces of the de Branges space $\mathcal{H}(E)$ is described by $V(t)$ as follows.

Theorem 5.3. Assume that the RH is true. Then, $E \mathrm{~F}(V(t))$ is a de Branges subspaces of $\mathcal{H}(E)$ for every $t \geq 0$ and is isometrically isomorphic to $\mathcal{H}_{W}(t)$ up to a constant multiple by the map of Theorem 1.1.

Proof. It is sufficient to prove the first half of the theorem, since the second half follows from Theorem 5.1 (2). We prove the claim for positive $t$ such that $V(t) \neq\{0\}$, since
the case of $t=0$ was proved in Lemma 5.1 and the claim is trivial if $V(t)=\{0\}$. The following is essentially the same as the proof of [10, Lemma 4.3].

We show that $\mathcal{H}:=E(z) \mathcal{F}(V(t))$ is a Hilbert space consisting of entire functions and satisfies the axiom of the de Branges spaces:
(dB1) For each $z \in \mathbb{C} \backslash \mathbb{R}$ the point evaluation $\Phi \mapsto \Phi(z)$ is a continuous linear functional on $\mathcal{H}$;
(dB2) If $\Phi \in \mathcal{H}, \Phi^{\sharp}$ belongs to $\mathcal{H}$ and $\|\Phi\|_{\mathcal{H}}=\left\|\Phi^{\sharp}\right\|_{\mathcal{H}}$;
(dB3) If $w \in \mathbb{C} \backslash \mathbb{R}, \Phi \in \mathcal{H}$ and $\Phi(w)=0$,

$$
\frac{z-\bar{w}}{z-w} \Phi(z) \in \mathcal{H} \quad \text { and } \quad\left\|\frac{z-\bar{w}}{z-w} \Phi(z)\right\|_{\mathcal{H}}=\|\Phi\|_{\mathcal{H}},
$$

where the Hilbert space structure is the one induced from $V(t)$ that is equivalent to $\langle F, G\rangle_{\mathcal{H}}=\int_{\mathbb{R}} F(z) \overline{G(z)}|E(z)|^{-2} d z$ for $F, G \in \mathcal{H}$.

Let $\Phi(z)=E(z)(\mathrm{F} f)(z) \in \mathcal{H}$ with $f \in V(t)$. First, we prove that $\mathcal{H}$ consists of entire functions. We see that $\Phi(z)$ is holomorphic in $\mathbb{C}_{+}$by $f \in L^{2}(t, \infty)$. If we write $\left(\mathrm{J}_{\sharp} f\right)(x):=\overline{f(-x)}$, the commutative relation $\mathrm{JF}=\mathrm{F}_{\sharp}$ holds. Therefore, using (5.1) and $\mathrm{K}^{2}=1$, we have $\Phi(z)=E(z)(\mathrm{F} f)(z)=E^{\sharp}(z)\left(\mathrm{F}_{\sharp} \mathrm{K} f\right)(z)$. This shows that $\Phi(z)$ is also holomorphic in $\mathbb{C}_{-}$. Furthermore, $\mathrm{J}_{\sharp} \mathrm{K} f \in L^{2}(-\infty,-t)$, because the tempered distribution kernel $k:=\mathrm{F}^{-1} \Theta$ of K has support in $[0, \infty)$ by [7, Theorems 1.1 and 1.2]. On the real line, $\lim _{z \rightarrow x}(\mathrm{~F} f)(z)=(\mathrm{F} f)(x)$ and $\lim _{z \rightarrow x}\left(\mathrm{FJ}_{\sharp} \mathrm{K} f\right)(z)=\lim _{z \rightarrow x}(\mathrm{FK} f)^{\sharp}(z)=$ $u^{\sharp}(x)(\mathrm{F} f)(x)$ for almost all $x \in \mathbb{R}$, where $z$ is allowed to tends to $x$ non-tangentially from $\mathbb{C}_{+}$and $\mathbb{C}_{-}$, respectively. Hence, $(\mathrm{F} f)(z)$ is also holomorphic in a neighborhood of each point of $\mathbb{R}$. By the above, $\Phi(z)$ is an entire function.

We confirm (dB1). For $z \in \mathbb{C}_{+}, \Phi \mapsto \Phi(z)=E(z) \int_{t}^{\infty} f(x) e^{i z x} d x$ is a continuous linear form. On the other hand, for $z \in \mathbb{C}_{-}, \Phi \mapsto \Phi(z)=E^{\sharp}(z) \int_{-\infty}^{-t} \overline{(\mathrm{~K} f)(-x)} e^{i z x} d x$ is a continuous linear functional. Finally, for $z \in \mathbb{R}$, the continuity follows by the Banach-Steinhaus theorem.

We confirm (dB2). We have $\Phi^{\sharp}(z)=E(z)(\mathrm{FK} f)(z)$. Since $\mathrm{K} f \in V(t), \Phi^{\sharp}$ belongs to $\mathcal{H}$. Since K is isometric, the equality of norms in (dB2) holds.

We confirm (dB3). The equality of norms in (dB3) is trivial by the definition of the norm of $\mathcal{H}$. From (dB2), it is sufficient to show only the case of $w \in \mathbb{C}_{+}$. Suppose that $\Phi(w)=0$ for some $w \in \mathbb{C}_{+}$. Then $(\mathrm{F} f)(w)=0$, since $E(z)$ has no zeros on $\mathbb{C}_{+}$. We put $f_{w}(x):=f(x)-i(w-\bar{w}) \int_{0}^{x-t} f(x-y) e^{-i w y} d y$. Then we easily find that $f_{w} \in L^{2}(t, \infty)$ and $\left(\mathrm{F}_{w}\right)(z)=((z-\bar{w}) /(z-w))(\mathrm{F} f)(z)$ for $z \in \mathbb{C}_{+}$. Hence we complete the proof if it is shown that $\mathrm{K} f_{w}$ has support in $[t, \infty)$, since $\mathrm{K} f_{w} \in L^{2}(\mathbb{R})$ by $f_{w} \in L^{2}(t, \infty)$. We put $g_{w}(x):=(\mathrm{K} f)(x)-i(\bar{w}-w) \int_{0}^{x-t}(\mathrm{~K} f)(x-y) e^{-i \bar{w} y} d y$. Then $g_{w}$ has support in $[t, \infty)$ by $\mathrm{K} f \in L^{2}(t, \infty)$ and $\left(\mathrm{F} g_{w}\right)(z)=((z-w) /(z-\bar{w}))(\mathrm{FK} f)(z)=\left(\mathrm{FK} f_{w}\right)(z)$ for $z \in \mathbb{C}_{+}$. Hence $g_{w}=\mathrm{K} f_{w}$ and the proof is completed.

We expect $V(t) \neq\{0\}$ to hold for all $t \geq 0$, but we do not discuss it in this paper. However, there exists $0<t_{0} \leq \infty$ (possibly $t_{0}=\infty$ ) such that $V(t) \neq\{0\}$ holds for all $0 \leq t<t_{0}$, since $V\left(t_{1}\right) \subset V\left(t_{2}\right)$ if $t_{1} \geq t_{2}$ by definition (5.2).
5.1. A weaker variant of Corollary 1.1. Since the space $V(0)$ can be constructed unconditionally as well as $V^{\circ}(0)$ in Corollary 1.1, it can be used to state an equivalence condition for the RH. However, since the construction of $V(0)$ is simpler than that of $V^{\circ}(0)$, more conditions are required for the equivalence condition.

Proposition 5.1. Let $V(0)=L^{2}(0, \infty) \cap \mathrm{K} L^{2}(0, \infty)$ be as above. Then the $R H$ is true if and only if the following two conditions hold:
(1) $\|\psi\|_{L^{2}(\mathbb{R})}^{2}=2^{-1}\langle\psi, \psi\rangle_{W}$ for every $\psi \in V(0)$.
(2) For a given $\gamma \in \Gamma$ and any $\epsilon>0$, there exists $\psi \in V(0)$ such that

$$
\widehat{\psi}(\gamma)=1, \quad\left|\widehat{\psi}\left(-\gamma^{\prime}\right)\right| \leq \frac{\epsilon}{\left|\gamma-\gamma^{\prime}\right|^{1+\delta}} \quad \text { for every } \gamma^{\prime} \in \Gamma \backslash\{\gamma\}
$$

for some $\delta>0$ independent of $\gamma, \epsilon$, and $\psi$.
Proof. Assuming the RH, (1) follows from Theorem [5.1(1). Also, (2) holds, since $\psi_{\gamma}=$ $\mathrm{F}^{-1}\left(F_{\gamma}\right)$ in $V(0)$ satisfies $\widehat{\psi_{\gamma}}(\gamma) \neq 0$ and $\widehat{\psi_{\gamma}}\left(\gamma^{\prime}\right)=0$ for $\gamma^{\prime} \in \Gamma \backslash\{\gamma\}$.

Conversely, we assume that (1) and (2) are satisfied. Then, we show that a contradiction arises if the RH is false. We take a non-real $\gamma_{0} \in \Gamma$. For any $\epsilon>0$, there exists $\psi_{1}, \psi_{2} \in V(0)$ such that $\widehat{\psi_{1}}\left(-\gamma_{0}\right)=i, \widehat{\psi_{2}}\left(-\overline{\gamma_{0}}\right)=-i,\left|\widehat{\psi_{1}}(-\gamma)\right| \leq \epsilon\left|\gamma_{0}-\gamma\right|^{-1-\delta}$ for every $\gamma \in \Gamma \backslash\left\{\gamma_{0}\right\}$, and $\left|\widehat{\psi_{2}}(-\gamma)\right| \leq \epsilon\left|\overline{\gamma_{0}}-\gamma\right|^{-1-\delta}$ for every $\gamma \in \Gamma \backslash\left\{\overline{\gamma_{0}}\right\}$ by (2). Then, for $\psi:=\psi_{1}+\psi_{2}(\neq 0)$, we have $\langle\psi, \psi\rangle_{W}=\sum_{\gamma \in \Gamma} m_{\gamma} \widehat{\psi}(-\gamma)(\widehat{\psi})^{\sharp}(-\gamma)=-m_{\gamma_{0}}+O(\epsilon)$, since $\sum_{\gamma \in \Gamma}|\gamma|^{-1-\delta}<\infty$. Therefore, $\langle\psi, \psi\rangle_{W}$ is negative for a sufficiently small $\epsilon>0$, but it contradicts (1). Hence the RH holds.

## 6. Hilbert-Pólya space

One of attractive strategies for proving the RH is the construction of a Hilbert-Pólya space, which is a pair of a Hilbert space and a self-adjoint operator acting on it such that all non-trivial zeros of the Riemann zeta-function are eigenvalues of the self-adjoint operator. In this section, we state that $\mathcal{H}_{W}$ is one of Hilbert-Pólya spaces. Note that $\mathcal{H}_{W}$ is unconditionally defined as $\mathcal{H}_{0}$ by Theorem 5.2.

We assume the RH and denote $E=E_{\xi}$ as in Section 廻. In this case, the domain $\mathfrak{D}(\mathrm{M})$ of the multiplication operator M on $\mathcal{H}(E)$ is dense in $\mathcal{H}(E)$, because $S_{\theta}(z)$ does not belongs to $\mathcal{H}(E)$ for all $\theta \in[0, \pi)$ by the estimate $\left|S_{\theta}(i y) / E(i y)\right| \gg(\log y)^{-1}(y \rightarrow+\infty)$ obtained by the Stirling formula for the gamma-function and [8, Proposition 2.1]. Using M , we define the operator $\mathrm{A}:=\mathrm{F}^{-1} \mathrm{MF}$ on $V(0)$ with the domain $\mathfrak{D}(\mathrm{A})=\mathrm{F}^{-1}(\mathfrak{D}(\mathrm{M}))$. If $\psi \in V(0)$ is differentiable and $\psi^{\prime}$ also belongs to $V(0)$, then $\mathrm{A} \psi=i \psi^{\prime}$. Further, we define the operator $\mathrm{A}_{W}$ on $\mathcal{H}_{W}$ as follows.

By Theorem [5.2, the inverse of (5.9) from $\mathcal{H}_{W}$ to $V(0)$ is given by $[\psi] \mapsto \mathrm{F}^{-1} \widehat{\mathcal{P}_{D \psi}}$. Further, if we choose the representative of $\psi$ from $V(0)$, it is possible and uniquely determined by Theorem $5.1(2)$, and therefore $\psi=\mathrm{F}^{-1} \widehat{\mathcal{P}_{D \psi}}$. By choosing representatives in this way, we define $\mathrm{A}_{W}$ on $\mathcal{H}_{W}$ by $\mathrm{A}_{W}[\psi]=[\mathrm{A} \psi]$. By the same procedure as above, the family of self-adjoint extensions $M_{\theta}$ of $M$ determines the corresponding families of selfadjoint extensions of A and $\mathrm{A}_{W}$. By this correspondence, the orthogonal basis $\left\{\left[\psi_{\gamma}\right]\right\}_{\gamma \in \Gamma}$ of $\mathcal{H}_{W}$ consists of eigenvectors $\left[\psi_{\gamma}\right]$ of $A_{W, \pi / 2}$ with eigenvalues $\gamma \in \Gamma$, since $\left\{E F_{\gamma}\right\}_{\gamma \in \Gamma}$ for (3.5) is an orthogonal basis of $\mathcal{H}(E)$ consists of eigenfunctions of $\mathrm{M}_{\pi / 2}$ with eigenvalues $\Gamma$ (see Seciton (2.3). Therefore, the pair $\left(\mathcal{H}_{W}, \mathrm{~A}_{W, \pi / 2}\right)$ is a Hilbert-Pólya space.

## 7. Special values of the screw line $\mathfrak{S}_{t}(z)$

The screw line $\mathfrak{S}_{t}(z)$ has the following unconditional relations with the screw function $g(t)$. It is interesting that they are not a special case of equations obtained from the general theory of screw functions.
Theorem 7.1. Let $g_{\xi}(t)$ and $\mathfrak{P}_{t}(z)$ be functions of (4.3) and (1.6), respectively. Then the following equations hold independently of the truth of the $R H$ :

$$
\begin{gather*}
\mathfrak{P}_{t}(0)=-g_{\xi}(t)  \tag{7.1}\\
\lim _{y \rightarrow+\infty}\left[y \mathfrak{B}_{t}(-i y)-\frac{1}{2} \frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}+\frac{y}{2}\right)+\frac{1}{2} \log \pi\right]=-g_{\xi}^{\prime}(t), \tag{7.2}
\end{gather*}
$$

where we assume $t \neq \log n$ for any $n \in \mathbb{N}$ in (7.2).

Proof. Equality (7.1) follows from (3.2), Proposition [3.1) and [11, Theorem 1.1 (2)], but it follows directly from (4.3) and (1.6) as follows. By $\Phi(z, s, a)=\sum_{n=0}^{\infty} z^{n}(n+a)^{-s}$ and (2.6),

$$
\begin{gathered}
\lim _{z \rightarrow 0} \frac{1}{i z}\left[\Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-i z\right)\right)-\Phi\left(e^{-2 t}, 1,1 / 4\right)\right]=-\frac{1}{2} \Phi\left(e^{-2 t}, 2,1 / 4\right), \\
\lim _{z \rightarrow 0} \frac{1}{i z}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}-\frac{i z}{2}\right)-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)\right]=\frac{1}{2} \psi_{1}\left(\frac{1}{4}\right)
\end{gathered}
$$

where $\psi_{1}(z)$ is the polygamma function of order one. The expansion $\psi_{1}(w)=\sum_{n=0}^{\infty}(w+$ $n)^{-2}$ gives $\psi_{1}(1 / 4)=\Phi(1,2,1 / 4)$. Taking $s=1 / 2$ in the logarithmic derivative of $\xi(s)=\xi(1-s)$ and using

$$
\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)=-\gamma_{0}-3 \log 2-\frac{\pi}{2}
$$

we have

$$
\frac{\zeta^{\prime}}{\zeta}\left(\frac{1}{2}\right)=\frac{1}{2}\left(\gamma_{0}+3 \log 2+\log \pi+\frac{\pi}{2}\right) .
$$

Hence, by taking the limit $z \rightarrow 0$ in (1.6), we obtain the minus of (4.3).
To show (7.2), we multiply (1.6) by $y$ and substitute $-i y$ for $z$ :

$$
\begin{aligned}
y \mathfrak{P}_{t}(-i y): & =\frac{4 y\left(e^{t / 2}-1\right)}{1+2 y}+\frac{4 y\left(e^{-t / 2}-1\right)}{1-2 y} \\
& +\left(e^{-y t}-1\right) \frac{\zeta^{\prime}}{\zeta}\left(\frac{1}{2}-y\right)+\sum_{n \leq e^{t}} \frac{\Lambda(n)}{\sqrt{n}}\left(e^{-y(t-\log n)}-1\right) \\
& +\frac{1}{2}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)-\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}-\frac{y}{2}\right)\right] \\
& +\frac{1}{2} e^{-t / 2}\left[\Phi\left(e^{-2 t}, 1,1 / 4\right)-\Phi\left(e^{-2 t}, 1, \frac{1}{2}\left(\frac{1}{2}-y\right)\right)\right]
\end{aligned}
$$

Therefore, for positive $t>0$,

$$
\begin{aligned}
& \lim _{y \rightarrow+\infty}\left[y \mathfrak{B}_{t}(-i y)-\frac{1}{2} \frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}+\frac{y}{2}\right)+\frac{1}{2} \log \pi\right] \\
& =2\left(e^{t / 2}-e^{-t / 2}\right)-\sum_{n \leq e^{t}} \frac{\Lambda(n)}{\sqrt{n}}+\frac{1}{2}\left[\frac{\Gamma^{\prime}}{\Gamma}\left(\frac{1}{4}\right)-\log \pi\right]+\frac{1}{2} e^{-t / 2} \Phi\left(e^{-2 t}, 1,1 / 4\right)
\end{aligned}
$$

by using the logarithmic derivative of $\xi(s)=\xi(1-s)$ at $s=1 / 2-y$. The righthand side equals to $-g^{\prime}(t)$ if $t \neq \log n$ by (4.3) and $(d / d t)\left(e^{-t / 2} \Phi\left(e^{-2 t}, 2,1 / 4\right)\right)=$ $-2 e^{-t / 2} \Phi\left(e^{-2 t}, 2,1 / 4\right)$ follows form $\Phi(z, s, a)=\sum_{n=0}^{\infty} z^{n}(n+a)^{-s}$.

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