

# RATIONAL SINGULARITIES AND $q$ -BIRATIONAL MORPHISMS

DONGHYEON KIM

**ABSTRACT.** In this paper, we generalize the notion of rational singularities for any reflexive sheaf of rank 1, link our notion of rational singularities with the notion of rational singularities in [Kov11], and prove generalizations of standard facts about rational singularities. Moreover, by using a definition of non-rational locus, we introduce the notion of  $(B_{q+1})$  as a dual notion of well-known Serre's notion of  $(S_{q+1})$ , and prove a theorem about  $q$ -birational morphisms.

## 1. INTRODUCTION

In this paper,  $q \geq 2$  is an integer, and  $k$  is any algebraically closed field of arbitrary characteristic. Any variety is quasi-projective, separated, finite type and integral scheme over  $k$ . Any divisor is assumed to be Cartier unless otherwise stated.

Throughout the paper, we will assume that for any normal variety  $X$ , there is a proper birational morphism  $f : X' \rightarrow X$  where  $X'$  is smooth and  $f$  is an isomorphism outside of the singular locus of  $X$ . We call such a map a *resolution* of  $X$ . If  $k$  is of characteristic 0, our assumption follows from Hironaka's resolution of singularities, and if  $k$  is of positive characteristic, only the threefold case is known to be affirmative (see [Hir64] and [Cut09]). Moreover, for a pair  $(X, \Delta)$ , a *log resolution*  $f : X' \rightarrow X$  of  $(X, \Delta)$  is a resolution  $f : X' \rightarrow X$  of  $X$  such that  $\text{Supp } f_*^{-1}\Delta \cup \text{Exc}(f)$  is simple normal crossing.

A variety  $X$  is said to have *rational singularities* if for any resolution  $f : X' \rightarrow X$ ,  $f_*\mathcal{O}_{X'} = \mathcal{O}_X$ ,  $R^if_*\mathcal{O}_{X'} = 0$  and  $R^if_*\omega_{X'} = 0$  for  $i \geq 1$  (see [Kol13], Definition 2.76, and [Kov17], ). The notion of rational singularities plays an important role in algebraic geometry. For instance, if  $X$  is a variety with rational singularities and  $f : X' \rightarrow X$  is a resolution of  $X$ , we have

$$H^i(X', f^*\mathcal{E}) \cong H^i(X, \mathcal{E})$$

for any  $i \geq 0$  and any vector bundle  $\mathcal{E}$  on  $X$ . In this paper, we generalize the notion to any reflexive sheaf of rank 1.

The fundamental property of the notion of rational singularities is that if  $X$  has rational singularities, then  $X$  is CM. In this paper, from this perspective, we will define another analogous notion of rational singularities for a reflexive sheaf  $\mathcal{F}$  of rank 1 on a normal variety  $X$ .

---

*Date:* March 28, 2024.

*2010 Mathematics Subject Classification.* 14B05, 14E05, 14G17.

*Key words and phrases.* rational singularities,  $q$ -birational morphisms.

**Definition 1.1.** Let  $X$  be any normal variety and  $\mathcal{F}$  any coherent sheaf on  $X$ . Let us define *dual* of  $\mathcal{F}$  by  $\mathcal{F}^D := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ , and *double dual* by  $\mathcal{F}^{DD} := ((\mathcal{F})^D)^D$ .

**Definition-Theorem 1.2.** Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . Suppose that  $f : X' \rightarrow X$  is a generically finite morphism with  $X'$  smooth. Then we have a natural map

$$\theta_{\mathcal{F},f} : Rf_*(f^*\mathcal{F})^D \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)[- \dim X].$$

Moreover, if  $f$  is a resolution, the following are equivalent:

- (a)  $\theta_{\mathcal{F},f}$  is a quasi-isomorphism.
- (b)  $R^i f_*(f^*\mathcal{F})^{DD} = 0$  for  $i \geq 1$ .
- (c) The natural map

$$H_x^i(X_x, \mathcal{F}_x) \rightarrow H_{f^{-1}(x)}^i(X'_x, (f^*\mathcal{F})_x^{DD})$$

is an isomorphism for each  $i \geq 0$  and each point  $x \in X$ .

We say that  $f$  is a *weakly rational resolution* of  $\mathcal{F}$  if one of the above is true for  $f$ , and  $\mathcal{F}$  has *weakly rational singularities* if one of the resolutions of  $X$  is weakly rational resolution of  $\mathcal{F}$ .

**Definition 1.3.** Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ .

- (a) If  $\mathcal{F}$  has weakly rational singularities and is CM, then we say  $\mathcal{F}$  has *rational singularities*.
- (b) We say that a Weil divisor  $D$  on  $X$  has *rational singularities* if  $\mathcal{O}_X(D)$  has rational singularities.

It is worth noting that  $\mathcal{F}$  is  $(KV_{\dim X})$  if and only if for some resolution  $f : X' \rightarrow X$ ,  $R^i f_*(f^*\mathcal{F})^D = 0$  for  $i > 0$ .

Note that our definition bears a similarity to Definition 2.5 in [Kov11] and Definition 2.78 in [Kol13]. Let us recall the definition of rational singularities for a reduced pair  $(X, D)$ .

**Definition 1.4** (See Definition 2.5, and Theorem 2.9 in [Kov11]). Let  $X$  be a normal variety, and  $D$  a reduced Weil divisor on  $X$ . We say  $(X, D)$  has *rational singularities* if there is a log resolution  $f : (X', D_{X'}) \rightarrow (X, D)$  such that

- (a)  $(X, D)$  is a normal pair in the sense that the natural map  $\mathcal{O}_X(-D) \rightarrow f_*\mathcal{O}_{X'}(-D_{X'})$  is an isomorphism, and
- (b)  $R^i f_*\mathcal{O}_{X'}(-D_{X'}) = 0$  for  $i > 0$ ,

where  $D_{X'}$  is the strict transform of  $D$  along  $f$ .

Here, we highlight the distinctions between our definition and those in [Kov11] and [Kol13]:

- We used double dual to define the notion of rational singularities instead of strict transform. The advantage of using double dual is that the notion of normal pair

in [Kov11] is unnecessary, and if we consider Proposition 4.7, our definition of rational singularities is minimal in sense of the requirement of the vanishing of higher direct image and CM-ness. See Corollary 4.8.

In particular, if a pair  $(X, D)$  has rational singularities in the sense of [Kov11] and [Kol13], then  $-D$  has rational singularities in the our sense.

- It is obvious that some Cartier divisor on  $X$  has rational singularities in our sense if and only if  $X$  has rational singularities, and in contrast to our notion, it is far from obvious that for some reduced, effective and Cartier divisor  $D$  on  $X$ ,  $(X, D)$  has rational singularities in the sense of [Kov11] if and only if  $X$  has rational singularities. For the only if assertion, see Corollary 2.13 in [Kov11]. Note that the proof of Corollary 2.13 in [Kov11] implicitly used our notion of rational singularities.
- We defined the notion of rational singularities for any reflexive sheaf of rank 1, not only for a reduced Weil divisor.

One of the fundamental properties of klt and strongly  $F$ -regular varieties is that klt varieties over a characteristic 0 field and strongly  $F$ -regular varieties over a positive characteristic field have rational singularities (see [Elk81], [KMM87] and [HW19]). The same principle holds in our definition of rational singularities for reflexive sheaves of rank 1.

**Theorem 1.5.** *Let  $X$  be any normal variety over a characteristic 0 field,  $\Delta$  any effective  $\mathbb{Q}$ -Weil divisor such that  $(X, \Delta)$  is dlt and  $D$  any  $\mathbb{Q}$ -Cartier Weil divisor on  $X$ . Then  $D$  has rational singularities.*

**Theorem 1.6.** *Let  $X$  be any strongly  $F$ -regular variety over a positive characteristic field, and  $D$  any Weil divisor on  $X$  such that there is an integer  $r$  which is relatively prime to  $p$ , and  $rD$  is Cartier. Then  $D$  has rational singularities.*

Note that Theorem 4.10 is a generalization of Corollary 5.25 in [KM98] and the proof does not require the cyclic covering theory.

One of the advantages of our notion is that our notion of rational singularities is well-behaved under finite morphisms under mild conditions, and it provides a partial answer to a question posed by Kollár (see Remark 2.81 (3) in [Kol13]).

**Theorem 1.7.** *Let  $X, Y$  be normal varieties over a characteristic 0 field,  $p : Y \rightarrow X$  any finite étale morphism and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . If  $p^*\mathcal{F}$  has (resp. weakly) rational singularities, then  $\mathcal{F}$  has (resp. weakly) rational singularities.*

**Theorem 1.8.** *Let  $X, Y$  be normal varieties over a characteristic 0 field,  $p : Y \rightarrow X$  any finite morphism and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . If  $\mathcal{F}$  is  $(KV_{\dim X})$ , and  $p^*\mathcal{F}$  has rational singularities, then  $\mathcal{F}$  has rational singularities.*

For a normal variety  $X$ , one may want to measure how far a non-rational singularity is from being rational. For this purpose, the notion of non-rational locus is defined in [AH12] and [Kov11]. We also consider an analogous notion here.

**Definition 1.9.** Let  $X$  be any normal variety,  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ , and  $f : X' \rightarrow X$  a resolution.

- (a) We say that  $f$  is  $(KV_q)$  about  $\mathcal{F}$  if the support of  $R^i f_*(f^* \mathcal{F})^D$  has codimension  $\geq i + q + 1$  for any  $i \geq 1$ . We call  $\mathcal{F}$   $(KV_q)$  if one of the resolutions of  $X$  is  $(KV_q)$  about  $\mathcal{F}$ .
- (b) We say that  $f$  is  $(RS_q)$  about  $\mathcal{F}$  if
  - (i)  $f$  is  $(KV_q)$  about  $\mathcal{F}$ , and
  - (ii) for any point  $x \in X$  with  $\text{codim}_X x \leq q$ ,  $\theta_{\mathcal{F},f}$  is a quasi-isomorphism after localizing  $\theta_{\mathcal{F},f}$  at  $x$ .

We call  $\mathcal{F}$   $(RS_q)$  if there is a resolution of  $X$  which is  $(RS_q)$  about  $\mathcal{F}$ .

*Remark 1.10.* In Definition 1.9,  $\mathcal{F}$  is  $(RS_{\dim X})$  if and only if  $\mathcal{F}$  has rational singularities. Also in Definition 1.9, condition (b) is equivalent to following: For any  $i \geq 1$ ,  $R^i f_*(f^* \mathcal{F})^{DD}$  has the support of codimension  $\geq q + 1$ .

Note that for a normal variety  $X$  and a coherent sheaf  $\mathcal{F}$  on  $X$ ,  $\mathcal{F}$  is reflexive if and only if the natural map  $\mathcal{F} \rightarrow \mathcal{F}^{DD}$  is an isomorphism. See Lemma 3.4.

Let us say  $\mathcal{F}$  is  $(R_q)$  if there is an open subscheme  $U \subseteq X$  of  $X$  such that  $\text{codim}_X(X \setminus U) \geq q + 1$ ,  $U$  is smooth and  $\mathcal{F}|_U$  is a vector bundle on  $U$ . The codimension of singular locus of  $X$  determines the property of  $(RS_q)$  of a reflexive sheaf  $\mathcal{F}$  on  $X$  of rank 1.

**Lemma 1.11.** *Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$  with  $(KV_q)$ . If  $X$  is  $(R_q)$ , then  $\mathcal{F}$  is  $(RS_q)$ .*

**Example 1.12.** We can give examples of reflexive sheaves of rank 1 with  $(RS_q)$ . Let  $X$  be any normal variety over  $\mathbb{C}$  and suppose  $\Delta$  is an effective Weil divisor such that  $(X, \Delta)$  is log canonical. If  $D$  is a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$ , then  $D$  is  $(RS_{q+1})$ , where  $q$  is the codimension of the union of the non-klt centers of  $(X, \Delta)$ .

It would be interesting to know whether any closed subvariety defined by the points  $x \in X$  in which  $(\theta_{\mathcal{F},f})_x$  is not a quasi-isomorphism for some resolution  $f : X' \rightarrow X$  is a non-klt center of  $(X, \Delta)$  or not. Note that if  $D = 0$ , then there is an affirmative answer. See Theorem 1.2 in [AH12].

By adopting the idea of proof of Lemma 3.3 in [Kov99], we can define the notion of  $(B_{q+1})$  as a dual of the notion of  $(S_{q+1})$  and prove a duality theorem about a reflexive sheaf of rank 1 with rationality condition.

We say  $\mathcal{F}$  is  $(S_{q+1})$  if  $H_x^i(X, \mathcal{F}) = 0$  for  $i < \min\{q+1, \text{codim}_X x\}$  and any point  $x \in X$ . In (4.3) in [Koll11], Kollár remarked that  $\mathcal{O}_X$  is  $(S_3)$  does not imply the condition that  $\omega_X$  is  $(S_3)$ , and therefore it is hard to say that the depth condition on  $\omega_X$  is a formal consequence of the depth condition on  $\mathcal{O}_X$ . Our result might answer what the formal consequence of the depth condition on  $\mathcal{O}_X$  about condition of  $\omega_X$  should be.

**Definition 1.13.** Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . We say  $\mathcal{F}$  is  $(B_{q+1})$  if  $H_x^i(X, \mathcal{F}) = 0$  for any  $\dim X - q < i < \dim X$  and any closed point  $x \in X$ .

**Theorem 1.14.** *Let  $X$  be any normal projective variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ , which is  $(RS_q)$ . Then the following are equivalent:*

- (a)  $\mathcal{F}$  is  $(S_{q+1})$ .
- (b) For any resolution  $f : X' \rightarrow X$  with  $f$   $(RS_q)$  about  $\mathcal{F}$ ,  $R^i f_*(f^* \mathcal{F})^{DD} = 0$  for any  $1 \leq i < q$ .
- (c)  $\mathcal{F}^D$  is  $(B_{q+1})$ .

Using the notions of  $(B_{q+1})$  and  $(RS_q)$ , we can prove the following theorem about the CM-ness of normal  $\mathbb{Q}$ -factorial varieties.

**Corollary 1.15.** *Let  $X$  be any normal  $\mathbb{Q}$ -factorial variety with  $(R_q)$ . Assume that any Weil divisor on  $X$  is  $(KV_q)$ , and  $(S_{q+1})$ . If  $q \geq \lceil \frac{\dim X + 1}{2} \rceil$ , then any Weil divisor on  $X$  is CM. In particular,  $X$  is CM itself.*

Let us define the notion of  $q$ -birational morphism which is firstly defined in [Kim22].

**Definition 1.16.** Let  $X, X'$  be any normal varieties and  $f : X' \rightarrow X$  any proper birational morphism.

- (a) The *center* of  $f$  is the reduced closed subscheme  $C$  of  $X$  which is the image of exceptional locus along  $f$ .
- (b) We say  $f$  is a  $q$ -birational morphism if the exceptional locus has codimension 1 and the center of  $f$  has codimension  $\geq q + 1$ .

*Remark 1.17.* For any  $q$ -birational morphism  $f : X' \rightarrow X$  between normal varieties, if  $X'$  is smooth, then  $X$  is  $(R_q)$ .

The argument of the proof of Theorem 1.14 can be used to generalize Theorem 3.4 in [Kim22] as follows:

**Theorem 1.18** (See Theorem 6.3). *Let  $X, X'$  be any varieties over a characteristic 0 field such that  $X$  is normal and  $X'$  is smooth. Let  $f : X' \rightarrow X$  be any  $q$ -birational morphism. Suppose that  $D$  is any anti  $f$ -nef divisor on  $X'$  such that  $f_* D$  is  $\mathbb{Q}$ -Cartier and  $(S_{q+1})$ . Then  $R^i f_* \mathcal{O}_{X'}(D) = 0$  for  $1 \leq i < q$ .*

The rest of the paper is organized as follows. We begin Section 2 by defining some basic notions and by stating and proving basic theorems. Section 3 is devoted to proving basic results about reflexive sheaves. In Section 4, we define (weakly) rational singularities and prove some basic theorems. In Section 5, we define the notion of  $(B_{q+1})$  and prove the main theorem about that notion. Section 6 is devoted to defining  $q$ -birational morphisms and proving the main theorems about  $q$ -birational morphisms.

*Acknowledgments.* The author thanks his advisor Sung Rak Choi for his comments, questions, and discussions. He is grateful for his encouragement and support. The author is grateful to Fabio Bernasconi and Sándor Kovács for their helpful comments on earlier versions of this paper. The author thanks the referee for careful reading, comments, and corrections of the previous version of this paper. We would like to acknowledge the assistance of ChatGPT in polishing the wording.

## 2. PRELIMINARIES

The section is devoted to collecting basic definitions and facts used in the paper.

*Notation 2.1.* Given two normal varieties  $X, X'$ , and a proper morphism  $f : X' \rightarrow X$ , and a (not necessarily closed) point  $x \in X$ , we use the following notation:

- $X'_x := X' \times_X \operatorname{Spec} \mathcal{O}_{X,x}$ ,
- $\dim x$  denotes the dimension of the closure of  $x$ ,
- $\operatorname{codim}_X x := \dim X - \dim x$ .

For any coherent sheaves  $\mathcal{F}, \mathcal{F}'$  on  $X, X'$  respectively and the inclusion  $j : X'_x \rightarrow X'$ , we denote

- $\mathcal{F}'_x := j^* \mathcal{F}'$ ,
- $\mathcal{O}_{X',x} := (\mathcal{O}_{X'})_x$ ,  $\omega_{X',x} := (\omega_{X'})_x$ .

Note that if  $f : X' \rightarrow X$  is a resolution of  $X$ , then  $X'_x$  is a regular scheme.

For simplicity, if  $i \geq 0$ , a point  $x \in X$  and a coherent sheaf  $\mathcal{F}$  on  $X$  are given, set

$$H_x^i(X, \mathcal{F}) := H_x^i(X_x, \mathcal{F}_x).$$

Also, under the same conditions as above, if a resolution  $f : X' \rightarrow X$  and a coherent sheaf  $\mathcal{F}'$  on  $X'$  are given, let

$$H_{f^{-1}(x)}^i(X', \mathcal{F}') := H_{f^{-1}(x)}^i(X'_x, \mathcal{F}'_x).$$

**Definition 2.2.** Let  $X$  be any normal variety and  $\mathcal{F}$  any torsion-free sheaf on  $X$ .

- We say  $\mathcal{F}$  is  $(R_q)$  if there is an open subscheme  $U \subseteq X$  such that  $\operatorname{codim}_X(X \setminus U) \geq q + 1$ ,  $U$  is smooth and  $\mathcal{F}|_U$  is a vector bundle on  $U$ .
- We say  $X$  is  $(R_q)$  if  $\mathcal{O}_X$  is  $(R_q)$ .
- We say  $\mathcal{F}$  is  $(S_{q+1})$  if  $H_x^i(X, \mathcal{F}) = 0$  for  $i < \min\{q + 1, \operatorname{codim}_X x\}$  and any point  $x \in X$ .
- We say  $\mathcal{F}$  is *Cohen-Macaulay (CM)* if  $\mathcal{F}$  is  $(S_{\dim X})$ .
- We say  $X$  is *Cohen-Macaulay (CM)* if  $\mathcal{O}_X$  is CM.
- We say  $\mathcal{F}$  is *reflexive* if  $\mathcal{F}$  is  $(S_2)$ .

*Remark 2.3.* If  $X$  is any normal variety with  $(R_q)$ , then any reflexive sheaf of rank 1 on  $X$  is also  $(R_q)$ . An analogue for  $(S_{q+1})$  does not hold. Indeed, let  $X := \operatorname{Spec} k[x, y, z, w]/(xy - zw)$ . Then  $X$  is CM but not any reflexive sheaf of rank 1 on  $X$  is CM. See 3.15 in [Koll13].

We will use derived category machinery in the paper, especially in Section 4 and Section 5. Hence, it is worth stating the *local duality* and the *Grothendieck duality*. For a separated scheme  $X$  over  $k$ , let us define the notion of *normalized dualizing complex*  $\omega_X^\bullet$  of  $X$  by  $f^! \omega_{\operatorname{Spec} k}$ , where  $f : X \rightarrow \operatorname{Spec} k$  is the structure morphism.

**Theorem 2.4** (See Lemma 0A85 in [Stacks]). *Let  $X$  be any variety,  $x \in X$  any closed point and  $E$  the injective hull of the residue field of  $\mathcal{O}_{X,x}$ . Write  $\mathfrak{m}_{X,x}$  the maximal ideal of  $\mathcal{O}_{X,x}$ , and  $Z = V$ . Then*

$$R\operatorname{Hom}_{\mathcal{O}_{X,x}}(K, \omega_X^\bullet)_x^\wedge \cong R\operatorname{Hom}_{\mathcal{O}_{X,x}}(R\Gamma_Z(K), E[0])$$

for any  $i$  and  $K \in D(X)$ , where  $(-)_x^\wedge$  denotes the derived completion along  $\mathfrak{m}_{X,x}$ , and  $D(X)$  denotes the derived category of bounded complexes of coherent sheaves on  $X$ .

**Theorem 2.5** (See 0AU3, (4) in [Stacks]). *Let  $X, X'$  be any noetherian separated schemes over  $k$ ,  $f : X' \rightarrow X$  any proper morphism of varieties. For any  $K \in D(X')$ ,*

$$R\mathcal{H}om_{\mathcal{O}_X}(Rf_*K, \omega_X^\bullet) \cong Rf_*R\mathcal{H}om_{\mathcal{O}_{X'}}(K, \omega_{X'}^\bullet) \text{ in } D(X).$$

Let us prove the following duality result. For a similar lemma, see Proposition 11.6 in [Kol97].

**Lemma 2.6.** *Let  $f : X' \rightarrow X$  be any proper birational morphism between normal varieties and  $\mathcal{E}$  any vector bundle on  $X'$ . Fix any point  $x \in X$ . Suppose  $X'$  is Gorenstein. Then for  $i \geq 0$ ,  $(R^i f_* \mathcal{E})_x = 0$  if and only if  $H_{f^{-1}(x)}^{\text{codim}_X x - i}(X', \omega_{X'} \otimes \mathcal{E}^\vee) = 0$ .*

*Proof.* Let  $k$  be the residue field of  $\mathcal{O}_{X,x}$  and  $E$  the injective hull of  $k$  as an  $\mathcal{O}_{X,x}$ -module. We have

$$\begin{aligned} R\Gamma_{f^{-1}(x)}(X', \omega_{X'} \otimes \mathcal{E}^\vee) &\cong R\Gamma_x(X, Rf_*(\omega_{X'} \otimes \mathcal{E}^\vee)) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}(R\mathcal{H}om_{\mathcal{O}_X}(Rf_*(\omega_{X'} \otimes \mathcal{E}^\vee)_x, \omega_{X,x}^\bullet), E) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}(Rf_*R\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_{X'} \otimes \mathcal{E}^\vee, \omega_{X'})_x[\text{codim}_X x], E) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}(Rf_*\mathcal{H}om_{\mathcal{O}_{X'}}(\omega_{X'} \otimes \mathcal{E}^\vee, \omega_{X'})_x[\text{codim}_X x], E) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}((Rf_*\mathcal{E})_x[\text{codim}_X x], E), \end{aligned}$$

where we used the Leray spectral sequence in the first equality, the local duality in the second equality, the Grothendieck duality in the third equality, the fact that  $\omega_{X'}$  is a line bundle on  $X'$  in the forth equality. Hence, since  $E$  is an injective  $\mathcal{O}_{X,x}$ -module, we have the assertion.  $\square$

Let us prove the following corollary.

**Corollary 2.7** (See Lemma 3.5.10 in [Nak04] and Corollary 11.7 in [Kol97]). *Let  $X$  be any normal projective variety over a field of characteristic 0,  $f : X' \rightarrow X$  any resolution of  $X$  and  $D$  any  $\mathbb{Q}$ -Cartier Weil divisor on  $X'$ , which is anti  $f$ -nef. Then  $f_*\mathcal{O}_{X'}(D) = \mathcal{O}_X(f_*D)$ .*

*Proof.* Note that both  $\mathcal{O}_X(f_*D)$  and  $(f_*\mathcal{O}_{X'}(D))^{\vee\vee}$  are reflexive, and they are isomorphic in codimension 1. Hence,  $\mathcal{O}_X(f_*D) \cong (f_*\mathcal{O}_{X'}(D))^{\vee\vee}$  by Theorem 1.12 in [Har94], and thus it suffices to show that  $f_*\mathcal{O}_{X'}(D)$  is reflexive. Let  $x \in X$  be any point of codimension  $\geq 2$ .

Consider the following spectral sequence

$$E_2^{st} = H_x^s(X, R^t f_* \mathcal{O}_{X'}(D)) \implies H_{f^{-1}(x)}^{s+t}(X', \mathcal{O}_{X'}(D)).$$

Inspecting the spectral sequence, we have  $E_2^{10} = E_\infty^{10}$ . Hence, by considering the edge map  $E_\infty^{10} \rightarrow E^1$ , it suffices to show that  $H_{f^{-1}(x)}^1(X', \mathcal{O}_{X'}(D)) = 0$ . Note that by the relative Kawamata-Viehweg vanishing theorem (Theorem 1-2-3 in [KMM87]),  $R^i f_* \mathcal{O}_X(K_{X'} - D) = 0$  for  $i \geq 1$ . Thus, by Lemma 2.6, we have the assertion.  $\square$



*Remark 2.8.* Note that it is a special case of [Nak04], Lemma 3.5.10. Nakayama proved the lemma using the relative Zariski decomposition.

We will use the following lemma without any mention.

**Lemma 2.9** (See Lemma 2.2 in [BMP+20]). *For any normal variety  $X$  and any coherent sheaf  $\mathcal{F}$  on  $X$ , we have*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)[- \dim X] = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X).$$

*Proof.* We may consider the following exact triangle

$$\omega_X[- \dim X] \rightarrow \omega_X^\bullet \rightarrow C \rightarrow$$

for some complex  $C$  in  $D(X)$ . If we apply  $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$  to this triangle, then we obtain the following exact triangle

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X[- \dim X]) \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet) \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, C) \rightarrow .$$

Note that  $C$  has cohomological degree  $\geq 1$  and hence  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, C) = 0$ . Thus, the long exact cohomology sequence gives us the assertion.  $\square$

### 3. REFLEXIVE SHEAVES

In this section, we collect important facts about reflexive sheaves. For a variety  $X$ , the generic point  $\eta \in X$  and a coherent sheaf  $\mathcal{F}$  on  $X$ ,  $\text{rank } \mathcal{F}$  denotes the dimension of  $\mathcal{F}_\eta$  over the function field of  $X$ .

**Definition 3.1** (See [HL10], Definition 1.1.7). Let  $X$  be any variety and  $\mathcal{F}$  any coherent sheaf on  $X$ . We define  $\mathcal{F}^D := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ .

*Remark 3.2.* For any normal Gorenstein variety  $X$  and any reflexive sheaf  $\mathcal{F}$  of rank 1 on  $X$ ,  $\mathcal{F}^D = \mathcal{F}^\vee \otimes \omega_X$ . Moreover, we can construct a natural map  $\mathcal{F} \rightarrow \mathcal{F}^{DD}$ . Indeed, for any local section  $a \in \mathcal{F}$ , consider a map

$$\varphi : \mathcal{F} \rightarrow \mathcal{F}^{DD}, a \mapsto (f \mapsto f(a)) \text{ for } f \in \mathcal{F}^D := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X).$$

Note that if  $X$  is normal, the map  $\mathcal{F}^D \rightarrow \mathcal{F}^{DDD}$  is an isomorphism, because  $\mathcal{F}^D$  is a reflexive sheaf. See Lemma 0AY4 in [Stacks], Lemma 0AWE in [Stacks], and Lemma 3.4 below.

Now, let us prove the following two lemmas.

**Lemma 3.3.** *Let  $X$  be any normal variety and  $\mathcal{F}$  any coherent sheaf on  $X$ . Then the following are equivalent:*

- (a)  $\mathcal{F}$  is torsion-free.
- (b) The map  $\mathcal{F} \rightarrow \mathcal{F}^{DD}$  is injective.



*Proof.* It is worth noting that for any variety  $X$ ,  $\omega_X$  is  $(S_2)$ . See Lemma 0AWE in [Stacks]. Hence, if  $X$  is normal, then  $\omega_X$  is reflexive.

If  $\mathcal{F}$  is torsion-free, then the kernel  $\mathcal{K}$  of the map is torsion-free because any subsheaf of a torsion-free sheaf is also torsion-free. Furthermore, since  $\omega_X$  is reflexive,  $\mathcal{F}^{DD}$  is torsion-free by Lemma 0AY4 in [Stacks]. Consider the fact that for every point  $x \in X$  of codimension 1,  $\mathcal{F}_x \rightarrow \mathcal{F}_x^{DD}$  is an isomorphism (which follows from Lemma 0CC4 in [Stacks], and the fact that  $\mathcal{O}_{X,x}$  is a discrete valuation ring). Then  $\mathcal{K} = 0$  because  $\text{rank } \mathcal{K} = 0$ .

For the converse,  $\mathcal{F}^{DD}$  is torsion-free. By our assumption,  $\mathcal{F}$  is torsion-free because any subsheaf of a torsion-free sheaf is also torsion-free.  $\square$

**Lemma 3.4.** *Let  $X$  be any normal variety and  $\mathcal{F}$  any coherent sheaf on  $X$ . Then the following are equivalent:*

- (a)  $\mathcal{F}$  is reflexive.
- (b) The natural map  $\mathcal{F} \rightarrow \mathcal{F}^{DD}$  is an isomorphism.

*Proof.* Let us assume that  $\mathcal{F}$  is reflexive. By Lemma 0AY4 in [Stacks], we know that  $\mathcal{F}^{DD}$  is also reflexive and  $\mathcal{F} \rightarrow \mathcal{F}^{DD}$  is an isomorphism outside a codimension  $\geq 2$  closed subscheme of  $X$ . Hence the map is an isomorphism by [Har94], Proposition 1.11.

For the converse, let  $\mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F}^D \rightarrow 0$  be any resolution of  $\mathcal{F}$  in which  $\mathcal{E}_0, \mathcal{E}_1$  are vector bundles. By taking  $\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$ , we have the following exact sequence

$$0 \rightarrow \mathcal{F} \cong \mathcal{F}^{DD} \rightarrow \mathcal{E}_0^D \rightarrow \mathcal{E}_1^D,$$

and  $\mathcal{E}_i^D$  are reflexive for  $i = 0, 1$ . Hence,  $\mathcal{F}$  is reflexive (see Lemma 0EBG in [Stacks]).  $\square$

*Remark 3.5.* If  $X$  is not normal, then such an equivalence is false. See Remark 0AY1 in [Stacks].

Let us note an important fact about reflexive sheaves. For any normal variety  $X$  and any reflexive sheaf  $\mathcal{F}$  on  $X$ ,  $\mathcal{F}$  is a vector bundle outside codimension  $\geq 2$  closed subscheme of  $X$ . See Lemma 0AY6 in [Stacks].

We will use the following three lemmas without any mention. We believe that they are well-known to experts but we cannot find any reference about them. Hence, we include their proof.

**Lemma 3.6.** *Let  $X$  be any variety and  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  any map between torsion-free sheaves on  $X$ . Suppose that  $\varphi$  is generically injective. Then  $\varphi$  is an injection.*

*Proof.* Let  $\mathcal{K}$  be the kernel of  $\varphi$ . Note that  $\mathcal{K}$  is a subsheaf of a torsion-free sheaf  $\mathcal{F}$ . By the condition on  $\varphi$ ,  $\text{rank } \mathcal{K} = 0$  and hence  $\mathcal{K} = 0$ , because any subsheaf of torsion-free sheaf is also torsion-free.  $\square$

**Lemma 3.7.** *Let  $X, X'$  be any normal varieties and  $f : X' \rightarrow X$  any proper birational morphism. Suppose that  $\mathcal{F}$  is any reflexive sheaf on  $X$ . Then  $\mathcal{F} \cong f_*(f^*\mathcal{F})^{DD}$ .*

*Proof.* Let us consider the adjunction property

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f_*(f^*\mathcal{F})^{DD}) \cong f_*\mathcal{H}om_{\mathcal{O}_{X'}}(f^*\mathcal{F}, (f^*\mathcal{F})^{DD})$$

and the map  $f^*\mathcal{F} \rightarrow (f^*\mathcal{F})^{DD}$ . Then we can construct the following map  $\mathcal{F} \rightarrow f_*(f^*\mathcal{F})^{DD}$ . Note that the map is an injection because it is an isomorphism on the regular locus of  $X$  and  $\mathcal{F}, f_*(f^*\mathcal{F})^{DD}$  are torsion-free sheaves.

Let  $\mathcal{Q}$  be the cokernel. Then we have the following exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow f_*(f^*\mathcal{F})^{DD} \rightarrow \mathcal{Q} \rightarrow 0.$$

Note that the support of  $\mathcal{Q}$  has codimension  $\geq 2$ . Let  $x \in X$  be any point of codimension  $\geq 2$ . Then the local cohomology exact sequence tells us that  $H_x^0(X, \mathcal{Q}) = 0$ . Hence  $x$  is not an associated point of  $\mathcal{Q}$  (cf. Lemma 0EEZ in [Stacks]). Thus  $\mathcal{Q} = 0$  by Lemma 05AG in [Stacks].  $\square$

**Lemma 3.8.** *Let  $X, Y$  be any normal varieties and  $p : X' \rightarrow X$  any proper flat morphism. Suppose that  $\mathcal{F}$  is any coherent sheaf on  $X$ . Then  $(p^*\mathcal{F})^{DD} = p^*(\mathcal{F}^{DD})$ .*

*Proof.* Note that the pullback of a reflexive sheaf along a flat morphism is also reflexive. See Proposition 1.8 in [Har80].

Taking the natural map  $\mathcal{F} \rightarrow \mathcal{F}^{DD}$ , we have a map  $\varphi : (p^*\mathcal{F})^{DD} \rightarrow p^*(\mathcal{F}^{DD})$ . We know that both  $(p^*\mathcal{F})^{DD}, p^*(\mathcal{F}^{DD})$  on  $Y$  are reflexive and  $\varphi$  is an isomorphism outside codimension  $\geq 2$  closed subscheme of  $Y$  (see Lemma 3.4). Hence by [Har94], Proposition 1.11, we have the assertion.  $\square$

#### 4. RATIONAL SINGULARITIES

Throughout the remainder of this paper, for simplicity, assume that any reflexive sheaf has rank 1, unless explicitly mentioned otherwise. Note that the condition of being rank 1 is only used in the proof of Definition-Theorem 4.1 to ensure that

$$\mathcal{E}xt_{\mathcal{O}_{X'}}^i((f^*\mathcal{F})^{DD}, \omega_{X'}) = 0 \text{ for } i > 0$$

if we use the notations in Definition-Theorem 4.1 below, and hence we believe such condition can be loosened easily.

**Definition-Theorem 4.1.** Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . Suppose that  $f : X' \rightarrow X$  is a generically finite morphism with  $X'$  smooth. Then we have a natural map

$$\theta_{\mathcal{F}, f} : Rf_*(f^*\mathcal{F})^D \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)[- \dim X].$$

Moreover, if  $f$  is a resolution, the following are equivalent:

- (a)  $\theta_{\mathcal{F}, f}$  is a quasi-isomorphism.
- (b)  $R^i f_*(f^*\mathcal{F})^{DD} = 0$  for  $i \geq 1$ .

(c) The natural map

$$H_x^i(X_x, \mathcal{F}_x) \rightarrow H_{f^{-1}(x)}^i(X'_x, (f^*\mathcal{F})_x^{DD})$$

is an isomorphism for each  $i \geq 0$  and each point  $x \in X$ .

We say that  $f$  is a *weakly rational resolution* of  $\mathcal{F}$  if one of the above is true for  $f$ , and that  $\mathcal{F}$  has *weakly rational singularities* if one of the resolutions of  $X$  is weakly rational resolution of  $\mathcal{F}$ .

*Proof.* The following proof is inspired by the argument in the proof of Lemma 3.7 in [ST08].

Let us construct  $\theta_{\mathcal{F},f}$ . Consider a composition  $\mathcal{F} \rightarrow f_*(f^*\mathcal{F})^{DD} \rightarrow Rf_*(f^*\mathcal{F})^{DD}$ . We may take  $R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$  on the map and it gives

$$R\mathcal{H}om_{\mathcal{O}_X}(Rf_*(f^*\mathcal{F})^{DD}, \omega_X^\bullet) \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet).$$

Now, we may compute the left-hand side by the Grothendieck duality as follows:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{O}_X}(Rf_*(f^*\mathcal{F})^{DD}, \omega_X^\bullet) &= Rf_*R\mathcal{H}om_{\mathcal{O}_{X'}}((f^*\mathcal{F})^{DD}, \omega_{X'}^\bullet)[\dim X] \\ &= Rf_*\mathcal{H}om_{\mathcal{O}_{X'}}((f^*\mathcal{F})^{DD}, \omega_{X'}^\bullet)[\dim X] \\ &= Rf_*(f^*\mathcal{F})^D[\dim X], \end{aligned}$$

where we used the fact that  $(f^*\mathcal{F})^{DD}$  is a line bundle on  $X'$  on the second equality. Therefore, we have a map

$$\theta_{\mathcal{F},f} : Rf_*(f^*\mathcal{F})^D \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)[- \dim X].$$

For (a)  $\iff$  (b), if (b) is true, then  $\mathcal{F} \cong Rf_*(f^*\mathcal{F})^{DD}$ . Thus,  $\theta_{\mathcal{F},f}$  is a quasi-isomorphism and (a) is true. For the converse, we may take  $R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$  on  $\theta_{\mathcal{F},f}$ . See 0AU3 (3) in [Stacks].

For (a)  $\iff$  (c), we may take the local duality to  $\theta_{\mathcal{F},f}$ . Note that the dual of  $\theta_{\mathcal{F},f}$  is

$$R\Gamma_x(X_x, \mathcal{F}_x) \rightarrow R\Gamma_x(X_x, Rf_*(f^*\mathcal{F})^{DD})$$

and the right hand side is  $R\Gamma_{f^{-1}(\{x\})}(X'_x, (f^*\mathcal{F})^{DD})$  by the Leray spectral sequence.  $\square$

*Remark 4.2.* If we use the notions of Definition-Theorem 4.1, then  $\mathcal{H}^0(\theta_{\mathcal{F},f})$  is

$$\mathcal{H}^0(\theta_{\mathcal{F},f}) : f_*(f^*\mathcal{F})^D \rightarrow \mathcal{F}^D = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X).$$

If we let  $\mathcal{F} = \mathcal{O}_X$ , then  $\mathcal{H}^0(\theta_{\mathcal{F},f})$  is the trace map introduced in [KM98], Proposition 5.77.

Let  $f : X' \rightarrow X, g : X'' \rightarrow X'$  be generically finite morphisms between normal varieties, and  $\mathcal{F}$  any coherent sheaf on  $X$ . If  $X', X''$  are smooth, then the map

$$\mathcal{H}^0(\theta_{\mathcal{F},f \circ g}) : (f \circ g)_*((f \circ g)^*\mathcal{F})^D \rightarrow \mathcal{F}^D$$

can be factored as follows:

$$(f \circ g)_*((f \circ g)^*\mathcal{F})^D \xrightarrow{\mathcal{H}^0(\theta_{(f^*\mathcal{F})^{DD}, g})} f_*(f^*\mathcal{F})^D \xrightarrow{\mathcal{H}^0(\theta_{\mathcal{F},f})} \mathcal{F}^D.$$

Indeed, let  $u : (f^*\mathcal{F})^{DD} \rightarrow Rg_*((f \circ g)^*\mathcal{F})^{DD}$  be the natural map. Then by definition,

$$\theta_{g,(f^*\mathcal{F})^{DD}} = R\mathcal{H}om_{\mathcal{O}_{X'}}(u, \omega_{X'}^\bullet)[- \dim X].$$

If we consider

$$\mathcal{F} \rightarrow Rf_*(f^*\mathcal{F})^{DD} \xrightarrow{Rf_*(u)} R(f \circ g)_*((f \circ g)^*\mathcal{F})^{DD},$$

by taking  $R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$  on that, then we have the maps

$$(4.1) \quad R(f \circ g)_*((f \circ g)^*\mathcal{F})^D \xrightarrow{\alpha} Rf_*(f^*\mathcal{F})^D \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)[- \dim X].$$

If we use the Grothendieck duality, then we obtain

$$\alpha = Rf_* \circ R\mathcal{H}om_{\mathcal{O}_{X'}}(u, \omega_{X'}^\bullet)[- \dim X] = Rf_* \circ \theta_{g,(f^*\mathcal{F})^{DD}}.$$

If we apply  $\mathcal{H}^0$  on (4.1), then we have the assertion.

**Proposition 4.3.** *Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$  which has weakly rational singularities. Then the following are equivalent:*

- (a)  $\mathcal{F}$  is  $(S_{q+1})$ .
- (b) For some weakly rational resolution  $f : X' \rightarrow X$ , the support of  $R^i f_*(f^*\mathcal{F})^D$  has codimension  $\geq i + q + 1$  for any  $i \geq 1$ .
- (c) The support of  $R^i f_*(f^*\mathcal{F})^D$  has codimension  $\geq i + q + 1$  for any  $i \geq 1$  and any weakly rational resolution  $f : X' \rightarrow X$  of  $X$ .

*Proof.* For (a)  $\implies$  (c), given any weakly rational resolution  $f : X' \rightarrow X$ , by our assumption, there is an isomorphism

$$\theta_{\mathcal{F},f} : Rf_*(f^*\mathcal{F})^D \cong R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet)[- \dim X].$$

Moreover, for any point  $x \in X$  with  $\text{codim}_X x \leq i + q$ ,  $H_x^{\text{codim}_X x - i}(X, \mathcal{F}) = 0$  holds, because  $\mathcal{F}$  is  $(S_{q+1})$ . Hence, by the local duality,  $\text{Ext}_{\mathcal{O}_{X,x}}^{-\text{codim}_X x + i}(\mathcal{F}_x, \omega_{X,x}^\bullet) = 0$  for such  $i$ . Thus,  $R^i f_*(f^*\mathcal{F})_x^D = 0$  and the support of  $R^i f_*(f^*\mathcal{F})^D$  does not contain  $x$ .

(c)  $\implies$  (b) is trivial.

For (b)  $\implies$  (a), consider a point  $x \in X$ . If  $\text{codim}_X x \leq q$ , then  $R^i f_*(f^*\mathcal{F})_x^D = 0$  and  $\text{Ext}_{\mathcal{O}_{X,x}}^{-\text{codim}_X x + i}(\mathcal{F}_x, \omega_{X,x}^\bullet) = 0$  hold for any  $i \geq 1$ . Therefore, the local duality gives

$$H_x^{\text{codim}_X x - i}(X, \mathcal{F}) = 0 \text{ for any } i \geq 1.$$

If  $\text{codim}_X x \geq q + 1$  and  $R^i f_*(f^*\mathcal{F})_x^D = 0$ , by the local duality,  $H_x^{\text{codim}_X x - i}(X, \mathcal{F}) = 0$ . Furthermore, if  $\text{codim}_X x - i < q$ , then  $R^i f_*(f^*\mathcal{F})_x^D = 0$ . Hence, for  $0 \leq j < q$ ,  $H_x^j(X, \mathcal{F}) = 0$ .  $\square$

Now, we can define the notion of *rational singularities* for any reflexive sheaf of rank 1.

**Definition 4.4.** Let  $X$  be any normal variety,  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ .

- (a) We say that  $\mathcal{F}$  has *rational singularities* if  $\mathcal{F}$  has weakly rational singularities and  $\mathcal{F}$  is CM.

- (b) We say that a Weil divisor  $D$  on  $X$  has *rational singularities* if  $\mathcal{O}_X(D)$  has rational singularities.
- (c) Let  $f : X' \rightarrow X$  be a resolution. We say  $f$  is  $(KV_q)$  *about*  $\mathcal{F}$  if the support of  $R^i f_*(f^* \mathcal{F})^D$  has codimension  $\geq i + q + 1$  for any  $i \geq 1$ . We call  $\mathcal{F}$   $(KV_q)$  if there is a resolution of  $X$  which is  $(KV_q)$  about  $\mathcal{F}$ .

**Example 4.5.** Having weakly rational singularities does not ensure having rational singularities.

For example, take  $X := \text{Spec } k[x, y, z, w]/(xy - zw)$ ,  $A := \{x = z = 0\}$ , the blowup  $f : X' \rightarrow X$  along  $A$  and the exceptional divisor  $E$  on  $X'$ . We know that  $f$  is a small resolution and  $(f^* \mathcal{O}_X(-mA))^{DD} = \mathcal{O}_{X'}(-mE)$ . Then for  $m \gg 0$ ,  $R^i f_*(f^* \mathcal{O}_X(-mA))^{DD} = 0$  for  $i \geq 1$ . Thus,  $-mA$  has weakly rational singularities for such  $m$ . However, for  $m \geq 2$ ,  $-mA$  is not CM as in (3.15) in [Kol13] and hence  $-mA$  does not have rational singularities.

Let us prove the following lemma. Similar statements are proved in Lemma 3.2 in [Ale08] and Theorem 7.1.1 in [Fuj17].

**Lemma 4.6.** *Let  $f : X' \rightarrow X$  be any proper birational morphism of varieties and  $\mathcal{F}, \mathcal{F}'$  any coherent sheaves on  $X'$ . For any given positive integer  $n \geq 1$ , suppose the three conditions hold:*

- (1) *There is an injection  $\iota : \mathcal{F} \rightarrow \mathcal{F}'$ ,*
- (2)  *$\iota$  induces an isomorphism  $f_* \mathcal{F} \cong f_* \mathcal{F}'$ , and*
- (3)  *$R^i f_* \mathcal{F}' = 0$  for  $1 \leq i < n$ .*

*Then we have  $R^i f_* \mathcal{F} = 0$  for  $1 \leq i < n$ .*

*Proof.* Consider any point  $x \in X$ . We have the following two spectral sequences

$$\begin{aligned} {}^1E_2^{st} &= H_x^s(X, R^t f_* \mathcal{F}) \implies H_{f^{-1}(x)}^{i+j}(X', \mathcal{F}), \\ {}^2E_2^{st} &= H_x^s(X, R^t f_* \mathcal{F}') \implies H_{f^{-1}(x)}^{i+j}(X', \mathcal{F}'). \end{aligned}$$

From (3), we deduce  ${}^2E_2^{i0} = {}^2E_\infty^{i0}$ , making the edge map  ${}^2E_2^{i0} \rightarrow {}^2E^i$  injective for all  $0 \leq i \leq n$ . Using (1), we have the following diagram:

$$\begin{array}{ccc} {}^1E^i = H_{f^{-1}(x)}^i(X', \mathcal{F}) & \longrightarrow & {}^2E^i = H_{f^{-1}(x)}^i(X', \mathcal{F}') \\ \alpha_i \uparrow & & \beta_i \uparrow \\ {}^1E_2^{i0} = H_x^i(X, f_* \mathcal{F}) & \xrightarrow{\gamma_i} & {}^2E_2^{i0} = H_x^i(X, f_* \mathcal{F}'). \end{array}$$

Given (2), for any  $i$ ,  $\gamma_i$  is an isomorphism. Since  $\beta_i$  is an injection for  $1 \leq i \leq n$ ,  $\alpha_i$  is also an injection for such  $i$ .

Now, we may use induction. Indeed, suppose for a positive integer  $1 < n' \leq n$ ,  $R^i f_* \mathcal{F} = 0$  for  $1 \leq i < n'$ . Our goal is to show  $R^{n'} f_* \mathcal{F} = 0$ . By the induction hypothesis, we have  ${}^1E_2^{0n'} = {}^1E_n^{0n'}$  and  ${}^1E_2^{(n'+1)0} = {}^1E_n^{(n'+1)0}$ . Hence, there is an exact sequence

$$0 \rightarrow {}^1E_2^{0n'} \rightarrow {}^1E_2^{(n'+1)0} \xrightarrow{\beta_{n'+1}} {}^1E^{n'+1}.$$

Since  $\beta_{n'+1}$  is an isomorphism, we obtain  $H_x^0(X, R^{n'+1}f_*\mathcal{F}) = {}^1E_2^{(n'+1)0} = 0$  and hence  $R^{n'+1}f_*\mathcal{F}$  does not have (the arbitrary point)  $x \in X$  as an associated point. Thus  $R^{n'+1}f_*\mathcal{F} = 0$  and we have the assertion.  $\square$

Now, we have an alternative description of the notion of rational singularities.

**Proposition 4.7.** *Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . Then  $\mathcal{F}$  has rational singularities if and only if for a resolution  $f : X' \rightarrow X$ , there is a line bundle  $\mathcal{L}_{\mathcal{F},f}$  on  $X'$  such that the three following conditions hold:*

- (i)  $f_*\mathcal{L}_{\mathcal{F},f} = \mathcal{F}$ ,
- (ii)  $R^if_*\mathcal{L}_{\mathcal{F},f} = 0$  for  $i \geq 1$ , and
- (iii)  $R^if_*\mathcal{L}_{\mathcal{F},f}^D = 0$  for  $i \geq 1$ .

*Proof.* For the only if direction, set  $\mathcal{L}_{\mathcal{F},f} := (f^*\mathcal{F})^{DD}$ . Now, apply Lemma 3.7 and Definition-Theorem 4.1.

For the if direction, let us consider the counit map  $f^*f_*\mathcal{L}_{\mathcal{F},f} \rightarrow \mathcal{L}_{\mathcal{F},f}$ . By taking double dual, we obtain an injection  $(f^*f_*\mathcal{L}_{\mathcal{F},f})^{DD} \rightarrow \mathcal{L}_{\mathcal{F},f}$ . Since (i) holds,  $(f^*f_*\mathcal{L}_{\mathcal{F},f})^{DD} = (f^*\mathcal{F})^{DD}$ . Thus, there is an injection  $(f^*\mathcal{F})^{DD} \rightarrow \mathcal{L}_{\mathcal{F},f}$ .

Considering Lemma 3.7, Lemma 4.6, (i) and (ii), we have  $R^if_*(f^*\mathcal{F})^{DD} = 0$  for  $i \geq 1$ . Let us use the notations in Lemma 4.6. By (iii) and Lemma 2.6, we obtain

$$H_{f^{-1}(x)}^i(X', \mathcal{L}_{\mathcal{F},f}) = 0 \text{ for } i < \text{codim}_X x.$$

Since  $\beta_i$  is an isomorphism for  $i \geq 1$ , we deduce

$$H_x^i(X, \mathcal{F}) = H_{f^{-1}(x)}^i(X', \mathcal{L}_{\mathcal{F},f}) = 0 \text{ for } i < \text{codim}_X x$$

and hence  $\mathcal{F}$  is CM. Thus  $\mathcal{F}$  has rational singularities.  $\square$

Now, we can see that our notion of rational singularities is weaker than the notion of rational singularities in [Kov11] and [Kol13].

**Corollary 4.8.** *Let  $X$  be a normal variety over a characteristic 0 field, and  $D$  a reduced Weil divisor on  $X$ . If  $(X, D)$  has rational singularities in the sense of [Kov11], then  $-D$  has rational singularities.*

*Proof.* Let  $f : (X', D_{X'}) \rightarrow (X, D)$  be a log resolution of  $(X, D)$ . Then we may put  $\mathcal{L}_{\mathcal{O}_X(-D),f} := \mathcal{O}_{X'}(-D_{X'})$  and use Proposition 4.7. Now, use Theorem 2.9 in [Kov11].  $\square$

Before proving any  $\mathbb{Q}$ -Cartier Weil divisor on a dlt pair  $(X, \Delta)$  over a characteristic 0 field has rational singularities, let us prove the following lemma.

**Lemma 4.9.** *Let  $X$  be a normal variety,  $f : X' \rightarrow X$  a resolution and  $D$  a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$ . For a Cartier divisor  $L$  on  $X'$  such that  $\mathcal{O}_{X'}(L) = (f^*\mathcal{O}_X(D))^{DD}$  holds, there is an effective  $f$ -exceptional divisor  $E$  such that  $L + E \sim_{\mathbb{Q}} f^*D$  holds.*

*Proof.* Let  $d$  be a positive integer such that  $dD$  is Cartier. Let us consider the following map

$$\mathcal{O}_X(D) \otimes \cdots \otimes \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(dD).$$

Then by applying  $f^*$  to both sides, and reflexing, we have an injection

$$\mathcal{O}_{X'}(dL) = (f^*\mathcal{O}_X(D) \otimes \cdots \otimes f^*\mathcal{O}_X(D))^{dD} \rightarrow \mathcal{O}_{X'}(f^*(dD)).$$

Thus  $f^*(dD) - dL$  is effective. We may choose  $L$  so that  $f_*L = D$  holds. Moreover, since  $f_*(f^*(dD) - dL) = dD - dD = 0$  holds,  $f^*(dD) - dL$  is  $f$ -exceptional. Now, put  $E := \frac{1}{d}(f^*(dD) - dL)$ .  $\square$

Now, we prove the following two properties.

**Theorem 4.10.** *Let  $X$  any normal variety over a characteristic 0 field,  $\Delta$  any effective  $\mathbb{Q}$ -Weil divisor such that  $(X, \Delta)$  is dlt and  $D$  any  $\mathbb{Q}$ -Cartier Weil divisor on  $X$ . Then  $D$  has rational singularities.*

*Proof.* It suffices to prove the case of  $(X, \Delta)$  klt. In fact, by Proposition 2.43 in [KM98], for any ample Cartier divisor  $H$  on  $X$ , there is  $0 < \varepsilon \ll 1$ , a rational number  $c > 0$ , and an effective  $\mathbb{Q}$ -divisor  $D \equiv \Delta + cH$  such that  $(X, (1 - \varepsilon)\Delta + D)$  is klt. Now, replace  $\Delta$  with  $(1 - \varepsilon)\Delta + D$ .

Suppose  $f : X' \rightarrow X$  is any log resolution of  $(X, \Delta)$ , and write

$$K_{X'} = f^*(K_X + \Delta) + F - F',$$

where  $F$  is an effective,  $f$ -exceptional Cartier divisor and  $F'$  is a simple normal crossing divisor with  $[F'] = 0$ . Then  $F \sim_{\mathbb{Q}} K_{X'} - f^*(K_X + \Delta) + F'$ . If we pick a divisor  $L$  on  $X'$  such that  $\mathcal{O}_{X'}(L) \cong (f^*\mathcal{O}_X(D))^{dD}$ , then by Lemma 4.9, there is an effective  $f$ -exceptional divisor  $E$  such that  $L + E \sim_{\mathbb{Q}} f^*D$  holds, and we have

$$F + L + [E] \sim_{\mathbb{Q}} K_{X'} + F' - \{E\} + (f^*D - f^*(K_X + \Delta)).$$

Now, let us use Proposition 4.7 to prove this assertion with  $\mathcal{L}_{\mathcal{F}, f} = \mathcal{O}_{X'}(L + [E])$ .

For (i), if we consider an injection  $\varphi : f_*\mathcal{O}_{X'}(L) \rightarrow f_*\mathcal{O}_{X'}(L + [E])$  which is induced by  $\mathcal{O}_{X'}(L) \rightarrow \mathcal{O}_{X'}(L + [E])$ ,  $\varphi$  is an isomorphism outside of codimension  $\geq 2$  closed subset of  $X$ . If we let  $\mathcal{Q}$  to be the cokernel of  $\varphi$ , then we have an exact sequence

$$0 \rightarrow f_*\mathcal{O}_{X'}(L) \cong \mathcal{O}_X(D) \rightarrow f_*\mathcal{O}_{X'}(L + [E]) \rightarrow \mathcal{Q} \rightarrow 0.$$

The first isomorphism is by Lemma 3.7. As the proof of Lemma 3.7, by taking the local cohomology spectral sequence, we obtain  $H_x^0(X, \mathcal{Q}) = 0$  for a point  $x \in X$  with codimension  $\geq 2$ , and we can prove  $\mathcal{Q} = 0$ .

For (ii), let  $E_i$  be the prime  $f$ -exceptional divisors, and let us define

$$a_i := \begin{cases} 1 & \text{if } \text{mult}_{E_i}(F' - \{E\}) < 0, \text{ and} \\ 0 & \text{elsewhere,} \end{cases}$$



and

$$E' := \sum_i a_i E_i.$$

Then

$$F + L + \lfloor E \rfloor + E' \sim_{\mathbb{Q}, f} K_{X'} + F' - \{E\} + E',$$

and  $\lfloor F' - \{E\} + E' \rfloor = 0$ . Therefore, by the relative Kawamata-Viehweg vanishing theorem (Theorem 1-2-3 in [KMM87]),  $R^i f_* \mathcal{O}_X(F + L + \lfloor E \rfloor + E') = 0$ . If we consider the injection  $\mathcal{O}_X(L + \lfloor E \rfloor) \rightarrow \mathcal{O}_X(F + L + \lfloor E \rfloor + E')$ , then we have  $R^i f_* \mathcal{O}_X(L + \lfloor E \rfloor) = 0$  by Lemma 4.6.

For (iii), by the relative Kawamata-Viehweg vanishing theorem,

$$R^i f_* \mathcal{O}_{X'}(K_{X'} - (L + \lfloor E \rfloor)) = R^i f_* \mathcal{O}_{X'}(K_{X'} + \{E\}) = 0 \text{ for } i > 0.$$

Thus, we achieved all the conditions listed in Proposition 4.7, and thus we deduce the assertion.  $\square$

**Theorem 4.11.** *Let  $X$  any strongly  $F$ -regular variety over a field of characteristic  $p > 0$  and  $D$  any Weil divisor such that there is an integer  $r$  such that  $rD$  is linearly equivalent to a Cartier divisor and  $r$  is relatively prime to  $p$ . Then  $D$  has rational singularities.*

*Proof.* Since having rational singularities is a local property, we may assume  $X := \text{Spec } R$  is an affine scheme and  $rD \sim 0$ . Moreover, by multiplying some integer on  $r$ , we may take  $r = p^e - 1$  for sufficiently large  $e$ . Let  $x \in X$  be a point of  $X$  and  $f : X' \rightarrow X$  any resolution. We follow the proof of Theorem 3.1 in [PS14].

We may equip an endomorphism  $F_*^e : H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D)) \rightarrow H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$  on  $H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$  as an endomorphism of an abelian group. Indeed, if we consider the inclusion  $\mathcal{O}_X \subseteq F_*^e \mathcal{O}_X$ , then by tensoring  $\mathcal{O}_X(D)$  and reflexing, we have a map  $\mathcal{O}_X(D) \rightarrow F_*^e \mathcal{O}_X(p^e D)$ . Since  $(p^e - 1)D \sim 0$  holds, we obtain a map  $\mathcal{O}_X(D) \rightarrow F_*^e \mathcal{O}_X(D)$ . Taking local cohomology gives the desired map. Let us call a submodule  $K \subseteq H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$   $F$ -stable if  $F_*^e(K) \subseteq K$ .

Let  $R^\wedge$  be the completion of  $R$  along  $x$ ,  $K \subsetneq H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$  any  $F$ -stable submodule of  $H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$ . It is worth noting that  $H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$ , and its submodules are also  $R^\wedge$ -modules.

Suppose  $c \in \text{Ann}_{R^\wedge} K$  is any nontrivial element of  $\text{Ann}_{R^\wedge} K$ . Note that such an element exists because of the Matlis duality. Indeed, if we let  $N^\wedge := R^\wedge N$  for any  $R$ -module  $N$ , and  $N'$  is the  $R$ -module corresponding to  $\mathcal{O}_X(D)$ , then the inclusion  $K \subsetneq H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D))$  gives us a surjection

$$\text{Hom}_{R^\wedge}((N')^\wedge, \omega_{R^\wedge}) \rightarrow \text{Hom}_{R^\wedge}(K, E),$$

where  $E$  is the injective hull of the residue field of  $R$  (see Theorem 3.5.8 in [BH98]). Since  $\omega_{R^\wedge}, (N')^\wedge$  are torsion-free,

$$0 \neq \text{Ann}_{R^\wedge}(\text{Hom}_{R^\wedge}(K, E)) = \text{Ann}_{R^\wedge} K.$$

Note that  $R^\wedge$  is also strongly  $F$ -regular. There is a positive integer  $e'$  and a map  $\varphi : F_*^{e'} R^\wedge \rightarrow R^\wedge$  such that the composition

$$(4.2) \quad R^\wedge \rightarrow F_*^{e'} R^\wedge \xrightarrow{F_*^{e'}(\cdot^c)} F_*^{e'} R^\wedge \xrightarrow{\varphi} R^\wedge$$

is the identity. We may assume  $e|e'$  and hence  $(p^e - 1)|(p^{e'} - 1)$ . Twisting by  $\mathcal{O}_X(D)$  and reflexing on (4.2), we have the following composition which is the identity

$$(N')^\wedge \rightarrow F_*^{e'}(N')^\wedge \rightarrow F_*^{e'}(N')^\wedge \rightarrow (N')^\wedge.$$

Now, taking local cohomology gives us

$$\begin{aligned} & H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D)) \\ & \rightarrow H_x^{\text{codim}_X x}(X, F_*^{e'} \mathcal{O}_X(D)) \\ & \xrightarrow{F_*^{e'}(\cdot^c)} H_x^{\text{codim}_X x}(X, F_*^{e'} \mathcal{O}_X(D)) \\ & \rightarrow H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D)). \end{aligned}$$

Since  $K$  is  $F$ -stable and  $cK = 0$  holds,  $K = 0$ .

We may mimic the argument of Smith (see [Smi97], Theorem 3.1). Let  $f : X' \rightarrow X$  be a resolution. Now, consider the Leray spectral sequence

$$(4.3) \quad E_2^{st} = H_x^s(X, R^t f_*(f^* \mathcal{O}_X(D))^{DD}) \implies H_{f^{-1}(x)}^{s+t}(X', (f^* \mathcal{O}_X(D))^{DD})$$

and its edge map  $\delta_D : H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D)) \rightarrow H_{f^{-1}(x)}^{\text{codim}_X x}(X', (f^* \mathcal{O}_X(D))^{DD})$ . Let  $K_D$  be the kernel of  $\delta_D$ . If we consider Proposition 1.12 in [Smi97] and set  $\psi$  by

$$\psi : (f^* \mathcal{O}_X(D))^{DD} \rightarrow F_*^e(f^* \mathcal{O}_X(p^e D))^{DD},$$

then we have a diagram

$$\begin{array}{ccc} H_{f^{-1}(x)}^{\text{codim}_X x}(X', (f^* \mathcal{O}_X(D))^{DD}) & \xrightarrow{F_*^e} & H_{f^{-1}(x)}^{\text{codim}_X x}(X', (f^* \mathcal{O}_X(D))^{DD}) \\ \delta_D \uparrow & & \uparrow \delta_D \\ H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D)) & \xrightarrow{F_*^e} & H_x^{\text{codim}_X x}(X, \mathcal{O}_X(D)) \end{array}$$

by considering the argument in the proof of Theorem 3.1 in [Smi97] almost unchangingly. For any  $c \in K_D$ ,  $\delta_D(F_*^e(c)) = F_*^e(\delta_D(c)) = 0$  and hence  $K_D$  is  $F$ -stable. Therefore  $K_D = 0$ . It means  $\delta_D$  is an injection.

We may use induction. Suppose  $n \geq 2$  is an integer and assume  $R^i f_*(f^* \mathcal{O}_X(D))^{DD} = 0$  for  $1 \leq i < n - 1$ . Then by inspecting (4.3), we obtain an exact sequence

$$0 \rightarrow H_x^0(X, R^{n-1} f_*(f^* \mathcal{O}_X(D))^{DD}) \rightarrow H_x^n(X, \mathcal{O}_X(D)) \xrightarrow{\delta_{n,D}} H_{f^{-1}(x)}^n(X', (f^* \mathcal{O}_X(D))^{DD}).$$

For  $n < \text{codim}_X x$ , we know  $\mathcal{O}_X(D)$  is CM by Theorem 3.2 in [PS14] and hence  $\delta_{n,D}$  is an injection. Moreover, we have proven that  $\delta_{\text{codim}_X x, D}$  is an injection. Thus,  $H_x^0(X, R^n f_*(f^* \mathcal{O}_X(D))^{DD}) = 0$  and  $R^n f_*(f^* \mathcal{O}_X(D))^{DD}$  does not have an associated

point as  $x$ . This argument works for any point  $x \in X$  and therefore  $R^{n-1}f_*(f^*\mathcal{O}_X(D))^{DD} = 0$ . By considering Definition-Theorem 4.1,  $D$  has weakly rational singularities.

The remaining is to show  $R^i f_*(f^*\mathcal{O}_X(D))^D = 0$  for  $i \geq 1$ . By Lemma 2.6, it suffices to show  $H_{f^{-1}(x)}^{\text{codim}_X x-i}(X', (f^*\mathcal{O}_X(D))^{DD}) = 0$  for  $i \geq 1$ . If we consider (4.3), then  $E_2^{st} = 0$  for  $s + t < \text{codim}_X x$  and hence  $E^{\text{codim}_X x-i} = 0$  and that is the assertion.  $\square$

**Example 4.12.** Consider klt varieties over fields with positive characteristic.

On the positive side, let  $X$  be a normal threefold over a field with characteristic  $p > 5$ . Suppose there exists an effective  $\mathbb{Q}$ -divisor  $\Delta$  on  $X$  such that  $(X, \Delta)$  is klt. Referring to Theorem 3 in [BK21] and following a similar approach as in Theorem 4.10, it can be shown that any  $\mathbb{Q}$ -Cartier Weil divisor  $D$  on  $X$  possesses rational singularities.

On the negative side, when the characteristic of  $k$  is 3, there exists a klt  $\mathbb{Q}$ -factorial threefold  $X$  where  $X$  lacks rational singularities, as seen in Theorem 1.2 of [Ber21]. Additionally, for any prime  $p > 2$  and any field  $k$  of characteristic  $p$ , an example of a variety exhibiting terminal singularities without rational singularities is given in Corollary 2.2 of [Tot19].

Let us the following lemma before proving Theorem 1.7, and Theorem 1.8.

**Lemma 4.13.** *Let  $X, Y$  be normal varieties,  $p : Y \rightarrow X$  any finite flat morphism, and  $\mathcal{F}$  any coherent sheaf on  $X$ . If  $p^*\mathcal{F}$  is CM, then  $\mathcal{F}$  is CM.*

*Proof.* Since  $p$  is finite flat,  $p$  is a surjection, and thus  $p$  is faithfully flat. By Lemma 00LM in [Stacks], for any point  $x \in X$ , any regular sequence of  $\mathcal{F}_x$  comes from a regular sequence of  $\mathcal{F}_y$ , where  $p(y) = x$ . Thus, we have the assertion.  $\square$

We aim to establish that the property of having rational singularities remains stable under finite étale morphisms. This may provide a partial resolution to Remark 2.81 (3) in [Kol13].

**Theorem 4.14.** *Let  $X, Y$  be normal varieties over characteristic 0 field,  $p : Y \rightarrow X$  any finite étale morphism and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . If  $p^*\mathcal{F}$  has (resp. weakly) rational singularities, then  $\mathcal{F}$  has (resp. weakly) rational singularities.*

*Proof.* By the functoriality of resolution of singularities (see Theorem 3.36 in [Kol07]), for some resolution  $f : X' \rightarrow X$ , there is a resolution  $f' : Y' \rightarrow Y$  such that  $f, f'$  fit in the following Cartesian diagram:

$$\begin{array}{ccc} Y' \cong X' \times_X Y & \xrightarrow{f'} & Y \\ p' \downarrow & & \downarrow p \\ X' & \xrightarrow{f} & X. \end{array}$$

Note that  $p'$  is finite étale because being finite étale is stable under base change (see Lemma 01TS and Lemma 02GO in [Stacks]). Moreover, we have a split injection  $(f^*\mathcal{F})^{DD} \rightarrow (p')_*(p')^*(f^*\mathcal{F})^{DD}$  by Lemma 3.17 in [KTT+22].

Note that for any coherent sheaf  $\mathcal{G}$  on  $Y$ ,  $R^i p_* \mathcal{G} = 0$  for  $i \geq 1$  by Lemma 02OE in [Stacks]. Now, we consider the following two Grothendieck spectral sequences

$$\begin{aligned} {}^1 E_2^{st} &= R^s p_* (R^t (f')_* ((f')^* (p^* \mathcal{F}))^{DD}) \implies R^{s+t} (p \circ f')_* ((f')^* (p^* \mathcal{F}))^{DD} \\ {}^2 E_2^{st} &= R^s f_* (R^t (p')_* ((p')^* (f^* \mathcal{F}))^{DD}) \implies R^{s+t} (f \circ p')_* ((p')^* (f^* \mathcal{F}))^{DD}. \end{aligned}$$

Note that

$$((f')^* (p^* \mathcal{F}))^{DD} = ((p \circ f')^* \mathcal{F})^{DD} = ((f \circ p')^* \mathcal{F})^{DD} = ((p')^* (f^* \mathcal{F}))^{DD} = (p')^* (f^* \mathcal{F})^{DD}.$$

Hence, by inspecting the spectral sequences, we obtain

$$p_* (R^i (f')_* ((f')^* (p^* \mathcal{F}))^{DD}) = R^i f_* ((p')_* ((p')^* (f^* \mathcal{F}))^{DD})$$

for any  $i \geq 0$ .

Suppose  $p^* \mathcal{F}$  has weakly rational singularities so that  $R^i (f')_* ((f')^* (p^* \mathcal{F}))^{DD} = 0$  for  $i \geq 1$ . Then  $R^i f_* ((p')_* ((p')^* (f^* \mathcal{F}))^{DD}) = 0$  for  $i \geq 1$ . By the splitting

$$(4.4) \quad (f^* \mathcal{F})^{DD} \rightarrow (p')_* (p')^* (f^* \mathcal{F})^{DD} \rightarrow (f^* \mathcal{F})^{DD}$$

and taking  $R^i f_*$  on (4.4), we also have  $R^i f_* (f^* \mathcal{F})^{DD} = 0$  for  $i \geq 1$ . Therefore  $\mathcal{F}$  has weakly rational singularities. If  $p^* \mathcal{F}$  is CM, then  $\mathcal{F}$  is CM because of Lemma 4.13, and thus if  $p^* \mathcal{F}$  has rational singularities, then  $\mathcal{F}$  has rational singularities.  $\square$

*Remark 4.15.* We hope that Theorem 4.14 is true for any field  $k$  of positive characteristic. The difficulty is that there may be no splitting  $(f^* \mathcal{F})^{DD} \rightarrow (p')_* (p')^* (f^* \mathcal{F})^{DD} \rightarrow (f^* \mathcal{F})^{DD}$  if the degree of  $p$  is coprime to the characteristic of  $k$ . Also we hope that Theorem 4.14 is true only when  $p$  is finite but not necessarily étale.

In Theorem 4.14, if  $p$  is a quotient by a finite group which may not be étale, then the major problem for proving Theorem 4.14 is we cannot ensure the splitting of  $(f^* \mathcal{F})^{DD} \rightarrow (p')_* (p')^* (f^* \mathcal{F})^{DD}$  if  $p'$  is generically finite.

Using the notation introduced in Theorem 4.14, suppose that  $\mathcal{F}$  is  $(KV_{\dim X})$ . In this case, the requirement that  $p$  be étale can be relaxed to be flat.

**Theorem 4.16.** *Let  $X, Y$  be normal varieties over characteristic 0 field,  $p : Y \rightarrow X$  any finite flat morphism and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ . If  $\mathcal{F}$  is  $(KV_{\dim X})$ , and  $p^* \mathcal{F}$  has rational singularities, then  $\mathcal{F}$  has rational singularities.*

*Proof.* We borrow the proof of Theorem 1 in [Kov00]. We have proven if  $p^* \mathcal{F}$  is CM, then  $\mathcal{F}$  is CM in Lemma 4.13.

Let

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \downarrow p' & & \downarrow p \\ X' & \xrightarrow{f} & X \end{array}$$

be a diagram, where  $f, f'$  are proper birational morphisms,  $p'$  is a proper morphism and  $X', Y'$  are smooth.

Consider the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\alpha} & Rp_*(p^*\mathcal{F}) \\ \downarrow \beta & & \downarrow \gamma \\ Rf_*(f^*\mathcal{F})^{DD} & \xrightarrow{\delta} & R(p \circ f')_*((p \circ f')^*\mathcal{F})^{DD}. \end{array}$$

By Lemma 3.17 in [KTT+22], there is a left inverse  $\alpha'$  of  $\alpha$ . Moreover,  $\gamma$  is a quasi-isomorphism because  $p^*\mathcal{F}$  has rational singularities. Thus,  $\beta$  has a left inverse  $\beta' := \alpha' \circ \gamma^{-1} \circ \delta$ .

Now, we may consider a resolution  $f : X' \rightarrow X$  which is  $(KV_{\dim X})$  about  $\mathcal{F}$ . If we take  $R\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X^\bullet)$  on

$$\mathcal{F} \xrightarrow{\beta} Rf_*(f^*\mathcal{F})^{DD} \xrightarrow{\beta'} \mathcal{F},$$

then we have

$$\mathcal{F}^D \rightarrow f_*(f^*\mathcal{F})^D \rightarrow \mathcal{F}^D$$

because  $\mathcal{F}$  is CM and  $f$  is  $(KV_{\dim X})$  about  $\mathcal{F}$ . Thus  $f_*(f^*\mathcal{F})^D \cong \mathcal{F}^D$ . If we consider Definition-Theorem 4.1,  $\theta_{\mathcal{F},f}$  is a quasi-isomorphism and, therefore,  $\mathcal{F}$  has rational singularities.  $\square$

Now, we define the following notion.

**Definition 4.17.** Let  $X$  be any normal variety,  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ , and  $f : X' \rightarrow X$  a resolution. We say that  $f$  is  $(RS_q)$  about  $\mathcal{F}$  if

- (a)  $f$  is  $(KV_q)$  about  $\mathcal{F}$ , and
- (b) for any point  $x \in X$  with  $\text{codim}_X x \leq q$ ,  $\theta_{\mathcal{F},f}$  is a quasi-isomorphism after localizing  $\theta_{\mathcal{F},f}$  at  $x$ .

We call  $\mathcal{F}$   $(RS_q)$  if there is a resolution of  $X$  which is  $(RS_q)$  about  $\mathcal{F}$ .

**Lemma 4.18.** Let  $X$  be any normal variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$  with  $(KV_q)$ . If  $X$  is  $(R_q)$ , then  $\mathcal{F}$  is  $(RS_q)$ .

*Proof.* Let  $f : X' \rightarrow X$  be any resolution. By dualizing  $\theta_{\mathcal{F},f}$ , it suffices to show that  $R^i f_*(f^*\mathcal{F})^{DD}$  has support of codimension  $\geq q+1$  for  $i \geq 1$ . For any point  $x \in X$  with  $\text{codim}_X x \leq q$ ,  $\mathcal{F}_x = \mathcal{O}_{X,x}$ . Hence  $(R^i f_*(f^*\mathcal{F}))_x^{DD} = (R^i f_* \mathcal{O}_{X'})_x$  for  $i \geq 0$  by Corollary 3.8.2 in [Har77]. Thus, by Theorem 1.1 in [CR15], we have the assertion.  $\square$

## 5. NOTION OF $(B_{q+1})$

The goal of this section is to introduce the notion of  $(B_{q+1})$ .

**Definition 5.1.** Let  $X$  be a normal variety and let  $\mathcal{F}$  be a reflexive sheaf of rank 1 on  $X$ . We define  $\mathcal{F}$  to be  $(B_{q+1})$  if, for every closed point  $x \in X$ , we have  $H_x^i(X, \mathcal{F}) = 0$  whenever  $\dim X - q < i < \dim X$ .

For any normal variety  $X$ ,  $X$  is CM if and only if  $\omega_X$  is CM. This equivalence is established in Lemma 0AWS of [Stacks]. Nonetheless, when  $q < \dim X - 1$ , the statement that  $X$  is  $(S_{q+1})$  if and only if  $\omega_X$  is  $(S_{q+1})$  does not hold. It appears to us that the more appropriate property for  $\omega_X$  might be  $(B_{q+1})$  rather than  $(S_{q+1})$ . For further discussion on this topic, one can refer to Section 4.3 in [Kol11].

**Theorem 5.2.** *Let  $X$  be any normal projective variety and  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$ , which is  $(RS_q)$ . Then the following are equivalent:*

- (a)  $\mathcal{F}$  is  $(S_{q+1})$ .
- (b) For any resolution  $f : X' \rightarrow X$  with  $f$   $(RS_q)$  about  $\mathcal{F}$ ,  $R^i f_*(f^* \mathcal{F})^{DD} = 0$  for any  $1 \leq i < q$ .
- (c)  $\mathcal{F}^D$  is  $(B_{q+1})$ .

*Proof.* Let  $f : X' \rightarrow X$  be any resolution of  $X$  which is  $(RS_q)$  about  $\mathcal{F}$ . For (a)  $\implies$  (b), we consider the following spectral sequence

$$(5.1) \quad E_2^{st} = H_x^s(X, R^t f_*(f^* \mathcal{F})^{DD}) \implies H_{f^{-1}(x)}^{s+t}(X', (f^* \mathcal{F})^{DD}).$$

We may use induction. Indeed, let us fix any positive integer  $2 \leq n \leq q$  and assume that  $R^i f_*(f^* \mathcal{F})^{DD} = 0$  for  $1 \leq i < n - 1$ .

Given any point  $x \in X$  with  $\text{codim}_X x \geq q + 1$ , since  $f$  is  $(KV_q)$  about  $\mathcal{F}$ , and Lemma 2.6 holds, we have  $H_{f^{-1}(x)}^i(X', (f^* \mathcal{F})^{DD}) = 0$  for  $i \leq q$ . By the spectral sequence (5.1), we obtain the following exact sequence

$$0 \rightarrow H_x^0(X, R^{n-1} f_*(f^* \mathcal{F})^{DD}) \rightarrow H_x^n(X, \mathcal{F}) \xrightarrow{\alpha_n} H_{f^{-1}(x)}^n(X', (f^* \mathcal{F})^{DD}).$$

From the above exact sequence,  $H_x^0(X, R^{n-1} f_*(f^* \mathcal{F})^{DD}) \cong H_x^n(X, \mathcal{F})$  and if we use (a), then  $H_x^0(X, R^{n-1} f_*(f^* \mathcal{F})^{DD}) = 0$  holds. Thus,  $R^{n-1} f_*(f^* \mathcal{F})^{DD}$  does not have  $x$  as an associated point, and thus any generic point of  $\text{Supp } R^{n-1} f_*(f^* \mathcal{F})^{DD}$  has codimension  $\geq q$  if there exists. Let us assume  $\text{codim}_X x \leq q$ . Then since  $f$  is  $(RS_q)$  about  $\mathcal{F}$ ,  $(R^i f_*(f^* \mathcal{F}))_x = 0$  for  $i \geq 1$ . Hence, we have  $R^{n-1} f_*(f^* \mathcal{F})^{DD} = 0$ .

For (b)  $\implies$  (a), since  $\theta_{\mathcal{F}, f}$  is a quasi-isomorphism at  $x \in X$  with  $\text{codim}_X x \leq q$ , we deduce  $(R^i f_*(f^* \mathcal{F})^D)_x \cong \mathcal{E}xt_{\mathcal{O}_{X,x}}^{\text{codim}_X x + i}((f^* \mathcal{F})_x^{DD}, \omega_{X,x}^\bullet)$ . Hence the fact that  $f$  is  $(KV_q)$  about  $\mathcal{F}$  and the local duality gives us  $H_x^i(X, \mathcal{F}) = 0$  for  $i < \text{codim}_X x$ .

If  $\text{codim}_X x \geq q + 1$ , by inspecting the spectral sequence (5.1),  $E_2^{s0} = E_\infty^{s0}$  for  $s \leq q$ . Considering the edge map  $E_\infty^{s0} \rightarrow E^s = H_{f^{-1}(x)}^s(X', (f^* \mathcal{F})^{DD})$  for such  $s$ , if we use Lemma 2.6, then we obtain the assertion.

For (a)  $\iff$  (c), let us consider the following exact sequence

$$0 \longrightarrow f_*(f^* \mathcal{F})^D \xrightarrow{\mathcal{H}^0(\theta_{\mathcal{F}, f})} \mathcal{F}^D \longrightarrow \mathcal{Q} \longrightarrow 0$$

for a coherent sheaf  $\mathcal{Q}$  on  $X$ . Since  $f$  is  $(RS_q)$  about  $\mathcal{F}$ , the support of  $\mathcal{Q}$  has codimension  $\geq q + 1$  and hence  $H_x^i(X, \mathcal{Q}) = 0$  for  $i > \dim X - q$ .

By taking local cohomology on the above exact sequence, we obtain

$$H_x^i(X, f_*(f^* \mathcal{F})^D) \cong H_x^i(X, \mathcal{F}^D) \text{ for any } \dim X - q < i < q \text{ and any closed point } x \in X.$$

Hence,  $\mathcal{F}^D$  is  $(B_{q+1})$  if and only if  $H_x^i(X, f_*(f^*\mathcal{F})^D) = 0$  for any  $\dim X - q < i < \dim X$  and any closed point  $x \in X$ .

Let  $x \in X$  be any closed point and consider the Leray spectral sequence

$$E_2^{st} = H_x^s(X, R^t f_*(f^*\mathcal{F})^D) \implies H_{f^{-1}(x)}^{s+t}(X', (f^*\mathcal{F})^D).$$

Then, by the fact that  $f$  is  $(KV_q)$  about  $\mathcal{F}$  and the dimension counting,  $E_2^{st} = 0$  for  $s + t \geq \dim X - q$ . Hence, by inspecting the spectral sequence,  $E_2^{s0} = E_\infty^{s0}$  for  $s > \dim X - q$ . Moreover,  $E_\infty^{s0} = E^s$  for  $s > \dim X - q$ . Thus,  $\mathcal{F}^D$  is  $(B_{q+1})$  if and only if  $H_{f^{-1}(x)}^i(X', (f^*\mathcal{F})^D) = 0$  for any  $\dim X - q < i < \dim X$  and any closed point  $x \in X$ . If we use Lemma 2.6, then we have the assertion.  $\square$

*Remark 5.3.* In Theorem 5.2, the argument of the proof of (a)  $\iff$  (b) is similar to the argument of the proof of Lemma 3.1 in [Ale08] and Proposition 7.1.7 in [Fuj17]. Moreover, the idea of the proof of (a)  $\iff$  (c) is inspired by the proof of Lemma 3.3 in [Kov99].

There is an interesting corollary of Theorem 5.2.

**Corollary 5.4.** *Assume that  $q \geq \lceil \frac{\dim X + 1}{2} \rceil$ . Let  $X$  be any normal  $\mathbb{Q}$ -factorial variety over a characteristic 0 field and any Weil divisor on  $X$  is  $(RS_q)$  and  $(S_{q+1})$ . Then any Weil divisor on  $X$  is CM. In particular,  $X$  is CM itself.*

*Proof.* Note that any reflexive sheaf  $\mathcal{F}$  on  $X$  of rank 1 is  $(RS_q)$  because  $\mathcal{F}$  is  $(KV_q)$  by assumption,  $X$  is  $(R_q)$ , and thus  $\mathcal{F}$  is  $(RS_q)$  by Lemma 4.18. Since  $(\mathcal{O}_X(D))^D$  is  $(S_q)$  and  $\mathcal{O}_X(D) \cong (\mathcal{O}_X(D))^{DD}$  is  $(B_q)$  by Theorem 5.2,  $\mathcal{O}_X(D)$  is CM.  $\square$

*Remark 5.5.* Corollary 5.4 proves that in the setting of  $\text{char } k = 0$ , for any factorial variety  $X$ , if  $X$  is  $(RS_{\lceil \frac{\dim X + 1}{2} \rceil})$  and  $(S_{\lceil \frac{\dim X + 3}{2} \rceil})$ , then  $X$  is CM. Indeed, any reflexive sheaf on  $X$  of rank 1 is a line bundle, and thus the Grauert-Riemenschneider vanishing theorem ensures  $R^i f_*(f^*\mathcal{F})^D = 0$  for any reflexive sheaf  $\mathcal{F}$  on  $X$  of rank 1, any resolution  $f : X' \rightarrow X$ , and any  $i \geq 1$ . Thus, Corollary 5.4 gives us the claim.

One may think that Theorem 1.6 in [HO74] is similar to Corollary 5.4. Hence, if we assume  $X$  is  $(R_q)$ , then one can believe that there is a simple proof of Corollary 5.4 using the local duality only.

## 6. $q$ -BIRATIONAL MORPHISMS

In this section, we always assume that the ground field  $k$  is of characteristic 0.

**Definition 6.1** (See Definition 3.1, [Kim22]). Let  $X, X'$  be any normal varieties over a characteristic 0 field and  $f : X' \rightarrow X$  any proper birational morphism.

- (a) The *center* of  $f$  is the reduced closed subscheme  $C$  of  $X$  which is the image of the exceptional locus of  $f$ .
- (b) We say  $f$  is a  *$q$ -birational morphism* if the exceptional locus has codimension 1 and the center of  $f$  has codimension  $\geq q + 1$ .



Let us prove the following lemma which can be regarded as a partial converse of Lemma 4.6.

**Lemma 6.2.** *Let  $X$  be any normal variety,  $\mathcal{F}$  any reflexive sheaf of rank 1 on  $X$  which is  $(S_{q+1})$ ,  $f : X' \rightarrow X$  any resolution, and  $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$  are line bundles on  $X'$  such that*

- (a) *there is an isomorphism  $\mathcal{F} \cong f_*\mathcal{L}$ , and  $f_*\varphi$  is an isomorphism, and*
- (b)  *$R^i f_*\mathcal{L}^D = 0$  for  $i \geq 1$ .*

*If  $f$  is  $q$ -birational, then  $R^i f_*\mathcal{L}' = 0$  for  $1 \leq i < q$ .*

*Proof.* We may use a similar argument as in the proof of (a)  $\iff$  (c) in Theorem 5.2.

By taking dual on  $\varphi$ , we obtain an injection  $\varphi^D : (\mathcal{L}')^D \rightarrow \mathcal{L}$ . Let  $\mathcal{Q}$  be the cokernel of  $f_*\varphi^D$ . Then we have the following exact sequence

$$(6.1) \quad 0 \rightarrow f_*(\mathcal{L}')^D \rightarrow f_*\mathcal{L}^D \rightarrow \mathcal{Q} \rightarrow 0.$$

Since  $f$  is  $q$ -birational, the support of  $\mathcal{Q}$  has codimension  $\geq q+1$ , and  $H_x^i(X, \mathcal{Q}) = 0$  for  $i > \dim X - q$ .

By taking local cohomology on (6.1), we have isomorphisms

$$H_x^i(X, f_*(\mathcal{L}')^D) \cong H_x^i(X, f_*\mathcal{L}^D) \text{ for any closed point } x \in X \text{ and any } \dim X - q < i < \dim X.$$

Let us prove  $H_x^i(X, f_*\mathcal{L}^D) = 0$  using the argument of the proof of (a)  $\iff$  (c) in Theorem 5.2. Consider

$$E_2^{st} = H_x^s(X, R^t f_*\mathcal{L}^D) \implies H_{f^{-1}(x)}^{s+t}(X', \mathcal{L}^D).$$

By (b) in the statement, we deduce that  $E_2^{st} = 0$  for  $s+t \geq \dim X - q$ . Hence, inspecting the spectral sequence,  $E_2^{s0} = E_\infty^{s0} = E^s$  for  $s > \dim X - q$ . Thus, it suffices to show

$$H_{f^{-1}(x)}^i(X', \mathcal{L}^D) = 0$$

for any  $\dim X - q < i < \dim X$ , and we can apply Lemma 2.6 to prove the claim.

Consider the Leray spectral sequence

$$E_2^{st} = H_x^s(X, R^t f_*(\mathcal{L}')^D) \implies H_{f^{-1}(x)}^{s+t}(X', (\mathcal{L}')^D).$$

By inspecting the spectral sequence, we obtain  $H_{f^{-1}(x)}^i(X', (\mathcal{L}')^D) = 0$  for any  $\dim X - q < i < \dim X$  and any closed point  $x \in X$ . Now, we may apply Lemma 2.6 and obtain the assertion.  $\square$

Using Lemma 6.2, we have the following theorem.

**Theorem 6.3.** *Let  $X, X'$  be any normal varieties,  $X'$  smooth and  $f : X' \rightarrow X$  any  $q$ -birational morphism. Suppose that  $D$  is any anti  $f$ -nef Cartier divisor on  $X'$  such that  $f_*D$  is  $\mathbb{Q}$ -Cartier and  $(S_{q+1})$ . Then  $R^i f_*\mathcal{O}_{X'}(D) = 0$  for  $1 \leq i < q$ .*

*Proof.* Since  $f_*D$  is  $\mathbb{Q}$ -Cartier, we may consider the  $\mathbb{Q}$ -divisor  $f^*f_*D$ . If we define a divisor  $L$  on  $X'$  such that  $\mathcal{O}_{X'}(L) = (f^*\mathcal{O}_X(f_*D))^{DD}$ , there is an effective  $f$ -exceptional divisor  $E$  such that  $f^*f_*D \sim_{\mathbb{Q}} L + E$  by Lemma 4.9 and hence  $D - (L + E) \sim_{\mathbb{Q}} D - f^*f_*D$

is anti  $f$ -nef. Thus  $D - (L + E)$ , and  $D - (L + \lfloor E \rfloor) = D - (L + E) + \{E\}$  are effective  $f$ -exceptional divisors on  $X'$  by the Negativity lemma.

For applying Lemma 6.2, put  $\mathcal{L} := \mathcal{O}_{X'}(L + \lfloor E \rfloor)$  and  $\mathcal{L}' = \mathcal{O}_{X'}(D)$ . Then there is an injection  $\mathcal{L} \rightarrow \mathcal{L}'$ , and (a) in Lemma 6.2 is satisfied. For (b) in Lemma 6.2, as in the proof of Theorem 4.10, we can prove  $R^i f_* \mathcal{O}_{X'}(L + \lfloor E \rfloor) = 0$  for  $i \geq 1$ . Thus, we can apply Lemma 6.2 to prove the assertion.  $\square$

We can write Theorem 6.3 as an absolute cohomology vanishing. For a similar result, see Corollary 3.8 in [Kim22].

**Corollary 6.4.** *Let  $X, X'$  be any normal projective varieties,  $X'$  smooth and  $f : X' \rightarrow X$  any  $q$ -birational morphism. Suppose that  $D$  is any anti  $f$ -nef Cartier divisor on  $X'$  such that  $f_* D$  is a  $\mathbb{Q}$ -divisor and  $(S_{q+1})$  on  $X$  and  $E$  is any effective  $f$ -exceptional Cartier divisor on  $X'$ . Then*

$$H^i(X', \mathcal{O}_{X'}(D + E)) = H^i(X, \mathcal{O}_X(f_* D)) = H^i(X', \mathcal{O}_{X'}(D))$$

for  $0 \leq i < q$ .

*Proof.* We may use the Leray spectral sequence

$$E_2^{st} = H^s(X, R^t f_* \mathcal{O}_{X'}(D + E)) \implies H^{s+t}(X', \mathcal{O}_{X'}(D + E))$$

and hence  $H^i(X, \mathcal{O}_X(f_* D)) = H^i(X', \mathcal{O}_{X'}(D + E))$  for  $0 \leq i < q$  by Theorem 6.3.

Let  $\mathcal{O}_{X'}(D) \rightarrow \mathcal{O}_{X'}(D + E)$  be an injection. Then by taking  $f_*$ , and applying Corollary 2.7, we obtain an injection

$$\mathcal{O}_{X'}(f_* D) \rightarrow f_* \mathcal{O}_{X'}(D + E).$$

By the same argument to Lemma 3.7, we have that the above map is an isomorphism, and thus we obtain the assertion.  $\square$

## REFERENCES

- [Ale08] V. Alexeev: *Limits of stable pairs*, Pure Appl. Math. Q. **4** (2008), no. 3, 767–783.
- [AH12] V. Alexeev and C. Hacon: *Non-rational centers of log canonical singularities*, J. Algebra. **369** (2012), 1–15.
- [Ber21] F. Bernasconi: *Kawamata-Viehweg vanishing fails for log del Pezzo surfaces in characteristic 3*, J. Pure Appl. Algebra **225** (2021), no. 11.
- [BH98] W. Bruns and H. Herzog: *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1998.
- [BK21] F. Bernasconi, J. Kollár: *Vanishing theorems for threefolds in characteristic  $p > 5$* , Int. Math. Res. Not (2021), 1073–7928.
- [BMP+20] B. Bhatt, L. Ma, Z. Patakfalvi, K. Schwede, K. Tucker, J. Waldron, Joe, and J. Witaszek: *Globally  $+$ -regular varieties and the minimal model program for threefolds in mixed characteristic*, Publ. Math. Inst. Hautes Études Sci. **138** (2023), 69–227.
- [Cut09] S. Cutkosky: *Resolution of singularities for 3-folds in positive characteristic*, Amer. J. Math. **131** (2009), no. 1 59–127.
- [CR12] A. Chatzistamatiou and K. Rülling: *Higher direct images of the structure sheaf in positive characteristic*, Algebra & Number Theory **5** (2012), no. 6, 693–775.

- [CR15] A. Chatzistamatiou and K. Rülling: *Vanishing of the higher direct images of the structure sheaf*, Compositio. Mathematica. **151** (2015), no. 11, 2131–2144.
- [Elk81] R. Elkik: *Rationalité des singularités canoniques*, Invent. Math. **64** (1981) no. 1 1–6.
- [Fuj17] O. Fujino: *Foundations of the minimal model program*, MSJ Memoirs, vol. 35, Mathematical Society of Japan, Tokyo, 2017.
- [Har77] R. Hartshorne: *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52. Springer, 1977.
- [Har80] R. Hartshorne: *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176.
- [Har94] R. Hartshorne: *Generalized divisors on Gorenstein schemes*, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), *K-Theory* **8** (1994), no. 3 287–339.
- [Hir64] H. Hironaka: *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; *ibid.* (2) **79** (1964) 205–326.
- [HO74] R. Hartshorne and A. Ogus: *On the factoriality of local rings of small embedding codimension*, Comm. Algebra **1** (1974), 415–437.
- [HL10] D. Huybrechts and M. Lehn: *The geometry of moduli spaces of sheaves*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, (2010).
- [HW19] C. Hacon and J. Witaszek: *On the rationality of Kawamata log terminal singularities in positive characteristic*, Algebr. Geom. **6** (2019), no. 5 516–529.
- [KMM87] Y. Kawamata, K. Matsuda and K. Matsuki: *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10** (1987), 283–360.
- [Kim22] D. Kim:  *$q$ -birational morphisms and divisors*, arXiv preprint arXiv:2209.07720.
- [Kol97] J. Kollár: *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math. **62** (1997), 221–287.
- [Kol07] J. Kollár: *Lectures on Resolution of Singularities*, volume 166 of Annals of Mathematics Studies, Princeton University Press, Princeton, 2007.
- [Kol11] J. Kollár: *A local version of the Kawamata-Viehweg vanishing theorem*, Pure Appl. Math. Q. **7** (2011), no. 4, 1477–1494.
- [Kol13] J. Kollár: *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, (2013).
- [KM98] J. Kollár and S. Mori: *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge, 1998.
- [Kov99] S. Kovács: *Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink*, Compositio Math **118** (1999), no. 2, 123–133,
- [Kov00] S. Kovács: *A characterization of rational singularities*, Duke Math. J. **102** (2000), no. 2 187–191.
- [Kov11] S. Kovács: *Irrational centers*, Pure Appl. Math. Q. **7** (2011) no. 4, Special Issue: In memory of Eckart Viehweg, 1495–1515.
- [Kov17] S. Kovács: *Rational singularities*, arXiv preprint arXiv:1703.02269.
- [KTT+22] T. Kawakami, T. Takamatsu, H. Tanaka, J. Witaszek, F. Yobuko, and S. Yoshikawa: *Quasi- $F$ -splittings in birational geometry.*, arXiv preprint arXiv:2208.08016 (2022).
- [Nak04] N. Nakayama: *Zariski-decomposition and abundance*, Mathematical Society of Japan, vol. 14, Tokyo, 2004.
- [PS14] Z. Patakfalvi and K. Schwede: *Depth of  $F$ -singularities and base change of relative canonical sheaves*, J. Inst. Math. Jussieu. **13** (2014), no. 1, 43–63.
- [Smi97] K. Smith,  *$F$ -rational rings have rational singularities*, Amer. J. Math. **119** (1997), no. 1, 159–180.
- [ST08] K. Schwede and S. Takagi: *Rational singularities associated to pairs*, Special volume in honor of Melvin Hochster, Michigan Math. J. **57** (2008), 625–658.

- [Stacks] A. J. de Jong et al., *The Stacks Project*, Available at <http://stacks.math.columbia.edu>.
- [Tot19] B. Totaro: *The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen-Macaulay*, J. Algebraic Geom. **28** (2019), no. 4, 751–771.

DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, 50 YONSEI-RO, SEODAEMUN-GU, SEOUL  
03722, KOREA

*Email address:* `whatisthat@yonsei.ac.kr`