Understanding the chiral and parity anomalies without Feynman diagrams

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Abstract

We review the construction of the adiabatic expansion for Bose and Fermi systems and show how it may be used to explore the chiral and parity anomalies for Dirac fermions without the need to compute Feynman diagrams.

Contents

1 Introduction

Feynman diagrams are the standard tool for computing quantities in perturbative quantum field theory. After one has learned the rules for converting diagram to integral, and the often formidable techniques necessary for evaluating the resulting integral, their power is such they provide a magical black box into which one inserts a problem and extracts an answer. What is often lost in the process is a picture of what physics the mathematical machinery is capturing. This is true even at the level of one-loop diagrams whose mathematics can output non-obvious physical effects such as the ABJ chiral anomaly [1, 2, 3], the parity-anomaly in odd space-time dimensions [4, 5], and the related current inflow from higher dimensions that provides the anomalous chiral charge [6].

When using one-loop diagrams to evaluate the vacuum expectation of operators such as currents, charges, and energy fluxes we are basically exploring the physics of systems whose hamiltonians are quadratic in annihilation and creation operators and with coefficients that depend on whatever perturbations are represented by the external legs on the diagram. These perturbations can be electromagnetic fields coupling to the bilinear current operator, or perhaps the gravitational effects of curved space that couple to the energy-momentum tensor. In such a case the effects of slowly varying external fields can be captured by a gradient or derivative expansion of the one-loop effective action [7, 8, 9]. In particular, if the external fields depend on only *one* space-time dimension a powerful tool is provided by the adiabatic expansion of ordinary quantum mechanics, which does not require the full machinery of quantum field theory. Indeed much of the work on field theory in curved space uses exactly this tool [10, 11].

In this paper we will use versions of the adiabatic expansion to exhibit special cases of the anomaly-related effects mentioned above, and in doing so hope to achieve some insights that are denied in the diagram derivations. We set the stage in section 2 by reviewing how time-varying parameters in a harmonic oscillator causes the ground-state to evolve. We relate this evolution to vacuum squeezing in both Bose and Fermi systems and establish the basic recurrence relations that allow us to mechanically compute the slow-squeeze adiabatic series to arbitrary order. Then, in section 3, we apply what we have learned to field theory. We use the fermion version of the adiabatic series in $1 + 1$ spacetime dimensions to show how the standard spectral-flow picture of the chiral anomaly for massless fermions is affected by the inclusion of a fermion mass. Similar methods are then used to obtain the related parity anomaly in $1 + 2$ spacetime dimensions, and in $1 + 3$ dimensions to compute the gradient expansion for the current induced by an external spatially constant electric field. *En passant* we obtain the one-loop beta function for QED. Finally we extend the chiral anomaly results to four-dimensional spacetime. In the appendices we verify, when possible, the output of our asymptotic expansions by comparing them with one-loop results obtained by other methods.

We use units in which $\hbar = \epsilon_0 = \mu_0 = 1$.

2 Quantum Mechanics

2.1 Time-dependent harmonic oscillators

The quantum harmonic oscillator with Hamiltonian

$$H_0 = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\Omega^2\hat{x}^2 \tag{1}$$

is considered in every introductory textbook — not only because it is easily solved and therefore a pedagogically useful illustration, but also because many real world systems are well approximated as harmonic oscillators. An oscillator driven by a time-varying linear term

$$H(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\Omega^2\hat{x}^2 + F(t)\hat{x} \tag{2}$$

is also useful and straightforward to solve (see appendix A) but when it is the oscillator *frequency* Ω that is allowed to depend on time the problem is more challenging, and the physical effects more exotic. A period of rapid frequency change will leave the oscillator in a *squeezed state* — a superposition of excited states that has many applications in quantum optics [12], and even in gravitational wave detection [13]. We will see, however, that much can also be learned by tracking what happens *during* a slow frequency change that leaves little permanent excitation.

2.2 Schrödinger-picture wavefunctions

The wavefunction $\psi(x,t)$ of a variable-frequency harmonic oscillator obeys the time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega^2(t)x^2\right)\psi, \quad \Omega(t) \in \mathbb{R}. \tag{3}$$

This equation has a Gaussian solution [14]

$$\psi(x,t) = \chi^{-1/2}(t)\exp\left\{-\frac{1}{2}\omega(t)x^2\right\} \tag{4}$$

provided that $\chi(t)$, $\omega(t)$ obey the evolution equations

$$\begin{aligned} \dot{\chi}/\chi &= i\omega, \\ \omega^2 - i\dot{\omega} &= \Omega^2(t). \end{aligned} \tag{5}$$

Here the dot denotes a time derivative: $\dot{\omega} \equiv \partial_t \omega$. With $\omega = \omega_R + i\omega_I$, the parameter-evolution equations imply that

$$\begin{aligned} \partial_t \ln(|\chi|) &= -\omega_I, \\ \partial_t \ln(\omega_R) &= +2\omega_I, \end{aligned} \tag{6}$$

and so ensure that the normalization $\propto |\chi|\omega_R^{1/2}$ is preserved.

The equations (5) together with the Riccati identity

$$-(-\partial_t - i\omega)(\partial_t - i\omega) = \partial_{tt}^2 + (\omega^2 - i\dot{\omega}) \tag{7}$$

show that

$$\frac{d^2\chi}{dt^2} + \Omega^2(t)\chi = 0, \tag{8}$$

where

$$\chi(t) = \exp\left\{ i \int_{-\infty}^{t} \omega(\tau)\, d\tau \right\}. \tag{9}$$

We can rewrite (8) as

$$\left(-\frac{d^2}{dt^2} + [\Omega_0^2 - \Omega^2(t)] \right) \chi = \Omega_0^2 \chi, \tag{10}$$

and if $\Omega^2(t) \to \Omega_0^2$ as $t \to \pm\infty$, regard it as a scattering problem with the frequency excursion away from Ω_0^2 providing the scattering potential. Consider boundary conditions for which the asymptotic solution is of the form

$$\chi(t) \to \begin{cases} Te^{i\Omega_0 t}, & t \to -\infty, \\ e^{i\Omega_0 t} + Re^{-i\Omega_0 t}, & t \to +\infty, \end{cases} \tag{11}$$

with $|T|^2 = 1 - |R|^2$. For t in the pre-excursion region the asymptotic form for χ gives

$$\omega(t) = -i \left(\frac{\dot{\chi}}{\chi} \right) \to \Omega_0 \tag{12}$$

so the "transmission coefficient" T does not affect $\omega(t)$. For t in the post-excursion asymptotic region we have

$$\omega(t) = -i \left(\frac{\dot{\chi}}{\chi} \right) \to \Omega_0 \left(\frac{1 - R\, e^{-2i\Omega_0 t}}{1 + R\, e^{-2i\Omega_0 t}} \right). \tag{13}$$

The oscillations in $\omega(t)$ reveal that the gaussian wavefunction is breathing in and out — *i.e.* getting narrower and wider — at frequency $2\Omega_0$.

Setting $y = 0$ and $s = i\sqrt{R}e^{-i\Omega_0 t}$ in Mehler's formula

$$\sum_{n=0}^{\infty} s^n \varphi_n(x)\varphi_n(y) = \frac{1}{\sqrt{\pi(1-s^2)}} \exp\left\{\frac{4xys - (x^2+y^2)(1+s^2)}{2(1-s^2)}\right\}, \quad 0 \le |s| < 1, \tag{14}$$

where

$$\varphi_n(x) \equiv \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2} \tag{15}$$

is the normalized $\Omega_0 = 1$ harmonic oscillator wavefunction, we find that

$$\frac{1}{(\pi\Omega_0)^{1/4}} \chi^{-1/2}(t) \exp\left\{-\frac{1}{2}\omega(t)x^2\right\} \stackrel{t\to+\infty}{=} \pi^{1/4} \sum_{n=0}^{\infty} e^{-i(n+1/2)\Omega_0 t} \varphi_n(0)(i\sqrt{R})^n \frac{\varphi_n(\sqrt{\Omega_0}x)}{(\Omega_0)^{1/4}}. \tag{16}$$

Now $\varphi_n(0)$ vanishes if n is odd, and

$$\pi^{1/4}\varphi_{2n}(0) = \frac{1}{\sqrt{4^n(2n)!}} \frac{(2n)!}{n!}(-1)^n. \tag{17}$$

Comparing with the wavefunction for $t \to -\infty$ we see that the amplitude for being excited from the ground state to the $2n$-th eigenstate is

$$A_{2n} = \sqrt{T}(R\,e^{-2i\Omega_0 t})^n \frac{1}{\sqrt{4^n(2n)!}} \frac{(2n)!}{n!}. \tag{18}$$

As a check we may evaluate

$$\sum_{n=0}^{\infty} |A_{2n}|^2 = |T| \sum_{n=0}^{\infty} |\tfrac{1}{2}R|^{2n} \frac{(2n)!}{(n!)^2} = \frac{|T|}{\sqrt{1-|R|^2}} = 1. \tag{19}$$

The probabilities of excitation therefore sum to unity as they should.

2.3 Squeezed vacuum states

We can appreciate the formula for A_{2n} by relating it to the generalized coherent states associated with the non-compact group $\mathrm{SU}(1,1) \simeq \mathrm{Sp}(2,\mathbb{R})$ or, more accurately, with its metaplectic double cover $\mathrm{MSp}(2,\mathbb{R})$. In quantum optics these coherent states are known as *squeezed vacuum states*.

Let $\hat{a}$, $\hat{a}^\dagger$ be bosonic annihilation and creation operators with their usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, and vacuum state $|0\rangle$ defined by $\hat{a}|0\rangle = 0$.

A unitary infinite-dimensional Fock-space representation of the Lie algebra $\mathfrak{su}(1,1) \simeq \mathfrak{sp}(2,\mathbb{R})$ is then generated by the quadratic operators a^2, ${a^\dagger}^2$ and $\hat{a}^\dagger\hat{a} + \frac{1}{2}$ whose commutators are

$$\begin{aligned} [(\hat{a}^\dagger)^2, \hat{a}^2] &= -4(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}), \\ [(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}), \hat{a}^2] &= -2\hat{a}^2, \\ [(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}), (\hat{a}^\dagger)^2] &= +2(\hat{a}^\dagger)^2. \end{aligned} \tag{20}$$

By exponentiating these generators we construct a unitary *squeezing* operator [15]

$$S(z) \stackrel{\text{def}}{=} \exp\left\{\tfrac{1}{2}(z(\hat{a}^\dagger)^2 - z^*\hat{a}^2)\right\}, \tag{21}$$

which implements the Bogoliubov-Valatin transformation

$$S^\dagger(z)\begin{bmatrix}\hat{a}\\ \hat{a}^\dagger\end{bmatrix} S(z) \;=\; \begin{bmatrix}\cosh|z| & e^{i\theta}\sinh|z|\\ e^{-i\theta}\sinh|z| & \cosh|z|\end{bmatrix}\begin{bmatrix}\hat{a}\\ \hat{a}^\dagger\end{bmatrix}. \tag{22}$$

Here the angle θ is defined by $z = |z|e^{i\theta}$.

There is also a faithful but non-unitary representation of $\mathfrak{sp}(2,\mathbb{R})$ in terms of the two-by-two Pauli matrices in which

$$\begin{aligned} a^2 &\mapsto 2i\sigma_-, \\ (\hat{a}^\dagger)^2 &\mapsto 2i\sigma_+, \\ (\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) &\mapsto \sigma_3. \end{aligned} \tag{23}$$

Because the representation is faithful, the resulting group-element map

$$\exp\left\{\tfrac{1}{2}(z(\hat{a}^\dagger)^2 - z^*\hat{a}^2)\right\} \mapsto \exp\left\{(iz\sigma_+ - iz^*\sigma_-)\right\} = \exp\left\{\begin{pmatrix}0 & iz\\ -iz^* & 0\end{pmatrix}\right\}. \tag{24}$$

is an isomorphism. Consequently the Gauss-Bruhat factorization

$$\begin{aligned} \exp\left\{\begin{pmatrix}0 & iz\\ -iz^* & 0\end{pmatrix}\right\} &\equiv \begin{pmatrix}\cosh|z| & ie^{i\theta}\sinh|z|\\ -ie^{-i\theta}\sinh|z| & \cosh|z|\end{pmatrix}, \\ &= \begin{pmatrix}1 & ie^{i\theta}\tanh|z|\\ 0 & 1\end{pmatrix}\begin{pmatrix}1/\cosh|z| & 0\\ 0 & \cosh|z|\end{pmatrix}\begin{pmatrix}1 & 0\\ -ie^{-i\theta}\tanh|z| & 1\end{pmatrix}, \\ &= \exp\{ie^{i\theta}\tanh|z|\sigma_+\}\exp\{-\ln(\cosh|z|)\sigma_3\}\exp\{-ie^{-i\theta}\tanh|z|\sigma_-\} \end{aligned} \tag{25}$$

of the two-by-two matrix establishes [16] the normal-ordered factorization of the infinite-dimensional Fock-space operator

$$\begin{aligned} S(z) &= \exp\left\{\tfrac{1}{2}(z(\hat{a}^\dagger)^2 - z^*\hat{a}^2)\right\} \\ &= \exp\left\{e^{i\theta}\tfrac{1}{2}\tanh|z|\,(\hat{a}^\dagger)^2\right\}\exp\left\{-\ln\cosh|z|(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\right\}\exp\left\{-e^{-i\theta}\tfrac{1}{2}\tanh|z|\,\hat{a}^2\right\}. \end{aligned} \tag{26}$$

The normal-ordering shows that

$$\begin{aligned} S(z)|0\rangle &= \frac{1}{\sqrt{\cosh|z|}}\sum_{n=0}^{\infty}\frac{1}{n!}(e^{i\theta}\tfrac{1}{2}\tanh|z|)^n(\hat{a}^\dagger)^{2n}|0\rangle \\ &= \frac{1}{\sqrt{\cosh|z|}}\sum_{n=0}^{\infty}\frac{1}{n!}(e^{i\theta}\tfrac{1}{2}\tanh|z|)^n\sqrt{(2n)!}|2n\rangle \\ &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty}A_{2n}|2n\rangle. \end{aligned} \tag{27}$$

After identifying $|n\rangle$ with the n-th eigenstate of our oscillator and $e^{i\theta}\tanh|z|$ with $Re^{-2i\Omega_0 t}$ we recognize the oscillator's post-excursion excited state as a squeezed vacuum state, and so understand the combinatoric origin of

$$\begin{aligned} \sum_{n=0}^{\infty}|A_{2n}|^2 &= \frac{1}{|\cosh|z||}\sum_{n=0}^{\infty}(\tfrac{1}{2}\tanh|z|)^{2n}\frac{(2n)!}{(n!)^2} \\ &= \frac{1}{|\cosh|z||}\frac{1}{\sqrt{1-\tanh^2|z|}} \\ &= 1. \end{aligned} \tag{28}$$

2.4 Two-mode squeezed vacua

Given *two* frequency-Ω harmonic oscillators with ladder operators $\hat{a}$ and $\hat{b}$ we can similarly construct an operator

$$\begin{aligned} S_2(\xi) &\equiv \exp\{\xi^*\hat{a}\hat{b} - \xi\hat{a}^\dagger\hat{b}^\dagger\} \\ &= \exp\{-e^{i\theta}\tanh|\xi|\hat{a}^\dagger\hat{b}^\dagger\}\exp\{-\ln\cosh|\xi|((\hat{a}^\dagger\hat{a}+\tfrac{1}{2})+(\hat{b}^\dagger\hat{b}+\tfrac{1}{2}))\}\exp\{e^{-i\theta}\tanh|\xi|\hat{a}\hat{b}\} \end{aligned} \tag{29}$$

that creates a two-mode squeezed vacuum state

$$\begin{aligned} S_2(\xi)|0\rangle &= \frac{1}{\cosh|\xi|}\exp\{-e^{-i\theta}\tanh|\xi|\hat{a}^\dagger\hat{b}^\dagger\}|0\rangle \\ &= \frac{1}{\cosh|\xi|}\sum_{n=0}^{\infty}\frac{1}{n!}(-e^{i\theta}\tanh|\xi|)^n(\hat{a}^\dagger\hat{b}^\dagger)^n|0\rangle \\ &= \frac{1}{\cosh|\xi|}\sum_{n=0}^{\infty}(-e^{i\theta}\tanh|\xi|)^n|n,n\rangle. \end{aligned} \tag{30}$$

When we observe only the $\hat{a}$ mode, the probability of being in the n-th excited state is

$$p_n = \frac{1}{\cosh^2|z|}(\tanh^2|z|)^n \tag{31}$$

which is classical thermal Bose distribution with $e^{-\beta\Omega} = \tanh^2|z|$. If we observe both the $\hat{a}$ and $\hat{b}$ modes together we will find, however, that they are non-classically quantum entangled.

We introduced the two-mode operator (33) because we will later have cause to refer to its fermionic cousin. When $\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, $\hat{b}^\dagger$ obey the fermion algebra

$$\{\hat{a},\hat{a}^\dagger\} = \{\hat{b},\hat{b}^\dagger\} = 1, \quad \{\hat{a},\hat{a}\} = \{\hat{b},\hat{b}\} = \{\hat{a},\hat{b}\} = \{\hat{a},\hat{b}^\dagger\} = 0, \tag{32}$$

we have $\hat{a}^2 = (\hat{a}^\dagger)^2 = 0$, so there is no fermion analogue of a single-mode squeezing operator. We can, however, still construct a two-mode operator

$$\begin{aligned} U[z] &= \exp\{z\hat{a}^\dagger\hat{b}^\dagger - z^*\hat{b}\hat{a}\} \\ &= \exp\{(e^{i\theta}\tan|z|)\hat{a}^\dagger\hat{b}^\dagger\}\exp\{(\ln\cos|z|)[(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})+(\hat{b}^\dagger\hat{b}+\tfrac{1}{2})]\}\exp\{(-e^{-i\theta}\tan|z|)\hat{b}\hat{a}\} \end{aligned} \tag{33}$$

which also implements a Bogoliubov-Valatin transformation

$$\begin{aligned} U[z]\hat{a}U^\dagger[z] &= (\cos|z|)\hat{a} - (e^{i\theta}\sin|z|)\hat{b}^\dagger, \\ U[z]\hat{b}U^\dagger[z] &= (e^{i\theta}\sin|z|)\hat{a}^\dagger + (\cos|z|)\hat{b}. \end{aligned} \tag{34}$$

The right-hand-side of (34) is now a compact SU(2) rotation rather than a non-compact SU(1,1) transformation.

The factored form shows that $U[z]$ acts on the vacuum to create a squeezed state of the form

$$\begin{aligned} U[z]|0\rangle &= \cos|z| \exp\{(e^{i\theta}\tan|z|)\hat{a}^\dagger\hat{b}^\dagger\}|0\rangle \\ &= \cos|z|\{1+(e^{i\theta}\tan|z|)\hat{a}^\dagger\hat{b}^\dagger\}|0\rangle. \end{aligned} \tag{35}$$

If we ascribe an energy ϵ to the $\hat{a}$ mode and define a real number β so that

$$\begin{aligned} \tan^2|z| &= e^{-\beta\epsilon}, \\ \sin^2|z| &= \frac{e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}}, \\ \cos^2|z| &= \frac{1}{1+e^{-\beta\epsilon}}, \end{aligned} \tag{36}$$

the probabilities of observing the $\hat{a}$ mode as being unoccupied or occupied are respectively

$$p_0 = \frac{1}{1+e^{-\beta\epsilon}}, \quad p_1 = \frac{e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}}. \tag{37}$$

This is again a thermal distribution [17], but now a Fermi one. As before the $\hat{a}$ and $\hat{b}$ modes are quantum entangled.

2.5 Squeezing in the Heisenberg picture

The rhythmic in-and-out out breathing of the time-dependent Shrödinger wavefunction gives a concrete physical picture of the effect of squeezing on an oscillator ground state. For applications to field theories, however, it is more convenient to work in the Heisenberg-picture language where the states do not evolve but instead the Hermitian position and momentum operators $\hat{x}$, $\hat{p}$ depend on time and obey both their classical equations of motion and the quantum equal-time commutation relation

$$[\hat{x}(t), \hat{p}(t)] = i. \tag{38}$$

For the variable-frequency harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\Omega^2(t)\hat{x}^2 \tag{39}$$

the classical equation of motion is

$$\frac{d^2\hat{x}}{dt^2} + \Omega^2(t)\hat{x} = 0, \tag{40}$$

and $\hat{p}(t) \equiv \dot{\hat{x}}(t)$ so the commutation relation is $[\hat{x}(t), \dot{\hat{x}}(t)] = i$.

The equation of motion is linear, so we can expand the Hermitian operator $\hat{x}(t)$ as a sum

$$\hat{x}(t) = f(t)\hat{a} + f^*(t)\hat{a}^\dagger, \tag{41}$$

where the constant coefficients $\hat{a}$ and $\hat{a}^\dagger$ are operators and the complex-valued c-number function f obeys

$$\ddot{f} + \Omega^2(t) f = 0. \tag{42}$$

The condition $[\hat{x}(t), \dot{\hat{x}}(t)] = i$ requires the coefficients $\hat{a}$, $\hat{a}^\dagger$ to obey

$$\langle f, f\rangle [\hat{a}, \hat{a}^\dagger] = 1, \quad \text{where} \quad \langle f, g\rangle \stackrel{\text{def}}{=} i(f^*\partial_t g - (\partial_t f^*) g). \tag{43}$$

Being proportional to the Wronskian, the non-positive-definite "inner product" $\langle f, g\rangle$ is independent of t, so there is no contradiction with $\hat{a}$ and $\hat{a}^\dagger$ being constants.

For constant Ω the appropriate choice for making $[\hat{a}, \hat{a}^\dagger] = 1$ is to take f as the *positive-frequency* solution

$$f(t) = \sqrt{\frac{1}{2\Omega}} e^{-i\Omega t}. \tag{44}$$

With this choice

$$H = \Omega(\hat{a}^\dagger \hat{a} + \tfrac{1}{2}), \tag{45}$$

so the ground state $|0\rangle$ obeys $\hat{a}|0\rangle = 0$ and we have the Heisenberg-picture expansions

$$\begin{aligned} \hat{x}(t) &= \sqrt{\frac{1}{2\Omega}}(\hat{a}^\dagger e^{i\Omega t} + \hat{a} e^{-i\Omega t}), \\ \hat{p}(t) &= i\sqrt{\frac{\Omega}{2}}(\hat{a}^\dagger e^{i\Omega t} - \hat{a} e^{-i\Omega t}), \end{aligned} \tag{46}$$

$$\begin{aligned} \hat{a}\, e^{-i\Omega t} &= \frac{1}{\sqrt{2}}\left(\sqrt{\Omega}\,\hat{x}(t) + \frac{i}{\sqrt{\Omega}}\hat{p}(t)\right), \\ \hat{a}^\dagger e^{i\Omega t} &= \frac{1}{\sqrt{2}}\left(\sqrt{\Omega}\,\hat{x}(t) - \frac{i}{\sqrt{\Omega}}\hat{p}(t)\right). \end{aligned} \tag{47}$$

Now consider a frequency excursion with Ω_{in}, Ω_{out} as the initial and final asymptotic values of $\Omega(t)$. If we start in the initial Heisenberg-picture ground state $|0\rangle_{\rm in}$ neither the state nor the $a_{\rm in}$ and $a^\dagger_{\rm in}$ coefficients change but a c-number solution that starts off as

$$f(t) = \sqrt{\frac{1}{2\Omega_{\rm in}}} e^{-i\Omega_{\rm in} t} \tag{48}$$

in the distant past will evolve to

$$\alpha\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{-i\Omega_{\rm out} t} + \beta\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{i\Omega_{\rm out} t} \tag{49}$$

after the frequency has ceased to change. It is natural to define new expansion coefficients $\hat{a}_{\rm out}$ and $\hat{a}^\dagger_{\rm out}$ by writing

$$\begin{aligned} \hat{x}(t) &= \hat{a}_{\rm in}\left(\alpha\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{-i\Omega_{\rm out} t} + \beta\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{i\Omega_{\rm out} t}\right) \\ &\quad + \hat{a}^\dagger_{\rm in}\left(\alpha^*\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{i\Omega_{\rm out} t} + \beta^*\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{-i\Omega_{\rm out} t}\right) \\ &\stackrel{\rm def}{=} \hat{a}_{\rm out}\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{-i\Omega_{\rm out} t} + \hat{a}^\dagger_{\rm out}\sqrt{\frac{1}{2\Omega_{\rm out}}} e^{i\Omega_{\rm out} t}. \end{aligned} \tag{50}$$

Comparison of the last two lines shows that

$$\begin{bmatrix} \hat{a}_{\rm out} \\ \hat{a}^\dagger_{\rm out} \end{bmatrix} = \begin{bmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} \hat{a}_{\rm in} \\ \hat{a}^\dagger_{\rm in} \end{bmatrix}. \tag{51}$$

The commutation relation for the "out" operators require that $|\alpha^2| - |\beta|^2 = 1$, which holds true because the Wronskian is constant. Using this we can solve for inverse transformation

$$\begin{bmatrix} \hat{a}_{\rm in} \\ \hat{a}^\dagger_{\rm in} \end{bmatrix} = \begin{bmatrix} \alpha^* & -\beta^* \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} \hat{a}_{\rm out} \\ \hat{a}^\dagger_{\rm out} \end{bmatrix}. \tag{52}$$

The initial state $|0\rangle_{\rm in}$ now appears as a squeezed version

$$|0\rangle_{\rm in} = \exp\left\{\tfrac{1}{2}(z(\hat{a}^\dagger_{\rm out})^2 - z^*\hat{a}^2_{\rm out})\right\}|0\rangle_{\rm out}, \tag{53}$$

of the $\Omega_{\rm out}$ ground state defined by $\hat{a}_{\rm out}|0\rangle_{\rm out} = 0$. Here

$$\begin{aligned} \alpha &= \cosh|z|, \\ \beta^* &= e^{i\theta}\sinh|z|. \end{aligned} \tag{54}$$

That the oscillator is not in the state $|0\rangle_{\rm out}$ manifests itself through the computation of the energy expectation

$$
\begin{aligned}
{}_{\rm in}\langle 0|\tfrac{1}{2}\dot{\hat{x}}^2 + \tfrac{1}{2}\Omega_{\rm out}^2\hat{x}^2|0\rangle_{\rm in} &= \tfrac{1}{2}\Omega_{\rm out}(|\alpha|^2+|\beta|^2) \\
&= \Omega_{\rm out}\left(|\beta|^2+\tfrac{1}{2}\right) \\
&> \tfrac{1}{2}\Omega_{\rm out}, \qquad (55)
\end{aligned}
$$

and matrix elements such as

$$
\begin{aligned}
{}_{\rm in}\langle 0|\hat{x}^2|0\rangle_{\rm in} &= \left|\alpha\sqrt{\frac{1}{2\Omega_{\rm out}}}e^{-i\Omega_{\rm out}t} + \beta\sqrt{\frac{1}{2\Omega_{\rm out}}}e^{i\Omega_{\rm out}t}\right|^2 \\
&= \frac{1}{2\Omega_{\rm out}}\left\{|\alpha|^2+|\beta|^2+(\alpha^*\beta e_+^2+\alpha\beta^* e_-^2)\right\}, \qquad (56)
\end{aligned}
$$

where

$$
e_\pm(t) = \exp\left\{\pm i\Omega_{\rm out}t\right\}. \qquad (57)
$$

Similarly

$$
\begin{aligned}
{}_{\rm in}\langle 0|\dot{\hat{x}}^2|0\rangle_{\rm in} &= \frac{\Omega_{\rm out}}{2}\left\{|\alpha|^2+|\beta|^2-(\alpha^*\beta e_+^2+\alpha\beta^* e_-^2)\right\}, \\
{}_{\rm in}\langle 0|\tfrac{1}{2}(\hat{x}\dot{\hat{x}}+\dot{\hat{x}}\hat{x})|0\rangle_{\rm in} &= \frac{i}{2}(\alpha^*\beta e_+^2-\alpha\beta^* e_-^2). \qquad (58)
\end{aligned}
$$

The expectation values therefore show the same $2\Omega_{\rm out}$ pulsations as in the wavefunction description and we can identify the transmission and reflection coefficient from section 2.2 as $T = 1/\alpha$ and $R = \beta/\alpha$.

At intermediate times one can seek a solution of the form

$$
f(t) = \alpha(t)\frac{1}{\sqrt{2\Omega(t)}}e_-(t) + \beta(t)\frac{1}{\sqrt{2\Omega(t)}}e_+(t) \qquad (59)
$$

where

$$
e_\pm(t) \stackrel{\rm def}{=} \exp\left\{\pm i\int^t \Omega(t')dt'\right\}, \qquad (60)
$$

is a generalization of (57) to admit a variable frequency. Given such a solution we are invariably tempted to interpret the quantity $|\beta(t)|^2$ as the average occupation number of the excited states above the ground state of $H(t)$. We may, however, swap terms $\propto e_\pm^2$ between $\alpha(t)$ and $\beta(t)$ and as a result the decomposition of $f(t)$ into positive and negative frequency terms is not unique.

This non-uniqueness makes any physical interpretation of $|\beta|^2$ unclear, and complicates any interpretation of $|0\rangle_{\rm in}$ as a squeezed ground state of $H(t)$ [18]. What *is* well defined at all times are the equal-time expectation values ${}_{\rm in}\langle 0|\ldots|0\rangle_{\rm in}$ of functions of the Heisenberg-picture operators. These can be extracted from the well-defined $f(t)$ alone, and hence from the $\alpha(t)$ and $\beta(t)$ coefficients despite their individual ambiguity.

2.6 Hyperbolic Bloch equations

One of many ways of defining α and β coefficients during the evolution of the system as $\Omega(t)$ varies is that of Zeldovich and Starobinskii (ZS) [19] who use Lagrange's method of variation of parameters to solve

$$\ddot{\chi} + \Omega^2(t)\chi = 0. \tag{61}$$

ZS start by assuming the ambiguous form (59)

$$\chi(t) = \alpha(t)\frac{1}{\sqrt{2\Omega}}e_-(t) + \beta(t)\frac{1}{\sqrt{2\Omega}}e_+(t), \quad e_\pm(t) \stackrel{\rm def}{=} \exp\left\{\pm i\int^t \Omega(t')dt'\right\}, \tag{62}$$

but follow Lagrange by demanding that

$$\dot{\chi}(t) = -i\Omega\left(\alpha(t)\frac{1}{\sqrt{2\Omega}}e_- - \beta(t)\frac{1}{\sqrt{2\Omega}}e_+\right). \tag{63}$$

This expression is what we would obtain from differentiating $\chi(t)$ while taking α, β, and Ω to be constants — but as these quantities vary with t the demand imposes the condition

$$0 = \left(-\frac{1}{2}\frac{\dot{\Omega}}{\Omega}\alpha + \dot{\alpha}\right)e_- + \left(-\frac{1}{2}\frac{\dot{\Omega}}{\Omega}\beta + \dot{\beta}\right)e_+. \tag{64}$$

This condition serves to uniquely specify $\alpha(t)$ and $\beta(t)$ and hence to disambiguate the decomposition of $\chi(t)$ into positive and negative frequency modes. In particular, the time independence of the Wronskian of χ and χ^* constructed using (62) and (63) shows that $|\alpha|^2 - |\beta|^2 = 1$ at all times.

Inserting the $\dot{\chi}(t)$ defined by (63) into (61) gives

$$0 = \left(\frac{1}{2}\frac{\dot{\Omega}}{\Omega}\alpha + \dot{\alpha}\right)e_- - \left(\frac{1}{2}\frac{\dot{\Omega}}{\Omega}\beta + \dot{\beta}\right)e_+. \tag{65}$$

Adding and subtracting the two conditions (64) and (65) we find

$$\begin{aligned}\dot{\alpha} &= \frac{1}{2}\frac{\dot{\Omega}}{\Omega}\beta e_+^2, \\ \dot{\beta} &= \frac{1}{2}\frac{\dot{\Omega}}{\Omega}\alpha e_-^2. \end{aligned} \tag{66}$$

It is difficult to get a sense of what the equations (66) imply for the evolution of α and β because of the rapidly varying phase factors $e_\pm^2$. To deal with this ZS introduce the real quantities

$$\begin{aligned}\sigma(t) &= |\beta|^2, \\ \tau(t) &= i(\alpha\beta^* e_-^2 - \alpha^*\beta e_+^2), \\ \upsilon(t) &= (\alpha\beta^* e_-^2 + \alpha^*\beta e_+^2), \end{aligned} \tag{67}$$

which have already appeared in equations (56, 58). These combinations obey

$$(1+2\sigma)^2 - \tau^2 - \upsilon^2 = 1,$$

which is the equation for a hyperboloid of two sheets. Indeed the manifold of squeezed vacuum states possesses an inherent hyperbolic geometry arising from it being a coset $K = \mathrm{Sp}(2,R)/\mathrm{U}(1)$ which can be identified as the upper sheet of a two-dimensional hyperboloid embedded in 2+1 dimensional Minkowski space – a classic model for Bolyai-Lobachevskii space. The map (67) taking α, β to points on the coset is a hyperbolic version of the $\mathrm{SU}(2) \to S^3$ Hopf map.

The advantage of the quantities σ, τ, υ is that when $\alpha \approx 1$ and β is small the rapidly varying phases $e_\pm^2(t)$ almost cancel their rapid phase variation in $\beta(t)$ and $\beta^*(t)$ and allow σ, τ, and υ be slowly varying.

Using the equations for $\dot{\alpha}$, $\dot{\beta}$ shows that

$$\begin{aligned}\dot{\sigma} &= \frac{1}{2}\left(\frac{\dot{\Omega}}{\Omega}\right)\upsilon, \\ \dot{\upsilon} &= \left(\frac{\dot{\Omega}}{\Omega}\right)(1+2\sigma) - 2\Omega\tau, \\ \dot{\tau} &= 2\Omega\upsilon. \end{aligned} \tag{68}$$

This set of three equations is a hyperbolic analogue of the Bloch equations describing the interaction of a spin with a time dependent magnetic field. The initial conditions $\alpha = 1$, $\beta = 0$ corresponds to conditions on σ, τ, υ that they are all zero in the distant past.

2.7 Adiabatic expansion

Much of the effort in investigating the squeezed oscillator has focused on the permanent excitation of the system after the frequency excursion. Such an excitation is most efficiently achieved by a period in which $\Omega(t)$ oscillates at a frequency near $2\Omega_0$ (parametric resonance) or by relatively violent changes in the frequency such as occur in Landau-Zener tunneling [20]. For applications and a review of techniques see [21, 22]. In the rest of this paper, however, we will focus on the behavior of systems *during* a relatively slow *adiabatic* frequency excursion in which little or no permanent excitation occurs.

There are a number of methods for obtaining a systematic adiabatic expansion of the time evolution of $\chi(t)$, and hence for quantities such as $\sigma(t)$. A textbook route [10, 11] starts from a single-exponential WKB-like solution

$$\chi(t) = \frac{1}{\sqrt{2W(t)}} \exp\left\{ -i \int^t W(\tau)d\tau \right\}, \tag{69}$$

and generates a series expansion for $W(t)$. We will, however, continue with the "two-exponential" method of [19] as it is computationally simpler.

We rearrange the evolution equations (68) as

$$\begin{aligned} \sigma &= \frac{1}{2}\int_0^t (\dot{\Omega}/\Omega)\upsilon dt, \\ \tau &= \frac{(\dot{\Omega}/\Omega)(1+2\sigma) - \dot{\upsilon}}{2\Omega}, \\ \upsilon &= \frac{\dot{\tau}}{2\Omega}, \end{aligned} \tag{70}$$

and expand in inverse powers of $\Omega(t)$ as

$$\begin{aligned} \sigma &= \sigma_{\{2\}} + \sigma_{\{4\}} + \ldots \\ \tau &= \tau_{\{1\}} + \tau_{\{3\}} + \ldots \\ \upsilon &= \upsilon_{\{2\}} + \upsilon_{\{4\}} + \ldots. \end{aligned} \tag{71}$$

Regarding $(\dot{\Omega}/\Omega)$ as being of $O[\Omega^0]$, we obtain recursion relations

$$\begin{aligned} \sigma_{\{n\}} &= \frac{1}{2}\int_0^t (\dot{\Omega}/\Omega)\upsilon_{\{n\}} dt, \\ \upsilon_{\{n\}} &= \dot{\tau}_{\{n-1\}}/2\Omega, \\ \tau_{\{n+1\}} &= \frac{2(\dot{\Omega}/\Omega)\sigma_{\{n\}} - \dot{\upsilon}_{\{n\}}}{2\Omega}, \end{aligned} \tag{72}$$

with starting condition $\tau_{\{1\}} = \dot{\Omega}/2\Omega^2$.

By hand we find, for example,

$$\sigma_{\{2\}} = \int_0^t \frac{\dot{\Omega}}{4\Omega^2}\frac{d}{dt}\left[\frac{\dot{\Omega}}{2\Omega^2}\right] dt = \frac{1}{16}\frac{\dot{\Omega}^2}{\Omega^4}. \tag{73}$$

Using Mathematica to automate the labour, we can compute higher order terms

$$\sigma_{\{4\}} = \frac{\ddot{\Omega}^2}{64\Omega^6} - \frac{45\dot{\Omega}^4}{256\Omega^8} - \frac{\Omega^{(3)}\dot{\Omega}}{32\Omega^6} + \frac{5\dot{\Omega}^2\ddot{\Omega}}{32\Omega^7}, \tag{74}$$

$$\begin{aligned} \sigma_{\{6\}} \quad = \quad & \frac{(\Omega^{(3)})^2}{256\Omega^8} + \frac{7\ddot{\Omega}^3}{128\Omega^9} + \frac{4725\dot{\Omega}^6}{2048\Omega^{12}} + \frac{\Omega^{(5)}\dot{\Omega}}{128\Omega^8} - \frac{\Omega^{(4)}\ddot{\Omega}}{128\Omega^8} - \frac{7\Omega^{(4)}\dot{\Omega}^2}{64\Omega^9} \\ & + \frac{217\Omega^{(3)}\dot{\Omega}(x)^3}{256\Omega^{10}} - \frac{945\dot{\Omega}^4\ddot{\Omega}}{256\Omega^{11}} + \frac{441\dot{\Omega}^2\ddot{\Omega}^2}{512\Omega^{10}} - \frac{21\Omega^{(3)}\dot{\Omega}\ddot{\Omega}}{128\Omega^9}, \end{aligned} \tag{75}$$

and so on.

All terms in these expressions contain derivatives of $\Omega(t)$. Consequently they revert to being zero when Ω ceases to change. In particular, despite the appearance of the factor $\beta(t)\exp\{+i\int^t \Omega(t')dt'\}$ in $\chi(t)$, the recursion process does not generate the "reflected" wave that indicates a permanent excitation of the oscillator. Such a persistent excitation is a non-perturbative effect [23] so the expansion is, at best, an asymptotic series. Furthermore because of the rapid $e_-(t)^2$ phase variation in $\beta(t)$ the factor $\beta(t)\exp\{+i\int^t \Omega(t')dt'\}$ is close in time evolution to $\alpha(t)\exp\{-i\int^t \Omega(t')dt'\}$. This means the "two-exponential" expansion used by Zeldovich and Starobinskii [19] is consistent with the "one-exponential" WKB expansion of [10, 11] and nicely illustrates the ambiguity in the notion of positive or negative frequency.

The ambiguity means our asymptotic expressions for $\sigma = |\beta|^2$ are not in themselves physically meaningful — they depend on the choice we made when we imposed Lagrange's condition (64). However, when used as ingredients for computing quantities such as (56),(58) and the expectation values of the currents that will appear in section 3.1, the results *are* independent of this choice.

We have been rather vague as to what exactly is the small parameter in the adiabatic series. We can make it explicit by replacing $\Omega(t)$ by $\Omega(\varepsilon t)$ where ε is to be assumed small. Then $\sigma_{\{2n\}}$ is replaced by $\varepsilon^{2n}\sigma_{\{2n\}}$ and ε becomes the small parameter defining the resulting asymptotic expansion. We prefer, though, to simply count the number of derivatives of $\Omega(t)$ in our expressions as this amounts to the same thing.

3 Field Theory

3.1 Dirac fermions

Now we turn to the effect of changing oscillator frequencies in fermion systems. In particular we will apply the adiabatic expansion to Quantum electrodynamics (QED), the theory of a spin-1/2 Dirac Fermi field coupled to electromagnetism.

With the space-time signature $(+,-,-,-)$ the four-vector gauge field decomposes into time and space parts as $A^\mu = (\phi, \mathbf{A})$ and $A_\mu = (\phi, -\mathbf{A})$, where $\mathbf{A} = (A_x, A_y, A_z)$ is the usual three-vector potential in terms of which $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$.

On its own, the Maxwell gauge field is described by the action functional

$$S_{\mathrm{Maxwell}}[A] = -\frac{1}{4e^2}\int F_{\mu\nu}F^{\mu\nu} d^d x = \frac{1}{2e^2}\int (\mathbf{E}^2 - \mathbf{B}^2) d^d x. \tag{76}$$

After being integrated-out in a path-integral formalism, the Fermi field adds to $S_{\mathrm{Maxwell}}[A]$ the fermionic *effective action*

$$S_F[A] = -i \ln \mathrm{Det}(\not{D}[A] + m), \tag{77}$$

where $\not{D}[A] = i\gamma^\mu(\partial_\mu + iA_\mu)$.

The effective action is a sophisticated object, being shorthand for an infinite sum of one-loop Feynman diagrams with an arbitrary number of $\gamma^\mu A_\mu$ vertices. Even if we take the A_μ to be a non-fluctuating external field $S_F[A]$ captures much interesting physics. For a static magnetic field $S_F[A]$ is real number equal to minus the energy of the of the electrons in the field — a relativistic analogue of the De Haas-Van Alphen effect. For a static electric field $2S_F[A]$ gains an imaginary part that gives the rate of electron-positron pairs created per unit volume. For non-constant fields S_F captures the 1-loop renormalization effects and the non-linear effect of scattering of light by light. With an extra γ^5 vertex inserted, we also uncover the ABJ chiral anomaly [1, 2, 3].

It is not easy to compute $S_F[A]$ for general space-time dependent A_μ, but with some effort one can compute the first few terms in a systematic expansion in powers of derivatives of A_μ [7, 8, 9]. When, however, we restrict ourselves to A_μ fields that depend on only *one* space-time coordinate we can — after some small changes to take into account that we are dealing with fermions rather than bosons — exploit the adiabatic expansion methods from section 2.7 and calculate many terms with relative ease. As an illustration we will consider the specific case of a spatially uniform, but time dependent, electric field $\mathbf{E}(t)$ in one, two and three space dimensions.

3.2 One space dimension: chiral anomaly

To describe our uniform electric field we will use a gauge in which $A_0 = 0$ and $\mathbf{E} = -\partial_t \mathbf{A}$. In the one-dimensional case the Hamiltonian form of the Dirac equation in this field becomes

$$i\partial_t \hat{\psi} = H(t)\hat{\psi} \tag{78}$$

where

$$H(t) = -i\sigma_3(\partial_x - iA_x(t)) + \sigma_1 m \tag{79}$$

is a differential operator involving the two-by-two Pauli matrices.

The field equations for the Heisenberg-picture operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ are

$$\begin{aligned} i\partial_t \hat{\psi} &= -i\sigma_3(\partial_x - iA_x)\hat{\psi} + \sigma_1 m\hat{\psi}, \\ -i\partial_t \hat{\psi}^\dagger &= +i(\partial_x + iA_x)\hat{\psi}^\dagger \sigma_3 + m\hat{\psi}^\dagger \sigma_1. \end{aligned} \tag{80}$$

The equations (80) are linear so, as for the harmonic oscillator, the operators can be expanded linear combinations

$$\hat{\psi}(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Big(\hat{a}_p \boldsymbol{\psi}_+(p,x,t) + \hat{b}^\dagger_{-p} \boldsymbol{\psi}_-(p,x,t) \Big) \tag{81}$$

of two linearly-independent c-number solutions

$$\boldsymbol{\psi}_\pm(x,t) = \begin{bmatrix} u_\pm(x,t) \\ v_\pm(x,t) \end{bmatrix} \tag{82}$$

of the equation of motion

$$i\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -i(\partial_x - iA_x(t)) & m \\ m & +i(\partial_x - iA_x(t)) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{83}$$

As we are considering only spatially uniform systems we have $\boldsymbol{\psi}_\pm(p,x,t) = \boldsymbol{\psi}_\pm(p,t)e^{ipx}$. In the Heisenberg picture the operator-valued expansion coefficients $\hat{a}_p$ and $\hat{b}^\dagger_{-p}$ are time-independent and obey the standard Fermi anti-commutation relations $\{\hat{a}_p, \hat{a}^\dagger_q\} = 2\pi\delta(p-q)$, *etc.*

By formal manipulation of the field equations (80) we obtain the particle-number current conservation equation

$$\partial_t(\hat{\psi}^\dagger\hat{\psi}) + \partial_x(\hat{\psi}^\dagger\sigma_3\hat{\psi}) = 0, \tag{84}$$

and, more interestingly, the chiral current (non)-conservation equation in the form

$$\partial_t(\hat{\psi}^\dagger\sigma_3\hat{\psi}) + \partial_x(\hat{\psi}^\dagger\hat{\psi}) \stackrel{?}{=} 2m(\hat{\psi}^\dagger\sigma_2\hat{\psi}). \tag{85}$$

The "?" is there because, while last equation is valid for any *c-number* solution $\boldsymbol{\psi}(x,t)$ of the field equations, the actual equation obeyed by the *operator-valued* chiral charge $\hat{\psi}^\dagger\sigma_3\hat{\psi}$ should be

$$\partial_t(\hat{\psi}^\dagger\sigma_3\hat{\psi}) + \partial_x(\hat{\psi}^\dagger\hat{\psi}) = 2m(\hat{\psi}^\dagger\sigma_2\hat{\psi}) + \frac{1}{\pi}E(t), \tag{86}$$

where extra term $E(t)/\pi$ is the 1+1 dimensional version of the chiral anomaly [1, 2, 3].

To understand the source of the E/π term it helps to visualize the initial many-body ground state as a filled Dirac sea in which all negative-energy states are occupied and all positive-energy states vacant[1]. The antiparticle creation operator $\hat{b}^\dagger_{-p}$ is then to be thought of as an operator annihilating a negative-energy positively charged particle that was occupying a momentum $+p$ mode and as a result creating a negatively-charged particle excitation with *positive* energy $\sqrt{p^2+m^2}$ and momentum $-p$.

When $m=0$ and the field E time independent this view of the ground state allows a simple physical picture [24, 25, 26] of the how the anomaly arises. The Hamiltonian for the massless two component spinor is now diagonal and the equation of motion

$$i\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p+Et & 0 \\ 0 & -(p+Et) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{87}$$

can be interpreted as stating that the energy of a right-going mode which possessed momentum p at $t=0$ becomes $p+Et$. Similarly the energy of a

[1] The Dirac sea is rather out of fashion, but the principal objection to it — the associated infinite vacuum charge — is rendered moot by the observation that each generation in the standard model has a charge-neutral sea: there are three (for colour) up-type quarks each with $q=+2/3$, three down-type quarks with $q=-1/3$ and one electron-like lepton with $q=-1$. These charges sum to zero.

left-going mode becomes $p - Et$. For positive E there is therefore a steady flow of occupied right-going states into the positive energy continuum and an equal flow of empty left-going states down into the negative continuum. The latter leaves behind a growing number of holes and so, compared to the ground state at $t = 0$, a negative left-going particle number. If the "volume" of the system is L the density of momentum modes is $dn/dp = L/2\pi$ so the chiral particle number $n_R - n_L$ is changing according to

$$\frac{1}{L}\frac{d(n_R - n_L)}{dt} = 2 \times \frac{E}{2\pi} = \frac{E}{\pi}. \tag{88}$$

Taking the chiral charge density operator to be $\hat{\psi}_R^\dagger \hat{\psi}_R - \hat{\psi}_L^\dagger \hat{\psi}_L = \psi^\dagger \sigma_3 \hat{\psi}$ we have a c-number version

$$\frac{d\langle \hat{\psi}^\dagger \sigma_3 \hat{\psi}\rangle}{dt} = \frac{E}{\pi} \tag{89}$$

of Eq. (86).

This simple picture requires massless fermions. A mass term thwarts the spectral flow: the negative energy right-going single-particle modes still rise adiabatically in energy from deep in the sea but in the neighbourhood of the mass gap they mix with, and mutate into, left-going modes which descend again into the depths. When we ignore the possibility of Zener tunneling across the gap, the negative energy states remain filled and the positive energy ones empty, so $d(n_R - n_L)/dt$ is zero. The corresponding c-number version of the anomaly equation should therefore reduce to

$$2m\langle \hat{\psi}^\dagger \sigma_2 \hat{\psi}\rangle + \frac{E}{\pi} = 0. \tag{90}$$

We explore how this comes about, and how the equation is modified when the electric field E depends on time.

3.3 Adiabatic expansion for Dirac fields

Despite much work on the adiabatic expansion for scalar Bose fields, the construction of systematic adiabatic expansions for the time-dependent Dirac equation appears to be relatively recent. Landete, Navarro-Salas and Torrenti [28] developed techniques similar to those of [10, 11], but we will use the later method due to Gosh [29] which is closer in spirit to [19] and appears to be simpler and more efficient.

The time independent "in" vacuum state $|0\rangle$ is defined by $\hat{a}_p|0\rangle = \hat{b}_p|0\rangle = 0$, and, for any fixed p and before the $A_x(t)$ field appears, the corresponding solutions $\boldsymbol{\psi}_+(p,t)$, $\boldsymbol{\psi}_-(p,t)$ are those that have time dependence

$$\boldsymbol{\psi}_+(p,t) = \boldsymbol{\psi}_+(p)e^{-i\epsilon t}, \quad \boldsymbol{\psi}_-(p,t) = \boldsymbol{\psi}_-(p)e^{+i\epsilon t}, \quad \epsilon = \sqrt{p^2+m^2}. \tag{91}$$

The initial $\boldsymbol{\psi}_\pm(p)$ therefore coincide with the positive and negative energy eigenvectors $\boldsymbol{\chi}_\pm(p)$ of the hamiltonian matrix H with $A_x = 0$.

In the course of the time evolution caused by the field, an occupied initially-negative-energy mode $\boldsymbol{\psi}_-(p,t) = \boldsymbol{\chi}_-(p)e^{i\epsilon t}$ will acquire a positive energy component and become

$$\boldsymbol{\psi}_-(p,t) = \alpha(t)\boldsymbol{\chi}_-(t)e_+ + \beta(t)\boldsymbol{\chi}_+(t)e_-, \tag{92}$$

where

$$e_\pm(t) = e^{\pm i\int_{-\infty}^t \epsilon(\tau)d\tau}. \tag{93}$$

The $\boldsymbol{\chi}_\pm(t)$ are most conveniently chosen to be normalized positive and negative energy eigenvectors of the instantaneous Hamiltonian matrix $H(t)$. The linear independence of $\boldsymbol{\chi}_\pm(t)$ then uniquely defines $\alpha(t)$ and $\beta(t)$.

The normalized instantaneous eigenvectors are

$$\boldsymbol{\chi}_+ = \frac{1}{\sqrt{2\epsilon}}\begin{bmatrix}\sqrt{\epsilon+\tilde{p}}\\ \mathrm{sgn}(m)\sqrt{\epsilon-\tilde{p}}\end{bmatrix}, \quad \boldsymbol{\chi}_- = \frac{1}{\sqrt{2\epsilon}}\begin{bmatrix}\sqrt{\epsilon-\tilde{p}}\\ -\mathrm{sgn}(m)\sqrt{\epsilon+\tilde{p}}\end{bmatrix} \tag{94}$$

where $\tilde{p}(t) = p - A_x(t)$. They obey

$$H(t)\boldsymbol{\chi}_\pm = \pm\epsilon\boldsymbol{\chi}_\pm \tag{95}$$

with $\epsilon = \sqrt{\tilde{p}^2+m^2}$, and $|\boldsymbol{\chi}_\pm|^2 = 1$, $\boldsymbol{\chi}_+^\dagger\boldsymbol{\chi}_- = \boldsymbol{\chi}_-^\dagger\boldsymbol{\chi}_+ = 0$. The instantaneous eigenvectors do not obey the time evolution equation, but they still depend on t as a parameter. Because they are normalized and have real entries they are orthogonal to their derivative with respect to t. We can therefore most easily compute these derivatives by using the eigenstate perturbation formula

$$\langle m|\delta n\rangle = \frac{\langle m|\delta H|n\rangle}{E_n - E_m} \tag{96}$$

to find the projection of $\dot{\boldsymbol{\chi}}_\pm$ on the other eigenvector, and hence compute $\dot{\boldsymbol{\chi}}_\pm$

itself. For example

$$
\begin{aligned}
\langle \chi_+ | \dot{\chi}_- \rangle &= -\frac{\langle \chi_+ | \dot{H} | \chi_- \rangle}{2\epsilon}, \\
&= -\frac{1}{2\epsilon}\, \boldsymbol{\chi}_+^T \begin{bmatrix} E(t) & 0 \\ 0 & -E(t) \end{bmatrix} \boldsymbol{\chi}_-, \\
&= -\frac{|m|E(t)}{2\epsilon^2}. \qquad (97)
\end{aligned}
$$

The result is

$$
\frac{d}{dt}\boldsymbol{\chi}_+ = +\left(\frac{E|m|}{2\epsilon^2}\right)\boldsymbol{\chi}_-, \quad \frac{d}{dt}\boldsymbol{\chi}_- = -\left(\frac{E|m|}{2\epsilon^2}\right)\boldsymbol{\chi}_+. \qquad (98)
$$

Insert

$$
\boldsymbol{\psi}(t) = \alpha(t)\boldsymbol{\chi}_- e_+ + \beta(t)\boldsymbol{\chi}_+ e_- \qquad (99)
$$

into the equation of motion (83) to get

$$
\begin{aligned}
0 &= \dot{\alpha}\boldsymbol{\chi}_- e_+ + \alpha\dot{\boldsymbol{\chi}}_- e_+ + \dot{\beta}\boldsymbol{\chi}_+ e_- + \beta\dot{\boldsymbol{\chi}}_+ e_- \\
&= \dot{\alpha}\boldsymbol{\chi}_- e_+ - \alpha\left(\frac{E|m|}{2\epsilon^2}\right)\boldsymbol{\chi}_+ e_+ + \dot{\beta}\boldsymbol{\chi}_+ e_- + \beta\left(\frac{E|m|}{2\epsilon^2}\right)\boldsymbol{\chi}_- e_-. \qquad (100)
\end{aligned}
$$

From the coefficients of the of the linearly independent $\boldsymbol{\chi}_\pm$ we read off that

$$
\dot{\alpha} = -\left(\frac{E|m|}{2\epsilon^2}\right)\beta e_-^2, \quad \dot{\beta} = +\left(\frac{E|m|}{2\epsilon^2}\right)\alpha e_+^2. \qquad (101)
$$

Except for the relative minus sign — due to preserving $|\alpha|^2+|\beta|^2 = 1$ instead of $|\alpha|^2 - |\beta|^2 = 1$ — this is of the same form as the Bose case (66).

When $\alpha \approx 1$ the β coefficient has a very rapid $\propto e_+^2$ phase evolution so that the time dependence of the adiabatically small $\beta(t)\boldsymbol{\chi}_+ e_-$ term is close to that of the leading $\alpha(t)\boldsymbol{\chi}_- e_+$ term — just as it is in the bosonic case.

The slowly varying quantities

$$
\sigma = |\beta|^2, \quad \tau = i(\alpha\beta^* e_+^2 - \alpha^*\beta e_-^2), \quad \upsilon = (\alpha\beta^* e_+^2 + \alpha^*\beta e_-^2), \qquad (102)
$$

now label points on the Bloch sphere

$$
(1-2\sigma)^2 + \tau^2 + \upsilon^2 = 1 \qquad (103)
$$

and obey the spherical Bloch equations

$$\begin{aligned}
\dot{\sigma} &= F\upsilon, \\
\dot{\tau} &= -2\epsilon\upsilon, \\
\dot{\upsilon} &= 2F(1-2\sigma)+2\epsilon\tau, \qquad (104)
\end{aligned}$$

where $F = E|m|/2\epsilon^2 = \dot{\tilde{p}}|m|/2\epsilon^2$, and in the distant zero-field past $\sigma = \tau = \upsilon = 0$.

Just as the bosonic squeezed states are parameterized by points on a hyperboloid, the 2-mode fermionic squeezed state of the form (35) correspond to points on the Bloch sphere. The map (102) taking α, β to points on the sphere is now the classic Hopf map $\mathrm{SU}(2) \to S^3$. The equations (104) can therefore be interpreted as $\boldsymbol{\psi}_-(p,t)$ being a 2-mode (positive and negative energy) squeezed version of the ground state of one particle Hamiltonian $H(t)$, and $|0\rangle_{\text{in}}$ being a squeezed version of the ground state of the corresponding instantaneous many-body Hamiltonian.

As we are interested in the low energy response to the external field, we will focus on the case in which the time evolution of the hamiltonian is relatively slow compared to the mass gap. To obtain an adiabatic expansion we again rearrange the Bloch equations (104) as

$$\begin{aligned}
\sigma &= \int_0^t F[t']\upsilon dt' \\
\tau &= -\frac{2F(1-2\sigma)-\dot{\upsilon}}{2\epsilon} \\
\upsilon &= -\frac{\dot{\tau}}{2\epsilon}, \qquad (105)
\end{aligned}$$

and expand in inverse powers of $\epsilon(t)$

$$\begin{aligned}
\sigma &= \sigma_{\{2\}} + \sigma_{\{4\}} + \dots \\
\tau &= \tau_{\{1\}} + \tau_{\{3\}} + \dots \\
\upsilon &= \upsilon_{\{2\}} + \upsilon_{\{4\}} + \dots . \qquad (106)
\end{aligned}$$

Regarding F as being $O[\epsilon^0]$ the recursion relations are

$$\begin{aligned}
\sigma_{\{n\}} &= \int_0^t F[t']\upsilon_{\{n\}} dt', \\
\upsilon_{\{n\}} &= -\dot{\tau}_{\{n-1\}}/2\epsilon, \\
\tau_{\{n+1\}} &= \frac{4F\sigma_{\{n\}}+\dot{\upsilon}_{\{n\}}}{2\epsilon},
\end{aligned}$$

with $\tau_{\{1\}} = -F/\epsilon$.

From the matrix elements

$$
\begin{aligned}
\boldsymbol{\chi}_+^T\sigma_1\boldsymbol{\chi}_+ = -\boldsymbol{\chi}_-^T\sigma_1\boldsymbol{\chi}_- &= \frac{m}{\epsilon} \\
\boldsymbol{\chi}_+^T\sigma_2\boldsymbol{\chi}_+ = \quad \boldsymbol{\chi}_-^T\sigma_2\boldsymbol{\chi}_- &= 0 \\
\boldsymbol{\chi}_+^T\sigma_3\boldsymbol{\chi}_+ = -\boldsymbol{\chi}_-^T\sigma_3\boldsymbol{\chi}_- &= \frac{\tilde{p}}{\epsilon}.
\end{aligned}
\tag{107}
$$

and

$$
\begin{aligned}
\boldsymbol{\chi}_+^T\sigma_1\boldsymbol{\chi}_- = \quad \boldsymbol{\chi}_-^T\sigma_1\boldsymbol{\chi}_+ &= -\frac{\tilde{p}}{\epsilon}, \\
\boldsymbol{\chi}_+^T\sigma_2\boldsymbol{\chi}_- = -\boldsymbol{\chi}_-^T\sigma_2\boldsymbol{\chi}_+ &= i\,\mathrm{sgn}(m), \\
\boldsymbol{\chi}_+^T\sigma_3\boldsymbol{\chi}_- = \quad \boldsymbol{\chi}_-^T\sigma_3\boldsymbol{\chi}_+ &= \frac{|m|}{\epsilon}
\end{aligned}
\tag{108}
$$

we find, for each p-mode,

$$
\begin{aligned}
\boldsymbol{\psi}^\dagger\sigma_2\boldsymbol{\psi} &= \alpha^*\beta e_-^2\boldsymbol{\chi}_-\sigma_2\boldsymbol{\chi}_+ + \alpha\beta^* e_+^2\boldsymbol{\chi}_+\sigma_2\boldsymbol{\chi}_- \\
&= -i\,\mathrm{sgn}(m)(\alpha^*\beta e_-^2 - \alpha\beta^* e_+^2) \\
&= \mathrm{sgn}(m)\tau(t) \\
&= -\frac{mE}{2\epsilon^3} + \text{higher order.}
\end{aligned}
\tag{109}
$$

To compute the expectation values of the field operators we need to sum the contributions of the filled negative-energy sea by integrating over p. For a time independent E field we only need the lowest order contribution to τ and find that

$$
\begin{aligned}
2m\langle 0|\hat{\psi}^\dagger\sigma_2\hat{\psi}|0\rangle &= -\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{m^2E}{\epsilon^3} \\
&= -E\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{m^2}{((p+Et)^2+m^2)^{3/2}} \\
&= -\frac{E}{\pi}.
\end{aligned}
\tag{110}
$$

Thus we find the anticipated result that

$$
2m\langle 0|\hat{\psi}^\dagger\sigma_2\hat{\psi}|0\rangle + \frac{1}{\pi}E = 0, \quad E \text{ constant.} \tag{111}
$$

Now consider a time-dependent E field. Using the equations above we find that for any p mode the matrix elements are given in terms of σ, τ, υ as

$$\begin{aligned}
\boldsymbol{\psi}^\dagger\sigma_1\boldsymbol{\psi} &= (2\sigma-1)\frac{m}{\epsilon} - \upsilon\frac{\tilde{p}}{\epsilon} \\
\boldsymbol{\psi}^\dagger\sigma_2\boldsymbol{\psi} &= \mathrm{sgn}(m)\tau. \\
\boldsymbol{\psi}^\dagger\sigma_3\boldsymbol{\psi} &= (2\sigma-1)\frac{\tilde{p}}{\epsilon} + \upsilon\frac{|m|}{\epsilon}
\end{aligned} \tag{112}$$

So, using

$$\begin{aligned}
F &= |m|\dot{\tilde{p}}/2\epsilon^2 \\
\frac{d\epsilon}{dt} &= \frac{\tilde{p}\dot{\tilde{p}}}{\epsilon}, \\
\frac{d}{dt}\frac{1}{\epsilon} &= -\frac{\tilde{p}\dot{\tilde{p}}}{\epsilon^3}, \\
\frac{d}{dt}\frac{\tilde{p}}{\epsilon} &= \frac{\dot{\tilde{p}}m^2}{\epsilon^3},
\end{aligned} \tag{113}$$

and the Bloch equations (104) we find that the combination

$$\mathcal{A} \stackrel{\text{def}}{=} \partial_t(\boldsymbol{\psi}^\dagger\sigma_3\boldsymbol{\psi}) - 2m\boldsymbol{\psi}^\dagger\sigma_2\boldsymbol{\psi}, \tag{114}$$

which is expected to give the contribution of the p mode to the anomaly, actually evaluates to

$$\begin{aligned}
\mathcal{A} &= \partial_t\left((2\sigma-1)\frac{\tilde{p}}{\epsilon} + \upsilon\frac{|m|}{\epsilon}\right) - 2|m|\tau \\
&= 2\dot{\sigma}\frac{\tilde{p}}{\epsilon} + (2\sigma-1)\frac{\dot{\tilde{p}}m^2}{\epsilon^3} + \dot{\upsilon}\frac{|m|}{\epsilon} - \upsilon|m|\frac{\dot{\tilde{p}}\tilde{p}}{\epsilon^3} - 2|m|\tau \\
&= 2F\upsilon\frac{\tilde{p}}{\epsilon} + (2\sigma-1)2F\frac{|m|}{\epsilon} + (2F(1-2\sigma)+2\epsilon\tau)\frac{|m|}{\epsilon} - 2F\upsilon\frac{\tilde{p}}{\epsilon} - 2|m|\tau \\
&= 0.
\end{aligned} \tag{115}$$

That this expression is zero is inevitable. The Bloch equations encode the original field equations and the c-number version of the chiral charge non-conservation equation follows from these field equations. Consequently, when considered *mode-by-mode*, the anomaly appears to be zero. The anomaly is non-zero, however, because the operations of integration over p and taking the time derivative do not necessarily commute.

For the rest of this section we will assume that $m > 0$, so $|m| = m$ and $\mathrm{sgn}(m) = 1$, and again expand[2]

$$j \stackrel{\text{def}}{=} \boldsymbol{\psi}^\dagger \sigma_3 \boldsymbol{\psi} \tag{116}$$

in inverse powers of ϵ

$$j = (2\sigma - 1)\frac{\tilde{p}}{\epsilon} + \upsilon\frac{m}{\epsilon} = j_{\{1\}} + j_{\{3\}} + j_{\{5\}} + \dots \tag{117}$$

As $\sigma_{\{0\}} = \upsilon_{\{0\}} = 0$, we have

$$j_{\{1\}} = -\frac{\tilde{p}}{\epsilon}. \tag{118}$$

For $n \geq 2$, we have

$$j_{\{n+1\}}(\tilde{p}, t) = 2\sigma_{\{n\}}(\tilde{p}, t)\left(\frac{\tilde{p}}{\epsilon}\right) + \upsilon_{\{n\}}(\tilde{p}, t)\left(\frac{m}{\epsilon}\right). \tag{119}$$

The corresponding contributions to the chiral charge are are given by summing over the occupied p-modes as

$$J_{\{n+1\}}(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{dp}{2\pi} j_{\{n+1\}}(\tilde{p}, t). \tag{120}$$

For example the recurrence relations give

$$j_{\{3\}}(\tilde{p}, t) = -\frac{5}{8}\frac{m^2\tilde{p}E^2}{(m^2+\tilde{p}^2)^{7/2}} + \frac{1}{4}\frac{m^2\dot{E}}{(m^2+\tilde{p}^2)^{5/2}} \tag{121}$$

The integral over p is sufficiently convergent at large p that we can shift the integration variable $p \to \tilde{p} = p - A(t)$ without altering the value of the integral, and so find

$$J_{\{3\}} = \frac{\dot{E}}{6\pi m^2}. \tag{122}$$

We can similarly shift the integration variable and integrate

$$\tau_{\{3\}}(\tilde{p}) = \frac{m\ddot{E}}{8\left(m^2+\tilde{p}^2\right)^{5/2}} - \frac{5mpE\dot{E}}{4\left(m^2+\tilde{p}^2\right)^{7/2}} - \frac{3mE^3}{8\left(m^2+\tilde{p}^2\right)^{7/2}} + \frac{9m\tilde{p}^2E^3}{4\left(m^2+\tilde{p}^2\right)^{9/2}} + \frac{m^3E^3}{16\left(m^2+\tilde{p}^2\right)^{9/2}} \tag{123}$$

[2]We use the symbol j for for the per-mode chiral-charge density and J for the total because in 1+1 dimensions the chiral-charge density coincides with the particle-number current.

to find

$$T_{\{3\}} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tau_{\{3\}}(\tilde{p}) = \frac{\ddot{E}}{12\pi m^3}. \tag{124}$$

Consequently $\partial_t J_{\{3\}} = 2mT_{\{3\}}$.

The same is true of

$$T_{\{5\}} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tau_5(p) = \frac{4E^2\ddot{E} + 8E\dot{E}^2 - m^2E^{(4)}}{60\pi m^7} \tag{125}$$

and

$$J_{\{5\}} = \frac{4E^2\dot{E} - m^2E^{(3)}}{30\pi m^6}, \tag{126}$$

so $\partial_t J_{\{5\}} = 2mT_{\{5\}}$.

The interchange of derivative and integral is indeed legitimate for any $n > 1$, but an issue occurs for $j_{\{1\}}$ and $\tau_{\{1\}}$. As we have already seen

$$T_{\{1\}} = -\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{Em}{2(p^2+m^2)^{3/2}} = -\frac{E}{2\pi m}. \tag{127}$$

The problem is that the integral for the expectation of the chiral charge

$$J_{\{1\}} \stackrel{?}{=} \int_{\infty}^{\infty} \frac{dp}{2\pi} j_1(\tilde{p}) = -\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\tilde{p}}{\sqrt{\tilde{p}^2+m^2}} \tag{128}$$

is convergent, but only *conditionally* convergent[3]. The value of the integral depends on how we treat the large-p limits, and is therefore ambiguous. If we elect to remove the ambiguity by choosing limits that are symmetric about $p = 0$

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} j_1(\tilde{p}) \quad \stackrel{\text{def}}{=} \quad \lim_{\Lambda\to\infty} \left\{ \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} j_1(\tilde{p}) \right\} = \lim_{\Lambda\to\infty} \left\{ -\int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} \frac{\tilde{p}}{\sqrt{\tilde{p}^2+m^2}} \right\} \tag{129}$$

then, for $A(t) = 0$,

$$\lim_{\Lambda\to\infty} \left\{ -\int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} \frac{p}{\sqrt{p^2+m^2}} \right\} \tag{130}$$

[3]The same is true in the diagram calculation. The associated Feynman integral is linearly divergent by power counting, but becomes conditionally convergent after evaluating the gamma-matrix traces.

is zero by the $p \leftrightarrow -p$ symmetry. The $A(t) \neq 0$ integral, however, requires shifting the integration variable $p \to \tilde{p} = p - A$ and leads to

$$\lim_{\Lambda\to\infty}\left\{-\int_{-\Lambda}^{\Lambda}\frac{dp}{2\pi}\frac{(p-A)}{\sqrt{(p-A)^2+m^2}}\right\} = \lim_{\Lambda\to\infty}\left\{-\int_{-\Lambda-A}^{\Lambda-A}\frac{dp'}{2\pi}\frac{p'}{\sqrt{p'^2+m^2}}\right\} = \frac{A}{\pi}. \tag{131}$$

If we accept this symmetric-cutoff definition of $J_{\{1\}}$ we end up with the anomaly-free $\partial_t J_{\{1\}} = 2mT_{\{1\}}$ — but the arbitrariness is unsatisfying. More concerning is that a time-independent shift in $A_x(t)$ is a gauge transformation. If we insist on preserving gauge invariance we must ensure that the physical value of $J_{\{1\}}$ is unaffected when A_x is augmented by a constant. We can arrange this gauge invariance by re-defining the physical chiral charge operator to be

$$\hat{J}_{\rm phys} \stackrel{\rm def}{=} \hat{\psi}^\dagger\sigma_3\hat{\psi} - \frac{A_x}{\pi}. \tag{132}$$

This new definition must be used even when A_x becomes time dependent, so making $J_{\{1\}} \equiv 0$ and giving the anomalous $0 = \partial_t J_{\{1\}} = 2mT_{\{1\}} + E/\pi$.

The necessary $-A_x/\pi$ c-number subtraction in the definition of the physical current is the source of the difference between the mode-by-mode chiral-charge evolution equation and the anomalous equation for the mode-summed current. It is also this subtraction that allows the simple physical interpretation of the energy levels that cross zero in the $m = 0$ case as being newly created particles and holes — even though the occupation numbers of the Heisenberg states, labeled by their initial "p" and counted by $\hat{\psi}^\dagger\sigma_3\hat{\psi}$, remain unchanged.

Understanding that the E/π comes only from the lowest order of the adiabatic expansion reveals why the total charge created by the anomaly is insensitive to how rapidly the time evolution occurs.

3.4 Two space dimensions: the parity anomaly

The interesting physics in 2+1 dimensions is the "parity anomaly" [4] which arises from the fact that in odd spacetime dimensions a Dirac mass term violates space-inversion symmetry. The result is a current at right angles to the applied electric field with the direction of the current depending on the sign of the mass term. This effect is usually derived by computing a one-loop triangle diagram and extracting from it a Chern-Simons effective action [5, 30]. We can, however, also obtain it by a small modification of the results from the previous section.

Keeping the electric field parallel to the x axis the 2+1 dimensional Dirac hamiltonian is

$$\begin{aligned} H(t) &= \sigma_3(p_x - A_x) + p_y\sigma_2 + m\sigma_1 \\ &= \begin{bmatrix} p_x - A_x(t) & m - ip_y \\ m + ip_y & -(p_x - A_x(t)) \end{bmatrix}. \end{aligned} \tag{133}$$

We have

$$\begin{aligned} &= \sigma_3(p_x - A_x) + M\sigma_1 e^{i\sigma_3\phi} \\ &= \sigma_3(p_x - A_x) + M(\sigma_1\cos\phi + \sigma_2\sin\phi) \\ &= e^{-i\sigma_3\phi/2}(\sigma_3(p_x - A_x) + M\sigma_1)e^{i\sigma_3\phi/2} \end{aligned} \tag{134}$$

so if we define $M = \sqrt{p_y^2 + m^2}$ and an angle ϕ by setting $M\cos\phi = m$, $M\sin\phi = p_y$, we find that the solutions to

$$i\partial_t\boldsymbol{\psi} = H(p_x, p_y, t)\boldsymbol{\psi} \tag{135}$$

are simply

$$\boldsymbol{\psi} = e^{-i\sigma_3\phi/2}\boldsymbol{\psi}_{1d} \tag{136}$$

where $\boldsymbol{\psi}_{1d}$ are the solutions for the one-dimensional case, but with the m appearing there replaced by $M = \sqrt{p_y^2 + m^2}$.

For fixed p_x, p_y we have

$$j_x = \boldsymbol{\psi}^\dagger\sigma_3\boldsymbol{\psi} = \boldsymbol{\psi}_{1d}^\dagger\sigma_3\boldsymbol{\psi}_{1d} \tag{137}$$

but the more interesting transverse component j_y is given by

$$\begin{aligned} j_y &= \boldsymbol{\psi}^\dagger\boldsymbol{\sigma}_2\boldsymbol{\psi} = \boldsymbol{\psi}_{1d}^\dagger\left(e^{i\sigma_3\theta/2}\boldsymbol{\sigma}_2 e^{-i\sigma_3\theta/2}\right)\boldsymbol{\psi}_{1d} \\ &= \cos\phi\,\boldsymbol{\psi}_{1d}^\dagger\boldsymbol{\sigma}_2\boldsymbol{\psi}_{1d} + \sin\phi\,\boldsymbol{\psi}_{1d}^\dagger\boldsymbol{\sigma}_1\boldsymbol{\psi}_{1d} \\ &= \frac{m}{\sqrt{p_y^2 + m^2}}\boldsymbol{\psi}_{1d}^\dagger\boldsymbol{\sigma}_2\boldsymbol{\psi}_{1d} + \frac{p_y}{\sqrt{p_y^2 + m^2}}\boldsymbol{\psi}_{1d}^\dagger\boldsymbol{\sigma}_1\boldsymbol{\psi}_{1d} \\ &= \frac{m}{\sqrt{p_y^2 + m^2}}\mathrm{sgn}(M)\tau(t) + \frac{p_y}{\sqrt{p_y^2 + m^2}}\left((2\sigma(t) - 1)\frac{\sqrt{p_y^2 + m^2}}{\sqrt{\tilde{p}_x^2 + p_y^2 + m^2}} - \upsilon(t)\frac{\tilde{p}_x}{\sqrt{\tilde{p}_x^2 + p_y^2 + m^2}}\right). \end{aligned} \tag{138}$$

To compute the parity anomaly current we simply substitute our results from one dimension and perform the p_x, p_y integrals. The p_y integral of the last term $\boldsymbol{\psi}_{1d}^{\dagger}\boldsymbol{\sigma}_1\boldsymbol{\psi}_{1d}$ is zero, so we only need the τ term

$$\frac{m}{\sqrt{p_y^2+m^2}}\tau. \tag{139}$$

In particular, at lowest order, we have

$$j_y^{\{1\}}(p_x,p_y)=\frac{m}{\sqrt{p_y^2+m^2}}\tau_1=\frac{m}{M}\left(-\frac{E_x M}{2(p^2+m^2)^{3/2}}\right) \tag{140}$$

so

$$\begin{aligned} J_y^{\{1\}} &= \int \frac{d^2p}{(2\pi)^2} j_y^{\{1\}}(p_x,p_y), \\ &= -\frac{mE_x}{4\pi}\int_0^\infty \frac{p\,dp}{(p^2+m^2)^{3/2}}, \\ &= -\frac{mE_x}{4\pi|m|}, \\ &= -\mathrm{sgn}(m)\frac{E_x}{4\pi}, \end{aligned} \tag{141}$$

which is the correct coefficient.

Higher terms in the adiabatic expansion are all total derivatives

$$\begin{aligned} J_y^{\{3\}} &= \mathrm{sgn}(m)\frac{\partial}{\partial t}\left(\frac{1}{48\pi m^2}\dot{E}_x\right), \\ J_y^{\{5\}} &= \mathrm{sgn}(m)\frac{\partial}{\partial t}\left(-\frac{1}{320\pi m^4}E_x^{(3)}+\frac{1}{96\pi m^6}E_x^2\dot{E}_x\right), \\ J_y^{\{7\}} &= \mathrm{sgn}(m)\frac{\partial}{\partial t}\left(\frac{1}{1792\pi m^6}E_x^{(5)}-\frac{1}{768\pi m^8}(5\dot{E}_x^3+20E_x\dot{E}_x\ddot{E}_x+5E_x^2E_x^{(3)})+\frac{3}{128\pi m^{10}}E_x^4\dot{E}_x\right), \end{aligned} \tag{142}$$

and so on.

The parity anomaly provides the simplest illustration of the general Callan-Harvey anomaly-inflow mechanism [6]. In 2+1 dimensions a domain wall across which the fermion mass changes sign traps a 1+1 dimensional chiral fermion which possesses a charge-conservation anomaly

$$\partial_t\rho_{\mathrm{wall}}=\frac{E_\parallel}{2\pi}, \tag{143}$$

that is one-half of the E/π Dirac-particle chiral anomaly. Here $E_\|$ is the component of the two dimensional electric field parallel to the domain wall. The charge $\rho_{\rm wall}$ is not appearing from nowhere, but is supplied by twice (because the wall has two sides) the Hall-effect-like parity-anomaly current (141) which is perpendicular to the wall. The change in sign of the fermion mass across the wall means that the currents from the two sides are in opposite directions and therefore add. If the wall is parallel to the x axis, and if E_x is zero at $t = \pm\infty$ then *total* charge per unit length that appears on the domain wall is given by

$$\Delta\rho^0_{\rm wall} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_x(t)dt. \tag{144}$$

This expression for the accumulated charge is exact because the total derivatives in the higher order contributions to J_y make no net contribution – everything comes from the leading order term only.

3.5 Three dimensional vector currents and vacuum polarization

In 3+1 dimensions, and with the external electric field parallel to the z axis, it turns out to convenient to change the spinor basis so that the 4-by-4 Dirac Hamiltonian

$$H(t) = \alpha^1 p_1 + \alpha^2 p_2 + \alpha^3(p_3 - A_z(t)) + \beta m \tag{145}$$

becomes

$$H(t) = \begin{bmatrix} p_3 - A_z(t) & M \\ M^\dagger & -(p_3 - A_z(t)) \end{bmatrix}, \quad M = M^\dagger = p_1\sigma_1 + p_2\sigma_2 + m\sigma_3. \tag{146}$$

The $\beta \equiv \gamma^0$ matrix must be the coefficient of m and so

$$\gamma^0 = \begin{bmatrix} & \sigma_3 \\ \sigma_3 & \end{bmatrix} = \sigma_1 \otimes \sigma_3. \tag{147}$$

Similarly

$$\begin{aligned} \alpha^1 &\equiv \gamma^0\gamma^1 = \sigma_1 \otimes \sigma_1 \Rightarrow \gamma^1 = \mathbb{I} \otimes \sigma_3\sigma_1 \\ \alpha^2 &\equiv \gamma^0\gamma^2 = \sigma_1 \otimes \sigma_2 \Rightarrow \gamma^2 = \mathbb{I} \otimes \sigma_3\sigma_2 \\ \alpha^3 &\equiv \gamma^0\gamma^3 = \sigma_3 \otimes \mathbb{I} \ \Rightarrow \gamma^3 = \sigma_1\sigma_3 \otimes \sigma_3. \end{aligned} \tag{148}$$

We see that distinct γ's mutually anticommute, and

$$(\gamma^0)^2 = 1, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1, \tag{149}$$

so we have a valid, if non-standard, representation of the Dirac algebra.

The chiral symmetry operation should rotate m and so must be

$$\begin{aligned}\begin{bmatrix} e^{-i\sigma_3\theta} & \\ & e^{i\sigma_3\theta}\end{bmatrix} \begin{bmatrix} p_3 - A_z(t) & p_1\sigma_1 + p_2\sigma_2 + m\sigma_3 \\ p_1\sigma_1 + p_2\sigma_2 + m\sigma_3 & -(p_3 - A_z(t))\end{bmatrix} \begin{bmatrix} e^{i\sigma_3\theta} & \\ & e^{-i\sigma_3\theta}\end{bmatrix} \\ = \begin{bmatrix} p_3 - A_z(t) & p_1\sigma_1 + p_2\sigma_2 + (me^{-2i\theta})\sigma_3 \\ p_1\sigma_1 + p_2\sigma_2 + (me^{2i\theta})\sigma_3 & -(p_3 - A_z(t))\end{bmatrix}.\end{aligned} \tag{150}$$

Thus

$$\begin{bmatrix} e^{i\sigma_3\theta} & \\ & e^{-i\sigma_3\theta}\end{bmatrix} \to e^{i\theta\gamma^5} \tag{151}$$

and

$$\gamma^5 = \begin{bmatrix} \sigma_3 & \\ & -\sigma_3\end{bmatrix} = \sigma_3 \otimes \sigma_3. \tag{152}$$

We see that the γ^μ and γ^5 anticommute as they should.

The p-mode contribution to the expectation of the vector current is then

$$j_z = \boldsymbol{\psi}^\dagger(\sigma_3 \otimes \mathbb{I})\boldsymbol{\psi}, \tag{153}$$

and the chiral charge density and chiral current density are respectively

$$\begin{aligned} j^5_0 &= \boldsymbol{\psi}^\dagger\gamma^5\boldsymbol{\psi} = \boldsymbol{\psi}^\dagger(\sigma_3 \otimes \sigma_3)\boldsymbol{\psi}, \\ j^5_3 &= \boldsymbol{\psi}^\dagger\gamma^0\gamma^3\gamma^5\boldsymbol{\psi} = \boldsymbol{\psi}^\dagger(\sigma_3 \otimes \mathbb{I})(\sigma_3 \otimes \sigma_3)\boldsymbol{\psi} = \boldsymbol{\psi}^\dagger(\mathbb{I} \otimes \sigma_3)\boldsymbol{\psi}. \end{aligned} \tag{154}$$

Similarly, the bilinear that appears on the right-hand-side of the chiral charge (non)-conservation law is

$$i\bar{\boldsymbol{\psi}}\gamma^5\boldsymbol{\psi} = i\boldsymbol{\psi}^\dagger\gamma^0\gamma^5\boldsymbol{\psi} = i\boldsymbol{\psi}^\dagger(\sigma_1 \otimes \sigma_3)(\sigma_3 \otimes \sigma_3)\boldsymbol{\psi} = \boldsymbol{\psi}^\dagger(\sigma_2 \otimes \mathbb{I})\boldsymbol{\psi}. \tag{155}$$

The advantage of the unconventional basis choice is that the t-parameterized snapshot eigenfunctions of $H(t)$ factorize. They are a tensor product of our previous 2-spinor eigenstates with a second orthonormal set of 2-spinor eigenstates $\boldsymbol{\lambda}_\alpha$, $\alpha = \pm$, of the two-by-two matrix M. These have eigenvalues $\mu = \pm q$ where

$$q = \sqrt{m^2 + p_1^2 + p_2^2} = \sqrt{m^2 + p_\perp^2}. \tag{156}$$

We find that

$$
\begin{aligned}
\boldsymbol{\chi}_+(t) &= \frac{1}{\sqrt{2\epsilon}}\begin{bmatrix}\sqrt{\epsilon+\tilde{p}}\,\boldsymbol{\lambda}_\alpha \\ \mathrm{sgn}(\mu)\sqrt{\epsilon-\tilde{p}}\,\boldsymbol{\lambda}_\alpha\end{bmatrix} = \frac{1}{\sqrt{2\epsilon}}\begin{bmatrix}\sqrt{\epsilon+\tilde{p}} \\ \mathrm{sgn}(\mu)\sqrt{\epsilon-\tilde{p}}\end{bmatrix}\otimes\boldsymbol{\lambda}_\alpha \\
\boldsymbol{\chi}_-(t) &= \frac{1}{\sqrt{2\epsilon}}\begin{bmatrix}\sqrt{\epsilon-\tilde{p}}\,\boldsymbol{\lambda}_\alpha \\ -\mathrm{sgn}(\mu)\sqrt{\epsilon+\tilde{p}}\,\boldsymbol{\lambda}_\alpha\end{bmatrix} = \frac{1}{\sqrt{2\epsilon}}\begin{bmatrix}\sqrt{\epsilon-\tilde{p}} \\ -\mathrm{sgn}(\mu)\sqrt{\epsilon+\tilde{p}}\end{bmatrix}\otimes\boldsymbol{\lambda}_\alpha.
\end{aligned}
\tag{157}
$$

Here again $\tilde{p} = p_3 - A(t)$, but now $\epsilon = \sqrt{\tilde{p}_3(t)^2 + \mu^2}$.

Again we expand

$$
\boldsymbol{\psi}(t) = \alpha(t)\boldsymbol{\chi}_-(t)e_+ + \beta(t)\boldsymbol{\chi}_+e_-. \tag{158}
$$

The evolution equations for α, β, σ, τ, υ are the same as in 1+1 dimensions except that ϵ is the 3+1 energy, and $|m|$ must be replaced by $|\mu|$. Also the the vector-current expectations need to be multiplied by 2 to take into account both $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_-$.

The vector current density is

$$
\begin{aligned}
j_z &= \boldsymbol{\psi}^\dagger(\sigma_3\otimes\mathbb{I})\boldsymbol{\psi}, \\
&= 2\left((2\sigma-1)\frac{\tilde{p}}{\epsilon} + \upsilon\frac{|\mu|}{\epsilon}\right).
\end{aligned}
\tag{159}
$$

We find that

$$
j_z^{\{3\}} = 2\left(-\frac{5}{8}\frac{\mu^2 p_3 E^2}{(\mu^2+p_3^2)^{7/2}} + \frac{1}{4}\frac{\mu^2\dot{E}}{(\mu^2+p_3^2)^{5/2}}\right). \tag{160}
$$

Because of the p_3 in its numerator, the first term cancels in the momentum integration. The second contributes

$$
\begin{aligned}
J_z^{\{3\}} &= \int\frac{d^3p}{(2\pi)^3}j^{(3)} \\
&= \dot{E}\cdot\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p_1^2+p_2^2+m^2}{(p_1^2+p_2^2+p_3^2+m^2)^{5/2}} \\
&= \dot{E}\cdot\frac{1}{2}\left(\frac{2}{3}\int\frac{d^3p}{(2\pi)^3}\frac{p_1^2+p_2^2+p_3^2}{(p_1^2+p_2^2+p_3^2+m^2)^{5/2}} + \int\frac{d^3p}{(2\pi)^3}\frac{m^2}{(p_1^2+p_2^2+p_3^2+m^2)^{5/2}}\right) \\
&= \dot{E}\cdot\frac{1}{4\pi^2}\left(\frac{2}{3}\int_0^\infty\frac{p^4\,dp}{(p^2+m^2)^{5/2}} + \int_0^\infty\frac{m^2p^2\,dp}{(p^2+m^2)^{5/2}}\right)
\end{aligned}
\tag{161}
$$

The second integral in the parentheses in the last line of (161) is independent of m and equal to $1/3$. Scaling $p \to mp$ suggests that the first term is also independent of m — but as the integral is logarithmically divergent this is an unsafe conclusion. To get a meaningful result we should regulate $J_z^{\{3\}}$ by introducing a suitable high energy cutoff. The simplest Lorentz- and gauge-invariant way to do this is to follow Pauli-Villars and subtract from the integrand in (161) the same expression but with m^2 everywhere replaced by Λ^2. Then

$$\int_0^\infty \frac{p^4\,dp}{(p^2+m^2)^{5/2}} \to I \stackrel{\text{def}}{=} \int_0^\infty \left(\frac{p^4}{(p^2+m^2)^{5/2}} - \frac{p^4}{(p^2+\Lambda^2)^{5/2}}\right) dp. \qquad (162)$$

The new integrand falls off as $1/p^2$ at large p so the integral is convergent and easily evaluated by Frullani's method

$$\begin{aligned}
I &= \lim_{M\to\infty}\left\{\int_0^M \left(\frac{p^4}{(p^2+m^2)^{5/2}} - \frac{p^4}{(p^2+\Lambda^2)^{5/2}}\right) dp\right\} \\
&= \lim_{M\to\infty}\left\{\int_0^{M/m} \frac{\rho^4\,d\rho}{(\rho^2+1)^{5/2}} - \int_0^{M/\Lambda} \frac{\rho^4\,d\rho}{(\rho^2+1)^{5/2}}\right\} \\
&= \lim_{M\to\infty}\left\{\int_{M/\Lambda}^{M/m} \frac{\rho^4\,dp}{(\rho^2+1)^{5/2}}\right\} \\
&= \lim_{M\to\infty}\left\{\int_{1/\Lambda}^{1/m} \frac{\rho^4\,d\rho}{(\rho^2+M^{-2})^{5/2}}\right\} \\
&= \int_{1/\Lambda}^{1/m} \frac{d\rho}{\rho} \\
&= \ln\left(\Lambda/m\right). \qquad (163)
\end{aligned}$$

The resulting cut-off-dependent current

$$\mathbf{J}(\Lambda) == \frac{\dot{\mathbf{E}}}{8\pi^2}\left(\frac{2}{3}\ln\left(\frac{\Lambda^2}{m^2}\right)\right) \qquad (164)$$

can be interpreted as $\mathbf{J}(\Lambda) = \dot{\mathbf{P}}(\Lambda)$ where

$$\mathbf{P}(\Lambda) = \frac{\mathbf{E}}{8\pi^2}\left(\frac{2}{3}\ln\left(\frac{\Lambda^2}{m^2}\right)\right) \qquad (165)$$

is the polarization of the vacuum. The associated cut-off-dependent dielectric constant is the source of charge renormalization (see appendix B for details).

Higher order terms in the adiabatic series are all finite — but more complicated. For example

$$
\begin{aligned}
j_z^{\{5\}}(\mathbf{p}) \;=\; & 2\Bigg(-\frac{\mu^2 E^{(3)}}{16\,(\mu^2+p_3^2)^{7/2}}+\frac{21\mu^2 p_3 \dot{E}^2}{32\,(\mu^2+p_3^2)^{9/2}}+\frac{7\mu^2 p_3 E\ddot{E}}{8\,(\mu^2+p_3^2)^{9/2}} \\
& +\frac{19\mu^2 E^2\dot{E}}{16\,(\mu^2+p_3^2)^{9/2}}-\frac{117\mu^2 p_3^2 E^2\dot{E}}{16\,(\mu^2+p_3^2)^{11/2}}-\frac{3\mu^4 E^2\dot{E}}{32\,(\mu^2+p_3^2)^{11/2}} \\
& -\frac{57\mu^2 p_3 E^4}{16\,(\mu^2+p_3^2)^{11/2}}+\frac{303\mu^2 p_3^3 E^4}{32\,(\mu^2+p_3^2)^{13/2}}+\frac{57\mu^4 p_3 E^4}{128\,(\mu^2+p_3^2)^{13/2}}\Bigg).
\end{aligned}
\tag{166}
$$

The expression for $j_z^{\{7\}}(\mathbf{p})$ contains 69 terms and $j_z^{\{9\}}(\mathbf{p})$ nearly 5,000, but the resulting expressions for the current

$$
J_z^{\{n\}} = \int \frac{d^3p}{(2\pi)^3} j^{\{n\}}(\mathbf{p}) \tag{167}
$$

are relative compact:

$$
\begin{aligned}
J_z^{\{5\}} &= \frac{1}{2\pi^2}\left(\frac{1}{15}\frac{E^2\dot{E}}{m^4}-\frac{1}{30}\frac{E^{(3)}}{m^2}\right), \\
J_z^{\{7\}} &= \frac{1}{2\pi^2}\left(\frac{2}{21}\frac{E^4\dot{E}}{m^8}-\frac{2}{63}\frac{(\dot{E})^3}{m^6}-\frac{8}{63}\frac{E\dot{E}\ddot{E}}{m^6}-\frac{2}{63}\frac{E^2E^{(3)}}{m^6}+\frac{1}{280}\frac{E^{(5)}}{m^4}\right), \\
J_z^{\{9\}} &= \frac{1}{2\pi^2}\Bigg(-\frac{E^{(7)}}{1890m^6}+\frac{E^{(5)}E^2}{84m^8}-\frac{8E^{(3)}E^4}{75m^{10}}+\frac{16E^6\dot{E}}{45m^{12}}-\frac{16E^2(\dot{E})^3}{25m^{10}}+\frac{E^{(4)}E\dot{E}}{14m^8} \\
&\quad +\frac{5E^{(3)}E\ddot{E}}{42m^8}+\frac{2E^{(3)}(\dot{E})^2}{21m^8}-\frac{64E^3\dot{E}\ddot{E}}{75m^{10}}+\frac{11\dot{E}(\ddot{E})^2}{84m^8}\Bigg).
\end{aligned}
\tag{168}
$$

The expressions in Eq. (168) contain contributions to the current that can be verified by comparison with other methods. For example, if we retain only the terms linear in E and its derivatives $E^{(n)} \equiv d^n E/dt^n$ we find

$$
\begin{aligned}
j_z^{\{5\}}(\mathbf{p}) &= 2\left(-\frac{1}{16}\frac{\mu^2 E^{(3)}}{(\mu^2+p_3^2)^{7/2}} + O(E^2)\right),\\
j_z^{\{7\}}(\mathbf{p}) &= 2\left(+\frac{1}{64}\frac{\mu^2 E^{(5)}}{(\mu^2+p_3^2)^{9/2}} + O(E^2)\right),\\
j_z^{\{9\}}(\mathbf{p}) &= 2\left(-\frac{1}{256}\frac{\mu^2 E^{(7)}}{(\mu^2+p_3^2)^{11/2}} + O(E^2)\right),
\end{aligned}
$$

where the pattern of rational-number coefficients is clear. In the resulting currents

$$
\begin{aligned}
J_z^{\{5\}} &= \int \frac{d^3p}{(2\pi)^3} j_z^{\{5\}} = -\frac{1}{30}\frac{1}{2\pi^2}E^{(3)} + O(E^2),\\
J_z^{\{7\}} &= \int \frac{d^3p}{(2\pi)^3} j_z^{\{7\}} = +\frac{1}{280}\frac{1}{2\pi^2}E^{(5)} + O(E^2),\\
J_z^{\{9\}} &= \int \frac{d^3p}{(2\pi)^3} j_z^{\{9\}} = -\frac{1}{1890}\frac{1}{2\pi^2}E^{(7)} + O(E^2),
\end{aligned} \tag{169}
$$

the sequence of numerical fractions is more obscure, but, as we will show in in appendix B, they are the coefficients occurring in the the series expansion of the two-point vacuum polarization diagram in powers of the frequency.

Another set of terms that can be confirmed by other methods are those containing an arbitrary power of E, one power of $\dot{E}$, and no higher derivatives of E.

For example the coefficient of $E^2\dot{E}$ in $j^{\{5\}}(\mathbf{p})$ is

$$
\begin{aligned}
\mathrm{coef}(E^2\dot{E})^{\{5\}} &= 2\left(\frac{19\mu^2}{16\,(\mu^2+p_3^2)^{9/2}} - \frac{117\mu^2 p_3^2}{16\,(\mu^2+p_3^2)^{11/2}} - \frac{3\mu^4}{32\,(\mu^2+p_3^2)^{11/2}}\right)\\
&= 2\left(+\frac{19(p^2\sin^2\theta+m^2)}{16\,(p^2+m^2)^{9/2}} - \frac{117(p^2\sin^2\theta+m^2)p^2\cos^2\theta}{16\,(p^2+m^2)^{11/2}} - \frac{3(p^2\sin^2\theta+m^2)^2}{32\,(p^2+m^2)^{11/2}}\right).
\end{aligned} \tag{170}
$$

where θ is the angle between $\mathbf{p}$ and the z-axis. The integration measure becomes

$$
\frac{d^3p}{(2\pi)^3} = \frac{1}{(2\pi)^2}\sin\theta d\theta p^2 dp \tag{171}
$$

and gives

$$J_z^{\{5\}} = \dot{E}\frac{1}{2\pi^2}\frac{1}{15}\frac{E^2}{m^4} + \dots. \tag{172}$$

The analogous calculation with $j^{\{7\}}$ gives a term

$$J_z^{\{7\}} = \dot{E}\frac{1}{2\pi^2}\frac{2}{21}\frac{E^4}{m^8} + \dots. \tag{173}$$

Both these terms agree with those in the current calculated from the Euler-Heisenberg-Schwinger one-loop effective action (see appendix B).

3.6 Three-dimensional chiral currents and their anomaly

In three space dimensions the uniform electric field version of the anomalous chiral current (non)-conservation is expected be

$$\partial_t\, \hat{\psi}^\dagger(\sigma_3 \otimes \sigma_3)\hat{\psi} = 2m\, \hat{\psi}^\dagger(\sigma_2 \otimes \mathbb{I})\hat{\psi} + \frac{1}{2\pi^2}\mathbf{E}\cdot\mathbf{B}. \tag{174}$$

In the absence of a magnetic field, the vacuum expectation of this equation is satisfied by all three terms being zero. The expectation of the current on the LHS is zero because, from (154), each p_3 mode has a factor of

$$\boldsymbol{\lambda}_+^\dagger \sigma_3 \boldsymbol{\lambda}_+ + \boldsymbol{\lambda}_-^\dagger \sigma_3 \boldsymbol{\lambda}_- \tag{175}$$

where $\boldsymbol{\lambda}_\pm$ are the normalized eigenvectors of

$$M = \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 m \tag{176}$$

with eigenvalue $\mu = \pm\sqrt{|p_\perp|^2 + m^2}$. This sum of expectations is zero because sandwiching the identity

$$\sigma_3 M + M\sigma_3 = 2m\mathbb{I} \tag{177}$$

between the $\pm$ eigenvectors and subtracting gives

$$|\mu|(\boldsymbol{\lambda}_+^\dagger \sigma_3 \boldsymbol{\lambda}_+ + \boldsymbol{\lambda}_-^\dagger \sigma_3 \boldsymbol{\lambda}_-) = 0. \tag{178}$$

The expectation of the RHS is zero because, from (155) ,

$$\boldsymbol{\psi}^\dagger(\sigma_2 \otimes \mathbb{I})\boldsymbol{\psi} \propto \mathrm{sgn}(\mu) \tag{179}$$

and we must sum over both signs of the eigenvalue μ when we include the $\boldsymbol{\lambda}_\pm$ factors.

To get a non-trivial result we must include a magnetic field. If we take the vector potential to be

$$\mathbf{A}(x,y) = (By/2, -Bx/2, 0) \tag{180}$$

then $\mathbf{B} = (0,0,-B)$ and a classical positively charged particle will orbit anticlockwise about the z direction. It is well known [31, 32] that in such a field the spectrum of the operator M is highly degenerate, being an ensemble of Landau levels in which each level eigenvalue μ has degeneracy $|\mathbf{B}|/2\pi$ per unit area perpendicular to the magnetic field. One set of eigenfunctions of M is

$$\boldsymbol{\lambda}_{l,\pm} \propto \begin{bmatrix} (\mu+|m|)r^{-1}e^{-i\theta} \\ iB \end{bmatrix} r^l e^{il\theta} \exp\left\{-\frac{Br^2}{4}\right\}, \quad l>0 \tag{181}$$

with eigenvalues $\mu = \pm\sqrt{2lB+m^2}$. These eigenfunctions correspond to the positively charged particles circling the z axis at a distance $\langle r\rangle = \sqrt{2l/B}$. The other eigenstates in the same Landau level can be thought of as as describing orbits of the same radius but with different centers. There are no $l<0$ modes because they would be singular at the origin.

The features that lead all terms in the chiral current to cancel still hold, so the $l>0$ eigenmodes make no contribution to the anomaly equation. The $l=0$ case is special, however. The $\mu = -|m|$ eigenvector

$$\boldsymbol{\lambda}_0 = \frac{1}{\sqrt{2\pi}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \exp\left\{-\frac{Br^2}{4}\right\} \tag{182}$$

is finite at $r=0$ but the $\mu = +|m|$ is not because of the r^{-1} in the upper component. As only one of the $\mu = \pm|m|$ pair is allowed, there is no cancellation in the currents and we recover exactly the situation explored in sections 3.2 and 3.3 — except that the RHS of the one-dimensional anomaly equation (86) we must replace

$$\frac{E_z}{\pi} \to \frac{\mathbf{E}\cdot\mathbf{B}}{2\pi^2} \tag{183}$$

to take into account the $B/2\pi$ per-unit-area Landau-level degeneracy. Note that the fermions still have a mass-gap given by m and so the same interplay between time derivative and $2m\langle\hat{\psi}^\dagger\sigma_2\hat{\psi}\rangle$ occurs as in 3.3.

4 Conclusion

We have shown that a simple quantum mechanics adiabatic expansion can be used to capture the non-trivial physics of the chiral and parity anomalies for massive fermions. Although similar in spirit to the well known spectral flow interpretation of the chiral anomaly, the presence of a fermion mass both thwarts the flow and at the same time controls the accuracy of the expansion, so allowing us to see that in both cases the anomaly arises only from the lowest term in the expansion.

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Appendices

A The driven oscillator

For completeness we include a brief discussion of the well-known coherent state solution of the constant-frequency harmonic oscillator driven by a time-dependent external force. The hamiltonian can be written in terms of $\hat{a}$, $\hat{a}^\dagger$ as

$$H(t) = \Omega \hat{a}^\dagger \hat{a} + F(t)\hat{a}^\dagger + F^*(t)\hat{a}. \tag{184}$$

The associated unitary evolution operator $U(t) = \mathcal{T}\exp\{-i\int_0^t H(t)dt\}$ obeys the equation

$$i\partial_t U = \{\Omega \hat{a}^\dagger \hat{a} + F(t)\hat{a}^\dagger + F^*(t)\hat{a}\}U \tag{185}$$

and we may seek a solution in the form

$$\begin{aligned} U &= e^{i\theta} e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} e^{-i\omega\hat{a}^\dagger\hat{a}}, \\ U^{-1} &= e^{-i\theta} e^{+\frac{1}{2}|z|^2} e^{i\omega\hat{a}^\dagger\hat{a}} e^{z^*\hat{a}} e^{-z\hat{a}^\dagger}. \end{aligned} \tag{186}$$

Then, using

$$\begin{aligned} e^{-i\varphi\hat{a}^\dagger\hat{a}} \begin{bmatrix} \hat{a} \\ \hat{a}^\dagger \end{bmatrix} e^{i\varphi\hat{a}^\dagger\hat{a}} &= \begin{bmatrix} \hat{a}e^{+i\varphi} \\ \hat{a}^\dagger e^{-i\varphi} \end{bmatrix}, \\ e^{-\lambda\hat{a}^\dagger}\, \hat{a}\, e^{+\lambda\hat{a}^\dagger} &= \hat{a} + \lambda, \\ e^{+\lambda^*\hat{a}} \hat{a}^\dagger e^{-\lambda^*\hat{a}} &= \hat{a}^\dagger + \lambda^*, \end{aligned} \tag{187}$$

we find

$$iU^{-1}\partial_t U = ie^{i\omega\hat{a}^\dagger\hat{a}} \left[\frac{1}{2}(-\dot{z}z^* - \dot{z}^*z) + \dot{z}(\hat{a}^\dagger + z^*) - \dot{z}^*\hat{a} - i\dot{\omega}\hat{a}^\dagger\hat{a} + i\dot{\theta}\right] e^{-i\omega\hat{a}^\dagger\hat{a}}, \tag{188}$$

and

$$U^{-1}\{\Omega\hat{a}^\dagger\hat{a} + F(t)\hat{a}^\dagger + F^*(t)\hat{a}\}U = e^{i\omega\hat{a}^\dagger\hat{a}}[\Omega(\hat{a}^\dagger + z^*)(\hat{a}+z) + F(t)(\hat{a}^\dagger + z^*) + F^*(t)(\hat{a}+z)]e^{-i\omega\hat{a}^\dagger\hat{a}}. \tag{189}$$

Comparing coefficients of $\hat{a}^\dagger\hat{a}$, $\hat{a}^\dagger$, $\hat{a}$ and 1, we read off that [33]

$$\begin{aligned} \dot{\omega} &= \Omega, \\ i\dot{z} &= \Omega z + F, \\ -i\dot{z}^* &= \Omega z^* + F^*, \\ \dot{\theta} &= \tfrac{i}{2}(\dot{z}z^* - \dot{z}^*z) - (\Omega z^* z + F z^* + F^* z). \end{aligned} \tag{190}$$

The accumulating phase

$$\theta(t) = \int_0^t \left\{ \frac{i}{2}(\dot{z}z^* - \dot{z}^*z) - (\Omega z^* z + F z^* + F^* z) \right\} dt \tag{191}$$

considered as a functional $\theta = S[z(t)]$ of the path $z(t)$ is the classical action whose variation gives the equations of motion for z and z^* in (188). Thus $S[z(t)]$ for the actual trajectory is Hamilton's principal function solution of the Hamilton-Jacobi equation [34, 35].

The unitary displacement operator

$$D(z) \stackrel{\text{def}}{=} e^{z\hat{a}^\dagger - z\hat{a}} = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} \tag{192}$$

acts on the ground state to create a conventional (unsqueezed) coherent state

$$|z\rangle = e^{z\hat{a}^\dagger - z^*\hat{a}}|0\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^\dagger}|0\rangle \tag{193}$$

that obeys $\hat{a}|z\rangle = z|z\rangle$. Under our time evolution

$$|0\rangle \mapsto U[t]|0\rangle = e^{i\theta(t)}|z(t)\rangle, \tag{194}$$

so if we start in the ground state $|0\rangle$ the external driving force will always leave the system in a coherent state. As

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\Omega}\,\hat{x} + \frac{i}{\sqrt{\Omega}}\hat{p}\right), \tag{195}$$

and the coherent state maps $\hat{a} \mapsto z$, the real and imaginary parts of the complex parameter $z(t)$ track the classical motion in x, p phase space.

Note that allowing the parameter Ω appearing in (184) to become time dependent does *not* have the same effect as the time dependence of the frequency $\Omega(t)$ appearing in the original oscillator Hamiltonian (1). We need to include additional a^2 and $(a^\dagger)^2$ terms in (184) to change the oscillator frequency because changing the frequency while keeping the operators $\hat{p}$ and $\hat{x}$ fixed redefines $\hat{a}$ and $\hat{a}^\dagger$.

B Dirac effective action and renormalization in 1+3 dimensions

Throughout the main text we have focussed on non-interacting fermions and scaled the A_μ field so as to set the particle charge to unity. We do this because in a general non-abelian gauge theory the coefficient of A_μ becomes the Lie-algebra matrix generator corresponding to the group representation in which the fermion lives — and for the U(1) group of electromagnetism the representations are labelled by integers. This rescaling has the effect that in the QED path integral

$$Z = \int d[A]d[\bar{\psi}]d[\psi]\exp\{iS[A]\} \tag{196}$$

the pure-gauge contribution to the action becomes

$$S_{\text{Maxwell}}[A] = -\frac{1}{4e_0^2}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2e_0^2}\int d^4x\left(\mathbf{E}^2 - \mathbf{B}^2\right). \tag{197}$$

The (bare) coupling constant e_0^2 now appears in its natural location where it governs the magnitude of gauge-field fluctuations. The integration over $\bar{\psi}$, ψ adds to the pure-gauge action the fermionic effective action

$$S_F = -i\ln\mathrm{Det}(\not{D}[A] + m). \tag{198}$$

where $\not{D} = i\gamma^\mu(\partial_\mu + iA_\mu)$, $A_\mu = (\phi, -\mathbf{A})$.

One special case in which S_F can be evaluated in closed form is when the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is constant [36, 37]. For a constant electric field we have

$$S_F[E] = VT\,\mathcal{L}_F[E] \tag{199}$$

where V and T are the volume and total time, respectively, and the effective Lagrangian density is

$$\begin{aligned}\mathcal{L}_{\text{F}}[E] &= -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\{sE\cot(sE)\}e^{-m^2 s}\\ &= \frac{1}{8\pi^2}\left(\frac{1}{3}E^2\ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{1}{45}\frac{E^4}{m^4} + \frac{4}{315}\frac{E^6}{m^8} + \frac{8}{315}\frac{E^8}{m^{12}}\cdots\right).\end{aligned} \tag{200}$$

In obtaining the second line we have expanded

$$(sE)\cot(sE) = 1 - \frac{(sE)^2}{3} - \frac{(sE)^4}{45} - \frac{2(sE)^6}{945} - \frac{(sE)^8}{4725} + \ldots, \tag{201}$$

and dropped the infinite negative Dirac sea contribution from the first term. The second term has then been regulated[4] using the Frullani integral

$$\int_0^\infty \frac{e^{-m^2 s} - e^{-\Lambda^2 s}}{s}ds = \ln\left(\frac{\Lambda^2}{m^2}\right). \tag{202}$$

[4]Strictly we need *three* Pauli-Villars massive particles to gauge-invariantly regulate both divergent terms.

The remaining terms are all finite.

The cutoff dependent term is proportional to $\mathbf{E}^2$, and this would become $\mathbf{E}^2 - \mathbf{B}^2$ had we included a constant magnetic field. We can therefore combine the cut-off dependent term with the free-field Maxwell action by replacing

$$\frac{1}{2e_0^2}\int(\mathbf{E}^2-\mathbf{B}^2)d^dx \to \frac{1}{2e_R^2}\int(\mathbf{E}^2-\mathbf{B}^2)d^dx \tag{203}$$

where the renormalized coupling constant e_R^2 is defined by

$$\frac{1}{e_R^2} = \frac{1}{e_0^2} + \frac{1}{8\pi^2}\frac{2}{3}\ln\left(\frac{\Lambda^2}{m^2}\right). \tag{204}$$

This relation can be written as

$$e_R^2 = \frac{e_0^2}{1+\frac{e_0^2}{8\pi^2}\frac{2}{3}\ln\left(\frac{\Lambda^2}{m^2}\right)}, \quad e_0^2 = \frac{e_R^2}{1-\frac{e_R^2}{8\pi^2}\frac{2}{3}\ln\left(\frac{\Lambda^2}{m^2}\right)}. \tag{205}$$

In terms of the fine-structure constant $\alpha \equiv e_R^2/4\pi$ and its bare value $\alpha_0 = e_0^2/4\pi$ Eq. (204) becomes

$$\alpha = \frac{\alpha_0}{1+\frac{\alpha_0}{3\pi}\ln\left(\frac{\Lambda^2}{m^2}\right)}, \tag{206}$$

which gives us the one-loop beta function for QED

$$\beta_{\text{one-loop}}(\alpha) \stackrel{\text{def}}{=} \left(\frac{\partial\alpha}{\partial\ln m}\right)_{\Lambda,\alpha_0} = \frac{2}{3}\frac{\alpha^2}{\pi}. \tag{207}$$

Although (200) was derived under the assumption that E is constant, we can use $J_z = \delta S_F/\delta A_z$ and

$$\delta\int E^n d^4x = \int nE^{n-1}(-\partial_t\delta A_z)d^4x = n(n-1)\int(E^{n-2}\dot{E})\delta A_z d^4x \tag{208}$$

to compute the current $\propto \dot{E}$ induced by a slowly varying electric field. The reason is that higher-order $\dot{E}$ corrections to (200) give rise in J_z to terms proportional to $\ddot{E}$ or $(\dot{E})^2$ or higher. Exploiting this observation we find from (200) the following contribution to the induced current

$$J_z = \frac{\dot{E}}{2\pi^2}\left(\frac{2}{12}\ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{1}{15}\frac{E^2}{m^4} + \frac{2}{21}\frac{E^4}{m^8} + \frac{16}{45}\frac{E^6}{m^{12}}\dots\right) \tag{209}$$

which agrees with the corresponding terms in the adiabatic expansion in Eqs. (172) and (173)

A second special case is the two-point diagram with arbitrary space-time dependent $A_\mu(\mathbf{x},t)$ fields

$$S_{\rm F2} \equiv \frac{1}{2}\int d^4x d^4x' A_\mu(x)\Pi_{\mu\nu}(x,x')A_\nu(x'). \tag{210}$$

In momentum space the kernel $\Pi_{\mu\nu}$ is given by [38]

$$\Pi_{\mu\nu}(k^2) = (g_{\mu\nu}k^2 - k_\mu k_\nu)\Omega(k^2,m,\Lambda), \tag{211}$$

where Λ is a momentum-space cut-off and

$$\Omega(k^2,m,\Lambda) = -\frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{-\ln\frac{\Lambda^2}{m^2} + \ln\left(1 - x(1-x)\frac{k^2}{m^2}\right)\right\}. \tag{212}$$

For a time-dependent electric field $\propto e^{i\omega t}$ we have $k_\mu = (\omega,0,0,0)$ and $g_{zz}k^2 = -\omega^2$, and so in frequency space

$$\mathcal{L}_{\rm F2} = \frac{1}{4\pi^2}A_z(\omega)A_z(-\omega)\int_0^1 dx\, x(1-x)\left\{-\ln\frac{\Lambda^2}{m^2} + \ln\left(1 - x(1-x)\frac{\omega^2}{m^2}\right)\right\} \tag{213}$$

Recalling that $E_z = -\partial_t A_z$ we see that this gives rise to a current

$$J_z(\omega) = \dot{E}_z(\omega)\frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\frac{\Lambda^2}{m^2} - \ln\left(1 - x(1-x)\frac{\omega^2}{m^2}\right)\right\}. \tag{214}$$

As

$$\int_0^1 dx\, x(1-x) = \frac{1}{6} \tag{215}$$

we have the cut-off dependent contribution to the vacuum polarization current

$$J_{z,\rm divergent} = \dot{\mathbf{P}} = \frac{\dot{E}}{8\pi^2}\left(\frac{2}{3}\ln\left(\frac{\Lambda^2}{m^2}\right) + \dots\right) \tag{216}$$

that we obtained in Eq. (164) and (209).

The non-divergent parts of $\mathcal{L}_{\mathrm{F}2}[E]$ come from

$$
\begin{aligned}
I\left(\frac{\omega^2}{m^2}\right) &= -\int_0^1 dx\, x(1-x)\ln\left(1 - x(1-x)\frac{\omega^2}{m^2}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{\omega^2}{m^2}\right)^n \frac{1}{n}\int_0^1 dx\, x^{n+1}(1-x)^{n+1} \\
&= \sum_{n=1}^{\infty}\left(\frac{\omega^2}{m^2}\right)^n \frac{1}{n}\frac{[(n+1)!]^2}{(2n+3)!} \\
&= \frac{1}{30}\left(\frac{\omega^2}{m^2}\right) + \frac{1}{280}\left(\frac{\omega^2}{m^2}\right)^2 + \frac{1}{1890}\left(\frac{\omega^2}{m^2}\right)^3 + \ldots
\end{aligned} \tag{217}
$$

After one takes into account that each factor of ω^2 corresponds to a $-d^2/dt^2$, the sequence of rational coefficients agree with those found in Eq. (169). The resulting series has radius of convergence $\omega^2 = 4m^2$, which is determined by a branch-point in the function $\Pi_{\mu\nu}(\omega^2, m^2)$. The discontinuity across the branch cut for $\omega^2 > 4m^2$ gives the cross-section for particle-hole pair creation *via* parametric resonance.

C Vacuum polarization and energy density

In Euclidean signature a time-independent effective-action density has a physical interpretation as the vacuum-energy density. In Minkowski signature the action density becomes a Lagrangian density whose interpretation in terms of energy is not so clear. It is interesting, therefore, to relate the constant-field effective Lagrangian density $\mathcal{L}_F[E]$ in (200) to the vacuum energy expressed as a function of the polarization.

Recall that in a Minkowski-signature action functional the coupling of the gauge field to the current is given by $\int \mathbf{A}\cdot\mathbf{J}\, d^dx$. Therefore, if $S_F = \int \mathcal{L}_F d^dx$ then the space component of the current is given by the functional derivative

$$J_z(\mathbf{x}, t) = \frac{\delta S_F}{\delta A_z(\mathbf{x}, t)}. \tag{218}$$

For spatially constant E

$$\delta S_F[E] = \int d^dx \mathcal{L}'_F[E](-\partial_t \delta A_z(t)) = \int d^dx \mathcal{L}''_F[E]\dot{E}\delta A_z, \tag{219}$$

where the prime denotes a derivative with respect to E. The corresponding current density is therefore

$$\mathbf{J} = \mathcal{L}_F''(E)\dot{\mathbf{E}}. \tag{220}$$

As the current is related to the vacuum polarization by $\mathbf{J} = \dot{\mathbf{P}}$, Equation (220) implies that the vacuum polarization is related to the action-density $\mathcal{L}_F[E]$ by $P[E] = \mathcal{L}_F'[E]$.

The associated vacuum energy-density $\mathcal{E}$ as function of P is found from the work required for an external E field to adiabatically create the given polarization density over a period of time τ. This work is

$$\begin{aligned}
\mathcal{E} &= \int_0^\tau JEdt \\
&= \int_0^\tau \mathcal{L}_F''(E)E\dot{E}dt \\
&= \int_0^E \mathcal{L}_F''(\varepsilon)\varepsilon d\varepsilon \\
&= \int_0^E (-\mathcal{L}_F'[\varepsilon])d\varepsilon + [\varepsilon\mathcal{L}_F'[\varepsilon]]_0^E \\
&= E\mathcal{L}_F'[E] - \mathcal{L}_F[E].
\end{aligned} \tag{221}$$

In the last line we assumed that that $\mathcal{L}_F$ is zero when $E = 0$. The result (221) is a Legendre transformation, so $\mathcal{E} \equiv E\mathcal{L}_F' - \mathcal{L}_F$ is naturally expressed in terms of $P = \mathcal{L}_F'[E]$ as $\mathcal{E}[P]$. We may then recover the electric field from the derivative

$$\begin{aligned}
\frac{d\mathcal{E}}{dP} &= \frac{dE}{dP}P + E - \frac{d\mathcal{L}_F}{dP} \\
&= \frac{dE}{dP}P + E - \frac{dE}{dP}\frac{d\mathcal{L}_F}{dE} \\
&= \frac{dE}{dP}P + E - \frac{dE}{dP}P \\
&= E.
\end{aligned} \tag{222}$$

These transformations are just a slightly disguised version of the standard relation between the Lagrangian and Hamiltonian.

The stability of the vacuum requires that $\mathcal{E}[P]$ be a convex function of P and this is property assured by the signs of the terms in (200) and that the Legendre transform of a convex function is convex.