ON FINITE PARTS OF DIVERGENT COMPLEX GEOMETRIC INTEGRALS AND THEIR DEPENDENCE ON A CHOICE OF HERMITIAN METRIC

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Abstract. Let X be a reduced complex space of pure dimension. We consider divergent integrals of certain forms on X that are singular along a subvariety defined by the zero set of a holomorphic section of some holomorphic vector bundle $E \to X$. Given a choice of Hermitian metric on E we define a finite part of the divergent integral. Our main result is an explicit formula for the dependence on the choice of metric of the finite part.

1. Introduction

Let X be a reduced complex analytic space of pure dimension n and let $V \subset X$ be an analytic subvariety. Consider an (n,n)-form ω which is smooth in $X \setminus V$ with singularities along V and such that $\overline{\operatorname{supp}\omega}$ is compact in X. We are interested in studying finite parts of the divergent integral $\int_X \omega$, inspired by the process of regularization and renormalization in perturbative quantum field theory. In general, the finite part of a given divergent integral is not uniquely defined, rather, it depends on the choice of regularization data. It is a fundamental problem to describe this dependence.

In this paper we consider the setting when the variety V is the vanishing locus of a global holomorphic section $s : X \to E$ of some holomorphic vector bundle $E \to X$. Given a (smooth) Hermitian metric $\|\cdot\|$ on E we consider the space $\mathcal{A}_{s,\|\cdot\|}(X)$ of smooth differential forms ω on $X \setminus V$ such that for each compact subset $K \subset X$ there exists some integer $N \geq 0$ such that $\|s\|^{2N}\omega$ extends to a smooth form across $V \cap K$. Let $\mathcal{A}_s(X)$ be the union over metrics of all such $\mathcal{A}_{s,\|\cdot\|}(X)$. Note that if s defines a Cartier divisor, then $|s|^2/\|s\|^2$ is smooth and non-vanishing for any two metrics $\|\cdot\|$ and $|\cdot|$ on E. Thus, in that case we have that $\mathcal{A}_{s,\|\cdot\|}(X) = \mathcal{A}_{s,|\cdot|}(X) = \mathcal{A}_s(X)$. In the general case $|s|^2/\|s\|^2$ is only locally bounded and there may be different *conformal classes* $\mathcal{A}_{s,\|\cdot\|}(X) \subset \mathcal{A}_s(X)$.

Any $\omega \in \mathcal{A}_s^{p,q}(X)$ defines a current on $X \setminus V$, that is, a continuous linear functional on the space $\mathscr{D}^{n-p,n-q}(X \setminus V)$ of test forms on $X \setminus V$ of complementary bidegree, by

$$\xi \mapsto \int_X \omega \wedge \xi.$$

To find a current extension of ω across V, following a classical idea, we consider the function

$$\Gamma_{\|\cdot\|}(\lambda) = \int_X \|s\|^{2\lambda} \omega \wedge \xi, \tag{1.1}$$

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defined for $\mathfrak{Re}\,\lambda$ sufficiently large. Differentiation under the integral sign shows that $\Gamma_{\|\cdot\|}(\lambda)$ is holomorphic for $\mathfrak{Re}\,\lambda \gg 0$. It is clear that if there exists a metric $\|\cdot\|$ on E such that $\|s\|^{2N}\omega$ is smooth for some $N \geq 0$, then for any other metric $|\cdot|$ on E, $|s|^{2N}\omega$ is locally bounded. Thus (1.1) is defined and holomorphic for any $\omega \in \mathcal{A}_s^{p,q}(X)$ and any choice of Hermitian metric on E if $\mathfrak{Re}\,\lambda \gg 0$. It is well known that (1.1) has a meromorphic continuation to $\mathbb{C}$, see, e.g., [2] and [8]. The Laurent series of $\Gamma_{\|\cdot\|}(\lambda)$ about the origin is of the form

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=0}^{\kappa} \frac{1}{\lambda^j}\langle \mu_j(\omega), \xi\rangle + \mathcal{O}(\lambda),$$

where $0 \leq \kappa \leq n$ and $\mu_j(\omega)$ are currents on X. Moreover, $\operatorname{supp}\mu_j(\omega) \subseteq V$ for $j \geq 1$. See Theorem 4.1 below for details. It follows that $\mu_0(\omega) = \omega$ as currents on $X \setminus V$. Thus $\mu_0(\omega)$ is a current extension of ω across V. For ω of top degree, and with $\operatorname{supp}\omega \subset\subset X$, it is therefore natural to define the finite part of $\int_X \omega$ as

$$\operatorname{fp}\int_X \omega := \langle \mu_0(\omega), 1\rangle. \tag{1.2}$$

This definition depends on the choice of metric on E, as well as on the choice of section s defining V. In this paper our focus is the metric dependence, keeping the section fixed. In some situations, however, a change of sections can be realized a change of metrics, see Example 5.1 below. The following theorem is the main result of this article. It describes the metric dependence of $\mu_j(\omega)$ for each $j = 0, \dots, \kappa$.

Theorem 1.1. *Let $\omega \in \mathcal{A}_s^{p,q}(X)$. For any two Hermitian metrics $\|\cdot\|$ and $|\cdot|$ on E, let $\mu_j^{\|\cdot\|}(\omega)$ and $\mu_j^{|\cdot|}(\omega)$ denote the currents defined by the coefficient of the $-j^{th}$ order term in the Laurent series expansion around 0 of $\Gamma_{\|\cdot\|}$ and $\Gamma_{|\cdot|}$, respectively. We have that*

$$\mu_j^{|\cdot|}(\omega) = \sum_{\ell=0}^{n-j} \frac{1}{\ell!}\left(\log\frac{|s|^2}{\|s\|^2}\right)^{\ell} \mu_{j+\ell}^{\|\cdot\|}(\omega). \tag{1.3}$$

A version of this theorem, in the special case when X and V are smooth, is a central result in [12, 13, 14], see Example 1.3 below and the paragraph preceding it. There are also partial results in [12, 13, 14] in the case when V is a normal crossings divisor. The key idea of the proof is to consider a particular function of two complex parameters, see (3.1), and use it to interpolate between the functions $\Gamma_{\|\cdot\|}$ and $\Gamma_{|\cdot|}$.

Note that the factor $\log\frac{|s|^2}{\|s\|^2}$ appearing in (1.3) is locally integrable, but not smooth in general. This means that the products on the right-hand side of (1.3) are not canonically defined. However, the proof of Theorem 1.1 shows that these products have a natural meaning. In the special case where s defines a Cartier divisor, $\log\frac{|s|^2}{\|s\|^2}$ is smooth and the products on the right hand side of (1.3) are canonically defined. An immediate consequence of this is the following result, which generalizes some results in [13, 14].

Corollary 1.2. *Assume that s defines a Cartier divisor, and let κ be the order of the pole of $\Gamma_{\|\cdot\|}(\lambda)$ at 0. Then κ and $\mu_\kappa^{\|\cdot\|}(\omega)$ are independent on the choice of metric.*

There is another standard way to regularize divergent integrals, such as $\int_X \omega$, which is to introduce a *cut-off* parameter $\epsilon > 0$, integrate ω over the locus $\{\|s\|^2 \geq \epsilon\}$

and then study the asymptotic behavior of the integral as $\epsilon \to 0$. For $\omega \in \mathcal{A}_s^{p,q}(X)$, $\xi \in \mathscr{D}^{n-p,n-q}(X)$, and any smooth Hermitian metric $\|\cdot\|$ on E, we let

$$\mathcal{I}_{\|\cdot\|}(\epsilon) = \int\limits_{\|s\|^2 \geq \epsilon} \omega \wedge \xi. \tag{1.4}$$

The functions $\mathcal{I}_{\|\cdot\|}(\epsilon)$ and $\Gamma_{\|\cdot\|}(\lambda)$ are related via the *Mellin transform.* If the limit of $\mathcal{I}_{\|\cdot\|}(\epsilon)$ as $\epsilon \to 0$ exists, we find that

$$\lim_{\epsilon \to 0} \mathcal{I}_{\|\cdot\|}(\epsilon) = \langle \mu_0^{\|\cdot\|}(\omega), \xi \rangle.$$

On the other hand, if $\lim_{\epsilon \to 0} \mathcal{I}_{\|\cdot\|}(\epsilon)$ does not exist, then, using standard techniques we find that

$$\mathcal{I}_{\|\cdot\|}(\epsilon) = \langle \mu_0^{\|\cdot\|}(\omega), \xi \rangle + \frac{|\log \epsilon|^q}{\epsilon^p} \phi(\epsilon) + \mathcal{O}(\epsilon^\delta), \tag{1.5}$$

for some $\delta > 0$, $p, q \in \mathbb{N}$ and $\phi \in \mathscr{C}^0([0, \infty))$ such that $\phi(0) \neq 0$. Clearly ϕ depends on ω and ξ, see Proposition 6.1 below for a more precise formula.

For ω of top degree with $\operatorname{supp} \omega \subset\subset X$ we have defined a finite part of $\int_X \omega$ in (1.2). Another natural definition of a finite part of $\int_X \omega$ is as the limit as $\epsilon \to 0$ of $\mathcal{I}_{\|\cdot\|}(\epsilon)$ (with $\xi = 1$) after having subtracted possible divergent terms. In view of (1.2) and (1.5) we find that they are the same, that is,

$$\lim_{\epsilon \to 0} \left(\mathcal{I}_{\|\cdot\|}(\epsilon) - \frac{|\log \epsilon|^q}{\epsilon^p} \phi(\epsilon) \right) = \operatorname{fp} \int\limits_X \omega. \tag{1.6}$$

Since the finite part extracted from $\mathcal{I}_{\|\cdot\|}(\epsilon)$ is the same as the one coming from $\Gamma_{\|\cdot\|}(\lambda)$, the metric dependence of the former thus is given by Theorem 1.1. Proving this metric dependence directly, without considering $\Gamma_{\|\cdot\|}(\lambda)$, seems, to the author, more involved.

1.1. **Relations to previous results.** This work is inspired by work by Felder–Kazhdan in [12, 13], where the authors investigate finite parts of divergent integrals of differential forms with singularities along a submanifold Y in the real setting. The singularities considered are determined by a *conformal class of non-negative Morse–Bott functions*. These are smooth non-negative functions vanishing precisely on Y with non-degenerate Hessian in the normal directions of Y. They consider regularizations of $\int_X \omega$ that closely resemble our $\Gamma_{\|\cdot\|}$ and $\mathcal{I}_{\|\cdot\|}$ and they investigate the dependence on the representative Morse–Bott function within a given conformal class. This is similar to the way we consider the spaces $\mathcal{A}_{s,\|\cdot\|}(X)$ and describe the metric dependence given a section s.

Example 1.3. Let X be a (complex) manifold, V a (complex) submanifold and suppose that s defines the radical ideal of V. Then $\kappa \leq 1$ and

$$\mu_0^{|\cdot|}(\omega) - \mu_0^{\|\cdot\|}(\omega) = \log \frac{|s|^2}{\|s\|^2} \mu_1(\omega),$$

which is a version of a main result in [12, 13, 14]. Note that $\mu_1(\omega)$ here is independent of the choice of metric.

The formula for $\mu_0^{|\cdot|}(\omega) - \mu_0^{\|\cdot\|}(\omega)$ follows directly from Theorem 1.1 and Corollary 1.2 if $\kappa \leq 1$. The fact that $\kappa \leq 1$ follows from Theorem 4.1 (i) below, since $V_{\text{sing}} = \varnothing = X_{\text{sing}}$.

Example 1.4. Suppose X is a compact complex manifold and let $\omega = \alpha \wedge \bar{\beta}$, where α and β are meromorphic forms of bidegree $(n, 0)$, that is, locally of the form $\alpha = f_\alpha / g_\alpha$ and $\beta = f_\beta / g_\beta$ where f_α and f_β are holomorphic $(n, 0)$-forms and where g_α and g_β are holomorphic functions. Then $\omega \in \mathcal{A}_s(X)$, where $V = \{g_\alpha g_\beta = 0\}$ locally. The problem of extracting a finite part of $\int_X \omega$ arises in perturbative superstring theory, see [17, Section 7.6]. This problem is considered in [12] in the case where V is a smooth hypersurface and in [13] when V has normal crossings singularities.

Meromorphic functions of the form (1.1) also appear in a number theoretic context in [10, Section 4]. More precisely, in [10, Section 4] it is assumed that E is a line bundle and ω is of the form $\|s\|^{-2c} \mathrm{d}V$, for a volume form $\mathrm{d}V$ on X, where c is the corresponding integrability threshold. An explicit expression for the corresponding measure $\mu_\kappa(\omega)$ is given in [10, Proposition 4.3], when the divisor D cut out by s has simple normal crossings, expressed in terms of the Clemens complex of D.

2. Preliminaries

2.1. **Smooth forms on reduced complex analytic spaces.** We will briefly mention how one defines smooth forms on spaces with singularities, specifically reduced analytic spaces. Recall that an analytic subspace $(Z, \mathcal{O}_Z)$, or simply Z when there is no risk of confusion, of a domain $\Omega \subseteq \mathbb{C}^n$, is a ringed space where Z is given by the common vanishing locus of a collection of holomorphic functions $f_1, \dots, f_k : \Omega \to \mathbb{C}$ and where the structure sheaf $\mathcal{O}_Z = \mathcal{O}_\Omega / \mathcal{J}_Z$, where $\mathcal{J}_Z$ is the ideal sheaf generated by $f_1, \dots, f_k$. The space $(Z, \mathcal{O}_Z)$ is reduced if $\mathcal{J}_Z$ is radical. For Z reduced, Z_{reg} is the set of points z such that Z is a manifold in a neighborhood of z, and Z_{reg} is dense in Z. When Z is reduced, we define the sheaf $\mathscr{E}_Z$ of smooth forms on Z as the quotient sheaf $\mathscr{E}_\Omega / \mathscr{N}_{Z,\Omega}$, where $\mathscr{E}_\Omega$ is the sheaf of smooth forms on Ω, and $\mathscr{N}_{Z,\Omega} \subseteq \mathscr{E}_\Omega$ is the subsheaf of forms whose pullback to Z_{reg} vanishes.

A reduced analytic space $(X, \mathcal{O}_X)$ is a ringed space such that for any point $x \in X$ there exists a *local model* consisting of an open neighborhood U of x and an isomorphism of ringed spaces $U \to Z$ where $Z \subset \Omega \subseteq \mathbb{C}^n$ is a reduced analytic subspace. The sheaf of smooth forms $\mathscr{E}_U$ on U, as defined above, is independent of the choice of local model. For a reduced analytic space X, the sheaf of smooth forms $\mathscr{E}_X$ is defined as the sheaf obtained from gluing the sheaves of smooth forms on the local models of X. For a more substantial treatment, see, e.g., [9, 11].

2.2. **Currents.** On a smooth manifold M of real dimension n, a current ν of degree k is a continuous linear functional $\xi \mapsto \langle \nu, \xi \rangle$ on the space $\mathscr{D}^{n-k}(M)$ of smooth $(n-k)$-forms with compact support. We define the current $\mathrm{d}\nu$, where d is the exterior derivative, by duality; for $\xi \in \mathscr{D}^{n-k-1}(M)$ we let

$$\langle \mathrm{d}\nu, \xi \rangle := (-1)^{k+1} \langle \nu, \mathrm{d}\xi \rangle. \tag{2.1}$$

Thus d takes k-currents to $(k+1)$-currents.

If M is a complex manifold the complex structure induces a decomposition of the spaces of smooth differential k-forms into bigraded (p, q)-forms, and the exterior derivative decomposes as $\mathrm{d} = \partial + \bar{\partial}$. By duality, the space of k-currents have a similar decomposition into bigraded objects: A current of bidegree (p, q) on M acts trivially on the space $\mathscr{D}^{n-p', n-q'}(M)$ of compactly supported forms of bidegree $(n-p', n-q')$ when $(p', q') \neq (p, q)$. For a (p, q)-current ν, we define the $(p+1, q)$ and $(p, q+1)$

currents $\partial\nu$ and $\bar\partial\nu$ by

$$\langle \partial\nu, \xi\rangle := (-1)^{p+q+1}\langle \nu, \partial\xi\rangle \quad \text{and} \quad \langle \bar\partial\nu, \xi\rangle := (-1)^{p+q+1}\langle \nu, \bar\partial\xi\rangle,$$

respectively.

We define the *support* $\operatorname{supp}\nu$ of a (p,q)-current ν as the smallest closed subset $U \subset M$ such that $\langle \nu, \xi\rangle = 0$ for each $\xi \in \mathscr{D}^{n-p,n-q}(M \setminus U)$.

For a (p,q)-current ν and a smooth (p',q')-form β, we define the $(p+p', q+q')$-current $\nu \wedge \beta$ by

$$\langle \nu\wedge\beta, \xi\rangle := \langle \nu, \beta\wedge\xi\rangle, \tag{2.2}$$

for $\xi \in \mathscr{D}^{n-p-p',n-q-q'}(M)$. We let $\beta\wedge\nu := (-1)^{(p+q)(p'+q')}\nu\wedge\beta$.

If X is a reduced analytic space, a current on X is a continuous linear functional on the space $\mathscr{D}(X)$ of smooth forms with compact support. The properties of currents presented above all generalize to this setting. For a modification $f : Y \to X$ of X, and a current ν on Y, we define the *push-forward* $f_*\nu$ of ν by

$$\langle f_*\nu, \xi\rangle := \langle \nu, f^*\xi\rangle, \tag{2.3}$$

for $\xi \in \mathscr{D}(X)$. The push-forward operator is continuous and commutes with ∂ and $\bar\partial$. If β is a smooth form on X we have that

$$\beta \wedge f_*\nu = f_*(f^*\beta\wedge\nu). \tag{2.4}$$

We can generalize this product as follows: Suppose that β is generically smooth on X with $f^*\beta$ smooth on Y. Moreover, let μ be a current on X such that $\mu = f_*\nu$ for some current ν on Y. Then we define

$$\beta \wedge_{f,\nu} \mu := f_*(\pi^*\beta\wedge\nu). \tag{2.5}$$

Note that, if β is smooth, $\beta\wedge_{f,\nu}\mu = \beta\wedge\mu$ by (2.4). Also note that the product in (2.5) is ill-defined in general since it depends on the choice of modification f and current ν. As hinted at in the introduction, products of the type (2.5) appear when we look at the metric dependence of the currents $\mu_j(\omega)$, cf. Theorem 1.1 and the subsequent comments. However, as it turns out, there are canonical choices of f and ν in this case, see the proof of Theorem 1.1 below.

The following example shows that $\beta\wedge_{f,\nu}\mu$ may be non-zero even though $\mu = 0$.

Example 2.1. Consider the blowup of $\mathbb{C}^2$ at the origin, $\pi : \mathrm{Bl}_0\mathbb{C}^2 \to \mathbb{C}^2$, where

$$\mathrm{Bl}_0\mathbb{C}^2 = \{(z_1, z_2, [w_0 : w_1]) \in \mathbb{C}^2_z \times \mathbb{P}^1_{[w]} : z_1w_1 - z_2w_0 = 0\}, \tag{2.6}$$

and π is the restriction of the natural projection $\Pi : \mathbb{C}^2\times\mathbb{P}^1 \to \mathbb{C}^2$ to $\mathrm{Bl}_0\mathbb{C}^2$. Let

$$\beta = \frac{i}{2\pi}\partial\bar\partial\log(|z_1|^2 + |z_2|^2).$$

Then $\widetilde\beta = \pi^*\beta = \omega_{FS}(w)|_{\mathrm{Bl}_0\mathbb{C}^2}$, that is, the Fubini–Study form on $\mathbb{P}^1_{[w]}$, extended to $\mathbb{C}^2_z \times \mathbb{P}^1_{[w]}$ and restricted to $\mathrm{Bl}_0\mathbb{C}^2$.

One way to see this is as follows: Away from the origin, π is a biholomorphism, so β and $\widetilde\beta$ are related via a holomorphic change of coordinates. The Fubini–Study form on $\mathbb{P}^1$ with homogeneous coordinates $[w_0 : w_1]$ is given by

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar\partial\log(|w_0|^2 + |w_1|^2).$$

Away from $z_1 = 0$ and $[w_0 : w_1] = [0 : 1]$ we see from (2.6) that

$$\frac{z_2}{z_1} = \frac{w_1}{w_0}.$$

Since $\partial\bar{\partial}\log|g|^2 = 0$ if g is holomorphic and non-vanishing it follows that

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|w_0|^2 + |w_1|^2) = \frac{i}{2\pi}\partial\bar{\partial}\log(|z_1|^2 + |z_2|^2).$$

By a symmetrical argument, $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|z_1|^2 + |z_2|^2)$ away from $z_2 = 0$ and $[w_0 : w_1] = [1 : 0]$.

Now, let $\nu = [E]$ be the integration current for the exceptional divisor $E = \pi^{-1}(0)$ on $\mathrm{Bl}_0\mathbb{C}^2$. Then, e.g., by the dimension principle, $\mu := \pi_*\nu = 0$. However, for $\xi \in \mathscr{D}^{0,0}(\mathbb{C}^2)$, we have by (2.3) and (2.5) that

$$\langle \beta \wedge_{\pi,\nu} \mu, \xi\rangle = \langle \widetilde{\beta} \wedge [E], \pi^*\xi\rangle = \int_E \omega_{FS}\,\pi^*\xi = \xi(0)\int_E \omega_{FS} = \xi(0).$$

Thus, we conclude that $\varphi \wedge_{\pi,\nu} \mu = \delta_0$, where δ_0 is the Dirac distribution.

3. Meromorphic continuation

In this section we show the existence of meromorphic continuations of functions, closely related to (1.1), which we will make use of in the proof of Theorem 1.1. Recall that X is a reduced analytic space, $E \to X$ is a holomorphic vector bundle, and s is a holomorphic section of E with $V = \{s = 0\}$.

Proposition 3.1. *Let $\omega \in \mathcal{A}_s^{n,n}(X)$ with $\operatorname{supp}\omega \subset\subset X$, and let $|\cdot|$ and $\|\cdot\|$ be two Hermitian metrics on E. Then the function*

$$(\lambda, \tau) \mapsto \int_X \|s\|^{2\lambda}\left(\frac{|s|}{\|s\|}\right)^{2\tau}\omega, \tag{3.1}$$

a priori defined and holomorphic for $\mathfrak{Re}\,\lambda \gg 0$, has a meromorphic continuation to $\mathbb{C}^2$, and there is a discrete subset $P \subset \mathbb{Q} \cap (-\infty, N]$, for some $N \geq 0$, such that the polar locus $\subseteq P \times \mathbb{C}_\tau$.

One can show Proposition 3.1 using Berstein–Sato theory in a standard way, see, e.g., [6, 7, 8]. We choose here instead to use Hironaka's theorem to reduce the proof to an elementary calculation. This approach is common in residue calculus, see, e.g., [1].

Note that $|s|^2/\|s\|^2$ is not only locally bounded but everywhere positive. Thus we can find constants $C_1, C_2 > 0$ such that $C_1 < |s|^2/\|s\|^2 < C_2$ on $\overline{\operatorname{supp}\omega}$. This implies that (3.1) is defined and holomorphic for any $\tau \in \mathbb{C}$ provided that $\mathfrak{Re}\,\lambda \gg 0$.

Our proof of Proposition 3.1 relies on the following lemma.

Lemma 3.2. *Let Ψ be a smooth compactly supported function on $\mathbb{C}^n_z$, let v and w be smooth positive functions defined in a neighborhood of $\operatorname{supp}\Psi$, let $1 \leq \kappa \leq n$ and let $m_1, \ldots, m_\kappa$ be positive integers. Then, for any non-negative integer N, the function*

$$\Gamma(\tau, \lambda) = \int_{\mathbb{C}^n} |z_1^{m_1}\cdots z_\kappa^{m_\kappa}|^{2(\lambda - N)} v^\lambda w^\tau \Psi \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where $\mathrm{d}z \wedge \mathrm{d}\bar{z} = \mathrm{d}z_1 \wedge \bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$, is holomorphic for $\mathfrak{Re}\,\lambda \gg 0$, and has a meromorphic continuation to $\mathbb{C}^2$. Moreover, there is a discrete subset $P \subset \mathbb{Q} \cap (-\infty, N]$ such that the polar locus is contained in $P \times \mathbb{C}_\tau$, $\forall \Psi, v, w$.

A computation similar to the following proof can be found in [14]. We provide our adapted version for future reference.

Proof of Lemma 3.2. For $\mathfrak{Re}\,\lambda \gg 0$, we have that

$$\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} |z_1^{m_1}|^{2\lambda} = m_1^2 \lambda^2 \frac{|z_1^{m_1}|^{2\lambda}}{|z_1|^2}.$$

By an induction argument it follows that

$$|z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda - N)} = \frac{h(\lambda)}{\lambda^{2\kappa}} \frac{\partial^{2N \sum_{j=1}^{\kappa} m_j}}{\partial z_1^{Nm_1} \partial \bar{z}_1^{Nm_1} \cdots \partial z_\kappa^{Nm_\kappa} \partial \bar{z}_\kappa^{Nm_\kappa}} |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2\lambda}, \tag{3.2}$$

where

$$h(\lambda) = \prod_{i=1}^{\kappa} \frac{1}{m_i^2} \prod_{j=1}^{Nm_i - 1} \frac{1}{(m_i \lambda - j)^2}. \tag{3.3}$$

By writing

$$\frac{\partial^{2N \sum_{i=1}^{\kappa} m_i}}{\partial z_1^{Nm_1} \partial \bar{z}_1^{Nm_1} \cdots \partial z_\kappa^{Nm_\kappa} \partial \bar{z}_\kappa^{Nm_\kappa}} = P\bar{P} \quad \text{where} \quad P = \frac{\partial^{N \sum_{i=1}^{\kappa} m_i}}{\partial z_1^{Nm_1} \cdots \partial z_\kappa^{Nm_\kappa}},$$

(3.2) then becomes

$$|z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda - N)} = \frac{h(\lambda)}{\lambda^{2\kappa}} P\bar{P} |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2\lambda}. \tag{3.4}$$

Using (3.4) and integration by parts, and the fact that $(P\bar{P})^* = P\bar{P}$, we find that

$$\Gamma(\lambda, \tau) = \frac{h(\lambda)}{\lambda^{2\kappa}} \int\limits_{\mathbb{C}^n} |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2\lambda} P\bar{P}(v^\lambda w^\tau \Psi) \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{3.5}$$

for $\mathfrak{Re}\,\lambda \gg 0$. The integral on the right-hand side of (3.5) is an analytic function of (λ, τ) for $\mathfrak{Re}\,\lambda > -\epsilon$ for a small enough $\epsilon > 0$, and $h(\lambda)$ is a meromorphic function on $\mathbb{C}_\lambda$ with poles at

$$\lambda = \frac{1}{m_i}, \frac{2}{m_i}, \cdots, \frac{Nm_i - 1}{m_i}, \quad i = 1, \ldots, \kappa.$$

It follows that $\Gamma(\lambda, \tau)$ can be meromorphically continued to $\{\mathfrak{Re}\,\lambda > -\epsilon\} \times \mathbb{C}_\tau$.

For any integer $M \geq 0$ and $\mathfrak{Re}\,\lambda \gg 0$, it follows from (3.2) by changing λ to $\lambda + M$ and N to $N + M$ that

$$|z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda - N)} = \frac{h_M(\lambda)}{\lambda^{2\kappa}} P_M \bar{P}_M |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda + M)}, \tag{3.6}$$

where

$$h_M(\lambda) = \prod_{i=1}^{\kappa} \frac{1}{m_i^2} \prod_{j=1}^{(N+M)m_i - 1} \frac{1}{(m_i(\lambda + M) - j)^2}, \tag{3.7}$$

and

$$P_M = \frac{\partial^{(N+M) \sum_{i=1}^{\kappa} m_i}}{\partial z_1^{(N+M)m_1} \cdots \partial z_\kappa^{(N+M)m_\kappa}}.$$

Analogously to (3.5) we then have that

$$\Gamma(\lambda, \tau) = \frac{h_M(\lambda)}{\lambda^{2\kappa}} \int\limits_{\mathbb{C}^n} |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda + M)} P_M \bar{P}_M (v^\lambda w^\tau \Psi) \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{3.8}$$

for $\mathfrak{Re}\,\lambda \gg 0$. The integral on the right-hand side of (3.8) is an analytic function of (λ, τ), now for $\mathfrak{Re}\,\lambda > -M - \epsilon$, and $h_M(\lambda)$ is a meromorphic function on $\mathbb{C}_\lambda$ with poles at

$$\lambda = \frac{1}{m_i} - M, \frac{2}{m_i} - M, \ldots, \frac{(N+M)m_i - 1}{m_i} - M, \quad i = 1, \ldots, \kappa.$$

Since M is arbitrary, it follows that $\Gamma(\lambda, \tau)$ has a meromorphic continuation to $\mathbb{C}^2$. We also see that there is a discrete subset $P \subseteq \mathbb{Q} \cap (-\infty, N]$, such that the polar locus of $\Gamma(\lambda, \tau)$ is contained in $P \times \mathbb{C}_\tau$, independent of v, w and Ψ. $\square$

Proof of Proposition 3.1. We note that we can find constants $C_1, C_2 > 0$ such that $C_1 < |s|^2/\|s\|^2 < C_2$ on $\overline{\operatorname{supp}\omega}$, and that (3.1) is analytic for $\mathfrak{Re}\,\lambda$ sufficiently large. Let $\pi : \widetilde{X} \to X$ be a modification such that $\widetilde{X}$ is smooth and $\pi^* s$ defines a normal crossings divisor on $\widetilde{X}$. Since π is a biholomorphism outside a set of measure 0 we have, for $\mathfrak{Re}\,\lambda \gg 0$,

$$\int\limits_X \|s\|^{2\lambda} \left(\frac{|s|}{\|s\|}\right)^{2\tau} \omega = \int\limits_{\widetilde{X}} \|\pi^* s\|^{2\lambda} \left(\frac{|\pi^* s|}{\|\pi^* s\|}\right)^{2\tau} \pi^* \omega. \tag{3.9}$$

We can find an open cover $\{U_j\}$ such that, in each U_j, there are local holomorphic coordinates $z = (z_1, \ldots, z_n)$ such that either $\pi^*\omega$ is smooth or there is some $1 \leq \kappa \leq n$ such that $\|\pi^* s\|^2 = |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^2 e^{-\phi}$ and $|\pi^* s|^2 = |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^2 e^{-\psi}$ for some $m_1, \ldots, m_\kappa \geq 1$ and $\phi, \psi \in \mathscr{C}^\infty(U_j, \mathbb{R})$. It follows that $\|\pi^* s\|^{2\lambda} \pi^* \omega$ is smooth for $\mathfrak{Re}\,\lambda$ sufficiently large. Thus, we can find an integer $N \geq 0$ such that

$$\pi^* \omega = \frac{\Psi \mathrm{d}z \wedge \mathrm{d}\bar{z}}{|z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2N}}, \tag{3.10}$$

where Ψ is a smooth function. By introducing a partition of unity (ρ_j) subordinate to $\{U_j\}$ we find that the right-hand side of (3.9) is a finite sum of terms of the form

$$\int\limits_{\mathbb{C}^n} |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda - N)} e^{-\lambda\phi} e^{-\tau(\psi - \phi)} \rho_j \Psi \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Note that the constants $\kappa, m_1, \ldots, m_\kappa$ depend on the local chart U_j, although we have suppressed this dependence in the notation. The proof now follows by Lemma 3.2, with $v = e^{-\phi}$ and $w = e^{-(\psi - \phi)}$. By the uniqueness of meromorphic continuation, it is independent of the particular choice of modification. $\square$

4. The currents $\mu_j(\omega)$ associated with $\Gamma_{\|\cdot\|}$

In this section we prove the following theorem, which is a collection of know results together with applications of classical ideas, see, e.g., [2, 3, 6, 7, 8, 12, 13, 14], and also, e.g., [1, 5, 15, 16] and references therein for analogous results in residue theory. We supply details of the proof for completeness, and to gather and organize these results and techniques in our setting.

Theorem 4.1. *Let $\omega \in \mathcal{A}_s^{p,q}(X)$, $\xi \in \mathscr{D}^{n-p,n-q}(X)$ and let $\|\cdot\|$ be a Hermitian metric on E.*

(i) The function

$$\Gamma_{\|\cdot\|}(\lambda) = \int\limits_X \|s\|^{2\lambda} \omega \wedge \xi,$$

a priori defined and holomorphic for $\mathfrak{Re}\,\lambda \gg 0$*, extends to a meromorphic function on* $\mathbb{C}$ *with polar set contained in* $\mathbb{Q}$*. Moreover, there exists a* $\kappa \leq n$ *such that the Laurent series expansion of* $\Gamma_{\|\cdot\|}$ *in a neighborhood of* 0 *is given by*

$$\sum_{j=0}^{\kappa} \frac{1}{\lambda^j} \langle \mu_j(\omega), \xi \rangle + \mathcal{O}(\lambda), \tag{4.1}$$

where $\mu_j(\omega)$ *are currents on* X *satisfying* $\operatorname{supp} \mu_\kappa(\omega) \subseteq \operatorname{supp} \mu_{\kappa-1}(\omega) \subseteq \cdots \subseteq \operatorname{supp} \mu_1(\omega) \subseteq \operatorname{supp} \mu_0(\omega) = \overline{\operatorname{supp} \omega}$*. Moreover,* $\operatorname{supp} \mu_1(\omega) \subseteq V$ *and if* s *defines the radical ideal of* V *then* $\operatorname{supp} \mu_j(\omega) \subseteq V_{\text{sing}} \cup (X_{\text{sing}} \cap V)$ *for* $j \geq 2$*.*

(ii) Suppose that $\omega \in \mathcal{A}_{s,\|\cdot\|}(X)$*. Then we have that* $\mathrm{d}\omega, \frac{\mathrm{d}\|s\|^2}{\|s\|^2} \wedge \omega \in \mathcal{A}_{s,\|\cdot\|}(X)$*, and, for any* j*,*

$$\mathrm{d}\mu_j(\omega) = \mu_j(\mathrm{d}\omega) + \mu_{j+1}\left(\frac{\mathrm{d}\|s\|^2}{\|s\|^2} \wedge \omega\right). \tag{4.2}$$

4.1. **Proof of Theorem 4.1 (i).** To begin with we consider the case where ω is of top degree and s defines a normal crossings divisor. We have the following lemma.

Lemma 4.2. *Suppose that* X *is a manifold and that* s *defines a normal crossings divisor with support* $V = \{s = 0\}$*. Let* $\omega \in \mathcal{A}_s^{n,n}(X)$ *and* $\|\cdot\|$ *be any Hermitian metric on* E*. For any test function* $\xi \in \mathscr{D}^{0,0}(X)$ *we let*

$$\Gamma_{\|\cdot\|}(\lambda) = \int_X \|s\|^{2\lambda} \omega \xi,$$

for $\mathfrak{Re}\,\lambda \gg 0$*. Then* $\Gamma_{\|\cdot\|}$ *has a meromorphic continuation to* $\mathbb{C}_\lambda$ *with polar set given by a discrete subset* $P \subset \mathbb{Q} \cap (-\infty, N]$ *for some* $N \geq 0$ *independent of* $\|\cdot\|$ *and* ξ*. Moreover, there exists a* $0 \leq \kappa \leq n$ *such that the Laurent series expansion of* $\Gamma_{\|\cdot\|}$ *around* 0 *is given by*

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=0}^{\kappa} \frac{1}{\lambda^j} \langle \mu_j(\omega), \xi \rangle + \mathcal{O}(\lambda),$$

where $\mu_j(\omega)$*, for* $j = 0, \ldots, \kappa$*, are* (n,n)*-currents on* X *satisfying* $\operatorname{supp} \mu_\kappa(\omega) \subseteq \operatorname{supp} \mu_{\kappa-1}(\omega) \subseteq \cdots \subseteq \operatorname{supp} \mu_1(\omega) \subseteq \operatorname{supp} \mu_0(\omega) = \overline{\operatorname{supp} \omega}$*,* $\operatorname{supp} \mu_1(\omega) \subseteq V$ *and* $\operatorname{supp} \mu_j(\omega) \subseteq V_{\text{sing}}$ *for* $j \geq 2$*.*

Proof. The statement that $\Gamma_{\|\cdot\|}$ has a meromorphic continuation with the prescribed polar set follows immediately from Proposition 3.1 by setting $\tau = 0$. Now, consider the Laurent series expansion of $\Gamma_{\|\cdot\|}(\lambda)$ around $\lambda = 0$,

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=0}^{N_0} \frac{1}{\lambda^j} c_j + \mathcal{O}(\lambda),$$

where $c_j \in \mathbb{C}$, for some $N_0 \geq 0$. Since being a current is a local property, we may assume that ξ has support in some neighborhood where we can find local holomorphic coordinates $z = (z_1, \ldots, z_n)$ such that $\|s\|^2 = |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^2 e^{-\phi}$ for some $1 \leq \kappa \leq n$ and $\phi \in \mathscr{C}^\infty(\operatorname{supp} \xi, \mathbb{R})$. Since $\|s\|^{2\lambda}\omega$ is smooth for $\mathfrak{Re}\,\lambda$ sufficiently large, we can find an integer $N \geq 0$ and a smooth function Ψ such that ω is given by the right-hand side of (3.10). Thus, in a neighborhood of $\lambda = 0$ we know from the proof of Lemma 3.2, cf. (3.5), that we may write

$$\Gamma_{\|\cdot\|}(\lambda) = \frac{h(\lambda)}{\lambda^{2\kappa}} I(\lambda),$$

where $h(\lambda)$ is given by (3.3), and where

$$I(\lambda) = \int_X |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2\lambda} P\bar{P}\big(e^{-\lambda\phi}\Psi\xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{4.3}$$

with P as in the proof of Lemma 3.2. In particular, both h, and I are holomorphic in a neighborhood of 0. We have that

$$\begin{aligned} c_j &= \operatorname*{Res}_{\lambda=0} \big\{\lambda^{j-1}\Gamma_{\|\cdot\|}(\lambda)\big\} = \operatorname*{Res}_{\lambda=0} \left\{\frac{1}{\lambda^{2\kappa-j+1}} h(\lambda) I(\lambda)\right\} \\ &= \frac{1}{(2\kappa-j)!} \frac{\mathrm{d}^{2\kappa-j}}{\mathrm{d}\lambda^{2\kappa-j}} \big(h(\lambda)I(\lambda)\big)\Big|_{\lambda=0} \\ &= \frac{1}{(2\kappa-j)!} \sum_{\ell=0}^{2\kappa-j} \binom{2\kappa-j}{\ell} h^{(\ell)}(0) I^{(2\kappa-j-\ell)}(0). \end{aligned} \tag{4.4}$$

Let $k = 2\kappa - j - \ell$, and consider $I^{(k)}(0)$. A standard computation with Leibniz rule gives that

$$\begin{aligned} I^{(k)}(0) &= \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} I(\lambda)\big|_{\lambda=0} \\ &= \sum_{r=0}^{k} \binom{k}{r} (-1)^r \int_{\mathbb{C}^n} \big(\log |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^2\big)^{k-r} P\bar{P}\big(\phi^r \Psi\xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \sum_{r=0}^{k} \binom{k}{r} (-1)^r \int_{\mathbb{C}^n} \big(\log |z_1^{m_1}|^2 + \cdots + \log |z_\kappa^{m_\kappa}|^2\big)^{k-r} P\bar{P}\big(\phi^r \Psi\xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}. \end{aligned}$$

By the multinomial theorem we have that

$$\big(\log |z_1^{m_1}|^2 + \cdots + \log |z_\kappa^{m_\kappa}|^2\big)^{k-r} = \sum_{\substack{\alpha \in \mathbb{Z}^\kappa_{\geq 0} \\ |\alpha| = k-r}} \frac{(k-r)!}{\alpha_1! \cdots \alpha_\kappa!} \prod_{t=1}^{\kappa} \big(\log |z_t^{m_t}|^2\big)^{\alpha_t}.$$

We see that if $k - r < \kappa$, each multi-index α will contain at least one 0 entry. Suppose, for simplicity, that $\alpha_1 = 0$ for a given term. Then we clearly have that

$$\frac{\partial}{\partial z_1} \prod_{t=1}^{\kappa} \big(\log |z_t^{m_t}|^2\big)^{\alpha_t} = \frac{\partial}{\partial z_1} \prod_{t=2}^{\kappa} \big(\log |z_t^{m_t}|^2\big)^{\alpha_t} = 0.$$

If $k - r < \kappa$, it follows that

$$\int_{\mathbb{C}^n} \big(\log |z_1^{m_1}|^2 + \cdots + \log |z_\kappa^{m_\kappa}|^2\big)^{k-r} P\bar{P}\big(\phi^r \Psi\xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z} = 0$$

by integration by parts; thus, $I^{(k)}(0) = 0$ if $k < \kappa$. From (4.4) it follows that $c_j = 0$ if $2\kappa - j < \kappa$, that is, for $j > \kappa$. Thus, we have that

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=1}^{\kappa} \frac{1}{\lambda^j} c_j + \mathcal{O}(\lambda).$$

We see from (4.4) and the expansion of $I^{(k)}(0)$ that c_j, for each $j = 0, \ldots, \kappa$, consists of a finite sum of integrals of the form

$$\int_{\mathbb{C}^n} \big(\log |z_1^{m_1}|^2 + \cdots + \log |z_\kappa^{m_\kappa}|^2 \big)^{k-r} P\bar{P}\big(\phi^r \Psi \xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Since $(\log |z_1^{m_1}|^2 + \cdots + \log |z_\kappa^{m_\kappa}|^2)^{k-r}$ is locally integrable in $\mathbb{C}^n$, it follows by the product rule that c_j consists of a finite sum of integrals, where the integrands consist of derivatives on the test function ξ multiplied by L^1_{loc}-functions. This immediately implies that c_j defines the action of a (n,n)-current on ξ, which we denote by $c_j = \langle \mu_j(\omega), \xi \rangle$.

In [14] it is shown that $\operatorname{supp} \mu_j(\omega) \subseteq \operatorname{supp} \mu_{j-1}(\omega)$ for each $j = 1, \ldots, \kappa$. For convenience we sketch an argument. Let

$$I_{k,r} = \sum_{\substack{\alpha \in \mathbb{Z}^\kappa_{\geq 0} \\ |\alpha| = k-r}} \frac{(k-r)!}{\alpha_1! \cdots \alpha_\kappa!} \int_{\mathbb{C}^n} \prod_{t=1}^{\kappa} \big(\log |z_t^{m_t}|^2 \big)^{\alpha_t} P\bar{P}\big(\phi^r \Psi \xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{4.5}$$

Then

$$I^{(k)}(0) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r I_{k,r}. \tag{4.6}$$

By the above, we know that $I_{k,r} = 0$ if $k - r < \kappa$. If $k - r = \kappa$, then partial integration shows that each term in the right-hand side of (4.5) except for the $\alpha = (1, \ldots, 1)$ term vanishes. It follows that

$$I_{k,r} = \kappa! \int_{\mathbb{C}^n} \prod_{t=1}^{\kappa} m_t \frac{\partial^2}{\partial z_t \partial \bar{z}_t} \big(\log |z_t|^2 \big) \frac{\partial^{2N \sum_{i=1}^{\kappa} m_i - 2\kappa}}{\partial z_1^{N m_1 - 1} \cdots \partial \bar{z}_\kappa^{N m_\kappa - 1}} \big(\phi^r \Psi \xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where $\frac{\partial^2}{\partial z_t \partial \bar{z}_t} \log |z_t|^2$ is to be regarded as a distribution. Thus, if $k - r = \kappa$, by repeated use of the Poincaré–Lelong formula,

$$\partial \bar{\partial} \log |z_j|^2 = -2\pi i [z_j = 0],$$

for $j = 1, \ldots, \kappa$, we have that

$$I_{k,r} = \kappa! (-2\pi i)^\kappa \prod_{t=1}^{\kappa} m_t \int_{\{z_1 = \cdots = z_\kappa = 0\}} \frac{\partial^{2N \sum_{i=1}^{\kappa} m_i - 2\kappa}}{\partial z_1^{N m_1 - 1} \cdots \partial \bar{z}_\kappa^{N m_\kappa - 1}} \big(\phi^r \Psi \xi\big) \mathrm{d}z' \wedge \mathrm{d}\bar{z}',$$

where $\mathrm{d}z' \wedge \mathrm{d}\bar{z}' = \mathrm{d}z_{\kappa+1} \wedge \mathrm{d}\bar{z}_{\kappa+1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$. By (4.4) and (4.6) it follows that

$$c_\kappa = h(0) I_{\kappa,0},$$

whence $\operatorname{supp} \mu_\kappa(\omega) \subseteq \{z_1 = \cdots = z_\kappa = 0\} \cap \operatorname{supp} \Psi$. Similarly, by (4.4) and (4.6) it follows that

$$c_{\kappa-1} = \frac{1}{(\kappa+1)!} h(0) I_{\kappa+1,0} - \frac{1}{\kappa!} h(0) I_{\kappa+1,1} + \frac{1}{\kappa!} h'(0) I_{\kappa,0}.$$

From (4.5), setting $k = \kappa+1$ and $r = 1$, we find that only the term with $\alpha = (1, \dots, 1)$ gives a non-zero contribution to $I_{\kappa+1,1}$. Thus, by the same argument as above, $I_{\kappa+1,1}$ is an integral over the locus $\{z_1 = \cdots = z_\kappa = 0\}$. Looking at the expression for $I_{\kappa+1,0}$, we find that the only terms that contribute are $\alpha = (2, 1, \dots, 1), (1, 2, 1, \dots, 1), \dots, (1, \dots, 1, 2)$. Consider for example the term with $\alpha = (2, 1, \dots, 1)$,

$$\begin{aligned}
(I_{\kappa+1,0})_\alpha &= \frac{(\kappa+1)!}{2!} \int\limits_{\mathbb{C}^n} \big(\log|z_1^{m_1}|^2\big)^2 \prod_{t=2}^{\kappa} \log|z_t^{m_t}|^2 \frac{\partial^{2N\sum_{i=1}^{\kappa} m_i}}{\partial z_1^{Nm_1} \cdots \partial \bar{z}_\kappa^{Nm_\kappa}} \big(\Psi\xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{(\kappa+1)!}{2!} \int\limits_{\mathbb{C}^n} \big(\log|z_1^{m_1}|^2\big)^2 \prod_{t=2}^{\kappa} \frac{\partial^2}{\partial z_t \partial \bar{z}_t} \big(\log|z_t^{m_t}|^2\big) \times \\
&\qquad \times \frac{\partial^{2N\sum_{i=1}^{\kappa} m_i - 2(\kappa-1)}}{\partial z_1^{Nm_1} \partial \bar{z}_1^{Nm_1} \partial z_2^{Nm_2-1} \cdots \partial \bar{z}_\kappa^{Nm_\kappa-1}} \big(\Psi\xi\big) \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{(\kappa+1)!}{2!} (-2\pi i)^{\kappa-1} \prod_{t=1}^{\kappa} m_t \int\limits_{\{z_2=\cdots=z_\kappa=0\}} \big(\log|z_1|^2\big)^2 \times \\
&\qquad \times \frac{\partial^{2N\sum_{i=1}^{\kappa} m_i - 2(\kappa-1)}}{\partial z_1^{Nm_1} \partial \bar{z}_1^{Nm_1} \partial z_2^{Nm_2-1} \cdots \partial \bar{z}_\kappa^{Nm_\kappa-1}} \big(\Psi\xi\big) \mathrm{d}z' \wedge \mathrm{d}\bar{z}',
\end{aligned}$$

where $\mathrm{d}z' \wedge \mathrm{d}\bar{z}' = \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_{\kappa+1} \wedge \mathrm{d}\bar{z}_{\kappa+1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$. Thus, $(I_{\kappa+1,0})_\alpha$ is an integral over the locus $\{z_2 = \cdots = z_\kappa = 0\}$. By symmetry, it follows that $I_{\kappa+1,0}$ is an integral over the locus

$$\bigcup_{i=1}^{\kappa} \bigcap_{j\neq i} \{z_j = 0\},$$

whence

$$\operatorname{supp} \mu_{\kappa-1}(\omega) \subseteq \bigcup_{i=1}^{\kappa} \bigcap_{j\neq i} \{z_j = 0\} \cap \operatorname{supp} \Psi.$$

Furthermore, since the integral $I_{\kappa,0}$ appears in both the expression for c_κ and $c_{\kappa-1}$, we have that

$$\operatorname{supp} \mu_\kappa(\omega) \subseteq \operatorname{supp} \mu_{\kappa-1}(\omega)$$

By analogous arguments for $k - r = \kappa + 2, \dots, 2\kappa$ we find that

$$\operatorname{supp} \mu_j(\omega) \subseteq \bigcup_{\substack{\ell_1,\dots,\ell_j=1 \\ \ell_1<\cdots<\ell_j}}^{\kappa} \{z_{\ell_1} = \cdots = z_{\ell_j} = 0\},$$

and that $\operatorname{supp} \mu_j(\omega) \subseteq \operatorname{supp} \mu_{j-1}(\omega)$ for each $j = 1, \dots, \kappa$.

It is clear that $\Gamma_{\|\cdot\|}$ is holomorphic if $\operatorname{supp} \xi \subseteq X \setminus V$. Thus, $\operatorname{supp} \mu_j(\omega) \subseteq V$ for $j = 1, \dots, \kappa$. It follows that $\mu_0(\omega)$ is a current extension of ω across V, and we have that $\operatorname{supp} \mu_0(\omega) = \overline{\operatorname{supp} \omega}$. It is shown in [14] that if V is smooth and s defines the radical ideal of V then $\kappa \leq 1$. Thus, if $\operatorname{supp} \xi \subseteq X \setminus V_{\mathrm{sing}}$, $\Gamma_{\|\cdot\|}$ has at most a pole of order 1 at $\lambda = 0$. This implies that $\operatorname{supp} \mu_j(\omega) \subseteq V_{\mathrm{sing}}$ for $j \geq 2$. $\square$

Now we generalize Lemma 4.2 to $\omega \in \mathcal{A}_s^{p,q}(X)$. In this setting, we note that $\omega \wedge \xi \in \mathcal{A}_s^{n,n}(X)$ for any $\xi \in \mathscr{D}^{n-p,n-q}(X)$. Lemma 4.2 then implies that there is some $0 \leq \kappa \leq n$ and currents $\mu_j(\omega \wedge \xi)$ with compact support, for $0 \leq j \leq \kappa$, which depend on ω (a priori on $\omega \wedge \xi$) such that

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=0}^{\kappa} \frac{1}{\lambda^j} \langle \mu_j(\omega \wedge \xi), 1\rangle + \mathcal{O}(\lambda).$$

For $\omega \in \mathcal{A}_s^{p,q}(X)$ we then define

$$\langle \mu_j(\omega), \xi\rangle := \langle \mu_j(\omega \wedge \xi), 1\rangle. \tag{4.7}$$

It is clear from the definition of $\mu_j(\omega \wedge \xi)$ that (4.7) defines a linear functional on $\mathscr{D}^{n-p,n-q}(X)$. Furthermore, if $\omega \in \mathcal{A}_s^{n,n}(X)$, it follows by Lemma 4.2 that, if ξ is a test function, $\mu_j(\omega\xi) = \xi\mu_j(\omega)$, which agrees with (4.7).

To see that (4.7) defines a (p, q)-current $\mu_j(\omega)$, it remains to check continuity. Since being a current is a local statement, we may assume that ξ has support in a small neighborhood with local coordinates $z = (z_1, \dots, z_n)$ and that

$$\xi = \sum_{J,K} \xi_{JK} \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K,$$

where the sum is over all multi-indices J,K consisting of ordered subsets of $\{1, \dots, n\}$ of size $n - p$ and $n - q$, respectively. Since $\mu_j(\omega)$ is a linear functional, we can fix some indices (J, K) and consider $\langle \mu_j(\omega), \xi_{JK}\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K\rangle$. By (4.7), we have

$$\begin{aligned}\langle \mu_j(\omega), \xi_{JK}\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K\rangle &= \langle \mu_j(\omega \wedge \xi_{JK}\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K), 1\rangle \\ &= \langle \xi_{JK}\mu_j(\omega \wedge \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K), 1\rangle \\ &= \langle \mu_j(\omega \wedge \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K), \xi_{JK}\rangle,\end{aligned}$$

where we used that $\mu_j(\omega\xi) = \xi\mu_j(\omega)$ for $\omega \in \mathcal{A}_s^{n,n}(X)$ and $\xi \in \mathscr{D}^{0,0}(X)$. Since we know that $\mu_j(\omega \wedge \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K)$ is a continuous linear functional on $\mathscr{D}^{0,0}(X)$, it follows that $\mu_j(\omega)$ is a continuous linear functional on $\mathscr{D}^{n-p,n-q}(X)$.

Thus, Lemma 4.2 holds for $\omega \in \mathcal{A}_s^{p,q}(X)$ with $\mu_j(\omega)$ defined as in (4.7). We have the following formula.

Lemma 4.3. *Let $\omega \in \mathcal{A}_s^{p,q}(X)$. For each $j = 0, \dots, \kappa$, $\mu_j(\omega)$ satisfies*

$$\mu_j(\omega) \wedge \xi = \mu_j(\omega \wedge \xi), \tag{4.8}$$

for any smooth (p', q')-form ξ.

Proof. Let $\eta \in \mathscr{D}^{n-p-p',n-q-q'}(X)$. Since $\mu_j(\omega)$ is a (p, q)-current, by (2.2) we have that

$$\langle \mu_j(\omega) \wedge \xi, \eta\rangle = \langle \mu_j(\omega), \xi \wedge \eta\rangle.$$

By (4.7) we have

$$\langle \mu_j(\omega), \xi \wedge \eta\rangle = \langle \mu_j(\omega \wedge \xi \wedge \eta), 1\rangle.$$

Since $\omega \wedge \xi \in \mathcal{A}_V^{p+p',q+q'}(X)$, again by (4.7), we have that

$$\langle \mu_j(\omega \wedge \xi \wedge \eta), 1\rangle = \langle \mu_j(\omega \wedge \xi), \eta\rangle.$$

Thus, $\langle \mu_j(\omega) \wedge \xi, \eta\rangle = \langle \mu_j(\omega \wedge \xi), \eta\rangle$ for all η which proves the lemma. □

Proof of Theorem 4.1 (i). Let $\pi : \widetilde{X} \to X$ be a modification such that $\widetilde{X}$ is smooth and $\pi^* s : \widetilde{X} \to \pi^* E$ defines a normal crossings divisor. As in the proof of Proposition 3.1, with $\tau = 0$, we have, for $\mathfrak{Re}\, \lambda \gg 0$,

$$\Gamma_{\|\cdot\|}(\lambda) = \int_X \|s\|^{2\lambda} \omega \wedge \xi = \int_{\widetilde{X}} \|\pi^* s\|^{2\lambda} \pi^* \omega \wedge \pi^* \xi.$$

Since $\pi^*\omega \wedge \pi^*\xi \in \mathcal{A}^{n,n}_{\pi^* s}(\widetilde{X})$, by Lemma 4.2 $\Gamma_{\|\cdot\|}(\lambda)$ has a meromorphic continuation to $\mathbb{C}_\lambda$, with polar set given by a discrete subset $P \subset \mathbb{Q} \cap (-\infty, N]$ for some $N \geq 0$. Moreover, there is some $0 \leq \kappa \leq n$ such that, in a neighborhood of $\lambda = 0$,

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=1}^{\kappa} \frac{1}{\lambda^j} \langle \mu_j(\pi^*\omega \wedge \pi^*\xi), 1 \rangle + \mathcal{O}(\lambda),$$

where $\mu_j(\pi^*\omega \wedge \pi^*\xi)$ define (n,n)-currents on $\widetilde{X}$. By (4.7) we may write

$$\Gamma_{\|\cdot\|}(\lambda) = \sum_{j=1}^{\kappa} \frac{1}{\lambda^j} \langle \mu_j(\pi^*\omega), \pi^*\xi \rangle + \mathcal{O}(\lambda),$$

where $\mu_j(\pi^*\omega)$ are (p,q)-currents on $\widetilde{X}$. Since π is proper, by (2.3) we have that

$$\langle \mu_j(\pi^*\omega), \pi^*\xi \rangle = \langle \mu_j(\omega), \xi \rangle,$$

where

$$\mu_j(\omega) := \pi_* \mu_j(\pi^*\omega) \tag{4.9}$$

is a current on X, for each $j = 0, \ldots, \kappa$.

By Lemma 4.2 we have that $\operatorname{supp} \mu_0(\pi^*\omega) = \overline{\operatorname{supp} \pi^*\omega}$, $\operatorname{supp} \mu_1(\pi^*\omega) \subseteq \pi^{-1}V$, and $\operatorname{supp} \mu_\kappa(\pi^*\omega) \subseteq \cdots \subseteq \operatorname{supp} \mu_2(\pi^*\omega) \subseteq (\pi^{-1}V)_{\text{sing}}$. Furthermore, it follows immediately by taking direct images that $\operatorname{supp} \mu_0(\omega) = \operatorname{supp} \pi_* \mu_0(\pi^*\omega) = \overline{\operatorname{supp} \omega}$, and $\operatorname{supp} \mu_\kappa(\omega) \subseteq \cdots \subseteq \operatorname{supp} \mu_0(\omega)$.

It is shown in [14] that if X is smooth, and V is a submanifold, then $\Gamma_{\|\cdot\|}(\lambda)$ has a pole of order at most 1 at the origin. Thus, it follows that $\operatorname{supp} \mu_1(\omega) \subset V$ and $\operatorname{supp} \mu_j(\omega) \subseteq V_{\text{sing}} \cup (X_{\text{sing}} \cap V)$ for each $j \geq 2$.

□

A priori Lemma 4.3 holds in the case when X is smooth and s defines a normal crossings divisor. The corresponding statement in the general setting follows by Lemma 4.3 and (2.4).

4.2. **Proof of Theorem 4.1 (ii).**

Lemma 4.4. *For $\omega \in \mathcal{A}_{s,\|\cdot\|}(X)$ we have that $\mathrm{d}\omega$, $\frac{\mathrm{d}\|s\|^2}{\|s\|^2} \wedge \omega \in \mathcal{A}_{s,\|\cdot\|}(X)$.*

Proof. Since $\omega \in \mathcal{A}_{s,\|\cdot\|}(X)$, for each compact $K \subset X$ we can find an integer $N \geq 0$ such that $\|s\|^{2N}\omega$ extends smoothly across $V \cap K$. Thus, we may write

$$\omega = \frac{\widetilde{\omega}}{\|s\|^{2N}},$$

where $\widetilde{\omega}$ is smooth across $K \cap V$. On $X \setminus V$ we have that

$$\mathrm{d}\omega = \mathrm{d}\frac{\widetilde{\omega}}{\|s\|^{2N}} = \frac{\mathrm{d}\widetilde{\omega}}{\|s\|^{2N}} - N\frac{\mathrm{d}\|s\|^2}{\|s\|^2} \wedge \frac{\widetilde{\omega}}{\|s\|^{2N}}.$$

Since $\mathrm{d}\widetilde{\omega}$ and $\mathrm{d}\|s\|^2$ are smooth across $V\cap K$, it is clear that

$$\|s\|^{2(N+1)}\mathrm{d}\omega = \|s\|^2\mathrm{d}\widetilde{\omega} - N\mathrm{d}\|s\|^2\wedge\widetilde{\omega}$$

extends smoothly across $V\cap K$. It is also clear that $\|s\|^{2(N+1)}\frac{\mathrm{d}\|s\|^2}{\|s\|^2}\wedge\omega$ extends smoothly across $V\cap K$. □

Remark 4.5. For $\omega\in\mathcal{A}_{s,\|\cdot\|}(X)$ and $|\cdot|$ some different metric on E, it is not true in general that $\frac{\mathrm{d}|s|^2}{|s|^2}\wedge\omega\in\mathcal{A}_s(X)$. However, we can always find an integer $N\geq 0$ such that $|s|^{2N}\frac{\mathrm{d}|s|^2}{|s|^2}\wedge\omega$ extends to a locally bounded form on X, and for a modification $\pi:\widetilde{X}\to X$ such that π^*s defines a divisor, the pullback of $|s|^{2N}\frac{\mathrm{d}|s|^2}{|s|^2}\wedge\omega$ is smooth for large N, that is, $\pi^*\Big(\frac{\mathrm{d}|s|^2}{|s|^2}\wedge\omega\Big)\in\mathcal{A}_{\pi^*s}(\widetilde{X})$.

Proof of Theorem 4.1 (ii). Let $\xi\in\mathscr{D}^{n-p,n-q}(X)$. Then $\exists N\geq 0$ such that $\|s\|^{2N}\omega$ extends smoothly across $V\cap\operatorname{supp}\xi$. Using integration by parts and Stokes' theorem, we have, for $\mathfrak{Re}\,\lambda\gg 0$,

$$\begin{aligned}\int_X\|s\|^{2\lambda}\omega\wedge\mathrm{d}\xi &= (-1)^{p+q+1}\int_X\mathrm{d}(\|s\|^{2\lambda}\omega)\wedge\xi\\ &= (-1)^{p+q+1}\lambda\int_X\|s\|^{2\lambda}\frac{\mathrm{d}\|s\|^2}{\|s\|^2}\wedge\omega\wedge\xi+(-1)^{p+q+1}\int_X\|s\|^{2\lambda}\mathrm{d}\omega\wedge\xi.\end{aligned}$$

By Lemma 4.4, $\mathrm{d}\omega$, $\frac{\mathrm{d}\|s\|^2}{\|s\|^2}\wedge\omega\in\mathcal{A}_s(X)$. Thus, by Theorem 4.1 (i), and by uniqueness of meromorphic continuation, we obtain the following equality of Laurent series expansions about 0,

$$\sum_{j=0}^{\kappa}\frac{1}{\lambda^j}\langle\mathrm{d}\mu_j(\omega),\xi\rangle=\sum_{j=1}^{\kappa'}\frac{1}{\lambda^{j-1}}\langle\mu_j\left(\frac{\mathrm{d}\|s\|^2}{\|s\|^2}\wedge\omega\right),\xi\rangle+\sum_{j=0}^{\kappa''}\frac{1}{\lambda^j}\langle\mu_j(\mathrm{d}\omega),\xi\rangle+\mathcal{O}(\lambda),$$

where we have used (2.1) on the left-hand side. Collecting the terms by order in λ, we obtain the equality (4.2) for each j. □

5. Proof of Theorem 1.1

In this section we give the proof of our main result, Theorem 1.1.

Proof of Theorem 1.1. Recall that $\omega\in\mathcal{A}_s(X)$ and that $\|\cdot\|$ and $|\cdot|$ are two smooth Hermitian metrics on E. Let ξ be a test form of complementary bidegree to ω and consider

$$\Gamma(\lambda,\tau)=\int_X\|s\|^{2\lambda}\left(\frac{|s|}{\|s\|}\right)^{2\tau}\omega\wedge\xi.$$

By Proposition 3.1, $\Gamma(\lambda,\tau)$ is holomorphic if $\mathfrak{Re}\,\lambda\gg 0$ and extends to a meromorphic function on $\mathbb{C}^2$. Furthermore, there is a discrete subset $P\subset\mathbb{Q}\cap(-\infty,N]$, for some $N\geq 0$ such that the polar locus of $\Gamma(\lambda,\tau)$ lies in $P\times\mathbb{C}_\tau$.

Suppose first that X is smooth and that s defines a normal crossings divisor. Then $|s|^2/\|s\|^2$ is a smooth positive function on X. By Theorem 4.1 (i), for each fixed $\tau \in \mathbb{C}$ and $\mathfrak{Re}\,\lambda \gg 0$, there is some $\kappa' \leq n$ such that

$$(5.1)\qquad \int_X \|s\|^{2\lambda}\left(\frac{|s|^2}{\|s\|^2}\right)^{\tau}\omega\wedge\xi = \sum_{j=0}^{\kappa'}\frac{1}{\lambda^j}\langle\mu_j^{\|\cdot\|}\left(\left(\frac{|s|^2}{\|s\|^2}\right)^{\tau}\omega\right),\xi\rangle + F(\lambda,\tau),$$

where $\lambda \mapsto F(\lambda,\tau)$ is meromorphic in $\mathbb{C}_\lambda$, holomorphic for λ near 0 and $F(0,\tau) = 0$. By Lemma 4.3 we have that

$$(5.2)\quad \sum_{j=0}^{\kappa'}\frac{1}{\lambda^j}\langle\mu_j^{\|\cdot\|}\left(\left(\frac{|s|^2}{\|s\|^2}\right)^{\tau}\omega\right),\xi\rangle + F(\lambda,\tau) = \sum_{j=0}^{\kappa'}\frac{1}{\lambda^j}\langle\left(\frac{|s|^2}{\|s\|^2}\right)^{\tau}\mu_j^{\|\cdot\|}(\omega),\xi\rangle + F(\lambda,\tau).$$

The left hand side of (5.1) is meromorphic by Proposition 3.1 with polar set $P \times \mathbb{C}_\tau$. Each term in the sum in the right hand side of (5.2) is meromorphic in $\mathbb{C}^2$ with polar set $\{0\} \times \mathbb{C}_\tau$. It follows that $F(\lambda,\tau)$ is meromorphic in $\mathbb{C}^2$ with polar set $(P \setminus \{0\}) \times \mathbb{C}_\tau$. On the line $\tau = \lambda$ in $\mathbb{C}^2$ we obtain an equality of meromorphic functions

$$\int_X |s|^{2\lambda}\omega\wedge\xi = \sum_{j=0}^{\kappa'}\frac{1}{\lambda^j}\langle\left(\frac{|s|^2}{\|s\|^2}\right)^{\lambda}\mu_j^{\|\cdot\|}(\omega),\xi\rangle + F(\lambda,\lambda),$$

where $F(0,0) = 0$, and $F(\lambda,\lambda)$ is holomorphic for λ near 0. Thus, the sum on the right hand side contains the principal part of the Laurent series expansion of the left hand side around $\lambda = 0$. But, by Theorem 4.1 (i), the Laurent series expansion of the left hand side is given by

$$\sum_{j=0}^{\kappa}\frac{1}{\lambda^j}\langle\mu_j^{|\cdot|}(\omega),\xi\rangle + \mathcal{O}(\lambda),$$

for some $\kappa \leq n$. Thus, since

$$\left(\frac{|s|^2}{\|s\|^2}\right)^{\lambda} = \sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\left(\log\frac{|s|^2}{\|s\|^2}\right)^{\ell},$$

by uniqueness of Laurent series expansions we have that

$$(5.3)\qquad \mu_j^{|\cdot|}(\omega) = \sum_{\ell=0}^{\kappa'-j}\frac{1}{\ell!}\left(\log\frac{|s|^2}{\|s\|^2}\right)^{\ell}\mu_{j+\ell}^{\|\cdot\|}(\omega).$$

It immediately follows that $\kappa' = \kappa$, that is, κ is independent of the metric when s defines a divisor, and, as a consequence $\mu_\kappa(\omega) := \mu_\kappa^{|\cdot|}(\omega)$ is independent of the choice of metric.

Now, for the general case: Let $\pi : \widetilde{X} \to X$ be a modification such that $\widetilde{X}$ is smooth and π^*s defines a normal crossings divisor. Then (5.3) holds with ω and s replaced by $\pi^*\omega$ and π^*s, respectively. In view of (4.9), we have that $\mu_j^{|\cdot|}(\omega) = \pi_*\mu_j^{|\cdot|}(\pi^*\omega)$ for $j = 0,\dots,\kappa$ and $\mu_j^{\|\cdot\|}(\omega) = \pi_*\mu_j^{\|\cdot\|}(\pi^*\omega)$ for $j = 0,\dots,\kappa'$. It follows that, for each $j = 0,\dots,\kappa$,

$$(5.4)\quad \mu_j^{|\cdot|}(\omega) = \pi_*\sum_{\ell=0}^{\kappa'-j}\frac{1}{\ell!}\left(\log\frac{|\pi^*s|^2}{\|\pi^*s\|^2}\right)^{\ell}\mu_{j+\ell}^{\|\cdot\|}(\pi^*\omega) = \sum_{\ell=0}^{\kappa'-j}\frac{1}{\ell!}\left(\log\frac{|s|^2}{\|s\|^2}\right)^{\ell}\mu_{j+\ell}^{\|\cdot\|}(\omega),$$

where

$$\left(\log \frac{|s|^2}{\|s\|^2}\right)^{\ell} \mu_{j+\ell}^{\|\cdot\|}(\omega) := \pi_*\left(\left(\log \frac{|\pi^* s|^2}{\|\pi^* s\|^2}\right)^{\ell} \mu_{j+\ell}^{\|\cdot\|}(\pi^*\omega)\right)$$

according to (2.5).

□

Note that if $\kappa > \kappa'$, even though $\mu_{j+\ell}^{\|\cdot\|}(\omega) = 0$ for $\ell > \kappa' - j$, this does not immediately imply that

$$\left(\log \frac{|s|^2}{\|s\|^2}\right)^{\ell} \mu_{j+\ell}^{\|\cdot\|}(\omega) = 0,$$

for $\kappa' - j < \ell \leq \kappa - j$, cf. Example 2.1. Thus, it is not clear in general whether $\kappa(\leq n)$ is independent of the choice of metric, unless $V = \{s = 0\}$ is a hypersurface, in which case $\log \frac{|s|^2}{\|s\|^2}$ is smooth.

The dependence of the currents $\mu_j(\omega)$ on the choice of section defining V is, in fact, essentially described by Theorem 1.1, in a sense which we try to illustrate with the following example.

Example 5.1. Suppose that V is a hypersurface and that there are (holomorphic) line bundles E and F and (holomorphic) sections $s : X \to E$ and $\sigma : X \to F$ such that $V = \{s = 0\} = \{\sigma = 0\}$ and such that σ and $s^{\otimes k}$ define the same divisor, for some $k \in \mathbb{N}$. Moreover, let $|\cdot|_E$ and $|\cdot|_F$ be Hermitian metrics on E and F, respectively, and suppose that $\omega \in \mathcal{A}_s(X) = \mathcal{A}_\sigma(X)$.

The metric $|\cdot|_E$ naturally induces a metric $|\cdot|_{E^{\otimes k}}$ on $E^{\otimes k}$ satisfying $|s^{\otimes k}|^2_{E^{\otimes k}} = |s|^{2k}_E$. Thus, since σ and $s^{\otimes k}$ define the same divisor, we have that

$$\frac{|\sigma|_F^2}{|s^{\otimes k}|_{E^{\otimes k}}} = \frac{|\sigma|_F^2}{|s|_E^{2k}}$$

is a smooth positive function on X. Thus, we can define a new metric $\|\cdot\|_E$ on E by

$$\|v\|_E^2 := |v|_E^2 \frac{|\sigma|_F^{2/k}}{|s|_E^2},$$

for $v \in H^0(X, E)$. For $\mathfrak{Re}\,\lambda \gg 0$, we then have that

$$\begin{aligned}\int_X |\sigma|_F^{2\lambda}\, \omega \wedge \xi &= \int_X |s^{\otimes k}|^{2\lambda}_{E^{\otimes k}} \left(\frac{|\sigma|_F^2}{|s^{\otimes k}|^2_{E^{\otimes k}}}\right)^{\lambda} \omega \wedge \xi \\ &= \int_X |s|_E^{2k\lambda} \left(\frac{|\sigma|_F^2}{|s|_E^{2k}}\right)^{\lambda} \omega \wedge \xi = \int_X \|s\|_E^{2k\lambda} \omega \wedge \xi.\end{aligned}$$

Thus, we see that the change of sections, from s to σ, can be realized as a change in metrics on E, keeping the section s fixed, after a possible rescaling of λ.

6. Asymptotic expansion of $\mathcal{I}_{\|\cdot\|}(\epsilon)$

Recall that $\mathcal{I}_{\|\cdot\|}(\epsilon)$ is given by (1.4), where $\omega \in \mathcal{A}_s(X)$, $\xi \in \mathscr{D}(X)$, s is a holomorphic section of E such that $\{s = 0\} = V$ and $\|\cdot\|$ is a smooth Hermitian metric on E. In this section we relate the asymptotic expansion of $\mathcal{I}_{\|\cdot\|}(\epsilon)$ to the Laurent series expansion (4.1) of $\Gamma_{\|\cdot\|}(\lambda)$.

Proposition 6.1. *Let $P \subset \mathbb{Q}$ denote the polar set of $\Gamma_{\|\cdot\|}(\lambda)$ and let $P_+ = P \cap \{\mathfrak{Re}\,\lambda > 0\}$. We have that*
(6.1)
$$\mathcal{I}_{\|\cdot\|}(\epsilon) = \langle \mu_0^{\|\cdot\|}(\omega), \xi\rangle + \sum_{j=1}^{\kappa} \frac{1}{j!}\big(\log \epsilon^{-1}\big)^j \langle \mu_j^{\|\cdot\|}(\omega), \xi\rangle + \sum_{p\in P_+} \operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} + \mathcal{O}(\epsilon^{\delta}),$$
for some $\delta > 0$.

As we show below,
$$\operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} = \epsilon^{-p}\sum_{j=0}^{2\ell_p-1} \frac{1}{j!}\big(\log \epsilon^{-1}\big)^j c_{2\ell_p-1-j}$$
where $\ell_p \in \mathbb{N}$ and where the $c_{2\ell_p-1-j}$ are independent of ϵ. If V is a hypersurface, the existence of an asymptotic expansion of $\mathcal{I}_{\|\cdot\|}(\epsilon)$ of this form follows from [3, Theorem 4.3.1]. The proof of that theorem is based on [4] and the existence of Bernstein–Sato polynomials. It is reasonable to expect that Proposition 6.1 can be proven in a similar way. We have instead chosen to use the fact that $\mathcal{I}_{\|\cdot\|}(\epsilon)$ and $\Gamma_{\|\cdot\|}(\lambda)$ are related via the Mellin transform.

The first observation is that $\Gamma_{\|\cdot\|}(\lambda)$ satisfies a certain growth condition.

Lemma 6.2. *The function $\Gamma_{\|\cdot\|}(\lambda)$ is rapidly decreasing in $\mathfrak{Im}\,\lambda$, in the sense that the product $\lambda^{\ell}\Gamma_{\|\cdot\|}(\lambda)$, for any $\ell \in \mathbb{N}$, is a bounded function when $\lambda = \alpha + i\beta$ and $|\beta| \to \infty$, locally uniformly in α.*

The following proof is an adaptation of the proof of Lemma 6.1 in [1].

Proof. Let $\pi : \widetilde{X} \to X$ be a modification such that $\widetilde{X}$ is smooth and $\pi^* s$ defines a normal crossings divisor. Recall that then
$$\Gamma_{\|\cdot\|}(\lambda) = \int_{\widetilde{X}} \|\pi^* s\|^{2\lambda}\pi^*\omega \wedge \pi^*\xi.$$
Locally in $\widetilde{X}$ we can choose coordinates such that $\|\pi^* s\|^{2\lambda} = |z_1^{m_1}, \ldots, z_\kappa^{m_\kappa}|^{2\lambda} e^{-\lambda\phi}$, for some $1 \le \kappa \le n$, $m_1, \ldots, m_\kappa \ge 1$, and ϕ a local weight associated to $\|\cdot\|$, and
$$\pi^*\omega = \frac{\Psi\, \mathrm{d}z \wedge \mathrm{d}\bar{z}}{|z_1^{m_1}, \ldots, z_\kappa^{m_\kappa}|^{2N}},$$
for some smooth function Ψ and some integer $N \ge 0$. By introducing a partition of unity $\{\rho_j\}$ on $\widetilde{X}$, $\Gamma_{\|\cdot\|}(\lambda)$ can be written as a finite sum of terms of the form
$$\int_{\mathbb{C}^n} |z_1^{m_1}\cdots z_\kappa^{m_\kappa}|^{2(\lambda-N)} e^{-\lambda\phi}\Psi\pi^*\xi\rho_j\, \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$
Notice that κ, $m_1, \ldots, m_\kappa$, N, ϕ and Ψ all depend on j.

Consider the (non-holmorphic) change of variables, $\sigma_1 = e^{-\phi/2m_1}z_1$, $\sigma_\ell = z_\ell$ for $2 \le \ell \le n$. We have that $\mathrm{d}\sigma_\ell = \mathrm{d}z_\ell$ for $2 \le \ell \le n$, and
$$\mathrm{d}\sigma_1 = e^{-\phi/2m_1}\mathrm{d}z_1 - \frac{1}{2m_1}e^{-\phi/2m_1}z_1\sum_{\ell=1}^{n}\bigg(\frac{\partial\phi}{\partial z_\ell}\mathrm{d}z_\ell + \frac{\partial\phi}{\partial\bar{z}_\ell}\mathrm{d}\bar{z}_\ell\bigg).$$

It follows that

$$\mathrm{d}\sigma \wedge \mathrm{d}\bar{\sigma} = e^{-\phi/m_1}\Big(1 - \frac{1}{m_1}\mathfrak{Re}\, z_1 \frac{\partial \phi}{\partial z_1}\Big)\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

We can take ρ_j to be such that $\mathrm{d}\sigma \wedge \mathrm{d}\bar{\sigma} \neq 0$ on $\operatorname{supp}\rho_j$. We then have that

$$\int_{\mathbb{C}^n} |z_1^{m_1} \cdots z_\kappa^{m_\kappa}|^{2(\lambda-N)} e^{-\lambda\phi}\Psi\pi^*\xi\rho_j \,\mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{\mathbb{C}^n} |\sigma_1^{m_1} \cdots \sigma_\kappa^{m_\kappa}|^{2(\lambda-N)} \widetilde{\Psi}\pi^*\xi\rho_j \,\mathrm{d}\sigma \wedge \mathrm{d}\bar{\sigma},$$

where

$$\widetilde{\Psi} = \Big(1 - \frac{1}{m_1}\mathfrak{Re}\, z_1 \frac{\partial \phi}{\partial z_1}\Big)^{-1} e^{-(N-1/m_1)\phi}\Psi,$$

is smooth on $\operatorname{supp}\rho_j$. Following the steps in the proof of Lemma 3.2, for any positive integer M we have by (3.6) that

$$\int_{\mathbb{C}^n} |\sigma_1^{m_1} \cdots \sigma_\kappa^{m_\kappa}|^{2(\lambda-N)} \widetilde{\Psi}\pi^*\xi\rho_j \,\mathrm{d}\sigma \wedge \mathrm{d}\bar{\sigma} = \frac{h_M(\lambda)}{\lambda^{2\kappa}} \int_{\mathbb{C}^n} |\sigma_1^{m_1} \cdots \sigma_\kappa^{m_\kappa}|^{2(\lambda+M)} \times$$
$$\times P_M \bar{P}_M\big(\widetilde{\Psi}\pi^*\xi\rho_j\big)\mathrm{d}\sigma \wedge \mathrm{d}\bar{\sigma},$$

where

$$P_M\bar{P}_M = \frac{\partial^{2(N+M)\sum_{i=1}^{\kappa} m_i}}{\partial\sigma_1^{(N+M)m_1} \cdots \partial\bar{\sigma}_\kappa^{(N+M)m_\kappa}},$$

and where $h_M(\lambda)$ is given by (3.7). Notice that

$$\frac{|h_M(\lambda)|}{|\lambda|^{2\kappa}} = \mathcal{O}\Big(|\lambda|^{-2(N+M)\sum_{i=1}^{\kappa} m_i}\Big),$$

for $|\lambda| \gg 0$. For $\lambda = \alpha + i\beta$ with $\alpha > -M$, the integral

$$\int_{\mathbb{C}^n} |\sigma_1^{m_1} \cdots \sigma_\kappa^{m_\kappa}|^{2(\lambda+M)} P_M\bar{P}_M\big(\widetilde{\Psi}\pi^*\xi\rho_j\big)\mathrm{d}\sigma \wedge \mathrm{d}\bar{\sigma}$$

is finite, and it clearly remains finite if we let $|\beta| \to \infty$, locally uniformly in α. Thus, for $|\beta| \gg 0$, $|\Gamma_{\|\cdot\|}(\alpha + i\beta)| = \mathcal{O}(|\beta|^{-2(N+M)\sum_{i=1}^{\kappa} m_i})$. As M was chosen arbitrarily, it follows that $(\alpha + i\beta)^\ell\Gamma_{\|\cdot\|}(\alpha + i\beta)$ is a bounded function when $|\beta| \to \infty$ for any $\ell \in \mathbb{N}$. □

As mentioned, the functions $\Gamma_{\|\cdot\|}(\lambda)$ and $\mathcal{I}_{\|\cdot\|}(\epsilon)$ are related via the Mellin transform. The Mellin transform of a function f defined on $\mathbb{R}_+$ is given by

$$\{\mathcal{M}f\}(\lambda) = \int_0^\infty \epsilon^{\lambda-1} f(\epsilon)\mathrm{d}\epsilon.$$

Notice that if $|f(\epsilon)| \lesssim \epsilon^{-N}$ for some $N \geq 0$ as $\epsilon \to 0$ and $f(\epsilon) = 0$ if $\epsilon \gg 0$, then $\{\mathcal{M}f\}(\lambda)$ is holomorphic for $\mathfrak{Re}\,\lambda \gg 0$.

If a function $\varphi : \mathbb{C} \to \mathbb{C}$ is holomorphic in the strip $a < \mathfrak{Re}\,\lambda < b$, and if it tends to zero uniformly as $|\mathfrak{Im}\,\lambda| \to \infty$, for $\mathfrak{Re}\,\lambda = c$, where $c \in (a, b)$, such that its integral along such a line is absolutely convergent, then φ has an inverse Mellin transform, given by

$$\{\mathcal{M}^{-1}\varphi\}(\epsilon) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \epsilon^{-\lambda}\varphi(\lambda)\mathrm{d}\lambda.$$

Lemma 6.3. *We can find an integer $N \geq 0$ such that $|\mathcal{I}_{\|\cdot\|}(\epsilon)| \lesssim \epsilon^{-2N}$ as $\epsilon \to 0$; additionally we have that $\mathcal{I}_{\|\cdot\|}(\epsilon) = 0$ for $\epsilon \gg 0$. For $\mathfrak{Re}\,\lambda \gg 0$, we have that*

$$\{\mathcal{M}\mathcal{I}_{\|\cdot\|}\}(\lambda) = \frac{1}{\lambda}\Gamma_{\|\cdot\|}(\lambda).$$

This relation between the two regularization methods considered is well known. It frequently appears in the context of residue theory, see [1, 15, 16], but it has also been recognized in the context of divergent integrals, see, e.g., [3, 13].

Proof. Since $\omega \in \mathcal{A}_s(X)$ we can find an integer $N \geq 0$ such that $\omega = \widetilde{\omega}/\|s\|^{2N}$ where $\widetilde{\omega}$ is bounded on $\operatorname{supp}\xi$. Thus,

$$|\mathcal{I}_{\|\cdot\|}(\epsilon)| \leq \int\limits_{\|s\|^2 \geq \epsilon} \frac{|\widetilde{\omega} \wedge \xi|}{\|s\|^{2N}} \lesssim \frac{1}{\epsilon^{2N}}.$$

Since $\widetilde{\omega} \wedge \xi$ has compact support, $\mathcal{I}_{\|\cdot\|}(\epsilon) = 0$ for $\epsilon \gg 0$.

By Fubini's theorem

$$\begin{aligned}\{\mathcal{M}\mathcal{I}_{\|\cdot\|}\}(\lambda) &= \int\limits_0^\infty \epsilon^{\lambda-1} \int\limits_{\|s\|^2 \geq \epsilon} \omega \wedge \xi \, \mathrm{d}\epsilon \\ &= \int_X \int\limits_0^{\|s\|^2} \epsilon^{\lambda-1}\mathrm{d}\epsilon \, \omega \wedge \xi = \frac{1}{\lambda}\int\limits_X \|s\|^{2\lambda}\omega \wedge \xi = \frac{1}{\lambda}\Gamma_{\|\cdot\|}(\lambda),\end{aligned}$$

for $\mathfrak{Re}\,\lambda \gg 0$. □

By Lemmas 6.2 and 6.3 it follows that $\mathcal{I}_{\|\cdot\|}(\epsilon)$ can be recovered from $\Gamma_{\|\cdot\|}(\lambda)$ via the inverse Mellin transform as follows,

$$\mathcal{I}_{\|\cdot\|}(\epsilon) = \big\{\mathcal{M}^{-1}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\big\}(\epsilon) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\mathrm{d}\lambda, \tag{6.2}$$

for $c \gg 0$. With (6.2), we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. It follows from Theorem 4.1 (i) that $\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)$ defines a meromorphic function with polar set P contained in $\mathbb{Q} \cap (-\infty, N]$ for some $N \geq 0$. Let $\delta > 0$ such that $\Gamma_{\|\cdot\|}(\lambda)$ has no poles in the interval $[-\delta, 0)$ and let $c \gg 1$ such that (6.2) holds. Let $B = \{-\delta < \mathfrak{Re}\,\lambda < c\} \subset \mathbb{C}$ and let ∂B be the positively oriented boundary of B. By the Residue theorem and Lemma 6.2 we have that

$$\frac{1}{2\pi i}\oint\limits_{\partial B} \epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\mathrm{d}\lambda = \sum_{p \in P \cap B} \operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\}. \tag{6.3}$$

By a straightforward computation

$$\frac{1}{2\pi i}\oint\limits_{\partial B} \epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\mathrm{d}\lambda = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\mathrm{d}\lambda + \mathcal{O}(\epsilon^\delta).$$

Thus, by (6.2) and (6.3), it follows that

$$\mathcal{I}_{\|\cdot\|}(\epsilon) = \sum_{p \in P \cap B} \operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} + \mathcal{O}(\epsilon^\delta). \tag{6.4}$$

Let $P_+ = P \cap \{\mathfrak{Re}\,\lambda > 0\}$; we write

$$\sum_{p\in P\cap B} \operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} = \operatorname*{Res}_{\lambda=0}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} + \sum_{p\in P_+} \operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\},$$

where, by Theorem 4.1 (i), we have that

$$\begin{aligned}\operatorname*{Res}_{\lambda=0}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} &= \operatorname*{Res}_{\lambda=0}\left\{\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\lambda^{\ell-1}\big(\log\epsilon^{-1}\big)^{\ell}\left(\sum_{j=0}^{\kappa}\lambda^{-j}\langle\mu_j^{\|\cdot\|}(\omega),\xi\rangle + \mathcal{O}(\lambda)\right)\right\}\\ &= \operatorname*{Res}_{\lambda=0}\left\{\sum_{j=0}^{\kappa}\langle\mu_j^{\|\cdot\|}(\omega),\xi\rangle\sum_{\ell=0}^{j}\frac{1}{\ell!}\lambda^{\ell-j-1}\big(\log\epsilon^{-1}\big)^{\ell} + \mathcal{O}(1)\right\}\\ &= \sum_{j=0}^{\kappa}\frac{1}{j!}\big(\log\epsilon^{-1}\big)^{j}\langle\mu_j^{\|\cdot\|}(\omega),\xi\rangle.\end{aligned}$$

Proposition 6.1 now follows in view of (6.4). □

We can look more closely at the residues

$$\operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\},$$

for $p \in P_+$. Following the proof of Lemma 4.2 and Theorem 4.1, let $\pi : \widetilde{X} \to X$ be a modification such that $\widetilde{X}$ is smooth and $\pi^* s$ defines a normal crossings divisor. By introducing a partition of unity, $\Gamma_{\|\cdot\|}(\lambda)$ can be written as a finite sum of terms of the form

$$\frac{h(\lambda)}{\lambda^{2\kappa}} I(\lambda),$$

where $h(\lambda)$ is given by (3.3) and $I(\lambda)$ by (4.3). Since $I(\lambda)/\lambda^{2\kappa}$ is holomorphic on $\mathfrak{Re}\,\lambda > 0$, by inspection of (3.3), we find that $(\lambda-p)^{2\ell_p}\Gamma_{\|\cdot\|}(\lambda)$ is holomorphic in a neighborhood of p, where

$$\ell_p = \#\Big\{(i,j) : i \in \{1,\ldots,\kappa\},\ j\in\{1,\ldots,Nm_i-1\},\ \frac{j}{m_i} = p\Big\} \geq 1 \text{ for } p \in P_+.$$

Thus, we have that

$$\begin{aligned}\operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} &= \operatorname*{Res}_{\lambda=p}\left\{\epsilon^{-p}\sum_{j=0}^{\infty}\frac{1}{j!}\big(\log\epsilon^{-1}\big)^{j}(\lambda-p)^{j-2\ell_p}\frac{(\lambda-p)^{2\ell_p}\Gamma_{\|\cdot\|}(\lambda)}{\lambda}\right\}\\ &= \operatorname*{Res}_{\lambda=p}\left\{\epsilon^{-p}\sum_{j=0}^{\infty}\frac{1}{j!}\big(\log\epsilon^{-1}\big)^{j}(\lambda-p)^{j-2\ell_p}\sum_{k=0}^{\infty}c_k(\lambda-p)^k\right\},\end{aligned}$$

where

$$c_k = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\left(\frac{(\lambda-p)^{2\ell_p}\Gamma_{\|\cdot\|}(\lambda)}{\lambda}\right)\Bigg|_{\lambda=p}.$$

We obtain

$$\operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma_{\|\cdot\|}(\lambda)\Big\} = \epsilon^{-p}\sum_{j=0}^{2\ell_p-1}\frac{1}{j!}\big(\log\epsilon^{-1}\big)^{j}c_{2\ell_p-1-j}.$$

The coefficients $c_{2\ell_p-1-j}$ can be interpreted as the action of currents similar to $\mu_j^{\|\cdot\|}(\omega)$ on ξ.

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