THE GYSIN BRAID FOR S^3-ACTIONS ON MANIFOLDS

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Abstract. In a previous work, we constructed a Gysin sequence that relates the cohomology of a manifold M to that of the orbit space M/S^3, where the sphere S^3 acts smoothly on M. This sequence includes an exotic term that depends on M^{S^1}, the subset of points fixed by the action of the subgroup S^1.

The orbit space is a stratified pseudomanifold, which is a type of singular space where intersection cohomology can be applied. When the action is semi-free, the second author has already constructed a Gysin sequence that relates the cohomology of M to the intersection cohomology of M/S^3.

However, what happens when the action is not semi-free? This is the main focus of this work. The situation becomes more complex, and we do not find just a Gysin sequence. Instead, we construct a Gysin braid that relates the cohomology of M to the intersection cohomology of M/S^3. This braid also contains an exotic term that depends on the intersection cohomology of the fixed point subset M^{S^1}.

Given a smooth free action of the sphere S^3 on a smooth manifold M, we have a sphere bundle and the Gysin sequence

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-4}\left(M/S^3\right) \longrightarrow H^{*}\left(M/S^3\right) \longrightarrow H^{*}(M) \longrightarrow \cdots \tag{1}$$

which relates the cohomologies of the manifold M and that of its orbit space M/S^3 (see for example [9, 2]). In the case of a semi-free action, we do not have a sphere bundle, but we have the Gysin sequence

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-4}\left(M/S^3, M^{S^3}\right) \longrightarrow H^{*}\left(M/S^3\right) \longrightarrow H^{*}(M) \longrightarrow \cdots . \tag{2}$$

In the general case, an exotic term appears:

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-4}\left(M/S^3, \Sigma/S^3\right) \oplus \left(H^{*-3}\left(M^{S^1}\right)\right)^{-\mathbb{Z}_2} \longrightarrow H\left(M/S^3\right) \longrightarrow H^{*}(M) \longrightarrow \cdots \tag{3}$$

(cf. [13]). In this context, $\Sigma \subset M$ denotes the subset of points in M whose isotropy group is infinite. The group $\mathbb{Z}_2$ acts on M^{S^1} by $j \in S^3$.

When the action is free, the orbit space is a manifold. In the more general case, it is a pseudomanifold. The second author constructed in [15] the following exact sequence

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-4}_{\overline{p}-\overline{e}}\left(M/S^3\right) \longrightarrow H^{*}_{\overline{p}}\left(M/S^3\right) \longrightarrow H^{*}(M) \longrightarrow \cdots \tag{4}$$

This sequence establishes a connection between the cohomology of the space M and the intersection cohomology of its orbit space M/S^3 in the case of a semi-free action. This exact sequence holds when $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$. The Euler perversity $\overline{e}$ takes the value 4 on singular strata. In the special case where $\overline{p} = \overline{0}$, the exact sequence (4) simplifies to (2).

In this paper, we establish a connection between the cohomology of a manifold M and the intersection cohomology of its orbit space M/S^3, for any smooth action $\Phi\colon S^3 \times M \to M$. Of particular interest is the case where the action has three-dimensional orbits, also known as a *mobile action*.

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Given a mobile action, we obtain the following Gysin braid which relates the cohomologies $H^*(M)$ and $H^*_{\overline{p}}(M/S^3)$

$$
\begin{tikzcd}[column sep=small]
H^*_{\overline{p}}(M/S^3) \arrow[rr, bend left, "①"] \arrow[dr, "④"] & & H^*(M) \arrow[rr, bend left, "③"] \arrow[dr, "①"] & & H^{*-3}_{\overline{p}-\overline{e}}(M/S^3) \arrow[rr, bend left, "②"] \arrow[dr, "③"] & & \bigoplus_{S\in\mathscr{S}_1} H^{*-1-2p_S}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \\
& H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "③"] \arrow[dr, "④"] & & H^*\left(G^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "②"] \arrow[dr, "①"] & & H^{*+1}\left(K^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "④"] \arrow[dr, "③"] & \\
H^{*-4}_{\overline{p}-\overline{e}}(M/S^3) \arrow[ur, "③"] \arrow[rr, bend right, "②"'] & & \bigoplus_{S\in\mathscr{S}_1} H^{*-2p_S-2}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \arrow[ur, "②"] \arrow[rr, bend right, "④"'] & & H^{*+1}_{\overline{p}}(M/S^3) \arrow[ur, "④"] \arrow[rr, bend right, "①"'] & & H^{*+1}(M),
\end{tikzcd}
$$

where

- $\mathscr{S}_1$ (resp. $\mathscr{S}_3$) is the family of singular strata S of M with $\dim S^3_x = 1$ (resp. 3) for any $x \in S$. Here S^3_x denotes the isotropy subgroup.
- the Euler perversity $\overline{e}$ takes the value 4 (resp. 2) on the strata of $\mathscr{S}_1$ (resp $\mathscr{S}_3$),
- the perversity $\overline{p}$ lies between $\overline{0}$ and the top perversity $\overline{t}$ on M,
- the number p_S denotes the integer part of $\overline{p}(S)/2$,
- the perversity $\overline{P_S}$ on the filtered space $\overline{S}^{S^1}$ is defined by $\overline{P_S}(Q) = \overline{p}(Q) - 2p_S - 2$ for any $Q \in \mathscr{S}_3$ with $Q \subset \overline{S}$,
- the Gysin term $G^*_{\overline{p}}(M)$ represents the cokernel of the map induced by the natural projection $\pi\colon M \to M/S^3$ and
- the co-Gysin term $K^*_{\overline{p}}(M)$ denotes the kernel of the map induced by integrating along the fibers of π

(see Theorem C). The braid consists of four long exact sequences, denoted by ①, ②, ③, and ④. All the triangles and diamonds in the braid are commutative. The top and bottom sequences in the braid are semi-exact and both have the same exactness defaults (cf. Remark 4.3 (c)). The cohomologies of the Gysin and co-Gysin terms are interconnected through the long exact sequences of Remark 4.3 (b).

Let's analyze the four exact sequences that make up the Gysin braid.

In the classical framework of a free action, there exist two methods for constructing the Gysin Sequence (1). One approach involves employing the pullback induced by the natural projection $\pi\colon M \to M/S^3$, while the other entails integrating along the fibers of π. Remarkably, both methodologies yield identical outcomes, resulting in the Gysin sequence. Meanwhile, the Gysin sequence (3) is derived via the former method.

In the broader context of this paper's discussion on mobile actions, we utilize both techniques, and they yield distinct results. This fundamental difference is the primary reason why the Gysin braid appears instead of a Gysin sequence.

- The pull back associated to the projection π induces the long exact sequence ①:

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-1}\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*_{\overline{p}}(M/S^3) \longrightarrow H^*(M) \longrightarrow \cdots$$

  where the *Gysin term* $G^*_{\overline{p}}(M)$ is the cokernel of π^*. This is the first Gysin sequence associated to the action Φ.

  We can determine the cohomology of the Gysin-term through the sequence ②, which employs integration along the fibers of π. This method is employed in [13], where we implicitly work with the perversity $\overline{p} = \overline{0}$. In this context, we obtain the Gysin sequence (3) since the sequence ② splits at the position of the connecting map. However, it's important to note that for other perversities, the sequence ② may not necessarily split as demonstrated by the example in Remark 4.3 (d).

- Employing the integration along the fibers of π, we obtain the long exact sequence ③:

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-4}_{\overline{p}-\overline{e}}(M/S^3) \longrightarrow H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*(M) \longrightarrow \cdots,$$

  where the *co-Gysin term* $K^*_{\overline{p}}(M)$ is the kernel of the integration operator along the fibers of π. This is the second Gysin sequence associated to the action Φ.

We compute the cohomology of the co-Gysin term using the sequence ④, which employs the pullback operator induced by π. It's important to note that unlike the previous case, sequence ③ does not necessarily split, even for the perversity $\overline{p} = \overline{0}$, as demonstrated in the example in Remark 4.3 (d).

- If the exotic term $\bigoplus_{S \in \mathscr{S}_1} H^*_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$ vanishes, then the Gysin braid simplifies to the sequence (4). In particular, this happens when the action Φ is semi-free, as noted in Remark 4.3 (a).

- Another approach to constructing a Gysin sequence involves utilizing the Leray-deRham spectral sequence. Recall that in the case of a differentiable action $\Phi\colon G \times M \to M$ of a connected compact Lie group G on a manifold M, there exists a spectral sequence $E_r^{i,j}$ converging to $H^{i+j}(M)$, where $E_2^{i,j} = H^i(M/G) \otimes H^j(G)$. When $G = S^3$ and the action is free, almost-free, or semifree, this spectral sequence degenerates into a Gysin sequence (cases (1), (2), and (4)). This is because the second term of the spectral sequence contains only two levels: $j = 0, 3$. However, the situation becomes significantly more complex when the action is not free, and computing the second term of this spectral sequence becomes challenging (see [14, 16]).

  In the case of a mobile action of S^3, and using singular cohomology, we have shown in [13] that the second term of the spectral sequence possesses three levels ($j = 0, 2, 3$) and that the spectral sequence degenerates into the Gysin sequence (3). In this paper, we prove that that this phenomenon persists within the framework of intersection cohomology. The Leray-deRham spectral sequence ${}_{\overline{p}}E_r^{i,j}$ depends on a perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M and satisfies the following properties.
  - ⋆ It converges: ${}_{\overline{p}}E_r^{i,j} \Rightarrow H^{i+j}(M)$.
  - ⋆ The second page is given by

$$
{}_{\overline{p}}E_2^{i,j} = \begin{cases} H^i_{\overline{p}}(M/S^3) & \text{if } j = 0 \\ \bigoplus_{S \in \mathscr{S}_1} H^{i-2p_S}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} & \text{if } j = 2 \\ H^i_{\overline{p}-\overline{e}}(M/S^3) & \text{if } j = 3. \end{cases}
$$

    It is 0 otherwise.
  - ⋆ The Gysin term appears in this spectral sequence through the long exact sequence ②

$$
\cdots \longrightarrow H^{i-1}\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow {}_{\overline{p}}E_2^{i-4,3} \xrightarrow{d_2} {}_{\overline{p}}E_2^{i-2,2} \longrightarrow H^i\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow \cdots .
$$

  - ⋆ This spectral sequence degenerates at the third page and produces the long exact sequence ①.

  In other words, the information in the Leray-deRham spectral sequence beyond page ${}_{\overline{p}}E_2$ is contained within the Gysin braid.

- The non-mobile actions are simpler, and we obtain $H^*(M) = H^*_{\overline{p}}(M/S^3) \oplus H^{*-2}_{\overline{p}-\overline{2}}\left(M^{S^1}\right)^{-\mathbb{Z}_2}$ (see Section 5).

In the following, we consider a smooth action $\Phi\colon S^3 \times M \to M$, where M is a second countable, Hausdorff, smooth manifold of dimension m without boundary. For the definitions and properties related to compact Lie group actions, we refer the reader to [2].

The first section of this work is devoted to studying the intersection cohomology of the orbit space M/S^3. We demonstrate how to compute this cohomology using differential forms defined on an open subset of M. The complex of invariant intersection forms of M is a key tool for constructing the Gysin braid, which is discussed in the second section. The final two sections of this work focus on constructing and analyzing the Gysin braid associated to the action. This braid arises from integrating along the orbits of the action discussed in section three.

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1. INTERSECTION COHOMOLOGY

The intersection cohomology of the orbit space M/S^3 is originally defined using singular simplices. In this section, we demonstrate an alternative method for computing this cohomology by utilizing differential forms defined on the regular part of M.

1.1. **Filtered spaces** [6]**.** The *orbit type stratification* $\mathscr{S}$ of M is the partition obtained by defining an equivalence relation in M as follows:

$$x \sim y \Leftrightarrow \dim S^3_x = \dim S^3_y.$$

This condition is equivalent to $(S^3_x)_0$ and $(S^3_y)_0$ being conjugated, where $(-)_0$ denotes the connected component containing the unity. The elements of $\mathscr{S}$ are called *strata*, and they correspond to the connected components of the partition induced by $\sim$.

There are four possible isotropy subgroups of a point in M, up to conjugacy: a finite subgroup of S^3, S^1, the normalizer $N = N(S^1)$ of S^1 in S^3, and S^3 itself (cf. [2, Th. 9.5,pag.153]). Recall that $N = S^1 \sqcup j\,S^1 \cong O(2)$.

We define $\mathscr{S} = \mathscr{S}_0 \sqcup \mathscr{S}_1 \sqcup \mathscr{S}_3$ as follows:

$$\underset{\textit{Mobile strata}}{\mathscr{S}_0 = \{S \in \mathscr{S} \mid \dim S^3_x = 0, x \in S\}} \quad \underset{\textit{Semi-mobile strata}}{\mathscr{S}_1 = \{S \in \mathscr{S} \mid \dim S^3_x = 1,\ x \in S\}} \quad \underset{\textit{Fixed strata}}{\mathscr{S}_3 = \{S \in \mathscr{S} \mid \dim S^3_x = 3,\ x \in S\}}$$

The action is considered *mobile* if $\mathscr{S}_0 \neq \varnothing$. If $\mathscr{S}_0 = \varnothing$ and $\mathscr{S}_1 \neq \varnothing$, we say that the action is *semi-mobile*. The remaining case is the *trivial* action. The set of *singular strata* is denoted by $\mathscr{S}^{sing}$. If the action is mobile, then $\mathscr{S}^{sing} = \mathscr{S}_1 \sqcup \mathscr{S}_3$. If the action is semi-mobile, then $\mathscr{S}^{sing} = \mathscr{S}_3$.

We define $F_3 = \sqcup_{S \in \mathscr{S}_3} S = M^{S^3}$ and $F_1 = \sqcup_{S \in \mathscr{S}_1} S$, which are S^3-invariant submanifolds of M[1]. Note that F_1 is actually the twisted product $S^3 \times_N M^{S^1}$, where S^3 acts on the left of the left factor. Furthermore, if $S \in \mathscr{S}_1$ we have

$$S = S^3 \times_N S^{S^1}.$$

The union of singular strata is $\Sigma = F_1 \sqcup F_3$ (resp. F_3) when the action is mobile (resp. semi-mobile).

Filtered spaces provide the essential framework for defining singular intersection cohomology, which is the dual of the intersection homology introduced in [8] (see, for example, [5]).

Proposition 1.1. *The strata of $\mathscr{S}$ are invariant submanifolds. For each integer i we define $M_i = \sqcup \left\{S \in \mathscr{S} \mid \dim S \leqslant i\right\}$. The filtration*

$$\varnothing = M_{-1} \subset M_0 \subseteq \cdots \subseteq M_i \subseteq \cdots \subseteq M_n = M.$$

defines a filtered space.

For each integer i we define $\left(M/S^3\right)_i = \sqcup \left\{\pi(S) \mid S \in \mathscr{S} \text{ and } \dim \pi(S) \leqslant i\right\}$, where $\pi\colon M \to M/S^3$ denotes the canonical projection. The filtration

$$\varnothing = \left(M/S^3\right)_{-1} \subset \left(M/S^3\right)_0 \subseteq \cdots \subseteq \left(M/S^3\right)_i \subseteq \cdots \subseteq \left(M/S^3\right)_m = M/S^3$$

defines a filtered structure in M/S^3.

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[1]The next Proposition shows that they are manifolds, in fact, these manifolds may have connected components with different dimensions.

Proof. Let S be a stratum of $\mathscr{S}$. Each point $x \in S$ possesses an open neighborhood S^3-equivariantely diffeomorphic to the twisted product $S^3 \times_H \mathbb{R}^a$ where the isotropy subgroup $H = S^3_x$ acts orthogonally on $\mathbb{R}^a$. The point x becomes the class $\langle 1, 0\rangle$. Recall that the isotropy subgroup of a point $\langle g, u\rangle \in S^3 \times_H \mathbb{R}^a$ is $gH_u g^{-1}$. So, the trace of S in tis neighborhood is $S^3 \times_H \mathbb{R}^b$, where $\mathbb{R}^b = \{u \in \mathbb{R}^a \mid \dim H_u = \dim H\} = \{u \in \mathbb{R}^a \mid H_0 \cdot u = u\}$. The stratum S is an invariant submanifold with $\dim S = 3 + b - \dim H$.

It remains to prove that each M_i and $\left(M/S^3\right)_i$ are closed subsets. It suffices to verify that the maps $\dim\colon M \to \mathbb{Z}$ and $i\colon M \to \mathbb{Z}$, defined by $\dim(x) = \dim S$ and $i(x) = \dim \pi(S) = \dim S/S^3 = \dim S + \dim S^3_x - 3$, with $S \in \mathscr{S}$ and $x \in S$, are lower semi-continuous. Since the problem is a local question then we can suppose that M is $S^3 \times_H \mathbb{R}^a$. We prove that the functions dim and i are bigger than $\dim(x)$ and $i(x)$ respectively. Notice that the map $\langle g, u\rangle \mapsto -\dim S^3_{\langle g,u\rangle}$ is a lower semi-continuous map since $-\dim S^3_{\langle g,u\rangle} = -\dim H_u \geqslant -\dim H = -\dim S^3_x$. So, it remains to study the function dim.

Considering the G-equivariant covering $S^3 \times_{H_0} \mathbb{R}^a \to S^3 \times_H \mathbb{R}^a$, we can suppose that H is connected. Let $\mathbb{R}^a = \mathbb{R}^b \times \mathbb{R}^c$ be the H-equivariant orthogonal decomposition of $\mathbb{R}^a$. This gives $S^3 \times_H \mathbb{R}^a = \mathbb{R}^b \times \left(S^3 \times_H \mathbb{R}^c\right)$. Given a point $y = \langle g, u\rangle \in S^3 \times_H \mathbb{R}^a$ we consider $Q \in \mathscr{S}$ the stratum containing this point. In fact, $Q = \mathbb{R}^b \times \left(S^3 \times_H \mathbb{R}^d\right)$ where $\mathbb{R}^d = \{v \in \mathbb{R}^a \mid \dim H_v = \dim H_u\}$. We have finished since $\dim(y) = \dim Q = b + 3 + d - \dim H \geqslant b + 3 - \dim H = \dim S = \dim(x)$. □

The dimension m of the filtered space M is $\dim M$. The dimension n of the filtered space M/S^3 is $m - 3$ (resp. $m - 1$) when the action is mobile (resp. semi-mobile).

Brylinski-Goresky-MacPherson showed how to compute intersection cohomology with differential forms (cf. [4]). To this effect, they use the Thom-Mather systems.

1.2. **Thom-Mather systems.** Since F_1 and F_3 are S^3-invariant sub-manifolds of M, we can consider $\tau_k\colon T_k \to F_k$ two S^3-invariant tubular neighborhoods of F_k in M, $k = 1, 3$. Associated to these tubular neighborhoods we have the following maps:

⇝ The *radius map* $\nu_k\colon T_k \to [0, \infty[$ s defined fiberwise by $u \mapsto \|u\|$. This map is invariant and smooth.

⇝ The *dilatation map* $\partial_k\colon [0, \infty[\times T_k \to T_k$, defined fiberwise by $(t, u) \mapsto t \cdot u$. It is a smooth equivariant map.

Given $S \in \mathscr{S}$ contained in F_k for $k = 1, 3$, we can define $T_S = \tau_k^{-1}(S)$ and $\tau_S : T_S \to S$ as the restriction of τ_k. We can define the maps ν_S and ∂_S analogously. The *soul* of T_S is defined as the open subset $D_S = \nu_S^{-1}([0, 2[)$.

The family of tubular neighborhoods $\mathfrak{T}_M = T_1, T_3$ is called a *Thom-Mather system of M* when:

$$\left\{\begin{array}{l} \tau_3 = \tau_3 \circ \tau_1 \\ \nu_3 = \nu_3 \circ \tau_1 \end{array}\right\} \text{ on } T_1 \cap T_3 = \tau_1^{-1}(T_3 \cap F_1)$$

We have proved in [13] that there exists an S^3-invariant Thom-Mather system of M.

Consider the induced maps $\widetilde{\tau_k}\colon T_k/S^3 \to F_k/S^3$, $\widetilde{\nu_k}\colon \widetilde{T_k} \to [0, \infty[$ and $\widetilde{\partial}_k\colon [0, \infty[\times T_k/S^3 \to T_k/S^3$. The family of tubular neighborhoods $\mathfrak{T}_{M/S^3} = \{T_1/S^3, T_3/S^3\}$ is a *Thom-Mather system of M/S^3*.

We need a more precise description of the atlas of the bundle τ_3. The open tubular neighborhood T_3 can be chosen as a disjoint union $T_3 = \sqcup\{T_S \mid S \in \mathscr{S}_3\}$ with $T_S \cap T_{S'} = \emptyset$ if $S \neq S'$. There exists an S^3-equivariant atlas $\mathcal{A} = \{\varphi\colon \tau_S^{-1}(U) \to U \times \mathbb{R}^{b+1}\}$ relatively to an orthogonal action $\Phi_S : S^3 \times \mathbb{R}^{b+1} \to \mathbb{R}^{b+1}$, having the origin as the only fixed point.

1.3. **Intersection differential forms on** M**.** The *perverse degree* of a differential form $\omega \in \Omega^*(M\backslash\Sigma)$ relatively to a singular stratum $S \in \mathscr{S}$ is the number

$$||\omega||_S = \min\left\{\ell \in \mathbb{N} \mid \omega(v_0, \ldots, v_\ell, -) = 0 \text{ where } v_0, \ldots, v_\ell \text{ are vectors tangent to the fibers of } \tau_S : D_S \to S\right\}$$

if $\omega \neq 0$ on D_S. If $\omega = 0$ on D_S we define $||\omega||_S = -\infty$. The condition $||\omega||_S = ||d\omega||_S = 0$ is equivalent to stating that the restriction of ω to D_S is a τ_S-basic form.

We shall need the following properties of the perverse degree:

$$(5)\qquad \begin{array}{lll} ||\omega||_S \leqslant |\omega|, \text{degree of } \omega, & ||d\omega||_S \leqslant ||\omega||_S + 1 & ||g^*\omega||_S = ||\omega||_S \quad \text{for each } g \in S^3. \\ ||\omega + \eta||_S \leqslant \max(||\omega||_S, ||\eta||_S) & ||\omega \wedge \eta||_S \leqslant ||\omega||_S + ||\eta||_S & ||i_X\omega||_S \leqslant ||\omega||_S \text{ or each vector field } X \end{array}$$

These properties come directly from the definition of perverse degree. We have also used the fact that the Thom-Mather system is S^3-invariant.

A *perversity* is a map $\overline{p}\colon \mathscr{S}^{sing} \to \overline{\mathbb{Z}} = \mathbb{Z} \sqcup \{-\infty, \infty\}$. The *constant perversity* is $\overline{\ell}(S) = \ell$, with $\ell \in \overline{\mathbb{Z}}$, for any singular stratum. The *top perversity* is defined by $\overline{t}(S) = \text{codim } S - 2$ for any singular stratum $\mathscr{S}$.

The complex of *intersection differential forms* of M, relatively to the perversity $\overline{p}$ is defined by

$$\Omega^*_{\overline{p}}(M) = \left\{ \omega \in \Omega^*(M\backslash\Sigma) \mid \max(||\omega||_S, ||d\omega||_S) \leqslant \overline{p}(S)\ \forall S \in \mathscr{S} \right\}.$$

The complex $\Omega^*_{\overline{p}}(M)$ computes the cohomology $H^*(M)$ for Goresky-MacPherson perversities [4] or for perversities verifying $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ [1, 17]. The cohomology of this complex is $H^*(M, \Sigma)$ (resp. $H^*(M\backslash\Sigma)$) when $\overline{p} < \overline{0}$ (resp. $\overline{p} > \overline{t}$).

We can compute the intersection cohomology of the orbit space M/S^3 by using differential forms defined on $M\backslash\Sigma$. To see how, first notice that the natural projection π establishes a bijection between the strata of M and those of M/S^3. Therefore, a perversity $\overline{p}$ on M/S^3 determines a perversity $\overline{p}$ on M as well, which we will still denote by $\overline{p}$, following the formula $\overline{p}(S) = \overline{p}(S/S^3)$. The reverse is also true. Throughout this work, we will consider all perversities on M. Let us set such a perversity as $\overline{p}$. Its *dual perversity* is $D\overline{p} = \overline{t} - \overline{p}$.

A differential form $\omega \in \Omega^*(M\backslash\Sigma)$ is a *basic form* if it verifies the following condition: $\omega(v, -) = d\omega(v, -) = 0$ for each vector tangent v to the fibers of $\pi\colon M\backslash\Sigma \to (M\backslash\Sigma)/S^3$. It has been proved in [18] that the sub-complex $\Omega^*_{\overline{p}}(M/S^3)$ of basic forms of $\Omega^*_{\overline{p}}(M)$ computes the intersection cohomology $H^*_{D\overline{p}}(M/S^3)$. Notice that this cohomology does not depend on the Thom-Mather system we have chosen.

Given two perversities $\overline{q} \leqslant \overline{p}$ on M the *step complex* $\Omega^*_{\overline{p}/\overline{q}}(M/S^3)$ is the quotient $\Omega^*_{\overline{p}}(M/S^3)/\Omega^*_{\overline{q}}(M/S^3)$ and its cohomology is denoted by $H^*_{\overline{p}/\overline{q}}(M/S^3)$ (cf. [11]).

1.4. **Closure of a stratum.** The exotic term of the Gysin braid we construct in this work uses a particular filtered space we describe now. Consider a non-closed stratum $S \in \mathscr{S}_1$. The closure of S is of the form $\overline{S} = S \sqcup_{\ell \in J} Q_\ell$ where $\{Q_\ell \in \mathscr{S}_3 \mid \ell \in J\} = \{Q \in \mathscr{S}_3 \mid Q \subset \overline{S}\}$. It is a filtered space whose regular stratum is S. Any perversity $\overline{p}$ on M induces a perversity on $\overline{S}$, still denoted by $\overline{p}$, which is defined by the numbers $\overline{p}(Q_\ell)$, $\ell \in J$. The Thom-Mather system $\mathfrak{T}_M$ of M induces the Thom-Mather system $\mathfrak{T}_{\overline{S}} = \{T_{Q_\ell} \cap \overline{S} \mid \ell \in J\}$.

In fact, we need to go a step further and consider the space $\overline{S}^{S^1}$ which is the union $S^{S^1} \sqcup_{\ell \in J} Q_\ell$. It is a filtered space whose regular part is S^{S^1}. Any perversity $\overline{p}$ on M induces a perversity on $\overline{S}^{S^1}$, still denoted by $\overline{p}$, which is defined by the numbers $\overline{p}(Q_\ell)$, $\ell \in J$. The Thom-Mather system $\mathfrak{T}_M$ of M induces the Thom-Mather system $\mathfrak{T}_{\overline{S}^{S^1}} = \left\{ T_{Q_\ell} \cap \overline{S}^{S^1} \mid \ell \in J \right\}$.

The complex of *intersection differential forms* of $\overline{S}^{S^1}$, relatively to the perversity $\overline{p}$, can be defined as in the previous Section. It computes the intersection cohomology $H^*_{\overline{p}}\left(\overline{S}^{S^1}\right)$. Since $j^2 = -1 \in S^1$, the group $\mathbb{Z}_2$ acts on $\overline{S}^{S^1}$ by $g \cdot x = j(x)$, where g denotes the generator of $\mathbb{Z}_2$. Then he group $\mathbb{Z}_2$ also acts on this cohomology. We shall use the notation

$$H^*_{\overline{p}}\left(\overline{S}^{S^1}\right)^{-\mathbb{Z}_2} = \left\{ \omega \in H^*_{\overline{p}}\left(\overline{S}^{S^1}\right) \mid g \cdot \omega = -\omega \right\}.$$

In fact, we are going to use a particular perversity on this space. Associated to any perversity $\overline{p}$ on M we have the perversity $\overline{P_S}$ on $\overline{S}^{S^1}$ defined by

$$\overline{P_S}(Q) = \begin{cases} \overline{p}(Q) - 2p_S - 2 & \text{if } Q = Q_\ell \text{ for some } \ell \in J. \\ 0 & \text{otherwise} \end{cases}$$

where p_S is the integer part of $\overline{p}(S)/2$.

1.5. S^1**-actions.** A similar study can be done for a smooth action $\Psi\colon S^1 \times M \to M$. In this case the orbit stratification type is $\mathscr{S} = \mathscr{S}_0 \sqcup \mathscr{S}_1$ where

$$\underset{\textit{Mobile strata (regular strata)}}{\mathscr{S}_0 = \{S \in \mathscr{S} \mid \dim S^1_x = 0, x \in S\}} \quad \underset{\textit{Fixed strata (singular strata)}}{\mathscr{S}_1 = \{S \in \mathscr{S} \mid \dim S^1_x = 1,\ x \in S\}}$$

We suppose that the action is mobile, meaning $\mathscr{S}_0 \neq \emptyset$ or non trivial.

The family of singular strata is denoted by $\mathscr{S}^{sing}$ and its union is Σ. The manifold M and the orbit space M/S^1 are filtered spaces. Thom-Mather systems also exist. In this case $\mathfrak{T}_M = \{T_S \mid S \in \mathscr{S}_1\}$ where each $\tau_S : T_S \to S$ is a S^1-fiber bundle. The elements of $\mathfrak{T}_M$ can be chosen to be disjoint.

We need a more precise description of the atlas of the bundle τ_S. There exists an S^1-equivariant atlas $\mathcal{A} = \{\varphi\colon \tau_S^{-1}(U) \to U \times \mathbb{R}^{2b+2}\}$ relatively to an orthogonal action $\Phi_S : S^1 \times \mathbb{R}^{2b+2} \to \mathbb{R}^{2b} + 2$, $b \geqslant 0$, having the origin as the only fixed point.

A perversity is a map $\overline{p}\colon \mathscr{S}^{sing} \to \overline{\mathbb{Z}}$. The *top perversity* is defined by $\overline{t}(S) = \operatorname{codim} S - 2$ on $\mathscr{S}_1$. The *dual perversity* of $\overline{p}$ is the perversity $D\overline{p} = \overline{t} - \overline{p}$.

The *perverse degree* $||-||_S, S \in \mathscr{S}_1$, is defined relatively to this Thom-Mather system. The complex

$$\Omega^*_{\overline{p}}(M) = \{\omega \in \Omega^*(M\backslash\Sigma) \mid \max(||\omega||_S, ||d\omega||_S) \leqslant \overline{p}(S)\}$$

computes the cohomology $H^*(M)$, where $\overline{p}$ is a perversity verifying $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ (cf. [17]). On the other hand, the complex

$$\Omega^*_{\overline{p}}\left(M/S^1\right) = \{\omega \in \Omega^*\left((M\backslash\Sigma)/S^1\right) \mid \max(||\omega||_S, ||d\omega||_S) \leqslant \overline{p}(S)\},$$

where $\Omega^*\left((M\backslash\Sigma)/S^1\right) = \{\omega \in \Omega^*(M\backslash\Sigma) \mid \omega(v,-) = \omega(v,-) = 0$ for each vector tangent v to the fibers of $\pi\colon M\backslash\Sigma \to (M\backslash\Sigma)/S^1\}$, *basic forms*, computes the intersection cohomology $H^*_{D\overline{p}}\left(M/S^1\right)$. Notice that this cohomology does not depend on the Thom-Mather system chosen.

Given two perversities $\overline{q} \leqslant \overline{p}$ on M the *step complex* $\Omega^*_{\overline{p}/\overline{q}}\left(M/S^1\right)$ is the quotient $\Omega^*_{\overline{p}}\left(M/S^1\right)/\Omega^*_{\overline{q}}\left(M/S^1\right)$ and its cohomology is denoted by $H^*_{\overline{p}/\overline{q}}\left(M/S^1\right)$ (cf. [11]). This cohomology fits into the long exact sequence

$$\cdots \longrightarrow H^\ell_{\overline{q}}\left(M/S^1\right) \longrightarrow H^\ell_{\overline{p}}\left(M/S^1\right) \longrightarrow H^\ell_{\overline{p}/\overline{q}}\left(M/S^1\right) \longrightarrow H^{\ell+1}_{\overline{q}}\left(M/S^1\right) \longrightarrow \cdots . \tag{6}$$

1.6. *N***-actions.** A smooth action $\Theta\colon N \times M \to M$ induces a circle action $\Psi\colon S^1 \times M \to M$. Since the stratification $\mathscr{S}_1$ of this last action is N-invariant then we can choose an N-invariant Thom-Mather system $\mathfrak{T}_M = \{T_S \mid S \in \mathscr{S}_1\}$, that is, the map $g\colon T_S \to T_{g(S)}$ is an S^1-morphism bundle for each $g \in N$.

Since $j^2 = -1 \in S^1$, the singular part of the S^1-action $\Sigma = \sqcup\{S \in \mathscr{S}_1\}$ is $\mathbb{Z}_2$-invariant relatively to the action $g \cdot x = j(x)$. Also, the union $S \cup j(S)$ is $\mathbb{Z}_2$-invariant for any $S \in \mathscr{S}_1$. Notice that we have two possibilities $j(S) = S$ or $j(S) \cap S = \emptyset$.

The induced family $\mathfrak{T}_{M/S^1} = \{T_S/S^1 \mid S \in \mathscr{S}_1\}$ is a Thom-Mather system on the orbit space M/S^1. The element $j \in S^3$ induces the map $j\colon M/S^1 \to M/S^1$ preserving $\mathfrak{T}_{M/S^1}$. So, it induces the map $j^*\colon \Omega^*_{\overline{p}}\left(M/S^1\right) \to \Omega^*_{\overline{p}}\left(M/S^1\right)$. Since $j^2 = -1 \in S^1$ then $j^* \circ j^*$ is the identity. So, the group $\mathbb{Z}_2$ acts on $\Omega^*_{\overline{p}}\left(M/S^1\right)$ by $g \cdot \omega = j^*\omega$. We shall write

$$\Omega^*_{\overline{p}}\left(M/S^1\right)^{-\mathbb{Z}_2} = \{\omega \in \Omega^*_{\overline{p}}\left(M/S^1\right) \mid g \cdot \omega = -\omega\}.$$

The space $H^*_{\overline{p}}\left(M/S^1\right)^{-\mathbb{Z}_2}$ is defined in a similar way.

2. Invariant differential forms

A key ingredient in this paper is the complex of S^3-invariant forms of $\Omega^*_{\overline{p}}(M)$. It is a simpler sub-complex computing the same cohomology.

For the rest of this Section we assume that the action $\Phi\colon S^3 \times M \to M$ is a mobile action. In particular, the action of S^3 on $M\backslash\Sigma$ is *almost free*, that is, the isotropy subgroup of any point of $M\backslash\Sigma$ is finite.

2.1. **The Lie algebra $\mathfrak{su}(2)$.** We shall consider $\{u_1, u_2, u_3\}$ an orthogonal basis of the Lie algebra $\mathfrak{su}(2)$ of S^3, relatively to a bi-invariant metric κ of S^3, where

- u_1 generates the Lie algebra of the subgroup S^1.
- $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$, $[u_3, u_1] = u_2$ and
- $\operatorname{Ad}(j)\, u_1 = -u_1, \operatorname{Ad}(j)\, u_2 = u_2, \operatorname{Ad}(j)\, u_3 = -u_3$.

Consider the action $\Psi\colon S^3 \times S^3 \to S^3$, defined by $\Psi(g,k) = k \cdot g^{-1}$. We have on S^3 the fundamental vector fields Y_i associated to u_i, $i = 1,2,3$. They are left invariant vector fields verifying $j_*Y_1 = -Y_1$, $j_*Y_2 = Y_2$ and $j_*Y_3 = -Y_3$.

We shall write $\gamma_i \in \Omega^1(S^3)$ the dual form of Y_i relatively to the metric κ: $\gamma_i = i_{Y_i}\kappa$, $i = 1,2,3$. They are left invariant differential forms verifying $j^*\gamma_1 = -\gamma_1$, $j^*\gamma_2 = \gamma_2$ and $j^*\gamma_3 = -\gamma_3$. The differentials verify $d\gamma_1 = \gamma_2 \wedge \gamma_3$, $d\gamma_2 = -\gamma_1 \wedge \gamma_3$ and $d\gamma_3 = \gamma_1 \wedge \gamma_2$.

2.2. **Fundamental vector fields and characteristic forms.** The *fundamental vector field* associated to $u \in \mathfrak{su}(2)$ is X_u. For the sake of simplicity, we shall write $X_i = X_{u_i}$ with $i = 1,2,3$. This vector field is defined on M but we are going to work with its restriction to $M\backslash\Sigma$. Since the action is mobile then these vector fields are non-vanishing on $M\backslash\Sigma$. Moreover, the family $\{X_1(x), X_2(x), X_2(x)\}$ is a basis of the tangent space of the orbit $S^3(x)$ for any $x \in M\backslash\Sigma$. We have the equalities: $j_*X_1 = j_*X_{u_1} = X_{\mathrm{Ad}\,(j)u_1} = -X_{u_1} = -X_1$ and $j_*X_2 = X_2$, $j_*X_3 = -X_3$ in the same way.

An *adapted metric* on $M\backslash\Sigma$ is a S^3-invariant Riemannian metric μ on $M\backslash\Sigma$ verifying

$$\mu(X_{v_1}(x), X_{v_2}(x)) = \kappa(v_1, v_2) \qquad \forall x \in M\backslash\Sigma \text{ and } v_1, v_2 \in \mathfrak{su}(2). \tag{7}$$

It always exists since the fundamental vector fields are non-vanishing.

We denote by $\chi_u = i_{X_u}\mu \in \Omega^1(M\backslash\Sigma)$ the *characteristic form* associated to $u \in \mathfrak{su}(2)$. Notice that, for each $g \in S^3$, we have

$$g^*\chi_u = \chi_{\mathrm{Ad}\,(g^{-1})\cdot u}. \tag{8}$$

For the sake of simplicity, we shall write $\chi_i = \chi_{u_i}$, for $i = 1,2,3$. Since $L_{X_u}\chi_v = \chi_{[u,v]}$, for each $u, v \in \mathfrak{su}(2)$, then we have

$$\begin{aligned} &L_{X_1}\chi_1 = L_{X_2}\chi_2 = L_{X_3}\chi_3 = 0, && L_{X_1}\chi_2 = -L_{X_2}\chi_1 = -\chi_3 \\ &L_{X_1}\chi_3 = -L_{X_3}\chi_1 = \chi_2 && L_{X_2}\chi_3 = -L_{X_3}\chi_2 = -\chi_1. \end{aligned} \tag{9}$$

Since $\chi_k(X_\ell) = \mu(X_\ell, X_k) = \delta_{\ell k}$, each differential form $\omega \in \Omega^*(M\backslash\Sigma)$ possesses a unique writing,

$$\omega = \omega_0 + \chi_1 \wedge \omega_1 + \chi_2 \wedge \omega_2 + \chi_3 \wedge \omega_3 + \chi_1 \wedge \chi_2 \wedge \omega_{12} + \chi_1 \wedge \chi_3 \wedge \omega_{13} + \chi_2 \wedge \chi_3 \wedge \omega_{23} + \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \omega_{123}, \tag{10}$$

where the coefficients $\omega_\bullet \in \Omega^*(M\backslash\Sigma)$ are *horizontal forms*, that is, they verify $i_{X_\ell}\omega_\bullet = 0$ for each $\ell = 1,2,3$. This is the *canonical decomposition* of ω.

The canonical decomposition of the differential of a characteristic form is

$$d\chi_1 = e_1 + \chi_2 \wedge \chi_3 \qquad d\chi_2 = e_2 - \chi_1 \wedge \chi_3 \qquad d\chi_3 = e_3 + \chi_1 \wedge \chi_2 \tag{11}$$

for some horizontal forms $e_1, e_2, e_3 \in \Omega^2(M\backslash\Sigma)$, called the *Euler forms*. Notice that

$$j^*e_1 = -e_1 \qquad j^*e_2 = e_2 \qquad j^*e_3 = -e_3. \tag{12}$$

2.3. **Invariant differential forms.** A differential form ω of $M\backslash\Sigma$ is an *invariant form* when $g^*\omega = \omega$ for each $g \in S^3$ or, equivalently, $L_{X_\ell}\omega = 0$ for each $\ell = 1,2,3$. In fact, invariant differential forms are characterized by the following conditions:

$$\begin{aligned} &\omega_0 \text{ and } \omega_{123} \text{ are basic forms,} && L_{X_\ell}\omega_\ell = 0, \ell = 1,2,3 && L_{X_1}\omega_{23} = L_{X_2}\omega_{13} = L_{X_3}\omega_{12} = 0 \\ &L_{X_1}\omega_2 = -L_{X_2}\omega_1 = -\omega_3 && L_{X_1}\omega_3 = -L_{X_3}\omega_1 = \omega_2, && L_{X_2}\omega_3 = -L_{X_3}\omega_2 = -\omega_1, \\ &L_{X_1}\omega_{13} = L_{X_2}\omega_{23} = \omega_{12} && L_{X_1}\omega_{12} = -L_{X_3}\omega_{23} = -\omega_{13} && L_{X_2}\omega_{12} = L_{X_3}\omega_{13} = -\omega_{23} \end{aligned}$$

(see (9)).

The complex of invariant forms is denoted by $\underline{\Omega}^*(M\backslash\Sigma)$. The complex of invariant intersection differential forms is $\underline{\Omega}^*_{\overline{p}}(M) = \Omega^*(M\backslash\Sigma) \cap \underline{\Omega}^*_{\overline{p}}(M)$. Some cohomological computations are simpflied by replacing the complex $\Omega^*_{\overline{p}}(M)$ by its subcomplex $\underline{\Omega}^*_{\overline{p}}(M)$, since proceeding as in [9, Theorem I, pag. 151], we have

Proposition 2.1. *The inclusion $\underline{\Omega}^*_{\overline{p}}(M) \hookrightarrow \Omega^*_{\overline{p}}(M)$ is a quasi-isomorphism for any perversity $\overline{p}$.*

Notice that $\Omega^*_{\overline{p}}(M/S^3) = \{\omega \in \underline{\Omega}^*_{\overline{p}}(M) \mid i_{X_\ell}\omega = 0 \text{ for each } \ell = 1,2,3\}$.

2.4. **Perverse degree of characteristic forms.** Notice that, for any singular stratum $S \in \mathscr{S}^{sing}$, we have:

(13) $$||\omega||_S = \max\left\{||\omega_0||_S, ||\mathcal{X}_\ell \wedge \omega_i||_S, ||\mathcal{X}_\ell \wedge \mathcal{X}_k \wedge \omega_{\ell k}||_S, ||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \omega_{123}|| \;\Big|\; 1 \leqslant \ell < k \leqslant 3\right\}$$

for any $\omega \in \Omega^*(M\backslash\Sigma)$ (cf. (5)). When $S \in \mathscr{S}_3$ is a fixed stratum then the orbits of the action are tangent to the fibers of $\tau_S : D_S \to S$. So, we have

(14) $$||\mathcal{X}_\ell \wedge \alpha||_S - 1 = ||\mathcal{X}_\ell \wedge \mathcal{X}_k \wedge \alpha||_S - 2 = ||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \alpha||_S - 3 = ||\alpha||_S,$$

for each $\alpha \in \Omega^*(M\backslash\Sigma)$ and each $1 \leqslant \ell < k \leqslant 3$.

In order to control the perverse degree of characteristic forms relatively to mobile strata we need richer metrics than adapted metrics.

Definition 2.2. *An adapted metric μ on $M\backslash\Sigma$ is an* adjusted metric *if*

(15) $$\mu(X_v(x), w) = 0$$

whenever

- *$x \in D_S\backslash\Sigma$ for some $S \in \mathscr{S}_1$,*
- *w is a vector tangent to the fibers of $\tau_S : D_S\backslash\Sigma \to S$ at x, and*
- *$v \in \mathfrak{su}(2)$ belongs to the κ-orthogonal of $\mathfrak{su}(2)_y$, Lie algebra of S^3_y with $y = \tau_S(x)$.*

Proposition 2.3. *Every mobile action admits an adjusted metric.*

Proof. A convex combination of adapted metrics is an adapted metric. So, by using partitions of unity, we can reduce the problem to the following two cases:

• $M = T_S$ for some $S \in \mathscr{S}_1$. In this case, $\Sigma = S$. We set μ' an adapted metric on $T_S\backslash S$.

We put $\mathcal{K}$ (resp. $\mathcal{G}$) the sub-bundle of $T_S\backslash S$ tangent to the fibers of τ_S (resp. the orbits of the action). The bundle $\mathcal{G} \cap \mathcal{K}$ is of constant rank equal to one. In fact, we have

$$\mathcal{G}_x \cap \mathcal{K}_x = \{X_v(x) \mid v \in \mathfrak{su}(2)_y\}$$

for each $x \in T_S\backslash\Sigma$ with $y = \tau_S(x)$. Let us consider the S^3-invariant decomposition

$$T\left(T_S\backslash\Sigma\right) = \mathcal{D} \oplus \mathcal{K} \oplus (\mathcal{G} + \mathcal{K})^{\perp_{\mu'}},$$

where $\mathcal{D} = (\mathcal{G} \cap \mathcal{K})^{\perp_{\mu'}} \cap \mathcal{G}$. Since $\mu' = \kappa$ on $\mathcal{G}$ (cf. (7)) then we have $\mathcal{D}_x = \{X_v(x) \mid v \in \mathfrak{su}(2)_y^\perp\}$ for each $x \in T_S\backslash\Sigma$ with $y = \tau_S(x)$. We denote by μ'_1, μ'_2, and μ'_3 the restrictions of μ to each term of the above decomposition. The Riemannian metric μ defined by:

$$\mu = \mu'_1 + \mu'_2 + \mu'_3,$$

is an adapted metric. It also satisfies (15) since $w \in \mathcal{K}_x$ and $X_v(x) \in \mathcal{D}_x$.

• $M = T_Q$ for some $Q \in \mathscr{S}_3$. The open subset $T_Q\backslash\Sigma$ is S^3-equivariantly diffeomorphic to $(D_Q\backslash\Sigma)\times]0, \infty[$. The action of S^3 on $D_Q\backslash\Sigma$ has no fixed points. The previous step gives an adjusted metric μ on $D_Q\backslash\Sigma$. As the tubular neighborhood T_S of any stratum $S \in \mathscr{S}_1$ is the product $T_{S\cap(D_Q\backslash\Sigma)}\times]0, \infty[$, the metric $\mu + dr^2$ is an adjusted metric on $T_Q\backslash\Sigma$. □

For a such metric we can compute the terms appearing in formula (13).

Proposition 2.4. *Let us suppose that $M\backslash\Sigma$ is endowed with an adjusted metric. Given a stratum $S \in \mathscr{S}_1$ and a horizontal form $\alpha \in \Omega^*(M\backslash\Sigma)$ we have*

(16) $$||\mathcal{X}_\ell \wedge \alpha||_S = ||\mathcal{X}_\ell \wedge \mathcal{X}_k \wedge \alpha||_S = ||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \alpha||_S = ||\alpha||_S + 1,$$

for each $1 \leqslant \ell < k \leqslant 3$.

Proof. Without loss of generality, we can suppose $M = T_S$ and $\Sigma = S$. We proceed in two steps.

Step $\leqslant$. Following (5) it suffices to prove that $||\mathcal{X}_\ell \wedge \mathcal{X}_k||_S \leqslant 1$ and $||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3||_S \leqslant 1$. We deal with the first inequality, the second one can be approached in the same way. If $||\mathcal{X}_\ell \wedge \mathcal{X}_k||_S = 2$ then there exists $x \in T_S\backslash S$ and $v, w \in \mathcal{K}_x$ with $\mathcal{X}_\ell \wedge \mathcal{X}_k(v, w) \neq 0$. Since $\dim[u_\ell, u_k]^2 = 2 = \dim \mathfrak{su}(2)_y^\perp$ and $\dim \mathfrak{su}(2) = 3$, then there exist $v_1 \in \mathfrak{su}(2)_y^\perp$ and $v_2 \in \mathfrak{su}(2)$ with $\mathcal{X}_{v_1} \wedge \mathcal{X}_{v_2}(v, w) \neq 0$. This is impossible since $\mathcal{X}_{v_1}(v) = \mathcal{X}_{v_1}(w) = 0$ (cf. (15)).

[2]Notation $[-]$ stands for the vector subespace generated by v.

Step $\geqslant$. Since the result is clear for $\alpha = 0$. Let us suppose $||\alpha||_S = a > 0$. So, there exist $x \in T_S\backslash\Sigma$ and $\{w_0, \ldots, w_{a-1}\} \subset \mathcal{K}_x$, with $\alpha(w_0, \ldots, w_{a-1}, -) \neq 0$. Here, $\mathcal{K}$ denotes the sub-bundle of $T(T_S\backslash S)$ tangent to the fibers of τ_S as defined on the proof of the previous Proposition. Since the perverse degree is S^3-invariant, we can suppose that $S^3_{y=\tau_S(x)} \supset S^1$ (cf. (5)). This gives $\mathfrak{su}(2)_y = \lfloor u_1 \rfloor$ and therefore $X_1(x) \in \mathcal{K}_x$.

The adjoint map associated to the group S^3 is the covering $S^3 \to SO(3)$. Since there exists a rotation sending u_1 to u_ℓ then there exists $g \in S^3$ with $\mathrm{Ad}(g)(u_1) = u_\ell$. Since the perverse degree is S^3-invariant then it suffices to prove $||\mathcal{X}_1 \wedge \alpha||_S \geqslant a + 1$, $||\mathcal{X}_1 \wedge \mathcal{X}_u \wedge \alpha||_S \geqslant a + 1$ and $||\mathcal{X}_1 \wedge \mathcal{X}_u \wedge \mathcal{X}_v \wedge \alpha||_S \geqslant a + 1$, where $u, v \in \mathfrak{su}(2)$ (cf. (5) and (8)). Without loss of generality we can suppose that $u, v \in \lfloor u_1 \rfloor^\perp$. The inequality comes from:

$$\begin{aligned} 0 &\neq \alpha(w_0, \ldots, w_{a-1}, -) = (\mathcal{X}_1 \wedge \mathcal{X}_u \wedge \mathcal{X}_v \wedge \alpha)(X_1(x), X_u(x), X_v(x), w_0, \ldots, w_{a-1}, -) \\ &= (\mathcal{X}_1 \wedge \mathcal{X}_u \wedge \alpha)(X_1(x), X_u(x), w_0, \ldots, w_{a-1}, -) = (\mathcal{X}_1 \wedge \alpha)(X_1(x), w_0, \ldots, w_{a-1}, -), \end{aligned}$$

since $\{w_0, \ldots, w_{a-1}, X_1(x)\} \subset \mathcal{K}_x$.

If $a = 0$, we just have $\alpha(-) \neq 0$ and the same argument applies by just omitting the vectors $w_0, \ldots, w_{a-1}$. □

2.5. **Circle actions.** In [10] a Gysin sequence is obtained for any mobile smooth circle action by doing a similar study and using more restrictive perversities. For the convenience of the reader, in this section we obtain that Gysin sequence for general perversities in a shorter way, using the techniques and presentation to be applied later for the case of mobile S^3-actions.

Fix a mobile smooth action $\Psi\colon S^1 \times M \to M$. Here, we just have a fundamental vector field X and a characteristic form $\mathcal{X}$ relatively to an adapted metric (the notion of adjusted metric does not apply here). This form is S^1-invariant and verifies $||\chi||_S = 1$ on $\mathcal{S}_1$ (the family of fixed strata). Its differential $e = d\chi$, the *Euler form*, belongs to $\Omega^2_{\overline{e}}(M/S^1)$ where the *Euler perversity* $\overline{e}$ is defined by $\overline{e}(S) = 2$ on $\mathcal{S}_1$. We also use the *characteristic perversity* χ defined by $\overline{\chi}(S) = 1$ on $\mathcal{S}_1$.

For any perversity $\overline{p}$, the complex $\underline{\Omega}^*_{\overline{p}}(M)$ is

$$\left\{\alpha + \mathcal{X} \wedge \beta \mid \alpha \in \Omega^*\left((M\backslash\Sigma)/S^1\right), \beta \in \Omega^*_{\overline{p}-\overline{\chi}}\left((M\backslash\Sigma)/S^1\right) \text{ with } \left\{ \begin{array}{l} ||\alpha|| \leqslant \overline{p}(S), \text{ and} \\ ||d\alpha + e \wedge \beta||_S \leqslant \overline{p}(S) \end{array} \right. \forall S \in \mathcal{S}_1 \right\}. \tag{17}$$

The *integration operator* is the differential operator $⨍\colon \underline{\Omega}^*_{\overline{p}}(M) \to \Omega^*_{\overline{p}-\overline{\chi}}(M/S^3)$ defined by $⨍\omega = i_X\omega$, that is, $⨍(\alpha + \chi \wedge \beta) = \beta$. Associated to this operator, we have the short exact sequence

$$0 \to K^*_{\overline{p}}(M) = \Omega^*_{\overline{p}}(M/S^1) \to \underline{\Omega}^*_{\overline{p}}(M) \to I^*_{\overline{p}}(M) \to 0,$$

which induces the following Gysin sequence (see [10]).

Proposition 2.5. *For each perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ we have the long exact sequence*

$$\cdots \to H^*_{\overline{p}}(M/S^1) \to H^*(M) \to H^{*-1}_{\overline{p}-\overline{e}}(M/S^1) \to H^{*+1}_{\overline{p}}(M/S^1) \to \cdots.$$

Proof. For each $\beta \in \Omega^*_{\overline{p}-\overline{e}}(M/S^1)$ we have $\chi \wedge \beta \in \underline{\Omega}^*_{\overline{p}}(M)$. Since $⨍(\chi \wedge \beta) = \beta$ then it suffices to prove that the inclusion $I\colon \Omega^*_{\overline{p}-\overline{e}}(M/S^1) \hookrightarrow I^*_{\overline{p}}(M)$ is a quasi-isomorphism. Notice that

$$I^*_{\overline{p}}(M) = \{\beta \in \Omega^*_{\overline{p}-\overline{\chi}}(M/S^1) \mid \exists \alpha \in \Omega^*(M\backslash\Sigma) \text{ with } ||\alpha|| \leqslant \overline{p}(S), \text{ and } ||d\alpha + e \wedge \beta||_S \leqslant \overline{p}(S) \forall S \in \mathcal{S}_1\}.$$

We proceed in three steps.

• *Step 1: $\mathcal{S}_1 = \varnothing$.* The action is almost-free. In this case $\Sigma = \varnothing$ and therefore $\Omega^*\left((M\backslash\Sigma)/S^1\right) = \Omega^*_{\overline{p}}(M/S^1) \subset I^*_{\overline{p}-\overline{e}}(M) \subset \Omega^*_{\overline{p}-\overline{\chi}}(M/S^1) = \Omega^*\left((M\backslash\Sigma)/S^1\right)$. In other words, the map I itself is an isomorphism.

• *Step 2: $M = T_S$ for some $S \in \mathcal{S}_1$.* Recall that $\tau_S : T_S \to S$ is an S^1-invariant smooth bundle whose fiber is $\mathbb{R}^{2b+2}$ for some $b \geqslant 0$. In fact, the group S^1 acts trivially on S and orthogonally on the fiber $\mathbb{R}^{2b+2}$ having the origin as the only fixed point. Notice that the action of S^1 on the unit sphere S^{2b+1} is almost-free.

Consider a good covering $\mathcal{U}$ of S and $\{f_U \mid U \in \mathcal{U}\}$ a subordinated partition of unity. The family $\{\tau_S^{-1}(U), \mid u \in \mathcal{U}\}$ is an open covering of T_S having $\{f_U \circ \tau_S \mid U \in \mathcal{U}\}$ a subordinated partition of unity. These maps are S^1-invariant smooth maps constant on the fibers of τ_S. This last property implies that $||f_U \circ \tau_S||_S = ||d(f_U \circ \tau_S)||_S = 0$. So, the covering $\mathcal{U}$ possesses a subordinated partition of unity living in $\underline{\Omega}^*_{\overline{0}}(M)$.

By applying Bredon's trick [3, p. 289], we can reduce the problem to the case where $M = \mathbb{R}^{\dim S} \times \mathbb{R}^{2b+2}$, with τ_S being the projection onto the first factor. The action of the group S^1 is trivial on the first factor.

Contracting this factor to a point, we reduce the problem to the case $M = \mathbb{R}^{2b+2} = \mathring{c}S^{2b+1} = (S^{2b+1}\times[0,\infty[)/(S^{2b+1}\times\{0\})$. Here, the stratum S is the apex of the cone. We have $\overline{\chi}(S) = 1$ and $\overline{e}(S) = 2$. The number $p \in \overline{\mathbb{Z}}$ is defined by $\overline{p}(S) = p$. We need to prove that the inclusion

$$I\colon \underline{\Omega}^*_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right) \hookrightarrow I^*_{\overline{p}}\left(\mathring{c}S^{2b+1}\right) = \{i_Z\omega \mid \omega \in \Omega^{*+1}_{\overline{p}}\left(\mathring{c}S^{2b+1}\right)\} \tag{18}$$

is a quasi-isomorphism. Notice first that

$$\begin{array}{lcl}
\Omega^{*<p-2}_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right) & = & \Omega^{*<p-2}\left(\left(S^{2b+1}/S^1\right)\times]0,\infty[\right)\\
\Omega^{p-2}_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right) & = & \left\{\beta\in\Omega^{p-2}\left(\left(S^{2b+1}/S^1\right)\times]0,\infty[\right) \mid d\beta\equiv 0 \text{ on } \left(S^{2b+1}/S^1\right)\times]0,2[\right\},\\
\Omega^{*>p-2}_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right) & = & \Omega^{*>p-2}\left(\left(S^{2b+1}/S^1\right)\times]0,\infty[,\left(S^{2b+1}/S^1\right)\times]0,2[\right),\\
I^{*<p-1}_{\overline{p}}\left(\mathring{c}S^{2b+1}\right) & = & \left\{i_X\omega \mid \omega\in\underline{\Omega}^{*+1<p}\left(S^{2b+1}\times]0,\infty[\right)\right\} =_{(1)} \underline{\Omega}^{*<p-1}\left(\left(S^{2b+1}/S^1\right)\times]0,\infty[\right)\\
I^{p-1}_{\overline{p}}\left(\mathring{c}S^{2b+1}\right) & = & \left\{i_X\omega \mid \omega\in\underline{\Omega}^{p}\left(S^{2b+1}\times]0,\infty[\right) \mid d\omega\equiv 0 \text{ on } S^{2b+1}\times]0,2[\right\}, \text{ and}\\
I^{*>p-1}_{\overline{p}}\left(\mathring{c}S^{2b+1}\right) & = & \left\{i_X\omega \mid \omega\in\underline{\Omega}^{*+1>p}\left(S^{2b+1}\times]0,\infty[,S^{2b+1}\times]0,2[\right)\right\}\\
 & =_{(2)} & \Omega^{*>p-1}\left(\left(S^{2b+1}/S^1\right)\times]0,\infty[,\left(S^{2b+1}/S^1\right)\times]0,2[\right),
\end{array}$$

where $=_{(1)}$ is given by the previous step and $=_{(2)}$ comes from the fact that $\beta \equiv 0$ on $\left(S^{2b+1}/S^1\right)\times]0,2[$ implies $\omega = \chi\wedge\beta \equiv 0$ on $S^{2b+1}\times]0,2[$. Since $\Omega^{p-2}_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right)\cap d^{-1}(0) = I^{p-2}_{\overline{p}}\left(\mathring{c}S^{2b+1}\right)\cap d^{-1}(0)$ then it suffices to study the degrees $*\geqslant p-1$.

$\boxed{* = p-1}$ Since $H^{p-1}_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right) = 0$ then we need to prove $H^{p-1}\left(I^*_{\overline{p}}\left(\mathring{c}S^{2b+1}\right)\right) = 0$, that is:

$$\left\{\begin{array}{l}\omega\in\underline{\Omega}^{p}\left(S^{2b+1}\times]0,\infty[\right)\\ \text{with } d\omega\equiv 0 \text{ on } S^{2b+1}\times]0,2[\text{ and } di_X\omega = 0\end{array}\right. \Longrightarrow \left\{\begin{array}{l}\exists\eta\in\underline{\Omega}^{p-1}\left(S^{2b+1}\times]0,\infty[\right)\\ \text{with } i_X\omega = di_X\eta.\end{array}\right.$$

If $p = 0$ then we can consider $\eta = 0$ since ω is constant. Let us suppose $p \geqslant 1$. Consider $\eta' = \int_1^- \omega$. It is an element of $\underline{\Omega}^{p-1}\left(S^{2b+1}\times]0,\infty[\right)$ since the action of S^1 on the $]0,\infty[$-factor is trivial. A straightforward calculation gives $\omega = \mathrm{pr}^*\,\omega(1) + d\eta' + \int_1^- d\omega$. Here, $\omega(1)$ is the restriction of ω to $S^{2b+1}\times\{1\}$ and $\mathrm{pr}: S^{2b+1}\times]0,\infty[\to S^{2b+1}\times\{1\}$ is the map defined by $\mathrm{pr}\,(x,t) = (x,1)$. A straightforward calculation gives

$$i_X\omega = i_X\,\mathrm{pr}^*\,\omega(1) + i_Xd\eta' + i_X\int_1^- d\omega = i_X\,\mathrm{pr}^*\,\omega(1) - di_X\eta' - \int_1^- di_X\omega = i_X\,\mathrm{pr}^*\,\omega(1) - di_X\eta'.$$

By hypothesis the differential form $\mathrm{pr}^*\,\omega(1)$ is a cycle of $\underline{\Omega}^{p}\left(S^{2b+1}\times]0,\infty[\right)$. Condition $\overline{p}\leqslant\overline{t}$ implies $p\leqslant \operatorname{codim} S - 2 = 2b$. Since $p\geqslant 1$, this gives the existence of $\eta''\in\underline{\Omega}^{p-1}\left(S^{2b+1}\times]0,\infty[\right)$ with $\mathrm{pr}^*\,\omega(1) = d\eta''$. We end the proof taking $\eta = -\eta' - \eta''$.

$\boxed{*\geqslant p}$ Since $H^{*\geqslant p}_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right) = 0$ then we need to prove $H^{*\geqslant p}\left(I^*_{\overline{p}}\left(\mathring{c}S^{2b+1}\right)\right) = 0$, that is:

$$\left\{\begin{array}{l}\omega\in\underline{\Omega}^{*+1\geqslant p+1}\left(S^{2b+1}\times]0,\infty[,S^{2b+1}\times]0,2[\right)\\ \text{with } di_X\omega = 0\end{array}\right. \Longrightarrow \left\{\begin{array}{l}\exists\eta\in\underline{\Omega}^{*\geqslant p}\left(S^{2b+1}\times]0,\infty[\right)\\ \text{with } d\eta\equiv 0 \text{ on } S^{2b+1}\times]0,2[\text{and } i_X\omega = di_X\eta.\end{array}\right.$$

Same proof as before with $\omega(1) = 0$.

• *Final Step*. Consider the invariant open covering $\mathcal{V} = \{T_S \mid S\in\mathcal{S}_1\}\sqcup\{M\backslash\Sigma\}$ of M. We fix a smooth map $\lambda\colon[0,\infty[\to[0,1]$ verifying $\lambda = 1$ on $[0,2]$ and $\lambda = 0$ on $[3,\infty[$. The map $f_S : T_S \to [0,\infty[$ is defined by $f_S(x) = \lambda(\nu_S(x))$. It is an S^1-invariant smooth map, constant on the fibers of $\tau_S : D_S\to S$, which gives $||f_S||_S = ||df_S||_S = 0$. So, the family $\{f_S \mid S\in\mathcal{S}_1\}\sqcup\{1-\sum f_S\}$ is a partition of unity, subordinated to $\mathcal{V}$, living in $\underline{\Omega}^*_{\overline{0}}(M)$. Now, it suffices to apply Bredon's trick [3, pag. 289] and the previous cases. □

Remark 2.6. *In fact, we have proved that the operator*

$$f\colon (\Omega^*_{\overline{p}}(M/S^1) \oplus \Omega^{*-1}_{\overline{p}-\overline{e}}(M/S^1), D) \longrightarrow (\underline{\Omega}^*_{\overline{p}}(M), d),$$

defined by $f(\tau, \lambda) = \tau + \chi \wedge \lambda$, where $D(\tau, \lambda) = (d\tau + e \wedge \lambda, -d\lambda)$, is a quasi-isomorphism.

2.6. **Twisted product.** The model of the tubular neighborhood of a semi-mobile stratum is given by twisted products. We present this notion. First of all, we consider a smooth action $\Theta\colon N \times E \to E$ of the normalizer N on a manifold E. It induces the action $\Phi\colon S^3 \times (S^3 \times_N E) \to (S^3 \times_N E)$ defined by $g \cdot \langle k, x\rangle = \langle g \cdot k, x\rangle$.

This action is a mobile action and the strata are of the form $S^3 \times_N S$ with $S \in \mathscr{S}$, stratification induced by Θ. A perversity $\overline{p}$ on E determines a perversity on the twisted product, still denoted $\overline{p}$, defined by $\overline{p}(S^3 \times_N S) = \overline{p}(S)$.

Given an N-invariant Thom-Mather system $\mathfrak{T}_E = \{T_S \mid S \in \mathscr{S}_1\}$ on E we can consider the following Thom-Mather systems

- $\mathfrak{T}_{S^3\times E} = \{(S^3 \times T_S) \mid S \in \mathscr{S}_1\}$ on the product, and
- $\mathfrak{T}_{S^3\times_N E} = \{(S^3 \times_N T_S) \mid S \in \mathscr{S}_1\}$ on the twisted product.

Relatively to these Thom-Mather systems we have the equality

$$||\Pi^*\omega||_{S^3\times S} = ||\omega||_{S^3\times_N S}$$

where $S \in \mathscr{S}_1$, $\omega \in \Omega^*(S^3 \times_N (E\backslash\Sigma)_E)$ and $\Pi\colon S^3 \times E \to S^3 \times_N E$ is the canonical projection. This map is an N-bundle and verifies $\pi(g, x) = \pi(g \cdot h^{-1}, h \cdot x)$ where $(g, x) \in S^3 \times E$ and $h \in N$.

The goal of this Section is to write the S^3-invariant intersection forms of the twisted product $S^3 \times_N E$ in terms of the intersection forms of E.

First we establish some notation.

(a) $Y_u \in \mathfrak{X}(S^3)$ is the fundamental vector field associated to $u \in \mathfrak{su}(2)$ relatively to the right action: $S^3 \times S^3 \to S^3; (g, k) \mapsto k \cdot g^{-1}$. It is a left invariant vector field . For the sake of simplicity we shall write $Y_{u_\ell} = Y_\ell$ for $\ell \in \{1, 2, 3\}$.
(b) $Z \in \mathfrak{X}(E)$ is the fundamental vector field of the action: $\Psi\colon S^1 \times E \to E$, induced by Θ. It verifies $j_*Z = -Z$. Let ρ be an N-invariant metric on E. The *characteristic form* $\zeta = \iota_Z\rho$ verifies $j^*\zeta = -\zeta$ and the associated *Euler form* $e = d\zeta$ verifies $j^*e = e$.
(c) Let $\gamma_u \in \Omega^1(S^3)$ be the dual form of Y_u, that is, $\gamma_u = i_{Y_u}\kappa$, $u \in \mathfrak{su}(2)$. Notice that $\kappa(u, v) = \gamma_u(Y_v)$. These forms are invariant by the left action of S^3. For the sake of simplicity we shall write $\gamma_{u_k} = \gamma_k$ for $\ell \in \{1, 2, 3\}$. They verify $L_{Y_1}\gamma_1 = 0, L_{Y_1}\gamma_2 = -\gamma_3, L_{Y_1}\gamma_3 = \gamma_2, d\gamma_1 = \gamma_2 \wedge \gamma_3, d\gamma_2 = -\gamma_1 \wedge \gamma_3$ and $d\gamma_3 - \gamma_1 \wedge \gamma_2$ (cf. (9)).
(d) The group N acts on the complex of differential forms $\Omega^*(S^3)$ by the left. So, the group $\mathbb{Z}_2 = N/S^1$ acts on the complex of S^1-left invariant forms of S^3, which is $\bigwedge^*(\gamma_1, \gamma_2, \gamma_3)$. This action is given by

$$g \cdot \gamma_\ell = (-1)^\ell \gamma_\ell, \tag{19}$$

for $\ell = 1, 2, 3$.

Proposition 2.7. *Using the natural projection Π we get the identification*

$$\underline{\Omega}^*_{\overline{p}}(S^3 \times_N E) = \left\{\omega \in \bigwedge\nolimits^*(\gamma_1, \gamma_2, \gamma_3) \otimes \underline{\Omega}^*_{\overline{p}}(E) \;\middle|\; \begin{array}{l} i_{Y_1}\omega = -i_Z\omega \\ L_{Y_1}\omega = -L_Z\omega \end{array}\right\}^{\mathbb{Z}_2}$$

Proof. Since the map Π is S^3-invariant then Π^* induces a monomorphism between

$$\Pi^*\colon \underline{\Omega}^*(S^3 \times E) = \{\omega \in \Omega^*(S^3 \times E) \mid g^*\omega = \omega \;\forall g \in S^3\} = \bigwedge\nolimits^*(\gamma_1, \gamma_2, \gamma_3) \otimes \Omega^*(E)$$

and

$$\underline{\Omega}^*(S^3 \times_N E) = \{\omega \in \Omega^*(S^3 \times_N E) \mid g^*\omega = \omega \;\forall g \in S^3\}.$$

So, we can identify $\underline{\Omega}^*(S^3_N \times E)$ with

$$\left\{\omega \in \bigwedge\nolimits^*(\gamma_1, \gamma_2, \gamma_3) \otimes \Omega^*(E) \;\middle|\; \begin{array}{l} i_{Y_1}\omega = -i_Z\omega \\ L_{Y_1}\omega = -L_Z\omega \\ g^*\omega = \omega \end{array}\right\} = \left\{\omega \in \bigwedge\nolimits^*(\gamma_1, \gamma_2, \gamma_3) \otimes \Omega^*(E) \;\middle|\; \begin{array}{l} i_{Y_1}\omega = -i_Z\omega \\ L_{Y_1}\omega = -L_Z\omega \end{array}\right\}^{\mathbb{Z}_2}$$

A similar identification is obtained for $E\backslash\Sigma_E$ instead E. Using (2.6) we get (2.7). □

Corollary 2.8. *We have the identification*

$$\underline{\Omega}^*_{\overline{p}}\left(S^3 \times_N E\right) = \underline{\Omega}^*_{\overline{p}}(E)^{\mathbb{Z}_2} \oplus \underline{\Omega}^{*-2}_{\overline{p}}(E)^{-\mathbb{Z}_2} \oplus \{\xi \in \Omega^{*-1}_{\overline{p}}(E)^{-\mathbb{Z}_2} \mid L_Z L_Z \xi = -\xi\}.$$

where the differential becomes $D'(\alpha,\beta,\xi) = (d\alpha, d\beta - i_Z\alpha, -d\xi)$. The third term of this direct sum is acyclic.

Proof. The equality (2.7) gives that an element of $\underline{\Omega}^*_{\overline{p}}\left(S^3 \times_N E\right)$ is of the form

$$\alpha - \gamma_1 \wedge i_Z\alpha + \gamma_2 \wedge L_Z\xi + \gamma_3 \wedge \xi + \gamma_1 \wedge \gamma_2 \wedge i_Z L_Z \xi + \gamma_1 \wedge \gamma_3 \wedge i_Z\xi + \gamma_2 \wedge \gamma_3 \wedge \beta - \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge i_Z\beta.$$

with $\alpha \in \underline{\Omega}^*_{\overline{p}}(E), \beta \in \underline{\Omega}^{*-2}_{\overline{p}}(E), \xi \in \Omega^{*-1}_{\overline{p}}(E)$, $g \cdot \alpha = \alpha$, $g \cdot \xi = -\xi$, $g \cdot \beta = -\beta$, and $L_Z L_Z \xi = -\xi$. For the calculation of D we compute the differential of the previous expression:

$$d\alpha - \gamma_1 \wedge i_Z d\alpha - \gamma_2 \wedge L_Z d\xi - \gamma_3 \wedge d\xi - \gamma_1 \wedge \gamma_2 \wedge \left(i_Z L_Z d\xi + \xi\right) + \gamma_1 \wedge \gamma_3 \wedge \left(L_Z\xi - i_Z d\xi\right) +$$

$$\gamma_2 \wedge \gamma_3 \wedge \left(d\beta - i_Z\alpha\right) + \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge i_Z d\beta.$$

We verify the the acyclicity property. Let $\xi \in \Omega^*_{\overline{p}}(E)^{-\mathbb{Z}_2}$ be a cycle. The differential form $\eta = i_Z L_Z \xi \in \Omega^*(E \backslash \Sigma)$ verifies

- $g \cdot \eta = j^*\eta = j^* i_Z L_Z \xi = -i_Z j^* L_Z \xi = i_Z L_Z j^* \xi = -i_Z L_Z \xi = -\eta$.
- $L_Z L_Z \eta = L_Z L_Z i_Z L_Z \xi = i_Z L_Z L_Z L_Z \xi = -i_Z L_Z \xi = -\eta$.
- $d\eta = d i_Z L_Z \xi = L_Z L_Z \xi = -\xi$, since $d\xi = 0$.
- If Q is a singular stratum of E_s we have $||\eta||_Q = ||i_Z L_Z \xi||_Q \overset{(5)}{\leqslant} ||\xi||_Q \leqslant \overline{p}(Q)$.

So, the complex $\{\xi \in \Omega^{*-1}_{\overline{p}}(E)^{-\mathbb{Z}_2} \mid L_Z L_Z \xi = -\xi\}$ is acyclic. □

The following calculations will be used in the next Section. We use the N-action presented in Section 1.6.

Corollary 2.9. *Let $\Theta\colon N \times M \to M$ be a smooth action. We have*

$$H^*_{\overline{p}/\overline{p}-\overline{e}}\left(M/S^1\right)^{-\mathbb{Z}_2} = H^{*-2q}(\Sigma)^{-(-1)^q\mathbb{Z}_2}.$$

where

- *$\overline{p}$ is a constant perversity p on M/S^1 verifying $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$,*
- *q denotes the integer part of $p/2$, and*
- *Σ is the singular part of the induced S^1-action,*

Proof. Consider an N-invariant Thom-Mather system (cf. Section 1.6). Using Mayer-Vietoris, we can suppose $M = T_S \cap T_{j(S)}$, for some $S \in \mathscr{S}_1$. We write $\tau = \tau_S \cup \tau_{j(S)}$. Since $\Sigma = S \cup j(S)$ is $\mathbb{Z}_2$-invariant the we can consider the operator

$$J\colon \Omega^{*-2q}(\Sigma)^{-(-1)^q\mathbb{Z}_2} \longrightarrow \Omega^*_{\overline{p}/\overline{p}-\overline{e}}\left(M/S^1\right)^{-\mathbb{Z}_2},$$

defined by $J(\alpha) =< \tau_S^*\alpha \wedge e^q >$, where e is the Euler form of the induced S^1-action relatively to an N-invariant Riemannian metric on M. This metric always exists since the group N is compact. It is a well defined differential operator since

- $||\tau^*\alpha \wedge e^q||_S \leqslant ||\tau^*\alpha||_S + ||e^q||_S \leqslant 0 + ||e^q||_S \leqslant^{2b+1} 2q \leqslant \overline{p}(S)$, and similarly $||\tau^*\alpha \wedge e^q||_S \leqslant \overline{p}(j(S))$,
- $d(\tau^* d\alpha \wedge e^q) = \tau^* d\alpha \wedge e^q$ and the operator is a differential operator,
- $i_X(\tau^*\alpha \wedge e^q) = i_X d(\tau^*\alpha \wedge e^q) = 0$, which gives $\tau^*\alpha \wedge e^q \in \Omega^*\left((M \backslash S)/S^1\right)$, and
- $j^* e = -e$.

This last property comes from $j_* X = -X$, where X is the fundamental vector field of the circle action Ψ.

We claim that this operator is a quasi-isomorphism. Proceeding as in the proof of the Proposition 2.5 we can reduce the question to the case where the stratum S (resp. $j(S)$) is the apex $\mathtt{v}$ (resp. $\mathtt{w}$) of the cone $T_S = \mathring{c}_{\mathtt{v}} S^{2b+1}$ (resp. $T_{j(S)} = \mathring{c}_{\mathtt{w}} S^{2b+1}$) where the circle S^1 acts orthogonally and almost freely on the sphere S^{2b+1}. We also have $H^*\left(S^{2b+1}/S^1\right) = H^*\left(\mathbb{C}P^b\right)$. Recall that $\overline{e}(S) = 2$ and $\overline{p}(S) = p$. We distinguish two cases.

$\boxed{* \leqslant p-2 \text{ or } * > p}$. We have the equality $\Omega^*_{\overline{p}-\overline{e}}\left(\mathring{c}_{\mathtt{v}} S^{2b+1}/S^1\right) = \Omega^*_{\overline{p}}\left(\mathring{c}_{\mathtt{v}} S^{2b+1}/S^1\right)$, similarly for $\mathtt{w}$, and therefore $\Omega^*_{\overline{p}/p-\overline{e}}\left((T_S \cup T_{j(S)})/S^1\right) = 0 = \Omega^{*-2q}(\{\mathtt{v},\mathtt{w}\})^{(-1)^q\mathbb{Z}_2} = \Omega^{*-2q}(S \cup j(S))^{(-1)^q\mathbb{Z}_2}$.

$\boxed{* = p-1 \text{ or } p}$. This gives

$$H^{*-2q}(S \cup j(S))^{-(-1)^q\mathbb{Z}_2} = \begin{cases} H^0(S)^{-(-1)^q\mathbb{Z}_2} = 0 \text{ (if } q \text{ even) or } \mathbb{R} \text{ (if } q \text{ odd)} & \text{if } j(S) = S \\ H^0(S) = \mathbb{R} & \text{if } j(S) \cap S = \varnothing \end{cases}$$

In the $\mathbb{R}$-case the cohomology is generated by 1 in each term. On the other hand, the typical conical calculations of the intersection cohomology (see for example [5]) and the sequence (6) give $H^*_{p/p-\overline{e}}\left(\mathring{c}_{\mathrm{v}} S^{2b+1}/S^1\right) = H^{2q}\left(\mathbb{C}P^b\right)$ generated by e^q, since $0 \leqslant p = \overline{p}(S) \leqslant \overline{t}(S) = 2b$. And similarly for $\mathtt{w}$. So,

$$H^*_{\overline{p}/\overline{p}-\overline{e}}\left((\mathring{c}_{\mathrm{v}} S^{2b+1} \cup \mathring{c}_{\mathrm{v}} S^{2b+1})/S^1\right)^{-\mathbb{Z}_2} = \begin{cases} H^{2q}\left(\mathbb{C}P^b\right)^{-\mathbb{Z}_2} = 0 \text{ (if } q \text{ even) or } \mathbb{R} \text{ (if } q \text{ odd)} & \text{if } j(S) = S \\ H^{2q}\left(\mathbb{C}P^b\right) = \mathbb{R} & \text{if } j(S) \cap S = \varnothing. \end{cases}$$

We get that J is a quasi-isomorphism. □

3. The integration operator $⨍$.

The main tool we use in this work is the integration operator

$$⨍ : \underline{\Omega}^*_{\overline{p}}(M) \longrightarrow \Omega^{*-3}_{\overline{p}-\overline{\chi}}\left(M/S^3\right),$$

defined by $⨍_3 \omega = (-1)^{\deg \omega}\, i_{X_3} i_{X_2} i_{X_1} \omega$, where $\overline{\chi}$ is the *characteristic perversity* defined by $\overline{\chi}(S) = \begin{cases} 1 & \text{if } S \in \mathscr{S}_1 \\ 3 & \text{if } S \in \mathscr{S}_3 \end{cases}$.

The operator $⨍$ is a well defined differential operator since

- $L_A i_B = i_B L_A + i_{[A,B]}$ when A, B are vector fields of $M\backslash\Sigma$ and
- $\overline{p}(S) \geqslant ||\omega||_S \overset{(10),(13)}{\geqslant} ||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge i_{X_3} i_{X_2} i_{X_1}\omega||_S \overset{(14),(16)}{=} ||i_{X_3} i_{X_2} i_{X_1}\omega||_S + \overline{\chi}(S) = ||⨍\omega||_S + \overline{\chi}(S)$ for each $S \in \mathscr{S}$,

where we have considered an adjusted metric μ on $M\backslash\Sigma$.

The goal of this Section is the computation of the cohomology of the complexes $I^*_{\overline{p}}(M)$ and $K^*_{\overline{p}}(M)$. For the first one, we need to introduce the *Euler perversity* $\overline{e}$, defined by $\overline{e}(S) = \begin{cases} 2 & \text{if } S \in \mathscr{S}_1 \\ 4 & \text{if } S \in \mathscr{S}_3. \end{cases}$

For the sake of simplicity we shall write

$$\begin{aligned} \ker ⨍ &= K^*_{\overline{p}}(M) = \left\{\omega \in \underline{\Omega}^*_{\overline{p}}(M) \mid i_{X_3} i_{X_2} i_{X_1}\omega = 0\right\} \\ \mathrm{Im}\, ⨍ &= I^*_{\overline{p}}(M) = \left\{i_{X_3} i_{X_2} i_{X_1}\omega \mid \omega \in \underline{\Omega}^{*+1}_{\overline{p}}(M)\right\}. \end{aligned}$$

Proposition 3.1. *Let $\overline{p} \leqslant \overline{t}$ be a perversity on M. The natural inclusion $I\colon \Omega^*_{\overline{p}-\overline{e}}\left(M/S^3\right) \hookrightarrow I^*_{\overline{p}}(M)$ is a quasi-isomorphism.*

Proof. The inclusion makes sense if we prove that $\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \alpha \in \underline{\Omega}^*_{\overline{p}}(M)$ for each $\alpha \in \Omega^{*-3}_{\overline{p}-\overline{e}}\left(M/S^3\right)$. This comes from

$$L_{X_\ell}(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \alpha) = 0 \text{ for each } \ell \in \{1,2,3\} \text{ (cf. (9))},$$

and for each $S \in \mathscr{S}_1 \sqcup \mathscr{S}_3$:

$$\begin{aligned} ||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \alpha||_S &\overset{(14),(16)}{=} ||\alpha||_S + \overline{\chi}(S) \leqslant \overline{p}(S) - \overline{e}(S) + \overline{\chi}(S) \leqslant \overline{p}(S) \\ ||d(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \alpha)||_S &\overset{(10),(13)}{\leqslant} \max(||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge d\alpha||_S, ||d(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3) \wedge \alpha||_S) \leqslant \max(\overline{p}(S), ||\alpha||_S \\ &\quad + ||d(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3)||_S) \overset{(9)}{\leqslant} \max(\overline{p}(S), \overline{p}(S) - \overline{e}(S) + ||\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3||_S + 1) \\ &\quad \overset{(14),(16)}{\leqslant} \max(\overline{p}(S), \overline{p}(S) - \overline{e}(S) + \overline{e}(S)) = \overline{p}(S) \end{aligned}$$

(cf. (5)).

In order to prove that I is a quasi-isomorphism, we proceed in several steps. We use the S^3-invariant Thom-Mather system of Section 1.2.

• *Step 1: $\mathscr{S}_1 = \mathscr{S}_3 = \emptyset$.* The action is almost-free. In this case $\Sigma = \emptyset$ and therefore $\Omega^*\left((M\backslash\Sigma)/S^1\right) = \Omega^*_{\overline{p}}\left(M/S^1\right) \subset I^*_{\overline{p}-\overline{e}}(M) \subset \Omega^*_{\overline{p}-\overline{\chi}}\left(M/S^1\right) = \Omega^*\left((M\backslash\Sigma)/S^1\right)$. In other words, the map I itself is an isomorphism.

• *Step 2: $M = T_S$ for some $S \in \mathscr{S}_1$.* We have seen that $S = S^3 \times_N S^{S^1}$. The restriction $\tau_S : E_S = \tau_S^{-1}(S^{S^1}) \to S^{S^1}$ is a N-invariant bundle. Notice that E_S is a filtered space whose singular strata are the connected component of S^{S^1}. The fiber of this bundle is an $\mathbb{R}^{2b+2}$, for some $b \in \mathbb{N}$. The group S^1 acts trivially on S^{S^1} and S^1 orthogonally on $\mathbb{R}^{2b+2}$ having the origin as the only fixed point. Notice that the action of S^1 on the unit sphere S^{2b+1} is almost-free.

We identify T_S with the twisted product $S^3 \times_N E_S$ and we use the calculations of Section 2.6. We have

$$\Omega^*_{\overline{p}}\left(M/S^3\right) = \Omega^*_{\overline{p}}(E_S/N) = \Omega^*_{\overline{p}}\left(E_S/S^1\right)^{\mathbb{Z}_2} \tag{20}$$

for any perversity $\overline{p}$. The integration $\fint$ becomes the map

$$\fint : \underline{\Omega}^*_{\overline{p}}\left(S^3 \times_N E_S\right) \longrightarrow \Omega^{*-3}_{\overline{p}-\overline{\chi}}\left(E_S/S^1\right)^{\mathbb{Z}_2}, \tag{21}$$

defined by $(\alpha,\beta,\xi) \mapsto -i_Z\beta$ (cf. Corollary 2.8). The map I becomes the inclusion

$$I\colon \Omega^*_{\overline{p}-\overline{e}}\left(M/S^3\right) = \Omega^*_{\overline{p}-\overline{e}}\left(E_S/S^1\right)^{\mathbb{Z}_2} \longrightarrow I^*_{\overline{p}}(M) = \left\{ i_Z\beta \mid \beta \in \underline{\Omega}^{*+1}_{\overline{p}}(E_S)^{-\mathbb{Z}_2} \right\} \tag{22}$$

since for each $\lambda \in \Omega^{*-3}_{\overline{p}-\overline{e}}\left(E_S/S^1\right)^{\mathbb{Z}_2}$ we have $\zeta \wedge \lambda \in \underline{\Omega}^{*-2}_{\overline{p}}(E_S)^{-\mathbb{Z}_2}$ and $i_Z(\zeta \wedge \lambda) = \lambda$.

Consider now a good covering $\mathcal{U}$ of S^{S^1} and $\{f_U \mid U \in \mathcal{U}\}$ a subordinated partition of unity. The family $\{\tau_S^{-1}(U), \mid u \in \mathcal{U}\}$ is an open covering of E_S having $\{\tau_S \circ f_U \mid U \in \mathcal{U}\}$ a subordinated partition of unity. These maps are N-invariant smooth maps constant on the fibers of τ_S. This last property implies that $||f_U \circ \tau_S||_S = ||d(f_U \circ \tau_S)||_S = 0$. So, the covering $\mathcal{U}$ possesses a subordinated partition of unity living in $\underline{\Omega}^*_{\overline{0}}(E_S)$.

Using Bredon's trick [3, pag. 289] one reduces the problem to the case $E_S = \mathbb{R}^{\dim S^{S^1}} \times \mathbb{R}^{2b+2}$. where τ_S becomes the projection on the first factor. The action of the group S^1 is trivial on the first factor.

Contracting this factor to a point, we reduce he problem to the case $E_S = \mathbb{R}^{2b+2} = \mathring{c}S^{2b+1} = (S^{2b+1} \times [0,\infty[)/(S^{2b+1} \times \{0\})$ as filtered space. Here, S is the apex of the cone. We have $\overline{\chi}(P) = 1, \overline{e}(P) = 2$ and $\overline{p}(P) = \overline{p}(S) = p$, for any connected component P of S^{S^1} and $p \in \overline{\mathbb{Z}}$. We need to prove that the inclusion

$$I\colon \underline{\Omega}^*_{\overline{p}-\overline{e}}\left(\mathring{c}S^{2b+1}/S^1\right)^{\mathbb{Z}_2} \hookrightarrow I^*_{\overline{p}}\left(\mathring{c}S^{2b+1}\right) = \left\{ i_Z\beta \mid \beta \in \underline{\Omega}^{*+1}_{\overline{p}}\left(\mathring{c}S^{2b+1}\right)^{-\mathbb{Z}_2} \right\},$$

is a quasi-isomorphism. This comes directly from (18) with the equality $j_*Z = -Z$ (cf. Section 2.6 (b)).

• *Step 3: $\mathscr{S}_3 = \emptyset$.* Consider the invariant open covering $\mathcal{V} = \{T_S \mid S \in \mathscr{S}_1\} \sqcup \{M\backslash F_1\}$ of M. We fix a smooth map $\lambda\colon [0,\infty[\to [0,1]$ verifying $\lambda = 1$ on $[0,2]$ and $\lambda = 0$ on $[3,\infty[$. The map $f_S : T_S \to [0,\infty[$ is defined by $f_S(x) = \lambda(\nu_S(x))$. It is an S^3-invariant smooth map, constant on the fibers of $\tau_S : D_S \to S$, which gives $||f_S||_S = ||df_S||_S = 0$. So, the family $\{f_S \mid S \in \mathscr{S}_1\} \sqcup \{1 - \sum f_S\}$ is a partition of unity, subordinated to $\mathcal{V}$, living in $\underline{\Omega}^*_{\overline{0}}(M)$. Now, it suffices to apply the Bredon's trick [3, pag. 289] and the previous cases.

• *Step 4: $M = T_Q$ for some $Q \in \mathscr{S}_3$.* Recall that $\tau_Q\colon T_S \to Q$ is an S^3-invariant smooth bundle whose fiber is $\mathbb{R}^{f+1}$ for some $f \geqslant 3$. In fact, the group S^3 acts trivially on S and orthogonally on the fiber $\mathbb{R}^{f+1}$ having the origin as the only fixed point. Notice that the action of S^3 on the sphere S^f is a mobile action.

Consider now a good covering $\mathcal{U}$ of Q and $\{f_U \mid U \in \mathcal{U}\}$ a subordinated partition of unity. The family $\{\tau_Q^{-1}(U), \mid u \in \mathcal{U}\}$ is an open covering of T_Q having $\{f_U \circ \tau_Q \mid U \in \mathcal{U}\}$ a subordinated partition of unity. These maps are S^3-invariant smooth maps constant on the fibers of τ_Q. This last property implies that $||f_U \circ \tau_Q||_S = ||d(f_U \circ \tau_Q)||_S = 0$ for each singular stratum S (cf. (1.2)).

Using the Bredon's trick [3, pag. 289] one reduces the problem to the case $M = \mathbb{R}^{\dim Q} \times \mathbb{R}^{f+1}$. The action of the group S^3 is trivial on the first factor.

Contracting this factor to a point, we reduce the problem to the case $M = \mathbb{R}^{f+1} = \mathring{c}S^f = (S^f \times [0,\infty[)/(S^f \times \{0\})$. Here, the stratum Q is the apex of the cone. We have $\overline{\chi}(Q) = 3$ and $\overline{e}(Q) = 4$. The number $p \in \overline{\mathbb{Z}}$ is defined by $\overline{p}(Q) = p$. We need to prove that the inclusion

$$I\colon \underline{\Omega}^*_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right) \hookrightarrow I^*_{\overline{p}}\left(\mathring{c}S^f\right) = \left\{ i_{X_3}i_{X_2}i_{X_1}\omega \mid \omega \in \underline{\Omega}^{*+3}_{\overline{p}}\left(\mathring{c}S^f\right) \right\}.$$

is a quasi-isomorphism. Notice first that

$$\begin{array}{lcl}
\Omega^{*<p-4}_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right) &=& \Omega^{*<p-4}_{\overline{p}-\overline{e}}\left(\left(S^f/S^3\right)\times]0,\infty[\right)\\
\Omega^{p-4}_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right) &=& \left\{\beta\in\Omega^{p-4}_{\overline{p}-\overline{e}}\left(\left(S^f/S^3\right)\times]0,\infty[\right)\mid d\beta\equiv 0 \text{ on } \left(S^f/S^3\right)\times]0,2[\right\},\\
\Omega^{*>p-2}_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right) &=& \Omega^{*>p-2}_{\overline{p}-\overline{e}}\left(\left(S^f/S^3\right)\times]0,\infty[,\left(S^f/S^3\right)\times]0,2[\right),\\
I^{*<p-3}_{\overline{p}}\left(\mathring{c}S^f\right) &=& \left\{i_{X_3}i_{X_2}i_{X_1}\omega\mid\omega\in\underline{\Omega}^{*+3<p}_{\overline{p}}\left(S^f\times]0,\infty[\right)\right\}\\
I^{p-3}_{\overline{p}}\left(\mathring{c}S^f\right) &=& \left\{i_{X_3}i_{X_2}i_{X_1}\omega\mid\omega\in\underline{\Omega}^{p}_{\overline{p}}\left(S^f\times]0,\infty[\right)\mid d\omega\equiv 0 \text{ on } S^f\times]0,2[\right\}, \text{ and}\\
I^{*>p-3}_{\overline{p}}\left(\mathring{c}S^f\right) &=& \left\{i_{X_3}i_{X_2}i_{X_1}\omega\mid\omega\in\underline{\Omega}^{*+3>p}_{\overline{p}}\left(S^f\times]0,\infty[,S^f\times]0,2[\right)\right\}.
\end{array}$$

Since $\Omega^{p-4}_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right)\cap d^{-1}(0)=\Omega^{p-4}_{\overline{p}-\overline{e}}\left(\left(S^f/S^3\right)\times]0,\infty[\right)\cap d^{-1}(0)$ then Step 3 gives that $I^*\colon H^*_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right)\to H^*\left(I^{\cdot}_{\overline{p}}\left(\mathring{c}S^f\right)\right)$ is an isomorphism for $*\geqslant p-3$.

$\boxed{*=p-3}$ Since $H^{p-3}_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right)=0$ (cf. [17]) then we need to prove $H^{p-3}\left(I^*_{\overline{p}}\left(\mathring{c}S^f\right)\right)=0$, that is:

$$\left\{\begin{array}{l}\omega\in\underline{\Omega}^{p}_{\overline{p}}\left(S^f\times]0,\infty[\right)\\ \text{with } d\omega\equiv 0 \text{ on } S^f\times]0,2[\text{ and } di_X\omega=0\end{array}\right. \Longrightarrow \left\{\begin{array}{l}\exists\eta\in\underline{\Omega}^{p-1}_{\overline{p}}\left(S^f\times]0,\infty[\right)\\ \text{with } i_X\omega=di_X\eta.\end{array}\right.$$

If $p=0$ then we can consider $\eta=0$ since ω is constant. Let us suppose $p\geqslant 1$. Consider $\eta'=\int_1^-\omega$. It is an element of $\underline{\Omega}^{p-1}\left(S^f\times]0,\infty[\right)$ since the action of S^3 on the $]0,\infty[$-factor is trivial. A straightforward calculation gives

$$\omega=\operatorname{pr}^*\omega(1)+d\eta'+\int_1^- d\omega.$$

Here, $\omega(1)$ is the restriction of ω to $S^f\times\{1\}$ and $\operatorname{pr}\colon S^f\times]0,\infty[\to S^f\times\{1\}$ is the map defined by $\operatorname{pr}(x,t)=(x,1)$. A well known calculation gives

$$\begin{array}{rcl}
i_{X_3}i_{X_2}i_{X_1}\omega &=& i_{X_3}i_{X_2}i_{X_1}\operatorname{pr}^*\omega(1)+i_{X_3}i_{X_2}i_{X_1}d\eta'+i_{X_3}i_{X_2}i_{X_1}\displaystyle\int_1^- d\omega\\
&=& i_{X_3}i_{X_2}i_{X_1}\operatorname{pr}^*\omega(1)-di_{X_3}i_{X_2}i_{X_1}\eta'-\displaystyle\int_1^- di_{X_3}i_{X_2}i_{X_1}\omega=i_{X_3}i_{X_2}i_{X_1}\operatorname{pr}^*\omega(1)-di_{X_3}i_{X_2}i_{X_1}\eta'.
\end{array}$$

By hypothesis the differential form $\operatorname{pr}^*\omega(1)$ is a cycle of $\underline{\Omega}^{p}_{\overline{p}}\left(S^f\times]0,\infty[\right)$. Condition $\overline{p}\leqslant\overline{t}$ implies $p\leqslant\operatorname{codim}Q-2=f-1$. This gives the existence of $\eta''\in\underline{\Omega}^{p-1}\left(S^f\times]0,\infty[\right)$ with $\operatorname{pr}^*\omega(1)=d\eta''$. We need the proof taking $\eta=-\eta'-\eta''$.

$\boxed{*\geqslant p-2}$. Since $H^{*\geqslant p-2}_{\overline{p}-\overline{e}}\left(\mathring{c}S^f/S^3\right)=0$ (cf. [17]) then we need to prove $H^{*\geqslant p-2}\left(I^*_{\overline{p}}\left(\mathring{c}S^f\right)\right)=0$, that is:

$$\left\{\begin{array}{l}\omega\in\underline{\Omega}^{*+3\geqslant p+1}_{\overline{p}}\left(S^f\times]0,\infty[\right) \text{ with } di_{X_3}i_{X_2}i_{X_1}\omega=0\\ \text{and } \omega\equiv 0 \text{ on } S^f\times]0,2[\end{array}\right. \Longrightarrow \left\{\begin{array}{l}\exists\eta\in\underline{\Omega}^{*+2\geqslant p}_{\overline{p}}\left(S^f\times]0,\infty[\right) \text{ with } i_{X_3}i_{X_2}i_{X_1}\omega=di_{X_3}i_{X_2}i_{X_1}\eta\\ \text{and } d\eta\equiv 0 \text{ on } S^f\times]0,2[.\end{array}\right.$$

Same proof as before with $\omega(1)=0$.

• *Final Step*. Consider the invariant open covering $\mathcal{V}=\{T_Q\mid Q\in\mathcal{S}_3\}\sqcup\{M\backslash F_3\}$ of M. We fix a smooth map $\lambda\colon[0,\infty[\to[0,1]$ verifying $\lambda=1$ on $[0,2]$ and $\lambda=0$ on $[3,\infty[$. The map $f_Q\colon T_Q\to[0,\infty[$ is defined by $f_Q(x)=\lambda(\nu_Q(x))$. It is an S^3-invariant smooth map, constant on the fibers of $\tau_Q\colon D_Q\to Q$, which gives $||f_Q||_S=||df_Q||_S=0$ for each singular stratum S (cf. (1.2)). So, the family $\{f_Q\mid Q\in\mathcal{S}_3\}\sqcup\{1-\sum f_Q\}$ is a partition of unity, subordinated to $\mathcal{V}$, living in $\underline{\Omega}^*_{\overline{0}}(M)$. Now, it suffices to apply the Bredon's trick [3, pag. 289] and the previous cases. □

Let us study the complex $K^*_{\overline{p}}(M)$. Since the complex $\Omega^*_{\overline{p}}\left(M/S^3\right)$ is included in *ker* $\oint$ we have the short exact sequence

$$0\to\Omega^*_{\overline{p}}\left(M/S^3\right)\hookrightarrow K^*_{\overline{p}}(M)\to\frac{K^*_{\overline{p}}(M)}{\Omega^*_{\overline{p}}(M/S^3)}\to 0.$$

This allows us to compute the the cohomology of $K^*_{\overline{p}}(M)$ in terms of the intersection cohomology $H^*_{\overline{p}}\left(M/S^3\right)$ and the cohomology of the complex $C^*_{\overline{p}}(M) = \dfrac{K^*_{\overline{p}}(M)}{\Omega^*_{\overline{p}}(M/S^3)}$. In fact, we are going to prove that this cohomology is residual relatively to the strata of $\mathscr{S}_1$. The calculation of $H^*\left(C^{\cdot}_{\overline{p}}(M)\right)$ is carried out in several steps through the following restrictions

$$M \rightsquigarrow T_Q \sqcup T_S \rightsquigarrow Q \sqcup T_S \rightsquigarrow Q \sqcup S^{S^1} = \overline{S}^{S^1}$$

where S ranges over the strata of $\mathscr{S}_1$ and $Q \in \mathscr{S}_3$ with $\overline{Q} \subset S$. We proceed in several steps.

Lemma 3.2. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action with $\mathscr{S}_3 = \emptyset$. For any perversity $\overline{p}$ on M with $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ we have*

$$H^*\left(C^{\cdot}_{\overline{p}}(M)\right) = \bigoplus_{S \in \mathscr{S}_1} H^{*-2p_S-2}\left(S^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2},$$

where the number p_S denotes the integer part of $\overline{p}(S)/2$.

Proof. We proceed in several steps.

• *Step 1: $\mathscr{S}_1 = \emptyset$.* We do no have any singular stratum. We need to prove that the LHS of the equality (3.2) is 0. Let $\langle\omega\rangle$ be a cycle of $C^*_{\overline{p}}(M)$. The canonical decomposition of ω is

$$\omega = \omega_0 + \mathcal{X}_1 \wedge \omega_1 + \mathcal{X}_2 \wedge \omega_2 + \mathcal{X}_3 \wedge \omega_3 + \mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \omega_{12} + \mathcal{X}_1 \wedge \mathcal{X}_3 \wedge \omega_{13} + \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \omega_{23}$$

(cf. (10)). The differential form $\eta = -\mathcal{X}_1 \wedge \omega_{23} + \mathcal{X}_2 \wedge \omega_{13} - \mathcal{X}_3 \wedge \omega_{12}$ belongs to $K^*_{\overline{p}}(M)$ since $\eta \in \underline{\Omega}^*(M\backslash\Sigma)$ (cf. (9) and (2.3)), and $i_{X_3} i_{X_2} i_{X_1}\eta = 0$.

The differential form $\omega' = \omega - d\eta$ verifies $\omega'_{12} = \omega'_{13} = \omega'_{23} = \omega'_{123} = 0$. Since $d\omega' = d\omega \in \Omega^*_{\overline{p}}\left(M/S^3\right)$ then $\omega'_1 = \omega'_2 = \omega'_3 = 0$ and therefore $\omega' = \omega'_0 \in \Omega^*_{\overline{p}}\left(M/S^3\right)$. So, $[\langle\omega\rangle] = [\langle\omega'\rangle] = [0] = 0$.

• *Step 2: $M = T_S$ for some $S \in \mathscr{S}_1$.* In this case $\mathscr{S}_1 = \{S\}$ where S is a closed stratum. Using (21) we get

$$\begin{aligned} K^*_{\overline{p}}(M) &= \underline{\Omega}^*_{\overline{p}}(E_S)^{\mathbb{Z}_2} \oplus \left\{\beta \in \underline{\Omega}^{*-2}_{\overline{p}}(E_S)^{-\mathbb{Z}_2} \mid i_Z\beta = 0\right\} \oplus \left\{\xi \in \Omega^{*-1}_{\overline{p}}(E_S)^{-\mathbb{Z}_2} \mid L_Z L_Z \xi = -\xi\right\} \\ &\overset{(17)}{=} \underline{\Omega}^*_{\overline{p}}(E_S)^{\mathbb{Z}_2} \oplus \Omega^{*-2}_{\overline{p}}\left(E_S/S^1\right)^{-\mathbb{Z}_2} \oplus \left\{\xi \in \Omega^{*-1}_{\overline{p}}(E_S)^{-\mathbb{Z}_2} \mid L_Z L_Z \xi = -\xi\right\} \end{aligned}$$

and $\Omega^*_{\overline{p}}\left(M/S^3\right) \overset{(20)}{=} \Omega^*_{\overline{p}}\left(E_S/S^1\right)^{\mathbb{Z}_2}$. Following Proposition 2.8 it suffices to compute the cohomology of the quotient

$$\frac{\underline{\Omega}^*_{\overline{p}}(E_S)^{\mathbb{Z}_2} \oplus \underline{\Omega}^{*-2}_{\overline{p}}\left(E_S/S^1\right)^{-\mathbb{Z}_2}}{\Omega^*_{\overline{p}}(E_S/S^1)^{\mathbb{Z}_2}},$$

relatively to the differential $D''\langle(\alpha,\beta)\rangle = \langle(d\alpha, d\beta - i_Z\alpha)\rangle$. Following Remark 2.6 (a) this complex is quasi-isomorphic to

$$\frac{\Omega^*_{\overline{p}}\left(E_S/S^1\right)^{\mathbb{Z}_2} \oplus \Omega^{*-1}_{\overline{p}-\overline{e}}\left(E_S/S^1\right)^{-\mathbb{Z}_2} \oplus \Omega^{*-2}_{\overline{p}}\left(E_S/S^1\right)^{-\mathbb{Z}_2}}{\Omega^*_{\overline{p}}(E_S/S^1)^{\mathbb{Z}_2}} = \Omega^{*-1}_{\overline{p}-\overline{e}}\left(E_S/S^1\right)^{-\mathbb{Z}_2} \oplus \Omega^{*-2}_{\overline{p}}\left(E_S/S^1\right)^{-\mathbb{Z}_2}$$

endowed with the differential $D''(\lambda,\beta) = (-d\lambda, d\beta - \lambda)$. This complex is quasi-isomorphic to $\Omega^{*-2}_{\overline{p}/\overline{p}-\overline{e}}\left(E_S/S^1\right)^{-\mathbb{Z}_2}$. The induced action $\Theta\colon N \times E_S \to E_S$ and the induced perversity $\overline{p}$ on E_S verify the conditions of Corollary 2.9, with $\Sigma = S^{S^1}$ and $q = p_S$. So, we get $H^{*-2}_{\overline{p}/\overline{p}-\overline{e}}\left(E_S/S^1\right)^{-\mathbb{Z}_2} = H^{*-2p_S-2}\left(S^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$.

• *Final step.* Using Mayer-Vietoris as in the the Step 3 of the proof of the Proposition 3.1. □

The first step of (3) comes from the following result.

Lemma 3.3. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{p}$ on M with $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ the restriction induces the quasi-isomorphism $C^*_{\overline{p}}(M) \to C^*_{\overline{p}}(T_{\Sigma_1} \cup T_{\Sigma_3})$.*

Proof. Consider the invariant open covering $\{U = T_{\Sigma_1} \cup T_{\Sigma_3}, V = M \backslash \left(\nu_1^{-1}([0,3/4]) \cup \nu_3^{-1}([0,3/4])\right)\}$ of M. We have the Gysin sequence:

$$0 \longrightarrow C^*_{\overline{p}}(M) \longrightarrow C^*_{\overline{p}}(U) \oplus C^*_{\overline{p}}(V) \longrightarrow C^*_{\overline{p}}(U \cap V) \longrightarrow 0$$

Following Lemma 3.2 we know that the complexes $C^*_{\overline{p}}(V)$ and $C^*_{\overline{p}}(U \cap V)$ are acyclic. So, the restriction $C^*_{\overline{p}}(M) \to C^*_{\overline{p}}(U)$ is a quasi-isomorphism. □

The term $Q \sqcup T_S$ appearing in (3) is not a manifold, but it is possible to define the complex $C^*_{\overline{p}}(-)$ on it using the following notion.

Definition 3.4. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. We consider a perversity $\overline{p}$ on M. For each open subset $U \subset M$ we define*

$$\Xi^*_{\overline{p}}(U) = \{\omega \in \underline{\Omega}^*_{\overline{p}}(U \backslash \Sigma_3) \mid \omega \text{ and } d\omega \text{ verify condition (23)}\},$$

where this condition is

(23) $\quad \omega(v_0, \ldots, v_{\overline{p}(Q)}, -) = 0$ *where $v_0, \ldots, v_{\overline{p}(Q)}$ are vectors tangent to the fibers of $\tau_Q\colon (D_Q \cap U) \to Q$,*

for each $Q \in \mathscr{S}_3$. We analogously define $\Xi^*_{\overline{p}}\left(U/S^3\right)$ if $U \subset M$ is an S^3-invariant open subset.

We define

$$\widehat{C}^*_{\overline{p}}(T_S) = \frac{\widehat{K}^*_{\overline{p}}(T_S) = K^*_{\overline{p}}(T_S) \cap \Xi^*_{\overline{p}}(T_S)}{\Xi^*_{\overline{p}}(T_S/S^3)}.$$

We clearly have $\Xi^*_{\overline{p}}(U) = \underline{\Omega}^*_{\overline{p}}(U)$ and $\widehat{C}^*_{\overline{p}}(U) = C^*_{\overline{p}}(U)$ if $U \supset T_{\Sigma_3}$ or if $\mathscr{S}_3 = \emptyset$. In particular, we have

$$C^*_{\overline{p}}(T_{\Sigma_1} \cup T_{\Sigma_3}) = \widehat{C}^*_{\overline{p}}(T_{\Sigma_1} \cup T_{\Sigma_3}). \tag{24}$$

Lemma 3.5. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{p}$ on M with $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ the restriction induces the quasi-isomorphism $\widehat{C}^*_{\overline{p}}(T_{\Sigma_3} \cup T_{\Sigma_1}) \to \widehat{C}^*_{\overline{p}}(T_{\Sigma_1})$.*

Proof. Using Mayer-Vietoris, it suffices to prove that the restriction $\widehat{C}^*_{\overline{p}}(T_{\Sigma_3}) \to \widehat{C}^*_{\overline{p}}(T_{\Sigma_3} \cap T_{\Sigma_1})$ is a quasi-isomorphism. Proceeding as in the Step 4 of the proof of Proposition 3.1 we can suppose that $T_{\Sigma_3} = \mathring{c}S^f$ where S^3 acts orthogonally without fixed points on the sphere S^f and trivially on the radius of the cone. The subset Σ_3 is the apex v of the cone. Moreover, we have $\widehat{C}^*_{\overline{p}}(T_{\Sigma_3}) = \widehat{C}^*_{\overline{p}}\left(S^f \times]0,\infty[\right)$ and $\widehat{C}^*_{\overline{p}}(T_{\Sigma_3} \cap T_{\Sigma_1}) = \widehat{C}^*_{\overline{p}}\left(]0,\infty[\times (S^f \cap T_{\Sigma_1})\right)$. In this context, condition (23) becomes

$$\omega(v_0, \ldots, v_p, -) = 0 \text{ where } v_0, \ldots, v_p \text{ are vectors of } (S^f \backslash \Sigma_1) \times]0,2[\text{ (resp. } ((S^f \cap T_{\Sigma_1} \times]0,2[) \backslash \Sigma_1)).$$

Here, $p = \overline{p}(\mathsf{v})$. In order to prove that the restriction $\widehat{C}^*_{\overline{p}}\left(S^f \times]0,\infty[\right) \to \widehat{C}^*_{\overline{p}}\left((S^f \cap T_{\Sigma_1}) \times]0,\infty[\right)$ is a quasi-isomorphism we notice that

$$\begin{aligned}
\widehat{C}^{*<p}_{\overline{p}}\left(S^f \times]0,\infty[\right) &= C^{*<p}_{\overline{p}}\left(S^f \times]0,\infty[\right) \\
\widehat{C}^{p}_{\overline{p}}\left(S^f \times]0,\infty[\right) &= \left\{\langle\beta\rangle \in C^{*<p}_{\overline{p}}\left(S^f \times]0,\infty[\right) \mid d\beta \equiv 0 \text{ on } \left(S^f/S^3\right) \times]0,2[\right\}, \\
\widehat{C}^{*>p}_{\overline{p}}\left(S^f \times]0,\infty[\right) &= C^{*>p}_{\overline{p}}\left(S^f \times]0,\infty[, S^f \times]0,2[\right),
\end{aligned}$$

Let us consider the operator $\mathrm{pr} : C^*_{\overline{p}}\left(S^f\right) \to C^*_{\overline{p}}\left(S^f \times]0,\infty[\right)$ induced by the canonical projection. Following (3) we get $\langle\omega\rangle = \mathrm{pr}\,\langle\omega(1)\rangle + d\langle\int_-^1 \omega\rangle + \langle\int_-^1 d\omega\rangle$. We get that the operator $\mathrm{pr} : \tau_p C^*_{\overline{p}}\left(S^f\right) = C^{<p}_{\overline{p}}\left(S^f\right) \oplus \left(C^p_{\overline{p}}\left(S^f\right) \cap d^{-1}(0)\right) \to \widehat{C}^*_{\overline{p}}\left(S^f \times]0,\infty[\right)$, induced by the canonical projection, is a quasi-isomorphism. In a similar way we prove that the operator

$$\mathrm{pr} : \tau_p C^*_{\overline{p}}\left((S^f \cap T_{\Sigma_1})\right) \to \widehat{C}^*_{\overline{p}}\left((S^f \cap T_{\Sigma_1}) \times]0,\infty[\right), \tag{25}$$

induced by the canonical projection, is a quasi-isomorphism. So, the question becomes to prove that the restriction $\tau_p C^*_{\overline{p}}\left(S^f\right) \to \tau_p \widehat{C}^*_{\overline{p}}\left(S^f \cap T_{\Sigma_1}\right)$ induces a quasi-isomorphism, which is granted by Lemma 3.2. □

By the definition of the complex $\widehat{C}^*_{\overline{p}}(-)$ we have the equality $\widehat{C}^*_{\overline{p}}(T_{\Sigma_1}) = \oplus_{S \in \mathscr{S}_1} \widehat{C}^*_{\overline{p}}(T_S)$.

Lemma 3.6. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M and any stratum $S \in \mathscr{S}_1$ we have*
$$H^*\left(\widehat{C}_{\overline{p}}(T_S)\right) = H^{*-2p_S-2}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}.$$

Proof. Consider the family $\mathcal{Q} = \{Q \in \mathscr{S}_3 \mid Q \subset \overline{S}\}$. Recall that $\overline{S}^{S^1}$ (cf. Section 1.4) is a filtered space. The regular part (resp. singular strata) of $\overline{S}^{S^1}$ is S^{S^1} (resp. are $Q \in \mathcal{Q}$). So,
$$\Omega^*_{\overline{P_S}}\left(\overline{S}^{S^1}\right) = \left\{\alpha \in \Omega^*\left(S^{S^1}\right) \mid \alpha \text{ and } d\alpha \text{ verify condition (23) for } \overline{P}_S(Q) = \overline{p}(Q) - 2p_S - 2 \text{ where } Q \in \mathcal{Q}\right\}.$$
Let us define the operator
$$J_S : \Omega^{*-2p_S-2}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \to \widehat{C}^*_{\overline{p}}(T_S)$$
by

(26) $$J_S(\alpha) = \left\langle \gamma_1 \wedge \gamma_2 \wedge \tau_S^*\alpha \wedge \zeta_S^{p_S}\right\rangle,$$

where $\zeta_S \in \Omega^2_{\overline{2}}\left(E_S/S^1\right)^{-\mathbb{Z}_2}$ the Euler form of the S^1-action on E_S relatively to an N-invariant metric on T_S (cf. Section 2.6).

• *Step 1: The operator J_S is well defined.* Since $\tau_S : T_S \to S^{S^1}$ is an S^3-equivariant map then we have $\tau_S^*\alpha \in \underline{\Omega}^{*-2p_S-2}(E_S)^{-(-1)^{p_S}\mathbb{Z}_2}$ for each $\alpha \in \Omega^{*-2p_S-2}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$ and therefore $\tau_S^*\alpha \wedge \zeta_S^{p_S} \in \underline{\Omega}^{*-2}(E_S)^{-\mathbb{Z}_2}$. For the perverse degree, we have
$$||\tau_S^*\alpha \wedge \zeta_S^{p_S}||_S \leqslant ||\zeta_S^{p_S}||_S \leqslant 2p_S \leqslant \overline{p}(S),$$
and similarly for the differential $d(\tau_S^*\alpha \wedge \zeta_S^{p_S}) = \tau_S^* d\alpha \wedge \zeta_S^{p_S}$. We conclude that $\tau_S^*\alpha \wedge \zeta_S^{p_S} \in \underline{\Omega}^*_{\overline{p}}(E_S)^{-\mathbb{Z}_2}$. Applying (2.8) we get that $\gamma_2 \wedge \gamma_3 \wedge \tau_S^*\alpha \wedge e_S^{p_S} \in \underline{\Omega}^*_{\overline{p}}\left(T_S = S^3 \times_N E_S\right)$ with $⨍ \gamma_2 \wedge \gamma_3 \wedge e_S^{p_S} \wedge \tau_S^*\alpha = 0$.

It remains to prove that the differential form $\eta = \gamma_2 \wedge \gamma_3 \wedge \tau_S^*\alpha \wedge e_S^{p_S}$ and its differential $d\eta = \gamma_2 \wedge \gamma_3 \wedge \tau_S^* d\alpha \wedge e_S^{p_S}$ verify condition (23) for $\overline{p}(Q)$, where $Q \in \mathcal{Q}$. Consider a family $v_0, \ldots, v_{\overline{p}(Q)}$ of vectors tangent to the fibers of $\tau_Q\colon (D_Q \cap (T_S \backslash S)) \to Q$. Up to a reordering we get that $\eta(v_0, \ldots, v_{\overline{p}(Q)})$ is a multiple of $\alpha(\tau_{S,*}(v_0), \ldots, \tau_{S,*}(v_{\overline{p}(Q)-2p_S-2}))$. Condition (1.2) implies that the vectors $\tau_{S,*}(v_\bullet)$ are tangent to the fibers of $\tau_Q\colon D_Q \cap S \to Q$. Since α verifies condition (23) for $\overline{P_S}(Q)$ the we get $\alpha(\tau_{S,*}(v_0), \ldots, \tau_{S,*}(v_{\overline{p}(Q)-2p_S-2=\overline{P_S}(Q)})) = 0$. So, η verifies condition (23) for $\overline{p}(Q)$. Same argument applies to $d\eta$.

We conclude that $\gamma_2 \wedge \gamma_3 \wedge \tau_S^*\alpha \wedge e_S^{p_S} \in \ker ⨍$. The operator J_S is well defined.

• *Step 2: The operator J_S is a quasi-isomorphism when $M = T_Q$ for some $Q \in \mathscr{S}_3$ with $Q \subset \overline{S}$.* Proceeding as in the Step 4 of the proof of Proposition 3.1 we can suppose that $M = T_Q = \mathring{c}S^f$ where S^3 acts orthogonally without fixed points on the sphere S^f and trivially on the radius of the cone. Notice that the action of S^3 on the sphere S^f is a mobile action. The stratum Q is the apex $\mathtt{v}$ of the cone.

Since $\overline{S} = \mathring{c}(S \cap S^f)$ and $\overline{S}^{S^1} = \mathring{c}(S^{S^1} \cap S^f)$ then J_S becomes
$$J_S : \Omega^{*-2p_S-2}_{\overline{P_S}}\left(\mathring{c}(S^{S^1} \cap S^f)\right)^{-(-1)^{p_S}\mathbb{Z}_2} \to \widehat{C}^*_{\overline{p}}\left((S^f \cap T_S)\times]0,\infty[\right)$$
is a quasi-isomorphism. We know from [17] that the operator $\mathrm{pr} : \tau_q\Omega^*_{\overline{P_S}}\left(S^{S^1} \cap S^f\right) \to \Omega^*_{\overline{P_S}}\left(\mathring{c}(S^{S^1} \cap S^f)\right)$, induced by the canonical projection, is a quasi-isomorphism. Here, $q = \overline{P_S}(Q) = \overline{p}(\mathtt{v}) - 2p_S - 2 = p - 2p_S - 2$. Using (25) we conclude that it suffices to prove that
$$J_S : \Omega^{*-2p_S-2}_{\overline{P_S}}\left(S^{S^1} \cap S^f\right)^{-(-1)^{p_S}\mathbb{Z}_2} \to C^*_{\overline{p}}\left(S^f \cap T_S\right)$$
is a quasi-isomorphism. Following Lemma 3.2, we get the claim.

• *Final step.* Consider the invariant open covering $\mathcal{V} = \left\{T_S \cap T_3, T_S \backslash \nu_3^{-1}([0,2])\right\}$ of T_S. We fix a smooth map $\lambda\colon [0,\infty[\to [0,1]$ verifying $\lambda = 1$ on $[0,3]$ and $\lambda = 0$ on $[4,\infty[$. The map $f\colon T_S \to [0,\infty[$ is defined by $f(x) = \lambda(\nu_3(x))$.

It is an S^3-invariant smooth map, constant on the fibers of $\tau_3\colon D_3 \to \Sigma_3$, which gives $||f||_S = ||df||_S = ||f||_Q = ||df||_Q = 0$ for each $Q \in \mathscr{S}_3$ with $Q \subset \overline{S}$. So, the family $\{f, 1-f\}$ is a partition of unity, subordinated to $\mathcal{V}$, living in $\underline{\Omega}^*_{\overline{0}}(T_S)$. Now, it suffices to apply the Bredon's trick [3, pag. 289] and the previous cases. □

Proposition 3.7. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M we have*

$$H^*\left(C^*_{\overline{p}}(M)\right) = \bigoplus_{S \in \mathscr{S}_1} H^{*-2p_S-2}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}.$$

Proof. It suffices to consider Lemma 3.3, (24), Lemma 3.5 and Lemma 3.6. □

4. Gysin braid for a mobile action

We construct two Gysin sequences associated with a mobile action $\Phi\colon S^3 \times M \to M$. These sequences establish a relationship between the cohomology of the manifold M and the intersection cohomology of the orbit space M/S^3. The existence of two distinct approaches to the cohomology of M by the intersection cohomology of M/S^3 gives rise to two separate sequences: one from the left, and one from the right.

The first version, the left one, uses the short exact sequence

$$0 \to \Omega^*_{\overline{p}}(M/S^3) \to \underline{\Omega}^*_{\overline{p}}(M) \to \frac{\underline{\Omega}^*_{\overline{p}}(M)}{\Omega^*_{\overline{p}}(M/S^3)} \to 0,$$

where $\overline{p}$ is a perversity on M. The quotient $G^*_{\overline{p}}(M) = \dfrac{\underline{\Omega}^*_{\overline{p}}(M)}{\Omega^*_{\overline{p}}(M/S^3)}$ is the *Gysin term*.

Theorem A. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M we have the long exact sequence, known as a* Gysin Sequence*,*

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-1}\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*_{\overline{p}}(M/S^3) \longrightarrow H^*(M) \longrightarrow \cdots.$$

The cohomology of the Gysin term is determinated by the long exact sequence

$$\cdots \to H^{*-1}\left(G^{\cdot}_{\overline{p}}(M)\right) \to H^{*-4}_{\overline{p}-\overline{e}}(M/S^3) \to \bigoplus_{S \in \mathscr{S}_1} H^{*-2p_S-2}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \to H^*\left(G^{\cdot}_{\overline{p}}(M)\right) \to \cdots. \tag{27}$$

Proof. The first long exact sequence comes from (4) and from the fact that the complex $\underline{\Omega}^*_{\overline{p}}(M)$ computes the cohomology of M since $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ (cf. Section 1.3 and Proposition 2.1). We now consider the the short exact sequence

$$0 \longrightarrow C^*_{\overline{p}}(M) \longrightarrow G^*_{\overline{p}}(M) \xrightarrow{\overline{\oint}} I^{*-3}_{\overline{p}}(M) \longrightarrow 0, \tag{28}$$

where $\overline{\oint}\langle\omega\rangle = \oint\omega$. The second long exact sequence comes from the fact that the cohomology of $I^*_{\overline{p}}(M)$ is $H^*_{\overline{p}-\overline{e}}(M/S^3)$ (cf. Proposition 3.1) and from Proposition 3.7. □

The second version, the right one, uses the short exact sequence

$$0 \to K^*_{\overline{p}}(M) \to \underline{\Omega}^*_{\overline{p}}(M) \to I^{*-3}_{\overline{p}}(M) \to 0. \tag{29}$$

By symmetry, we say that the complex $K^*_{\overline{p}}(M)$ is the *co-Gysin term* of the action.

Theorem B. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M we have the long exact sequence, known as a* Gysin Sequence*,*

$$\cdots \longrightarrow H^{*-1}(M) \longrightarrow H^{*-4}_{\overline{p}-\overline{e}}(M/S^3) \longrightarrow H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*(M) \longrightarrow \cdots \tag{30}$$

The cohomology of the co-Gysin term is determinated by the long exact sequence

$$\cdots \longrightarrow H^{*-1}\left(K^*_{\overline{p}}(M)\right) \longrightarrow \bigoplus_{S\in\mathscr{S}_1} H^{*-3-2p_S}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \longrightarrow H^*_{\overline{p}}\left(M/S^3\right) \longrightarrow H^*\left(K^*_{\overline{p}}(M)\right) \longrightarrow \cdots . \tag{31}$$

Proof. The first long exact sequence is derived from (29) and the following two facts:

- The complex $\underline{\Omega}^*_{\overline{p}}(M)$ computes the cohomology of M, since $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ (cf. Section 1.3 and Proposition 2.1).
- The cohomology of the complex $I_{\overline{p}}(M)$ is $H_{\overline{p}-\overline{e}}\left(M/S^3\right)$ (cf. Proposition 3.1).

The second long exact sequence is derived from the short exact sequence

$$0 \to \Omega^*_{\overline{p}}\left(M/S^3\right) \to K^*_{\overline{p}}(M) \to C^*_{\overline{p}}(M) \to 0 \tag{32}$$

and Proposition 3.7. □

Four previous long exact sequences (A), (27), (30) and (31) can be gathered through in an exact braid diagram.

Definition 4.1. *Let us consider six chain complexes A^*, B^*, C^*, D^*, E^* and F^*. A* braid *is a diagram of chain maps of the form*

$$\begin{tikzcd}
A^* \arrow[rr, bend left, "\textcircled{1}"] \arrow[dr, "\textcircled{4}"] & & B^* \arrow[rr, bend left, "\textcircled{3}"] \arrow[dr, "\textcircled{1}"] & & C^* \arrow[rr, bend left, "\textcircled{2}"] \arrow[dr, "\textcircled{3}"] & & D^{*+1} \arrow[rr, bend left, "\textcircled{4}"] \arrow[dr, "\textcircled{2}"] & & A^{k+2} \\
& E^* \arrow[ur, "\textcircled{3}"] \arrow[dr, "\textcircled{4}"] & & F^* \arrow[ur, "\textcircled{2}"] \arrow[dr, "\textcircled{1}"] & \mathit{Txirikorda} & E^{*+1} \arrow[ur, "\textcircled{4}"] \arrow[dr, "\textcircled{3}"] & & F^{*+1} \arrow[ur, "\textcircled{1}"] \arrow[dr, "\textcircled{2}"] & \\
C^{*-1} \arrow[ur, "\textcircled{3}"] \arrow[rr, bend right, "\textcircled{2}"'] & & D^* \arrow[ur, "\textcircled{2}"] \arrow[rr, bend right, "\textcircled{4}"'] & & A^{*+1} \arrow[ur, "\textcircled{4}"] \arrow[rr, bend right, "\textcircled{1}"'] & & B^{*+1} \arrow[ur, "\textcircled{1}"] \arrow[rr, bend right, "\textcircled{3}"'] & & C^{*+1}.
\end{tikzcd}$$

It is a commutative braid *when all the triangles and diamonds are commutative. If the long sequences ①, ②, ③ and ④ are exact we say that braid is an* exact braid.

An exact and commutative braid possesses the two following properties.

B1- The following long sequence

$$\cdots \longrightarrow E^* \xrightarrow{(\textcircled{3},\textcircled{4})} B^* \oplus D^* \xrightarrow{\textcircled{1}-\textcircled{2}} F^* \xrightarrow{\textcircled{3}\textcircled{2}} E^{*+1} \longrightarrow \cdots$$

is exact (see for example [7, pag. 39-41]).

B2- The top and bottom sequences of the braid are semi-exact sequences and both have the same exactness defaults: $\ker \textcircled{3} / \operatorname{Im} \textcircled{1} = \ker \textcircled{4} / \operatorname{Im} \textcircled{2}, \ldots$ (see for example [12, pag.148]).

Theorem C. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M we have the exact commutative braid, the* Gysin braid*:*

$$\begin{tikzcd}
H^*_{\overline{p}}(M/S^3) \arrow[rr, bend left, "\textcircled{1}"] \arrow[dr, "\textcircled{4}"] & & H^*(M) \arrow[rr, bend left, "\textcircled{3}"] \arrow[dr, "\textcircled{1}"] & & H^{*-3}_{\overline{p}-\overline{e}}(M/S^3) \arrow[rr, bend left, "\textcircled{2}"] \arrow[dr, "\textcircled{3}"] & & \bigoplus\limits_{S\in\mathscr{S}_1} H^{*-1-2p_S}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \\
& H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "\textcircled{3}"] \arrow[dr, "\textcircled{4}"] & & H^*\left(G^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "\textcircled{2}"] \arrow[dr, "\textcircled{1}"] & & H^{*+1}\left(K^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "\textcircled{4}"] \arrow[dr, "\textcircled{3}"] & \\
H^{*-4}_{\overline{p}-\overline{e}}(M/S^3) \arrow[ur, "\textcircled{3}"] \arrow[rr, bend right, "\textcircled{2}"'] & & \bigoplus\limits_{S\in\mathscr{S}_1} H^{*-2p_S-2}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \arrow[ur, "\textcircled{2}"] \arrow[rr, bend right, "\textcircled{4}"'] & & H^{*+1}_{\overline{p}}(M/S^3) \arrow[ur, "\textcircled{4}"] \arrow[rr, bend right, "\textcircled{1}"'] & & H^{*+1}(M),
\end{tikzcd}$$

Proof. In [19], a braid is constructed from a triple. We follow this method to construct a braid associated with the following three complexes: $\Omega^*_{\overline{p}}\left(M/S^3\right) \subset K^*_{\overline{p}}(M) \subset \underline{\Omega}^*_{\overline{p}}(M)$. Recall that the cohomology of $\underline{\Omega}^*_{\overline{p}}(M)$ is $H^*(M)$ (cf. Section 1.3). To recognize the relative terms, we can do the following:

- The quotient $\dfrac{\Omega^*_{\overline{p}}(M)}{\Omega^*_{\overline{p}}(M/S^3)}$ is the Gysin term $G^{\cdot}_{\overline{p}}(M)$.
- We can determine the cohomology of $\dfrac{\Omega^*_{\overline{p}}(M)}{K^*_{\overline{p}}(M)}$ by referring to (29) and Proposition 3.1. It is given by $H^*_{\overline{p}-\overline{e}}\big(M/S^3\big)$.
- The cohomology of $\dfrac{K^*_{\overline{p}}(M)}{\Omega^*_{\overline{p}}(M/S^3)}$ is $\bigoplus_{S\in\mathscr{S}_1} H^{*-2p_S-2}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$, as stated in Proposition 3.7. (cf. Proposition 3.7).

□

In some cases, the sequence (27) splits and the sequence (A) is closer to the classical Gysin sequence. In particular, we find the Gysin sequence (3) of [13].

Corollary 4.2. *Let $\Phi\colon S^3\times M\to M$ be a mobile action. When $\overline{p}=\overline{0}$ the long exact sequence* (27) *splits on the connecting homomorphism and we have the long exact sequence*

$$\cdots\longrightarrow H^{*-1}(M)\longrightarrow H^{*-4}\big(M/S^3,\Sigma/S^3\big)\oplus H^{*-3}\left(M^{S^1}\right)^{-\mathbb{Z}_2}\longrightarrow H^*\big(M/S^3\big)\longrightarrow H^*(M)\longrightarrow\cdots.$$

Proof. Since $\overline{p}=\overline{0}$ then $p_S=0$ and $\overline{P_S}=\overline{-2}$ for each $S\in\mathscr{S}_1$. We have,

$$\begin{aligned}(33)\qquad \bigoplus_{S\in\mathscr{S}_1} H^{*-2p_S-3}_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} &= \bigoplus_{S\in\mathscr{S}_1} H^{*-3}_{\overline{-2}}\left(\overline{S}^{S^1}\right)^{-\mathbb{Z}_2} \overset{[5,\text{ Prop. } 13.5]}{=} \bigoplus_{S\in\mathscr{S}_1} H^{*-3}\left(\overline{S}^{S^1},\overline{S}^{S^3}\right)^{-\mathbb{Z}_2}\\ &= H^{*-3}\left(M^{S^1},M^{S^3}\right)^{-\mathbb{Z}_2} = H^{*-3}\left(M^{S^1}\right)^{-\mathbb{Z}_2}\end{aligned}$$

since $H^{*-3}\left(M^{S^3}\right)^{-\mathbb{Z}_2}=0$.

Consider a cycle $\omega\in I^*_{\overline{0}}(M)$ and compute the connecting homomorphism $\delta[\omega]$ of the sequence (27). We have seen in the proof of Proposition 3.1 that $\chi_3\wedge\chi_2\wedge\chi_1\wedge\omega\in\underline{\Omega}^{*+3}_{\overline{p}}(M)$ with $⨍(\chi_3\wedge\chi_2\wedge\chi_1\wedge\omega)=\omega$. Using (11) we get

$$d(\chi_3\wedge\chi_2\wedge\chi_1)\wedge\omega=(e_1^2+e_2^2+e_3^2)\wedge\omega-d((e_3\wedge\chi_3+e_2\wedge\chi_2+e_1\wedge\chi_1)\wedge\omega).$$

Since $\omega\in\Omega^*_{\overline{0}-\overline{\chi}}\big(M/S^3\big)$ then ω vanishes on $D_1\sqcup D_3$ and therefore, so does any multiple of ω. So, $(e_1^2+e_2^2+e_3^2)\wedge\omega\in\Omega^*_{\overline{0}-\overline{\chi}}\big(M/S^3\big)$ and $(e_3\wedge\chi_3+e_2\wedge\chi_2+e_1\wedge\chi_1)\wedge\omega\in\underline{\Omega}^*_{\overline{0}}(M)$. This gives

$$\begin{aligned} d\langle\chi_3\wedge\chi_2\wedge\chi_1\wedge\omega\rangle &= \langle d(\chi_3\wedge\chi_2\wedge\chi_1)\wedge\omega\rangle=\langle(e_1^2+e_2^2+e_3^2)\wedge\omega-d((e_3\wedge\chi_3+e_2\wedge\chi_2+e_1\wedge\chi_1)\wedge\omega\rangle\\ &= -\langle d(e_3\wedge\chi_3+e_2\wedge\chi_2+e_1\wedge\chi_1\wedge\omega))\rangle\end{aligned}$$

which gives $\delta[\omega]=0$. Now, the sequence (4.2) comes from (A) (cf. [5, Proposition 13.4)]), (28) and (33). □

Remark 4.3.

(a) The exotic term $\bigoplus_{S\in\mathscr{S}_1} H^*_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$. *When the exotic terms vanishes the Gysin Braid becomes the long exact sequence* (4). *This happens when $\mathscr{S}_1=\emptyset$, for example, when the acion Φ is almost-free (i.e., $\mathscr{S}_1=\mathscr{S}_3=\emptyset$) or semi-free.*

We have seen in the proof of the above Corollary that the exotic term can be simplified when $\overline{p}=\overline{0}$. In this case we have $\bigoplus_{S\in\mathscr{S}_1} H^*_{\overline{P_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}=H^*\left(M^{S^1}\right)^{-\mathbb{Z}_2}$.

The isotropy subgroup of a point of $M^{S^1}\backslash M^{S^3}$ is conjugated to S^1 or N. Let us suppose that the first situation does not appear, that is, the group $\mathbb{Z}_2$ acts trivially on M^{S^1}. This implies that $H^*\left(M^{S^1}\right)^{-\mathbb{Z}_2}=0$.

Equality (33) *is still true when $\overline{p}=4p,4p+1\mod 4$ on $\mathscr{S}_1$, $\overline{p}=q$ on $\mathscr{S}_3$ and $q-4p\leqslant 1$.*

We have

$$H^*\left(M^{S^1}\right)^{-\mathbb{Z}_2}=H^*\left(M^{S^1}\right)/H^*\big(M/S^3\big),$$

since $H^*\left(M^{S^1}\right) = H^*\left(M^{S^1}\right)^{\mathbb{Z}_2} \oplus H^*\left(M^{S^1}\right)^{-\mathbb{Z}_2} = H^*\left(M^{S^1}/\mathbb{Z}_2\right) \oplus H^*\left(M^{S^1}\right)^{-\mathbb{Z}_2} = H^*\left(M/S^3\right) \oplus H^*\left(M^{S^1}\right)^{-\mathbb{Z}_2}$.

Notice that $\bigoplus_{S\in\mathscr{S}_1} H^*_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} = \bigoplus_{S\in\mathscr{S}_1} H^*\left(S^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2}$ *when* $\mathscr{S}_3 = \varnothing$.

(*b*) *Property B1 yields the following two long exact sequences:*

$$\cdots \longrightarrow H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^{*-3}_{\overline{p}-\overline{e}}\left(M/S^3\right) \oplus H^{*+1}_{\overline{p}}\left(M/S^3\right) \longrightarrow H^{*+1}\left(K^{\cdot}_{\overline{p}}(M)\right) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{*-1}\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \longrightarrow H^*(M) \oplus \bigoplus_{S\in\mathscr{S}_1} H^{*-2p_S-2}_{\overline{p_S}}\left(\overline{S}^{S^1}\right)^{-(-1)^{p_S}\mathbb{Z}_2} \longrightarrow H^*\left(G^{\cdot}_{\overline{p}}(M)\right) \longrightarrow \cdots$$

relating the cohomologies of the Gysin and co-Gysin terms.

(*c*) *Consider the case* $\overline{p} = \overline{0}$. *Since the bottom map ② of the Gysin braid vanishes (cf. Corollary 4.2) a diagram chasing gives the short exact sequence*

$$0 \longrightarrow \mathrm{Im}\ ①_{top} \hookrightarrow \ker ③_{top} \xrightarrow{②^{-1}\circ ①} \ker ④_{bottom} \longrightarrow 0,$$

(cf. B2).

(*d*) *The splitting property given by this Corollary is not a general one as the following example shows.*

Let us consider the manifold $M = S^a \star S^2 \star S^3 = S^{a+7}$ *with* $a \geqslant 1$. *The action* $\Phi\colon S^3 \times M \to M$ *is defined in each factor:*

- S^3 *acts trivially on* S^a.
- S^3 *acts on the left of the homogeneous space* $S^2 = S^3/S^1$.
- S^3 *acts on the left of* S^3 *by multiplication on* S^3.

We put $\{b_1, b_2\}$ *the two points of* S^2 *whose isotropy subgroup is* S^1.

We have $\mathscr{S}_3 = \{Q = S^a\}$ *and* $\mathscr{S}_1 = \{S = (S^a \star S^2)\backslash S^a\}$. *Notice that*

$$M^{S^1} = \overline{S}^{S^1} = \left(S^a \star S^2\right)^{S^1} = S^a \star \left(S^2\right)^{S^1} = S^a \star \{b_1, b_2\},$$

the action of $g \in \mathbb{Z}_2$ *interchanges both points.*

The orbit space M/S^3 *is the filtered space* $S^a \star \left(S^2 \star S^3\right)/S^3 = S^a \star \Sigma S^2 = S^{a+4}$ *endowed with the filtration* $S^a \subset S^a \star \{P\} \subset S^a \star \Sigma S^2 = S^{a+4}$ *where* P *is one of the two apices of* ΣS^2.

The perversity $\overline{p}$ *is given by two numbers* $(p_1, p_3) = (\overline{p}(S), \overline{p}(Q))$. *Since* $\dim M = a+7$, $\dim Q = a$ *and* $\dim S = a+3$ *then the condition* $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ *becomes* $(0,0) \leqslant (p_1, p_3) \leqslant (2,5)$. *In particular, the perversities* $\overline{0} = (0,0)$ *and* $\overline{e} = (2,4)$ *satisfy this condition. If* $\overline{p} = \overline{0}$ *(resp.* $\overline{e}$ *) then* $\overline{P_S} = \overline{-2}$ *(resp.* $\overline{0}$*).*

A straightforward calculation using [5, Proposition 13.5] *gives*

$$\begin{array}{lcll}
+\ H^i(M) & = & \mathbb{R} & \text{if } i = 0, a+7, \\
+\ H^i_{-\overline{e}}(M/S^3) = H^i\left(S^{a+4}, S^a \star \{P\}\right) & = & \mathbb{R} & \text{if } i = a+4 \\
+\ H^i_{\overline{0}}(M/S^3) = H^i\left(S^{a+4}\right) & = & \mathbb{R} & \text{if } i = 0, a+4\,, \\
+\ H^i_{\overline{e}}(M/S^3) = H^i\left(S^{a+4}\backslash(S^a \star \{P\})\right) & = & \mathbb{R} & \text{if } i = 0, \\
+\ H^i_{\overline{0}}\left(\overline{S}^{S^1}\right)^{-\mathbb{Z}_2} = H^i\left(S^{a+1}\backslash S^a\right)^{-\mathbb{Z}_2} & = & \mathbb{R} & \text{if } i = 0 \\
+\ H^i_{\overline{-2}}\left(\overline{S}^{S^1}\right)^{\mathbb{Z}_2} = H^i\left(S^{a+1}, S^a\right)^{\mathbb{Z}_2} & = & \mathbb{R} & \text{if } i = a+1
\end{array}$$

and 0 for the other values of i.

One easily checks that the sequence the long exact sequence (27) *does not split on the connecting homorphism when* $\overline{p} = \overline{e}$. *Also, this example shows that the long exact sequence* (31) *does not split on the connecting homorphism, even in the case* $\overline{p} = \overline{0}$.

We end the Section by studying the behaviour of the Gysin and co-Gysin terms when the perversity changes.

Proposition 4.4. *Let $\Phi\colon S^3 \times M \to M$ be a mobile action. For any perversities $\overline{0} \leqslant \overline{p} \leqslant \overline{q} \leqslant \overline{t}$ on M we have the exact commutative braids,*

$$
\begin{tikzcd}[column sep=small]
H^*_{\overline{p}}(M/S^3) \arrow[rr, bend left, "1"] \arrow[dr, "4"] & & H^*(M) \arrow[rr, bend left, "3"] \arrow[dr, "1"] & & H^*\left(G^{\cdot}_{\overline{q}}(M)\right) \arrow[rr, bend left, "2"] \arrow[dr, "3"] & & H^{*+1}_{\overline{q}/\overline{p}}(M/S^3) \\
& H^*_{\overline{q}}(M/S^3) \arrow[ur, "3"] \arrow[dr, "4"] & & H^*\left(G^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "2"] \arrow[dr, "1"] & & H^{*+1}_{\overline{q}}(M/S^3) \arrow[ur, "4"] \arrow[dr, "3"] & \\
H^{*-1}\left(G^{\cdot}_{\overline{q}}(M)\right) \arrow[ur, "3"] \arrow[rr, bend right, "2"'] & & H^*_{\overline{q}/\overline{p}}(M/S^3) \arrow[ur, "2"] \arrow[rr, bend right, "4"'] & & H^{*+1}_{\overline{p}}(M/S^3) \arrow[ur, "4"] \arrow[rr, bend right, "1"'] & & H^{*+1}(M),
\end{tikzcd}
$$

and

$$
\begin{tikzcd}[column sep=small]
H^*\left(K^{\cdot}_{\overline{p}}(M)\right) \arrow[rr, bend left, "1"] \arrow[dr, "4"] & & H^*(M) \arrow[rr, bend left, "3"] \arrow[dr, "1"] & & H^{*-3}_{\overline{q}-\overline{e}}(M/S^3) \arrow[rr, bend left, "2"] \arrow[dr, "3"] & & H^{*-3}_{(\overline{q}-\overline{e})/(\overline{p}-\overline{e})}(M/S^3) \\
& H^*\left(K^{\cdot}_{\overline{q}}(M)\right) \arrow[ur, "3"] \arrow[dr, "4"] & & H^{*-3}_{\overline{p}-\overline{e}}(M/S^3) \arrow[ur, "2"] \arrow[dr, "1"] & & H^{*+1}\left(K^{\cdot}_{\overline{q}}(M)\right) \arrow[ur, "4"] \arrow[dr, "3"] & \\
H^{*-4}_{\overline{q}-\overline{e}}(M/S^3) \arrow[ur, "3"] \arrow[rr, bend right, "2"'] & & H^{*-4}_{(\overline{q}-\overline{e})/(\overline{p}-\overline{e})}(M/S^3) \arrow[ur, "2"] \arrow[rr, bend right, "4"'] & & H^{*+1}\left(K^{\cdot}_{\overline{p}}(M)\right) \arrow[ur, "4"] \arrow[rr, bend right, "1"'] & & H^{*+1}(M),
\end{tikzcd}
$$

Proof. The paper [19] presents a method for constructing a braid from a triple. We will use this method to construct two braids associated with the following pairs of triples: $\Omega^*_{\overline{p}}\left(M/S^3\right) \subset \Omega^*_{\overline{q}}\left(M/S^3\right) \subset \underline{\Omega}^*_{\overline{q}}(M)$ and $K^*_{\overline{p}}(M) \subset K^*_{\overline{q}}(M) \subset \underline{\Omega}^*_{\overline{q}}(M)$. It is important to recall that the cohomology of $\underline{\Omega}^*_{\overline{p}}(M)$ or $\underline{\Omega}^*_{\overline{q}}(M)$ is $H^*(M)$ (cf. Section 1.3). To identify the relative terms, we consider the following schema

$\dfrac{\Omega^*_{\overline{q}}\left(M/S^3\right)}{\Omega^*_{\overline{p}}(M/S^3)}$	$\dfrac{\Omega^*_{\overline{q}}(M)}{\Omega^*_{\overline{q}}(M/S^3)}$	$\dfrac{\Omega^*_{\overline{q}}(M)}{\Omega^*_{\overline{p}}(M/S^3)}$	$\dfrac{K^*_{\overline{q}}(M)}{K^*_{\overline{p}}(M)}$	$\dfrac{\underline{\Omega}^*_{\overline{q}}(M)}{K^*_{\overline{q}}(M)}$	$\dfrac{\underline{\Omega}^*_{\overline{q}}(M)}{K^*_{\overline{p}}(M)}$
$\Omega^*_{\overline{q}/\overline{p}}\left(M/S^3\right)$	$G^*_{\overline{q}}(M)$	$G^*_{\overline{p}}(M)$	$\Omega^{*-4}_{(\overline{q}-\overline{e})/(\overline{p}-\overline{e})}\left(M/S^3\right)$	$\Omega^*_{\overline{q}-\overline{e}}\left(M/S^3\right)$	$\Omega^*_{\overline{p}-\overline{e}}\left(M/S^3\right)$

The first two columns are actually equalities. In the other columns, we have complexes with the same cohomology. To prove this fact for the third column, we use the equality $\Omega^*_{\overline{q}}(M) \cap \Omega^*_{\overline{p}}\left(M/S^3\right) = \Omega^*_{\overline{q}}\left(M/S^3\right)$ and the fact that the inclusion $\Omega^*_{\overline{p}}(M) \hookrightarrow \Omega^*_{\overline{q}}(M)$ induces an isomorphism in cohomology, as shown in [1, 17] (this is property (a)).

For column 4, we use (29) and Proposition 3.1. Column 5 comes from the equality $K^*_{\overline{q}}(M) \cap \underline{\Omega}^*_{\overline{p}}(M) = K^*_{\overline{p}}(M)$ and property (a). The last column comes from the short exact sequence $0 \to K^*_{\overline{p}}(M) \to \Omega^*_{\overline{p}}(M) \to I^{*-3}_{\overline{p}}(M) \to 0$, property (a), and finally Proposition 3.1. ◻

5. Gysin sequence for a non-mobile action

"In this section, we consider a non-mobile non-trivial action $\Phi\colon S^3 \times M \to M$. The family of regular strata (resp. singular strata) is $\mathscr{S}_1 \neq \emptyset$ (resp. $\mathscr{S}_3$). A perversity is a map $\overline{p}\colon \mathscr{S}_3 \to \overline{\mathbb{Z}}$, that is, a family of numbers $\{\overline{p}(Q) \mid Q \in \mathscr{S}_3\} \subset \overline{\mathbb{Z}}$. We consider on $M/S^3, M^{S^1}$ the induced filtered space structure. Notice that the family of singular strata is still $\mathscr{S}_3$.

Theorem D. *Let $\Phi\colon S^3 \times M \to M$ be a non-mobile non-trivial action. For any perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ on M we have*

$$H^*(M) = H^*_{\overline{p}}\left(M/S^3\right) \oplus H^{*-2}_{\overline{p}-\overline{2}}\left(M^{S^1}\right)^{-\mathbb{Z}_2}. \tag{34}$$

Proof. Since the action of S^3 on $M\backslash\Sigma$ has no fixed points, the assignment $(g,x) \mapsto g\cdot x$, establishes an S^3-equivariant diffeomorphism between the twisted product $S^3 \times_N \left(M^{S^1}\backslash\Sigma\right)$ and $M\backslash\Sigma$. Here, $\Sigma = M^{S^3}$ is the union of singular strata. Following Corollary 2.8 we have

$$\underline{\Omega}^*(M\backslash\Sigma) = \Omega^*\left(M^{S^1}\backslash\Sigma\right)^{\mathbb{Z}_2} \oplus \Omega^{*-2}\left(M^{S^1}\backslash\Sigma\right)^{-\mathbb{Z}_2},$$

since $Z = 0$. A differential form $\omega \in \underline{\Omega}^*(M\backslash\Sigma)$ is $\alpha + \gamma_2 \wedge \gamma_3 \wedge \beta$ with $(\alpha,\beta) \in \Omega^*\left(M^{S^1}\backslash\Sigma\right)^{\mathbb{Z}_2} \oplus \Omega^{*-2}\left(M^{S^1}\backslash\Sigma\right)^{-\mathbb{Z}_2}$. It remains to compute the perverse degree $||\omega||_Q$.

We consider an S^3-invariant Thom-Mather system $\mathfrak{T}_M = \{T_Q \mid Q \in \mathscr{S}_3\}$. It induces the N-invariant Thom-Mather system $\mathfrak{T}_{M^{S^1}} = \{T_Q \cap M^{S^1} \mid Q \in \mathscr{S}_3\}$ on M^{S^1} (cf. Section 1.6). Notice that $T_Q\backslash Q = S^3 \times_N \left((T_Q \cap M^{S^1})\backslash Q\right)$. The map $\tau_Q \colon T_Q\backslash Q \to Q$ becomes $\langle g,x\rangle \mapsto \tau_Q(x)$. So, the fiber of τ_Q over a point $y \in Q$ is $S^3 \times_N \left((\tau_Q^{-1}(y) \cap M^{S^1})\backslash Q\right)$. This gives $||\omega||_Q = \max\{||\alpha||_Q, 2 + ||\beta||_Q\}$ and therefore

$$H^*(M) = H^*\left(\underline{\Omega}^{\cdot}_{\overline{p}}(M)\right) = H^*_{\overline{p}}\left(M^{S^1}\right)^{\mathbb{Z}_2} \oplus H^{*-2}_{\overline{p}-\overline{2}}\left(M^{S^1}\right)^{-\mathbb{Z}_2},$$

since the complex $\underline{\Omega}^*_{\overline{p}}(M)$ computes the cohomology of M for the perversity $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ (cf. Section 1.3 and Proposition 2.1). Finally, we get (34) from $(M^{S^1}\backslash\Sigma)/\mathbb{Z}_2 = (M\backslash\Sigma)/S^3$. □

Remark 5.1. *Considering the perversity $\overline{p} = \overline{0}$ we get*

$$H^*(M) = H^*\left(M/S^3\right) \oplus H^{*-2}\left(M^{S^1}\right)^{-\mathbb{Z}_2}$$

(cf. (33) *and* [5, Proposition 13.4)]*).*

References

[1] J.-P. Brasselet, G. Hector, and M. Saralegi, *Théorème de de Rham pour les variétés stratifiées*, Ann. Global Anal. Geom. **9** (1991), no. 3, 211–243. MR 1143404

[2] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York-London, 1972, Pure and Applied Mathematics, Vol. 46. MR 0413144 (54 #1265)

[3] ———, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1997, Corrected third printing of the 1993 original. MR 1700700 (2000b:55001)

[4] J.-L. Brylinski, *Equivariant intersection cohomology*, Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), Contemp. Math., vol. 139, Amer. Math. Soc., Providence, RI, 1992, pp. 5–32. MR 1197827 (94c:55010)

[5] D. Chataur, M. Saralegi-Aranguren, and D. Tanré, *Blown-up intersection cohomology*, An Alpine Bouquet of Algebraic Topology, Contemp. Math., vol. 708, Amer. Math. Soc., Providence, RI, 2018, pp. 45–102. MR 3807751

[6] ———, *Intersection cohomology, simplicial blow-up and rational homotopy*, Mem. Amer. Math. Soc. **254** (2018), no. 1214, viii+108. MR 3796432

[7] Samuel Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. No. **55** (1965), 39. MR 0178036

[8] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162. MR 572580 (82b:57010)

[9] Werner Greub, Stephen Halperin, and Ray Vanstone, *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973, Pure and Applied Mathematics, Vol. 47-II. MR 0336651 (49 #1424)

[10] G. Hector and M. Saralegi, *Intersection cohomology of S^1-actions*, Trans. Amer. Math. Soc. **338** (1993), no. 1, 263–288. MR 1116314 (93j:57021)

[11] Henry C. King, *Intersection homology and homology manifolds*, Topology **21** (1982), no. 2, 229–234. MR 642001 (83j:57010)

[12] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR 755006

[13] J.I. Royo Prieto and M. Saralegi Aranguren, *The Gysin sequence for $\mathbb{S}^3$-actions on manifolds*, Publ. Math. Debrecen **83** (2013), no. 3, 275–289. MR 3119667

[14] ———, *Leray-deRham spectral sequence associated to a regular action*, – (2023).

[15] M. Saralegi, *A Gysin sequence for semifree actions of* $\mathbf{S}^3$, Proc. Amer. Math. Soc. **118** (1993), no. 4, 1335–1345. MR 1184085 (94a:55005)

[16] M. Saralegi-Aranguren, *Cohomologie d'intersection des actions toriques simples*, Indag. Math. (N.S.) **7** (1996), no. 3, 389–417. MR 1621397

[17] ———, *de Rham intersection cohomology for general perversities*, Illinois J. Math. **49** (2005), no. 3, 737–758 (electronic). MR 2210257 (2006k:55013)

[18] M. Saralegi-Aranguren and R. Wolak, *Basic intersection cohomology of conical fibrations*, Mathematical Notes **77** (2005), no. 1, 213–231.

[19] C. T. C. Wall, *On the exactness of interlocking sequences*, Enseign. Math. (2) **12** (1966), 95–100. MR 206943

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