ON THE ORDER OF SEMIREGULAR AUTOMORPHISMS OF CUBIC VERTEX-TRANSITIVE GRAPHS

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ABSTRACT. We prove that, if Γ is a finite connected cubic vertex-transitive graph, then either there exists a semiregular automorphism of Γ of order at least 6, or the number of vertices of Γ is bounded above by an absolute constant.

1. INTRODUCTION

A fascinating old-standing question in the theory of group actions on graphs is the socalled *Polycirculant Conjecture*: non-identity 2-closed transitive permutation groups contain non-identity semiregular elements. This formulation of the conjecture was introduced by Klin [Kli98]. However, the question was previously posed independently by Marušič [Mar81, Problem 2.4] and Jordan [Jor88] in terms of graphs: vertex-transitive graphs having more than one vertex admit non-identity semiregular automorphisms.

In this paper, we focus our attention on cubic graphs. In [MS98], Marusič and Scappellato proved that, each cubic vertex-transitive graph admits a non-identity semiregular automorphism, settling the Polycirculant Conjecture for such graphs. Their proof did not take into account the order of the semiregular elements. In this direction, Cameron *et al.* proved in [CSS06] that, if Γ is a cubic vertex-transitive graph, then Aut(Γ) contains a semiregular automorphism of order at least 4. They also conjectured that, as the number of vertices of Γ tends to infinity, the maximal order of a semiregular automorphism tends to infinity. This was proven false by the third author in [Spi14] by building a family of cubic vertex-transitive graphs where such a maximum is precisely 6. In the light of these results, it is unclear whether 6 is optimal in the sense of minimizing the maximal order of a semiregular element. Broadly speaking, we are interested in

(1.1)
$$\liminf_{\substack{|V\Gamma| \to \infty \\ \Gamma \text{ cubic vertex-transitive}}} \max\{o(g) \mid g \in \operatorname{Aut}(\Gamma), g \text{ semiregular}\},$$

where we denote by o(g) the order of the group element g.

Theorem 1.1. The value of (1.1) is 6.

Theorem 1.1 is a consequence of the following result and the main result in [Spi14].

Theorem 1.2. Let (Γ, G) be a pair such that Γ is a connected cubic graph and G is a subgroup of the automorphism group of Γ acting vertex-transitively on $V\Gamma$. Then either G contains a semiregular automorphism of order at least 6 or the pair (Γ, G) appears in Table 1.

There is a considerable amount of work into the proof of Theorem 1.2. Broadly speaking, the proof divides into two main cases. In the first main case, the exponent of the group Gis very small, bounded above by 5, and we use explicit knowledge on the finite groups having exponent at most 5. The second main case is concerned with graphs admitting a normal quotient which is a cycle. Here, we need to refine our knowledge on the ubiquitous Praeger-Xu graphs

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and on the splitting and merging operators between cubic vertex-transitive graphs and 4-valent arc-transitive graphs defined in [PSV13].

Remark 1.3. The veracity of Theorem 1.2 for graphs with at most 1280 vertices has been proven computationally using the database of small cubic vertex-transitive graphs in [PSV13]. Therefore, in the course of the proof of Theorem 1.2 whenever we reduce to a graph having at most 1280 vertices we simply refer to this computation.

Table 1 consists of six columns. In the first column, we report the number of vertices of the exceptional cubic vertex-transitive graph Γ . In the second column, we report the order of the transitive subgroups G of $\operatorname{Aut}(\Gamma)$ with G not containing semiregular elements of order at least 6: each subgroup is reported up to $\operatorname{Aut}(\Gamma)$ -conjugacy class. In the third column, we report the cardinality of $\operatorname{Aut}(\Gamma)$. In the forth column, when $|V\Gamma| \leq 1280$, we report the number of the graph in the database of small cubic vertex-transitive graphs in [PSV13]. In the fifth column of Table 1, we write the symbol \checkmark when the graph is arc-transitive and the symbol \dagger when the graph is a split Praeger-Xu graph (see Section 2.5 for the definition of split Praeger-Xu graphs). Split Praeger-Xu graphs play an important role in our investigation and hence we are keeping track of this information in the forth column. In the sixth column, for the graphs not appearing in the database of small cubic vertex-transitive graphs, we give as much information as possible.

$ V\Gamma $	G	$ \operatorname{Aut}(\Gamma) $	DB	/ †	Comments
4	4, 4, 8, 12, 24	24	1	\checkmark	
6	6	12	1		
	6, 36	24	2	\checkmark	
8	8	16	1		
	8, 8, 8, 8, 16, 24, 24, 48	48	2	\checkmark	
10	10	20	1		
	10	20	2		
	20,60,120	120	3	\checkmark	
12	12, 24	24	2		
	24, 24	48	4	†	
16	16, 16, 32, 32, 64, 64	128	2	†	
	16	32	3		
	16, 48	96	4	\checkmark	
18	18, 108	216	4	\checkmark	
	36	72	5		
20	20	20	2		
	160, 160	320	3	† > >	
	60	120	6	\checkmark	
	120	240	7	\checkmark	
24	24	144	2	\checkmark	
	24	48	8		
	24	24	9		
	24	48	10		
	24, 24	48	11	†	
30	720	1440	8	\checkmark	
	60, 120	120	9		
	60	60	10		
32	32	64	2		
	32, 32, 64, 64	128	3	†	
	32, 96	192	4	\checkmark	
36	36	72	9		

SEMIREGULAR AUTOMORPHISMS

40	160, 160	320	12	†	
50	100	200	7	'	
	50, 150	300	8	\checkmark	
54	108	216	11		
60	60	360	2	\checkmark	
••	60, 120	120	3		
	60	60	4		
	60	120	5		
	60	120	6		
	60, 120	120	7		
	60	120	8		
	60	120	9		
	60	120	10		
64	64, 192	384	2	\checkmark	
	64	256	4		
	64, 64	128	11	†	
80	80, 160	160	29		
	160, 160	320	31	†	
90	720	1 4 4 0	20		
96	96	192	37		
100	100	200	19		
128	128	256	5		
160	160	160	89		
	160	160	90		
	160	320	91		
	160	320	92		
	160	320	93	†	
	160	320	94		
180	720		77		
	360, 720		78		
250	500		31		
256	256,768		30	\checkmark	
360	360	720	176		
	360	720	177		
	360	720	178		
	360	360	179		
	360	720	180		
	360	360	181		
	360	720	182		
	360	720	183		
	360	720	184		
	360	720	185		
	720	1440	268		
	720	1 4 4 0	270		
512	512	1024	734		
810	1 620	1 6 2 0	198		
1024	1024,3072	6144	3470	\checkmark	
1250	2500	2500	187		
1280	1 280	2500	2591		

2560	2560	5 1 2 0	
6250	12500	25 000	covers of the graph with 1250
	12500	12500	vertices, there are 2 graphs
31250	62500	125 000	covers of the graphs
	62500	125 000	with 6 250 vertices,
	62500	125 000	there are five graphs
	62500	62 500	
	62500	62500	
65610	131 220	?	cover of the graph with 810 ver-
			tices, only one graph
$2 \cdot 5^{\ell}$	$4 \cdot 5^{\ell}$		$7 \le \ell \le 34$

Table 1: Exceptional cases for Theorem 1.2

2. Main ingredients

2.1. **Permutations.** A permutation on the set Ω is a *derangement* if it fixes no elements in Ω . A permutation is *semiregular* if all of its cycles have the same length. For instance, any derangement of prime order is semiregular. A permutation group G on Ω is said to be *transitive* if it has a single orbit on Ω , and *semiregular* if the identity is the only element fixing some points. If G is both semiregular and transitive on Ω , then G is *regular* on Ω . Given a permutation group G, and an element $\alpha \in \Omega$, we denote by α^G the orbit of α under the action of G.

Lemma 2.1. Let G be a permutation group on Ω , and let p be a prime. If all the elements of G of order p are derangements, then all p-elements of G are semiregular.

Proof. Let $g \in G$ be an element of order p^k , for some positive integer k. Aiming for a contradiction, assume that g is not semiregular, that is, there exists $\alpha \in \Omega$ such that $|\alpha^{\langle g \rangle}| \leq p^{k-1}$. Hence $g^{p^{k-1}}$ fixes α , which implies $g^{p^{k-1}}$ is not a derangement, a contradiction.

Lemma 2.2. Let G be a permutation group acting on Ω , and let p and q be two distinct primes. If G has a semiregular element g of order p and a semiregular element h of order q with gh = hg, then gh is a semiregular element of order pq.

Proof. Since gh = hg, o(gh) = pq and hence it remains to prove that gh is semiregular. Note that $(gh)^p = h^p$ is semiregular, and also $(gh)^q = g^q$ is semiregular. Therefore, each orbit of $\langle gh \rangle$ has size pq, proving that gh is semiregular.

2.2. **Graphs.** A digraph is a binary relation $\Gamma = (V\Gamma, A\Gamma)$, where $A\Gamma \subseteq V\Gamma \times V\Gamma$. We refer to the elements of $V\Gamma$ as vertices and to the elements of $A\Gamma$ as arcs. In this paper, a graph is a finite simple undirected graph, that is, a pair $\Gamma = (V\Gamma, E\Gamma)$, where $V\Gamma$ is a set of vertices, and $E\Gamma$ is a set of unordered pairs of $V\Gamma$, called *edges*. In particular, a graph can be thought of as a digraph where the binary relation is symmetric and anti-reflexive.

The valency of a vertex $\alpha \in V\Gamma$ is the number of edges containing α . A graph is said to be *cubic* when all of its vertices have valency 3. A connected graph is a *cycle* when all of its vertices have valency 2.

Let Γ be a graph, and let G be a subgroup of the automorphism group $\operatorname{Aut}(\Gamma)$ of Γ . If G is transitive on $V\Gamma$, we say that G is *vertex-transitive*, similarly, if G is transitive on $A\Gamma$, we say that G is *arc-transitive*. Moreover, Γ is vertex- or arc-transitive when $\operatorname{Aut}(\Gamma)$ is vertex- or arc-transitive.

Let $\alpha, \beta \in V\Gamma$ be two adjacent vertices. We denote by G_{α} the *stabilizer* of the vertex α , by $G_{\{\alpha,\beta\}}$ the setwise stabilizer of the edge $\{\alpha,\beta\}$, by $G_{\alpha\beta}$ the pointwise stabilizer of the edge $\{\alpha,\beta\}$ (that is, the stabilizer of the arc (α,β) underlying the edge $\{\alpha,\beta\}$).

Let Γ be a graph, and let $N \leq \operatorname{Aut}(\Gamma)$. The normal quotient Γ/N is the graph whose vertices are the N-orbits of $V\Gamma$, and two N-orbits α^N and β^N are adjacent if there exists an edge $\{\alpha', \beta'\} \in E\Gamma$ such that $\alpha' \in \alpha^N$ and $\beta' \in \beta^N$. Note that the valency of Γ/N is at most the valency of Γ , and that, whenever Γ is conneted, so is Γ/N . Furthermore, if the group N is normal in some $G \leq \operatorname{Aut}(\Gamma)$, then G/N acts (possibly unfaithfully) on Γ/N . If the group G acts vertex- or arc-transitively on Γ , then G/N has the same property on Γ/N .

The following result is inspired by an analogous result for 4-valent graphs in [PS21, Lemma 1.13].

Lemma 2.3. Let Γ be a connected cubic graph, let α be a vertex of Γ , let G be a vertex-transitive subgroup of $\operatorname{Aut}(\Gamma)$ and let N be a semiregular normal subgroup of G. Suppose G_{α} is a non-identity 2-group and that the normal quotient Γ/N is a cycle of length $r \geq 3$, and denote by K the kernel of the action of G on the N-orbits on $V\Gamma$. Then either

- (1) G_{α} has order 2 and $|K_{\alpha}| = 1$, or
- (2) r is even and $G_{\alpha} = K_{\alpha}$ is an elementary abelian 2-group of order at most $2^{r/2}$.

Proof. Let $\Delta_0, \Delta_1, \ldots, \Delta_{r-1}$ be the orbits of N in its action on VT. Since Γ/N is a cycle, we may assume that Δ_i is adjacent to Δ_{i-1} and Δ_{i+1} with indices computed modulo r. Moreover, without loss of generality, we suppose that $\alpha \in \Delta_0$.

As G_{α} is a non-identity 2-group, by a connectedness argument, G_{α} induces a group of order 2 in its action on the neighbourhood of α . In particular, G_{α} fixes a unique neighbour of α . As usual, for each $\beta \in V\Gamma$, let β' be the unique neighbour of β fixed by G_{β} .

Suppose that $\{\alpha, \alpha'\}$ is contained in an N-orbit. Since $\alpha \in \Delta_0$, we deduce $\alpha' \in \Delta_0$. Let β and γ be the other two neighbours of α . As Γ/N is a cycle of length $r \geq 3$, we have $\beta \in \Delta_1$ and $\gamma \in \Delta_{r-1}$. Since $\operatorname{Aut}(\Gamma/N)$ is a dihedral group of order 2r and since G_α contains an element swapping β and γ , we deduce $|G_\alpha : K_\alpha| = 2$. Now, K_α fixes by definition each N-orbit and hence it fixes setwise Δ_1 and Δ_{r-1} . Therefore, K_α fixes β and γ , because β is the unique neighbour of α in Δ_1 and γ is the unique neighbour of α in Δ_{r-1} . This shows that K_α fixes pointwise the neighbourhood of α ; now, a connectedness argument shows that $K_\alpha = 1$. In particular, part (1) is satisfied. For the rest of the argument, we suppose that $\{\alpha, \alpha'\}$ is not contained in an N-orbit.

This means that α has two neighbours in an N-orbit, say Δ_1 , and only one neighbour in the other N-orbit, say Δ_{r-1} . (Thus $\alpha' \in \Delta_{r-1}$ and $\beta, \gamma \in \Delta_1$.) This implies that r is even and, for every $i \in \{0, \ldots, r/2 - 1\}$, each vertex in Δ_{2i} has two neighbours in Δ_{2i+1} and only one neighbour in Δ_{2i-1} . Therefore, G/K is a dihedral group of order r when $r \geq 8$ and G/Kis elementary abelian of order 4 when r = 4. Morever, G/K acts regularly on Γ/N and hence $G_{\alpha} = K_{\alpha}$. It remains to show that K_{α} is an elementary abelian 2-group of order at most 2^{r} .

Since N is normal in G, the orbits of N on the edge-set $E\Gamma$ form a G- invariant partition of $E\Gamma$. We claim that, no two edges incident to a fixed vertex of Γ belong to the same N-edge-orbit. We argue by contradiction and we suppose that α has two distinct neighbours v and w such that the edges $\{\alpha, v\}$ and $\{\alpha, w\}$ are in the same N-edge-orbit. In particular, there exists $n \in N$ with $\{\alpha, v\}^n = \{\alpha, w\}$. This gives $\alpha^n = \alpha$ and $v^n = w$, or $\alpha^n = w$ and $v^n = \alpha$. Since there are no edges inside an N-orbit, we cannot have $\alpha^n = w$ and $v^n = \alpha$. Therefore, $\alpha^n = \alpha$ and $v^n = w$. Since N acts semiregularly on $V\Gamma$, we have n = 1 and hence $v = v^n = w$, which is a contradiction.

Since G is vertex-transitive, the edges between Δ_{2i} and Δ_{2i+1} are partitioned into precisely two N-edge-orbits, let's call these two orbits Θ_{2i} and Θ'_{2i} ; whereas the edges between Δ_{2i} and Δ_{2i-1} form one N-edge-orbit, which we call Θ''_{2i} .

An element of K (fixing setwise the sets Δ_{2i} and Δ_{2i+1}) can map an edge in Θ_{2i} only to an edge in Θ_{2i} or to an edge in Θ'_{2i} . On the other hand, as G_{α} is not the identity group, for every vertex $v \in \Delta_{2i}$ there is an element $g \in G_v$ which maps an edge of Θ_{2i} incident to v to the edge of Θ'_{2i} incident to v; and this element g is clearly an element of K, because G/K acts semiregularly on Γ/N . This shows that the orbits of K on $E\Gamma$ are precisely the sets $\Theta_{2i} \cup \Theta'_{2i}, \Theta''_{2i}, i \in \{0, \ldots, r/2 - 1\}$. In other words, each orbit of the induced action of K on the set $E\Gamma/N = \{e^N : e \in E\Gamma\}$ has length at most 2. Consequently, if X denotes the kernel of the action of K on $E\Gamma$, then K/X embeds into $\operatorname{Sym}(2)^{r/2}$ and is therefore an elementary abelian 2-group of order at most $2^{r/2}$.

Let us now show that X = N. Clearly, $N \leq X$. Let $v \in \Delta_0$. Since N is transitive on Δ_0 , it follows that $X = NX_v$. Suppose that X_v is non-trivial and let g be a non-trivial element of X_v . Further, let w be a vertex which is closest to v among all the vertices not fixed by g, and let $v = v_0 \sim v_1 \sim \cdots \sim v_m = w$ be a shortest path from v to w. Then v_{m-1} is fixed by g. Since g fixes each N-edge-orbit setwise and since every vertex of Γ is incident to at most one edge in each N-edge-orbit, it follows that g fixes all the neighbours of v_{m-1} , thus also v_m . This contradicts our assumptions and proves that X_v is a trivial group, and hence that X = N. \Box

2.3. **Praeger-Xu graphs.** To introduce the infinite family of split Praeger-Xu graphs sC(r, s), we need two ingredients: the Praeger-Xu graphs and the splitting operation. This section is devoted to introduce the ubiquitous 4-valent Praeger-Xu graphs C(r, s) and their automorphism group. This infinite family was originally defined in [PX89], and it was studied in detail by Gardiner, Praeger and Xu in [PX89, GP94], and more recently in [JPW19]. Here, we introduce them through their directed counterparts defined in [Pra89].

Let r be an integer, $r \geq 3$. Then $\tilde{C}(r, 1)$ is the lexicographic product of a directed cycle of length r with an edgeless graph on 2 vertices. In other words, $V\vec{C}(r, 1) = \mathbb{Z}_r \times \mathbb{Z}_2$ with the out-neighbours of a vertex (x, i) being (x + 1, 0) and (x + 1, 1). We will identify the (s - 1)-arc

$$(x,\varepsilon_0) \sim (x+1,\varepsilon_1) \sim \ldots \sim (x+s-1,\varepsilon_{s-1})$$

with the pair (x; k) where $k = \varepsilon_0 \varepsilon_1 \dots \varepsilon_{s-1}$ is a string in \mathbb{Z}_2 of length s. For $s \ge 2$, let $V\vec{C}(r, s)$ be the set of all (s-1)-arcs of $\vec{C}(r, 1)$, let h be a string in \mathbb{Z}_2 of length s-1 and let $\varepsilon \in \mathbb{Z}_2$. The out-neighbours of $(x; \varepsilon h) \in V\vec{C}(r, s)$ are (x + 1; h0) and (x + 1; h1). The Praeger-Xu graph C(r, s) is then defined as the underlying graph of $\vec{C}(r, s)$. We have that C(r, s) is a connected 4-valent graph with $r2^s$ vertices (see [Pra89, Theorem 2.8]).

Let us now discuss the automorphisms of the graphs C(r, s). Every automorphism of $\dot{C}(r, 1)$ (C(r, 1), respectively) acts naturally as an automorphism of $\vec{C}(r, s)$ (C(r, s), respectively) for every $s \ge 2$. For $i \in \mathbb{Z}_r$, let τ_i be the transposition on $V\vec{C}(r, 1)$ swapping the vertices (i, 0) and (i, 1) while fixing every other vertex. This is clearly an automorphism of $\vec{C}(r, 1)$, and thus also of $\vec{C}(r, s)$ for $s \ge 2$. Let

$$K := \langle \tau_i \mid i \in \mathbb{Z}_r \rangle$$

and observe that $K \cong C_2^r$. Further, let ρ and σ be the permutations on $V\vec{C}(r,1)$ defined by

$$(x,i)^{\rho} := (x+1,i) \text{ and } (x,i)^{\sigma} := (x,-i).$$

Then ρ is an automorphism of $\vec{C}(r, 1)$ or order r, and σ is an involutory automorphism of C(r, 1)(but not of $\vec{C}(r, 1)$). Observe that the group $\langle \rho, \sigma \rangle$ normalises K. Let

$$H := K \langle \rho, \sigma \rangle \quad \text{and} \quad H^+ := K \langle \rho \rangle.$$

Then, for every $r \geq 3$ and $s \geq 1$,

$$C_2 \operatorname{wr} D_r \cong H \leq \operatorname{Aut}(\operatorname{C}(r,s))$$
 and $C_2 \operatorname{wr} C_r \cong H^+ \leq \operatorname{Aut}(\vec{\operatorname{C}}(r,s)).$

Moreover, H (H^+ , respectively) acts arc-transitively on C(r, s) ($\vec{C}(r, s)$, respectively) whenever $1 \leq s \leq r-1$. With three exceptions, the groups H and H^+ are in fact the full automorphism groups of C(r, s) and $\vec{C}(r, s)$, respectively.

Lemma 2.4 ([GP94, Theorem 2.13] and [Pra89, Theorem 2.8]). The automorphism group of a directed Praeger-Xu graph is

$$\operatorname{Aut}(\vec{\mathcal{C}}(r,s)) = H^+,$$

and, if $r \neq 4$, the automorphism group of a Praeger-Xu graph is

$$\operatorname{Aut}(\operatorname{C}(r,s)) = H.$$

Moreover,

$$Aut(C(4,1)): H| = 9, \quad |Aut(C(4,2)): H| = 3$$

and $|Aut(C(4,3)): H| = 2.$

The Praeger-Xu graphs also admit the following algebraic characterization.

Lemma 2.5 ([PS21, Lemma 1.11] or [BGS22b, Lemma 3.7]). Let Γ be a finite connected 4valent graph, let G be a vertex- and edge-transitive group of automorphisms of Γ , and let N be a minimal normal subgroup of G. If N is a 2-group and Γ/N is a cycle of length at least 3, then Γ is isomorphic to a Praeger-Xu graph C(r, s) for some positive integers $r \leq 3$ and $s \leq r - 1$.

For more details on Praeger-Xu graphs, we refer also to [JPW19, JPW22, BGS22a].

2.4. The splitting and merging operations. The operation of *splitting* were introduced in [PSV13, Construction 11]. Let Δ be a 4-valent graph, let C be a partition of $E\Delta$ into cycles. By applying the splitting operation to the pair (Δ, C) , we obtain the graph, denoted by $s(\Delta, C)$, whose vertices are

$$Vs(\Delta, \mathcal{C}) := \{ (\alpha, C) \in V\Delta \times \mathcal{C} \mid \alpha \in VC \},\$$

and such that two vertices (α, C) and (β, D) are declared adjacent if either $C \neq D$ and $\alpha = \beta$, or C = D and α and β are adjacent in Δ . Observe that, since Δ is 4-valent, there are precisely 2 cycles in C passing through α , thus $s(\Delta, C)$ is cubic and $|Vs(\Delta, C)| = 2|V\Delta|$.

Notice that, for any $G \leq \operatorname{Aut}(\Delta)$ such that its action is \mathcal{C} -invariant, $G \leq \operatorname{Aut}(\operatorname{s}(\Delta, \mathcal{C}))$. Moreover, if G is also arc-transitive on Δ (in particular, the action of G_{α} on the neighbourhood of α is either the Klein four group, or the cyclic group of order 4, or the dihedral group of order 8), then G is vertex-transitive on $\operatorname{s}(\Delta, \mathcal{C})$. For any vertex $(\alpha, C) \in \operatorname{s}(\Delta, \mathcal{C})$,

$$G_{(\alpha,C)} = G_{\alpha} \cap G_{\{C\}},$$

where $G_{\{C\}}$ is the setwise stabilizer of the cycle C. In particular, whenever G is arc-transitive on Δ , as G_{α} switches the two cycles passing through α , $|G_{\alpha}:G_{(\alpha,C)}|=2$.

Now, we introduce the tentative inverse of the splitting operator: the operation of merging (see [PSV13, Construction 7]). Let Γ be a connected cubic graph, and let $G \leq \operatorname{Aut}(\Gamma)$ be a vertex-transitive group such that the action of G_{α} on the neighbourhood of α is cyclic of order 2. In particular, G_{α} is a non-identity 2-group. Hence, G_{α} fixes a unique neighbour of α , which we denote by α' . Observe that $(\alpha')' = \alpha$ and $G_{\alpha} = G_{\alpha'}$. Thus, the set $\mathcal{M} := \{\{\alpha, \alpha'\} \mid \alpha \in V\Gamma\}$ is a complete matching of Γ , while the edges outside \mathcal{M} form a 2-factor, which we denote by \mathcal{F} . The group G in its action on $E\Gamma$ fixes setwise both \mathcal{F} and \mathcal{M} , and acts transitively on the arcs of each of these two sets. Let Δ be the graph with vertex-set \mathcal{M} and two vertices $e_1, e_2 \in \mathcal{M}$ are declared adjacent if they are (as edges of Γ) at distance 1 in Γ . We may also think of Δ as being obtained by contracting all the edges in \mathcal{M} . Let \mathcal{C} be the decomposition of $E\Delta$ into cycles given by the connected components of the the 2-factor \mathcal{F} . The merging operation applied to the pair (Γ, G) gives as a result the pair (Δ, \mathcal{C}) .

Two infinite families of cubic graph have degenerate merged graphs, namely the circular and Möbius ladders. For any $n \ge 3$, a *circular ladder graph* is a graph isomorphic to the Cayley graph

$$Cay(\mathbb{Z}_n \times \mathbb{Z}_2, \{(0,1), (1,0), (-1,0)\}),\$$

and, for any $n \geq 2$, a *Möbius ladder graph* is a graph isomorphic to the Cayley graph

$$Cay(\mathbb{Z}_{2n}, \{1, -1, n\}).$$

Observe that we consider the complete graph on 4 vertices to be a Möbius ladder graph.

Lemma 2.6. Let Λ be a (circular or Möbius) ladder, and let $G \leq \operatorname{Aut}(\Lambda)$ be a vertex-transitive group. Then either $|V\Lambda| \leq 10$ or G contains a semiregular element of order at least 6.

Lemma 2.7. Unless Λ is isomorphic to the skeleton of the cube or the complete graph on 4 vertices, the automorphism group of a (circular or Möbius) ladder Λ contains $N \leq \operatorname{Aut}(\Lambda)$, a normal cyclic subgroup of order 2, such that the normal quotient Λ/N is a cycle.

Remark 2.8. Let Γ be a connected cubic graph that is neither a circular nor a Möbius ladder, and let $G \leq \operatorname{Aut}(\Gamma)$ be a vertex-transitive group such that the action of G_{α} on the neighbourhood of α is cyclic of order 2. Then [PSV13, Lemma 9 and Theorem 10] imply that the merging operator applied to the pair (Γ, G) gives a pair (Δ, \mathcal{C}) such that Δ is 4-valent, and the action of G on Δ is faithful, arc-transitive and \mathcal{C} -invariant. This result motivates the use of the word degenerate when referring to the circular and Möbius ladders.

In view of [PSV13, Theorem 12], the merging operator is the right-inverse of the splitting one, or, more explicitly, unless Γ is a (circular or Möbius) ladder, splitting a pair (Δ, C) obtained via the merging operation on (Γ, G) results in the starting pair. For our purposes, we need to show that the merging operator is also the left-inverse of the splitting one.

Theorem 2.9. Let Δ be a 4-valent graph, let C be a partition of $E\Delta$ into cycles, and let $G \leq \operatorname{Aut}(\Delta)$ be an arc-transitive and C-invariant group. Then the merging operation can be applied to the pair $(s(\Delta, C), G)$ and it gives as a result (Δ, C) .

Proof. Let (α, C) be a generic vertex of $s(\Delta, C)$, let $D \in C$ be the other cycle of the partition passing through α , and let $\beta, \gamma \in V\Delta$ be the neighbours of α in C. Then, using the fact that G is arc-transitive on C,

$$(\alpha, D)^{G_{(\alpha,C)}} = \{(\alpha, D)\}$$
 and $(\beta, C)^{G_{(\alpha,C)}} = (\gamma, C)^{G_{(\alpha,C)}} = \{(\beta, C), (\gamma, C)\}.$

Therefore, for any vertex $(\alpha, C) \in Vs(\Delta, C)$, $G_{(\alpha, C)}$ acts on the neighbourhood of (α, C) as a cyclic group of order 2. Hence, we can apply the merging operation to the pair $(s(\Delta, C), G)$. Furthermore, we deduce that

$$\mathcal{M} = \{\{(\alpha, C), (\alpha, D)\} \mid \alpha \in VC \cap VD\}$$

is a complete matching for $(s(\Delta, C), G)$. Thus the connected components of the resulting 2-factor $\mathcal{F} = Es(\Delta, C) \setminus \mathcal{M}$ can be identified with the cycles of \mathcal{C} . Now, consider the map defined as

$$\theta: \mathcal{M} \to V\Delta, \{(\alpha, C), (\alpha, D)\} \mapsto \alpha.$$

Since a generic vertex $\alpha \in V\Delta$ belongs to precisely two distinct cycles, θ is bijective. Moreover, β is adjacent to α in Δ if, and only if, either $\{(\alpha, C), (\beta, C)\}$ or $\{(\alpha, D), (\beta, D)\}$ is an edge in $s(\Delta, C)$. In particular, θ also induces the bijection

$$\hat{\theta}: \mathcal{F} \to E\Delta, \{(\alpha, C), (\beta, C)\} \mapsto \{\alpha, \beta\},\$$

which sends the connected components of \mathcal{F} into disjoint cycles of \mathcal{C} . This shows that θ is a graph isomorphism between Δ and the 4-valent graph obtained by merging the pair $(s(\Delta, \mathcal{C}), G)$, and that the resulting cycle partition is isomorphic to \mathcal{C} .

Corollary 2.10. Let Δ be a 4-valent graph, let C be a partition of $E\Delta$ into cycles, and let $G \leq \operatorname{Aut}(\Delta)$ be an arc-transitive and C-invariant group (and so $G \leq \operatorname{Aut}(\operatorname{s}(\Delta, C))$). Suppose that $G \leq A \leq \operatorname{Aut}(\operatorname{s}(\Delta, C))$ is a vertex-transitive group such that, for any vertex $\alpha \in \operatorname{Vs}(\Delta, C)$, the action of A_{α} on the neighbourhood of α is cyclic of order 2, then $A \leq \operatorname{Aut}(\Delta)$.

Proof. Note that, as G is a subgroup of A, the actions of G and A on the neighbourhood of any vertex α coincide. In particular, applying the merging operation to the pair $(s(\Delta, C), A)$ yields the same result as doing it on the pair $(s(\Delta, C), G)$, that is, by Theorem 2.9, in both cases we obtain (Δ, C) . The result follows by Remark 2.8.

2.5. Split Praeger-Xu graphs. In this section, we bring together the information of Sections 2.3 and 2.4 to define and study the split Praeger-Xu graphs.

All the partitions of the edge set of a Praeger-Xu graph into disjoint cycles were classified in [JPW19, Section 6]. Regardless of the choice of the parameters r and s, there exists a decomposition into disjoint cycles of length 4 of the form

$$(x;0h) \sim (x+1;h0) \sim (x;1h) \sim (x+1;h1)$$

for some $x \in \mathbb{Z}_r$, and for some string h in \mathbb{Z}_2 of length s - 1. We denote this partition by \mathcal{S} . Moreover, observe that the only two neighbours of (x; 0h) in the K-orbit containing (x + 1; h0)are (x+1; h1) and (x+1; h0), and similarly the only two neighbours of (x+1; h0) in the K-orbit containing (x; 0h) are (x; 1h) and (x; 0h). Therefore, \mathcal{S} is the unique decomposition such that each cycle intersects exactly two K-orbits.

Definition 2.11. The split Praeger-Xu graph sC(r, s) is the cubic graph obtained from the pair (C(r, s), S) by applying the splitting operation.

Lemma 2.12. For some positive integers $r \ge 3$ and $s \le r - 1$, the automorphism group of the split Praeger-Xu graph is

$$\operatorname{Aut}(\operatorname{sC}(r,s)) = H,$$

and it acts transitively on VsC(r, s).

Proof. Note that H acts on the set of K-orbits in VC(r, s), thus each automorphism of H maps any cycle of S to a cycle intersecting exactly two K-orbits, that is, to an element of S. Thus, H is S-invariant, and so $H \leq \operatorname{Aut}(\operatorname{sC}(r, s))$. We now show the opposite inclusion. Let $\alpha \in V\operatorname{sC}(r, s)$ be a generic vertex, aiming for a contradiction we suppose that $\operatorname{Aut}(\operatorname{sC}(r, s))_{\alpha}$ does not act on the neighbourhood of α as a cycle of order 2. Let α', β, γ be the neighbours of α where α' is fixed by the action of H_{α} , and let δ be the unique vertex at distance 1 from both β and γ . Since $H_{\alpha} \leq \operatorname{Aut}(\operatorname{sC}(r, s))_{\alpha}$, our hypothesis implies that there exists an element $g \in \operatorname{Aut}(\operatorname{sC}(r, s))_{\alpha}$ such that $\beta^g = \alpha'$ and $\gamma^g = \gamma$. This yields a contradiction because δ^g is ill-defined: in fact there is no vertex of $\operatorname{sC}(r, s)$ at distance 1 from both γ^g and δ^g . Recall that, from Lemma 2.4, if $r \neq 4$, then $H = \operatorname{Aut}(\operatorname{C}(r, s))$, and so, by Corollary 2.10, $\operatorname{Aut}(\operatorname{sC}(r, s)) \leq H$. On the other hand, if r = 4, observe that H is vertex-transitive on $\operatorname{sC}(r, s)$ and $\operatorname{Aut}(\operatorname{sC}(r, s))_{\alpha} = H_{\alpha}$, hence the equality holds by Frattini's argument. \Box

Lemma 2.13. Let G be a vertex-transitive subgroup of Aut(sC(r, s)). Then either G contains a semiregular element of order at least 6, or (sC(r, s), G) is one of the examples in Table 1 marked with the symbol \dagger .

Proof. From Lemma 2.12, we have $G \leq H = K \langle \rho, \sigma \rangle$. Observe that $G/G \cap K \cong \langle \rho, \sigma \rangle$, otherwise G is not transitive on the vertices of the split graph sC(r, s). From this, it follows that $G = V \langle \rho f, \sigma g \rangle$, for some $f, g \in K$, where $V = G \cap K$. Since ρ has order r, we get that

$$\begin{aligned} (\rho f)^r &= \rho f \rho \dots (\rho f \rho) f \\ &= \rho f \rho \dots (\rho^2 \rho^{-1} f \rho) f \\ &= \rho f \rho \dots \rho^2 f^\rho f \\ &= \rho f \rho^{r-1} \dots f^\rho f \\ &= f^{\rho^{r-1}} \dots f^\rho f \end{aligned}$$

is an element of V. Since V is an elementary abelian 2-group, the element ρf has order either r or 2r. Recalling that $V \leq K$,

$$(\rho f)^r = \prod_{i=0}^{r-1} \tau_i^{a_i}$$

with $a_i \in \{0, 1\}$. Furthermore,

$$(\rho f)^r \rho = \rho(f\rho \dots \rho f\rho f\rho)$$

= $\rho(ff^{\rho} \dots f^{\rho^{r-2}} f^{\rho^{r-1}})$
= $\rho(f^{\rho^{r-1}} \dots f^{\rho} f)$
= $\rho(\rho f)^r$

thus ρ centralizes $(\rho f)^r$. From this, and from the fact that $\langle \rho \rangle$ acts transitively on $\{\tau_0, \ldots, \tau_{r-1}\}$, we deduce that

$$(\rho f)^r = \prod_{i=0}^{r-1} \tau_i^a$$

where a is either 0 or 1. If a = 0, then ρf is a semiregular element of order r. In particular, either $r \ge 6$, or the number of vertices of sC(r, s) is $r2^s$, which is bounded by $5 \cdot 2^5 = 160$, and we finish by Remark 1.3. On the other hand, if a = 1, ρf has order 2r, and it corresponds to the so-called *super flip* of the Praeger-Xu graph C(r, s). Since $(\rho f)^r$ does not fix any vertex in C(r, s), and since the vertex-stabilizers for a split graph has index 2 in the vertex-stabilizer of the starting graph, for any vertex $\alpha \in VsC(r, s)$, we obtain that $(\rho f)^r \notin G_{\alpha}$. Hence ρf is semiregular of order $2r \ge 6$.

To conclude this section, we show a result mimicking Lemma 2.5 for cubic graphs.

Lemma 2.14. Let Γ be a connected cubic vertex-transitive graph, let $G \leq \operatorname{Aut}(\Gamma)$ be a vertextransitive group such that the action of G_{α} on the neighbourhood of α is cyclic of order 2, and let N be a minimal normal subgroup of G. If N is a 2-group and Γ/N is a cycle of length at least 3, then Γ is isomorphic either to a circular ladder, or to a Möbius ladder, or to $\operatorname{sC}(r,s)$, for some positive integers $r \geq 3$ and $s \leq r - 1$.

Proof. We already know by Lemma 2.7 that both ladders admit a cyclic quotient graph, thus we can suppose that Γ is not isomorphic to a circulant ladder or to a Möbius ladder. By hypothesis, we can apply the merging operator to (Γ, G) , obtaining the pair (Δ, C) . Since we have excluded the possibility of Γ being a ladder, by Remark 2.8, Δ is 4-valent, and the action of G on Δ is faithful, arc-transitive and C-invariant. Since the action of N cannot map edges in \mathcal{M} to edges in \mathcal{F} , the quotient graph Γ/N retains a partition into two disjoint sets of edges, namely \mathcal{M}/N and \mathcal{F}/N . Moreover, since \mathcal{M} is a complete matching, each edge in \mathcal{M}/N is adjacent to precisely two edges in \mathcal{F}/N , and vice versa. This implies that the edges of Δ/N coincide with the elements of \mathcal{F}/N , two of which are adjacent if they share the same neighbour in \mathcal{M}/N . If $r \geq 6$, then Δ/N is a cycle of length r/2. From Lemma 2.5, we deduce that Δ is isomorphic to C(r, s), for some positive integers $r \geq 3$ and $s \leq r - 1$. Observe that, as \mathcal{C} coincides with the connected components of \mathcal{F} , each cycle in \mathcal{C} intersects precisely two K-orbits. This implies that $\mathcal{C} = \mathcal{S}$, and so [PSV13, Theorem 12] yields that Γ is isomorphic to

$$s(\Delta, C) = s(C(r, s), S) = sC(r, s).$$

Now, suppose that r = 4. In this case, we have that G is a 2-group, hence |N| = 2 and $|V\Gamma| = 8$, and so the only possibility is for Γ to be a (cirular or Möbius) ladder, which we already excluded.

3. Proof of Theorem 1.2

We aim to prove Theorem 1.2 by contradiction. In this section we will assume the following.

Hypothesis 3.1. Let Γ be a connected cubic graph, and let $G \leq \operatorname{Aut}(\Gamma)$ such that the pair (Γ, G) is a minimal counterexample to Theorem 1.2, first with respect to the cardinality of $V\Gamma$, and then to the order of G. From Remark 1.3, we have $|V\Gamma| > 1280$. Let α be an arbitrary vertex of Γ . Let N be a minimal normal subgroup of G.

Since Γ is connected, the stabilizer G_{α} is a $\{2,3\}$ -group. More generally, if Δ is a connected d-regular graph, then no prime bigger than d divides the order of a vertex stabilizer (this follows from an elementary connectedness argument, see for instance [Spi14, Lemma 3.1] or [MS98, Lemma 3.2]). Moreover, G must be a $\{2,3,5\}$ -group, otherwise we can find derangements of prime order at least 7, hence semiregular elements.

Since N is a minimal normal subgroup of G, N is a direct product of simple groups, any two of which are isomorphic. Clearly, N is a $\{2,3,5\}$ -group, and N_{α} is a $\{2,3\}$ -group. Thus N is a direct product S^l , for some positive integer l and for some simple $\{2,3,5\}$ -group S. Using the Classification of Finite Simple Groups, we see that the collection of simple $\{2,3,5\}$ -groups consists of

$$C_2, C_3, C_5, \text{Alt}(5), \text{Alt}(6), \text{PSp}(4,3),$$

see for instance [LW74].

Lemma 3.2. Under Hypothesis 3.1, if N_{α} is a 2-group, then N is an elementary abelian p-group, for some prime $p \in \{2, 3, 5\}$.

Proof. If N is abelian, then there is nothing to prove. Thus, suppose that $N = S^l$, where $S \in {Alt(5), Alt(6), PSp(4,3)}$ and $l \ge 1$.

Assume $l \geq 2$. Let S and T be two distinct direct factors of N. Then S_{α} and T_{α} are 2-groups, because so is N_{α} . Thus, by Lemma 2.1, all the 3- and 5-elements of S and T are semiregular. Applying Lemma 2.2, we obtain that $S \times T$, contains a semiregular element of order 15. Thus G contains a semiregular element of order exceeding 6, contradicting Hypothesis 3.1.

Assume l = 1. If N = PSp(4,3), then Lemma 2.1 implies that the 3-elements in N are semiregular. As PSp(4,3) contains elements of order 9, G contains a semiregular element of order 9, contradicting Hypothesis 3.1. Thus, N is either Alt(5) or Alt(6).

We claim that G is almost simple, that is, N is the unique minimal normal subgroup of G. Aiming for a contradiction, let M be a minimal normal subgroup of G distinct from N. If Γ/M is a cubic graph, then $M_{\alpha} = 1$, and hence each element of M is semiregular. Since [N, M] = 1, by Lemma 2.2, G contains a semiregular element of order at least 10, against Hypothesis 3.1. On the other hand, suppose that Γ/M is not cubic. Regardless of the valency of Γ/M , the group that G induces in its action on the vertices of Γ/M is a subgroup of a dihedral group, hence it is a soluble group. In particular, as N is a non-abelian simple group, N acts trivially on the vertices of Γ/M . This means that N fixes setwise each M-orbit. If M is abelian, then M acts regularly on each of its orbits. However, as N commutes with M and fixes each M-orbit, this contradicts the fact that N is non-abelian.¹ Therefore, M is not abelian. In particular, there is a prime $p \geq 5$ that divides the order of M, and the elements of M of order p are semiregular. As before, applying Lemma 2.2, we get that NM contains a semiregular element of order 3p, a contradiction. We conclude that N is the unique minimal normal subgroup of G.

Notice that $Alt(5) \leq G \leq Sym(5)$ or $Alt(6) \leq G \leq Aut(Alt(6))$. A computer computation in each of these cases shows that, if $G \leq Aut(\Gamma)$ has no semiregular elements of order at least 6, then $|V\Gamma| \in \{30, 60, 90, 180, 360\}$, which contradicts Hypothesis 3.1.

From here on, we divide the proof in five cases:

- $G_{\alpha} = 1;$
- $G_{\alpha} \neq 1$ and N is transitive on $V\Gamma$;
- $G_{\alpha} \neq 1$ and N has two orbits on $V\Gamma$;
- $G_{\alpha} \neq 1$ and Γ/N is a cycle of length at least 3;
- $G_{\alpha} \neq 1$ and Γ/N is a cubic graph.

¹Recall that, if $X \leq \text{Sym}(\Omega)$ is an abelian group and X acts regularly on Ω , then $X = \mathbf{C}_{\text{Sym}(\Omega)}(X)$.

3.1. $G_{\alpha} = 1$. In this case Γ is a Cayley graph over G. This means that there exists an inverseclosed subset I of G with $\Gamma \cong \operatorname{Cay}(G, I)$. We recall that $\operatorname{Cay}(G, I)$ is the graph having vertex set G where two vertices x and y are declared to be adjacent if and only if $yx^{-1} \in I$. Since Γ has valency 3, we have |I| = 3. Moreover, since Γ is connected, we have $G = \langle I \rangle$. In particular, G is generated by at most 3 elements. More precisely, either I consists of three involutions or Iconsists of an involution and an element of order greater than 2 together with its inverse.

In what follows we say that a finite group X satisfies \mathcal{P} if X is generated by either three involutions, or by an involution and by an element of order greater than 2. In particular, G satisfies \mathcal{P} .

Since each element of G is semiregular and since G has no semiregular elements of order at least 6, we deduce that each element of G has order at most 5. As customary, we let

$$\omega(G) := \{ o(g) \mid g \in G \}$$

be the spectrum of G. Observe that

$$\{1,2\} \subseteq \omega(G) \subseteq \{1,2,3,4,5\}.$$

Since G is generated by at most 3 elements, we deduce from Zelmanov's solution of the restricted Burnside problem that |G| is bounded above by an absolute constant. We divide the proof depending on $\omega(G)$.

Assume $\omega(G) = \{1, 2\}$. In this case, G is elementary abelian and, since G is generated by at most 3 elements, we deduce $|G| \leq 8$, which contradicts Hypothesis 3.1.

Assume $\omega(G) = \{1, 2, 4\}$. Here, either G is generated by an element of order 2 and an element of order 4, or G is generated by three involutions. We resolve these two cases with a computer computation. Suppose first that G is generated by an involution and by an element of order 4. We have constructed the free group $F := \langle x, y \rangle$ and we have constructed the set W of words in x, y of length at most 6. Then, we have constructed the finitely presented group $\overline{F} := \langle F | x^2, \{ w^4 : w \in W \} \rangle$. We use the "bar" notation for the projection of F onto \overline{F} . Now, \overline{x} has order 2 and \bar{y} has order 4. Furthermore, each element of \bar{F} that can be written as a word in \bar{x} and \bar{y} of length at most 6 has order at most 4. (The number 6 was chosen arbitrarily but large enough to guarantee to get an upper limit on the cardinality of G.) A computer computation shows that F has order 64 and exponent 4. This proves that the largest group of exponent 4 and generated by an involution and by an element of order 4 has order 64. Now, G is a quotient of F and hence |G| < |F| < 64, which contradicts Hypothesis 3.1. Next, suppose that G is generated by three involutions. The argument here is very similar. We have considered the free group $F = \langle x, y, z \rangle$, and we have considered the set W of words in x, y, z of length at most 6. We have verified that $\langle F|x^2, y^2, z^2, \{w^4: w \in W\}\rangle$ has order 1024 and exponent 4. This shows that $|G| \leq 1024$, which contradicts Hypothesis 3.1.

Assume $\omega(G) = \{1, 2, 3\}$. The groups having spectrum $\{1, 2, 3\}$ are classified in [Neu37]. Routine computations in the list of groups X classified in [Neu37, Theorem] show that, if X satisfies \mathcal{P} , then $|X| \leq 18$, which contradicts Hypothesis 3.1.

Assume $\omega(G) = \{1, 2, 5\}$. The groups having spectrum $\{1, 2, 5\}$ are classified in [New79]. As above, since G satisfies \mathcal{P} , we deduce from a case-by-case analysis in the groups appearing in [New79] that $|G| \leq 80$, which contradicts Hypothesis 3.1.

Assume $\omega(G) = \{1, 2, 3, 4\}$. The groups having spectrum $\{1, 2, 3, 4\}$ are classified in [BS91]. As above, since G satisfies \mathcal{P} , we deduce from a case-by-case analysis in the groups appearing in [BS91, Theorem] that $|G| \leq 96$, which contradicts Hypothesis 3.1.

Assume $\omega(G) = \{1, 2, 4, 5\}$. The groups having spectrum $\{1, 2, 4, 5\}$ are classified in [GM99]. This case is slightly more involved and hence we do give more details. We have three cases to consider:

(1) $G = T \rtimes D$ where T is a non-trivial elementary abelian normal 2-subgroup and D is a non-abelian group of order 10,

- (2) $G = F \rtimes T$ where F is an elementary abelian normal 5-subgroup and T is isomorphic to a subgroup of a quaternion group of order 8,
- (3) G contains a normal 2-subgroup T which is nilpotent of class at most 6 such that G/T is a 5-group.

Suppose that (1) holds. Clearly, D is the dihedral group of order 10 and T is a module for D over the field \mathbb{F}_2 of cardinality 2. The dihedral group D has two irreducible modules over \mathbb{F}_2 up to equivalence: the trivial module and a 4-dimensional module W. Since G has no elements of order 10, we deduce $V \cong W^{\ell}$, for some $\ell \ge 1$. We have verified with a computer computation that $W^3 \rtimes D$ does not satisfy \mathcal{P} and hence $G \cong W^{\ell} \rtimes D$ with $\ell \le 2$. We deduce that $|G| = |V\Gamma| \in \{10 \cdot 16, 10 \cdot 16^2\} = \{160, 2560\}$. From Hypothesis 3.1, we have $|V\Gamma| > 1280$ and hence $G \cong W^2 \rtimes D$. We have constructed all connected cubic Cayley graphs over $W^2 \rtimes D$ and we have found only one (up to isomorphism), therefore we obtain the example in Table 1.

Suppose that (2) holds. Since G satisfies \mathcal{P} , while the quaternion group of order 8 does not, we deduce that T is cyclic of order 4. Thus $G = F \rtimes \langle x \rangle$, for some x having order 4. As G satisfies \mathcal{P} , this means that $G = \langle x, y \rangle$, for some involution y. Clearly, $y = fx^2$ for some $f \in F$. As $G = \langle x, y \rangle = \langle x, fx^2 \rangle = \langle x, f \rangle$, we have $F = \langle f, f^x, f^{x^2}, f^{x^3} \rangle$. Since $y = fx^2$ has order 2 and x has order 4, we deduce

$$1 = y^2 = fx^2 fx^2 = ff^{x^2},$$

that is, $f^{x^2} = f^{-1}$. Now, $F = \langle f, f^x, f^{x^2}, f^{x^3} \rangle = \langle f, f^x, f^{-1}, (f^x)^{-1} \rangle = \langle f, f^x \rangle$. Thus $|F| \le 25$ and hence $|G| \le 100$, which contradicts Hypothesis 3.1.

Suppose that (3) holds. Since G satisfies \mathcal{P} , we deduce that G/T is cyclic of order 5. Thus $G = T \rtimes \langle x \rangle$, for some x having order 5. This means that $G = \langle x, y \rangle$, for some involution y. Clearly, $y \in T$. From Hypothesis 3.1, we have $|G| = |V\Gamma| > 1280$. Let N be a minimal normal subgroup of G. We have $N \leq T$ and N is an irreducible $\mathbb{F}_2\langle x \rangle$ -module. The cyclic group of order 5 has two irreducible modules over \mathbb{F}_2 up to equivalence: the trivial module and a 4-dimensional module. Since G has no elements of order 10, x does not centralize N and hence N is the irreducible 4-dimensional module for the cyclic group of order 5. In particular, $|N| = 2^4$. Consider $\overline{G} := G/N$. Now,

$$\{1, 2, 5\} \subseteq \omega(G) \subseteq \omega(G) = \{1, 2, 4, 5\}.$$

Assume $\omega(\bar{G}) = \{1, 2, 5\}$. From the discussion above (regarding the finite groups having spectrum $\{1, 2, 5\}$ and satisfying \mathcal{P}), we have $|\bar{G}| \leq 80$ and hence $|G| = |G : N||N| \leq 80 \cdot 16 = 1280$, which is a contradiction. Therefore, $\omega(\bar{G}) = \{1, 2, 4, 5\}$. Since (Γ, G) was chosen minimal in Hypothesis 3.1, we have $|\bar{G}| \leq 1280$. Therefore $(\Gamma/N, \bar{G})$ appears in Table 1. An inspection on the groups appearing in this table shows that there is only one group having spectrum $\{1, 2, 4, 5\}$ and is the group of order 1280. Thus we know precisely \bar{G} . Now, the group G is an extension of \bar{G} by N and hence it can be computed with the cohomology package in the computer algebra system magma. We have computed all the extensions E of \bar{G} via N and we have verified that none of the extensions E has the property that $\omega(E) = \{1, 2, 4, 5\}$ and with E satisfying \mathcal{P} .

Assume $\omega(G) = \{1, 2, 3, 5\}$. The groups having spectrum $\{1, 2, 3, 5\}$ are classified in [MZ99]. We deduce from [MZ99] that $G \cong A_5$, which contradicts Hypothesis 3.1.

Assume $\omega(G) = \{1, 2, 3, 4, 5\}$. The groups having spectrum $\{1, 2, 3, 4, 5\}$ are classified in [BS91]. We deduce from [BS91, Theorem] that either $G \cong A_6$ or $G \cong V^\ell \rtimes A_5$ where V is a 4-dimensional natural module over the finite field of size 2 for $A_5 \cong SL_2(4)$ and $\ell \ge 1$. The group $V^2 \rtimes A_5$ does not satisfy \mathcal{P} (this can be verified with a computer computation). Therefore, either $G \cong A_6$ or $G \cong V \rtimes A_5$. Thus $|G| = |V\Gamma| \le 960$, which contradicts Hypothesis 3.1.

3.2. $G_{\alpha} \neq 1$ and N is transitive on V Γ . By Hypothesis 3.1, (Γ, G) is a minimal counterexample. This minimality and the fact that N is transitive on V Γ imply that G = N. As N is a minimal normal subgroup of G, G is simple. Thus $G \in \{Alt(5), Alt(6), PSp(4, 3)\}$. A computer

computation in each of these cases shows that, if $G \leq \operatorname{Aut}(\Gamma)$ has no semiregular elements of order at least 6, then $|V\Gamma| \in \{10, 20, 30, 60, 90, 180, 360\}$, which contradicts Hypothesis 3.1.

3.3. $G_{\alpha} \neq 1$ and N has two orbits on VΓ. Suppose N is abelian. By [PS21, Lemma 1.15], either Γ is complete bipartite, or Γ is a bi-Cayley graph over N and the minimal number of generators of N is at most 4. (Here, it is not really relevant to introduce the definition of bi-Cayley graph, however, what is really relevant is the fact that N is generated by at most 4 elements.) Recalling that N is a $\{2,3,5\}$ -group, it follows that $|V\Gamma| = 2|N| \leq 2 \cdot 5^4 = 1250$, and the equality is realized for $N = C_5^4$. In particular, this contradicts Hypothesis 3.1.

Suppose N is not abelian. By Lemma 3.2, 3 divides the order of N_{α} . A fortiori, 3 divides the order of G_{α} , hence G acts arc-transitively on Γ . We can extract information on the local action of G by consulting the amalgams in [DM80, Section 4]. In particular, with a direct inspection (on a case-by-case basis) on these amalgams, it can be verified that, for any edge $\{\alpha, \beta\}$ of Γ , G contains an element y that swaps α and β and its order is either 2 or 4. As α and β belong to distinct N-orbits, y maps α^N to β^N . Moreover, as N has two orbits on $V\Gamma$, the subgroup $N\langle y \rangle$ is vertex-transitive on Γ . Therefore, by minimality of G, we have $G = N\langle y \rangle$.

Assume o(y) = 2. Thus |G : N| = 2. As $N = S^l$ is a minimal normal subgroup of G, $l \in \{1, 2\}$. If l = 1, then G is an almost simple group whose socle is either Alt(5), Alt(6) or PSp(4,3). A computer computation shows that (Γ, G) satisfies Theorem 1.2, a contradiction. If l = 2, then $\langle y \rangle$ permutes transitively the two simple direct factors of N. Let $s \in N$ be a 5-element in a simple direct factor of N, and notice that $t := s^y$ is a 5-element in the other simple direct factor of N. Thus [s, t] = 1. We claim that ys is a semiregular element of order 10. We get

$$(ys)^2 = ysys = ts \in N,$$

 $(ys)^5 = ysysysysys = ys(ts)^2 \in yN.$

We have that $(ys)^2$ is a 5-element in N, thus semiregular, and that $(ys)^5$ has order 2 and, being an element of yN = Ny, it has no fixed points, hence it is semiregular. Therefore ys is a semiregular element of order 10, contradicting Hypothesis 3.1.

Assume o(y) = 4. As |G:N| = 4 and N is a minimal normal subgroup of $G, l \in \{1, 2, 4\}$. Observe that a Sylow 3-subgroup of G_{α} has order 3, because Γ is cubic and G is arc-transitive. Let $x \in G_{\alpha}$ be an element of order 3. As |G:N| = 4, we have $x \in N \cap G_{\alpha} = N_{\alpha} \leq S^{l}$. In particular, we may write $x = (s_{1}, \ldots, s_{l})$, with $s_{i} \in S$. Let κ be the number of coordinates of x different from 1, we call κ the type of x. Since $\langle x \rangle$ is a Sylow 3-subgroup of G_{α} , from Sylow's theorem, we deduce that each element of order 3 in G fixing some vertex of Γ has type κ . Let $s \in S$ be an element of order 3 and let $t \in S$ be an element of order 5. Suppose l = 4. If $\kappa \neq 1$, then g = (s, t, 1, 1) has order 15 and is semiregular because $g^{5} = (s^{5}, 1, 1, 1)$ has order 3 but it is not of type κ . Similarly, if l = 4 and k = 1, then g = (s, s, t, 1) has order 15 and is semiregular. Analogously, when l = 2, if $\kappa \neq 1$, then g = (s, t) has order 15 and is semiregular. When l = 2, $\kappa = 1$ and S = PSp(4, 3), the group S contains an element r having order 9 and hence g = (r, r) is a semiregular element having order 9. Summing up, from these reductions, we may suppose that either l = 1, or l = 2 and $S \in \{Alt(5), Alt(6)\}$. These cases can be dealt with a computer computation: indeed, the invaluable help of a computer shows that no counterexample to Theorem 1.2 arises.

3.4. $G_{\alpha} \neq 1$ and Γ/N is a cycle of length $r \geq 3$. The full automorphism group of Γ/N is the dihedral group of order 2r. Let K be the kernel of the action of G on the N-orbits. The quotient G/K acts faithfully on Γ/N , that is, it is a transitive subgroup of the dihedral group of order 2r.

We claim that

(3.1)
$$G/K$$
 is regular in its action on the vertices of Γ/N .

Assume G/K acts on the vertices of Γ/N transitively but not regularly. In particular, G/K is isomorphic to the dihedral group of order 2r. Thus G has an index 2 subgroup M such that M is vertex-transitive and M/K is isomorphic to the cyclic group of order r. By minimality of G, we have G = M, which goes against the choice of M. Hence G/K is regular. In particular, either G/K is isomorphic to the cyclic group of order r, or r is even and G is isomorphic to the dihedral group of order r. Later in this proof we resolve this ambiguity and we prove that r is even and G/K is dihedral of order r, see (3.5).

As G/K acts regularly on the vertices of Γ/N , we have

$$1_{G/K} = \left(\frac{G}{K}\right)_{\alpha^N} = \frac{G_{\alpha}K}{K}.$$

Therefore

$$(3.2) K_{\alpha} = K \cap G_{\alpha} = G_{\alpha}$$

Assume G is arc-transitive. Let β be a neighbour of α and observe that $\alpha^N \neq \beta^N$. Since Γ is connected, we have

$$G = \langle G_{\alpha}, G_{\{\alpha,\beta\}} \rangle = \langle K_{\alpha}, G_{\{\alpha,\beta\}} \rangle \le \langle K, G_{\{\alpha,\beta\}} \rangle = KG_{\{\alpha,\beta\}},$$

and hence $G = KG_{\{\alpha,\beta\}}$. Recalling that K fixes all the N-orbits,

$$|G:K| = |KG_{\{\alpha,\beta\}}:K| = |G_{\{\alpha,\beta\}}:K_{\{\alpha,\beta\}}| = |G_{\{\alpha,\beta\}}:G_{\alpha\beta}| = 2.$$

Thus $G/K \cong C_2$ and r = 2, which is a contradiction. Therefore

G is not arc-transitive.

This implies that G_{α} does not act transitively on the neighbourhood of α , hence G_{α} is a 2-group. By (3.2), we deduce $G_{\alpha} = K_{\alpha}$ is a 2-group. Actually, Lemma 2.3 shows that

(3.3)
$$G_{\alpha} = K_{\alpha}$$
 is an elementary abelian 2-group.

If N is an elementary abelian 2-group, then, by Lemma 2.14, Γ is either a circular ladder, or a Möbius ladder, or a split Praeger-Xu graph sC(r/2, s). Now, in the former cases, the proof follows from Lemma 2.6, while, in the latter one, we conclude by Lemma 2.13. In particular, for the rest of the proof we may suppose that N is not an elementary abelian 2-group.

For any minimal normal subgroup M of G, $M_{\alpha} = M \cap G_{\alpha}$ is also a 2-group. Thus, in view of Lemma 3.2, M is an elementary abelian p-group, for some $p \in \{2, 3, 5\}$. This is true, in particular, for N. Let M be a minimal normal subgroup distinct from N. Since [N, M] = 1, Lemma 2.2 gives a contradiction unless N and M are both p-groups for the same prime p. Thus,

(3.4) the socle of G is an elementary abelian p-group, for some $p \in \{3, 5\}$.

Before going any further, we need some extra information on the local action of G on Γ . Since G_{α} is a non-identity 2-group, there exists a unique vertex $\alpha' \in V\Gamma$ adjacent to α that is fixed by the action of G_{α} . It follows that $\{\alpha, \alpha'\}$ is a block of imprimitivity for the action of G on the vertices. Hence,

 $G_{\alpha} \leq G_{\{\alpha,\alpha'\}}$ and $|G_{\{\alpha,\alpha'\}}: G_{\alpha}| = 2.$

We obtain that, for any $\beta \in V\Gamma$, neighbour of α distinct from α' ,

 $|G_{\{\alpha,\alpha'\}}:G_{\alpha\beta}|=4$ and $|G_{\{\alpha,\beta\}}:G_{\alpha\beta}|=2.$

Let $\{\alpha', \beta, \gamma\}$ be the neighbourhood of α .

Assume G/K is cyclic of order r. As Γ/N is a cycle of length r, this means that G/K acts transitively on the vertices and on the edges of Γ/N . Now, β and γ are in the same K-orbit because $K_{\alpha} = G_{\alpha}$ and G_{α} acts transitively on $\{\beta, \gamma\}$. In particular, each element in α^N has two neighbours in β^N . As G/K is transitive on edges, we reach a contradiction because each

element in α^N would have two neighbours in ${\alpha'}^N$, contradicting the fact that α has valency 3. Thus

(3.5) r is even and G/K is dihedral of order r.

Recall that N is an elementary abelian p-group with $p \in \{3, 5\}$. Thus N is semiregular. We consider $\mathbf{C}_K(N)$. Since $N \leq \mathbf{C}_K(N)$ and since $K = K_\alpha N$, we deduce $\mathbf{C}_K(N) = N \times Q$, for some subgroup Q of K_α . As K_α is a 2-group, so is Q. Therefore, Q is characteristic in $N \times Q = \mathbf{C}_K(N)$ and hence $Q \leq G$. Since G_α is a core-free subgroup of G, we get Q = 1 and $\mathbf{C}_K(N) = N$.

Since N is a minimal normal subgroup of G, G acts irreducibly by conjugation on it, that is, N is an irreducible \mathbb{F}_pG -module. As $K \leq G$, by Clifford's Theorem, N is a completely reducible \mathbb{F}_pK -module. As $K = NG_{\alpha}$ and N is abelian, N is a completely reducible \mathbb{F}_pG_{α} -module. As G_{α} is abelian, by Schur's Lemma, G_{α} induces on each irreducible \mathbb{F}_pG_{α} -submodule a cyclic group action. However, since G_{α} has exponent 2, we deduce that each irreducible \mathbb{F}_pG_{α} -submodule has dimension 1 and G_{α} induces on each irreducible \mathbb{F}_pG_{α} -submodule the scalars ± 1 . Therefore, G_{α} acts on N by conjugation as a group of diagonal matrices having eigenvalues in $\{\pm 1\}$. In other words, there exists a basis (n_1, \ldots, n_e) of N as a vector space over \mathbb{F}_p such that,

(3.6) for each
$$g \in G_{\alpha}$$
 and for each n_i , we have $n_i^g \in \{n_i, n_i^{-1}\}$

Furthermore, the action of G by conjugation on N preserves the direct product decomposition $N = \langle n_1 \rangle \times \cdots \times \langle n_e \rangle.$ We claim that

(3.7)
$$\mathbf{C}_{G_{\{\alpha,\beta\}}}(N) = 1$$
$$\mathbf{C}_{G_{I\alpha,\alpha'}}(N) = 1$$

In other words, $G_{\{\alpha,\beta\}}$ and $G_{\{\alpha,\alpha'\}}$ both act faithfully by conjugation on N. Let $\gamma \in \{\alpha',\beta\}$ and suppose, arguing by contradiction, that $\mathbf{C}_{G_{\{\alpha,\gamma\}}}(N) \neq 1$. Since $\mathbf{C}_K(N) = 1$ and $|G_{\{\alpha,\gamma\}} : K \cap G_{\{\alpha,\gamma\}}| = 2$, we deduce $\mathbf{C}_{G_{\{\alpha,\gamma\}}}(N) = \langle x \rangle$, where x is an involution. Since $x \notin K$, x acts semiregularly on Γ/N and hence x acts semiregularly on Γ . From this and from the fact that x centralizes N, we deduce that G contains semiregular elements of order $2p \geq 6$, which contradicts Hypothesis 3.1. Thus (3.7) is proven.

Observe that (3.7) implies that an element of $G_{\{\alpha,\alpha'\}}$ or of $G_{\{\alpha,\beta\}}$ is the identity if and only it its action on N by conjugation is trivial.

We show that

(3.8)
$$G_{\{\alpha,\beta\}} \setminus G_{\alpha\beta}$$
 contains an involution

Let H be the permutation group induced by $G_{\{\alpha,\alpha'\}}$ in its action on the four right cosets of $G_{\alpha\beta}$ in $G_{\{\alpha,\alpha'\}}$. Since H is a 2-group, H is isomorphic to either C_4 , or $C_2 \times C_2$, or to the dihedral group of order 8. In the first two cases, $G_{\alpha\beta}$ is a normal subgroup of both $G_{\{\alpha,\alpha'\}}$ and $G_{\{\alpha,\beta\}}$. As $G_{\alpha\beta}$ is core-free in G and

$$G = \langle G_{\{\alpha,\alpha'\}}, G_{\{\alpha,\beta\}} \rangle,$$

we have that $G_{\alpha\beta} = 1$. In particular, $G_{\{\alpha,\beta\}}$ is cyclic of order 2, hence it contains an involution and (3.8) follows in this case.

In the latter case, using the notation and the terminology in [Djo80], we have that the triple $(G_{\{\alpha,\alpha'\}}, G_{\alpha\beta}, G_{\{\alpha,\beta\}})$ is a locally dihedral faithful group amalgam of type (4, 2) and G is one of its realizations. Indeed, from the classification in [Djo80], we see that either $G_{\{\alpha,\alpha'\}} \setminus G_{\alpha}$ or $G_{\{\alpha,\beta\}} \setminus G_{\alpha\beta}$ contains an involution. If $G_{\{\alpha,\beta\}} \setminus G_{\alpha\beta}$ contains an involution, then (3.8) holds true also in this case. Therefore we suppose $\tau_1 \in G_{\{\alpha,\alpha'\}} \setminus G_{\alpha}$ is an involution. We investigate the action by conjugation of τ_1 on N. By (3.1), τ_1 is a semiregular automorphism of Γ/K , because $\tau_1 \notin K$. Therefore, τ_1 is a semiregular automorphism of Γ . Since no semiregular involution commutes with a non-identity element of N, τ_1 acts by conjugation on N without fixed points,

that is, for any $n \in N$, $n^{\tau_1} = n^{-1}$. It follows from (3.6) that τ_1 commutes with G_{α} and hence $G_{\{\alpha,\alpha'\}} = \langle G_{\alpha}, \tau_1 \rangle$ is an elementary abelian 2-group. Now, as $G_{\alpha\beta}$ is normal in both $G_{\{\alpha,\alpha'\}}$ and $G_{\{\alpha,\beta\}}$, we can conclude, as before, that $G_{\{\alpha,\beta\}}$ is cyclic of order 2, hence it contains an involution. Therefore, in any case, (3.8) holds true.

Let e be the positive integer such that $N = C_n^e$. We aim to show that

$$(3.9) e \in \{1, 2\}.$$

Let $\tau_2 \in G_{\{\alpha,\beta\}} \setminus G_{\alpha\beta}$ be an involution: the existence of τ_2 is guaranteed by (3.8). Now, we look at the action by conjugation of τ_2 on N. Observe $\tau_2 \notin K$ and hence τ_2 is a semiregular automorphism of Γ . Therefore, arguing as in the previous paragraph (with the involution τ_1 replaced by τ_2), we deduce that $n^{\tau_2} = n^{-1}$ for every $n \in N$. Let $L := \langle \tau_2^g \mid g \in G \rangle$. Since G/K is a dihedral group and τ_2 is an involution, we deduce that $|G/K : LK/K| \leq 2$, that is, $|G : LK| \leq 2$. Observe now that, for any $n \in N$, $n^{\tau_2^g} = n^{-1}$. Therefore, the group induced by the action by conjugation of L on N has order 2. This and (3.6) shows that the subgroup LK of G preserves the direct sum decomposition $N = \langle n_1 \rangle \times \cdots \times \langle n_e \rangle$. However, since G acts irreducibly on N and since $|G : LK| \leq 2$, we finally obtain $e \leq 2$, as claimed in (3.9). Observe that from this it follows that $|N| = p^e \in \{3, 9, 5, 25\}$.

We are now ready to conclude this case. Observe that G_{α} contains an element x with $n^x = n^{-1}$ for every $n \in N$. This is immediate from (3.6) when e = 1, or when e = 2 and $|G_{\alpha}| = 4$. When e = 2 and $|G_{\alpha}| < 4$, we have $|G_{\alpha}| = 2$ and hence the non-identity element of G_{α} acts by conjugation on N inverting each of its elements.

Now, x and τ_2 both induce the same action by conjugation on N, contradicting (3.7). This final contradiction has concluded the analysis of this case.

3.5. $G_{\alpha} \neq 1$ and Γ/N is a cubic graph. Under this assumption, any two distinct neighbours of α are in distinct *N*-orbits, thus $N_{\alpha} = 1$. In particular, Lemma 3.2 gives that *N* is elementary abelian. Set $\overline{\Gamma} := \Gamma/N$, $\overline{G} := G/N$ and $\overline{\alpha} := \alpha^N$. Since $|V\overline{\Gamma}| < |V\Gamma|$, by Hypothesis 3.1 the pair $(\overline{\Gamma}, \overline{G})$ is not a counterexample to Theorem 1.2 and hence $(\overline{\Gamma}, \overline{G})$ is one of the pairs appearing in Table 1. Moreover, since $G_{\alpha} \neq 1$, we have the additional information that a vertex-stabilizer $\overline{G}_{\overline{\alpha}} \cong G_{\alpha}$ is not the identity.

We have resolved this case with a computer computation. Since this computer computation is quite involved, we give some details. Let $(\bar{\Gamma}, \bar{G})$ be any pair in Table 1, except for the last row. For each prime $p \in \{2, 3, 5\}$, we have constructed all the irreducible modules of \bar{G} over the field \mathbb{F}_p having p elements. Let V be one of these irreducible modules. This module V corresponds to the putative minimal normal subgroup N of G. We have constructed all the distinct extensions of \bar{G} via V. Let E be one of these extensions and let $\pi : E \to \bar{G}$ be the natural projection with $\operatorname{Ker}(\pi) = V$. This extension E corresponds to the putative abstract group G. For each such extension, we have computed all the subgroups H of E with the property that $\pi_{|H}$ is an isomorphism between H and $\bar{G}_{\bar{\alpha}}$. This subgroup H is our putative vertex-stabilizer G_{α} . This computation can be performed in $\pi^{-1}(\bar{G}_{\bar{\alpha}})$. Next, we have constructed the permutation representation E_p of E acting on the right cosets of H in E. This permutation group E_p is our putative permutation group G. If E_p has semiregular elements of order at least 6, then we have discarded E from further consideration.

For each permutation group E_p as above, we have verified, by considering the orbital graphs of E_p , whether E_p acts on a connected cubic graph. This is our putative graph Γ . This step is by far the most expensive step in the computation.

This whole process had to be applied repeatedly starting with the pairs arising from the census of connected cubic graphs having at most 1 280 vertices.

For instance, the graphs having 65 610 vertices were found by applying this procedure starting with the graph having 810 vertices and its transitive group of automorphisms having 1 620 elements: here the elementary abelian cover N has cardinality $81 = 3^4$. Incidentally, we have

found only one pair up to isomorphism. Next, by applying this procedure to this pair, we found no new examples.

We give some further details of the computation when we applied the procedure with $\overline{\Gamma}$ having $1250 = 2 \cdot 5^4$ vertices and with its corresponding vertex-transitive subgroup \overline{G} having order $2500 = 2^2 \cdot 5^4$. When we applied this procedure, we have obtained graphs having $2 \cdot 5^5 = 6250$ vertices and admitting a group of automorphisms having $2^2 \cdot 5^5 = 12500$ elements. Actually, in this step, we have found only one pair up to isomorphism. We have repeated this procedure two more times, obtaining graphs having $2 \cdot 5^6 = 31250$ and $2 \cdot 5^7 = 156250$ vertices. We were not able to push this computation further. Therefore to complete the proof of Theorem 1.2, we need to show that any new pair (Γ, G) has the property that $|V\Gamma| = 2 \cdot 5^{\ell}$ and $|G| = 4 \cdot 5^{\ell}$, with $\ell \leq 34$.

From the discussion above we may suppose that $|V\bar{\Gamma}| = 2 \cdot 5^{\ell}$ and $|\bar{G}| = 4 \cdot 5^{\ell}$ with $\ell \leq 34$. Moreover, $\overline{\Gamma}$ is a regular cover of the graph, say Δ , having 1 250 vertices and \overline{G} is a quotient of the group of automorphisms of Δ , say H, with |H| = 2500. In particular, a Sylow 2-subgroup of \overline{G} is cyclic and \overline{G} has a normal Sylow 5-subgroup. (This information can be extracted from the analogous properties of H.) Let \overline{P} be a Sylow 5-subgroup of \overline{G} and observe that every non-identity element of \overline{P} has order 5 because every semiregular element of \overline{G} has order at most 6. Let P be the subgroup of G with $G/N = \overline{P}$. Assume N is not an elementary abelian 5-group. Then N is an elementary abelian p-group for some $p \in \{2,3\}$. Let Q be a Sylow 5-subgroup of P and observe that $P = N \rtimes Q$. The elements in P are semiregular and hence each element of P has order at most 6. This implies that the elements of P have order 1, 5 or p. This implies that the action, by conjugation, of Q on N is fixed-point-free and P is a Frobenius group with Frobenius kernel N and Frobenius complement Q. The structure theorem of Frobenius complements gives that Q is cyclic and hence |Q| = 5, which is a contradiction. This contradiction has shown that N is an elementary abelian 5-group and hence P is a Sylow 5-subgroup of G. Moreover, $G = P \rtimes \langle x \rangle$, where $\langle x \rangle$ is a cyclic group of order 4. We have shown that $|V\Gamma| = 2 \cdot 5^{\ell'}$ and $|G| = 2^2 \cdot 5^{\ell'}$. Therefore, it remains to show that $\ell' < 34$.

Since $|G_{\alpha}| = 2$, G_{α} fixes a unique neighbour of α . Let us call α' this neighbour. Now, $G_{\{\alpha,\alpha'\}}$ has order 4 because $\{\alpha, \alpha'\}$ is a block of imprimitivity for the action of G on $V\Gamma$. Therefore, by Sylow's theorem, we may suppose that

$$G_{\{\alpha,\alpha'\}} = \langle x \rangle.$$

In particular, $G_{\alpha} = \langle x^2 \rangle$.

Let β and γ be the neighbours of α with $\beta \neq \alpha' \neq \gamma$. Clearly, $|G_{\{\alpha,\beta\}}| = 2$ and hence, by Sylow's theorem,

$$G_{\{\alpha,\beta\}} = \langle (x^2)^y \rangle,$$

for some $y \in P$.

Since Γ is connected, we have

$$G = \langle G_{\{\alpha,\alpha'\}}, G_{\{\alpha,\beta\}} \rangle = \langle x, (x^2)^y \rangle = \langle x, y^{-1}y^{x^2} \rangle$$

As $P \leq G$ and o(x) = 4, we deduce

$$P = \langle y^{-1}y^{x^2}, (y^{-1}y^{x^2})^x, (y^{-1}y^{x^2})^{x^2}, (y^{-1}y^{x^2})^{x^3} \rangle.$$

Now,

$$(y^{-1}y^{x^2})^{x^2} = (y^{x^2})^{-1}y^{x^4} = (y^{x^2})^{-1}y = (y^{-1}y^{x^2})^{-1}$$

Therefore, $P = \langle y^{-1}y^{x^2}, (y^{-1}y^{x^2})^x \rangle$ is a 2-generated group of exponent 5. In view of the restricted Burnside problem (see [HWW74] and [Zel91]), the order of P is at most 5³⁴ and hence $\ell' \leq 34$.

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