PROJECTION CONSTANTS FOR SPACES OF DIRICHLET POLYNOMIALS

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ABSTRACT. Given a frequency sequence $\omega = (\omega_n)$ and a finite subset $J \subset \mathbb{N}$, we study the space $\mathscr{H}_\infty^J(\omega)$ of all Dirichlet polynomials $D(s) := \sum_{n\in J} a_n e^{-\omega_n s}$, $s \in \mathbb{C}$. The main aim is to prove asymptotically correct estimates for the projection constant $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big)$ of the finite dimensional Banach space $\mathscr{H}_\infty^J(\omega)$ equipped with the norm $\|D\| = \sup_{\operatorname{Re} s>0} |D(s)|$. Based on harmonic analysis on ω-Dirichlet groups, we prove the formula $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big) \;=\; \lim\limits_{T\to\infty} \dfrac{1}{2T}\displaystyle\int_{-T}^{T} \Big|\sum_{n\in J} e^{-i\omega_n t}\Big|\, dt$, and apply it to various concrete frequencies ω and index sets J. To see an example, combining with a recent deep result of Harper from probabilistic analytic number theory, we for the space $\mathscr{H}_\infty^{\le x}\big((\log n)\big)$ of all ordinary Dirichlet polynomials $D(s) = \sum_{n\le x} a_n n^{-s}$ of length x show the asymptotically correct order $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\le x}\big((\log n)\big)\big) \sim \sqrt{x}/(\log\log x)^{\frac{1}{4}}$.

INTRODUCTION

The study of complemented subspaces of a Banach space and their projection constants has a long history going back to the beginning of operator theory in Banach spaces. Recall that if X is a closed subspace of a Banach space Y, then the relative projection constant of X in Y is defined by

$$\boldsymbol{\lambda}(X,Y) = \inf\big\{\|P\| :\ P \in \mathscr{L}(Y,X),\ P|_X = \mathrm{id}_X\big\},$$

where id_X is the identity operator on X and as usual $\mathscr{L}(U,V)$ denotes the Banach space of all bounded linear operators between the Banach spaces U and V with the uniform norm. We use here the convention that $\inf\emptyset = \infty$.

The following straightforward result shows the intimate link between projection constants and extensions of linear operators: For every Banach space Y and its subspace X one has

$$\boldsymbol{\lambda}(X,Y) = \inf\big\{c>0 :\ \forall\, T \in \mathscr{L}(X,Z) \ \ \exists \ \text{an extension } \ \widetilde{T} \in \mathscr{L}(Y,Z) \ \text{with } \|\widetilde{T}\| \le c\,\|T\|\big\},$$

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where Z is any Banach space. Moreover, the (absolute) projection constant of a Banach space X is given by

$$\boldsymbol{\lambda}(X) := \sup \boldsymbol{\lambda}(I(X), Y),$$

where the supremum is taken over all Banach spaces Y for which it exists some isometric embedding $I\colon X \to Y$ such that $I(X)$ is complemented in Y.

A frequency $\omega = (\omega_n)_{n\in\mathbb{N}}$ is a strictly increasing, non-negative real sequence such that $\omega_n \to \infty$ as $n \to \infty$. Given a finite index set $J \subset \mathbb{N}$ and complex numbers $(a_n)_{n\in J}$, we say that

$$D(s) := \sum_{n\in J} a_n e^{-\omega_n s}, \quad s \in \mathbb{C}$$

is a ω-Dirichlet polynomial supported on the index set J. For the frequency $\omega = (n)_{n\in\mathbb{N}}$ we obtain (after the substitution $z = e^{-s}$) polynomials $\sum_{n\in J} a_n z^n$ in one complex variable, and in the case $\omega = (\log n)_{n\in\mathbb{N}}$ all ordinary Dirichlet polynomials $\sum_{n\in J} a_n n^{-s}$.

Denote by $\mathscr{H}_\infty^J(\omega)$ the (finite dimensional) Banach space of all ω-Dirichlet polynomials supported on the finite index subset $J \subset \mathbb{N}$, endowed with the norm

$$\|D\|_\infty := \sup_{t\in\mathbb{R}} \Big| \sum_{n\in J} a_n e^{-i\omega_n t} \Big| = \sup_{\operatorname{Re} s>0} \Big| \sum_{n\in J} a_n e^{-\omega_n s} \Big|,$$

where the last equality is a simple consequence of the maximum modulus principle.

Then the main goal of this article is to study the projection constant

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big)$$

for various 'natural' frequencies ω and various 'natural' finite index sets J of $\mathbb{N}$. Given $x \in \mathbb{N}$, we are particularly interested in the projection constant of the Banach space

$$\mathscr{H}_\infty^{\leq x}(\omega) = \mathscr{H}_\infty^{\{n\in\mathbb{N}:\ n\leq x\}}(\omega),$$

so all ω-Dirichlet polynomials $D(s) = \sum_{n\leq x} a_n e^{-\omega_n s}$ of length x.

Before we illustrate some of our main results, let us pause for a moment to say a few words about the modern theory of Dirichlet series. Within the last two decades, the theory of ordinary Dirichlet series $\sum a_n n^{-s}$ experienced a kind of renaissance. The study of these series in fact was one of the hot topics in mathematics at the beginning of the 20th century. Among others, H. Bohr, Besicovitch, Bohnenblust, Hardy, Hille, Landau, Perron, and M. Riesz.

However, this research took place before the modern interplay between function theory and functional analysis, as well as the advent of the field of several complex variables, and the area was in many ways dormant until the late 1990s. One of the main goals of the 1997 paper of Hedenmalm, Lindqvist, and Seip [21] was to initiate a systematic study of Dirichlet series from the point of view of modern operator-related function theory and harmonic analysis. Independently, at the same time, a paper of Boas and Khavinson [4] attracted renewed attention, in the context of several complex variables, to the original work of Bohr.

A new field emerged intertwining the classical work in novel ways with modern functional analysis, infinite dimensional holomorphy, probability theory as well as analytic number theory. As a consequence, a number of challenging research problems crystallized and were solved over the last decades. We refer to the monographs [9], [22], and [36], where many of the key elements of this new developments for ordinary Dirichlet series are described in detail.

Contemporary research in this field owes much to the following fundamental observation of H. Bohr [5], sometimes called Bohr's vision:

By the transformation $z_j = \mathfrak{p}_j^{-s}$ and the fundamental theorem of arithmetics, an ordinary Dirichlet series $\sum a_n n^{-s}$ may be thought of as a function $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha} z^\alpha$ of infinitely many complex variables $z_1, z_2, \ldots$, where $\mathfrak{p} = (\mathfrak{p}_j)$ stands for the sequence of prime numbers. By a classical approximation theorem of Kronecker, this is more than just a formal transformation: If, say, only a finite number of the coefficients a_n are nonzero (so that questions about convergence of the series are avoided), the supremum of the Dirichlet polynomial $\sum a_n n^{-s}$ in the half-plane $\operatorname{Re} s > 0$ equals the supremum of the corresponding polynomial on the infinite-dimensional circle group $\mathbb{T}^\infty$. Notice that a Dirichlet polynomial $\sum_{n=1}^{x} a_n n^{-s}$ corresponds to a polynomial $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} a_{\mathfrak{p}^\alpha} z^\alpha$ on the finite dimensional polytorus $\mathbb{T}^{\pi(x)}$, where $\pi(x)$ is the amount of prime numbers less or equal x. Thus, the supremum can be taken over $\mathbb{T}^\infty$ or $\mathbb{T}^{\pi(x)}$ indistinctly.

Let us sketch some of our main results, as well as some ideas for the strategies how to derive them.

Bohr's vision in its original form is the seed which allows to associate with each frequency ω a compact abelian group G (a so-called ω-Dirichlet group) as well as a sequence of characters h_{ω_n} on G, such that for each finite set $J \subset \mathbb{N}$ the mapping $\sum_J a_n e^{-\omega_n s} \mapsto \sum_J a_n h_{\omega_n}$ leads to a coefficient preserving identification of $\mathcal{H}_\infty^J(\omega)$ with the Banach space $\mathrm{Trig}_J(G)$ of all trigonometric polynomials on G having Fourier coefficients supported in $\{h_{\omega_n} : n \in J\}$ (see Section 3.1).

Using a famous averaging technique of Rudin (Theorem 2.1 and Theorem 2.2), we derive the following integral formula (see Theorem 3.6)

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^J(\omega)\big) = \int_G \Big|\sum_{n\in J} h_{\omega_n}\Big|\, d\mathrm{m},$$

where m stands for the Haar measure on G, and then equivalently

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^J(\omega)\big) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Big|\sum_{n\in J} e^{-i\omega_n t}\Big|\, dt$$

(Theorem 3.1). Our main applications for concrete frequencies ω and concrete index sets J offer concrete estimates for $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big)$, and their proofs are all mainly based on caculating one of the preceding two integrals.

For the three standard frequencies $\omega = (n)_{n\in\mathbb{N}_0}$, $\omega = (\log p_n)_{n\in\mathbb{N}}$ (where as above p_n is the nth prime number), and $\omega = (\log n)_{n\in\mathbb{N}}$, we in Section 3.2, Case I, get the following formulas which are asymptotically correct in x:

- $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\leq x}\big((n)\big)\big) = \frac{4}{\pi^2}\log(x+1) + o(1)$,
- $\lim_{x\to\infty} \frac{\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\leq x}\big((\log p_n)\big)\big)}{\sqrt{x}} = \frac{\sqrt{\pi}}{2}$,
- $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\leq x}\big((\log n)\big)\big) = O\left(\frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}\right)$.

To estimate the integrals for the first two statements is fairly standard – but for the third one this is highly non-trivial. Indeed, a recent deep theorem from probabilistic analytic number theory due to Harper from [19] shows that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\Big|\sum_{n=1}^{x}\frac{1}{n^{it}}\Big|\,dt = O\left(\frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}\right), \tag{1}$$

which is equivalent to

$$\int_{\mathbb{T}^\infty}\Big|\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}:1\leq n\leq x} z^\alpha\Big|\,dz = O\left(\frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}\right).$$

This resolved a long-standing problem of Helson from [23]. In fact, Helson had conjectured that the integral is of order $o(\sqrt{x})$ which would have disproved a certain generalisation of Nehari's theorem from harmonic analysis. We also mention that Harper's result gave a negative answer to the so-called embedding problem showing that for $0 < p < 2$ the L_p-integral of every ordinary Dirichlet polynomial $D = \sum_{n\leq x} a_n n^{-s}$ over any segment of fixed length on the vertical line $[\operatorname{Re} s = 1/2]$ can not be bounded by a universal constant times $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|D(it)|^p\,dt$ (see also Problem 2.1 in [43]).

Finally, we mention that we also study the projection constant of Banach spaces of ordinary Dirichlet polynomials supported on an index sets of natural numbers with a certain complexity of their prime number decompositions (see again Section 3.2, Case II and III).

We finish the article comparing and linking the results we obtained for projection constants, with some important estimates of the unconditional basis constant $\boldsymbol{\chi}_{\mathbf{mon}}(\mathscr{H}_\infty^J(\omega))$ (see Section 4 for the definition), for several natural frequencies $\omega = (\omega_n)$ and index sets $J \subset \mathbb{N}$. From the group point of view, this is related to the Sidon constant of specific sets of characters.

1. PRELIMINARIES

We use standard notation from Banach space theory as e.g., used in the monographs [11, 33, 35, 46, 48]. If not indicated differently, we consider complex Banach spaces.

Given two sequences (a_n) and (b_n) of non-negative real numbers we write $a_n \prec b_n$, if there is a constant $c > 0$ such that $a_n \le c\, b_n$ for all $n \in \mathbb{N}$, while $a_n \sim b_n$ means that $a_n \prec b_n$ and $b_n \prec a_n$ holds. In the case that an extra parameter m is also involved, for two sequences of non-negative real numbers $(a_{n,m})$ and $(b_{n,m})$, we write $a_{n,m} \prec_{C(m)} b_{n,m}$ whenever there is a constant $C(m) > 0$ (which depends exclusively on m and not on n) such that $a_{n,m} \le C(m) b_{n,m}$ for all $n, m \in \mathbb{N}$. We use the notation $a_{n,m} \sim_{C(m)} b_{n,m}$ if $a_{n,m} \prec_{C(m)} b_{n,m}$ and $b_{n,m} \succ_{C(m)} a_{n,m}$. We also write $a_{n,m} \prec_{C^m} b_{n,m}$ when there is a hypercontractive comparision, i.e., there is an absolute constant $C > 0$ such that $a_{n,m} \le C^m b_{n,m}$ for all $n, m \in \mathbb{N}$. Of course, if $a_{n,m} \prec_{C^m} b_{n,m}$ and $b_{n,m} \succ_{C^m} a_{n,m}$ we simply write $a_{n,m} \sim_{C^m} b_{n,m}$.

In the following two paragraphs we collect a few standard facts on projection constants as well as topological groups.

1.1. **Projection constants.** Any Banach space X can be embedded isometrically into $\ell_\infty(S)$, where S is a nonempty set which in general depends on X. Indeed, if S is the unit ball of the dual of X, then it follows from the Hahn–Banach theorem that the mapping $x \mapsto (\varphi(x))_{\varphi \in S}$ is an isometric embedding from X into $\ell_\infty(S)$. Moreover, for every separable X, we may choose $S = \mathbb{N}$ (see e.g. [1, Theorem 2.5.7]). Throughout the paper, we use the fact that, if S is a nonempty set for which the Banach space X is isometrically isomorphic to a subspace Z of $\ell_\infty(S)$, then $\boldsymbol{\lambda}(X) = \boldsymbol{\lambda}(Z, \ell_\infty(S))$. Indeed, this is due to the fact that $\ell_\infty(S)$ is isometrically injective (see [1, Definition 2.5.1. and Proposition 2.5.2.]). Thus finding $\boldsymbol{\lambda}(X)$ is equivalent to finding the norm of a minimal projection from $\ell_\infty(S)$ onto an isometric copy of X in $\ell_\infty(S)$. However, this is a non-trivial problem in general.

General bounds for projection constants of various finite dimensional Banach spaces were studied by many authors. The most fundamental general upper bound is due to Kadets and Snobar [24]: For every n-dimensional Banach space X_n one has

$$\boldsymbol{\lambda}(X_n) \le \sqrt{n}\,. \tag{2}$$

In contrast, König and Lewis [27] showed that for any Banach space X_n of dimension $n \ge 2$ the strict inequality $\boldsymbol{\lambda}(X_n) < \sqrt{n}$ holds, and this estimate was improved by Lewis [30] showing

$$\boldsymbol{\lambda}(X_n) \le \sqrt{n}\left(1 - \frac{1}{n^2}\left(\frac{1}{5}\right)^{2n+11}\right).$$

The exact values of $\boldsymbol{\lambda}(\ell_2^n)$ and $\boldsymbol{\lambda}(\ell_1^n)$ were computed by Grünbaum [18] and Rutovitz [42]: In the complex case

$$\boldsymbol{\lambda}\big(\ell_2^n(\mathbb{C})\big) = n\int_{\mathbb{S}_n(\mathbb{C})} |x_1|\,d\sigma = \frac{\sqrt{\pi}}{2}\frac{n!}{\Gamma(n+\frac{1}{2})}, \tag{3}$$

where $d\sigma$ stands for the normalized surface measure on the sphere $\mathbb{S}_n(\mathbb{C})$ in $\mathbb{C}^n$, and

$$\boldsymbol{\lambda}\big(\ell_1^n(\mathbb{C})\big) = \int_{\mathbb{T}^n}\Big|\sum_{k=1}^n z_k\Big|\,dz = \int_0^\infty \frac{1-J_0(t)^n}{t^2}\,dt, \tag{4}$$

where dz denotes the normalized Lebesgue measure on the distinguished boundary $\mathbb{T}^n$ in $\mathbb{C}^n$ and J_0 is the zero Bessel function defined by $J_0(t) = \frac{1}{2\pi}\int_0^\infty \cos(t\cos\varphi)\,d\varphi$. The corresponding real constants are different:

$$\boldsymbol{\lambda}\big(\ell_2^n(\mathbb{R})\big) = n\int_{\mathbb{S}_n(\mathbb{R})} |x_1|\,d\sigma = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})} \tag{5}$$

$$\boldsymbol{\lambda}\big(\ell_1^n(\mathbb{R})\big) = \begin{cases} \boldsymbol{\lambda}\big(\ell_2^n(\mathbb{R})\big), & n \text{ odd} \\ \boldsymbol{\lambda}(\ell_2^{n-1}(\mathbb{R})), & n \text{ even}. \end{cases} \tag{6}$$

Gordon [15] and Garling-Gordon [14] determined the asymptotic growth of $\boldsymbol{\lambda}\big(\ell_p^n\big)$ for $1 < p < \infty$ with $p \notin \{1,2,\infty\}$:

$$\boldsymbol{\lambda}\big(\ell_p^n\big) \sim n^{\min\left\{\frac{1}{2},\frac{1}{p}\right\}}. \tag{7}$$

König, Schütt and Tomczak-Jagermann [28] proved that for $1 \le p \le 2$

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}\big(\ell_p^n\big)}{\sqrt{n}} = \gamma, \tag{8}$$

where $\gamma = \sqrt{\frac{2}{\pi}}$ in the real and $\gamma = \frac{\sqrt{\pi}}{2}$ in the complex case. For an extensive treatment on all of this and more see the excellent monograph [46] of Tomczak-Jaegermann.

1.2. **Topological groups.** As usual a group G equipped with a topology τ is said to be a topological group whenever the mapping $(G,\tau)\times(G,\tau) \ni (a,b) \to ab^{-1} \in (G,\tau)$ is continuous. From here on G is assumed to be a compact group, that is, its topology is compact. In this case, G defines a natural set of maps $\{L_a\}_{a\in G}$ and $\{R_a\}_{a\in G}$ on $C(G)$, the complex-valued continuous functions on G, given for all $a,b \in G$ by

$$L_a f(b) := f(ab), \quad \text{and } R_a f(b) = f(ba), \quad f \in C(G).$$

It is well-known that for every compact group G there exists a unique Borel probability measure m which is left invariant, that is,

$$\int_G f(b)\,d\mathrm{m}(b) = \int_G L_a f(b)\,d\mathrm{m}(b), \quad a \in G,\ f \in C(G).$$

This m is called the Haar measure of G. If in addition m is also right invariant:

$$\int_G f(b)\,d\mathrm{m}(b) = \int_G R_a f(b)\,d\mathrm{m}(b), \quad a \in G,\ f \in C(G),$$

then the compact group G is called unimodular. Examples of unimodular groups are compact groups in which every one point set is closed.

Let m be the normalized Haar measure on G, and $\widehat{G}$ as usual the dual group of G (i.e., the set of all continuous characters on G). For any $f \in L^1(G) := L^1(G, \mathrm{m})$, the Fourier transform of f is given by

$$\widehat{f}(\gamma) := \int_G f(a)\,\overline{\gamma(a)}\,d\mathrm{m}(a), \quad \gamma \in \widehat{G}.$$

Recall that $L_1(G)$ forms a commutative Banach algebra, whenever it carries the convolution $f_1 * f_2$ as its multiplication, that is, for m-almost every $a \in G$

$$(f_1 * f_2)(a) := \int_G f_1(ab^{-1}) f_2(b)\,d\mathrm{m}(b).$$

Products of groups are going to be of particular interest for our purposes. Given compact abelian groups $G_1, \ldots, G_n$, each with the Haar measure $\mathrm{m}_j, 1 \le j \le n$, we denote by $G := G_1 \times \cdots \times G_n$ the product of these groups endowed its natural product operation and product topology . Given an n-tuple of characters $(\gamma_1, \ldots, \gamma_n) \in \widehat{G_1} \times \cdots \times \widehat{G_n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we write γ^α for the character in $\widehat{G}$ given by

$$\gamma^\alpha(x_1, \ldots, x_n) := \gamma_1(x_1)^{\alpha_1} \cdots \gamma_n(x_n)^{\alpha_n}, \quad (x_1, \ldots, x_n) \in G.$$

Of special interest is the n-dimensional circle group $G := \mathbb{T}^n$, where the Haar measure $\mathrm{m} =: dz$ on $\mathbb{T}^n$ acts on a Borel function $f\colon \mathbb{T}^n \to \mathbb{C}$ by the formula

$$\int_{\mathbb{T}^n} f(z)\,dz = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{it_1}, \ldots, e^{it_n})\,dt_1 \ldots dt_n.$$

Recall that $\widehat{\mathbb{T}^n} = \mathbb{Z}^n$, where the identification is given by the fact that for every character $\gamma \in \widehat{\mathbb{T}^n}$ there is a unique multi index $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ for which $\gamma(z) = z^\alpha$ for every $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$.

2. Trigonometric polynomials

For any compact group G and any nonempty finite set $E \subset \widehat{G}$ we denote $\operatorname{span} E$ by $\mathrm{Trig}_E(G)$, the space of all trigonometric polynomials P on G which are supported on E, that is, the Fourier transform $\widehat{P}$ is supported on E. In what follows, we consider this finite dimensional space with the supremum norm on G. Note that every polynomial $P \in \mathrm{Trig}_E(G)$ has the form

$$P(g) = \sum_{\gamma \in E} \widehat{P}(\gamma)\,\gamma(g), \quad g \in G.$$

The space of all trigonometric polynomials on G is denoted by $\mathrm{Trig}(G)$.

We need two fundamental consequences of the Peter-Weyl theorem (see, e.g., [36, Theorem 1.3.3]): For any compact abelian group G, the space $\mathrm{Trig}(G)$ is dense in the space $C(G)$ of complex-valued functions (see [36, Theorem 1.3.4]), and the dual group $\widehat{G}$ is an orthonormal basis of the Hilbert space $L^2(G,\mathrm{m})$ (see [36, Theorem 1.3.6]).

For the special case $G=\mathbb{T}^n$ with $\widehat{G}=\mathbb{Z}^n$ we are particularly interested in the index sets

$$J_p(m,n):=\Big\{\alpha\in\mathbb{Z}^n:\ \big(\sum_j|\alpha_j|^p\big)^{\frac{1}{p}}=m\Big\},\quad \Lambda_p(m,n):=\Big\{\alpha\in\mathbb{N}_0^n:\ \big(\sum_j\alpha_j^p\big)^{\frac{1}{p}}=m\Big\} \tag{9}$$

as well as

$$J_p(\le m,n):=\Big\{\alpha\in\mathbb{Z}^n:\ \big(\sum_j\alpha_j^p\big)^{\frac{1}{p}}\le m\Big\},\quad \Lambda_p(\le m,n):=\Big\{\alpha\in\mathbb{N}_0^n:\ \big(\sum_j|\alpha_j|^p\big)^{\frac{1}{p}}\le m\Big\}, \tag{10}$$

where $m,n\in\mathbb{N}$ and $p\in\{1,2,\infty\}$. For $p=1$ we write $|\alpha|=\sum_j|\alpha_j|$, usually called the order of α, and for $p=\infty$ we of course by $\big(\sum_j|\alpha_j|^p\big)^{1/p}$ mean $\max_j|\alpha_j|$.

For $m\in\mathbb{N}$ we write $\mathrm{Trig}_m(\mathbb{T}^n):=\mathrm{Trig}_{\Lambda_1(m,n)}(\mathbb{T}^n)$ for the Banach space of all analytic trigonometric polynomials which are m-homogeneous, i.e., all polynomials of the form

$$P(z)=\sum_{\alpha\in\Lambda_1(m,n)}c_\alpha z^\alpha\quad\text{for } z\in\mathbb{T}^n.$$

Similarly, $\mathrm{Trig}_{\le m}(\mathbb{T}^n):=\mathrm{Trig}_{\Lambda_1(\le m,n)}(\mathbb{T}^n)$, stands for the space of all analytic trigonometric polynomials of degree $\le m$.

The following integral formula for the projection constant of the Banach space of all trigonometric polynomials on a given compact abelian group supported on a priori given set of characters in $\widehat{G}$, is one of the main sources of what we intend to do.

Theorem 2.1. *Let G be a compact abelian group and $E:=\{\gamma_1,\dots,\gamma_N\}\subset\widehat{G}$ a finite set of characters. Then $\Pi\colon C(G)\to C(G)$, given by $\Pi f=\sum_{j=1}^N\widehat{f}(\gamma_j)\gamma_j$ for all $f\in C(G)$, is the unique projection onto $Trig_E(G)$ that commutes with the action of the group on $C(G)$. Moreover, Π is minimal:*

$$\boldsymbol{\lambda}\big(Trig_E(G)\big)=\big\|\Pi\colon C(G)\to C(G)\big\|=\int_G\Big|\sum_{j=1}^N\gamma_j(a)\Big|\,d\mathrm{m}(a)\,.$$

The following section is devoted to the proof of this result.

2.1. **Averaging projections.** As mentioned, one of the main tools we intend to use is a method due to Rudin (see the forthcoming Theorem 2.2). Roughly speaking, under certain assumptions, there is a somewhat universal averaging technique which allows to construct new projections with additional and somehow better properties from an a priori given projection.

Given a topological group G and a Banach space Y, we need to explain when all elements of G act as bounded linear operators on Y. Formally this means that there is a mapping

$$T\colon G \to \mathscr{L}(Y), \ a \mapsto T_a$$

such that

$$T_e = I_Y, \quad T_{ab} = T_a T_b, \quad a,\, b \in G$$

and all mappings

$$G \ni a \mapsto T_a(y) \in Y, \quad y \in Y \tag{11}$$

are continuous. Then G is said to act on Y through T (or simply, G acts on Y). If in addition all operators $T_a, a \in G$ are isometries, then we say that G acts isometrically on Y. We say that $S \in \mathscr{L}(Y)$ commutes with the action of G on Y through T whenever S commutes with all T_b, $b \in G$.

Theorem 2.2. *Let Y be a Banach space, X a complemented subspace of Y, and $\mathbf{Q}\colon Y \to Y$ a projection onto X. Suppose that G is a compact group with Haar measure m, which acts on Y through T such that X is invariant under the action of G, that is,*

$$T_a(X) \subset X, \quad a \in G. \tag{12}$$

Then $\mathbf{P}\colon Y \to Y$ given by

$$\mathbf{P}(y) := \int_G T_{a^{-1}} \mathbf{Q} T_a(y)\, d\mathrm{m}(a), \quad y \in Y, \tag{13}$$

is a projection onto X which commutes with the action of G on Y, i.e., $T_a\mathbf{P} = \mathbf{P}T_a$ for all $a \in G$, and satisfies

$$\|\mathbf{P}\| \le \|\mathbf{Q}\| \sup_{a\in G} \|T_a\|^2 .$$

Moreover, if there is a unique projection on Y onto X that commutes with the action of G on Y, and if G acts isometrically on Y, then $\mathbf{P}$ given in the formula (13) is minimal, i.e.,

$$\boldsymbol{\lambda}(X,Y) = \|\mathbf{P}\| .$$

This result (in one way or the other) found various applications in the literature (see, e.g., [26, 28, 29, 37, 32, 40, 41]. In the case of the circle group, it can be traced back to Faber's article [13]. For the sake of completeness we include a proof, which is inspired by [40] (and also [48, Theorem III.B.13]).

Proof of Theorem 2.2. Note first, that, by the Banach-Steinhaus theorem, $\sup_{a\in G} \|T_a\| < \infty$. Then, given $y \in Y$, the mapping $a \ni G \mapsto T_{a^{-1}}\mathbf{Q}T_a \in \mathscr{L}(Y)$ is bounded and by (11) measurable (being almost everywhere separably valued and weakly measurable), and hence Bochner integrable. Consequently, $\mathbf{P}$ defines an operator on Y. Moreover, from (12) we deduce for all $y \in Y$ and for all $x \in X$ that

$$T_{a^{-1}}\mathbf{Q}T_a(y) \in X, \quad T_{a^{-1}}\mathbf{Q}T_a(x) = x,$$

implying that $\mathbf{P}$ is a projection from Y onto X. The hypothesis that G acts on Y (through T) yields for all $b \in G$

$$T_{b^{-1}}\mathbf{P}T_b = \int_G T_{b^{-1}}T_{a^{-1}}\mathbf{Q}T_aT_b\,d\mathrm{m}(a) = \int_G T_{(ab)^{-1}}\mathbf{Q}T_{ab}\,d\mathrm{m}(a) = \mathbf{P},$$

so $\mathbf{P}$ commutes with the action of G on Y. Since for all $y \in Y$

$$\|\mathbf{P}y\|_Y \le \int_G \|T_{a^{-1}}\|\,\|\mathbf{Q}\|\,\|T_a\|\,\|y\|_Y\,d\mathrm{m}(a) \le \|\mathbf{Q}\| \sup_{a\in G}\|T_a\|^2\,\|y\|_Y,$$

the required estimate for $\|\mathbf{P}\|$ follows. If moreover there is a unique projection on Y onto X that commutes with the action of G on Y, and if G acts isometrically on X, then the projection $\mathbf{P}$ from (13) does not depend on $\mathbf{Q}$ and $\|\mathbf{P}\| \le \|\mathbf{Q}\|$ for all projections $\mathbf{Q}$ on Y onto X, i.e., $\|\mathbf{P}\| \le \boldsymbol{\lambda}(X,Y)$. This completes the proof. $\square$

Proof of Theorem 2.1. Note first that G in a natural way acts on $C(G)$ (in the sense of Section 2.1), where the action is given by the mapping $T: G \to \mathscr{L}(C(G))$, $a \mapsto T_a$ with

$$T_af(b) := f(ab), \quad f \in C(G),\ b \in G.$$

We claim that $\Pi\colon C(G) \to C(G)$ is the unique projection onto $\mathrm{Trig}_E(G)$ that commutes with all translation operators T_a, $a \in G$. To see this, assume that $\mathbf{Q}\colon C(G) \to C(G)$ onto $\mathrm{Trig}_E(G)$ is a projection that commutes with all translation operators. Then for all $\gamma, \gamma' \in \widehat{G}$ one has

$$\widehat{T_a\mathbf{Q}\gamma}(\gamma') = \widehat{\mathbf{Q}T_a\gamma}(\gamma').$$

It is easy to check that $\widehat{T_a\mathbf{Q}\gamma}(\gamma') = \gamma'(a)\,\widehat{\mathbf{Q}\gamma}(\gamma')$ and $\widehat{\mathbf{Q}T_a\gamma}(\gamma') = \gamma(a)\,\widehat{\mathbf{Q}\gamma}(\gamma')$. In consequence, we get

$$\gamma'(a)\widehat{\mathbf{Q}\gamma}(\gamma') = \gamma(a)\widehat{\mathbf{Q}\gamma}(\gamma'), \quad a \in G.$$

This implies that, for all $\gamma, \gamma' \in \widehat{G}$ with $\gamma \neq \gamma'$, we have $\widehat{\mathbf{Q}\gamma}(\gamma') = 0$. On the other hand, the Peter-Weyl theorem states that $\widehat{G}$ forms an orthonormal basis in the Hilbert space $L^2(G)$, hence

$$\mathbf{Q}\gamma = \sum_{\gamma'\in\widehat{G}} \widehat{\mathbf{Q}\gamma}(\gamma')\,\gamma', \quad \gamma \in \widehat{G},$$

and consequently, for every character $\gamma \in \widehat{G}$ there is a scalar c_γ such that $\mathbf{Q}\gamma = c_\gamma\gamma$.

Since $\mathbf{Q}$ is a projection onto $\mathrm{Trig}(G)$, $c_\gamma = 0$ for all $\gamma \in \widehat{G}\setminus E$, and $c_\gamma = 1$ for all $\gamma \in E$. In consequence, $\mathbf{Q}\gamma = \gamma$ for all $\gamma \in E$, and $\mathbf{Q}\gamma = 0$ for all $\gamma \in \widehat{G}\setminus E$. Hence the projection $\mathbf{Q}$, restricted to the algebra $\mathrm{Trig}(G)$ of all trigonometric polynomials on G, has the representation

$$\mathbf{Q}f = \sum_{j=1}^{N} \widehat{f}(\gamma_j)\gamma_j, \quad f \in \mathrm{Trig}(G). \tag{14}$$

Consequently, we conclude from the density of $\mathrm{Trig}(G)$ in $C(G)$ that the above formula holds for all $f \in C(G)$. Hence $\mathbf{Q} = \Pi$. This proves the claim.

Since Π is the unique projection onto $\mathrm{Trig}_E(G)$ that commutes with all translation operators, Π is fixed by the averaging technique introduced in Theorem 2.2. Now observe that G acts isometrically on $C(G)$, i.e., all mappings $T_a\colon C(G)\to C(G)$ are isometries on $C(G)$. Since for all $a\in G$

$$T_a\gamma=\gamma(a)\gamma,\quad \gamma\in\widehat{G},$$

it follows that $T_a(\mathrm{Trig}_E(G))\subset \mathrm{Trig}_E(G)$ for all $a\in G$. Then, by the moreover part of Theorem 2.2, Π is a minimal projection, that is,

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_E(G)\big)=\big\|\Pi\colon C(G)\to C(G)\big\|\,.$$

Finally, it remains to prove the integral formula for the norm of Π. Since $f*\gamma=\widehat{f}(\gamma)\gamma$ for all $f\in L^1(G)$ and $\gamma\in\widehat{G}$, we get by (14),

$$\Pi f=f*\Big(\sum_{j=1}^N\gamma_j\Big),\quad f\in C(G)\,.$$

Clearly, $\sum_{j=1}^N\gamma_j\in C(G)$, so it can readily be shown by direct computation that

$$\big\|\Pi\colon C(G)\to C(G)\big\|=\int_G\Big|\sum_{j=1}^N\gamma_j(a)\Big|\,d\mathrm{m}(a).$$

Indeed, we have

$$\begin{aligned}\big\|\Pi\colon C(G)\to C(G)\big\|&=\sup_{\|f\|_\infty=1}\ \sup_{b\in G}\ \Big|\int_G f(a)\Big(\sum_{j=1}^N\gamma_j\Big)(ba^{-1})d\mathrm{m}(a)\Big|\\&=\sup_{b\in G}\ \sup_{\|f\|_\infty=1}\ \Big|\int_G f(a)\Big(\sum_{j=1}^N\gamma_j\Big)(ba^{-1})d\mathrm{m}(a)\Big|\\&=\sup_{b\in G}\ \int_G\Big|\sum_{j=1}^N\gamma_j(ba^{-1})\Big|\,d\mathrm{m}(a)\,.\end{aligned}$$

Now note that by the translation invariance of the Haar measure m and the fact that it is both left and right invariant (G is unimodular since it is abelian), for each $b\in G$ we have,

$$\int_G\Big|\sum_{j=1}^N\gamma_j(ba^{-1})\Big|\,d\mathrm{m}(a)=\int_G\Big|\sum_{j=1}^N\gamma_j(a^{-1})\Big|\,d\mathrm{m}(a)=\int_G\Big|\sum_{j=1}^N\gamma_j(a)\Big|\,d\mathrm{m}(a)\,.$$

This completes the proof. □

2.2. **Projection constants - trigonometric polynomials.** Based on the integral formula from Theorem 2.1, we derive concrete formulas and asymptotically optimal estimates for the projection constants of spaces of trigonometric polynomials on compact abelian groups, which have Fourier expansions supported on an a priori fixed set of characters.

2.2.1. $\Lambda(2)$*-sets.* In order to show a very first application of Theorem 2.1, we need some further notation and preliminaries. Recall that Rudin in his classical paper [39] from 1960 introduced the notion of $\Lambda(p)$-sets within the setting of Fourier analysis on the circle group $\mathbb{T}$. In modern language, if G is a compact abelian group (with Haar measure m) and $p \in (1,\infty)$, then the subset $E \subset \widehat{G}$ is said to be a $\Lambda(p)$-set whenever there exists a constant $C > 0$ such that, for every trigonometric polynomial $P \in \mathrm{Trig}_E(G)$, one has

$$\|P\|_{L_p(G)} \le C\|P\|_{L_1(G)}\,. \tag{15}$$

In this case, the least such constant is called the $\Lambda(p)$-constant of E, and denoted by $C_p = C_p(E)$. Let us here remark that for $p > 2$ the validity of (15) is equivalent to the existence of a constant $A_p > 0$ such that

$$\|P\|_{L_p(G)} \le A_p\|P\|_{L_2(G)}, \quad P \in \mathrm{Trig}_E(G)\,.$$

The following almost immediate consequence of Theorem 2.1 shows that $\Lambda(2)$-sets are of particular importance for our purposes – see also Corollary 2.5 and Corollary 3.9.

Corollary 2.3. *Let G be a compact abelian group. Then, for any finite set $E = \{\gamma_1,\dots,\gamma_N\} \subset \widehat{G}$ of different characters with $\Lambda(2)$ constant C_2, we have*

$$C_2^{-1}\sqrt{N} \le \boldsymbol{\lambda}\big(Trig_E(G)\big) \le \sqrt{N}.$$

Proof. Let m be the Haar measure on G. Then from Theorem 2.1, we get

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_E(G)\big) = \int_G |\gamma_1(x) + \ldots + \gamma_N(x)|\, d\mathrm{m}(x)\,.$$

Since $\widehat{G}$ is an orthonormal basis in $L_2(G,\mathrm{m})$,

$$\Big(\int_G |\gamma_1(x) + \ldots + \gamma_N(x)|^2\, d\mathrm{m}(x)\Big)^{\frac{1}{2}} = \sqrt{N}.$$

The proof is completed by using (15) and the Cauchy-Schwarz inequality. □

In combination with the preceding corollary we need the following example.

Remark 2.4. Let G be a compact abelian group. Following [17], a set $E \subset \widehat{G}$ is said to be a B_2-set, whenever $\gamma_1\gamma_2 = \gamma_3\gamma_4$ for all $\gamma_1,\dots,\gamma_4 \in E$ if and only if $\{\gamma_3,\gamma_4\}$ is a permutation of $\{\gamma_1,\gamma_2\}$. It is worth noting that B_2-sets are $\Lambda(2)$-sets with $\Lambda(2)$-constant $\sqrt{2}$ (see [17, Proposition 6.3.11]).

In passing we observe that [17, Proposition 6.3.11], also implies the following: If $E \subset \widehat{G}$ fulfills that there exists $N \ge 2$ such that for every $\gamma \in \widehat{G}$ there are at most N pairs of the form $(\gamma_{i_1},\gamma_{i_2}) \subset E \times E$ with $\gamma_{i_1}\gamma_{i_2} = \gamma$, then E is $\Lambda(2)$-set with $C_2 \le \sqrt{N}$.

Another example of $\Lambda(2)$-sets, important for our purposes, follows from an inequality due to Weissler [47] (see also [9, Theorem 8.10]): For all $P \in \mathrm{Trig}_{\leq m}(\mathbb{T}^n)$

$$\frac{1}{\sqrt{2}^m}\Big(\int_{\mathbb{T}^n} |P(z)|^2\, dz\Big)^{\frac{1}{2}} \leq \int_{\mathbb{T}^n} |P(z)|\, dz. \tag{16}$$

In other terms, the set $\{z^\alpha : \alpha \in \Lambda_1(\leq m, n)\}$ of characters in $\widehat{\mathbb{T}^n} = \mathbb{Z}^n$ forms a $\Lambda(2)$-set with constant $C_2 \leq \sqrt{2^m}$ (recall from (10) the definition of $\Lambda_1(\leq m, n)$, which clearly may be identified with the set of characters z^α it generates).

Then the following result is an immediate consequence of Theorem 2.1, Corollary 2.3, and (16).

Corollary 2.5. *Let $J \subset \mathbb{Z}_0^n$ be a finite set. Then*

$$\boldsymbol{\lambda}\big(\mathit{Trig}_J(\mathbb{T}^n)\big) = \int_{\mathbb{T}^n} \Big|\sum_{\alpha\in J} z^\alpha\Big|\, dz,$$

and if $\Lambda \subset \Lambda_1(\leq m, n)$, then

$$\frac{1}{\sqrt{2^m}}\sqrt{|\Lambda|} \leq \boldsymbol{\lambda}\big(\mathit{Trig}_\Lambda(\mathbb{T}^n)\big) \leq \sqrt{|\Lambda|}.$$

In the context of this corollary the following formula and estimate

$$|\Lambda_1(m,n)| = \binom{n+m-1}{m} \leq e^m\Big(1+\frac{n}{m}\Big)^m \tag{17}$$

for the cardinality of $\Lambda_1(m, n)$ is very useful.

2.2.2. *Products of groups.* The aim is to prove various multivariate variants of a famous result by Lozinski-Kharshiladze (see [48, IIIB. Theorem 22] and [34]), which shows a precise estimate of the projection constant of $\mathrm{Trig}_{\leq m}(\mathbb{T})$, the space of trigonometric polynomials on $\mathbb{T}$ of degree at most m equipped with the sup-norm:

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{\leq m}(\mathbb{T})\big) = \frac{4}{\pi^2}\log(m+1) + o(1). \tag{18}$$

This is done by mostly routine applications of Theorem 2.1. We point out that these results in Section 3.2 are the key to obtain formulas and estimates for the projection constants of spaces of Dirichlet polynomials.

Proposition 2.6. *Let $G := G_1 \times \cdots \times G_n$ be a product of compact abelian groups $G_1, \ldots, G_n$ (with the Haar measures $\mathrm{m}_j, 1 \leq j \leq n$), and $E := \{\gamma^\alpha : \alpha \in J_\infty(\leq m, n)\} \subset \widehat{G}$, where $(\gamma_1, \ldots, \gamma_n) \in \widehat{G_1} \times \cdots \times \widehat{G_n}$ and the definition of $J_\infty(\leq m, n)$ is given in (10). Then*

$$\boldsymbol{\lambda}\big(\mathit{Trig}_E(G)\big) = \prod_{j=1}^n \int_{G_j} \Big|\sum_{k\in\mathbb{Z}:|k|\leq m} \gamma_j(x_j)^k\Big|\, d\mathrm{m}_j(x_j).$$

Proof. Let μ be the Haar measure on G, which is nothing else then the product of the Haar measures m_j. Then Theorem 2.1 yields

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_E(G)\big) = \int_G \Big| \sum_{\alpha\in J_\infty(\le m,n)} \gamma^\alpha(x) \Big| \, d\mu(x) \,.$$

Clearly, for all $x = (x_1,\dots,x_n) \in G_1 \times \cdots \times G_n$ one has

$$\sum_{\alpha\in J_\infty(\le m,n)} \gamma^\alpha(x) = \prod_{j=1}^n \sum_{|k|\le m} \gamma_j(x_j)^k \,,$$

and hence

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_E(G)\big) = \int_G \prod_{j=1}^n \Big| \sum_{|k|\le m} \gamma_j(x_j)^k \Big| \, d\mu(x) \,.$$

Fubini's theorem finishes the argument. ☐

We also use the following more general variant.

Proposition 2.7. *Let $G := G_1 \times \cdots \times G_n$ be a product of compact abelian groups $G_1,\dots,G_n$. Moreover, let $I := I_1 \times \cdots \times I_n \subset \mathbb{Z}^n$, with I_j finite subsets of $\mathbb{Z}$ for $j \in \{1,\dots,n\}$, and $E := \{\gamma^\alpha \colon \alpha \in I\}$. Then*

$$\boldsymbol{\lambda}\big(\mathit{Trig}_E(G)\big) = \prod_{j=1}^n \int_{G_j} \Big| \sum_{k\in I_j} \gamma_j(x_j)^k \Big| \, d\mathrm{m}_j(x_j) \,.$$

As announced above, we finally present multivariate variants of the the Lozinski-Kharshiladze result mentioned in (18). For each $m \in \mathbb{N}_0$, let D_m be the mth Dirichlet kernel

$$D_m(t) := \sum_{k=-m}^m e^{-ikt} \,,$$

and L_m the mth Lebesgue constant given by

$$L_m := \frac{1}{2\pi} \int_0^{2\pi} |D_m(t)| \, dt = \frac{1}{2\pi} \int_0^{2\pi} \Big| \frac{\sin(m+\frac{1}{2})t}{\sin\frac{t}{2}} \Big| \, dt \,.$$

We recall the well-known standard estimates

$$\frac{4}{\pi^2} \log(m+1) \;<\; L_m \;<\; 3 + \log m, \quad m \in \mathbb{N}. \tag{19}$$

Corollary 2.8. *For $\mathbf{d} = (d_1,\dots,d_n) \in \mathbb{N}_0^n$ let $I_{\mathbf{d}} := \big\{\alpha \in \mathbb{Z}^n \colon |\alpha_j| \le d_j,\ 1 \le j \le n\big\}$. Then*

$$\boldsymbol{\lambda}\big(\mathit{Trig}_{I_{\mathbf{d}}}(\mathbb{T}^n)\big) = \prod_{j=1}^n L_{d_j} \,.$$

Proof. Applying Proposition 2.7, we get

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{I_{\mathbf{d}}}(\mathbb{T}^n)\big) = \prod_{j=1}^n \int_{\mathbb{T}} \Big| \sum_{|\alpha_j|\le d_j} z^{\alpha_j} \Big| \, dz = \prod_{j=1}^n \int_{\mathbb{T}} |D_{d_j}(z_j)| \, dz_j = \prod_{j=1}^n L_{d_j} \,,$$

as required. ☐

In order to state the 'analytic counterpart' of Corollary 2.8, we define for each $m \in \mathbb{N}_0$,

$$D_m^+(t) := \sum_{k=0}^{m} e^{-ikt} \quad \text{and} \quad L_m^+ := \frac{1}{2\pi}\int_0^{2\pi} |D_m^+(t)|\, dt\,.$$

and recall from (10) the notation $\Lambda_\infty(\leq m, n) := \left\{\alpha \in \mathbb{N}_0^n : \max_j \alpha_j \leq m\right\}$.

Corollary 2.9. *For $m, n \in \mathbb{N}$*

(i) $\boldsymbol{\lambda}\big(Trig_{\Lambda_\infty(\leq m,n)}(\mathbb{T}^n)\big) = (L_m^+)^n$,

(ii) $\boldsymbol{\lambda}\big(Trig_{\Lambda_\infty(\leq 2m,n)}(\mathbb{T}^n)\big) = (L_m)^n$,

(iii) $(L_m - 1)^n \leq \boldsymbol{\lambda}\big(Trig_{\Lambda_\infty(\leq 2m+1,n)}(\mathbb{T}^n)\big) \leq (L_m + 1)^n$,

(iv) $\boldsymbol{\lambda}\big(Trig_{\Lambda_\infty(\leq m,n)}(\mathbb{T}^n)\big) \sim (1 + \log m)^n$.

Proof. (i) By Proposition 2.6, we have as desired

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{\Lambda_\infty(\leq m,n)}(\mathbb{T}^n)\big) = \prod_{j=1}^{n}\int_{\mathbb{T}}\Big|\sum_{0\leq\alpha_j\leq m} z^{\alpha_j}\Big|\, dz = \prod_{j=1}^{n}\int_{\mathbb{T}} |D_m^+(z_j)|\, dz_j = (L_m^+)^n\,.$$

(ii) It is easy to check that

$$D_{2m}^+(t) = e^{-imt}\,\frac{\sin(m+\frac{1}{2})t}{\sin\frac{t}{2}}, \qquad t \in (0, 2\pi)\,.$$

This implies that $L_m = \|D_m\|_{L_1(\mathbb{T})} = \|D_{2m}^+\|_{L_1(\mathbb{T})} = L_{2m}^+$, so the statement follows from (i).

(iii) The statement follows by (ii) combined with the obvious estimates

$$\|D_{2m}^+\|_{L_1(\mathbb{T})} - 1 \leq \|D_{2m+1}^+\|_{L_1(\mathbb{T})} \leq \|D_{2m}^+\|_{L_1(\mathbb{T})} + 1, \qquad m \in \mathbb{N}\,.$$

(iv) Combining the estimates from (19) with those from (ii) and (iii), we get the required equivalence. □

We conclude with the following two limit formulas; see again (10) for the definition of the index sets $J_\infty(\leq m, n)$ and $\Lambda_\infty(\leq m, n)$.

Corollary 2.10. *For each $m, n \in \mathbb{N}$*

$$\lim_{m\to\infty}\frac{\boldsymbol{\lambda}\big(Trig_{J_\infty(\leq m,n)}(\mathbb{T}^n)\big)}{\log^n m} = \lim_{m\to\infty}\frac{\boldsymbol{\lambda}\big(Trig_{\Lambda_\infty(\leq m,n)}(\mathbb{T}^n)\big)}{\log^n m} = \Big(\frac{4}{\pi^2}\Big)^n.$$

Proof. Recall the well-known formula

$$\lim_{m\to\infty}\frac{L_m}{\log m} = \frac{4}{\pi^2}\,. \tag{20}$$

Then the first limit follows from Corollary 2.8. For the proof of the second limit we distinguish the two sequences $\big(\boldsymbol{\lambda}\big(\mathrm{Trig}_{\Lambda_\infty(\le 2m,n)}(\mathbb{T}^n)\big)/\log(2m)\big)_m$ and $\big(\boldsymbol{\lambda}\big(\mathrm{Trig}_{\Lambda_\infty(\le 2m+1,n)}(\mathbb{T}^n)\big)/\log(2m+1)\big)_m$. Observe that both sequences converge to $\frac{4}{\pi^2}$. Indeed, by Corollary 2.9 (ii) we have that

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{\Lambda_\infty(\le 2m,n)}(\mathbb{T}^n)\big) = (L_m)^n .$$

So, using (20), gives the claim for the first sequence. The second sequence is handled the same way using Corollary 2.9 (iii). $\square$

2.2.3. *Lebesgue constants.* We point out that although we in Corollary 2.5 prove an integral formula for the projection constant of trigonometric polynomials on $\mathbb{T}^n$ (supported on the index set J), we usually will not be able to compute that integral explicitly.

In fact the integral from Corollary 2.5 appears naturally in multivariate Fourier analysis (due to different summation methods of multivariate Fourier series), and in this context it is usually called Lebesgue constant (of the index set J).

Let us illustrate this with an example of particular interest. Given $m,n\in\mathbb{N}$, the index set

$$J_2(\le m,n) = \Big\{\alpha\in\mathbb{Z}^n : \sum_{j=1}^n |\alpha_j|^2 \le m^2\Big\}, \tag{21}$$

of all spherical multi indices $\alpha\in\mathbb{Z}^n$ of 'Euclidean order' less or equal m (see again (10)) plays an outstanding role when multivariate Fourier series are summed up by their spherical partial sums. Obviously, we have that

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{J_2(\le m,n)}(\mathbb{T}^n)\big) \le \sqrt{\dim\big(\mathrm{Trig}_{J_2(\le m,n)}(\mathbb{T}^n)\big)} = \sqrt{|J_2(\le m,n)|};$$

this can either be seen as an immediate consequence of Corollary 2.5 or the Kadec-Snobar theorem.

However, to find a precise formula for the dimension of $\mathrm{Trig}_{J_2(\le m,n)}(\mathbb{T}^n)$, or equivalently the cardinality of $J_2(\le m,n)$, is in general a highly non-trivial problem.

The so-called n-dimensional sphere problem of classical lattice point theory concerns the number $\mathcal{N}_n(R)$ of lattice points in the closed Euclidean ball of radius R in $\mathbb{R}^n$. Let us mention here that in the case of $n>3$ one has

$$\mathcal{N}_n(R) = \omega_n R^n + O\big(R^{n-2}\big),$$

where $\omega_n := \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$. For $n=3$ Heath-Brown proves in [20] that

$$\mathcal{N}_3(R) = \frac{4}{3}\pi R^3 + O\big(R^{\frac{21}{16}+\varepsilon}\big),$$

where $\varepsilon>0$ is arbitrary. His proof is based on an explicit formula for the number of ways to write $n\in\mathbb{N}$ as a sum of three squares of integers. To get this, a Dirichlet L-series $L(1,\chi_n) = \sum_{m=1}^\infty \chi_n m^{-s}$

with $\chi_n(m) = \left(-\frac{4n}{m}\right)$ is studied. We also refer to the remarkable article [16] by Götze, where the lattice point problem is studied for the more complicated case of n-dimensional ellipsoids.

Under the above notation of (21) we are ready to state the following corollary.

Corollary 2.11. *There exist positive constants C_1 and C_2 with $C_1 < C_2$ depending on n such that*

$$C_1\, m^{\frac{n-1}{2}} \le \boldsymbol{\lambda}\big(Trig_{J_2(\le m,n)}(\mathbb{T}^n)\big) \le C_2\, m^{\frac{n-1}{2}}\,.$$

Proof. We use a two-sided estimate proved by Babenko [2], which (in our notation) states there exist constants C_1 and C_2 with $0 < C_1 < C_2$ depending on n, such that the following estimates hold for each $m \in \mathbb{N}$,

$$C_1\, m^{\frac{n-1}{2}} \le \int_{\mathbb{T}^n} \Big| \sum_{\alpha \in J_2(\le m,n)} z^\alpha \Big| \, dz \le C_2\, m^{\frac{n-1}{2}}\,.$$

Thus the required statement follows from Corollary 2.5. □

For the history of Babenko's important result, we refer to the interesting survey paper by Liflyand [31, Theorem 1.1]. We note that the proof given there, does not give precise asymptotic estimates in the dimension n for the constants C_1 and C_2. The estimates of both C_1 and C_2 depend deeply on the Fourier transform of the indicator function of the closed Euclidean ball of $\mathbb{R}^n$, which is expressed by the well-known Bessel function. It seems interesting to note that $C_1 = C_1(\varepsilon) := C_4\varepsilon^{(n-1)/2} - C_3\varepsilon^{n/2}$ for small $\varepsilon \in (0,1)$.

3. Dirichlet polynomials

Recall from the introduction that, given a frequency $\omega = (\omega_n)$ and a finite index set $J \subset \mathbb{N}$, we write $\mathscr{H}_\infty^J(\omega)$ for the Banach space of all ω-Dirichlet polynomials supported on J endowed with the supremum norm on the imaginary line. The following integral formula for the projection constant of $\mathscr{H}_\infty^J(\omega)$ is one of our main contributions, and crucial for all further descriptions of $\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big)$ for concrete frequencies and index sets. See also Theorem 3.6 for an equivalent formulation in terms of compact abelian groups.

Theorem 3.1. *Let $J \subset \mathbb{N}$ be a finite subset and $\omega = (\omega_n)$ a frequency. Then*

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \sum_{n\in J} e^{-i\omega_n t} \Big| \, dt\,.$$

The proof, given at the end of the following section, is mainly based on Theorem 2.1 and a reformulation of $\mathscr{H}_\infty^J(\omega)$ in terms of a Hardy type space of trigonometric polynomials on certain compact abelian groups associated to ω, which are all supported on a finite set of characters associated to J.

3.1. **Bohr's vision.** We need to adapt ideas from [10]. Our first aim is to describe the finite dimensional Banach space $\mathscr{H}_\infty^J(\omega)$ in terms of a Hardy type space of functions on a certain compact abelian group.

In what follows, a pair (G,β) is said to be a Dirichlet group if G is a compact abelian group and $\beta\colon \mathbb{R}\to G$ a continuous homomorphism with dense range; as before we denote the Haar measure on G by m. Then, by density, the dual map

$$\widehat{\beta}\colon \widehat{G}\to\widehat{\mathbb{R}}$$

is injective. Moreover, if we write $\mathbb{R}$ for the group $(\mathbb{R},+)$ endowed with its natural topology, then the homomorphism $\mathbb{R}=\widehat{\mathbb{R}},\ x\mapsto e^{ix\bullet}$ identifies $\mathbb{R}$ and $\widehat{\mathbb{R}}$ as topological groups, and hence $\widehat{\beta}(\widehat{G})$ may be interpreted as a subset of $\mathbb{R}$.

We in particular observe that if (G,β) is a Dirichlet group and $\gamma\colon G\to\mathbb{T}$ a character, then the composition $\gamma\circ\beta\colon\mathbb{R}\to\mathbb{T}$ is a character on $\mathbb{R}$. Thus there exists a unique $x\in\mathbb{R}$ such that $\gamma\circ\beta(t)=e^{ixt}$ for all $t\in\mathbb{R}$. The following result from [10, Proposition 3.10] is crucial - we include a proof for the sake of completeness.

Proposition 3.2. *For any Dirichlet group (G,β) one has*

$$\int_G f(a)\,d\mathrm{m}(a)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f\circ\beta(t)\,dt,\qquad f\in C(G).$$

Proof. Note first that for all $\gamma\in\widehat{G}$ we have

$$\int_G \gamma(a)\,d\mathrm{m}(a)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\gamma\circ\beta(t)\,dt.$$

Indeed, as explained above by the injectivity of the dual map $\widehat{\beta}$, we know that for every $\gamma\in\widehat{G}$ there is a unique $x=x(\gamma)\in\mathbb{R}$ such that $\gamma\circ\beta(t)=e^{ixt}$ for all $t\in\mathbb{R}$. Now recall that $\int_G\gamma\,d\mathrm{m}=0$ $\big($resp., $\int_G\gamma\,d\mathrm{m}=1\big)$ whenever $\gamma\neq 1$ (resp., $\gamma=1$). Clearly, $\gamma\neq1$ (resp., $\gamma=1$) yields $x\neq0$ (resp., $x=0$), so the above equality is obvious in this case. As a consequence, the claim also holds for all trigonometric polynomials $f\in C(G)$. But since all trigonometric polynomials are dense in $C(G)$, and moreover the set $\{\Phi_T\colon T>0\}$ of all linear functionals

$$\Phi_T\colon C(G)\to\mathbb{C},\quad \Phi_T(f):=\frac{1}{2T}\int_{-T}^{T} f\circ\beta(t)dt$$

is uniformly bounded, the conclusion is a consequence of the Banach-Steinhaus theorem. $\square$

Given a frequency $\omega=(\omega_n)$, we need Dirichlet groups which are adapted to ω. We call the pair (G,β) a ω-Dirichlet group, whenever $\{\omega_n\colon n\in\mathbb{N}\}\subset\widehat{\beta}(\widehat{G})$, that is, for every character $e^{-i\omega_n\bullet}:\mathbb{R}\to\mathbb{T}$

there is a character $h_{\omega_n} \in \widehat{G}$ (which then is unique) such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "h_{\omega_n}"] & \mathbb{T} \\ \mathbb{R} \arrow[u, "\beta"] \arrow[ur, "e^{-i\omega_n \bullet}"'] & \end{tikzcd}$$

For all needed information on ω-Dirichlet groups see again [10].

Let us collect some examples. The very first is the frequency $\omega = (n)$ for which the pair $(\mathbb{T}, \beta_{\mathbb{T}})$ with

$$\beta_{\mathbb{T}} : \mathbb{R} \to \mathbb{T}\,,\ t \mapsto e^{-it}$$

obviously forms a ω-Dirichlet group. Identifying $\widehat{\mathbb{T}} = \mathbb{Z}$ we get that $h_n(z) = z^n$ for $z \in \mathbb{T}, n \in \mathbb{Z}$.

The second example shows that ω-Dirichlet groups for any possible frequency ω always exist. In fact, given a frequency ω, the so-called Bohr compactification of the reals forms a ω-Dirichlet group.

Example 3.3. Denote by $\overline{\mathbb{R}} := \widehat{(\mathbb{R}, +, d)}$ the Bohr compactification of $\mathbb{R}$, where d stands for the discrete topology. This is a compact abelian group, which together with the mapping

$$\beta_{\overline{\mathbb{R}}} \colon \mathbb{R} \hookrightarrow \overline{\mathbb{R}},\ x \mapsto \left[t \mapsto e^{-ixt}\right]$$

forms a ω-Dirichlet group for every frequency ω.

But for concrete frequencies ω, there often are ω-Dirichlet groups which in a sense reflect their structure more naturally than the Bohr compactification.

Example 3.4. Let $\omega = (\log n)$. Then the infinite dimensional torus $\mathbb{T}^\infty$ together with the so-called Kronecker flow

$$\beta_{\mathbb{T}^\infty} : \mathbb{R} \to \mathbb{T}^\infty, \quad t \mapsto (\mathfrak{p}_k^{-it})\,,$$

where $\mathfrak{p}_k$ again denotes the kth prime, forms a ω-Dirichlet group. This is basically a consequence of Kronecker's approximation theorem (for an alternative 'harmonic analysis' argument see [10, Example 3.7]). Note that, identifying

$$\mathbb{Z}^{(\mathbb{N})} = \widehat{\mathbb{T}^\infty}, \quad \alpha \mapsto [z \mapsto z^\alpha]$$

($\mathbb{Z}^{(\mathbb{N})}$ all finite sequences of integers), for every $z \in \mathbb{T}^\infty$ one has

$$h_{\log n}(z) = h_{\sum \alpha_j \log \mathfrak{p}_j}(z) = z^\alpha\,, \tag{22}$$

where $n = \mathfrak{p}^\alpha$ with $\alpha \in \mathbb{Z}^{(\mathbb{N})}$.

As announced, we now reformulate the Banach space $\mathcal{H}_\infty^J(\omega)$ of all ω-Dirichlet polynomials supported on the finite index set $J \subset \mathbb{N}$ as a Hardy space of functions on a ω-Dirichlet group. Given a finite set $J \subset \mathbb{N}$ and a ω-Dirichlet group (G,β), we write

$$H_\infty^{\omega,J}(G)$$

for the subspace of all trigonometric polynomials $f = \sum_{n\in J} \hat{f}(h_{\omega_n})h_{\omega_n}$ in $L_\infty(G)$ (with respect to the Haar measure m on G).

The following equivalent description of $\mathcal{H}_\infty^J(\omega)$ is now obvious.

Proposition 3.5. *Let $J \subset \mathbb{N}$ be a finite subset, and (G,β) a ω-Dirichlet group. Then the Bohr map*

$$\mathscr{B} : \mathcal{H}_\infty^J(\omega) \to H_\infty^{\omega,J}(G), \quad \sum_{n\in J} a_n e^{-\omega_n s} \mapsto \sum_{n\in J} a_n h_{\omega_n}$$

defines an isometric linear bijection which preserves Bohr and Fourier coefficients.

Proof. The collections $(e^{-i\omega_n \bullet})_{n\in J}$ and $(h_{\omega_n})_{n\in J}$ are both linearly independent. Hence it suffices to check that for every collection $(a_n)_{n\in J}$ of complex scalars, we have

$$\sup_{t\in\mathbb{R}} \Big| \sum_{n\in J} a_n e^{-i\omega_n t} \Big| = \sup_{g\in G} \Big| \sum_{n\in J} a_n h_{\omega_n}(g) \Big|.$$

But this is clear by the fact that $e^{-i\omega_n \bullet} = h_{\omega_n} \circ \beta$ for each $n \in J$, and β moreover has dense range. $\square$

Finally, we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1. Choose some ω-Dirichlet group (G,β) (this is possible by Example 3.3). Then by Proposition 3.5 and Theorem 2.1 one has

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^J(\omega)\big) = \boldsymbol{\lambda}\big(H_\infty^{\omega,J}(G)\big) = \int_G \Big| \sum_{n\in J} h_{\omega_n} \Big| \, d\mathrm{m}.$$

Applying now Proposition 3.2, we get

$$\int_G \Big| \sum_{n\in J} h_{\omega_n} \Big| \, d\mathrm{m} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \sum_{n\in J} e^{-i\omega_n t} \Big| \, dt,$$

so this completes the proof. $\square$

We add a 'group version' of Theorem 3.1, which in view of Proposition 3.2 is immediate.

Theorem 3.6. *Let $J \subset \mathbb{N}$ be a finite subset, ω a frequency, and (G,β) a ω-Dirichlet group with Haar measure* m. *Then*

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^J(\omega)\big) = \int_G \Big| \sum_{n\in J} h_{\omega_n} \Big| \, d\mathrm{m}.$$

3.2. **Projection constants - Dirichlet polynomials.** Our aim now is to use the knowledge collected so far, to produce concrete applications. In fact, we distinguish three cases – each motivated slightly differently.

3.2.1. ***Case I.*** We consider Banach spaces $\mathcal{H}_\infty^{\leq x}(\omega)$ of Dirichlet polynomials of length x for the three standard frequencies $\omega = (n)_{n\in\mathbb{N}_0}$, $\omega = (\log p_n)_{n\in\mathbb{N}}$ (where as above p_n is the nth prime number), and $\omega = (\log n)_{n\in\mathbb{N}}$.

Starting with $\omega = (n)_{n\in\mathbb{N}_0}$, for $x \in \mathbb{N}$ the identification

$$\mathcal{H}_\infty^{\leq x}(\omega) \;=\; \mathrm{Trig}_{\{n\in\mathbb{N}_0\colon\, n\leq x\}}(\mathbb{T})\,, \quad \sum_{n=0}^{x} a_n e^{-\omega_n s} \mapsto \sum_{n=0}^{x} a_n z^n \tag{23}$$

is obviously isometric (doing the variable transformation $z = e^{-s}$), and hence by Theorem 3.6 we get the integral formula

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^{\leq x}(\omega)\big) = \int_{\mathbb{T}} \Big|\sum_{n=0}^{x} z^k\Big|\, dz\,. \tag{24}$$

Then a standard calculation leads to the following result, which in view of (23) is a counterpart of the Lozinski-Kharshiladze theorem from (18).

Theorem 3.7. *For the frequency* $\omega = (n)_{n\in\mathbb{N}_0}$

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^{\leq x}(\omega)\big) = \frac{4}{\pi^2}\log(x+1) + o(1)\,.$$

Next we consider the frequency $\omega = (\log p_n)$. This is the canonical example of a $\mathbb{Q}$-linearly independent frequency. It is well-known (see, e.g., [25, Theorem 9.2]) that, if $\omega_1, \ldots, \omega_n$ are real numbers, which are linearly independent over $\mathbb{Q}$, and $\omega_0 = 0$, then for any choice of complex numbers $a_0, a_1, \ldots, a_x$ we have

$$\sum_{n=0}^{x} |a_n| = \sup_{t\in\mathbb{R}} \Big|\sum_{n=0}^{x} a_n e^{-i\omega_n t}\Big|\,.$$

In other terms, for any frequency ω, which is linearly independent over $\mathbb{Q}$, and for any finite subset $J \subset \mathbb{N}$ the identification

$$\mathcal{H}_\infty^{J}(\omega) \;=\; \ell_1^{|J|}\,, \quad \sum_{n\in J} a_n e^{-\omega_n s} \mapsto (a_n)_{n\in J}$$

is isometric. Then the following integral and limit formula is an immediate consequences of (4) and (8).

Theorem 3.8. *Let* ω *be a* $\mathbb{Q}$*-linearly independent frequency and* $J \subset \mathbb{N}$ *a finite subset. Then*

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^{J}(\omega)\big) = \int_{\mathbb{T}^{|J|}} \Big|\sum_{n\in J} z_k\Big|\, dz\,.$$

Moreover,

$$\lim_{x\to\infty} \frac{\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\le x}(\omega)\big)}{\sqrt{x}} = \frac{\sqrt{\pi}}{2}.$$

We may, despite the precise constant, extend the preceding result within the setting of $\Lambda(2)$-sets explained in Section 2.2.1. Recall from Remark 2.4 that every B_2-set of characters on a compact abelian group is a $\Lambda(2)$-set with $\Lambda(2)$-constant $\le \sqrt{2}$.

Hence, Theorem 3.6, Corollary 2.3, and Remark 2.4 imply the following extension of Theorem 3.8.

Theorem 3.9. *Let ω be a frequency and (G,β) a ω-Dirichlet group with the property that all characters $h_{\omega_n} \in \widehat{G}$ form a $\Lambda(2)$-set with constant $C_2(\omega)$. Then*

$$\frac{1}{C_2(\omega)}\sqrt{x} \le \boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\le x}(\omega)\big) \le \sqrt{x}.$$

In particular, if ω satisfies the B_2-condition, then

$$\frac{1}{\sqrt{2}}\sqrt{x} \le \boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\le x}(\omega)\big) \le \sqrt{x}.$$

Finally, we handle the ordinary frequency $\omega = (\log n)$ - getting one of our main applications for free. Indeed, combining Harper's deep result from (1) with Theorem 3.1, we obtain the precise asymptotic order of the projection constant of the Banach space $\mathscr{H}_\infty^{\le x}$.

Theorem 3.10.

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\le x}\big((\log n)\big)\big) = O\left(\frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}\right)$$

It seems interesting to rephrase this result again in terms of trigonometric polynomials. We write

$$\Delta(x) = \big\{\alpha \in \mathbb{N}_0^{\pi(x)} : 1 \le \mathfrak{p}^\alpha \le x\big\},$$

where $x \ge 1$ and $\pi(x)$ as usual counts the primes $\mathfrak{p} \le x$. Then by Proposition 3.5 we have that

$$\mathscr{H}_\infty^{\le x}\big((\log n)\big) = \mathrm{Trig}_{\Delta(x)}\big(\mathbb{T}^{\pi(x)}\big), \tag{25}$$

where the identification is given by the Bohr transform – being isometric and coefficient preserving. Hence the following two results are immediate consequences of Theorem 3.10 (first statement) and Theorem 2.1 (second statement).

Corollary 3.11.

$$\boldsymbol{\lambda}\big(\mathit{Trig}_{\Delta(x)}\big(\mathbb{T}^{\pi(x)}\big)\big) = O\left(\frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}\right).$$

Moreover,

$$\boldsymbol{\lambda}\big(Trig_{\Delta(x)}\big(\mathbb{T}^{\pi(x)}\big)\big) = \big\|\mathbf{P}_{\Delta(x)} : C(\mathbb{T}^{\pi(x)}) \to Trig_{\Delta(x)}\big(\mathbb{T}^{\pi(x)}\big)\big\| = \int_{\mathbb{T}^{\pi(x)}} \Big|\sum_{\alpha\in\Delta(x)} z^{\alpha}\Big| dz\,,$$

where $\mathbf{P}_{\Delta(x)}$ *stands for the restriction of the orthogonal projection on* $L_2(\mathbb{T}^{\pi(x)})$ *onto* $Trig_{\Delta(x)}\big(\mathbb{T}^{\pi(x)}\big)$.

3.2.2. ***Case II***. For the ordinary frequency $\omega = (\log n)$ we look at Banach spaces $\mathcal{H}_\infty^J(\omega)$ of Dirichlet polynomials generated by index sets J of natural numbers k, which have an a priori specified complexity of their prime number decompositions. More precisely, the first two results consider index sets J, where each element $k = \mathfrak{p}^\alpha \in J$ incorporates not more than n different prime divisors with $\max \alpha_j \le m$, or similarly, n different prime divisors with $|\alpha| = \sum \alpha_j \le m$. Recall from (10) the definition of the index sets $\Lambda_1(\le m, n)$ and $\Lambda_\infty(\le m, n)$.

Theorem 3.12. *Let* $\omega = (\log n)$. *Then*

$$\boldsymbol{\lambda}\big(\mathcal{H}_\infty^{N_\infty(\le m,n)}(\omega)\big) \sim (1+\log m)^n\,,$$

and for each n

$$\lim_{m\to\infty} \frac{\boldsymbol{\lambda}\big(\mathcal{H}_\infty^{N_\infty(\le m,n)}(\omega)\big)}{\log^n m} = \Big(\frac{4}{\pi^2}\Big)^n,$$

where

$$N_\infty(\le m, n) = \big\{k \in \mathbb{N} \colon\, k = \mathfrak{p}^\alpha, \quad \alpha \in \Lambda_\infty(\le m, n)\,\big\}.$$

Proof. It follows from Proposition 3.5 and Example 3.4 together with (22) that the Bohr map

$$\mathcal{B} \colon \mathcal{H}_\infty^{N_\infty(\le m,n)}(\omega) \to \mathrm{Trig}_{\Lambda_\infty(\le m,n)}\big(\mathbb{T}^n\big), \quad \sum_{n\in J} a_n e^{-\omega_n s} \mapsto \sum_{\alpha\in\Lambda_\infty(\le m,n)} a_{\mathfrak{p}^\alpha} z^\alpha$$

defines an isometric linear bijection which preserves Bohr and Fourier coefficients. Hence the first asymptotic is a consequence of Corollary 2.9 (iv), and the formula for the limit of Corollary 2.10. $\square$

Replacing the use of Corollary 2.10 by Corollary 2.5 and (17), similar arguments lead to the following counterpart of the preceding theorem.

Theorem 3.13. *Let* $\omega = (\log n)$. *Then for each* $m, n \in \mathbb{N}$ *one has*

$$\frac{1}{\sqrt{2^m}}\left(\sum_{k=0}^{m}\binom{n+k+1}{k}\right)^{\frac12} \le \boldsymbol{\lambda}\big(\mathcal{H}_\infty^{N_1(\le m,n)}(\omega)\big) \le \left(\sum_{k=0}^{m}\binom{n+k+1}{k}\right)^{\frac12},$$

where

$$N_1(\le m, n) = \big\{k \in \mathbb{N} \colon\, k = \mathfrak{p}^\alpha,\ \textit{where}\ \alpha \in \Lambda_1(\le m, n)\,\big\}.$$

We finish this part with a result, which in view of Weissler's inequality from (16) is a formal extension of the last theorem. To do so, define for the finite index set $J \subset \mathbb{N}$, the numbers

$$\pi(J) = \max_{n \in J} \pi(n) \quad \text{and} \quad \Omega(J) = \max_{n \in J} \Omega(n)\,,$$

and

$$\Delta(J) = \left\{\alpha \in \mathbb{N}_0^{\pi(J)} \colon n \in J,\, \mathfrak{p}^\alpha = n\right\}.$$

Here as usual $\pi(n)$ counts all primes $\le n$ and $\Omega(n)$ is the number of prime divisors of n counted according to their multiplicities.

Theorem 3.14. *Let $\omega = (\log n)$ and $J \subset \mathbb{N}$ be a finite subset. Then*

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big) = \int_{\mathbb{T}^{\pi(J)}} \Big| \sum_{\alpha \in \Delta(J)} z^\alpha \Big| \, dz\,.$$

Moreover,

$$\frac{1}{\sqrt{2^{\Omega(J)}}}\sqrt{|J|} \;\le\; \boldsymbol{\lambda}\big(\mathscr{H}_\infty^J(\omega)\big) \le \sqrt{|J|}.$$

Proof. As above (Proposition 3.5), we see that the Bohr map

$$\mathscr{B}\colon \mathscr{H}_\infty^J(\omega) \to \mathrm{Trig}_{\Delta(J)}\big(\mathbb{T}^{\pi(J)}\big), \quad \sum_{n \in J} a_n e^{-\omega_n s} \mapsto \sum_{\alpha \in \Delta(J)} a_{\mathfrak{p}^\alpha} z^\alpha \tag{26}$$

is isometric and coefficient preserving. Again, the conclusion follows from Corollary 2.5. Indeed, the order $|\alpha|$ of each multi index $\alpha \in \Delta(J)$ is at most $\Omega(J)$, and the cardinalities of J and $\Delta(J)$ are the same. $\square$

3.2.3. ***Case III***. Finally, we produce an example, where we mix aspects from the preceding two cases. The following result considers ordinary Dirichlet polynomials of length x which are supported on all $1 \le n = \mathfrak{p}^\alpha \le x$ having precisely m prime devisors (all counted according to their multiplicities).

Theorem 3.15. *Let $\omega = (\log n)$. Then for each $m \in \mathbb{N}$ one has*

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{N_1(m,x)}(\omega)\big) \;\sim_{C(m)}\; \sqrt{\frac{x}{\log x}}\,\sqrt{\frac{(\log\log x)^{m-1}}{(m-1)!}}\,,$$

where the constants depend on m but not on x, and

$$N_1(m,x) = \left\{1 \le n \le x \colon n = \mathfrak{p}^\alpha \;\; \textit{with} \;\; |\alpha| = m\right\}.$$

Moreover, the constant $C(m)$ is independent of m, whenever $m \le \frac{\log\log x}{2e}$.

Let us here first look at the special case $m = 1$. Note first that for any m by Proposition 3.5,

$$\mathscr{H}_\infty^{N_1(m,x)}(\omega) = \mathrm{Trig}_{\Delta(N_1(m,x))}\big(\mathbb{T}^{\pi(N_1(m,x))}\big)\,,$$

where the identification, given by the Bohr map, is isometric and coefficient preserving. Then for $m=1$ we immediately deduce from Theorem 3.8 and the prime number theorem that

$$\lim_{x\to\infty}\frac{\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{N_1(1,x)}\big)}{\sqrt{\frac{x}{\log x}}}=\lim_{x\to\infty}\frac{\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{N_1(1,x)}\big)}{\sqrt{\pi(x)}}=\frac{\sqrt{\pi}}{2}.$$

So the preceding theorem extends this asymptotic estimate from $m=1$ to arbitrary $m>1$ (neglecting constants).

Proof of Theorem 3.15. Since $\pi(N(m,x))\le\pi(x)$ and $\Omega(N_1(m,x)))=m$, by Theorem 3.14

$$\frac{1}{\sqrt{2^m}}\sqrt{|N_1(m,x)|}\ \le\ \boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\le x,m}\big)\le\sqrt{|N_1(m,x)|}.$$

Then the first claim follows from a well-known result of Landau (see, e.g., [45, p. 200]) showing that

$$|N_1(m,x)|\ \sim_{C(m)}\ \frac{x}{\log x}\frac{(\log\log x)^{m-1}}{(m-1)!}\quad\text{as}\quad x\to\infty.$$

For the proof of the second claim we use again that by Theorem 3.14

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{N_1(m,x)}\big)=\int_{\mathbb{T}^\infty}\Big|\sum_{\alpha\in\Delta(N_1(m,x))}z^\alpha\Big|\,dz.$$

But in [7, Equation (13), p.107] it is shown that there is a universal constant $C\ge1$ such that for any x,m with $m\le\frac{\log\log x}{2e}$

$$\int_{\mathbb{T}^\infty}\Big|\sum_{\alpha\in\Delta(N_1(m,x))}z^\alpha\Big|\,dz\ \sim_C\ \bigg(\int_{\mathbb{T}^\infty}\Big|\sum_{\alpha\in\Delta(N_1(m,x))}z^\alpha\Big|^2dz\bigg)^{\frac12}=\sqrt{|N_1(m,x)|}.$$

On the other hand, applying the Sathe-Selberg formula (see, e.g., [12]), we conclude that for every $\delta>0$ there is $C(\delta)>0$ such that for $m\le(2-\delta)\log\log x$

$$|N_1(m,x)|\ \sim_{C(\delta)}\ \frac{x}{\log x}\frac{(\log\log x)^{m-1}}{(m-1)!}\quad\text{as}\quad x\to\infty.$$

This completes the argument. □

4. Comparing Sidon constants

Recall that, given a topological group G, the Sidon constant $\mathbf{Sid}(\Gamma)$ of a finite set Γ of characters in the dual group $\widehat{G}$ is given by the best constant constant $c\ge0$ such that for every trigonometric polynomial $f=\sum_{\gamma\in\Gamma}c_\gamma\gamma$ on G one has

$$\sum_{\gamma\in\Gamma}|c_\gamma|\le c\|f\|_\infty. \tag{27}$$

It is easily proved that this constant in fact equals the unconditional basis constant formed by all $\gamma\in\Gamma$ in the Banach space $\mathrm{Trig}_\Gamma(G)$.

Recall that the unconditional basis constant of a basis $(e_i)_{i\in I}$ of a Banach space X is given by the infimum over all $K>0$ such that for any finitely supported family $(\alpha_i)_{i\in I}$ of scalars and for any finitely supported family $(\varepsilon_i)_{i\in I}$ with $|\varepsilon_i|=1$, $i\in I$ we have

$$\Big\|\sum_{i\in I}\varepsilon_i\alpha_i e_i\Big\|\le K\Big\|\sum_{i\in I}\alpha_i e_i\Big\|;$$

moreover, in this case the unconditional basis constant of $(e_i)_{i\in I}$ is defined to be the infimum over all these constants K.

In particular, for any finite index set $\Gamma\subset\mathbb{Z}^n$ the unconditional basis constant of the collection of all monomials $(z^\alpha)_{\alpha\in\Gamma}$ in $\mathrm{Trig}_\Gamma(\mathbb{T}^n)$, here denoted by $\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_\Gamma(\mathbb{T}^n)\big)$, equals the Sidon constant $\mathbf{Sid}(\Gamma)$.

Given a finite dimensional Banach space X, the unconditional basis constant with respect to a basis of this space and its projection constant are two quite different objects (compare for example $\chi(\ell_2^n)=1$ with $\boldsymbol{\lambda}(\ell_2^n)\sim\sqrt{n}$).

But in the case of Banach spaces of Dirichlet polynomials and their associated spaces of multivariate trigonometric polynomials, it turns out that a better understanding of one of the two constants often leads to a better understanding of the other constant.

To illustrate this point of view, we start considering analytic trigonometric polynomials in one variable. Recall that $\mathrm{Trig}_{\{k:\,1\le k\le d\}}(\mathbb{T})$ stands for all analytic trigonometric polynomials of the form $P(z)=\sum_{k=1}^d a_k z^k$, $z\in\mathbb{T}$, so polynomials of degree $\le d$ without a constant term a_0 (following our notation from Section 2). Then Rudin [38] and Shapiro [44] (see also [9, Proposition 9.7]) proved that

$$\frac{1}{\sqrt{2}}\sqrt{d}\le\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_{\{k:\,1\le k\le d\}}(\mathbb{T})\big)\le\sqrt{d}.$$

If we allow constant terms, then the best known estimate is

$$\sqrt{d}-O(\log d)^{\frac{2}{3}+\varepsilon}\le\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_{\{k:\,0\le k\le d\}}(\mathbb{T})\big)\le\sqrt{d}.$$

This is a deep fact proved by Bombieri and Bourgain in [6], and it shows that at least from the technical point of view a seemingly small perturbation of the index set may change the situation drastically.

Let us compare these results with what we in Corollary 2.9 proved for projection constants, namely

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{\{k:\,1\le k\le d\}}(\mathbb{T})\big)\sim\boldsymbol{\lambda}\big(\mathrm{Trig}_{\{k:\,0\le k\le d\}}(\mathbb{T})\big)\sim 1+\log d.$$

This illustrates that unconditional basis constants (so Sidon constants) and projection constants of spaces of trigonometric polynomials in one variable behave quite differently.

Of course, the situation doesn't improve if we admit more variables - nevertheless we may recognize some in a sense systematic patterns. Indeed, it was proved in [8] (a result elaborated in [9, Theorem 9.10]) that

$$\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_{\leq m}(\mathbb{T}^n)\big) \sim_{C^m} \sqrt{\binom{n+m}{m}}.$$

On the other hand by Theorem 3.13 we know that

$$\boldsymbol{\lambda}\big(\mathrm{Trig}_{\leq m}(\mathbb{T}^n)\big) \sim_{C^m} \sqrt{\binom{n+m+1}{m}},$$

so in particular we have that

$$\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_{\leq m}(\mathbb{T}^n)\big) \sim_{C^m} \boldsymbol{\lambda}\big(\mathrm{Trig}_{\leq m-1}(\mathbb{T}^n)\big) \sim_{C^m} \left(\frac{n+m}{m}\right)^{\frac{m-1}{2}}. \tag{28}$$

Coming back to spaces of ordinary(!) Dirichlet polynomials, observe first that given a finite subset J of $\mathbb{N}$, it is obvious that the 'monomials' n^{-s}, $n \in J$ form a basis of the Banach space $\mathscr{H}_\infty^J\big((\log n)\big)$, in the following abbreviated by $\mathscr{H}_\infty^J$, and so we denote by

$$\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathscr{H}_\infty^J\big)$$

the unconditional basis constant of all n^{-s}, $n \in J$.

As already used in (26), there is an isometry $\mathscr{H}_\infty^J = \mathrm{Trig}_{\Delta(J)}(\mathbb{T}^{\pi(J)})$, which preserves coefficients, and hence by (28) we conclude that

$$\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathscr{H}_\infty^{N_1(m,n)}\big) \sim_{C^m} \boldsymbol{\lambda}\big(\mathscr{H}_\infty^{N_1(m-1,n)}\big) \sim_{C^m} \sqrt{\binom{n+m}{m}};$$

here we again use the notation fixed in Theorem 3.13.

What about the unconditional basis constants of $\mathscr{H}_\infty^{\leq x}$, the Banach space of all Dirichlet polynomials $D(s) = \sum_{n=1}^x a_n n^{-s}$ of length x, in comparison with their projection constants?

Again we use the notation from Section 3.2.2. The following asymptotic

$$\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathscr{H}_\infty^{\leq x}\big) = \boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_{\Delta(x)}(\mathbb{T}^{\pi(x)})\big) \sim_C \frac{\sqrt{x}}{e^{\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log x \log\log x}}}.$$

is taken from [8, Theorem 3] (see also [9, Theorem 9.1]), and it is the final outcome of a long lasting research project started by Queffélec, Konyagin, and de la Bretèche (see [9, Section 9.3] for more details on its history). In Theorem 3.10 and Corollary 3.11 we proved its counterpart for projection constants:

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{\leq x}\big) = \boldsymbol{\lambda}\big(\mathrm{Trig}_{\Delta(x)}(\mathbb{T}^{\pi(x)})\big) \sim_C \frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}.$$

In the m-homogeneous case analog results are known - by Balasubramanian-Calado-Queffélec [3] (a result elaborated in [9, Theorem 9.4]) one has (with the notation from Theorem 3.15)

$$\boldsymbol{\chi}_{\mathbf{mon}}\big(\mathscr{H}_\infty^{N_1(m,x)}\big) = \boldsymbol{\chi}_{\mathbf{mon}}\big(\mathrm{Trig}_{\Delta(N_1(m,x))}\big(\mathbb{T}^{\pi(N_1(m,x))}\big)\big) \sim_{C(m)} \frac{x^{\frac{m-1}{2m}}}{(\log x)^{\frac{m-1}{2}}},$$

whereas the corresponding result for projection constants from Theorem 3.15 reads

$$\boldsymbol{\lambda}\big(\mathscr{H}_\infty^{N_1(m,x)}\big) = \boldsymbol{\lambda}\big(\mathrm{Trig}_{\Delta(N_1(m,x))}\big(\mathbb{T}^{\pi(N_1(m,x))}\big)\big) \ \sim_{C(m)} \ \sqrt{\frac{x}{\log x}}\sqrt{\frac{(\log\log x)^{m-1}}{(m-1)!}}.$$

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