

Multidimensional Rogers–Ramanujan type identities with parameters

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Abstract. Via the contour integral method, we establish several multidimensional Rogers–Ramanujan type identities with parameters. As conclusions, some known formulas are recovered and several new identities are derived. The bisection method is also used to discover some new results.

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1 Introduction

For any complex numbers x, q with $|q| < 1$ and nonnegative integer n , define the q -shifted factorial to be

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k) \quad \text{and} \quad (x; q)_n = \frac{(x; q)_\infty}{(xq^n; q)_\infty}.$$

For simplicity, we usually adopt the compact notation

$$(x_1, x_2, \dots, x_m; q)_n = (x_1; q)_n (x_2; q)_n \cdots (x_m; q)_n,$$

where $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+ \cup \{0, \infty\}$. Following Gasper and Rahman [8], define the basic hypergeometric series as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q, b_1, \dots, b_r; q)_k} z^k.$$

The famous Rogers–Ramanujan identities are

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} = \frac{1}{(q, q^4; q^5)_\infty}, \quad (1.1)$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_k} = \frac{1}{(q^2, q^3; q^5)_\infty}. \quad (1.2)$$

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In 2019, Kanade and Russell [10] proposed nine conjectured multidimensional Rogers–Ramanujan type identities related to level 2 characters of the affine Lie algebra $A_9^{(2)}$. Five of them are confirmed by Bringmann, Jennings-Shaffer, and Mahlburg [4]. Rosengren [13] proved all of the nine formulas by the contour integral method.

Recently, Uncu and Zudilin [15] proved the following two interesting identities:

$$\sum_{j, k \geq 0} \frac{q^{j^2+2jk+2k^2}}{(q; q)_j (q^2; q^2)_k} = \frac{(q^3; q^3)_\infty^2}{(q; q)_\infty (q^6; q^6)_\infty}, \quad (1.3)$$

$$\sum_{j, k \geq 0} \frac{q^{j^2+2jk+2k^2+j+2k}}{(q; q)_j (q^2; q^2)_k} = \frac{(q^6; q^6)_\infty^2}{(q^2; q^2)_\infty (q^3; q^3)_\infty}. \quad (1.4)$$

Ole Warnaar has pointed that (1.3) and (1.4) are instances of Bressoud’s results (cf. [3]). Though the contour integral method, Wang [16] recovered (1.3) and (1.4) and Cao and Wang [6, Theorem 3.8] found the following two formulas:

$$\sum_{j, k \geq 0} \frac{q^{j^2+2jk+2k^2}}{(q; q)_j (q^2; q^2)_k} (-1)^j x^{j+k} = (qx; q^2)_\infty, \quad (1.5)$$

$$\sum_{j, k \geq 0} \frac{q^{j^2+2jk+2k^2+k}}{(q; q)_j (q^2; q^2)_k} x^{j+2k} = (-qx; q)_\infty, \quad (1.6)$$

where x is an arbitrary complex number. More conclusions can be seen in the papers [1, 5, 11, 12, 14]. Inspired by the works just mentioned, we shall establish the following theorem.

Theorem 1.1. *Let x, y be complex numbers. Then*

$$\sum_{j, k \geq 0} \frac{q^{j^2+2jk+2k^2-j-k}}{(q; q)_j (q^2; q^2)_k} x^j y^{2k} = (y; q)_\infty \sum_{k=0}^{\infty} \frac{(-x/y; q)_k}{(q; q)_k (y; q)_k} q^{\binom{k}{2}} y^k. \quad (1.7)$$

Choosing $(x, y) = (q, q^{\frac{1}{2}})$ in Theorem 1.1 and then calculating the series on the right-hand side by Ramanujan’s formula (cf. [1, Entry 5.3.2]):

$$\sum_{k=0}^{\infty} \frac{(-q; q^2)_k}{(q; q)_{2k}} q^{k^2} = \frac{(q^6; q^6)_\infty^2}{(q; q)_\infty (q^{12}; q^{12})_\infty},$$

we catch hold of (1.3). Fixing $(x, y) = (q^2, q^{\frac{3}{2}})$ in Theorem 1.1 and then evaluating the series on the right-hand side by Ramanujan’s another formula (cf. [1, Entry 3.4.4]):

$$\sum_{k=0}^{\infty} \frac{(-q; q^2)_k}{(q; q)_{1+2k}} q^{k^2+2k} = \frac{(q^{12}; q^{12})_\infty (-q^6; q^6)_\infty}{(q; q)_\infty (-q^2; q^2)_\infty},$$

we get hold of (1.4).

Taking $(x, y) \rightarrow (-qx, -q^{\frac{1}{2}}x^{\frac{1}{2}})$ in Theorem 1.1 and then computing the series on the right-hand side by Euler's q -exponential formula (cf. [8, Appendix (II.2)]):

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} z^k}{(q; q)_k} = (-z; q)_{\infty}, \quad (1.8)$$

we arrive at (1.5). Letting $(x, y) \rightarrow (qx, -qx)$ in Theorem 1.1, we are led to (1.6).

When $x = 0$, Theorem 1.1 produces the following special case of Berkovich and Warnaar [2, Equation (3.10)].

Corollary 1.2. *Let y be a complex number. Then*

$$\sum_{k=0}^{\infty} \frac{q^{2k^2-k} y^{2k}}{(q^2; q^2)_k} = (y; q)_{\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} y^k}{(q; q)_k (y; q)_k}.$$

Recall a generalization of (1.1) and (1.2) due to K. Garrett, Ismail, and Stanton [7]:

$$\sum_{k=0}^{\infty} \frac{q^{k^2+mk}}{(q; q)_k} = \frac{(-1)^m q^{-\binom{m}{2}} E_{m-2}(q)}{(q, q^4; q^5)_{\infty}} - \frac{(-1)^m q^{-\binom{m}{2}} D_{m-2}(q)}{(q^2, q^3; q^5)_{\infty}}, \quad (1.9)$$

where m is an integer and the Schur polynomials $D_m(q)$ and $E_m(q)$ are defined by

$$\begin{aligned} D_m(q) &= D_{m-1}(q) + q^m D_{m-2}(q), & D_0(q) &= 1, & D_1(q) &= 1 + q, \\ E_m(q) &= E_{m-1} + q^m E_{m-2}(q), & E_0(q) &= 1, & E_1(q) &= 1. \end{aligned}$$

Letting $(q, y) \rightarrow (q^2, q^{1+2m})$ or $(q^2, -q^{1+2m})$ in Corollary 1.2 and using (1.9), we obtain the following conclusion.

Corollary 1.3. *Let m be an integer. Then*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{q^{k^2+2mk}}{(q^2; q^2)_k (q^{1+2m}; q^2)_k} &= \frac{(-1)^m q^{2m-2m^2} E_{m-2}(q^4)}{(q^{1+2m}; q^2)_{\infty} (q^4, q^{16}; q^{20})_{\infty}} \\ &\quad - \frac{(-1)^m q^{2m-2m^2} D_{m-2}(q^4)}{(q^{1+2m}; q^2)_{\infty} (q^8, q^{12}; q^{20})_{\infty}}. \end{aligned} \quad (1.10)$$

Setting $m = 0$ or 1 in Corollary 1.3, we recover the two known results (cf. [14, Equations (98) and (96)]):

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_{2k}} = \frac{(q^{10}, q^8, q^2; q^{10})_{\infty} (q^{14}, q^6; q^{20})_{\infty}}{(q; q)_{\infty}}, \quad (1.11)$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+2k}}{(q; q)_{1+2k}} = \frac{(q^{10}, q^6, q^4; q^{10})_{\infty} (q^{18}, q^2; q^{20})_{\infty}}{(q; q)_{\infty}}. \quad (1.12)$$

It should be pointed that Gu and Prodinger [9, Theorem 2.6] gave one-parameter generalizations of (1.11) and (1.12), which are different from (1.10), several years ago.

Subsequently, we shall display the following triple-sum generalization of (1.5).

Theorem 1.4. *Let x, y be complex numbers. Then*

$$\sum_{j, k, \ell \geq 0} \frac{(x; q)_j (-x)^{k+2\ell} y^{k+\ell}}{(q; q)_j (q; q)_k (q^2; q^2)_\ell} q^{j+\binom{k}{2}+\binom{j+k+2\ell}{2}} = (qx, xy; q^2)_\infty (-q; q)_\infty. \quad (1.13)$$

When $x = 1$, Theorem 1.4 reduces to (1.5) thanks to the relation (cf. [8, P. 24]):

$$(q, -q, -q^2; q^2)_\infty = 1,$$

which will be utilized without indication elsewhere.

Similarly, we shall give the following two triple-sum generalizations of (1.6).

Theorem 1.5. *Let x, y be complex numbers. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x^2 y^2; q^2)_k x^j y^{2j+2\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} (-1)^{j+k} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+k} \\ &= \frac{(q; q^2)_\infty}{2} \{ (xy, -y; q)_\infty + (-xy, y; q)_\infty \}. \end{aligned} \quad (1.14)$$

Theorem 1.6. *Let x, y be complex numbers. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x^2 y^2; q^2)_k x^j y^{2j+2\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} (-1)^{j+k} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+3k} \\ &= \frac{(q; q^2)_\infty}{2(y/q - xy)} \{ (xy, -y/q; q)_\infty - (-xy, y/q; q)_\infty \}. \end{aligned} \quad (1.15)$$

When $xy = 1$, Theorems 1.5 and 1.6 both become (1.6). From the two theorems, we can also deduce some new multidimensional Rogers–Ramanujan type identities.

Taking $(x, y) \rightarrow (-xq^m, q^{-m})$ in Theorem 1.5, there is the following formula.

Corollary 1.7. *Let x be a complex number and let m be a nonnegative integer. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x^2; q^2)_k x^j (-1)^k}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+k-m(j+2\ell)} \\ &= (-q^{-m}; q)_m (-x; q)_\infty. \end{aligned} \quad (1.16)$$

Letting $(x, y) \rightarrow (1, x^{\frac{1}{2}})$, $(q, x^{\frac{1}{2}}/q)$ or $(1/q, qx^{\frac{1}{2}})$ in Theorem 1.5, we find the following three conclusions.

Corollary 1.8. *Let x be a complex number. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x; q^2)_k x^{j+\ell} (-1)^{j+k}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+k} = (q, x; q^2)_\infty, \\ & \sum_{j, k, \ell \geq 0} \frac{(x; q^2)_k x^{j+\ell} (-1)^{j+k}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2-j+k-2\ell} = (q, x; q^2)_\infty, \\ & \sum_{j, k, \ell \geq 0} \frac{(x; q^2)_k x^{j+\ell} (-1)^{j+k}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+j+k+2\ell} = (q, q^2 x; q^2)_\infty. \end{aligned}$$

Taking $(x, y) \rightarrow (-xq^{m-1}, q^{1-m})$ in Theorem 1.6, there holds the following formula.

Corollary 1.9. *Let x be a complex number and let m be a nonnegative integer. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x^2; q^2)_k x^j (-1)^k}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+3k-(m-1)(j+2\ell)} \\ &= \frac{(-q^{-m}; q)_m (-x; q)_\infty}{q^{-m} + x}. \end{aligned} \quad (1.17)$$

Letting $(x, y) \rightarrow (1, x^{\frac{1}{2}})$ or $(1/q^2, q^2 x^{\frac{1}{2}})$ in Theorem 1.6, we derive the following two conclusions.

Corollary 1.10. *Let x be a complex number. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x; q^2)_k x^{j+\ell} (-1)^{j+k}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+3k} = (q^3, x; q^2)_\infty, \\ & \sum_{j, k, \ell \geq 0} \frac{(x; q^2)_k x^{j+\ell} (-1)^{j+k}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+2j+3k+4\ell} = (q^3, q^2 x; q^2)_\infty. \end{aligned}$$

The rest of the paper is arranged as follows. In terms of the contour integral method, we shall prove Theorems 1.1 in Section 2. The proof of Theorems 1.4-1.6 will be displayed in Section 3. According to the bisection method, we shall establish several new multidimensional Rogers–Ramanujan type identities in Section 4.

2 Proof of Theorems 1.1

For the aim to prove Theorem 1.1, we need the following lemma (cf. [8, Equation (4.10.6)]).

Lemma 2.1. *Assume that*

$$P(z) = \frac{(a_1 z, \dots, a_A z, b_1/z, \dots, b_B/z; q)_\infty}{(c_1 z, \dots, c_A z, d_1/z, \dots, d_D/z; q)_\infty}$$

has only simple poles and $|a_1 \cdots a_A / c_1 \cdots c_A| < 1$. Then

$$\begin{aligned} \oint P(z) \frac{dz}{2\pi i z} &= \frac{(b_1 c_1, \dots, b_B c_1, a_1/c_1, \dots, a_A/c_1; q)_\infty}{(q, d_1 c_1, \dots, d_D c_1, c_2/c_1, \dots, c_A/c_1; q)_\infty} \\ &\times \sum_{k=0}^{\infty} \frac{(d_1 c_1, \dots, d_D c_1, q c_1/a_1, \dots, q c_1/a_A; q)_k}{(q, b_1 c_1, \dots, b_B c_1, q c_1/c_2, \dots, q c_1/c_A; q)_k} \left(\frac{a_1 \cdots a_A}{c_1 \cdots c_A} \right)^k \\ &+ \text{idem}(c_1; c_2, \dots, c_A), \end{aligned}$$

where the integration is over a positively oriented unit circle such that the origin and poles of $1/(d_1/z, \dots, d_D/z; q)_\infty$ lie inside the contour and the poles of $1/(c_1 z, \dots, c_A z; q)_\infty$ lie outside the contour.

Now we begin to prove Theorem 1.1.

Proof. Heine's transformation formulas of ${}_2\phi_1$ series (cf. [8, Appendix III.2 and III.3]) read

$${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, z \right] = \frac{(c/a, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} abz/c, a \\ az \end{matrix} ; q, \frac{c}{a} \right] \quad (2.1)$$

$$= \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} c/a, c/b \\ c \end{matrix} ; q, \frac{abz}{c} \right]. \quad (2.2)$$

By means of (2.2), we have

$${}_2\phi_1 \left[\begin{matrix} a, aq/c \\ aq/b \end{matrix} ; q, \frac{cq}{abz} \right] = \frac{(q/z; q)_\infty}{(cq/abz; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q/b, c/b \\ aq/b \end{matrix} ; q, \frac{q}{z} \right], \quad (2.3)$$

$${}_2\phi_1 \left[\begin{matrix} b, bq/c \\ bq/a \end{matrix} ; q, \frac{cq}{abz} \right] = \frac{(q/z; q)_\infty}{(cq/abz; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q/a, c/a \\ bq/a \end{matrix} ; q, \frac{q}{z} \right]. \quad (2.4)$$

Substituting (2.1), (2.3), and (2.4) into the three-term transformation formula of ${}_2\phi_1$ series (cf. [8, Appendix III.32]):

$$\begin{aligned} {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, z \right] &= \frac{(b, c/a, az, q/az; q)_\infty}{(c, b/a, z, q/z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} a, aq/c \\ aq/b \end{matrix} ; q, \frac{cq}{abz} \right] \\ &\quad + \frac{(a, c/b, bz, q/bz; q)_\infty}{(c, a/b, z, q/z; q)_\infty} {}_2\phi_1 \left[\begin{matrix} b, bq/c \\ bq/a \end{matrix} ; q, \frac{cq}{abz} \right], \end{aligned}$$

it is easy to show that

$$\begin{aligned} &(c/a, az; q)_\infty {}_2\phi_1 \left[\begin{matrix} abz/c, a \\ az \end{matrix} ; q, \frac{c}{a} \right] \\ &= \frac{(a, c/b, bz, q/bz; q)_\infty}{(a/b, cq/abz; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q/a, c/a \\ bq/a \end{matrix} ; q, \frac{q}{z} \right] \\ &\quad + \frac{(b, c/a, az, q/az; q)_\infty}{(b/a, cq/abz; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q/b, c/b \\ aq/b \end{matrix} ; q, \frac{q}{z} \right]. \end{aligned}$$

Take $(a, b, c, z) \rightarrow (-x/y, x/y, 0, -y^2/x)$ to obtain

$$\begin{aligned} &(y; q)_\infty \sum_{k=0}^{\infty} \frac{(-x/y; q)_k}{(q; q)_k (y; q)_k} q^{\binom{k}{2}} y^k \\ &= \frac{(y, x/y, q/y; q)_\infty}{(-1; q)_\infty} \sum_{k=0}^{\infty} \frac{(qy/x; q)_k}{(q^2; q^2)_k} \left(-\frac{qx}{y^2} \right)^k \\ &\quad + \frac{(-y, -x/y, -q/y; q)_\infty}{(-1; q)_\infty} \sum_{k=0}^{\infty} \frac{(-qy/x; q)_k}{(q^2; q^2)_k} \left(-\frac{qx}{y^2} \right)^k. \end{aligned} \quad (2.5)$$

Choose $(A, B, D) = (2, 1, 0)$ and $(a_1, a_2, b_1, c_1, c_2) = (x, q, 1, y, -y)$ in Lemma 2.1 to gain

$$\begin{aligned} & \oint \frac{(xz, q, qz, 1/z; q)_\infty}{(y^2 z^2; q^2)_\infty} \frac{dz}{2\pi iz} \\ &= \frac{(y, x/y, q/y; q)_\infty}{(-1; q)_\infty} \sum_{k=0}^{\infty} \frac{(qy/x; q)_k}{(q^2; q^2)_k} \left(-\frac{qx}{y^2}\right)^k \\ &+ \frac{(-y, -x/y, -q/y; q)_\infty}{(-1; q)_\infty} \sum_{k=0}^{\infty} \frac{(-qy/x; q)_k}{(q^2; q^2)_k} \left(-\frac{qx}{y^2}\right)^k. \end{aligned} \quad (2.6)$$

The combination of (2.5) and (2.6) engenders

$$\oint \frac{(xz, q, qz, 1/z; q)_\infty}{(y^2 z^2; q^2)_\infty} \frac{dz}{2\pi iz} = (y; q)_\infty \sum_{k=0}^{\infty} \frac{(-x/y; q)_k}{(q; q)_k (y; q)_k} q^{\binom{k}{2}} y^k. \quad (2.7)$$

Recall Euler's another q -exponential formula (cf. [8, Appendix (II.1)]) and Jacobi's product triple identity (cf. [8, Appendix (II.28)]) :

$$\sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty}, \quad (2.8)$$

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} z^k = (q, -z, -q/z; q)_\infty. \quad (2.9)$$

Employing (1.8), (2.8), and (2.9), it is not difficult to understand that

$$\begin{aligned} & \oint \frac{(xz, q, qz, 1/z; q)_\infty}{(y^2 z^2; q^2)_\infty} \frac{dz}{2\pi iz} \\ &= \oint \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}} (-xz)^j}{(q; q)_j} \sum_{k=0}^{\infty} \frac{(yz)^{2k}}{(q^2; q^2)_k} \sum_{\ell=-\infty}^{\infty} q^{\binom{\ell}{2}} (-1/z)^\ell \frac{dz}{2\pi iz} \\ &= \sum_{j, k \geq 0} \frac{q^{j^2 + 2jk + 2k^2 - j - k}}{(q; q)_j (q^2; q^2)_k} x^j y^{2k}. \end{aligned} \quad (2.10)$$

With the help of (2.7) and (2.10), we catch hold of (1.7). \square

3 Proof of Theorems 1.4-1.6

For proving Theorems 1.4-1.6, we draw support on the following lemma (cf. [13, Proposition 3.2]).

Lemma 3.1. *Let a, b, c, t be complex numbers such that $|t| < 1$. Then*

$${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, t \right] = \frac{(q; q)_\infty}{(c, t; q)_\infty} \oint \frac{(abz, cz, qz/t, t/z; q)_\infty}{(az, bz, cz/t; q)_\infty} \frac{dz}{2\pi iz},$$

where the integral is over a positively oriented contour separating the origin from all poles of the integrand.

Firstly, we start to prove Theorem 1.4.

Proof. Choosing $(a, q) \rightarrow (a^2, q^2)$ in the q -binomial theorem:

$${}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix}; q, t \right] = \frac{(at; q)_\infty}{(t; q)_\infty}, \quad (3.1)$$

it is routine to see that

$${}_2\phi_1 \left[\begin{matrix} a, -a \\ -q \end{matrix}; q, t \right] = \frac{(a^2t; q^2)_\infty}{(t; q^2)_\infty}.$$

Fix $(a, b, c, t) = (y^{\frac{1}{2}}, -y^{\frac{1}{2}}, -q, x)$ in Lemma 3.1 and use the above identity to deduce

$$\oint \frac{(-yz, -qz, q, qz/x, x/z; q)_\infty}{(yz^2; q^2)_\infty (-qz/x; q)_\infty} \frac{dz}{2\pi iz} = (-q; q)_\infty (qx, yx; q^2)_\infty. \quad (3.2)$$

Via (1.8), (2.8), (2.9), and (3.1), we have

$$\begin{aligned} & \oint \frac{(-yz, -qz, q, qz/x, x/z; q)_\infty}{(yz^2; q^2)_\infty (-qz/x; q)_\infty} \frac{dz}{2\pi iz} \\ &= \oint \sum_{j=0}^{\infty} \frac{(x; q)_j}{(q; q)_j} \left(-\frac{qz}{x} \right)^j \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (yz)^k}{(q; q)_k} \\ & \quad \times \sum_{\ell=0}^{\infty} \frac{(yz^2)^\ell}{(q^2; q^2)_\ell} \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{m}{2}} (x/z)^m \frac{dz}{2\pi iz} \\ &= \sum_{j, k, \ell \geq 0} \frac{(x; q)_j (-x)^{k+2\ell} y^{k+\ell}}{(q; q)_j (q; q)_k (q^2; q^2)_\ell} q^{j+\binom{k}{2}+\binom{j+k+2\ell}{2}}. \end{aligned} \quad (3.3)$$

The combination of (3.2) with (3.3) gives (1.13). \square

Secondly, we plan to prove Theorem 1.5.

Proof. Replace t by $-t$ in (3.1) to obtain

$${}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix}; q, -t \right] = \frac{(-at; q)_\infty}{(-t; q)_\infty}. \quad (3.4)$$

The sum of (3.1) and (3.4) produces

$${}_2\phi_1 \left[\begin{matrix} a, aq \\ q \end{matrix}; q^2, t^2 \right] = \frac{1}{2} \frac{(at; q)_\infty}{(t; q)_\infty} + \frac{1}{2} \frac{(-at; q)_\infty}{(-t; q)_\infty}.$$

Take $(q, a, b, c, t) \rightarrow (q^2, x, qx, q, y^2)$ in Lemma 3.1 and utilize the last equation to gain

$$\oint \frac{(qx^2z, qz, q^2, q^2z/y^2, y^2/z; q^2)_\infty}{(xz; q)_\infty (qz/y^2; q^2)_\infty} \frac{dz}{2\pi iz} = \frac{(q; q^2)_\infty}{2} \{(xy, -y; q)_\infty + (-xy, y; q)_\infty\}. \quad (3.5)$$

Through (1.8), (2.8), (2.9), and (3.1), we arrive at

$$\begin{aligned} & \oint \frac{(qx^2z, qz, q^2, q^2z/y^2, y^2/z; q^2)_\infty}{(xz; q)_\infty (qz/y^2; q^2)_\infty} \frac{dz}{2\pi iz} \\ &= \oint \sum_{j=0}^{\infty} \frac{(xz)^j}{(q; q)_j} \sum_{k=0}^{\infty} \frac{(x^2y^2; q^2)_k}{(q^2; q^2)_k} \left(\frac{qz}{y^2}\right)^k \\ & \quad \times \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\binom{\ell}{2}} (qz)^\ell}{(q^2; q^2)_\ell} \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{m}{2}} (y^2/z)^m \frac{dz}{2\pi iz} \\ &= \sum_{j, k, \ell \geq 0} \frac{(x^2y^2; q^2)_k x^j y^{2j+2\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} (-1)^{j+k} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+k}. \end{aligned} \quad (3.6)$$

Substituting (3.6) into (3.5), we get hold of (1.14). \square

Thirdly, we shall prove Theorem 1.6.

Proof. The difference of (3.1) and (3.4) can be expressed as

$${}_2\phi_1 \left[\begin{matrix} aq, aq^2 \\ q^3 \end{matrix}; q^2, t^2 \right] = \frac{1-q}{2(1-a)t} \left\{ \frac{(at; q)_\infty}{(t; q)_\infty} - \frac{(-at; q)_\infty}{(-t; q)_\infty} \right\}.$$

Let $(q, a, b, c, t) \rightarrow (q^2, q^2x, q^3x, q^3, y^2/q^2)$ in Lemma 3.1 and employ the upper formula to derive

$$\begin{aligned} & \oint \frac{(q^5x^2z, q^3z, q^2, q^4z/y^2, y^2/zq^2; q^2)_\infty}{(q^2xz; q)_\infty (q^5z/y^2; q^2)_\infty} \frac{dz}{2\pi iz} \\ &= \frac{(q; q^2)_\infty}{2(y/q - xy)} \{(xy, -y/q; q)_\infty - (-xy, y/q; q)_\infty\}. \end{aligned} \quad (3.7)$$

In terms of (1.8), (2.8), (2.9), and (3.1), there is

$$\begin{aligned} & \oint \frac{(q^5x^2z, q^3z, q^2, q^4z/y^2, y^2/zq^2; q^2)_\infty}{(q^2xz; q)_\infty (q^5z/y^2; q^2)_\infty} \frac{dz}{2\pi iz} \\ &= \oint \sum_{j=0}^{\infty} \frac{(q^2xz)^j}{(q; q)_j} \sum_{k=0}^{\infty} \frac{(x^2y^2; q^2)_k}{(q^2; q^2)_k} \left(\frac{q^5z}{y^2}\right)^k \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{2\binom{\ell}{2}} (q^3 z)^\ell}{(q^2; q^2)_\ell} \sum_{m=-\infty}^{\infty} (-1)^m q^{2\binom{m}{2}} (y^2/zq^2)^m \frac{dz}{2\pi iz} \\
& = \sum_{j, k, \ell \geq 0} \frac{(x^2 y^2; q^2)_k x^j y^{2j+2\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} (-1)^{j+k} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+3k}. \tag{3.8}
\end{aligned}$$

Substituting (3.8) into (3.7), we are led to (1.15). \square

4 The bisection method and multidimensional Rogers–Ramanujan type identities

In this section, we shall use the bisection method to establish several new multidimensional Rogers–Ramanujan type identities.

Theorem 4.1.

$$\sum_{j, k \geq 0} \frac{q^{4j^2+4jk+2k^2-j}}{(q; q)_{2j} (q^2; q^2)_k} = \frac{(q^8, -q^3, -q^5; q^8)_\infty}{(q^2; q^2)_\infty}, \tag{4.1}$$

$$\sum_{j, k \geq 0} \frac{q^{4j^2+4jk+2k^2+3j+2k}}{(q; q)_{1+2j} (q^2; q^2)_k} = \frac{(q^8, -q, -q^7; q^8)_\infty}{(q^2; q^2)_\infty}. \tag{4.2}$$

Proof. Replace x by $-x$ in (1.6) to achieve

$$\sum_{j, k \geq 0} \frac{q^{j^2+2jk+2k^2+k}}{(q; q)_j (q^2; q^2)_k} (-1)^j x^{j+2k} = (qx; q)_\infty. \tag{4.3}$$

The sum of (1.6) and (4.3) creates

$$\sum_{j, k \geq 0} \frac{q^{4j^2+4jk+2k^2+k}}{(q; q)_{2j} (q^2; q^2)_k} x^{2j+2k} = \frac{1}{2} \left\{ (-qx; q)_\infty + (qx; q)_\infty \right\}. \tag{4.4}$$

Notice a known relation (cf. [16, Equations (1.2a)]):

$$(-q; q^2)_\infty + (q; q^2)_\infty = \frac{2}{(q^4; q^4)_\infty} (q^{16}, -q^6, -q^{10}; q^{16})_\infty. \tag{4.5}$$

Combing the $x = q^{-\frac{1}{2}}$ case of (4.4) and (4.5), we obtain (4.1).

The difference of (1.6) and (4.3) engenders

$$\sum_{j, k \geq 0} \frac{q^{4j^2+4jk+2k^2+4j+3k+1}}{(q; q)_{1+2j} (q^2; q^2)_k} x^{1+2j+2k} = \frac{1}{2} \left\{ (-qx; q)_\infty - (qx; q)_\infty \right\}. \tag{4.6}$$

Notice another known relation (cf. [16, Equations (1.2b)]):

$$(-q; q^2)_\infty - (q; q^2)_\infty = \frac{2q}{(q^4; q^4)_\infty} (q^{16}, -q^2, -q^{14}; q^{16})_\infty. \quad (4.7)$$

Combing the $x = q^{-\frac{1}{2}}$ case of (4.6) with (4.7), we discover (4.2). \square

Theorem 4.2. *Let m be a nonnegative integer. Then*

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(q; q^2)_k (-1)^k}{(q; q)_{2j} (q^2; q^2)_k (q^2; q^2)_\ell} q^{(2j+k+\ell)(2j+k+\ell-1)+\ell^2+j+k-2m(j+\ell)} \\ &= \frac{(-q^{-m}; q)_m (q^8, -q^3, -q^5; q^8)_\infty}{(q^2; q^2)_\infty}, \end{aligned} \quad (4.8)$$

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(q; q^2)_k (-1)^k}{(q; q)_{1+2j} (q^2; q^2)_k (q^2; q^2)_\ell} q^{(1+2j+k+\ell)(2j+k+\ell)+\ell^2+j+k-m(1+2j+2\ell)} \\ &= \frac{(-q^{-m}; q)_m (q^8, -q, -q^7; q^8)_\infty}{(q^2; q^2)_\infty}. \end{aligned} \quad (4.9)$$

Proof. Replace x by $-x$ in (1.16) to gain

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(x^2; q^2)_k (-x)^j (-1)^k}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+k-m(j+2\ell)} \\ &= (-q^{-m}; q)_m (x; q)_\infty. \end{aligned} \quad (4.10)$$

According to (4.5) and the $x = q^{\frac{1}{2}}$ case of the sum of (1.16) and (4.10), we catch hold of (4.8). In accordance with (4.7) and the $x = q^{\frac{1}{2}}$ case of the difference of (1.16) and (4.10), we can verify (4.9). \square

Theorem 4.3.

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(q^{-1}; q^2)_k (-1)^k}{(q; q)_{2j} (q^2; q^2)_k (q^2; q^2)_\ell} q^{(2j+k+\ell)(2j+k+\ell-1)+\ell^2+j+3k+2\ell} \\ &= \frac{(q^8, -q^3, -q^5; q^8)_\infty}{(q^2; q^2)_\infty}, \end{aligned} \quad (4.11)$$

$$\begin{aligned} & \sum_{j, k, \ell \geq 0} \frac{(q^{-1}; q^2)_k (-1)^k}{(q; q)_{1+2j} (q^2; q^2)_k (q^2; q^2)_\ell} q^{(1+2j+k+\ell)(2j+k+\ell)+\ell^2+j+3k+2\ell} \\ &= \frac{(q^8, -q, -q^7; q^8)_\infty}{(q^2; q^2)_\infty}. \end{aligned} \quad (4.12)$$

Proof. The $m = 0$ case of (1.17) can be written as

$$\sum_{j, k, \ell \geq 0} \frac{(x^2; q^2)_k x^j (-1)^k}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+j+3k+2\ell} = (-qx; q)_\infty. \quad (4.13)$$

Replacing x by $-x$ in (4.13), we have

$$\sum_{j,k,\ell \geq 0} \frac{(x^2; q^2)_k (-x)^j (-1)^k}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+j+3k+2\ell} = (qx; q)_\infty. \quad (4.14)$$

Via (4.5) and the $x = q^{-\frac{1}{2}}$ case of the sum of (4.13) and (4.14), we find (4.11). Through (4.7) and the $x = q^{-\frac{1}{2}}$ case of the difference of (4.13) and (4.14), we can confirm (4.12). \square

Theorem 4.4. *Let x be a complex number. Then*

$$\begin{aligned} & \sum_{j,k,\ell \geq 0} \frac{(x; q^2)_k (-1)^k x^{-\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+j+k+2\ell} \\ &= \frac{(-qx, -q^3/x; q^4)_\infty}{(q^2; q^4)_\infty}, \end{aligned} \quad (4.15)$$

$$\begin{aligned} & \sum_{j,k,\ell \geq 0} \frac{(x; q^2)_k (-1)^k x^{-\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+2j+3k+4\ell} \\ &= \frac{(-q^3x, -q^5/x; q^4)_\infty}{(q^2; q^4)_\infty}. \end{aligned} \quad (4.16)$$

Proof. Choosing $(x, y) \rightarrow (-x/q, q/x^{\frac{1}{2}})$ in (1.14) and using (2.9), we obtain

$$\begin{aligned} & \sum_{j,k,\ell \geq 0} \frac{(x; q^2)_k (-1)^k x^{-\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+j+k+2\ell} \\ &= \frac{(q; q^2)_\infty}{2} \left\{ (-x^{\frac{1}{2}}, -q/x^{\frac{1}{2}}; q)_\infty + (x^{\frac{1}{2}}, q/x^{\frac{1}{2}}; q)_\infty \right\} \\ &= \frac{(q; q^2)_\infty}{2(q; q)_\infty} \left\{ \sum_{n=-\infty}^{\infty} q^{\binom{k}{2}} x^{\frac{k}{2}} + \sum_{n=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^{\frac{k}{2}} \right\} \\ &= \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{\binom{2k}{2}} x^k \\ &= \frac{(-qx, -q^3/x; q^4)_\infty}{(q^2; q^4)_\infty}. \end{aligned}$$

So we get hold of (4.15).

Taking $(x, y) \rightarrow (-x/q^2, q^2/x^{\frac{1}{2}})$ in (1.15) and utilizing (2.9), we achieve

$$\begin{aligned} & \sum_{j,k,\ell \geq 0} \frac{(x; q^2)_k (-1)^k x^{-\ell}}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_\ell} q^{(j+k+\ell)(j+k+\ell-1)+\ell^2+2j+3k+4\ell} \\ &= \frac{(q; q^2)_\infty}{2(q/x^{\frac{1}{2}} + x^{\frac{1}{2}})} \left\{ (-x^{\frac{1}{2}}, -q/x^{\frac{1}{2}}; q)_\infty - (x^{\frac{1}{2}}, q/x^{\frac{1}{2}}; q)_\infty \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{(q; q^2)_\infty}{2(q/x^{\frac{1}{2}} + x^{\frac{1}{2}})(q; q)_\infty} \left\{ \sum_{n=-\infty}^{\infty} q^{\binom{k}{2}} x^{\frac{k}{2}} - \sum_{n=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^{\frac{k}{2}} \right\} \\
&= \frac{1}{(q/x^{\frac{1}{2}} + x^{\frac{1}{2}})(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{\binom{1+2k}{2}} x^{\frac{1+2k}{2}} \\
&= \frac{(-q^3x, -q^5/x; q^4)_\infty}{(q^2; q^4)_\infty}.
\end{aligned}$$

Therefore, we complete the proof of (4.16). \square

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References

- [1] G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook, Part II*, Springer 2009.
- [2] A. Berkovich, S.O. Warnaar, Positivity preserving transformations for q -binomial coefficients, *Trans. Amer. Math. Soc.* 357 (2005) 2291-2351.
- [3] D.M. Bressoud, A generalization of the Rogers–Ramanujan identities for all moduli, *J. Combin. Theory Ser. A* 27 (1979), 64–68.
- [4] K. Bringmann, C. Jennings-Shaffer, K. Mahlburg, Proofs and reductions of various conjectured partition identities of Kanade and Russell, *J. Reine Angew. Math.* 766 (2020), 109–135.
- [5] S. Capparelli, On some representations of twisted affine Lie algebras and combinatorial identities, *J. Algebra* 154 (1993), 335–355.
- [6] Z. Cao, L. Wang, Multi-sum Rogers–Ramanujan type identities, *J. Math. Anal. Appl.* 522 (2023), Art. 126960.
- [7] K. Garrett, M.E.H. Ismail, D. Stanton, Variants of the Rogers–Ramanujan identities, *Adv. in Appl. Math.* 23 (1999), 274–299.
- [8] G. Gasper, M. Rahman, *Basic Hypergeometric Series* (2nd edition), Cambridge University Press, Cambridge, 2004.
- [9] Nancy S.S. Gu, H. Prodinger, One-parameter generalization of Rogers–Ramanujan type identities, *Adv. in Appl. Math.* 45 (2010), 149–196.
- [10] S. Kanade, M.C. Russell, Staircases to analytic sum-sides for many new integer partition identities of Rogers-Ramanujan type, *Electron. J. Combin.* 26 (2019), 1–6.
- [11] K. Kurşungöz, Andrews–Gordon type series for Capparelli's and Göllnitz–Gordon identities, *J. Combin. Theory Ser. A* 165 (2019), 117–138.
- [12] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi (1988).
- [13] H. Rosengren, Proofs of some partition identities conjectured by Kanade and Russell, *Ramanujan J.* (2022), <https://doi.org/10.1007/s11139-0>.

- [14] L.J. Slater, Further identities of the Rogers-Ramanujan type, *Proc. Lond. Math. Soc.* (2) 54 (1) (1952), 147–167.
- [15] A. Uncu, W. Zudilin, Reflecting (on) the modulo 9 Kanade–Russell (conjectural) identities, arXiv: 2106.02959v3.
- [16] L. Wang, New proofs of some double sum Rogers-Ramanujan type identities, *Ramanujan J.* (2022), <https://doi.org/10.1007/s11139-022-00654-5>.