

BOUNDEDNESS OF COMPOSITION OPERATORS ON HIGHER ORDER BESOV SPACES IN ONE DIMENSION

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ABSTRACT. This paper aims to characterize boundedness of composition operators on Besov spaces $B_{p,q}^s$ of higher order derivatives $s > 1 + 1/p$ on the one-dimensional Euclidean space. In contrast to the lower order case $0 < s < 1$, there were a few results on the boundedness of composition operators for $s > 1$. We prove a relation between the composition operators and pointwise multipliers of Besov spaces, and effectively use the characterizations of the pointwise multipliers. As a result, we obtain necessary and sufficient conditions for the boundedness of composition operators for general p, q , and s such that $1 < p \leq \infty$, $0 < q \leq \infty$, and $s > 1 + 1/p$. In this paper, we treat, as a map that induces the composition operator, not only a homeomorphism on the real line but also a continuous map whose number of elements of inverse images at any one point is bounded above. We also show a similar characterization of the boundedness of composition operators on Sobolev spaces.

1. INTRODUCTION

In this paper, we characterize a mapping $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ for which the composition operator

$$C_\varphi : f \mapsto f \circ \varphi$$

is bounded on the inhomogeneous Besov space $B_{p,q}^s = B_{p,q}^s(\mathbb{R})$ in one dimension*.

The composition operator appears in the context of a change of variables. Its boundedness has long been studied on various function spaces (see, e.g., [1–3, 7–12, 14, 16, 18, 20, 21, 24, 27, 29, 30, 32] and references therein). In addition, there have been applications to the analysis of dynamical systems (see [10, 15]), theory of function spaces on domains (see [17, 26, 27]), transport equations (see [3, 5, 10, 31]), and so on. For the composition operators on Besov spaces, Bourdaud and Sickel [2] provided the necessary and sufficient conditions on homeomorphisms φ for the boundedness of C_φ on homogeneous and inhomogeneous Besov spaces with low order derivatives $0 < s < 1$, and later, Bourdaud [1] weakened the assumption on φ and gave sufficient conditions on (not necessarily homeomorphisms) φ for the boundedness to hold. In contrast, the higher order case $s > 1$ is also important, but there needs to be more research. To the best of our knowledge, the characterization was mentioned only for

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*We think of the composition operator C_φ as the mapping to be defined by $f \mapsto f \circ \varphi$ for continuous functions f as canonical representative if $B_{p,q}^s$ is embedded in the space of continuous functions on \mathbb{R} , i.e., if “ $s > 1/p$ ” or “ $s = 1/p$ and $0 < q \leq 1$ ” (see Proposition 2.3). Otherwise, the mapping $f \mapsto f \circ \varphi$ is well-defined in the sense of equivalence classes with respect to equality almost everywhere if φ is non-singular (i.e. $|\varphi^{-1}(E)| = 0$ for any measurable null set $E \subset \mathbb{R}$).

$s \in \mathbb{N}$ with $s \geq 2$ in the framework of Sobolev spaces (see [2, Remark 1.14]). Our purpose is to study the boundedness of C_φ on inhomogeneous Besov spaces $B_{p,q}^s$ with $s > 1$, and our main results are its necessary and sufficient conditions in the case $s > 1+1/p$. In this paper, we do not deal with composition operators on homogeneous Besov spaces $\dot{B}_{p,q}^s$, because they are limited to the trivial ones in the case $s > 1$ (see, e.g., [2, Remark 1.14] and [1, Section 6]).

To state the results, let us give some notations and definitions. We employ the characterization of $B_{p,q}^s$ by differences, i.e., $B_{p,q}^s$ is the collection of all $f \in L_{\text{loc}}^1$ such that

$$\|f\|_{B_{p,q}^s} := \|f\|_{L^p} + \left(\int_{|h| \leq 1} |h|^{-sq} \|\Delta_h^m f\|_{L^p}^q \frac{dh}{|h|} \right)^{\frac{1}{q}} < \infty$$

(with the usual modification for $q = \infty$), where $m \in \mathbb{N}$ with $m > s$. Here, $L_{\text{loc}}^1 = L_{\text{loc}}^1(\mathbb{R})$ is the space of all measurable functions on \mathbb{R} such that $f \in L^1(K)$ for any compact set $K \subset \mathbb{R}$, and the difference operator Δ_h^m of order $m \in \mathbb{N}$ is defined by (2.2) below. See Subsection 2.1 for the details of $B_{p,q}^s$. We say that φ is Lipschitz if there exists a constant $L > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L|x - y|, \quad x, y \in \mathbb{R}.$$

We denote by $\#A$ the cardinality of a set A . For a measurable mapping φ , we define

$$U(\varphi) := \sup_{|I|=1} |\varphi^{-1}(I)| \quad \text{and} \quad \mathcal{M}(\varphi) := \sup_I |I|^{-1} |\varphi^{-1}(I)|,$$

where the supremum is taken over closed bounded intervals in \mathbb{R} . We denote by $M(X)$ the set of all pointwise multipliers of a quasi-normed space X , i.e., the set of all measurable functions f on \mathbb{R} such that

$$\|f\|_{M(X)} := \sup_{\|g\|_X \leq 1} \|fg\|_X < \infty.$$

The previous results on boundedness of C_φ in the case $0 < s < 1$ are summarized as follows:

Theorem 1.1 (Theorems 1.5 and 1.7 in [2] and Theorems 9 and 11 in [1]). *Let $0 < s < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Assume that*

$$\varphi : \mathbb{R} \rightarrow \mathbb{R} \text{ is a homeomorphism.} \tag{H}$$

Then C_φ is bounded on $B_{p,q}^s$ if and only if

$$\begin{cases} \varphi^{-1} \text{ is Lipschitz} & \text{if } 0 < s < \frac{1}{p}, \\ \varphi \text{ is Lipschitz and } U(\varphi) < \infty & \text{if } \frac{1}{p} < s < 1. \end{cases}$$

Theorem 1.2 (Theorems 7 and 8 in [1]). *Let $0 < s < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Assume that $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a mapping such that*

$$\sup_{x \in \mathbb{R}} \#\varphi^{-1}(x) < \infty. \tag{F}$$

Then C_φ is bounded on $B_{p,q}^s$ if

$$\begin{cases} \varphi \text{ is locally absolutely continuous and } \mathcal{M}(\varphi) < \infty & \text{if } 0 < s < \frac{1}{p}, \\ \varphi \text{ is Lipschitz and } U(\varphi) < \infty & \text{if } \frac{1}{p} < s < 1. \end{cases}$$

These theorems are obtained by interpolation from the borderline spaces: Lebesgue spaces L^p , Hölder spaces $B_{\infty,\infty}^s$, Sobolev spaces W_p^1 and Besov spaces $B_{1,1}^s$ for $1 \leq p \leq \infty$ and $0 < s < 1$. Theorem 1.1 was first proved by [2], and then, *a priori* condition on φ was improved in [1]. In addition, [1] also provided Theorem 1.2 under the weaker assumption (F) than (H). The results on the borderline spaces are well organized in [1, Section 3].

Remark 1.3. *Let us give remarks on the above theorems.*

- (a) *It is easy to see that (H) implies (F), as $\sup_{x \in \mathbb{R}} \#\varphi^{-1}(x) = 1$ for any homeomorphism φ .*
- (b) *The range of q can be easily extended to $0 < q \leq \infty$ in Theorems 1.1 and 1.2.*
- (c) *The critical case $s = 1/p$ has been only partially studied (see [1, 2, 14, 21, 29]), but it still has open problems.*

This paper addresses the higher order case $s > 1 + 1/p$. In this case, we may assume that φ is continuously differentiable in \mathbb{R} without loss of generality. In fact, it is necessary if C_φ is bounded on $B_{p,q}^s$ with $s > 1 + 1/p$ (see Lemma 2.10 below).

Our main result is the following:

Theorem 1.4. *Let $1 < p < \infty$, $0 < q \leq \infty$ and $s > 1 + 1/p$. Assume (F). Then C_φ is bounded on $B_{p,q}^s$ if and only if $U(\varphi) < \infty$ and $\varphi' \in M(B_{p,q}^{s-1})$.*

For the case $p = \infty$, we have the following:

Theorem 1.5. *Let $0 < q \leq \infty$ and $s > 1$. Then C_φ is bounded on $B_{\infty,q}^s$ if and only if $\varphi' \in M(B_{\infty,q}^{s-1})$.*

Let us here give some remarks on Theorems 1.4 and 1.5. The major difference from the lower order case $0 < s < 1$ is that the product of functions appears: $C_\varphi f \in B_{p,q}^s$ is roughly

$$(C_\varphi f)' = \varphi' \cdot C_\varphi f' \in B_{p,q}^{s-1}$$

in the higher order case $s \geq 1$, which implies that the composition operator C_φ on $B_{p,q}^s$ is related to the pointwise multiplier φ' of $B_{p,q}^{s-1}$. From this, the sufficient condition is derived from a combination of the result on the lower order case $1/p < s < 1$ (Theorem 1.2) with an inductive argument (see Lemmas 3.1, 3.7 and 3.10). The main part of our proofs is the necessity of $\varphi' \in M(B_{p,q}^{s-1})$ (see Lemmas 3.2 and 3.9). The critical tool is the characterizations of $M(B_{p,q}^{s-1})$. There are many works dealing with pointwise multipliers of Besov spaces and, in particular, Nguyen and Sickel [19] presented the characterizations for $s > 1 + 1/p$:

$$M(B_{p,q}^{s-1}) = \left\{ f \in L_{\text{loc}}^1 \left| \sup_{\|\{c_z\}_z\|_{\ell^p(\mathbb{Z})} \leq 1} \left\| f \sum_{z \in \mathbb{Z}} c_z \psi(\cdot - z) \right\|_{B_{p,q}^{s-1}} < \infty \right. \right\}$$

for $p \neq \infty$, and $M(B_{\infty,q}^{s-1}) = B_{\infty,q}^{s-1}$ (see also Subsection 2.3). Here, $\psi \in C_0^\infty$ is a non-negative function on \mathbb{R} defined by (2.6). The proofs of the theorems are given in Subsections 3.1 and 3.2, respectively.

Remark 1.6. *If either “ $0 < p \leq 1$, $0 < q \leq \infty$ and $s > 1 + 1/p$ ” or “ $0 < q \leq p \leq 1$ and $s = 1 + 1/p$ ”, then it is proved that the conditions $U(\varphi) < \infty$ and $\varphi' \in M(B_{p,q}^{s-1})$ are necessary for the boundedness of C_φ on $B_{p,q}^s$ under the assumption (F) (see Lemma 3.2 and Remark 3.6). It is still open if these are sufficient as long as we know. The case $1 \leq s \leq 1 + 1/p$ remains open.*

We also mention the case of Sobolev spaces H_p^s . In this case, we also obtain a similar result to Theorem 1.4. This result generalizes the previous result for Sobolev spaces of positive integer order derivatives $s \in \mathbb{N}$, $s \geq 2$, in [2, Remark 1.14]. The characterization of $M(H_p^{s-1})$ with $s > 1 + 1/p$ was presented by Strichartz [25] (see also [23]). See Section 4 for the details for H_p^s .

We also discuss necessary and sufficient conditions on homeomorphisms φ for automorphism of C_φ on $B_{p,q}^s$ to hold. Here, the automorphism means that C_φ is bijective and both $C_\varphi, C_\varphi^{-1} (= C_{\varphi^{-1}})$ are bounded on $B_{p,q}^s$. As corollaries of Theorems 1.4 and 1.5, we immediately have the following:

Corollary 1.7. *Let $1 < p \leq \infty$, $0 < q \leq \infty$ and $s > 1 + 1/p$. Assume (H). Then C_φ is automorphic on $B_{p,q}^s$ if and only if $\varphi', (\varphi^{-1})' \in M(B_{p,q}^{s-1})$.*

Remark 1.8. *Let us give remarks and related works on the automorphism of C_φ on Besov spaces and Sobolev spaces.*

- (a) *For $0 < s < 1$, the characterizations have been studied in [1, 2, 29].*
- (b) *A similar result for the Sobolev spaces to Corollary 1.7 can be also obtained from Theorem 4.1.*

Remark 1.9. *For all parameters $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ (except for $p = \infty$ for the Sobolev spaces), some sufficient conditions for boundedness and automorphism of C_φ on $B_{p,q}^s$ and H_p^s are known in [22, Theorem 4.46] and [27, Theorem in Subsection 4.3.2], respectively (see also the comments on Theorem 4.46 on page 563 in [22] for the boundedness). Our results improve these for $1 < p \leq \infty$, $0 < q \leq \infty$ and $s > 1 + 1/p$.*

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2. PRELIMINARIES

Let us introduce some notations and definitions used in this paper. For $a, b \geq 0$, the symbols $a \lesssim b$ and $b \gtrsim a$ mean that there exists a constant $C > 0$ such that $a \leq Cb$. The symbol $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ happen simultaneously. We

define χ_E by the characteristic function of $E \subset \mathbb{R}$. We use the notation $\|\cdot\|_{X \rightarrow Y}$ for the operator norm from a quasi-normed space X to another one Y , i.e.,

$$\|T\|_{X \rightarrow Y} := \sup_{\|f\|_X=1} \|Tf\|_Y$$

for an operator T from X into Y . For quasi-normed spaces X and Y , the notation $X \hookrightarrow Y$ means that X is continuously embedded in Y , i.e., X is a subset of Y and there exists a constant $C > 0$ such that

$$\|f\|_Y \leq C\|f\|_X \quad \text{for any } f \in X.$$

We denote by $BUC = BUC(\mathbb{R})$ the Banach space of all uniformly continuous and bounded functions on \mathbb{R} equipped with the supremum norm. For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $C^k = C^k(\mathbb{R})$ the space of k times continuously differentiable functions on \mathbb{R} , by $C_0^\infty = C_0^\infty(\mathbb{R})$ the space of smooth functions with compact support in \mathbb{R} , by $\mathcal{S} = \mathcal{S}(\mathbb{R})$ the Schwartz space, which consists of all rapidly decreasing infinitely differentiable functions on \mathbb{R} , and by $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ the space of all tempered distributions on \mathbb{R} . The space \mathcal{S}' is the dual space of \mathcal{S} . We denote by $L^p = L^p(\mathbb{R})$ the Lebesgue spaces and by $W_p^k = W_p^k(\mathbb{R})$ the Sobolev space for $k \in \mathbb{N}$ and $0 < p \leq \infty$.

2.1. Besov spaces. Let $\{\varphi_j\}_{j=0}^\infty$ be the Littlewood-Paley decomposition. More precisely, let $\varphi_0 \in C_0^\infty$ be a non-negative function with $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| \geq 2$, and define φ_j by

$$\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \quad x \in \mathbb{R}$$

for $j \in \mathbb{N}$. Then $\{\varphi_j\}_{j=0}^\infty$ is the partition of the unity such that

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \quad x \in \mathbb{R}.$$

Definition 2.1. Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then the Besov space $B_{p,q}^s = B_{p,q}^s(\mathbb{R})$ is defined by the collection of all tempered distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{B_{p,q,\varphi_0}^s} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\varphi_j \mathcal{F}f]\|_{L^p}^q \right)^{\frac{1}{q}} < \infty$$

(with the usual modification for $q = \infty$), where \mathcal{F} and \mathcal{F}^{-1} are the Fourier transform and its inverse, respectively.

The Besov space has the characterization by differences.

Proposition 2.2. Let $0 < p, q \leq \infty$ and $s > \max\{0, 1/p - 1\}$. Then the Besov space $B_{p,q}^s$ is the collection of all functions $f \in L_{\text{loc}}^1$ such that

$$\|f\|_{B_{p,q,m}^s} := \|f\|_{L^p} + \left(\int_{|h| \leq 1} |h|^{-sq} \|\Delta_h^m f\|_{L^p}^q \frac{dh}{|h|} \right)^{\frac{1}{q}} < \infty \quad (2.1)$$

(with the usual modification for $q = \infty$), where $m \in \mathbb{N}$ with $m > s$. Here, the difference operator Δ_h^m of order $m \in \mathbb{N}$ is defined by

$$\Delta_h^m f(x) := \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x + jh), \quad x, h \in \mathbb{R} \quad (2.2)$$

and Δ_h^0 is the identity operator.

Note that $B_{p,q}^s$ is independent of the choice of φ_0 and m . We write the notation $\|f\|_{B_{p,q}^s}$ as (2.1) and the notation $|f|_{B_{p,q}^s}$ as the second term in (2.1):

$$|f|_{B_{p,q}^s} := \left(\int_{|h| \leq 1} |h|^{-sq} \|\Delta_h^m f\|_{L^p}^q \frac{dh}{|h|} \right)^{\frac{1}{q}}.$$

The space $B_{p,q}^s$ is a quasi-Banach space (a Banach space if $p, q \geq 1$). The following are known.

Proposition 2.3. *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then the following three statements are equivalent:*

- (i) $B_{p,q}^s \hookrightarrow L^\infty$.
- (ii) $B_{p,q}^s \hookrightarrow BUC$.
- (iii) *There holds either “ $s > 1/p$ ” or “ $s = 1/p$ and $q \leq 1$ ”.*

Proposition 2.4. *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then the following statements are equivalent:*

- (i) $B_{p,q}^s$ is a multiplication algebra, i.e. $B_{p,q}^s \hookrightarrow M(B_{p,q}^s)$.
- (ii) *There holds either “ $s > 1/p$ ” or “ $s = 1/p$, $p \neq \infty$ and $q \leq 1$ ”.*

For the details of these propositions, we refer to [19, Lemma 2.2, Remark 2.3, Theorem 3.2, Remark 3.3] for instance.

2.2. Some necessary conditions and sufficient conditions. In this subsection, we give some preliminary lemmas on necessary conditions and sufficient conditions of boundedness of C_φ on $B_{p,q}^s$. These lemmas are already known, but we give the proofs for the reader’s convenience.

Lemma 2.5. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \max\{0, 1/p - 1\}$. Then, if C_φ is bounded on $B_{p,q}^s$, then $U(\varphi) < \infty$ and*

$$U(\varphi)^{\frac{1}{p}} \lesssim \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s}. \quad (2.3)$$

Proof. The proof is the same as in [2, Subsection 2.3]. Let us take a non-negative function $f \in C_0^\infty$ such that $f \equiv 1$ on $[0, 1]$, and define $f_a(x) := f(x - a)$ for $a \in \mathbb{R}$. By the assumption that C_φ is bounded on $B_{p,q}^s$, there exists a constant $C > 0$, independent of a , such that

$$\|C_\varphi f_a\|_{L^p} \leq \|C_\varphi f_a\|_{B_{p,q}^s} \leq \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s} \|f_a\|_{B_{p,q}^s} = \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s} \|f\|_{B_{p,q}^s}$$

for any $a \in \mathbb{R}$. Moreover,

$$\|C_\varphi f_a\|_{L^p}^p \geq \int_{\varphi^{-1}([a, a+1])} |f_a(\varphi(x))|^p dx = |\varphi^{-1}([a, a+1])|$$

for any $a \in \mathbb{R}$. Combining the above two estimates, we have $U(\varphi) < \infty$ and the inequality (2.3). \square

Lemma 2.6. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1/p$. Assume that $U(\varphi) < \infty$. Then*

$$\|C_\varphi f\|_{B_{p,q}^s \rightarrow L^p} \lesssim U(\varphi)^{\frac{1}{p}}.$$

Proof. The proof is the same as in the proof of [1, Theorem 5]. It is known that

$$\left(\sum_{j \in \mathbb{Z}} \|f\|_{L^\infty([j, j+1])}^p \right)^{\frac{1}{p}} \leq C \|f\|_{B_{p,q}^s} \quad (2.4)$$

for $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1/p$. The proof of (2.4) can be found in [2, Subsection 3.3]. By using (2.4), we obtain

$$\begin{aligned} \|C_\varphi f\|_{L^p}^p &= \sum_{j \in \mathbb{Z}} \int_{\varphi^{-1}([j, j+1])} |f(\varphi(x))|^p dx \\ &\leq U(\varphi) \sum_{j \in \mathbb{Z}} \|f\|_{L^\infty([j, j+1])}^p \\ &\leq CU(\varphi) \|f\|_{B_{p,q}^s}^p \end{aligned}$$

for any $f \in B_{p,q}^s$. \square

Lemma 2.7. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \max\{1, 1/p\}$. Assume that C_φ is bounded on $B_{p,q}^{s-1}$. Then*

$$\|C_\varphi\|_{B_{p,q}^s \rightarrow L^p} \lesssim \|C_\varphi\|_{B_{p,q}^{s-1} \rightarrow B_{p,q}^{s-1}}.$$

Proof. It follows from a combination of Lemmas 2.5 and 2.6. \square

Next, we shall prove the following result on a necessary condition for the boundedness of C_φ on $B_{p,q}^s$. It is one of the fundamental lemmas to prove the necessary condition in Theorem 1.4.

Proposition 2.8. *Let $0 < p < \infty$ and $0 < q \leq \infty$. Assume either “ $s > 1 + 1/p$ ” or “ $s > 1/p$ and (H)”’. Then, if C_φ is bounded on $B_{p,q}^s$, then φ is Lipschitz and*

$$\|\varphi'\|_{L^\infty}^{s-\frac{1}{p}} \lesssim \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s}. \quad (2.5)$$

To prove this, we prepare the following two lemmas:

Lemma 2.9 (Proposition 1 in [1]). *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and $A > 0$. Assume that for any $x_0 \in \mathbb{R}$, there exists a neighborhood V of x_0 such that*

$$|\varphi(x) - \varphi(y)| \leq A|x - y|$$

for any $x, y \in V$. Then φ is Lipschitz and $\|\varphi'\|_{L^\infty} \leq A$.

Lemma 2.10. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1/p$. Then, if C_φ is bounded on $B_{p,q}^s$, then φ belongs to $C^{k,\alpha}$ with $s - 1/p = k + \alpha$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1]$, where $C^{k,\alpha} = C^{k,\alpha}(\mathbb{R})$ is the space of all C^k -functions such that their k -th derivatives are Hölder continuous of order α .*

Proof. To begin with, by the Sobolev type embedding, we can see that φ is continuous in \mathbb{R} as C_φ is bounded on $B_{p,q}^s$ with $s > 1/p$. Let $I \subset \mathbb{R}$ be an open interval with $|I| < \infty$, and take $f \in C_0^\infty$ such that $f(x) = x$ for $x \in I$. Then $C_\varphi f(x) = \varphi(x)$ for $x \in \varphi^{-1}(I)$, and hence, $\varphi \in B_{p,q}^s(\varphi^{-1}(I))^\dagger$. Therefore, the Sobolev type embedding $B_{p,q}^s(\varphi^{-1}(I)) \subset C^{k,\alpha}(\varphi^{-1}(I))$ for $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1/p$ shows that

\dagger See e.g. [4, Section 2] for the definition of Besov spaces $B_{p,q}^s(I)$ on an open set $I \subset \mathbb{R}$.

$\varphi \in C^{k,\alpha}(\varphi^{-1}(I))$. Since $\mathbb{R} = \bigcup \varphi^{-1}(I)$, where the union is taken over all open intervals $I \subset \mathbb{R}$ with $|I| < \infty$, we conclude that $\varphi \in C^{k,\alpha}$. \square

Proof of Proposition 2.8. The proof of the case $1 \leq p < \infty$ and $1/p < s < 1$ under the assumption (H) can be found in [2, Subsection 2.2]. We also use the same idea to prove the other cases.

Case: $s > 1 + 1/p$. We give only the proof of the case $1 \leq s < 2$, as the other cases are similarly proved. By Lemma 2.10, we note that $\varphi \in C^1$. Then, for any $x \in \mathbb{R}$ with $\varphi'(x) \neq 0$, there exists an open interval $J \subset \mathbb{R}$ such that either $\varphi' > 0$ on J or $\varphi' < 0$ on J . Hence, let $b \in \mathbb{R}$ with $\varphi'(b) \neq 0$, and it suffices to prove that there exists a constant $C > 0$ independent of b such that $|\varphi'(b)| \leq C$. We give proof only in the case of $\varphi' > 0$ on J , as the proof of the other case is similar. For $\varepsilon > 0$, let us take a non-negative function $\eta_\varepsilon \in C_0^\infty$ such that

$$\eta'_\varepsilon = \frac{1}{\varepsilon}(\chi_{[-1-\varepsilon,-1]} - \chi_{[1,1+\varepsilon]}).$$

Then, $\eta_\varepsilon(x) = 1$ for $x \in [-1, 1]$, $\text{supp } \eta_\varepsilon \subset [-1 - \varepsilon, 1 + \varepsilon]$, and $|\eta_\varepsilon|_{B_{p,q}^s} \leq C\varepsilon^{1/p-s}$, where the constant C is independent of ε . Take $a, b, c \in J$ such that $a < b < c$ and $b - 1 \leq c \leq a + 1$ and

$$\varphi(c) - \varphi(b) \leq \frac{\varphi(b) - \varphi(a)}{2} \leq 1,$$

and define

$$r := \frac{\varphi(b) - \varphi(a)}{2}, \quad x_0 := \frac{\varphi(b) + \varphi(a)}{2}, \quad \varepsilon := \frac{\varphi(c) - \varphi(b)}{2}$$

and

$$f(x) := \eta_\varepsilon \left(\frac{x - x_0}{r} \right).$$

Then,

$$\|f\|_{L^p} = C(\varphi(c) - \varphi(b))^{\frac{1}{p}}$$

and

$$|f|_{B_{p,q}^s} = r^{-\frac{1}{p}+s} |\eta_\varepsilon|_{B_{p,q}^s} \leq Cr^{-\frac{1}{p}+s} \varepsilon^{\frac{1}{p}-s} = C(\varphi(c) - \varphi(b))^{\frac{1}{p}-s}.$$

By the assumption of boundedness,

$$\begin{aligned} \|C_\varphi f\|_{B_{p,q}^s} &\leq \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s} \|f\|_{B_{p,q}^s} \\ &\leq C \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s} \left((\varphi(c) - \varphi(b))^{\frac{1}{p}-s} + (\varphi(c) - \varphi(b))^{\frac{1}{p}} \right). \end{aligned}$$

Noting that

$$f(\varphi(x + 2h)) - 2f(\varphi(x + h)) + f(\varphi(x)) = 1$$

for any $x \in [c - h, b]$ and $h \in [c - b, c - a]$, we estimate from below

$$\begin{aligned} \|C_\varphi f\|_{B_{p,q}^s}^q &\geq \int_{c-b}^{c-a} \left(\int_{c-h}^b 1 \, dx \right)^{\frac{q}{p}} \frac{dh}{h^{1+sq}} \\ &\geq \int_{c-b}^{c-a} (b - c + h)^{\frac{q}{p}} \frac{dt}{h^{1+sq}} \geq C(c - b)^{q(\frac{1}{p}-s)}. \end{aligned}$$

Therefore, there exists a constant $C > 0$, independent of b and c , such that

$$\varphi(c) - \varphi(b) \leq C(c - b)$$

for sufficiently close b and c . Since $\varphi \in C^1$, we obtain $0 < \varphi'(b) \leq C$ as $c \rightarrow b$.

Case: $s > 1/p$ and (H). By the assumption (H), we may assume that φ is strictly increasing in \mathbb{R} without loss of generality. By the same argument as above, for any b and c sufficiently close, there exists a constant $C > 0$ independent of b and c such that $\varphi(c) - \varphi(b) \leq C(c - b)$. Therefore, the proof is completed by Lemma 2.9. \square

In the case $p = q = \infty$, we have the following:

Theorem 2.11 (Theorem 2 in [1]). *Let $0 < s < 1$. Then C_φ is bounded on $B_{\infty,\infty}^s$ if and only if φ is Lipschitz.*

As a corollary of Theorem 2.11, we have the following sufficient condition by the real interpolation argument $(B_{\infty,\infty}^{s_0}, B_{\infty,\infty}^{s_1})_{\theta,q} = B_{\infty,q}^s$ with $s = \theta s_0 + (1 - \theta) s_1$, $s_0 \neq s_1$ and $\theta \in (0, 1)$.

Corollary 2.12. *Let $0 < s < 1$ and $0 < q < \infty$. Then, if φ is Lipschitz, then C_φ is bounded on $B_{\infty,q}^s$.*

2.3. Characterization of $M(B_{p,q}^s)$. Let $\psi \in C_0^\infty$ be a non-negative function such that

$$\sum_{z \in \mathbb{Z}} \psi(\cdot - z) \equiv 1 \text{ on } \mathbb{R} \quad \text{and} \quad \text{supp } \psi = [-1, 1]. \quad (2.6)$$

We define

$$B_{p,q,\text{unif}}^s = B_{p,q,\text{unif}}^s(\mathbb{R}) := \left\{ f \in L_{\text{loc}}^1 \mid \sup_{z \in \mathbb{Z}} \|f\psi(\cdot - z)\|_{B_{p,q}^s} < \infty \right\}. \quad (2.7)$$

For $p \neq \infty$, we also define

$$M_{p,q}^s = M_{p,q}^s(\mathbb{R}) := \left\{ f \in L_{\text{loc}}^1 \mid \sup_{\|\{c_z\}_z\|_{\ell^p(\mathbb{Z})} \leq 1} \left\| f \sum_{z \in \mathbb{Z}} c_z \psi(\cdot - z) \right\|_{B_{p,q}^s} < \infty \right\}.$$

It is immediately seen that

$$M(B_{p,q}^s) \hookrightarrow B_{p,q,\text{unif}}^s.$$

for $0 < p, q \leq \infty$ and $s > \max\{0, 1/p - 1\}$. Moreover, we also have the following:

Lemma 2.13. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \max\{0, 1/p - 1\}$. Then*

$$M(B_{p,q}^s) \hookrightarrow M_{p,q}^s \hookrightarrow B_{p,q,\text{unif}}^s.$$

Proof. To show the second embedding, we have only to take $c_z = \delta_{z,\tilde{z}}$ for $\tilde{z} \in \mathbb{Z}$ (see also the statements before Theorem 1.5 in [19]). To see the first embedding, we show that $\sum_{z \in \mathbb{Z}} c_z \psi(\cdot - z)$ belongs to $B_{p,q}^s$. For this, we write $\psi_z := \psi(\cdot - z)$ and divide \mathbb{Z} into $\{\Omega_\ell\}_{\ell=1}^N$ such that

$$\text{dist}(\text{supp } \psi_{z_i}, \text{supp } \psi_{z_j}) \geq 3m$$

for any $z_i, z_j \in \Omega_\ell$ with $z_i \neq z_j$ and for $\ell = 1, \dots, N$. Here, we note that the finite number N depends only on m . Then

$$\begin{aligned} \left\| \sum_{z \in \Omega_\ell} c_z \psi_z \right\|_{L^p}^p &= \sum_{z \in \Omega_\ell} |c_z|^p \|\psi_z\|_{L^p}^p = \|\{c_z\}_z\|_{\ell^p(\Omega_\ell)}^p \|\psi\|_{L^p}^p, \\ \left\| \Delta_h^m \sum_{z \in \Omega_\ell} c_z \psi_z \right\|_{L^p}^p &= \sum_{z \in \Omega_\ell} |c_z|^p \|\Delta_h^m \psi_z\|_{L^p}^p = \|\{c_z\}_z\|_{\ell^p(\Omega_\ell)}^p \|\Delta_h^m \psi\|_{L^p}^p \end{aligned}$$

for $|h| \leq 1$ and $\ell = 1, \dots, N$. Hence,

$$\begin{aligned} \left\| \sum_{z \in \Omega_\ell} c_z \psi_z \right\|_{B_{p,q}^s} &= \left\| \sum_{z \in \Omega_\ell} c_z \psi_z \right\|_{L^p} + \left(\int_{|h| \leq 1} |h|^{-sq} \left(\left\| \Delta_h^m \sum_{z \in \Omega_\ell} c_z \psi_z \right\|_{L^p}^p \right)^{\frac{q}{p}} \frac{dh}{|h|} \right)^{\frac{1}{q}} \\ &= \|\{c_z\}_z\|_{\ell^p(\Omega_\ell)} \|\psi\|_{L^p} \\ &\quad + \left(\int_{|h| \leq 1} |h|^{-sq} \left(\|\{c_z\}_z\|_{\ell^p(\Omega_\ell)}^p \|\Delta_h^m \psi\|_{L^p}^p \right)^{\frac{q}{p}} \frac{dh}{|h|} \right)^{\frac{1}{q}} \\ &= \|\{c_z\}_z\|_{\ell^p(\Omega_\ell)} \|\psi\|_{B_{p,q}^s} \end{aligned}$$

for $\ell = 1, \dots, N$, which implies

$$\left\| \sum_{z \in \mathbb{Z}} c_z \psi_z \right\|_{B_{p,q}^s} \leq \|\{c_z\}_z\|_{\ell^p(\mathbb{Z})} \|\psi\|_{B_{p,q}^s}.$$

The proof of Lemma 2.13 is finished. \square

Combining Proposition 2.4 and Lemma 2.13, we see that

$$B_{p,q}^s \hookrightarrow M(B_{p,q}^s) \hookrightarrow M_{p,q}^s \hookrightarrow B_{p,q,\text{unif}}^s \quad (2.8)$$

for “ $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1/p$ ” or “ $0 < p < \infty$, $0 < q \leq 1$ and $s = 1/p$ ”, and that

$$B_{\infty,q}^s \hookrightarrow M(B_{\infty,q}^s) \hookrightarrow B_{\infty,q,\text{unif}}^s$$

for $0 < q \leq \infty$ and $s > 0$. The results of [19] are summarized as follows.

Theorem 2.14. *Let $0 < p, q \leq \infty$ and $s > 1/p$. Then the following assertions hold:*

- (i) $M(B_{p,q}^s) = B_{p,q,\text{unif}}^s$ if and only if $p \leq q$.
- (ii) Let $p \neq \infty$. Then $M(B_{p,q}^s) = M_{p,q}^s$.
- (iii) $M(B_{p,q}^s) = B_{p,q}^s$ if and only if $p = \infty$.

Proof. For the assertion (i), the sufficiency part was proved by [19, Theorem 1.2] and the necessary part was proved by [19, Corollary 3.18] (where $p, q \geq 1$ is imposed, but the proof shows this assumption can be removed when $s > 1/p$). The assertion (ii) was proved by [19, Theorem 1.5] when $q < p$ and by a combination of [19, Theorem 1.2] with (2.8) when $p \leq q$. For assertion (iii), the sufficiency part was proved by [19, Theorem 1.7]. The necessity immediately follows from the fact that $1 \in B_{p,q}^s$ if and only if $p = \infty$ (see e.g. [22, Example 2.7, page 241]) and it is clear that $1 \in M(B_{p,q}^s)$. Hence, $p = \infty$ if $M(B_{p,q}^s) = B_{p,q}^s$. \square

Remark 2.15. *From Proposition 2.4, Lemma 2.13 and Theorem 2.14, we also see the following relations between $B_{p,q}^s$, $B_{p,q,\text{unif}}^s$ and $M_{p,q}^s$ for $s > 1/p$:*

- (a) $B_{p,q,\text{unif}}^s = B_{p,q}^s$ if and only if $p = q = \infty$ (see [19, Remark 1.8 (iii)]).
- (b) Let $p \neq \infty$. Then $M_{p,q}^s = B_{p,q,\text{unif}}^s$ if and only if $0 < p \leq q \leq \infty$.
- (c) Let $p \neq \infty$. Then $B_{p,q}^s \neq M_{p,q}^s$.

In addition, we have the following result for $s = 1/p$.

Theorem 2.16 (Theorem 1.9 in [19]). *Let $0 < p \leq 1$ and $s = 1/p$. Then the following assertions hold:*

- (i) If $0 < p = q \leq 1$, then $M(B_{p,p}^{1/p}) = B_{p,p,\text{unif}}^{1/p}$.
- (ii) If $0 < q < p \leq 1$, then $M(B_{p,q}^{1/p}) = M_{p,q}^{1/p}$.

We also have the following embedding.

Proposition 2.17 (Theorem 2.21 in [28]). *Let $0 < p, q \leq \infty$ and $s > \max\{0, 1/p - 1\}$. Then*

$$B_{p,q,\text{unif}}^s \hookrightarrow L^\infty$$

holds if and only if either “ $s > 1/p$ ” or “ $s = 1/p$ and $q \leq 1$ ”.

3. PROOFS OF MAIN RESULTS

3.1. Proof of Theorem 1.4. We begin by showing the following:

Lemma 3.1. *Let $0 < p, q \leq \infty$ and $s > \max\{1, 1/p\}$. Assume that C_φ is bounded on $B_{p,q}^{s-1}$ and $\varphi' \in M(B_{p,q}^{s-1})$. Then C_φ is bounded on $B_{p,q}^s$.*

Proof. It follows from the chain rule and Lemma 2.7 that

$$\begin{aligned} \|C_\varphi f\|_{B_{p,q}^s} &\leq \|C_\varphi f\|_{L^p} + \|\varphi' \cdot C_\varphi f'\|_{B_{p,q}^{s-1}} \\ &\leq C \left(\|C_\varphi\|_{B_{p,q}^s \rightarrow L^p} + \|\varphi'\|_{M(B_{p,q}^{s-1})} \|C_\varphi\|_{B_{p,q}^{s-1} \rightarrow B_{p,q}^{s-1}} \right) \|f\|_{B_{p,q}^s} \\ &\leq C \left(1 + \|\varphi'\|_{M(B_{p,q}^{s-1})} \right) \|C_\varphi\|_{B_{p,q}^{s-1} \rightarrow B_{p,q}^{s-1}} \|f\|_{B_{p,q}^s} \end{aligned} \quad (3.1)$$

for any $f \in B_{p,q}^s$. □

Next, we prove the following:

Lemma 3.2. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > 1 + 1/p$. Assume (F). Then, if C_φ is bounded on $B_{p,q}^s$, then $U(\varphi) < \infty$ and $\varphi' \in M(B_{p,q}^{s-1})$.*

For this purpose, we give two auxiliary lemmas.

Lemma 3.3. *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous mapping satisfying (F) and $U(\varphi) < \infty$. Then, for any $a \in \mathbb{R}$ and $b > 0$, the closed set $\varphi^{-1}([a - b, a + b])$ is a finite disjoint union of closed intervals J_1, \dots, J_r . Moreover, we have*

$$\sum_{i=1}^r |J_i| \leq 2\lceil b \rceil U(\varphi), \quad (3.2)$$

$$r \leq \sup_{x \in \mathbb{R}} \#\varphi^{-1}(x), \quad (3.3)$$

where $\lceil x \rceil$ is the minimal integer greater than or equal to x .

Proof. Let $J := [a - b, a + b]$. We show that $\varphi^{-1}(J)$ is a finite disjoint union of closed intervals. It follows from the fact that $\partial\varphi^{-1}(J)$ is contained in the finite subset $\varphi^{-1}(\partial J) = \varphi^{-1}(\{a - b, a + b\})$, and thus $\varphi^{-1}(J)$ is necessarily a disjoint union of finite closed intervals. Let

$$\varphi^{-1}(J) = \bigsqcup_{j=1}^r J_j.$$

First, we show (3.2). Let $J_0 \supset J$ be a closed interval of width $2\lceil b \rceil$. Thus,

$$\sum_{j=1}^r |J_j| = |\varphi^{-1}(J)| \leq |\varphi^{-1}(J_0)| \leq 2\lceil b \rceil U(\varphi).$$

Next, we show (3.3). Since $\partial\varphi^{-1}(J) \subset \varphi^{-1}(\partial J) = \varphi^{-1}(\{a-b, a+b\})$, the inequality (3.3) follows from

$$2r = \#\partial\varphi^{-1}(J) \leq \#\varphi^{-1}(\partial J) \leq 2 \sup_{x \in \mathbb{R}} \#\varphi^{-1}(x).$$

□

Lemma 3.4. *Let $\{I_j\}_{j \in \mathbb{N}}$ be a countable sequence of closed intervals of \mathbb{R} . Assume that*

$$\sup_{j \in \mathbb{N}} \#\{i : I_j \cap I_i \neq \emptyset\} < \infty. \quad (3.4)$$

Then, there exists a finite partition S_1, \dots, S_r of \mathbb{N} such that $I_i \cap I_j = \emptyset$ for any $i, j \in S_k$ with $i \neq j$ ($k = 1, \dots, r$).

Proof. For any nonempty subset $S \subset \mathbb{N}$, we inductively define increasing finite subsets $\tau_1(S) \subset \tau_2(S) \subset \dots \subset S$ as follows: $\tau_0(\emptyset) := \emptyset$, $\tau_1(S) := \{\min S\}$, and for $n \geq 1$,

$$\tau_{n+1}(S) := \begin{cases} \tau_n(S) \cup \{\min \{j \in S \setminus \tau_n(S) : (\cup_{i \in \tau_n(S)} I_i) \cap I_j = \emptyset\}\} & (\tau_n(S) \subsetneq S), \\ \tau_n(S) & (\tau_n(S) = S). \end{cases}$$

Then, we define $\tau(S) := \cup_{n \geq 0} \tau_n(S)$. By the condition (3.4), for any nonempty subset $S \subset \mathbb{N}$, $\tau(S) \subset S$ is also nonempty and satisfies the following two conditions:

- (i) $I_i \cap I_j = \emptyset$ for any $i, j \in \tau(S)$, and
- (ii) $I_j \cap (\cup_{i \in \tau(S)} I_i) \neq \emptyset$ for any $j \in S \setminus \tau(S)$.

Then, we inductively construct disjoint sets $S_0, S_1, \dots \subset \mathbb{N}$ as follows:

$$S_0 := \emptyset, \\ S_{n+1} := \tau(\mathbb{N} \setminus \cup_{k=0}^n S_k)$$

for $n \in \mathbb{N}$. Let $L := \sup_{j \in \mathbb{N}} \#\{i : I_j \cap I_i \neq \emptyset\}$. We claim that $S_{L+2} = \emptyset$. In fact, suppose $S_{L+2} \neq \emptyset$. Fix $j \in S_{L+2}$. Then, by the construction of S_k 's, we have $j \in \mathbb{N} \setminus \cup_{k=0}^L S_k$ but $j \notin \tau(\mathbb{N} \setminus \cup_{k=0}^L S_k) = S_{L+1}$ for $\ell = 0, \dots, L$. Thus, by (ii), for any $\ell = 1, \dots, L+1$, there exists $i \in S_\ell$ such that $I_j \cap I_i \neq \emptyset$, but it implies $\#\{i : I_j \cap I_i \neq \emptyset\} \geq L+1$ that is contradictions. Therefore, we conclude that $S_{L+2} = \emptyset$. Then, Since $\mathbb{N} = \cup_{k=1}^\infty S_k$ and $S_\ell = \emptyset$ for all $\ell \geq L+2$, there exists $r \leq L+1$ such that $\mathbb{N} = \sqcup_{k=1}^r S_k$. □

Proof of Lemma 3.2. We note from Lemmas 2.5 and 2.8 that φ is Lipschitz and satisfies $U(\varphi) < \infty$. Hence, it is enough to show that $\varphi' \in M(B_{p,q}^{s-1}) = M_{p,q}^{s-1}$ (see Theorem 2.14 (ii)).

Recall that $\psi \in C_0^\infty$ is a non-negative function satisfying (2.6), and let $\psi_z := \psi(\cdot - z)$ for $z \in \mathbb{Z}$. Define

$$S_z := \bigcup_{j=1, \dots, m} \bigcup_{|h| \leq 1} \text{supp}(\Delta_h^{m-j} \psi_z(\cdot + jh)).$$

We take a positive integer $R > 0$ such that

$$2R \geq \|\varphi'\|_{L^\infty} \cdot \max\{\text{diam}(S_z), 2m\} + 2. \quad (3.5)$$

where $\text{diam}(S) := \sup_{x,y \in S} |x - y|$ for a set $S \subset \mathbb{R}$. We note that the right-hand side of (3.5) is independent of the choice of $z \in \mathbb{Z}$.

We divide \mathbb{Z} into $\{\Omega_\ell\}_{\ell=1}^N$ such that

$$\text{dist}(S_{z_i}, S_{z_j}) \geq 12RU(\varphi) \quad (3.6)$$

for any $z_i, z_j \in \Omega_\ell$ with $z_i \neq z_j$ and $\ell = 1, \dots, N$. Let us fix $\ell \in \{1, \dots, N\}$ for a while. Since φ is Lipschitz and (3.5) holds, we have

$$\text{diam}(\varphi(S_z)) \leq \|\varphi'\|_{L^\infty} \text{diam}(S_z) \leq 2R.$$

Thus, for $z \in \Omega_\ell$, there exists $a_z \in \mathbb{R}$ such that

$$S_z \subset \varphi^{-1}([a_z - R, a_z + R]), \quad (3.7)$$

and in particular, for any $z \in \Omega_\ell$, we have

$$\text{supp } \psi_z \subset \varphi^{-1}([a_z - R, a_z + R]). \quad (3.8)$$

Let $\mathcal{I}_a := [a - 2R, a + 2R]$. Then, we claim that

$$\#\{w \in \Omega_\ell : \mathcal{I}_{a_z} \cap \mathcal{I}_{a_w} \neq \emptyset\} \leq \sup_{x \in \mathbb{R}} \#\varphi^{-1}(x) < \infty \quad (3.9)$$

for any $z \in \Omega_\ell$. In fact, each connected component (closed interval) of $\varphi^{-1}([a_z - 6R, a_z + 6R])$ intersects with at most one S_z for $z \in \Omega_\ell$ by (3.2) in Lemma 3.3 and (3.6). Since $\mathcal{I}_{a_z} \cap \mathcal{I}_{a_w} \neq \emptyset$ implies that S_z intersects with $\varphi^{-1}([a_z - 6R, a_z + 6R])$, the formula (3.9) follows from (3.3). Thus, by Lemma 3.4, there exists a partition $\Omega_\ell^1, \dots, \Omega_\ell^{r_\ell}$ of Ω_ℓ such that $\mathcal{I}_{a_z} \cap \mathcal{I}_{a_w} = \emptyset$ for $z, w \in \Omega_\ell^k$ for $k = 1, \dots, r_\ell$.

We further fix $k \in \{1, \dots, r\}$. Let us take a function $f \in C_0^\infty$ such that $f(x) = x$ for $x \in [-R, R]$ and $\text{supp } f = [-R - 1, R + 1]$. For $a \in \mathbb{R}$, we define $f_a(x) := f(x - a)$ and

$$I_a := \varphi^{-1}([a - R, a + R]).$$

Then,

$$(C_\varphi f_a(x))' = \varphi'(x) \quad (3.10)$$

for $x \in I_a$. Moreover, since $\text{supp}(C_\varphi f_a) = \varphi^{-1}([a - R - 1, a + R + 1])$, we see that for any $z, w \in \Omega_\ell^k$ with $z \neq w$

$$\text{dist}(\text{supp}(C_\varphi f_{a_z}), \text{supp}(C_\varphi f_{a_w})) \geq \frac{2R - 2}{\|\varphi'\|_{L^\infty}} \geq 2m. \quad (3.11)$$

Let

$$\tilde{S}_z := \bigcup_{j=1, \dots, m} \bigcup_{|h| \leq 1} \text{supp}(\Delta_h^j C_\varphi f_{a_z}).$$

Then, by (3.5) with (3.11), for any $z, w \in \Omega_\ell^k$ with $z \neq w$, we have

$$\tilde{S}_z \cap \tilde{S}_w = \emptyset \quad (3.12)$$

for $z, w \in \Omega_\ell^k$ with $z \neq w$.

Then, by (3.6) and (3.12), we have the following formulas:

$$\left\| \sum_{z \in \Omega_\ell^k} c_z \psi_z \varphi' \right\|_{L^p}^p = \sum_{z \in \Omega_\ell^k} |c_z|^p \|\psi_z \varphi'\|_{L^p}^p, \quad (3.13)$$

$$\left\| \Delta_h^{m-j} \sum_{z \in \Omega_\ell^k} c_z \psi_z(\cdot + jh) \varphi' \right\|_{L^p}^p = \sum_{z \in \Omega_\ell^k} |c_z|^p \|\Delta_h^{m-j} \psi_z(\cdot + jh) \varphi'\|_{L^p}^p, \quad (3.14)$$

$$\left\| \sum_{z \in \Omega_\ell^k} (C_\varphi(|c_z| f_{a_z}))' \right\|_{L^p}^p = \sum_{z \in \Omega_\ell^k} |c_z|^p \|(C_\varphi f_{a_z})'\|_{L^p}^p, \quad (3.15)$$

$$\left\| \Delta_h^j \sum_{z \in \Omega_\ell^k} (C_\varphi(|c_z| f_{a_z}))' \right\|_{L^p}^p = \sum_{z \in \Omega_\ell^k} |c_z|^p \|\Delta_h^j (C_\varphi f_{a_z})'\|_{L^p}^p \quad (3.16)$$

for $|h| \leq 1$, $j = 1, \dots, m$, $\ell = 1, \dots, N$ and $k = 1, \dots, r_\ell$.

By the triangle inequality, we have

$$\left\| \varphi' \sum_{z \in \mathbb{Z}} c_z \psi_z \right\|_{B_{p,q}^{s-1}} \leq \sum_{\ell=1}^N \sum_{k=1}^{r_\ell} \left\| \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right\|_{B_{p,q}^{s-1}}.$$

Therefore, the proof of $\varphi' \in M_{p,q}^{s-1}$ is reduced to showing that

$$\sup_{\|\{c_z\}_z\|_{\ell^p(\mathbb{Z})} \leq 1} \left\| \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right\|_{B_{p,q}^{s-1}} \leq C (\|\varphi'\|_{L^\infty} + \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s} \|f\|_{B_{p,q}^s}) \quad (3.17)$$

for $\ell = 1, \dots, N$ and $k = 1, \dots, r_\ell$. First, it follows from (3.13), (3.8), (3.10) and (3.15) that

$$\begin{aligned} \left\| \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right\|_{L^p}^p &= \sum_{z \in \Omega_\ell^k} |c_z|^p \|\psi_z \varphi'\|_{L^p}^p \\ &\leq C \sum_{z \in \Omega_\ell^k} |c_z|^p \|\varphi'\|_{L^p(I_{a_z})}^p \\ &\leq C \sum_{z \in \Omega_\ell^k} |c_z|^p \|(C_\varphi f_{a_z})'\|_{L^p(I_{a_z})}^p \\ &\leq C \left\| \sum_{z \in \Omega_\ell^k} (C_\varphi(|c_z| f_{a_z}))' \right\|_{L^p}^p. \end{aligned} \quad (3.18)$$

Next, it follows from (3.14) that

$$\left\| \Delta_h^m \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right\|_{L^p}^p = \sum_{z \in \Omega_\ell^k} |c_z|^p \|\Delta_h^m \psi_z \varphi'\|_{L^p}^p. \quad (3.19)$$

Using the formula

$$\Delta_h^m (\psi_z \varphi')(x) = \sum_{j=0}^m \binom{m}{j} \Delta_h^{m-j} \psi_z(x + jh) \Delta_h^j \varphi'(x),$$

we have

$$\begin{aligned} \|\Delta_h^m \psi_z \varphi'\|_{L^p} &\leq C \left(\|\Delta_h^m \psi_z(\cdot + mh)\|_{L^p} \|\varphi'\|_{L^\infty} \right. \\ &\quad \left. + \sum_{j=1}^m \|\Delta_h^{m-j} \psi_z(\cdot + jh)\|_{L^\infty} \|\Delta_h^j \varphi'\|_{L^p(\text{supp}(\Delta_h^{m-j} \psi_z(\cdot + jh)))} \right). \end{aligned} \quad (3.20)$$

Here, we see from (3.7) and (3.10) that

$$\|\Delta_h^j \varphi'\|_{L^p(\text{supp}(\Delta_h^{m-j} \psi_z(\cdot + jh)))} \leq \|\Delta_h^j \varphi'\|_{L^p(I_{a_z})} \leq \|\Delta_h^j (C_\varphi f_{a_z})'\|_{L^p}. \quad (3.21)$$

Combining (3.19)–(3.21), we derive from (3.16) that

$$\begin{aligned} \left\| \Delta_h^m \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right\|_{L^p}^p &\leq C \sum_{z \in \Omega_\ell^k} |c_z|^p \left(\|\Delta_h^m \psi_z(\cdot + mh)\|_{L^p}^p \|\varphi'\|_{L^\infty}^p \right. \\ &\quad \left. + \sum_{j=1}^m \|\Delta_h^{m-j} \psi_z(\cdot + jh)\|_{L^\infty}^p \|\Delta_h^j (C_\varphi f_{a_z})'\|_{L^p}^p \right) \\ &\leq C \left(\|\{c_z\}_z\|_{\ell^p}^p \|\varphi'\|_{L^\infty}^p \|\Delta_h^m \psi\|_{L^p}^p \right. \\ &\quad \left. + \sum_{j=1}^m \|\Delta_h^{m-j} \psi\|_{L^\infty}^p \left\| \Delta_h^j \sum_{z \in \Omega_\ell^k} (C_\varphi(|c_z| f_{a_z}))' \right\|_{L^p}^p \right). \end{aligned}$$

Hence,

$$\begin{aligned} \left| \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right|_{B_{p,q}^{s-1}} &\leq C \|\{c_z\}_z\|_{\ell^p} \|\varphi'\|_{L^\infty} \|\psi\|_{B_{p,q}^{s-1}} \\ &\quad + \sum_{j=1}^m \|\psi\|_{B_{\infty,\infty}^{\frac{m-j}{m}(s-1)}} \left\| \sum_{z \in \Omega_\ell^k} (C_\varphi(|c_z| f_{a_z}))' \right\|_{B_{p,q}^{\frac{j}{m}(s-1)}}. \end{aligned} \quad (3.22)$$

Summarizing (3.18) and (3.22), we obtain

$$\begin{aligned} \left\| \varphi' \sum_{z \in \Omega_\ell^k} c_z \psi_z \right\|_{B_{p,q}^{s-1}} &\leq C \left(\|\{c_z\}_z\|_{\ell^p} \|\varphi'\|_{L^\infty} + \left\| \sum_{z \in \Omega_\ell^k} (C_\varphi(|c_z| f_{a_z}))' \right\|_{B_{p,q}^{s-1}} \right) \\ &\leq C \left(\|\{c_z\}_z\|_{\ell^p} \|\varphi'\|_{L^\infty} + \left\| \sum_{z \in \Omega_\ell^k} C_\varphi(|c_z| f_{a_z}) \right\|_{B_{p,q}^s} \right). \end{aligned}$$

By using

$$\left\| \sum_{z \in \Omega_\ell^k} |c_z| f_{a_z} \right\|_{L^p}^p \leq \sum_{z \in \Omega_\ell^k} |c_z|^p \|f_{a_z}\|_{L^p}^p$$

and

$$\left\| \Delta_h^m \sum_{z \in \Omega_\ell^k} |c_z| f_{a_z} \right\|_{L^p}^p \leq \sum_{z \in \Omega_\ell^k} |c_z|^p \|\Delta_h^m f_{a_z}\|_{L^p}^p,$$

we estimate

$$\begin{aligned}
\left\| \sum_{z \in \Omega_\ell^k} |c_z| f_{a_z} \right\|_{B_{p,q}^s} &\leq \left(\sum_{z \in \Omega_\ell^k} |c_z|^p \|f_{a_z}\|_{L^p}^p \right)^{\frac{1}{p}} \\
&\quad + \left\{ \sum_{k=0}^{\infty} 2^{ksq} \sup_{|h| < 2^{-k}} \left(\sum_{z \in \Omega_\ell^k} |c_z|^p \|\Delta_h^m f_{a_z}\|_{L^p}^p \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\
&\leq \|\{c_z\}_z\|_{\ell^p} \|f\|_{L^p} + \|\{c_z\}_z\|_{\ell^p} |f|_{B_{p,q}^s} \\
&= \|\{c_z\}_z\|_{\ell^p} \|f\|_{B_{p,q}^s}.
\end{aligned}$$

Thus, $\sum_{z \in \Omega_\ell^k} |c_z| f_{a_z}$ is a well-defined element of $B_{p,q}^s$. By the assumption that C_φ is bounded on $B_{p,q}^s$, we see that

$$\begin{aligned}
\left\| \sum_{z \in \Omega_\ell^k} C_\varphi(|c_z| f_{a_z}) \right\|_{B_{p,q}^s} &= \left\| C_\varphi \left(\sum_{z \in \Omega_\ell^k} |c_z| f_{a_z} \right) \right\|_{B_{p,q}^s} \\
&\leq \|C_\varphi\|_{B_{p,q}^s \rightarrow B_{p,q}^s} \left\| \sum_{z \in \Omega_\ell^k} |c_z| f_{a_z} \right\|_{B_{p,q}^s}.
\end{aligned}$$

By combining the above estimates, we conclude (3.17). The proof of Lemma 3.2 is finished. \square

Remark 3.5. When $p \leq q$, the assumption (F) in Lemma 3.2 can be removed, and the proof becomes simpler. In fact, the assumption (F) is essentially used to take appropriately the finite partition $\{\Omega_\ell^k\}_{\ell,k}$, but there is no need to take $\{\Omega_\ell^k\}_{\ell,k}$ if we use the characterization $M(B_{p,q}^{s-1}) = B_{p,q,\text{unif}}^{s-1}$ for $p \leq q$ from Theorem 2.14 (i).

Remark 3.6. The assertion of Lemma 3.2 also holds for $0 < q \leq p \leq 1$ and $s = 1 + 1/p$, since we have the same characterizations of $M(B_{p,q}^{s-1})$ for these parameters from Theorem 2.16.

Finally, we show the following:

Lemma 3.7. Let $1 < p < \infty$, $0 < q \leq \infty$ and $s > 1 + 1/p$. Assume (F), $U(\varphi^{-1}) < \infty$ and $\varphi' \in M(B_{p,q}^{s-1})$. Then C_φ is bounded on $B_{p,q}^{s-1}$.

The proof of this lemma is as follows. Combining Proposition 2.17 with (2.8), we see that φ is Lipschitz. Hence, when $1 + 1/p < s < 2$, the operator C_φ is bounded on $B_{p,q}^{s-1}$ by Theorem 1.2. Thus, the case $1 + 1/p < s < 2$ is proved. Next, to prove the case $2 \leq s \leq 2 + 1/p$, we use the following:

Lemma 3.8. Let $1 < p < \infty$, $0 < q \leq \infty$ and $1 \leq s \leq 1 + 1/p$. Assume (F), $U(\varphi^{-1}) < \infty$ and $\varphi' \in M(B_{p,q}^{\tilde{s}})$ for some $\tilde{s} > 1/p$. Then C_φ is bounded on $B_{p,q}^{s-1}$.

Proof. Lemma 3.8 follows from the interpolation argument between the case $1/p < s < 1$ (Theorem 1.2) and the case $1 + 1/p < s < 2$ of Lemma 3.7. \square

When $2 \leq s \leq 2 + 1/p$, it is clear that φ satisfies the assumptions of Lemma 3.8. Hence, C_φ is bounded on $B_{p,q}^{s-1}$. The same can be proved inductively for the higher order case $s > 2 + 1/p$. Thus, Lemma 3.7 is proved.

Proof of Theorem 1.4. Theorem 1.4 is immediately proved in a combination of Lemmas 3.1, 3.2, and 3.7. \square

3.2. Proof of Theorem 1.5. Theorem 1.5 is proved in a combination of the following two lemmas.

Lemma 3.9. *Let $s > 1$ and $0 < q \leq \infty$. Assume that C_φ is bounded on $B_{\infty,q}^s$. Then $\varphi' \in M(B_{\infty,q}^{s-1})$.*

Proof of Lemma 3.9. From Theorem 2.14 (iii), it suffices to show that $\varphi' \in B_{\infty,q}^{s-1}$. Let $s > 1$ and $m \in \mathbb{N}$ with $m > s - 1$. Let us take $f \in C_0^\infty(\mathbb{R})$ such that $f(x) = x$ on $[0, 1]$, and define $f_a(x) := f(x - a)$ for $a \in \mathbb{R}$. Then, since C_φ is bounded on $B_{\infty,q}^s$, we have

$$\|(C_\varphi f_a)'\|_{L^\infty} \leq C \|C_\varphi f_a\|_{B_{\infty,q}^s} \leq C \|C_\varphi\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^s} \|f\|_{B_{\infty,q}^s},$$

where the constant C is independent of a . For any $a \in \mathbb{R}$, we also have

$$\|(C_\varphi f_a)'\|_{L^\infty} = \|(C_\varphi f_a)' \cdot \varphi'\|_{L^\infty} \geq \|\varphi'\|_{L^\infty(\varphi^{-1}([a, a+1]))}.$$

Since $\mathbb{R} = \bigcup_{a \in \mathbb{R}} \varphi^{-1}([a, a+1])$, we obtain

$$\|\varphi'\|_{L^\infty} \leq C \|C_\varphi\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^s}.$$

Next, let us take $g \in B_{\infty,q}^s \cap C^\infty$ such that

$$g'(x) = (-1)^k \quad \text{for } x \in [2(4k-1)m, 2(4k+1)m]$$

for $k \in \mathbb{Z}$. Set

$$I_m := \bigcup_{k \in \mathbb{Z}} [2(4k-1)m + m, 2(4k+1)m - m].$$

Then we have

$$\sup_{x \in \varphi^{-1}(I_m)} |\Delta_h^m \varphi'(x)| = \sup_{x \in \varphi^{-1}(I_m)} |\Delta_h^m (C_\varphi g)'(x)|$$

for $|h| \leq 1$. Hence,

$$\begin{aligned} & \int_{|h| \leq 1} \sup_{x \in \varphi^{-1}(I_m)} |\Delta_h^m \varphi'(x)|^q \frac{dh}{|h|^{1+(s-1)q}} \\ &= \int_{|h| \leq 1} \sup_{x \in \varphi^{-1}(I_m)} |\Delta_h^m (C_\varphi g)'(x)|^q \frac{dh}{|h|^{1+sq}} \\ &= |(C_\varphi g)'|_{B_{\infty,q}^{s-1}}^q \leq \|C_\varphi g\|_{B_{\infty,q}^s}^q \leq \|C_\varphi\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^s}^q \|g\|_{B_{\infty,q}^s}^q. \end{aligned}$$

Similarly, if we translate g by $2m$, $4m$ and $6m$ and make the above argument, we get

$$\int_{|h| \leq 1} \sup_{x \in \varphi^{-1}(I_m + 2\ell m)} |\Delta_h^m \varphi'(x)|^q \frac{dh}{|h|^{1+(s-1)q}} \leq \|C_\varphi\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^s}^q \|g\|_{B_{\infty,q}^s}^q$$

for $\ell = 1, 2, 3$. Since

$$\mathbb{R} = \bigcup_{\ell=0}^3 (I_m + 2\ell m), \quad \text{i.e., } \mathbb{R} = \bigcup_{\ell=0}^3 \varphi^{-1}(I_m + 2\ell m),$$

we have

$$\begin{aligned} |\varphi'|_{B_{\infty,q}^{s-1}}^q &= \int_{|h|\leq 1} \sup_{x\in\mathbb{R}} |\Delta_h^m \varphi'(x)|^q \frac{dh}{|h|^{1+(s-1)q}} \\ &\leq \sum_{\ell=0}^3 \int_{|h|\leq 1} \sup_{x\in\varphi^{-1}(I_{m+2\ell m})} |\Delta_h^m \varphi'(x)|^q \frac{dh}{|h|^{1+(s-1)q}} \\ &\leq 4 \|C_\varphi\|_{B_{\infty,q}^s \rightarrow B_{\infty,q}^s}^q \|g\|_{B_{\infty,q}^s}^q. \end{aligned}$$

The proof of Lemma 3.9 is finished. \square

Lemma 3.10. *Let $s > 1$ and $0 < q \leq \infty$. Assume that $\varphi' \in M(B_{\infty,q}^{s-1})$. Then C_φ is bounded on $B_{\infty,q}^s$.*

Proof of Lemma 3.10. Since φ is Lipschitz by Proposition 2.17, we see that C_φ is bounded on $B_{\infty,q}^{s-1}$ with $1 < s < 2$ from Theorem 2.11 and Corollary 2.12. Hence, the proof can be done by a similar inductive argument to Lemma 3.7. \square

4. THE CASE OF SOBOLEV SPACES

In this section, we mention the composition operators C_φ on the Sobolev spaces H_p^s , which are defined by

$$H_p^s := \{f \in \mathcal{S}' \mid \|f\|_{H_p^s} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \mathcal{F}f]\|_{L^p} < \infty\}$$

for $s \in \mathbb{R}$ and $p > 1$. Similarly to the case of $B_{p,q}^s$, we have the following:

Theorem 4.1. *Let $1 < p < \infty$ and $s > 1 + 1/p$. Assume (H). Then C_φ is bounded on H_p^s if and only if $U(\varphi) < \infty$ and $\varphi' \in M(H_p^{s-1})$.*

The proof is done by a similar argument to that of Theorem 1.4 with the characterization of H_p^s by differences

$$\|f\|_{H_p^s} \sim \|f\|_{L^p} + \left\| \left(\int_0^1 t^{-2s} \left(\frac{1}{t} \int_{|h|\leq t} |\Delta_h^m f(x)| dh \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p}, \quad (4.1)$$

and the following two lemmas.

Lemma 4.2. *Let $1 < p < \infty$ and $1/p < s < 1$. Assume (H). Then C_φ is bounded on H_p^s if φ is Lipschitz and $U(\varphi) < \infty$.*

Proof. The proof is similar to [2, Subsections 3.2 and 3.3]. Similarly to the proof of Lemma 2.6, it can be shown that

$$\|C_\varphi f\|_{L^p} \leq CU(\varphi) \|f\|_{H_p^s}$$

for any $f \in H_p^s$ and some constant $C > 0$. For the second term in (4.1), we use the complex interpolation

$$[\text{BMO}, \dot{W}_{p_0}^1]_{\theta_1} = \dot{H}_p^{s_0}, \quad s_0 = \theta_1, \quad \frac{1}{p} = \frac{\theta_1}{p_0}, \quad 0 < \theta_1 < 1$$

for $1 < p_0 < \infty$, and further, the complex interpolation

$$[\dot{H}_p^{s_0}, \dot{W}_p^1]_{\theta_2} = \dot{H}_p^s, \quad s = (1 - \theta_2)s_0 + \theta_2, \quad 0 < \theta_2 < 1$$

(see [6, Corollary 8.3] for the complex interpolations). Combining these interpolations with the results on boundedness of C_φ on BMO ([12, Theorem]) and on \dot{W}_p^1 ([1, Theorem 4]), and taking p_0 close to 1, we see that C_φ is bounded from H_p^s to the homogeneous Sobolev space \dot{H}_p^s for any $1 < p < \infty$ and $1/p < s < 1$ if φ is Lipschitz and satisfies (H). The proof is finished. \square

Lemma 4.3. *Let $1 < p < \infty$ and $s > 1/p$. Then $M(H_p^s) = H_{p,\text{unif}}^s$. Here, the space $H_{p,\text{unif}}^s$ is similarly defined to (2.7).*

Lemma 4.3 is a famous result by [25, Corollary in Section 2 in Chapter II] (see also [13, (1.2) in Section 1] or [23, Theorem 2.5]).

Remark 4.4. *Theorem 4.1 includes the previous result for the Sobolev spaces W_p^k with $k \in \mathbb{N}$, $k \geq 2$ mentioned in [2, Remark 1.14], since $H_p^k = W_p^k$ for $k \in \mathbb{N}$ and $1 < p < \infty$.*

Remark 4.5. *We impose the assumption (H) in Theorem 4.1 and Lemma 4.2, because the monotonicity of φ is required to use the result on the boundedness on BMO by [12, Theorem].*

REFERENCES

- [1] G. Bourdaud, *Changes of variable in Besov spaces. II*, Forum Math. **12** (2000), no. 5, 545–563.
- [2] G. Bourdaud and W. Sickel, *Changes of variable in Besov spaces*, Math. Nachr. **198** (1999), 19–39.
- [3] D. Chae, *On the Euler equations in the critical Triebel-Lizorkin spaces*, Arch. Ration. Mech. Anal. **170** (2003), no. 3, 185–210.
- [4] R. A. DeVore and R. C. Sharpley, *Besov spaces on domains in \mathbb{R}^d* , Trans. Am. Math. Soc. **335** (1993), no. 2, 843–864.
- [5] L. C. F. Ferreira and D. F. Machado, *On the well-posedness in Besov-Herz spaces for the inhomogeneous incompressible Euler equations*, arXiv:2301.07765v1 (2023).
- [6] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990).
- [7] N. Hatano, M. Ikeda, I. Ishikawa, and Y. Sawano, *Boundedness of composition operators on Morrey spaces and weak Morrey spaces*, J. Inequal. Appl. **198** (2021), 19–39.
- [8] S. Hencl, L. Kleprlík, and J. Malý, *Composition operator and Sobolev-Lorentz spaces $WL^{n,q}$* , Studia Math. **221** (2014), 197–208.
- [9] M. Ikeda, I. Ishikawa, and Y. Sawano, *Boundedness of composition operators on reproducing kernel Hilbert spaces with analytic positive definite functions*, J. Math. Anal. Appl. **511** (2022), 126048.
- [10] M. Ikeda, I. Ishikawa, and C. Schlosser, *Koopman and Perron-Frobenius operators on reproducing kernel Banach spaces*, Chaos **32** (2022), 123143.
- [11] I. Ishikawa, *Bounded weighted composition operators on functional quasi-Banach spaces and stability of dynamical systems*, Adv. Math. **424** (2023), 109048.
- [12] P. W. Jones, *Homeomorphisms of the line which preserve BMO*, Arkiv Math. **21** (1983).
- [13] H. Koch and W. Sickel, *Pointwise multipliers of Besov spaces of smoothness zero and spaces of continuous functions*, Rev. Mat. Iberoamericana **18** (2002).
- [14] H. Koch, P. Koskela, E. Saksman, and T. Soto, *Bounded compositions on scaling invariant Besov spaces*, J. Funct. Anal. **266** (2014), no. 5, 2765–2788.
- [15] B. O. Koopman, *Hamiltonian systems and transformation in Hilbert space*, Proc. Natl. Acad. Sci. USA **17** (1931), no. 5, 315318.
- [16] P. Koskela, J. Xiao, Y. R.-Y. Zhang, and Y. Zhou, *A quasiconformal composition problem for the Q -spaces*, J. Eur. Math. Soc. **19** (2017), no. 4, 1159–1187.

- [17] V. G. Maz'ya and T. O. Shaposhnikova, *Theory of Sobolev multipliers*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 337, Springer-Verlag, Berlin, 2009. With applications to differential and integral operators.
- [18] A. Menovschikov and A. Ukhlov, *Composition operators on Hardy-Sobolev spaces and BMO-quasiconformal mappings*, J. Math. Sci. **258** (2021), no. 3, 313–325.
- [19] V. K. Nguyen and W. Sickel, *On a problem of Jaak Peetre concerning pointwise multipliers of Besov spaces*, Studia Math. **243** (2018), no. 2, 207–231.
- [20] M. Oliva and M. Prats, *Sharp bounds for composition with quasiconformal mappings in Sobolev spaces*, J. Math. Anal. Appl. **451** (2017), no. 2, 1026–1044.
- [21] D. Y. P. Koskela and Y. Zhou, *Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings*, Adv. Math. **226** (2011).
- [22] Y. Sawano, *Theory of Besov spaces*, Vol. 56, Springer, 2018.
- [23] W. Sickel, *Pointwise multipliers of Lizorkin-Triebel spaces*, The Maz'ya anniversary collection, 1999, pp. 295–321.
- [24] R. K. Singh, *Composition operators induced by rational functions*, Proc. Amer. Math. Soc. **59** (1976), no. 2, 329–333.
- [25] R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967).
- [26] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
- [27] ———, *Theory of function spaces II*, Monographs in Mathematics, Birkhäuser Verlag, Basel, 1992.
- [28] ———, *Theory of function spaces III*, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.
- [29] S. K. Vodop'yanov, *Mappings of homogeneous groups and embeddings of function spaces*, Sibirsk. Mat. Zh. **30** (1989), no. 5, 25–41, 215 (Russian).
- [30] ———, *Composition operators on Sobolev spaces*, Complex analysis and dynamical systems. II, Proceedings of the 2nd conference in honor of Professor Lawrence Zalcman's sixtieth birthday (Nahariya, Israel 2003), Contemp. Math., vol. 382 (Agranovsky M. et al., eds.), Amer. Math. Soc., Providence, RI; Bar-Ilan University, Ramat Gan, 2005, pp. 401–415.
- [31] J. Xiao, *The transport equation in the scaling invariant Besov or Essén-Janson-Peng-Xiao space*, J. Differ. Equ. **266** (2019), no. 11, 7124–7151.
- [32] D. Yang, W. Yuan, and Y. Zhou, *Sharp boundedness of quasiconformal composition operators on Triebel-Lizorkin type spaces*, J. Geom. Anal. **27** (2017), no. 2, 1548–1588.

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