Rainbow triangles sharing one common vertex or edge*

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Abstract

Let G be an edge-colored graph on n vertices. For a vertex v, the color degree of v in G, denoted by $d^c(v)$, is the number of colors appearing on the edges incident with v. Denote by $\delta^c(G) = \min\{d^c(v) : v \in V(G)\}$. By a theorem of H. Li, an n-vertex edge-colored graph G contains a rainbow triangle if $\delta^c(G) \geq \frac{n+1}{2}$. Inspired by this result, we consider two related questions concerning edge-colored books and friendship subgraphs of edge-colored graphs. Let $k \geq 2$ be a positive integer. We prove that if $\delta^c(G) \geq \frac{n+k-1}{2}$ where $n \geq 3k-2$, then G contains k rainbow triangles sharing one common edge; and if $\delta^c(G) \geq \frac{n+2k-3}{2}$ where $n \geq 2k+9$, then G contains k rainbow triangles sharing one common vertex. The special case k=2 of both results improves H. Li's theorem. The main novelty of our proof of the first result is a combination of the recent new technique for finding rainbow cycles due to Czygrinow, Molla, Nagle, and Oursler and some recent counting technique from [26]. The proof of the second result is with the aid of the machine implicitly in the work of Turán numbers for matching numbers due to Erdős and Gallai.

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1 Introduction

In 1907, Mantel [28] proved that every triangle-free graph on n vertices has size at most $\lfloor \frac{n^2}{4} \rfloor$. Rademacher (see [12, pp.91]) showed that there are indeed at least $\lfloor \frac{n}{2} \rfloor$ triangles in a graph G on n vertices and at least $\frac{n^2}{4}+1$ edges but not only one triangle. The k-fan (usually called friendship graph), denoted by F_k , is a graph which consists of k triangles sharing a common vertex. The $book\ B_k$ is a graph which consists of k triangles sharing a common edge. Erdős [11] extended Mantel's theorem and conjectured that there is a $B_{\lceil \frac{n}{6} \rceil}$ in G if $e(G) > \frac{n^2}{4}$, which was later confirmed by Edwards in an unpublished manuscript [9] and independently by Khadžiivanov and Nikiforov [23]. Erdős, Füredi, Gould, and Gunderson [12] also studied Turán numbers of F_k , and proved that $ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + k^2 - k$ if k is odd; and $ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + k^2 - \frac{3k}{2}$ if k is even, for $n \geq 50k^2$. These results immediately imply the fact that every graph on n vertices with minimum degree at least $\frac{n+1}{2}$ contains a B_k for $n \geq 6k$ and also a F_k for $n \geq 50k^2$. In this paper, we consider edge-colored versions of these extremal problems.

A subgraph of an edge-colored graph is *properly colored* (rainbow) if every two adjacent edges (all edges) have pairwise different colors. The rainbow and properly-colored subgraphs have been shown closely related to many graph properties and other topics, such as classical stability results on Turán functions [29], Bermond-Thomassen Conjecture [16], and Caccetta-Häggkvist Conjecture [1], etc. For more rainbow and properly-colored subgraphs under Dirac-type color degree conditions, we refer to [6, 8, 10, 17, 18].

The study of rainbow triangles has a rich history, and there are many classical open problems on them. In some classical problems, the host graph is complete. One conjecture due to Erdős and Sós [14] asserts that the maximum number of rainbow triangles in a 3-edge-coloring of the complete graph K_n , denoted by F(n), satisfied that F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd, where a + b + c + d = n and a, b, c, d are as equal as possible. By using Flag Algebra, Balogh et al. [2] confirmed this conjecture for n sufficiently large and $n = 4^k$ for any $k \ge 1$. The other example is a recent conjecture by Aharoni (see [1]), which can imply Caccetta-Häggkvist Conjecture [5], a big and fundamental open problem in digraph. The conjecture says that given any positive integer r, if G is an n-vertex edge-colored graph with n color classes, each of size at least n/r, then G contains a rainbow cycle of length at most r. For more recent developments on Aharoni's conjecture, we refer to the work [7, 20, 21] and more references therein. A special case of Aharoni's conjecture is about rainbow triangles. The relationship between directed triangles and rainbow triangles has been extensively used before (see [24, 25, 27]).

We would like to introduce a construction from Li [24]. Suppose that D is an n-vertex digraph satisfying out-degree of every vertex is at least n/3. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. We construct an edge-colored graph G such that: V(G) = V(D); for each arc $v_i v_j \in A(D)$, we color

the edge $v_i v_j$ with the color j. In this way, the number of colors appearing on edges incident with v_i different from i equals to $d_D^+(v_i)$. Thus, finding a directed triangle in D is equivalent to finding a rainbow triangle in such an edge-colored graph. More importantly for us, the idea of constructing the auxiliary digraph will also be an important constitution for our proofs in this paper.

Our theme of this paper is closely related to the color degree conditions for rainbow triangles. Let G be an edge-colored graph. For every vertex $v \in V(G)$, the color degree of v, denoted by $d_G^c(v)$, is the number of distinct colors appearing on the edges which are incident to v. The minimum color degree of G, denoted by $\delta^c(G)$ (or in short, δ^c), equals to $\min\{d^c(v):v\in V(G)\}$. It is an easy observation that every graph on n vertices contains a triangle if minimum degree is at least $\frac{n+1}{2}$. H. Li and Wang [27] considered a rainbow version and conjectured that the minimum color degree condition $\delta^c(G) \geq \frac{n+1}{2}$ ensures the existence of a rainbow triangle in G. This conjecture was confirmed by H. Li [24] in 2013.

Theorem 1 (H. Li [24]). Let G be an edge-colored graph on n vertices. If $\delta^c(G) \geq \frac{n+1}{2}$ then G contains a rainbow triangle.

Independently, B. Li, Ning, Xu and Zhang [25] proved a weaker condition $\sum_{v \in V(G)} d^c(v) \ge \frac{n(n+1)}{2}$ suffices for the existence of rainbow triangles, and moreover, characterized the exceptional graphs under the condition $\delta^c(G) \ge \frac{n}{2}$. Very recently, X. Li, Ning, Shi and Zhang [26] proved a counting version of Theorem 1, i.e., there are at least $\frac{1}{6}\delta^c(G)(2\delta^c(G)-n)n$ rainbow triangles in an edge-colored graph G, which is best possible.

Hu, Li and Yang [22] proposed the following conjecture: Let G be an edge-colored graph on $n \geq 3k$ vertices. If $\delta^c \geq \frac{n+k}{2}$ then G contains k vertex-disjoint rainbow triangles. Besides the work on Turán numbers of books and k-fans mentioned before, our another motivation is to study the converse of Hu-Li-Yang's conjecture, i.e., rainbow triangles sharing vertices or edges. As the selections for us, we shall study the existence of rainbow triangles sharing one common vertex or an edge under color degree conditions, in views of the famous books and friendship graphs in graph theory.

Our original result is the following one which improves Li's theorem. Indeed, we can go farther.

Theorem 2. Let G be an edge-colored graph on n vertices with $\delta^c(G) \geq \frac{n+1}{2}$.

- (i) If $n \ge 5$ then G contains two rainbow triangles sharing one common edge;
- (ii) If $n \geq 13$ then G contains two rainbow triangles sharing one common vertex.

This paper is organized as follows. In Section 2, we will list our main theorems. In Section 3, we introduce some necessary notations and terminology and prove some lemmas. In Section 4, we prove general versions of Theorem 2, i.e., Theorems 3 and 4. We conclude this paper

with some more discussions on the sharpness of our results, together some propositions on F_k and B_k on uncolored graphs.

2 Main theorems

Our main results are given as follows.

Theorem 3. Let $k \geq 2$ be a positive integer and G be an edge-colored graph on $n \geq 3k-2$ vertices. If $\delta^c(G) \geq \frac{n+k-1}{2}$ then G contains k rainbow triangles sharing one edge.

Theorem 4. Let $k \geq 2$ be a positive integer and G be an edge-colored graph on $n \geq 2k + 9$ vertices. If $\delta^c(G) \geq \frac{n+2k-3}{2}$ then G contains k rainbow triangles only sharing one common vertex.

Setting $\delta^c(G) = \frac{n+k-1}{2}$ in Theorem 3, the following example shows that the bound " $n \ge 3k-2$ " is sharp. Furthermore, it follows from Example 1 that the tight color degree should be at least $\delta^c \ge \frac{n+k}{2}$ when $n \le 3k-3$.

Example 1. Let G be a properly-colored balanced complete 3-partite graph $G[V_1, V_2, V_3]$ with |V(G)| = 3k - 3 and $|V_1| = |V_2| = |V_3| = k - 1$, where $k \ge 1$ is a positive integer. Then for each vertex $v \in V(G)$, $d^c(v) = d(v) = 2k - 2 = \frac{n+k-1}{2}$ while G contains no B_k and F_k (see Figure 2).

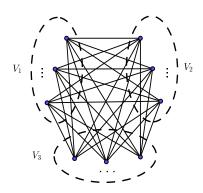


Figure 1: An extremal graph for Theorem 3

The main novelty of our proof of Theorem 3 is a combination of the recent new technique for finding rainbow cycles due to Czygrinow, Molla, Nagle, and Oursler [6] and some recent counting technique from [26]. In particular, Czygrinow et al. [6] extended H. Li's theorem by proving that for every integer ℓ , every edge-colored graph G on $n \geq 432\ell$ many vertices satisfying $\delta^c(G) \geq \frac{n+1}{2}$ admits a rainbow ℓ -cycle C_{ℓ} . One novel concept introduced in [6] is called restriction color, and it will be used in our proof. The proof of Theorem 4 is with the

aid of the machine implicitly in the work of Turán number for matching numbers due to Erdős and Gallai [13]. (For details, see the proof of Lemma 13.)

Meantime, both Theorem 3 and Theorem 4 improve Theorem 1 (by setting k = 2). On the other hand, Theorem 4 slightly improves Theorem 9 in [26] by a different method.

3 Additional notations and lemmas

Some of our notations come from [6, 26]. Let G be an edge-colored graph. Let $\mathcal{C}: E(G) \to \{1, 2, \ldots, k\}$ be an edge-coloring of G. For a color $\alpha \in \mathcal{C}(G)$ and a vertex $v \in V(G)$, define the α -neighborhood of v as

$$N_{\alpha}(v) = \{ u \in N(v) \mid c(uv) = \alpha \},\$$

 α -neighborhood in X of v as

$$N_{\alpha}(v, X) = \{ u \in N(v) \cap X \mid c(uv) = \alpha \}$$

where $X \subseteq V(G)$, and N(v) is the neighborhood of v in G. Define

$$N_!(v) = \bigcup_{\alpha \in \mathcal{C}(G)} \{ N_{\alpha}(v) \mid |N_{\alpha}(v)| = 1 \}.$$

For the sake of simplicity, let $d_{\alpha}(v) = |N_{\alpha}(v)|$ and $d_{\alpha}(v, X) = |N_{\alpha}(v, X)|$. Moreover, let $N[v] = N(v) \cup \{v\}$. The monochromatic degree of v, denoted by $d^{mon}(v)$, is the maximum number of edges incident to v colored with a same color (i.e., $d^{mon}(v) = \max\{d_{\alpha}(v) \mid \alpha \in \mathcal{C}(G)\}$.) For a graph G, we denote the monochromatic degree of G by $\Delta^{mon}(G) = \max\{d^{mon}(v) \mid v \in V(G)\}$.

W.l.o.g., assume that $d_1(v) \geq d_2(v) \geq \cdots \geq d_s(v)$ and $s := s(v) = d^c(v)$ for each vertex $v \in V(G)$. (When there is no danger of ambiguity, we use s for short.)

The following concept of restriction was firstly introduced in [6, Section 3].

Definition 5 (restriction color [6]). Let G be an edge-colored graph. Fix $v \in V(G)$ and $X \subseteq N(v)$. Suppose $\alpha = c(xy)$ for $x \in X \cap N(y)$ and $y \in V(G) \setminus \{v\}$. We say (v, X) restricts color α for y if vxy is a rainbow P_3 and $\alpha \notin \mathcal{C}(y, N(y) \setminus X)$. Denote by $\sigma_{v,X}(y)$ the number of colors $\alpha \in \mathcal{C}(E)$ restricted for y by (v, X).

Denote by rt(v) the number of rainbow triangles containing v; by rt(v, x) the number of rainbow triangles containing both v and x (i.e., containing the edge vx); and $rt(v, X) = \sum_{x \in X} rt(v, x)$.

According to the definition of restriction color, we have the following proposition.

Proposition 6. Let v be a vertex of G and $x \in N(v)$. Then $rt(v, x) \ge \sigma_{v,N(v)\setminus N_{c(vx)}(v)}(x)$ and $rt(v, N_{c(vx)}(v)) \ge \sum_{x \in N_{c(vx)}(v)} \sigma_{v,N(v)\setminus N_{c(vx)}(v)}(x)$.

Proof. For $y \in N(v) \setminus N_{c(vx)}(v)$, we have $c(vx) \neq c(vy)$. If c(xy) is a restriction color for x to v with respect to $N(v) \setminus N_{c(vx)}(v)$, then $c(vy) \neq c(xy)$ and $c(xy) \notin C(x, N(x) \setminus (N(v) \setminus N_{c(vx)}(x)))$, and $c(xy) \neq c(xv)$. Thus, vxyv is a rainbow triangle. It follows $rt(v, x) \geq \sigma_{v,N(v) \setminus N_{c(vx)}(v)}(x)$ and $rt(v, N_{c(vx)}(v)) \geq \sum_{x \in N_{c(vx)}(v)} \sigma_{v,N(v) \setminus N_{c(vx)}(v)}(x)$.

Remark 1. Throughout this paper, we say that an edge-colored graph G is edge-minimal if for any $e \in E(G)$, there exists a vertex v incident to e such that $d_{G-e}^c(v) < d_G^c(v)$. Hence, G contains neither monochromatic paths of length 3 nor monochromatic triangles.

The form of the following lemma is motivated by [26], but its proof is a mixture of techniques from [6, 26]. For the sake of simplicity, define the color set $C(G) = \{1, 2, \dots, c(G)\}$.

Lemma 7. Let G be an edge-colored graph which is edge-minimal. Then for $1 \le i \le s = d^c(v)$, we have,

$$rt(v, N_i(v)) \ge \sum_{x \in N_i(v)} \left(d^c(x) + d^c(v) - n \right) + d_i(v) \sum_{1 \le j \le s} \left(d_j(v) - 1 \right) - d_i(v) \left(d_i(v) - 1 \right) - \sum_{y \in N_i(v)} d_{c(vy)}(y, N_i(v)).$$

Proof of Lemma 7. For convenience, let $X_i = N_i(v)$, $Y_i = N(v) \setminus X_i$ for $1 \le i \le s$, and d_i for $d_i(v)$ in the following. Let $s = d^c(v)$. For $v \in V(G)$ and $1 \le i \le s$, define a directed graph D_i on $V(D_i) = X_i \cup Y_i$ as follows: the arc \overrightarrow{yx} exists if and only if the following hold: (1) $x \in X_i$ and $y \in Y_i$; and (2) c(xy) = c(vy). Since G is edge-minimal, the existence of \overleftarrow{xy} gives $d_{c(vy)}(v) = 1$. (Indeed, since the arc \overleftarrow{xy} exists, we have c(xy) = c(vy). If $d_{c(vy)}(v) \ge 2$, there exists a monochromatic path of length 3, a contradiction!) Hence $y \in N_!(v)$. Evidently, $d_D^+(y) \le d_{c(vy)}(y, X_i)$. Thus,

$$\sum_{x \in X_i} d_D^-(x) = \sum_{y \in Y_i} d_D^+(y) = \sum_{y \in N_1(v)} d_D^+(y) \le \sum_{y \in N_1(v)} d_{c(vy)}(y, X_i). \tag{1}$$

As for $x \in X_i$, there are at most $d_D^-(x) + \sigma_{v,Y_i}(x)$ colors which only appear on edges from x to Y_i . Then there are at least $d^c(x) - d_D^-(x) - \sigma_{v,Y_i}(x)$ vertices in $V(G) \setminus (Y_i \cup \{x\})$. Therefore,

$$n - |Y_i| - 1 \ge d^c(x) - d_D^-(x) - \sigma_{v,Y_i}(x)$$

$$\Rightarrow \quad \sigma_{v,Y_i}(x) \ge d^c(x) + |Y_i| + 1 - d_D^-(x) - n \tag{2}$$

Note that $|Y_i| = d(v) - d_i = \sum_{1 \le j \le s} d_j - d_i = d^c(v) + \sum_{1 \le j \le s} (d_j - 1) - d_i$. Combining Ineqs

(1) and (2), we can get that

$$\begin{split} & \sum_{x \in X_i} \sigma_{v,Y_i}(x) \\ & \geq \sum_{x \in X_i} \left(d^c(x) + |Y_i| + 1 - n \right) - \sum_{y \in N_!(v)} d_{c(vy)}(y, X_i) \\ & \geq \sum_{x \in X_i} \left(d^c(x) + d^c(v) + \sum_{1 \leq j \leq s} (d_j - 1) - d_i + 1 - n \right) - \sum_{y \in N_!(v)} d_{c(vy)}(y, X_i) \\ & \geq \sum_{x \in X_i} \left(d^c(x) + d^c(v) - n \right) + d_i \left(\sum_{1 \leq j \leq s} (d_j - 1) \right) - d_i \left(d_i - 1 \right) - \sum_{y \in N_!(v)} d_{c(vy)}(y, X_i). \end{split}$$

The proof is complete.

Since $rt(v) \ge \frac{1}{2} \sum_{1 \le i \le d^c(v)} rt(v, N_i(v))$, we have the following corollary.

Corollary 8. Let G be an edge-colored graph which is edge-minimal. Then for $1 \le i \le d^c(v) = s$, we have,

$$rt(v) \ge \frac{1}{2} \left(\sum_{x \in N(v)} \left(d^c(x) + d^c(v) - n \right) + d(v) \left(\sum_{1 \le j \le s} (d_j(v) - 1) \right) - \sum_{1 \le i \le s} d_i(v) \left(d_i(v) - 1 \right) - \sum_{y \in N_1(v)} d_{c(vy)}(y, N(v)) \right).$$

4 Edge-colored books and friendship graphs

4.1 Edge-colored books

The aim of this subsection is to prove Theorem 3.

Before the proof, it's better to introduce a notation, which indeed is some term from Corollary 8. For $i \in [1, s]$, let

$$B_i(v) = d_i(v) \left(\sum_{1 \le j \le s} (d_j(v) - 1) \right) - d_i(v) (d_i(v) - 1) - \sum_{y \in N_1(v)} d_{c(vy)}(y, N_i(v)).$$
 (3)

Set

$$B(v) = \sum_{1 \le i \le s} B_i(v)$$

$$= d(v) \sum_{1 \le j \le s} (d_j(v) - 1) - \sum_{1 \le i \le s} d_i(v) (d_i(v) - 1) - \sum_{y \in N_!(v)} d_{c(vy)}(y, N(v)).$$

$$= (d(v) - d_1(v)) (d_1(v) - 1) - \sum_{y \in N_!(v)} d_{c(vy)}(y, N(v)) + \sum_{2 \le i \le s} (d(v) - d_i(v)) (d_i(v) - 1).$$
(4)

For two disjoint subsets $V_1, V_2 \subseteq V(G)$, the set of colors appearing on the edges between V_1 and V_2 in G is denoted by $C(V_1, V_2)$. We say an edge $xy \in E(G)$ is a rainbow triangle edge of v if vxyv is a rainbow triangle. Denote by RE(v) the edge set of rainbow triangle edges of v.

First we prove a result on a vertex with maximum monochromatic degree, i.e., a vertex $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$. Recall that we assume $d_1(v) \geq d_2(v) \geq \ldots \geq d_s(v)$, where $s = d^c(v)$. Thus, $d_1(v) = d^{mon}(v)$. Also, $N_1(v)$ means the set of all vertices $u \in N(v)$ such that c(uv) = 1.

Lemma 9. Let G be an edge-colored graph. Then for a vertex $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$ we have $B(v) \geq 0$. If $\Delta^{mon}(G) \geq 2$ and B(v) = 0 then there hold:

- (a) $N_!(v) = N(v) \setminus N_1(v)$;
- (b) $d^{mon}(u) = \Delta^{mon}(G)$ for all $u \in N_!(v)$; and
- (c) if $B_1(v) = 0$, then $E[N_1(v), N_!(v)] \subseteq RE(v)$.

Proof of Lemma 9. If $\Delta^{mon}(G) = 1$, then obviously B(v) = 0. Suppose that $\Delta^{mon}(G) \geq 2$. Notice that $d_{c(vy)}(y, N(v)) \leq \Delta^{mon}(G) - 1 = d_1(v) - 1$ for $y \in N_!(v)$. From Ineq (4), we have

$$B(v) \ge \underbrace{\left(d(v) - d_1(v) - |N_1(v)|\right)\left(d_1(v) - 1\right)}_{\text{①}} + \underbrace{\sum_{2 \le i \le s} \left(d(v) - d_i(v)\right)\left(d_i(v) - 1\right)}_{\text{②}}$$

$$\tag{5}$$

 ≥ 0 .

If B(v) = 0, then $d_{c(vy)}(y, N(v)) = d_1(v) - 1$ for all $y \in N_!(v)$ and both ① and ② in Ineq (5) equal to 0. Since $\Delta^{mon}(G) = d^{mon}(v) = d_1(v) \ge 2$, we have $d(v) - d_1(v) - |N_!(v)| = 0$ and $d_i(v) = 1$ for $i \in [2, s]$. Therefore, both (a) and (b) hold.

Note that $d(v) - d^c(v) = d_1(v) - 1$. We have $B_1(v) = -\sum_{y \in N_!(v)} d_{c(vy)}(y, N_1(v)) = 0$. Hence, $c(vy) \notin \mathcal{C}(N_1(v), N_!(v))$ for $y \in N_!(v)$. Since G is edge-minimal and $d_1(v) \geq 2$, we have $1 \notin \mathcal{C}(N_1(v), N_!(v))$; indeed, if $1 \in \mathcal{C}(N_1(v), N_!(v))$, then there exists a monochromatic path of length 3 as $d_1(v) \geq 2$. Furthermore, for any edge $xy \in E[N_1(v), N_!(v)]$, xy is a rainbow triangle edge of v. Hence $E[N_1(v), N_!(v)] \subseteq RE(v)$. This proves (c). The proof is complete.

Lemma 10. Let $k \geq 2$ be a positive integer and G be a graph on $n \geq 3k-2$ vertices. If $\delta(G) \geq \frac{n+k-1}{2}$ then G contains a B_k .

Proof of Lemma 10. Suppose to the contrary that there is no B_k in G. If there exists a vertex $v \in V(G)$ with $d(v) \geq \frac{n+k}{2}$, then $|N(v) \cap N(u)| \geq \lceil \frac{n+k}{2} \rceil + \lceil \frac{n+k-1}{2} \rceil - n \geq k$ where $u \in N(v)$. In this case, there is a B_k . Suppose $d(v) = \frac{n+k-1}{2}$ for all $v \in V(G)$. Then for any edge $uv \in E(G)$, we have

$$\frac{n+k-1}{2} + \frac{n+k-1}{2} - n \le |N(v) \cap N(u)| \le k-1.$$

Hence $|N(v) \cap N(u)| = k - 1 \ge 1$ and $N(u) \cup N(v) = n$. It follows that $V(G) \setminus N(v) \subseteq N(u)$. Since each vertex in V(G) is of degree $\frac{n+k-1}{2}$, we have $|V(G) \setminus N(v)| = \frac{n-k+1}{2}$. Similarly, for $w \in N(v) \cap N(u)$, we also have $V(G) \setminus N(v) \subseteq N(w)$. Hence, $V(G) \setminus N(v) \subseteq N(u) \cap N(w)$. Then $|N(u) \cap N(w)| \ge \frac{n-k+1}{2}$. Since $\frac{n+k-1}{2}$ is an integer, $k-1 \ge \frac{n-k+1}{2}$, and so $n \le 3k-3$, a contradiction. This proves Lemma 10.

Now we are ready to prove Theorem 3.

Proof of Theorem 3. We prove the theorem by contradiction. Let G be a counterexample such that e(G) is as small as possible. We use d_i instead of $d_i(v)$ in the following. Let v be a vertex with $d^{mon}(v) = \Delta^{mon}(G)$. By Lemma 10, we can assume $\Delta^{mon}(G) \geq 2$.

If there exists an integer $i \in [1, s]$ such that $rt(v, N_i(v)) \ge (k-1)d_i + 1$, then there exists a vertex $x_0 \in N_i(v)$ satisfying $rt(v, x_0) \ge k$, a contradiction. Thus,

$$rt(v, N_i(v)) \le (k-1)d_i \le \sum_{x \in N_i(v)} (d^c(x) + d^c(v) - n)$$

(where the second inequality holds as the condition that $d^c(x) + d^c(v) \ge n + k - 1$). It follows that $B_i(v) \le 0$ for all $i \in [1, s]$ by Lemma 7. Therefore, $0 \le B(v) = \sum_{1 \le i \le s} B_i(v) \le 0$. That is, $B(v) = B_i(v) = 0$ for $i \in [1, s]$. By Lemma 9 we have $rt(v, N_i(v)) = (k - 1)d_i$ for all $i \in [1, s]$ and $d_i = 1$ for $i \in [2, s]$. Since G contains no k rainbow triangles sharing one common edge, rt(v, x) = k - 1 for all $x \in N(v)$.

Claim. There exists a vertex $x \in N_!(v)$ such that $d^c_{G[N[v]]}(x) \leq k$.

Proof. Choose $x \in N_!(v)$. If all edges in $G[N_!(v)]$ are rainbow edges of v, then as $E[N_1(v), N_!(v)] \subseteq RE(v)$ (recall Lemma 9(c)), $d^c_{G[N[v]]}(x) \leq d_{G[N[v]]}(x) \leq rt(v, x) + 1 \leq k$ for $x \in N_!(v)$. Now assume that there exists a vertex $u' \in N_!(v)$ such that c(xu') = c(vx). Since $x \in N_!(v)$, by (b) of Lemma 9, $d^{mon}(x) = \Delta^{mon}(G)$.

By a) of Lemma 9, $N_!(x) = N(x) \setminus N_{c(vx)}(x)$ and c(vx) is the maximum monochromatic color of x. For all $u \in N_!(v) \cap N(x)$, we have $u \in N_{c(xv)}(x)$ or $u \in N_!x$. By c) of Lemma 9, $E[N_{c(xv)}(x), N_!(x)] \subseteq RE(x)$. If $u \in N_!(x)$, then xvux is a rainbow triangle, and hence xu is a rainbow triangle edge of v. Therefore, for all $u \in N_!(v) \cap N(x)$, we have c(xu) = c(vx) or $xu \in RE(v)$. From c) of Lemma 9, we have $E[x, N_1(v)] \subseteq RE(v)$. Hence,

$$d_{G[N[v]]}^c(x) \le d_{N_1[v]}^c(x) + |\mathcal{C}(x, N_1(v))| \le rt(v, x) + 1 = k.$$

Then, we infer

$$\frac{n+k-1}{2} \le d^{c}(x) \le d^{c}_{G[N[v]]}(x) + n - \left(\Delta^{mon}(G) + d^{c}(v)\right)$$

$$= k + n - \Delta^{mon}(G) - d^{c}(v)$$

$$\le \frac{n+k+1}{2} - \Delta^{mon}(G),$$

that is, $\Delta^{mon}(G) = 1$, a contradiction.

4.2 Edge-colored friendship graphs

The aim of this subsection is to prove Theorem 4.

A matching of a graph consists of some vertex-disjoint edges. The matching number of a graph G is the maximum number of pairwise disjoint edges in G, denoted by $\alpha'(G)$. Erdős and Gallai [13] determined Turán number of a matching with given size.

A covering of a graph G is a subset K of V(G) such that every edge of G has at least one end vertex in K. A covering K^* is a minimum covering if G has no covering K with $|K| < |K^*|$. The number of vertices in a minimum covering of G is called the covering number of G, and is denoted by $\beta(G)$.

Lemma 11. For a graph G on n vertices, we have $\beta(G) \leq n-1$. Furthermore, G is complete if and only if $\beta(G) = n-1$.

Proof of Lemma 11. The sufficiency of condition is clear. We establish its necessity by the contradiction. Suppose that G is not complete. Then there exist two vertices $u, v \in V(G)$ such that $uv \notin E(G)$. Hence $V(G) \setminus \{u, v\}$ is a covering of G. Thus, $\beta(G) \leq n - 2$, a contradiction.

Now we will prove a useful bound on $\alpha'(G)$ and $\beta(G)$, which is partly inspirited by the proof of Turán numbers of matchings by Erdős and Gallai [13]. Following Berge [4, pp. 176] (see also Erdős and Gallai [13, pp. 354]), for a graph G with $\alpha'(G) = k$ and n > 2k, choose k independent edges and call them α' -edges and the remaining edges γ -edges. Add a vertex x to V(G) and connect x with every vertex which is not incident with α' -edges (i.e., any vertex not in the edges of the matching of size k) by an edge. For all the new edges (i.e., the edges we add) and the old α' -edges (i.e., the matching), we call them α -edges. A path is called alternating if its edges are alternately α -edges and γ -edges. For a vertex, we say that it is a γ -vertex if it is reachable from x by an alternating path which only ends with γ -edges. Apparently, x is a γ -vertex.

Let V_0 be the set of all γ -vertices of G, and V_1, \dots, V_p be the components of G obtained from G by deleting V_0 . There is a well-known fact ([3, 4, pp. 169-170], [19, pp. 141-142]) on the relationship among α -edges, γ -vertices and V_i .

Fact 12. Each α -edge incident to a γ -vertex is also incident to some odd V_i (i.e., $|V_i|$ is odd) and each odd V_i has exactly one α -edge incident to a γ -vertex.

It follows that each α -edge is either incident to a γ -vertex and an odd V_i , or is an edge of some $G[V_i]$ (see Figure 2, the edges colored blue are α' -edges). Furthermore, we can assert that

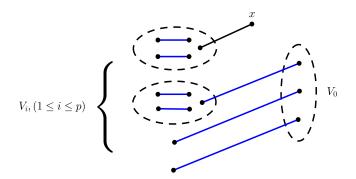


Figure 2: A partition (V_0, V_1, \dots, V_p) of a graph G

 $G[V_i]$ contains exactly $\lfloor \frac{|V_i|}{2} \rfloor$ α' -edges. If $\alpha'(G) < \lfloor \frac{|V_i|}{2} \rfloor$, then there exist at least two vertices in V_i incident to x, a contradiction to Fact 12. Note that each vertex in V_0 is incident with an α' -edge; otherwise the vertex is connected with x by an α -edge, a contradiction. The following equation on $\alpha'(G)$ can be deduced implicitly from Erdős and Gallai [13, pp. 355].

$$\alpha'(G) = |V_0| + \sum_{1 \le i \le p} \left\lfloor \frac{|V_i|}{2} \right\rfloor. \tag{6}$$

Since $\beta(V_i) \leq |V_i| - 1$ for $1 \leq i \leq p$, by Lemma 11, we have the following lemma.

Lemma 13. Let G be a graph on $n > 2\alpha'(G)$ vertices and $\{V_i : 0 \le i \le p\}$ be the partition as remarked above. Then

$$\beta(G) \le |V_0| + \sum_{1 \le i \le p} (|V_i| - 1) = n - p \le 2\alpha'(G) - |V_0|. \tag{7}$$

Lemma 14. Let G be a graph on $n \ge 2\alpha'(G) + 2$ vertices and $\{V_i : 0 \le i \le p\}$ be the partition as remarked above. Then $\beta(G) \le 2\alpha'(G) - 1$ and V_0 is not empty. Furthermore, if $\beta(G) = 2\alpha'(G) - 1$, then

- (1) V_i is complete and odd for $1 \le i \le p$;
- (2) $\beta(G) = n p$ and the covering of G consists of all vertices of V_0 and $|V_i| 1$ vertices of V_i for $1 \le i \le p$.

Proof of Lemma 14. Since $n \geq 2\alpha'(G) + 2$, G is not complete. If $V_0 = \emptyset$, then G contains exactly $\lfloor \frac{n}{2} \rfloor$ α' -edges, which implies that $n \leq 2\alpha'(G) + 1$, a contradiction. Hence $|V_0| \geq 1$. $\beta(G) \leq 2\alpha'(G) - 1$ follows from Ineq (7).

If $\beta(G) = 2\alpha'(G) - 1$, then all inequalities become equalities in Ineq (7). Hence $\beta(V_i) = |V_i| - 1$ for all $1 \le i \le p$. From Lemma 11, $G[V_i]$ is complete.

Recall RE(v) is the edge set of rainbow triangle edges of v. Let $G_{\triangle}(v)$ denote the subgraph induced by the edge set RE(v). Let $C_{\triangle}(v)$ denote a minimum covering of $G_{\triangle}(v)$. Thus $\beta(G_{\triangle}(v)) = |C_{\triangle}(v)|$. Now we are ready to prove Theorem 4.

Proof of Theorem 4. We prove the theorem by contradiction. Let G be a counterexample with the smallest order n, and then e(G) is as small as possible. Choose $v \in V(G)$ such that $d^{mon}(v) = \Delta^{mon}(G)$. Then $\alpha'(G_{\triangle}(v)) \leq k - 1$. Since $n \geq 2k + 9$, by Corollary 8 and Lemma 9 we have $e(G_{\triangle}(v)) = rt(v) \geq \frac{4k^2-9}{2} > {2k-1 \choose 2}$. Hence $|V(G_{\triangle}(v))| \geq 2k \geq 2\alpha'(G_{\triangle}(v)) + 2$. Then $\beta(G_{\triangle}(v)) \leq 2k - 3$ by Lemma 14.

Let α be a color satisfying $d^{mon}(v) = d_{\alpha}(v)$. Now we define two disjoint vertex subsets of N(v). If $\Delta^{mon}(G) \geq 2$, let U(v) be a maximum rainbow neighborhood of v in $N(v) \setminus (N_{\alpha}(v) \cup C_{\triangle}(v))$ and $W(v) = N_{\alpha}(v) \setminus C_{\triangle}(v)$. If $\Delta^{mon}(G) = 1$, let U(v) be a maximum rainbow neighborhood of v in $N(v) \setminus C_{\triangle}(v)$ and $W(v) = \emptyset$. Set $X(v) = U(v) \cup W(v)$, $T(v) = X(v) \cup C_{\triangle}(v)$ and $T[v] = T(v) \cup \{v\}$. Thus, $X(v) = (N(v) \cup N_{\alpha}(v)) \setminus C_{\triangle}(v) = N(v) \setminus C_{\triangle}(v)$. Apparently, $|U(v)| \geq d^c(v) - \beta(G_{\triangle}(v)) - 1$ and

$$|T[v]| \ge |N_{\alpha}(v)| + d^{c}(v) - 1 + 1 \ge d^{mon}(v) + d^{c}(v).$$
(8)

Define $q(u) = |\mathcal{C}(u, C_{\triangle}(v)) \setminus \mathcal{C}(u, X[v])|$ for $u \in X(v)$.

According to the definition of $C_{\triangle}(v)$, there is no rainbow triangle edge of v in $G[\{v\} \cup X(v)]$. For $u_1, u_2 \in X(v)$, if $u_1u_2 \in E(G)$, then we have

$$c(u_1u_2) \in \{c(vu_1), c(vu_2)\}. \tag{9}$$

Let $Y \subseteq X(v)$ and D[Y] be the oriented graph of G[Y] defined as follows: let V(D) = Y and orient each edge u_1u_2 in G[Y] by $\overrightarrow{u_1u_2}$ if $c(u_1u_2) = c(vu_2)$. Then for $u \in Y$, all in-arcs from u are assigned the color c(vu), and all out-arcs from u are assigned pairwise distinct colors which are different from c(vu) as G is edge-minimal. Hence $d^{mon}_{G[Y]}(u) = d^-_{D[Y]}(u)$ and $d^c_{G[Y]}(u) = d^+_{D[Y]}(u)$. Since $d^-_{D[Y]}(u) \le \Delta^{mon}(G) - 1$ for any $u \in Y$, there exists a vertex $u' \in Y$ such that

$$d_{G[Y]}^{c}(u') = d_{D[Y]}^{+}(u') \le \Delta^{mon}(G) - 1.$$
(10)

In the following, we prove a useful claim on a vertex $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$.

Claim. $q(u) = \beta(G_{\triangle}(v)) = 2k - 3$ for all $u \in X(v)$.

Proof. Apparently, we have

$$d^{c}(u) \le d^{c}_{T[v]}(u) + n - |T[v]|. \tag{11}$$

If $\Delta^{mon}(G) = 1$, then G is properly-colored. Therefore, G[X(v)] consists of isolated vertices by (9). Thus, for $u \in X(v)$, $d^c_{T[v]}(u) = |c(uv) \cup \mathcal{C}(\{u\}, C_{\triangle}(v))| \le 1 + q(u)$. Hence by Ineqs (8) and (11), we have $d^c(u) + d^c(v) - n \le q(u) \le \beta(G_{\triangle}(v)) \le 2k - 3$. Now we have $q(u) = |C_{\triangle}(v)| = 2k - 3$ for all $u \in X(v)$.

Assume that $\Delta^{mon}(G) \geq 2$. Then for $w \in W(v)$ and $u \in U(v)$, c(wu) = c(vu) if $wu \in E(G)$ (otherwise, there is a monochromatic P_3 , a contradiction). Thus for $u \in U(v)$, we have $d^c_{T[v]}(u) \leq d^+_{D[U(v)]}(u) + 1 + q(u)$. By Ineqs (8) and (11), we have

$$d_{D[U(v)]}^{+}(u) \ge d^{c}(u) + |T[v]| - n - q(u) - 1$$

$$\ge \Delta^{mon}(G) + 2k - 4 - q(u).$$
(12)

Since $q(u) \leq \beta(G_{\Delta}(v)) \leq 2k-3$, we have $d_{D[U(v)]}^+(u) \geq \Delta^{mon}(G)-1$. Since $d_{D[U(v)]}^-(u) \leq \Delta^{mon}(G)-1$, $d_{D[U(v)]}^+(u) = d_{D[U(v)]}^-(u) = \Delta^{mon}(G)-1$, which gives that q(u) = 2k-3 for all $u \in U(v)$ and all inequalities in Ineq (12) become equalities.

Meanwhile, we have $E[W(v), U(v)] = \emptyset$ as c(wu) = c(vu) if $wu \in E(G)$. Hence for $w \in W(v)$,

$$\frac{n+2k-3}{2} \le d^{c}(w) \le d^{c}_{T[v]}(w) + n - |T[v]|
\le \Delta^{mon}(G) + q(w) + n - \Delta^{mon}(G) - d^{c}(v)
\le \frac{n-2k+3}{2} + q(w).$$
(13)

Then q(w) = 2k - 3 for all $w \in W(v)$.

Let $\{V_i: 0 \leq i \leq p\}$ be the partition of $G_{\triangle}(v)$ as remarked in Lemma 13. Since $G_{\triangle}(v)$ is non-complete, by Claim, $2k-3=\beta(G_{\triangle}(v))\leq 2\alpha'(G_{\triangle}(v))-|V_0|\leq 2k-3$, which guarantees that $\alpha'(G_{\triangle}(v))=k-1$ and $|V_0|=1$. Since $|V(G_{\triangle}(v))|\geq 2k$, by Lemma 13, we have

$$p \ge |G_{\triangle}(v)| + |V_0| - 2\alpha'(G_{\triangle}(v)) \ge 2k + 1 - 2(k - 1) = 3.$$

We first consider the case of $V_0 \subsetneq C_{\triangle}(v)$. By Lemma 14(2), for $x \in V_i \cap C_{\triangle}(v)$, $1 \leq i \leq p$ there exists exactly one vertex $u' \in X(v)$ such that $xu' \in RE(v)$, as $|V_i \setminus C_{\triangle}(v)| = 1$. For any $u \in X(v) \setminus V(G_{\triangle})$ and $x \in C_{\triangle}(v)$, the fact $xu \notin RE(v)$ gives that $c(xu) \in \{c(vu), c(vx)\}$. Since $q(u) = |C(u, C_{\triangle}(v)) \setminus C(u, X[v])| = |C_{\triangle}(v)| = \beta_{\triangle}$, c(xu) = c(vx). Since $q(u) = |C_{\triangle}(v)| = \beta_{\triangle}$, by Claim, each vertex in X(v) is incident to all vertices in $C_{\triangle}(v)$. For $x \in V_i \cap C_{\triangle}(v)$,

there is only one vertex $u' \in X(v)$ such that $c(xu') \neq c(vx)$. Hence $d^{mon}(x) \geq d_{c(vx)}(x) \geq |X(v) \cup \{v\} - \{u'\}|$. Since $|X(v) \cup \{v\}| = |T[v]| - \beta_{\triangle}$, we have the following inequality:

$$d^{mon}(x) \ge |(X(v) \cup \{v\}) - \{u'\}|$$

$$\ge d^{c}(v) + \Delta^{mon}(G) - \beta(G_{\triangle}(v)) - 1$$

$$\ge \frac{n + 2k - 3}{2} + \Delta^{mon}(G) - (2k - 3) - 1$$

$$\ge \Delta^{mon}(G) + 1$$

as $n \ge 2k + 1$. This is a contradiction.

Next we consider the case that $C_{\triangle}(v) = V_0$. The goal of the coming part is to prove that $d^{mon}(u) = \Delta^{mon}(G)$ for any $u \in X(v)$.

Suppose that there exists a vertex $u' \in X(v)$ with $d^c_{G[X(v)]}(u') \leq \Delta^{mon}(G) - 2$. Recall that $T(v) = X(v) \cup C_{\Delta}(v)$, and X(v) and $C_{\Delta}(v)$ are disjoint. Then by Claim and (8), we have

$$\frac{n+2k-3}{2} \le d^{c}(u')$$

$$\le d^{c}_{G[X(v)]}(u') + d^{c}_{G[\{u'\}\cup C_{\triangle}(v)]}(u') + n - |T[v]| + |\{v\}|$$

$$= d^{c}_{G[X(v)]}(u') + q(u') + n - |T(v)|$$

$$\le \Delta^{mon}(G) - 2 + 2k - 3 + n - (d^{mon}(v) + d^{c}(v) - 1)$$

$$\le 2k - 4 + n - \frac{n+2k-3}{2},$$

a contradiction. Thus, for any vertex $u \in X(v)$, we have $d_{G[X(v)]}^c(u) \geq \Delta^{mon}(G) - 1$. Set Y = X(v). Recall the rule of the construction of D[Y] (above the proof of Claim). We have $d_{D[Y]}^-(u) \leq \Delta^{mon}(G) - 1$ for any $u \in Y$. Since $\sum_{u \in Y} d_{D[Y]}^-(u) = \sum_{u \in Y} d_{D[Y]}^+(u)$, each vertex $u \in X(v)$ satisfies that $d_{G[X(v)]}^c(u) = \Delta^{mon}(G) - 1$. Hence $d^{mon}(u) = d_{D[X(v)]}^-(u) + 1 = \Delta^{mon}(G)$. Since $C_{\triangle}(v) = V_0$ and $|V_0| = 1$, we have q(u) = 2k - 3 = 1, and hence k = 2. Note that $|G_{\triangle}(v) - C_{\triangle}(v)| = \sum_{i=1}^p |V_i| \geq p \geq 3$. To avoid the graph consisting of two rainbow triangles sharing one common vertex in G, we have $|RE(u)| \leq 1$ for $u \in G_{\triangle}(v) - C_{\triangle}(v)$. Since u is also a vertex with $d^{mon}(u) = \Delta^{mon}(G)$, we have $|RE(u)| \geq 2$, a contradiction.

5 Concluding remarks

One may wonder the sharpness of Theorems 3 and 4. For Theorem 3, by Example 1, we know when $k \geq \frac{n}{3} + 1$ (in this case, the subgraph B_k is with order $\Theta(n)$), the color degree guaranteeing a properly colored B_k should be larger than $\frac{n+k-1}{2}$.

For uncolored friendship subgraphs, we first prove the following result.

Proposition 15. Let $k \geq 2$ be a positive integer and G be a graph on $n \geq 3k-1$ vertices. If $\delta(G) \geq \frac{n+k-1}{2}$ then G contains a F_k .

Proof of Proposition 15. We proceed the proof by induction on k. The basic case k=2 is easily derived from Theorem 4. Let $k \geq 3$ and suppose the result holds for k-1. Suppose to the contrary that there is no F_k in G. Let v be the center of a F_{k-1} and $S = V(F_k - v)$. Since $\delta(G) \geq \frac{n+k-1}{2}$, each edge is contained in at least k-1 triangles. As $n \geq 3k-1$, there exist k vertices in $N(v) \setminus S$. According to pigeonhole principle, there exists a F_k in N[v].

As we have already mentioned in the introduction, as corollaries of the result of Erdős et al. [12] on k-fans and Erdős' conjecture on books, respectively, the following results hold.

Proposition 16. Let $k \geq 2$ be a positive integer and G be a graph on $n \geq 50k^2$ vertices. If $\delta(G) \geq \frac{n+1}{2}$ then G contains a F_k .

Proposition 17. Let $k \geq 2$ be a positive integer and G be a graph on $n \geq 6k$ vertices. If $\delta(G) \geq \frac{n+1}{2}$ then G contains a B_k .

So by reasoning the above results, Theorems 3 and 4 should be at least asymptotically tight when k = o(n).

For Theorem 4, the corresponding color degree condition may be acceptable when one considers a nearly spanning F_k . One evidence is that, if we consider the color degree condition for the spanning F_k (that is, $k = \frac{n-1}{2}$), then the sufficient condition is that $\delta^c \geq n-1$ which equals to $\frac{n+2k}{2} + O(1)$ (see the following).

Fact 18. Let G be an edge-colored graph on n vertices where n is odd. If $\delta^c \geq n-1$ then G contains a properly-colored $F_{\frac{n-1}{2}}$.

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