Minimal homogeneous and noncrossing chain decompositions of posets

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Abstract

Homogeneous chain decompositions (HCDs) and noncrossing chain decompositions (NcCDs) of a poset are studied here, the former having a close connection to linear sequential dynamical systems while the latter generalizing the well-known noncrossing partitions. There exists a unique HCD containing the minimum number of chains for any poset and we show the number is not necessarily Lipschitz. Making use of the unique HCD, we then identify a group that contains an isomorphic copy of the automorphism group of the poset. Finally, we prove some upper bounds for the minimum number of chains contained in an NcCD. In particular, the number of chains in the unique HCD provides an upper bound.

Keywords: partially ordered set; chain decomposition; noncrossing chains; automorphism group; 132-avoiding permutation; descent Mathematics Subject Classifications: 06A07, 06A11, 05D05

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1 Introduction

Partially ordered sets are well studied objects in discrete mathematics and we will basically follow the notation in Stanley [13]. A partially ordered set (poset) is a set P with a binary relation ' \leq ' among the elements in P, where the binary relation satisfies reflexivity, antisymmetry and transitivity. The poset will be denoted by (P, \leq) or P for short. For simplicity, all posets discussed in this paper are assumed to be finite.

If two elements x and y in P satisfy $x \leq y$, we say x and y are comparable. We write x < y if $x \leq y$ but $x \neq y$. A chain of P is a subset of elements such that

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any two elements there are comparable, while an antichain is a subset where any two elements are not comparable. A chain decomposition of P is a family of disjoint chains $\{C_1, C_2, \ldots, C_k\}$ such that $\bigcup_{i=1}^k C_i = P$. Let Min(P) denote the minimum number of chains that are contained in a chain decomposition of P, and let Anti(P) denote the maximum number of elements that can be contained in an antichain of P. The celebrated Dilworth's theorem [6] states that Min(P) = Anti(P) for all finite P.

Regarding chain decompositions of posets, various types (e.g., symmetric chain decomposition [7], canonical symmetric chain decomposition [8], etc.) have been studied. A new type of chain decomposition, called homogeneous chain decomposition (HCD), was introduced in [5] in the context of studying the interaction between incidence algebras of posets and linear sequential dynamical systems, where in particular, it was shown that the Möbius function of any poset can be efficiently computed via a sequential dynamical system and a cut theorem concerning HCDs of posets holds. However, HCDs have some nice structural properties thus deserve a close look for its own sake, which motivated this work. In addition, we discuss decomposing a poset into noncrossing chains which can be viewed as a generalization of the extensively studied noncrossing partitions (e.g., see [1, 10]).

The paper is organized as follows. In Section 2, we review some notation regarding HCDs and noncrossing chain decompositions (NcCDs). We also propose a new interpretation of the cut theorem in terms of counting certain chains. In Section 3, we show that there exists a unique HCD containing the minimum number $Min_h(P)$ of chains and that $Min_h(P)$ is not Lipschitz. We additionally show that the automorphism group Aut(P) of any poset P is isomorphic to a subgroup of the group $Aut(G_P) \bigcap O_P$, where G_P is a graph and O_P is a group, both induced by the unique minimum HCD. Finally, we prove some upper bounds for the minimum number of chains contained in an NcCD in Section 4. In particular, $Min_h(P)$ provides an upper bound.

2 HCDs and NcCDs of posets

In this section, we will review some notation regarding HCDs and the cut theorem of posets proved in Chen and Reidys [5]. The definition of NcCDs will be given too.

Definition 2.1. Let (P, \leq) be a poset. An HCD C of P is a collection of mutually disjoint chains C_1, C_2, \ldots, C_n such that $\bigcup_i C_i = P$, and if $s_i \in C_i$ and $s_j \in C_j$ are comparable, then all elements in C_i and C_j are pairwise comparable.

When all elements in two chains C_i and C_j are pairwise comparable, we say C_i and C_j are comparable for short. We also write $C = (\xi_1 < \xi_2 < \cdots < \xi_s)$ as a shorthand of that C is the chain $\{\xi_1, \xi_2, \ldots, \xi_s\}$ and $\xi_1 < \xi_2 < \cdots < \xi_s$.

Definition 2.2. Let $C = \{C_1, C_2, \ldots, C_n\}$ be an HCD of P. The C-graph of P is the graph G_C having C_i 's as vertices where C_i and C_j are adjacent if they are comparable.

Definition 2.3. A C-cut of P is a cut of each chain $C_i = (\xi_1 < \xi_2 < \cdots < \xi_s)$ into two subchains $C_i^{\downarrow} = (\xi_1 < \xi_2 < \cdots < \xi_{h(i)})$ and $C_i^{\uparrow} = (\xi_{h(i)+1} < \xi_{h(i)+2} < \cdots < \xi_s)$.

Note that C_i^{\downarrow} or C_i^{\uparrow} could be empty for some *i*. If none of C_i^{\downarrow} and C_i^{\uparrow} are empty for any *i*, the C-cut is called proper. Clearly, given an HCD C of P, a C-cut of P induces the sub-posets P^{\downarrow} and P^{\uparrow} , in which $C^{\downarrow} = \{C_1^{\downarrow}, \ldots, C_n^{\downarrow}\}$ and $C^{\uparrow} = \{C_1^{\uparrow}, \ldots, C_n^{\uparrow}\}$ are HCDs of P^{\downarrow} and P^{\uparrow} , respectively.

In the incidence algebra of a poset, a particular important function is known to be its Möbius function. The cut theorem proved in [5] is concerned with the relation among the Möbius functions of P, P^{\downarrow} and P^{\uparrow} . Here we present an alternative version of the cut theorem which may be easier to get the point.

For an increasing chain $(\xi_1 < \cdots < \xi_s)$, its length is defined as s - 1. The number of chains of length zero from x to y is one if x = y, otherwise zero. Let D be an $n \times n$ matrix, where the entry D_{ij} is the number of (increasing) chains of even length less the number of chains of odd length, starting with an element in C_i and ending with an element in C_j . The matrices D^{\downarrow} and D^{\uparrow} are defined analogously. Then, in the case of the cut C being proper and that elements in C_i^{\downarrow} are smaller than elements in C_j^{\uparrow} whenever C_i and C_j are comparable, we have

Proposition 2.1 (Cut theorem). Let $J = I + J_0$, where J_0 is the adjacency matrix of $G_{\mathcal{C}}$, i.e., $[J_0]_{ij} = 1$ if C_i and C_j $(i \neq j)$ are adjacent and $[J_0]_{ij} = 0$ otherwise. Then

$$DJ = D^{\downarrow}J + D^{\uparrow}J - D^{\downarrow}JD^{\uparrow}J.$$
⁽¹⁾

We refer to [5] for the details of the original form of the cut theorem and leave the proof of the new version here to the interested reader. Note that eq. (1) can be further simplified if J is invertible. It would be nice if a more intuitive, alternative explanation could be given to the cut theorem above, in particular the reason why the invertibility of J even matters.

Example 2.1. In Figure 1, $x \to y$ indicates that x < y, and the two elements on each vertical dashed line form a chain and these chains give an HCD C of the poset P. A C-cut will send the upper element on each chain to P^{\uparrow} and the lower element to P^{\downarrow} . All elements in P^{\downarrow} are assumed to be smaller than those in P^{\uparrow} if they are comparable although the corresponding arrows are not shown here for simplicity.

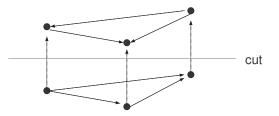


Figure 1: An example of a C-cut.

Definition 2.4. A collection of chains $\{C_1, C_2, \ldots, C_k\}$ is called a noncrossing chain decomposition of a poset P if there do not exist elements $a, b \in C_i$ and $c, d \in C_j$ $(i \neq j)$ such that a < c < b < d in P.

Let $[n] = \{1, 2, ..., n\}$. It is not hard to observe that a classical noncrossing partition [1, 10] of [n] is essentially a noncrossing chain decomposition of the poset [n] with the natural order. We denote by $Min_{nc}(P)$ the minimum number of chains contained in a noncrossing chain decomposition of P. Clearly, $Min_{nc}(P) = 1$ if and only if $P \sim [n]$.

3 Minimal HCDs

We denote by $|\mathcal{C}|$ the number of chains contained in \mathcal{C} . Let $Min_h(P) = \min_{\mathcal{C}} |\mathcal{C}|$, where the minimization is over all HCDs of P. An HCD of P containing exactly $Min_h(P)$ chains is called a minimal homogeneous chain decomposition (MHCD) of P. It turns out there is only one such a decomposition.

Proposition 3.1. For any poset P, there exists a unique MHCD of P.

Proof. Let $C = \{C_1, C_2, \ldots, C_m\}$ be a MHCD of P. If $|C_i| = 1$ for all $1 \leq i \leq m$, there is nothing to prove. Thus, we assume that there exists at least one i such that $|C_i| > 1$. Let $C' = \{C'_1, C'_2, \ldots, C'_m\}$ be a different MHCD of P. First, there exists k and $s_{k1}, s_{k2} \in C_k$ such that $s_{k1} \in C'_{j1}, s_{k2} \in C'_{j2}$ and $C'_{j1} \neq C'_{j2}$. Otherwise, it is not hard to argue C = C'. Next, since s_{k1} and s_{k2} are comparable, C'_{j1} and C'_{j2} are comparable. Thus, $C^* = C'_{j1} \bigcup C'_{j2}$ is a chain of P.

We claim $(\mathcal{C}' \setminus \{C'_{j1}, C'_{j2}\}) \bigcup \{C^*\}$ is an HCD of P. For any $j \notin \{j1, j2\}, C'_j$ is either comparable to C'_{j1} or not comparable to C'_{j1} . For the former case, there exists $s_j \in C'_j$ comparable to s_{k1} . Since s_{k1} and s_{k2} come from the same chain C_k , regardless of whether $s_j \in C_k, s_{k2}$ must be comparable to s_j as well. Hence, C'_j is also comparable to C'_{j2} so that C'_j is comparable to C^* . For the latter case, we can analogously show C'_j is not comparable to C^* . Thus, the claim holds. However, this contradicts the assumption that \mathcal{C}' is minimum. Hence, \mathcal{C} is the unique MHCD of P.

We denote by G_P the C-graph corresponding to the MHCD of P hereafter. The function Min(P) (and many other functions) on a poset P satisfies the Lipschitz property, i.e., for any element $z \in P$, $|Min(P) - Min(P \setminus \{z\})| \leq 1$. However, $Min_h(P)$ does not necessarily share this property. In fact, we have

Theorem 3.1. Let P be a poset and $z \in P$. Then we have the sharp bounds

$$Min_h(P \setminus \{z\}) \le Min_h(P) \le 2Min_h(P \setminus \{z\}) + 1.$$
(2)

Proof. Let $C = \{C_1, C_2, \ldots, C_k\}$ be the MHCD of P. Deleting z from the chain C_i containing it, we obtain an HCD $C' = \{C_1, \ldots, C_i \setminus \{z\}, \ldots, C_k\}$ of $P \setminus \{z\}$. Thus, $Min_h(P \setminus \{z\}) \leq Min_h(P)$. It is clear that if there is no edge in G_C and $|C_j| > 1$ for all $1 \leq j \leq k$, C' will be the MHCD of $P \setminus \{z\}$ whence the equality can be achieved.

For the second inequality, let \mathcal{C}' and \mathcal{C} be the MHCDs of $P \setminus \{z\}$ and P, respectively. We first claim that

Claim. The elements of each chain in \mathcal{C}' can be contained in at most two chains in \mathcal{C} . If otherwise, there exists a chain C'_i in \mathcal{C}' whose elements are contained in at least three different chains C_1 , C_2 , C_3 in \mathcal{C} . By the pigeonhole principle, there are at least two of them either comparable to z or not. W.l.o.g., suppose C_1 , C_2 are both comparable to z. Then C_1 , C_2 are both comparable to the chain containing z. Now, for any chain $C \in \mathcal{C}$ such that $z \notin C$, if C_1 is comparable to C, so is C_2 , since C_1 and C_2 are contained in the same chain C'_i , and vice versa. Thus, C_1 and C_2 can be combined in \mathcal{C} will induce a new HCD with less number of chains by merging C_1 and C_2 , which contradicts the assumption that \mathcal{C} is minimum. Hence, the claim is affirmed.

Note that z may possibly form a chain itself in \mathcal{C} . Therefore, we have

$$Min_h(P) \le 2Min_h(P \setminus \{z\}) + 1.$$

As far as the sharpness of the bound, consider the case: suppose $C' = \{C'_1, \ldots, C'_k\}$ with $|C'_i| = 2$ for $1 \le i \le k$, and there is no edge in $G_{P \setminus \{z\}}$. Assume for each $1 \le i \le k$, z is larger than the minimum element in C_i but not comparable to the maximum element in C_i . See Figure 2 for an illustration. Then, it is not difficult to check that each chain in the MHCD of P contains only one element. Thus, $Min_h(P) = 2k + 1 = 2Min_h(P \setminus \{z\}) + 1$. This completes the proof.

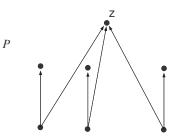


Figure 2: An example achieves the upper bound of $Min_h(P)$.

We proceed to prove a result about poset automorphisms. See the papers [2, 3, 11, 12] and references therein for previous studies. The following notation will be assumed through the end of the section. Let $\operatorname{Aut}(P)$ denote the automorphism group of P, and let $\operatorname{Aut}(G_P)$ denote the automorphism group of the graph G_P . Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be the MHCD of P. Without loss of generality, assume the chains in the set $S_i = \{C_{k_i+1}, C_{k_i+2}, \ldots, C_{k_{i+1}}\}$ have the same length l_i for $0 \leq i \leq t-1$, and $l_i \neq l_j$ if $i \neq j$, where $k_0 = 0, t \geq 1, \sum_{i=1}^t k_i = m$ and $\sum_{i=1}^t l_i k_i = |P|$. Let O_P be the group generated by all permutations π on $V(G_P)$ such that every cycle of π contains elements from the same set S_i for some $0 \leq i \leq t-1$.

Theorem 3.2. There exists an acyclic orientation $\operatorname{Asyc}(G_P)$ of G_P such that the automorphism group $\operatorname{Aut}(P)$ is isomorphic to a subgroup of the group

$$\operatorname{Aut}(\operatorname{Asyc}(G_P)) \bigcap O_P.$$

Proof. Note that both $\operatorname{Aut}(\operatorname{Asyc}(G_P))$ and O_P are groups of permutations on $V(G_p)$. Thus $\operatorname{Aut}(\operatorname{Asyc}(G_P)) \bigcap O_P$ is indeed a group. We first show that $\operatorname{Aut}(P)$ is isomorphic to a subgroup of O_P . Claim 1. Under any $g \in \operatorname{Aut}(P)$, two *P*-elements in the same chain C_i of the MHCD must be mapped to two *P*-elements in the same chain C_j of the MHCD with $|C_i| = |C_j|$. It is easy to see that any automorphism g will map a chain to a chain, thus a chain decomposition to a chain decomposition with the same number of chains. The resulting chain decomposition \mathcal{C}' induced by g acting on the MHCD must be an HCD. This is verified as follows. Let x'_i be an element from a chain C'_i of \mathcal{C}' and x'_j be an element from a chain C'_j of \mathcal{C}' . Suppose x'_i and x'_j are comparable. Note that $g^{-1} \in \operatorname{Aut}(P)$. Then their respective preimages $x_i = g^{-1}(x'_i)$ and $x_j = g^{-1}(x'_j)$ must be comparable as well. Let y'_j be any other element from C'_j and let $y_j = g^{-1}(y'_j)$. Note that by construction y_j is contained in the same chain as x_j . So x_i and y_j are comparable. Thus their images under g, x'_i and y'_j must be comparable too, whence C'_i and C'_j are comparable. Similarly, if x'_i and x'_j are incomparable, C'_i and C'_j are incomparable. Therefore, \mathcal{C}' is an HCD. By the uniqueness of the MHCD, each chain in the MHCD must be uniquely and exclusively mapped to a chain in the MHCD under g, whence Claim 1.

Following Claim 1, any chain in S_i will be mapped to a chain in S_i . Furthermore, the maximum element of a chain there must be mapped to the maximum element of a chain, and for each chain C_i of the MHCD, once the image of the maximum element in the chain under g is determined, then g is completely determined. Thus, g uniquely induces an element in O_P . For $g, g' \in \operatorname{Aut}(P)$, it is not hard to see that the induced element of $g \circ g'$ is also in O_P . Therefore, $\operatorname{Aut}(P)$ is isomorphic to a subgroup of O_P .

Let $C = \{C_1, C_2, \ldots, C_n\}$ be the MHCD of a poset P. Then, (C, \leq_h) is obviously a well-defined poset, where the relation \leq_h is defined as follows: $C_i \leq_h C_i$, and for $i \neq j$, $C_i \leq_h C_j$ iff $\min(C_i) < \min(C_j)$. Let $\operatorname{Asyc}(G_P)$ be the orientation induced by the poset (C, \leq_h) , i.e., the edge between C_i and C_j in G_P is oriented from C_i to C_j if $C_i \leq_h C_j$. We next show

Claim 2. $\operatorname{Aut}(P)$ is isomorphic to a subgroup of $\operatorname{Aut}(\operatorname{Asyc}(G_P))$.

To prove this, we show that each $g \in \operatorname{Aut}(P)$ uniquely induces an automorphism $\tilde{g} \in \operatorname{Aut}(\operatorname{Asyc}(G_P))$. From Claim 1, we know that g maps chain to chain so that it induces a bijection \tilde{g} on $V(G_P)$. It suffices to verify that \tilde{g} preserves directed edges in $\operatorname{Asyc}(G_P)$. Given a directed edge $C_i \to C_j$, by construction $\min(C_i) < \min(C_j)$. Thus, $g(\min(C_i)) < g(\min(C_j))$, and there is an edge between the chain C'_i containing $g(\min(C_i))$ and the chain C'_j containing $\min(C_j)$. By construction of the orientation, C'_i is directed to C'_j . Note under \tilde{g} , C_i and C_j will be mapped to C'_i and C'_j respectively. Obviously, \tilde{g} maps non-adjacent pairs to non-adjacent pairs. Hence, $\tilde{g} \in \operatorname{Aut}(\operatorname{Asyc}(G_P))$, whence the claim. Finally, for g, $g' \in \operatorname{Aut}(P)$, it is easy to check that $g \circ g'$ induces $\tilde{g} \circ \tilde{g'} \in \operatorname{Aut}(\operatorname{Asyc}(G_P))$. Therefore, Claim 2 holds, and the theorem follows.

Obviously $\operatorname{Aut}(\operatorname{Asyc}(G_P)) \subset \operatorname{Aut}(G_P)$, then the following corollary holds:

Corollary 3.1. The automorphism group Aut(P) is isomorphic to a subgroup of the group

$$\operatorname{Aut}(G_P) \bigcap O_P$$

4 Minimal NcCDs

An NcCD containing the minimum number $Min_{nc}(P)$ of chains is called a minimal NcCD of the poset P. There is no reason to expect the uniqueness of a minimal NcCD. Nevertheless, we shall prove some upper bounds for $Min_{nc}(P)$.

Another notion that we need is 132-avoiding permutations of a poset P and 132avoiding permutations with respect to a linear extension of P. A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ of the elements of P is said to be 132-avoiding if no three-element subsequence $\pi_{i_1} \pi_{i_2} \pi_{i_3}$ in π satisfies $i_1 < i_2 < i_3$ while $\pi_{i_1} < \pi_{i_3} < \pi_{i_2}$ in P. In the case of P = [n], it reduces to a conventional 132-avoiding permutation which is ubiquitous in combinatorics and computer science.

A linear extension of P is a permutation $e = e_1 e_2 \cdots e_n$ of P-elements such that $e_i < e_j$ implies i < j. There are more than one linear extension unless $P \sim [n]$. A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ of the elements of P is called 132^e -avoiding if there does not exist a subsequence $\pi_{i_1} \pi_{i_2} \pi_{i_3} = e_{j_1} e_{j_2} e_{j_3}$ such that $i_1 < i_2 < i_3$ and $j_1 < j_3 < j_2$. It is easily seen that a 132^e -avoiding permutation is a 132-avoiding permutation.

Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ of P, i is called a p-descent of π if $\pi_i > \pi_{i+1}$ or π_i is not comparable with π_{i+1} in P or i = n. The number of p-descents in π is denoted by $d_P(\pi)$. Let

$$Min_d(P) = \min\{d_P(\pi) : \pi \text{ is a } 132\text{-avoiding permutation of } P\},\$$

$$Min_d^e(P) = \min\{d_P(\pi) : \pi \text{ is a } 132^e \text{-avoiding permutation of } P\}.$$

Our main theorem in this section is the following quantitative relations.

Theorem 4.1. For any poset P, there exists a linear extension e of P such that

$$Min(P) \le Min_{nc}(P) \le Min_d(P) \le Min_d^e(P) \le Min_h(P).$$
(3)

We remark that the rightmost inequality is not necessarily true for an arbitrary linear extension and the leftmost inequality is trivial. A proof of the above theorem follows from a series of properties that we are about to present.

Proposition 4.1. Let π be a 132-avoiding permutation of a poset *P*. Then, the segments induced by the *p*-descents of π as sets give an NcCD of *P*.

Proof. Assume $\pi = \pi_1 \pi_2 \cdots \pi_n$. Read π from left to right and collect these elements between two consecutive p-descents, excluding the first one and including the second. By definition of p-descents, these elements comprise a chain. We next show these chains give an NcCD. If not, without loss of generality, suppose the first chain $\pi_1\pi_2$ and the second chain $\pi_3\pi_4$ cross, i.e., $\pi_1 < \pi_3 < \pi_2 < \pi_4$ or $\pi_3 < \pi_1 < \pi_4 < \pi_2$. Obviously, either case implies a 132 pattern in π , a contradiction whence the proposition.

As a result, we immediately have $Min_{nc}(P) \leq Min_d(P) \leq Min_d^e(P)$ for any linear extension e of P. If otherwise explicitly stated, we assume the following notation in the rest of the section. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be the MHCD of P, where

$$C_i = (s_{i1} < s_{i2} < \dots < s_{im_i}), \quad \sum_{i=1}^k m_i = n.$$

Lemma 4.1. If C_i and C_j are comparable, then there exists $0 \le l \le m_i$ such that

$$s_{i1} < s_{i2} < \dots < s_{il} < s_{j1} < s_{j2} < \dots < s_{jm_j} < s_{i(l+1)} < s_{i(l+2)} < \dots < s_{im_i}.$$

Proof. In order to prove the lemma, it suffices to show that there does not exist $0 < l_1 < m_i$ and $0 < l_2 < m_j$ such that

$$s_{il_2} < s_{jl_1} < s_{i(l_2+1)} < s_{j(l_1+1)}.$$

Assume by contradiction that such l_1 and l_2 exist. For any other chain C_k , if C_k is comparable to C_i and $x \in C_k$, then either $x < s_{i(l_2+1)}$ or $x > s_{i(l_2+1)}$. In any case, we conclude that an element in C_j is comparable to x whence C_j and C_k are comparable. By similar analysis, we can conclude that if C_k is not comparable to C_i , then C_k is not comparable to C_j either. Therefore, $(\mathcal{C} \setminus \{C_i, C_j\}) \bigcup \{C_i \cup C_j\}$ is an HCD of P. This contradicts the assumption that \mathcal{C} is the minimum and the lemma follows.

Consider the relation \leq_b on \mathcal{C} that $C_i \leq_b C_j$ if there exist elements $x, z \in C_j$ and $y \in C_i$ such that x < y < z or $\min(C_i) > \max(C_j)$. As for the first case, we say C_j wrap around C_i or C_i can be wrapped around by C_j . In view of Lemma 4.1, we leave it to the reader to verify that (\mathcal{C}, \leq_b) is a well-defined poset.

Proposition 4.2. Let $C_1C_2 \cdots C_k$ be a linear extension of (\mathcal{C}, \leq_b) . Then the following permutation π is 132-avoiding and has k p-descents:

$$\pi = s_{11}s_{12}\cdots s_{1m_1}s_{21}\cdots s_{2m_2}\cdots s_{k1}\cdots s_{km_k}.$$

Proof. By definition, it is easy to see there are exactly k p-descents in π . We prove the rest by contradiction. Suppose $\pi_{l_1}\pi_{l_2}\pi_{l_3}$ is a 132 pattern in π . Since each C_i appears as an increasing chain in π , we have only two possible cases:

- $\pi_{l_1}, \pi_{l_2} \in C_i, \pi_{l_3} \in C_j$, and i < j;
- $\pi_{l_1} \in C_i, \pi_{l_2} \in C_j, \pi_{l_3} \in C_k$, and i < j < k.

The first case cannot happen because the condition implies that $C_j <_b C_i$ in the light of Lemma 4.1, contradicting the assumption of the proposition. Next suppose the second case occurs. First, $\pi_{l_1} < \pi_{l_2}$ and $C_i <_b C_j$ imply that $\pi_{l_1} > x_j$ for some $x_j \in C_j$, i.e., C_j wrap around C_i . Analogously, C_k wrap around C_i . Secondly, $\pi_{l_2} > \pi_{l_3}$ and $C_j <_b C_k$ imply that either min $(C_j) > \max(C_k)$ or C_k wrap around C_j . Since both C_j and C_k can wrap around C_i , the former is absurd. On the other hand, that C_k wrap around C_j while C_j wrap around C_i makes it impossible to have a 132 pattern $\pi_{l_1}\pi_{l_2}\pi_{l_3}$ such that $\pi_{l_1} \in C_i, \pi_{l_2} \in C_j, \pi_{l_3} \in C_k$. Hence, no 132 patterns exist in π , completing the proof. \Box

From Lemma 4.1 and Proposition 4.2, we conclude $Min_{nc}(P) \leq Min_d(P) \leq Min_h(P)$. But we cannot conclude $Min_d^e(P) \leq Min_h(P)$ for a linear extension e. We proceed with further analysis below, where on the way we need to make use of plane trees. A plane tree T can be recursively defined as an unlabeled tree with one distinguished vertex called the root of T, where the unlabeled trees obtained by deleting the root as well as its adjacent edges from T are linearly ordered, and they are plane trees with the vertices adjacent to the root of T in T as their respective roots. These subtrees are pictured as locating below the root and appearing left to right. A non-root vertex without any child is called a leaf, and an internal vertex otherwise. A labelled plane tree is a plane tree where vertices carry mutually distinct labels from a certain set of labels. The preorder of the vertices in a labelled plane tree T is the sequence obtained by travelling T in a left-to-right depth-first manner and recording the label of a vertex when it is first visited. See Figure 3 for an example.

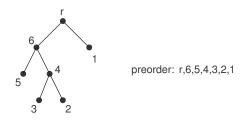


Figure 3: A labelled plane tree and the preorder of its vertices.

Proposition 4.3. Suppose $C_1C_2 \cdots C_k$ is a linear extension of the poset (\mathcal{C}, \leq_b) . Then, there exists a labelled plane tree T with non-root vertex labels from P such that π in Proposition 4.2 is 132^e -avoiding, where e is the reverse of the preorder of the vertices other than the root of T.

Proof. We actually build a sequence of trees T_1, T_2, \ldots, T_k so that T_k is the desired tree T. The tree T_1 is obtained by attaching the chain C_k (of vertices) to a vertex r such that the vertices reading from r along the path are $r, s_{km_k}, s_{k(m_k-1)}, \ldots, s_{k1}$. The vertex r will be the root of T_i 's. Next, we build T_2 from T_1 by attaching C_{k-1} to a certain vertex in T_1 from left: in the path from the leftmost leaf in T_1 to r, find the minimal vertex (excluding r) that is larger than $s_{(k-1)m_{k-1}}$ and attach the chain C_{k-1} of vertices to the found vertex (so that the found vertex and $s_{(k-1)m_{k-1}}$ are adjacent); if no such a vertex exists, then attach the chain C_{k-1} to r. The process of obtaining T_{i+1} from T_i is analogous, and we eventually obtain a tree $T_k = T$ with vertices other than r from P.

Denote the reverse of the preorder of the vertices other than the root of T by e. The consequence that π is a 132^e -avoiding permutation is implied in the Jani-Rieper bijection between plane trees and conventional 132-avoiding permutations. The reader is referred to [9] and [4] for discussion. This completes the proof.

While it may be hard to generate all 132-avoiding permutations of P, it is easy to generate all 132^e -avoiding permutations for any linear extension e of P as it is essentially generating all plane trees. It remains to prove that there exists a linear extension of (\mathcal{C}, \leq_b) of which the corresponding e is in fact a linear extension of P. We need one more lemma to that end.

Lemma 4.2. Suppose C_1, C_2, \ldots, C_m are the maximal elements of the poset (\mathcal{C}, \leq_b) . Then, any C_j for $m + 1 \leq j \leq k$ satisfies either one of the cases:

- (1) for at least one t $(1 \le t \le m)$, $\min(C_i) > \max(C_t)$;
- (2) for a unique t $(1 \le t \le m)$, C_t wrap around C_j .

In addition, two case (2) elements wrapped around by distinct maximal elements are not comparable, while a case (1) element is smaller than a case (2) element if comparable and the minimal (P-element) of the former is greater than the maximal of the latter.

Proof. For any $m + 1 \leq j \leq k$, it is clear that C_j is smaller than at least one maximal element. We first show that C_j cannot satisfy both cases. Suppose $\min(C_j) > \max(C_t)$ for some $1 \leq t \leq m$. If C_j can be wrapped around by another maximal element $C_{t'}$, then it is easy to see that C_t and $C_{t'}$ are comparable, a contradiction. Analogously, an element satisfy (2) cannot satisfy (1) at the same time. Moreover, an element cannot be wrapped around by more than one maximal elements.

If two case (2) elements wrapped around by distinct maximal elements are comparable, either the minimal P-element of one is greater than the maximal P-element of the other or one wrap around the other. Either case implies the two involved distinct maximal elements are comparable, contradicting the maximality. The remaining statement can be similarly verified, completing the proof.

Proposition 4.4. There exists a linear extension of (C, \leq_b) , still denoted by $C_1C_2 \cdots C_k$, of which the corresponding e refered to in Proposition 4.3 is a linear extension of P.

Proof. We first construct a linear extension of (\mathcal{C}, \leq_b) and then argue the corresponding e is a linear extension of P.

At the beginning, we arrange the maximal elements of (\mathcal{C}, \leq_b) in an arbitrary way. Next, we put each of those case (2) elements right before the maximal \mathcal{C} -element that wrap around it (and after the preceeding maximal element) and order those right before the same maximal element later. As for those case (1) elements, we arrange their maximal elements in front of the current partial sequence in an arbitrary way. Note that with respect to these case (1) maximal elements, other case (1) elements are either case (1)or case (2) which will be processed analogously. Iterate this procedure until no elements are in case (1) with respect to the most newly generated case (1) maximal elements. At this point, all elements of (\mathcal{C}, \leq_b) are grouped into disjoint ordered groups. The involved maximal \mathcal{C} -elements (w.r.t. a certain iteration) serve as a kind of group markers. The groups obtained so far will be referred to as type I groups. In view of Lemma 4.2, any \mathcal{C} -element in a left group is smaller than any \mathcal{C} -element in a right group if comparable, not violating the current sequence to possibly become a linear extension of (\mathcal{C}, \leq_b) . For each group of "roughly" placed elements right before a maximal element (w.r.t. a certain iteration), we do the same iteration to determine their relative order. It is kind of successive "refinement" until each non-empty "group" contains a single element. Eventually, we obtain a linear extension of (\mathcal{C}, \leq_b) .

Assume the resulting linear extension is $C_1C_2\cdots C_k$, and its corresponding tree is T. According to the construction of the linear extension $C_1C_2\cdots C_k$ and the tree T, P-elements contained in chains belonging to distinct type I groups (including respective

group markers) are contained in distinct subtrees of T. Thus, a P-element in a left subtree is greater than a *P*-element in a right subtree if comparable, leading to no contradiction for e being a linear extension of P. We next examine the stuctures of these subtrees. Note that all chains in the same type I group can be wrapped around by the corresponding group marker. As such, they naturally form a tree structure as follows. The group marker C_k is taken as an example. The corresponding subtree T_0 has s_{km_k} as its root, and the path from the rightmost leaf to the root of the subtree is exactly C_k . From the root s_{km_k} along the path down to s_{k1} , the first internal vertex having more than one child is the first place s_{kj} such that there are other chains that can be embedded between s_{kj_1} and $s_{k(j_1-1)}$, and the chains attaching to s_{kj_1} are those directly (i.e., adjacent in the Hasse diagram of (\mathcal{C}, \leq_b) wrapped around by C_k where the minimal P-element of a left chain is greater than the maximal of a right chain if comparable; Similarly, the second internal vertex having more than one child is the second place s_{kj_2} such that there are other chains that can be embedded between s_{kj_2} and $s_{k(j_2-1)}$, and so on. It is not difficult to observe that these chains attaching to s_{kj_t} will be group markers themselves, other branches will attach along these chains. As a consequence, we conclude that for any two subtrees incident to the same vertex in T_0 , the minimal P-element of the left subtree is greater than the maximal of the right subtree if comparable. It is also obvious that any vertex is greater than its descendants in T. These are enough for e being a linear extension of P, and the proof follows.

Now it is not hard to piece all properties above together to arrive at Theorem 4.1. We end this paper with some future study problems: (i) when can some of the equalities be achieved in Theorem 4.1, e.g., when $Min_{nc}(P) = Min_h(P)$? and (ii) following the notation of Theorem 3.2, consider the sufficient and/or necessary conditions such that

$$\operatorname{Aut}(P) \sim \operatorname{Aut}(\operatorname{Asyc}(G_P)) \bigcap O_P.$$

Declarations

Competing interests: None.

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