TILINGS OF DAMAGED HEXAGONS

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Contents

1. Introduction

In a recent paper [3], Byun presented nice formulas for the enumeration of lozenge tilings of certain hexagonal regions with "intrusions".

This paper attempts to generalise some of Byun's investigations. It is organised as follows:

- In section 2, we present the background needed for the considerations in this paper. This material is well–known to the expert, and the non–expert can easily conceive it from illustrations: Hence, in most cases we shall avoid lengthy formal definitions and present illustrative pictures instead.
- In section 3, we explain the bijection between lozenge tilings and nonintersecting lattice paths and recall
  - the Lindström–Gessel–Viennot method for counting nonintersecting lattice paths

– and Dodgson's condensation formula.

- In section 4, we apply these considerations to a generalisation of Byun's investigations, present solutions to (simple) special cases and formulate a conjecture for the general case. Moreover, by straightforward matrix manipulation we rewrite a simple special case as a summation formula.

Parts of the considerations involve lengthy manipulations of rational functions and polynomials: The software *Mathematica* and the Python–library *sympy* was used to help with such manipulations. Moreover, Zeilberger's algorithm [16] (which not only gives the result, but also an "automated proof") was employed; in the implementation of Paule, Schorn and Riese [14] .

2. Damaged hexagons and Byun's formulas

2.1. **Hexagons with intrusions in the triangular lattice.** In the *triangular lattice*, we consider (a, b, c)*–hexagons* with side lengths a, b, c, a, b, c (anti–clockwise, with $a, b, c \in \mathbb{N}$), see the upper left pictures in Figures 1 and 2.

We assume that the triangular lattice is drawn in a way that the hexagon's *baseline* of length a appears *horizontal*, and that an even number of *vertically* stacked triangles, adjacent to the baseline, is removed from the hexagon. Following Byun's wording, we call such stack of $2d$ removed triangles an *intrusion* of length d. Intrusions come in two flavours, namely

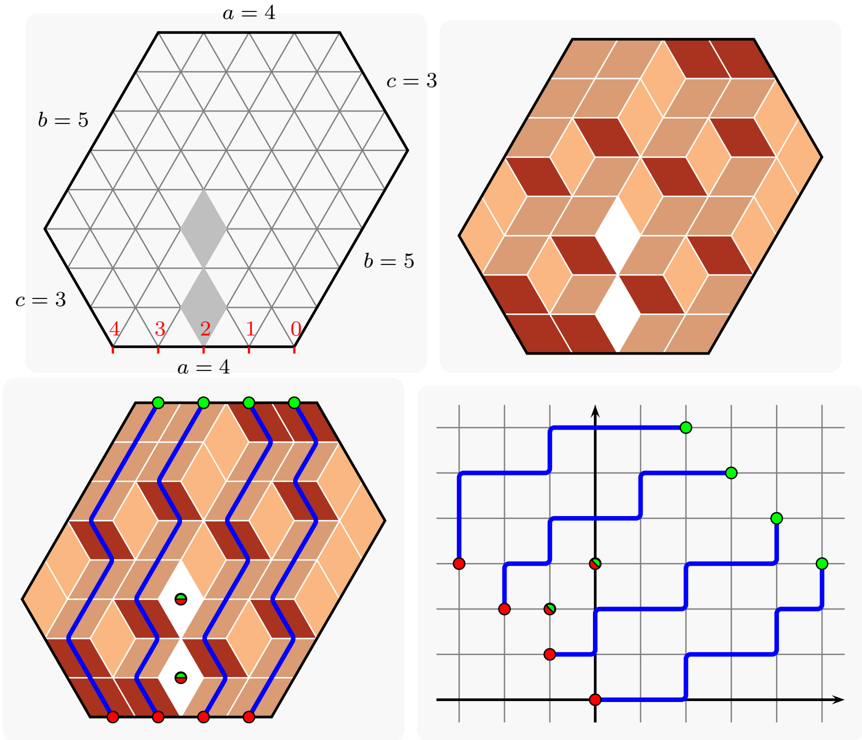
- starting with a triangle which has only a single vertex in common with the hexagon's baseline, see the upper left picture in Figure 1: We shall call this type an *even* intrusion;
- or starting with a triangle having an edge in common with the hexagon's baseline, see the upper left picture in Figure 2: We shall call this type an *odd* intrusion.

We count the horizontal *position* p of intrusions from right to left, starting with 0 for even intrusions and starting with 1 for odd intrusions, see the upper left pictures in Figures 1 and 2. We shall call such hexagon with an intrusion a *damaged hexagon*.

2.2. **Lozenge tilings and their enumeration.** A *lozenge* is a geometric shape in the triangular lattice which covers two triangles sharing a common edge.

A *lozenge tiling* of some (damaged) hexagon is a set of *pairwise disjoint* lozenges (in the sense that no two lozenges have a triangle in common) which together cover *all* triangles of the damaged hexagon, see the upper right pictures in Figures 1 and 2.

We denote by $\mathbf{e}(a,b,c,d,p)$ or $\mathbf{o}(a,b,c,d,p)$, respectively, the number of lozenge tilings of the damaged (a, b, c)–hexagon with an even or odd, respectively, intrusion of length d in position p.



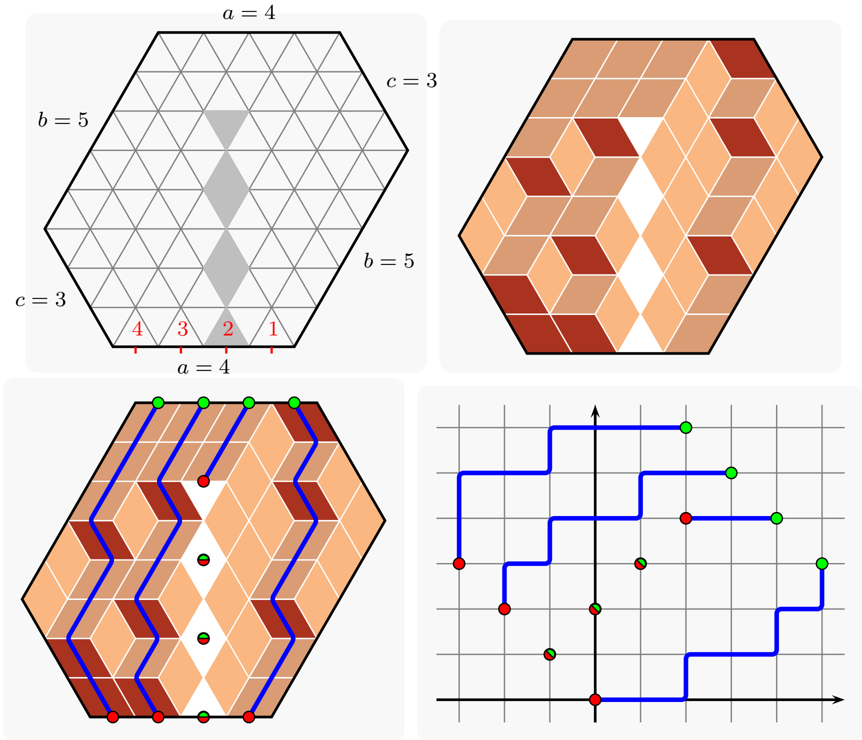
The upper left picture shows the hexagon with side lengths $(a, b, c) = (4, 5, 3)$ in the triangular lattice with an *even intrusion* of length $d = 2$ (marked as gray triangles) at position $p = 2$ (possible positions of intrusions are indicated by ticks at the base line of the hexagon). The upper right picture shows a *lozenge tiling* of this hexagonal region, and the lower left picture shows the *same* tiling together with the corresponding family of *nonintersecting lattice paths* (starting points of the paths are coloured red, and ending points are coloured green; points which are starting *and* ending points — these correspond to the "intrusion" — are coloured red *and* green): It is a well–known fact that this correspondence is a *bijection*. The lower right picture shows the family of nonintersecting lattice paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$: These are obtained by tilting the paths shown in the picture to the left and shifting them in the plane such that the lowest starting point coincides with the origin $(0, 0)$. Altogether, this gives $a = 4$ *lateral* starting points plus $d = 2$ *intrusive* starting points

$$\left(\underbrace{(0,0),(-1,1),(-2,2),(-3,3)}_{\text{lateral}},\underbrace{(-1,2),(0,3)}_{\text{intrusive}}\right),$$

and $a = 4$ *lateral* ending points plus $d = 2$ *intrusive* ending points

$$\left(\underbrace{(5,3),(4,4),(3,5),(2,6)}_{\text{lateral}},\underbrace{(-1,2),(0,3)}_{\text{intrusive}}\right).$$

FIGURE 1. The hexagon with side lengths $(a, b, c) = (4, 5, 3)$ and *even intrusion* of length $d = 2$ at position $p = 2$.



The upper left picture shows the hexagon with side lengths $(a, b, c) = (4, 5, 3)$ in the triangular lattice with an *odd intrusion* of length $d = 3$ (marked as gray triangles) at position 2 (possible positions of intrusions are indicated by ticks at the base line of the hexagon). The upper right picture shows a *lozenge tiling* of this hexagonal region, and the lower left picture shows the *same* tiling together with the corresponding family of *nonintersecting lattice paths* (starting points of the paths are coloured red, and ending points are coloured green; points which are starting *and* ending points — these correspond to the "intrusion" — are coloured red *and* green): It is a well–known fact that this correspondence is a *bijection*.
The lower right picture shows the family of nonintersecting lattice paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$: These are obtained by tilting the paths shown in the picture to the left and shifting them in the plane such that the lowest starting point coincides with the origin $(0, 0)$. Altogether, this gives $a = 4$ *lateral* starting points plus $d = 3$ *intrusive* starting points

$$\left(\underbrace{(0,0),(-1,1),(-2,2),(-3,3)}_{\text{lateral}},\underbrace{(0,2),(1,3),(2,4)}_{\text{intrusive}}\right),$$

and $a = 4$ *lateral* ending points plus $d = 3$ *intrusive* ending points

$$\left(\underbrace{(5,3),(4,4),(3,5),(2,6)}_{\text{lateral}},\underbrace{(-1,1),(0,2),(1,3)}_{\text{intrusive}}\right).$$

FIGURE 2. The hexagon with side lengths $(a, b, c) = (4, 5, 3)$ and *odd intrusion* of length $d = 3$ at position $p = 2$.

The *enumeration* of lozenge tilings of hexagonal regions in the triangular lattice often leads to interesting formulas, the most prominent of which is MacMahon's formula [12, § 429] giving the number of all lozenge tilings of the (a, b, c)–hexagon (without damage, i.e., with an intrusion of length 0). Denoting this number by $\mathbf{mm}(a,b,c)$, we have

$$\begin{aligned} \mathbf{mm}(a,b,c) = \mathbf{e}(a,b,c,0,p) &= \mathbf{o}(a,b,c,0,p) \\ &= \det\left[\binom{b+c}{b-i+j}\right]_{i,j=1}^{a} \\ &= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2} \\ &= \prod_{i=0}^{a-1}\frac{i!\,(b+c+i)!}{(b+i)!\,(c+i)!}. \end{aligned} \tag{1}$$

(The expression of this number as a determinant will become clear in section 3, and section 4.2.1 contains a short proof of MacMahon's formula.)

From now on, letters $a, b, c \in \mathbb{N}$ will always denote the side lengths of some hexagon, and letters $d \in \mathbb{N}$ and $p \in \mathbb{Z}$ will always denote the length and position of an intrusion.

2.3. **Byun's formulas.**

Byun found and proved nice formulas for the special cases

- $a = 2p$ for even intrusions [3, equation (2.1)],
- and $a = 2p + 1$ for odd intrusions [3, equation (2.2)].

In the notation just introduced, [3, equation (2.1)] is equivalent to

$$\mathbf{e}(2p,b,c,d,p) = \mathbf{mm}(2p,b,c)\cdot\prod_{k=1}^{d}4^{p}\frac{(1+b-k)_{p}\,(1+c-k)_{p}\left(-\frac{1}{2}+k\right)_{p}}{(2+b+c-2k)_{2p}\,(k)_{p}}, \tag{2}$$

and [3, equation(2.2)] reads

$$\mathbf{o}(2p{+}1,b,c,d,p) = \mathbf{mm}(2p{+}1,b,c)\cdot\frac{1}{4^{d}}\cdot$$
$$\prod_{k=0}^{d-1}\frac{(a+k+1)_{c-2k}\left(k+\frac{3}{2}\right)_{c-2k-2}(b-k)_{\left\lfloor\frac{c-b}{2}\right\rfloor}\left(c-k-\frac{1}{2}\right)_{-\left\lfloor\frac{c-b}{2}\right\rfloor}}{(k+1)_{c-2k-1}\left(a+k+\frac{3}{2}\right)_{c-2k-1}(a+b-k+1)_{\left\lfloor\frac{c-b}{2}\right\rfloor}\left(a+c-k+\frac{1}{2}\right)_{-\left\lfloor\frac{c-b}{2}\right\rfloor}}. \tag{3}$$

Byun's proofs of these formula involved results by Ciucu ([5, Theorem 3.1] and [4, Matching Factorisation Theorem]) and certain elegant recursions for the enumeration of perfect matchings (basically applications of Pfaffian identities to the Kasteleyn–Percus method [9, 15], for which Kuo [10] coined the name "graphical condensation").

3. Enumeration of lozenge tilings by determinants

3.1. **Bijection between lozenge tilings and nonintersecting lattice paths.** Lozenge tilings are in bijection with *nonintersecting lattice paths.* Instead of giving a formal description we point to the lower left picture in Figure 1: First, observe that a lozenge tilings of an (a, b, c)–hexagon with an *even* intrusion might be viewed as a "stack of cubes" fitting in a rectangular box with side lengths a, b, c, and that such "stack of cubes" is uniquely described by a family of lattice paths, where the intrusion of length d corresponds to d lattice paths of length 0. These lattice paths and their respective starting and ending points are indicated in the lower left picture of Figure 1 by blue lines and by red and green points, and it is easy to see that by tilting the picture, the paths appear in the integer lattice $\mathbb{Z} \times \mathbb{Z}$, with unit steps directed upwards and to the right (see the lower right picture in Figure 1).

Note that there are starting and ending points

- on the horizontal sides of the hexagon: We shall call these *lateral* points,
- and inside the intrusion's removed triangles: We shall call these *intrusive* points.

The situation is a little bit more complicated in the case of odd intrusion, since lozenge tilings do not correspond to a simple "stack of cubes" now: But it is easy to see that there is basically the same bijection with nonintersecting lattice paths, see Figure 2.

3.2. **Counting nonintersecting lattice paths with determinants.** Of course, we may shift the nonintersecting lattice paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ such that the *lowest lateral starting point* has coordinates $(0, 0)$: Then the coordinates of the *lateral* starting and ending points, counted from right to left, are the following:

- For the i–th lateral starting: $(1 - i, i - 1)$,
- and for the j–th lateral ending point: $(b + 1 - j, c + j - 1)$.

The coordinates of the *intrusive* starting and ending points are the following:

- for even intrusions, the i–th intrusive starting point coincides with the i–th intrusive ending point: $(-p + i, +i - 1)$,
- for p intrusions,
  - the i–th intrusive starting point: $(-p + i, p + i)$,
  - and the j–th intrusive ending point: $(-p + j - 1, p + j - 1)$.

(See again Figures 1 and 2.)

The well–known Lindström–Gessel–Viennot method [11, 7] counts the number of nonintersecting lattice paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ as a *determinant*, whose (i, j)–entry equals the number of lattice paths from the i–th starting point to the j–th starting point (under the assumption that all permutations π for which there actually *are* nonintersecting lattice paths from starting point i to ending point $\pi(i)$ have the same *positive* sign).

We shall consider the following order of starting and ending points of the nonintersecting lattice paths corresponding to lozenge tilings with even or odd intrusions:

- First, there come the *lateral* points, numbered from lower right to upper left,
- then, there come the *intrusive* points, numbered from lower left to upper right.

Note that there is precisely one permutation π admitting nonintersecting lattice paths running from starting point i to ending point $\pi(i)$: For even intrusions, this is simply the identity permutation, while for odd intrusions the corresponding permutation might have the negative sign.

Clearly, the number of all lattice paths in the integer lattice $\mathbb{Z}\times\mathbb{Z}$ starting in (x,y) and ending in (u,v), with unit steps to the right and upwards, is either zero or a binomial coefficient. By slight abuse of the standard notation, throughout this paper we adopt the convention

$$\binom{n}{k} \equiv 0 \text{ if } k<0 \text{ or } k>n$$

(i.e., $\binom{-1}{3}=0$, not -1) and set $n=(u-x)+(v-y)$ and $k=(u-x)$: Then this number of lattice paths is simply $\binom{n}{k}=\binom{u-x+v-y}{u-x}$.

Example 1. *For the damaged hexagon with parameters $(a,b,c,d,p)=(4,5,3,2,4)$ depicted in Figure 1, the determinant counting the nonintersecting lattice paths (and thus the lozenge tilings) is*

$$\det\left[\begin{array}{cccc|cc}
\binom{8}{5} & \binom{8}{4} & \binom{8}{3} & \binom{8}{2} & 0 & \binom{3}{0}\\
\binom{8}{6} & \binom{8}{5} & \binom{8}{4} & \binom{8}{3} & \binom{1}{0} & \binom{3}{1}\\
\binom{8}{7} & \binom{8}{6} & \binom{8}{5} & \binom{8}{4} & \binom{1}{1} & \binom{3}{2}\\
\binom{8}{8} & \binom{8}{7} & \binom{8}{6} & \binom{8}{5} & 0 & \binom{3}{3}\\
\hline
\binom{7}{6} & \binom{7}{5} & \binom{7}{4} & \binom{7}{3} & \binom{0}{0} & \binom{2}{1}\\
\binom{5}{5} & \binom{5}{4} & \binom{5}{3} & \binom{5}{2} & 0 & \binom{0}{0}
\end{array}\right]
=\det\left[\begin{array}{cccc|cc}
56 & 70 & 56 & 28 & 0 & 1\\
28 & 56 & 70 & 56 & 1 & 3\\
8 & 28 & 56 & 70 & 1 & 3\\
1 & 8 & 28 & 56 & 0 & 1\\
\hline
7 & 21 & 35 & 35 & 1 & 2\\
1 & 5 & 10 & 10 & 0 & 1
\end{array}\right]=12600.$$

Example 2. *For the damaged hexagon with parameters $(a,b,c,d,p)=(4,5,3,3,3)$ depicted in Figure 2, the determinant counting the nonintersecting lattice paths (and thus the lozenge tilings) is*

$$\det\left[\begin{array}{cccc|ccc}
\binom{8}{5} & \binom{8}{4} & \binom{8}{3} & \binom{8}{2} & 0 & \binom{2}{0} & \binom{4}{1}\\
\binom{8}{6} & \binom{8}{5} & \binom{8}{4} & \binom{8}{3} & \binom{0}{0} & \binom{2}{1} & \binom{4}{2}\\
\binom{8}{7} & \binom{8}{6} & \binom{8}{5} & \binom{8}{4} & 0 & \binom{2}{2} & \binom{4}{3}\\
\binom{8}{8} & \binom{8}{7} & \binom{8}{6} & \binom{8}{5} & 0 & 0 & \binom{4}{4}\\
\hline
\binom{6}{5} & \binom{6}{4} & \binom{6}{3} & \binom{6}{2} & 0 & \binom{0}{0} & \binom{2}{1}\\
\binom{4}{4} & \binom{4}{3} & \binom{4}{2} & \binom{4}{1} & 0 & 0 & \binom{0}{0}\\
0 & \binom{2}{2} & \binom{2}{1} & \binom{2}{0} & 0 & 0 & 0
\end{array}\right]
=\det\left[\begin{array}{cccc|ccc}
56 & 70 & 56 & 28 & 0 & 1 & 4\\
28 & 56 & 70 & 56 & 1 & 2 & 6\\
8 & 28 & 56 & 70 & 0 & 1 & 4\\
1 & 8 & 28 & 56 & 0 & 0 & 1\\
\hline
6 & 15 & 20 & 15 & 0 & 1 & 2\\
1 & 4 & 6 & 4 & 0 & 0 & 1\\
0 & 1 & 2 & 1 & 0 & 0 & 0
\end{array}\right]=-4032.$$

Note that this determinant is negative*: This does, of course, not mean that the number of lozenge tilings is negative, but that the permutation admitting nonintersecting lattice paths has the negative sign.*

3.3. **Symmetries and Dodgson's condensation formula.** Note that our definition of starting and ending points makes perfect sense also for intrusions in positions $p < 0$ or $p > a$ (but for *odd* intrusions, positions outside the range $[1, a]$ would give an endpoint which cannot be reached by *any* of the starting points, hence the number of nonintersecting lattice paths and the corresponding determinant is zero). From now on, we shall understand $\mathbf{e}(a,b,c,d,p)$ and $\mathbf{o}(a,b,c,d,p)$ as notations for the *determinants* described above (i.e., $p < 0$ or $p > a$ is now possible, and $\mathbf{o}(a,b,c,d,p)$ might give the *negative* of the number of corresponding lozenge tilings).

By reflecting the damaged hexagon at a vertical axis, we observe the following symmetries:

$$\mathbf{e}(a,b,c,d,p) = \mathbf{e}(a,c,b,d,a-p) \text{ and } \mathbf{o}(a,b,c,d,p) = \mathbf{o}(a,c,b,d,a-p+1). \tag{4}$$

Now recall *Dodgson's condensation formula* [6] (also known as Desnanot—Jacobi's Adjoint Matrix Theorem: According to [2], Lagrange discovered this Theorem for dimension $n = 3$, Desnanot proved it for dimensions $n \leq 6$, and Jacobi published the general theorem [8], see also [13, vol. I, pp. 142]): Let M be some $n \times n$ matrix. Consider row indices $1 \leq i_1 \neq i_2 \leq n$ and column indices $1 \leq j_1 \neq j_2 \leq n$, and denote by $M_{(r)|(c)}$ the matrix obtained from M by deleting rows and columns with indices in lists (r) and (c), respectively. Then there holds:

$$\det M \cdot \det M_{(i_1,i_2)|(j_1,j_2)} = \\ \det M_{(i_1)|(j_1)} \cdot \det M_{(i_2)|(j_2)} - \det M_{(i_1)|(j_2)} \cdot \det M_{(i_2)|(j_1)}. \tag{5}$$

Applying Dodgson's condensation formula (5) to the determinant $\mathbf{e}(a,b,c,d,p)$ for row and column indices $i_1 = j_1 = 1$ and $i_2 = j_2 = a$ gives the following functional equation

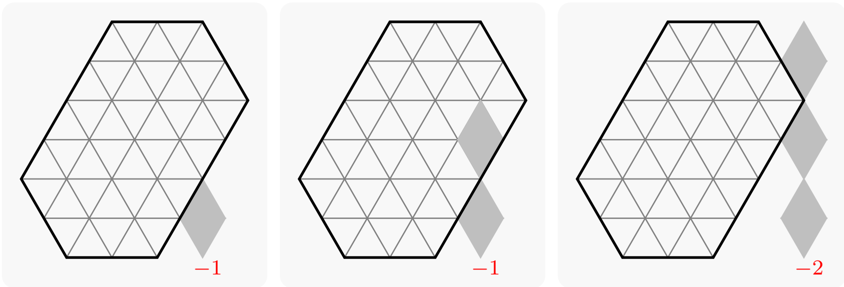
$$\mathbf{e}(a,b,c,d,p) \cdot \mathbf{e}(a-2,b,c,d,p-1) = \mathbf{e}(a-1,b,c,d,p-1) \cdot \mathbf{e}(a-1,b,c,d,p) \\ - \mathbf{e}(a-1,b+1,c-1,d,p-1) \cdot \mathbf{e}(a-1,b-1,c+1,d,p) \tag{6}$$

for all $d \in \mathbb{N}$, $p \in \mathbb{Z}$ and $a \geq 2$, and the analogous identity for $\mathbf{o}(a,b,c,d,p)$

$$\mathbf{o}(a,b,c,d,p) \cdot \mathbf{o}(a-2,b,c,d,p-1) = \mathbf{o}(a-1,b,c,d,p-1) \cdot \mathbf{o}(a-1,b,c,d,p) \\ - \mathbf{o}(a-1,b+1,c-1,d,p-1) \cdot \mathbf{o}(a-1,b-1,c+1,d,p). \tag{7}$$

By convention, empty determinants or products are equal to 1, which is perfectly in line with the fact that a (damaged) hexagon with side $a = 0$ has, in fact, precisely one lozenge tiling:

$$\mathbf{e}(0,b,c,d,p) \equiv 1.$$



All pictures show hexagons with side lengths $(a, b, c) = (2, 4, 2)$ and intrusions, not all of which actually cause a *damage* to the hexagon. The left picture shows the situation $d = 1$ and position $p = -1$, which illustrates the fact that intrusions with $d \le -p$ do not affect the number of tilings at all, and the same holds for $-p \ge \frac{b}{2}$, as is illustrated in the right picture (where $d = 3$ and $p = -2$). The central picture illustrates the case $-p < \min\left(d, \frac{b}{2}\right)$ (i.e., where the intrusion "actually causes damage". The special case $p = 1 - d$ of this situation is considered in Proposition 2.

FIGURE 3. Hexagons with side lengths $(a, b, c) = (2, 4, 2)$ and intrusions of length $d \le 3$ at positions $p < 0$.

4. EVEN INTRUSIONS

In the rest of this paper, we shall restrict our considerations to even intrusions.

It is clear that even intrusions at positions "too far away" from the hexagon's baseline are equivalent to "no intrusions at all" (as far as the counting of lozenge tilings or nonintersecting lattice paths is concerned). More precisely:

$$\mathbf{e}(a,b,c,d,p) = \mathbf{e}(a,b,c,0,p) = \mathbf{mm}(a,b,c)$$
$$\text{if } p \le \max\left(-d, -\left\lfloor \frac{b+1}{2} \right\rfloor\right) \text{ or } p \ge \min\left(a+d, a+\left\lfloor \frac{c+1}{2} \right\rfloor\right). \quad (8)$$

(See Figure 3 for an illustration.)

4.1. **Simple observations: Cancellations.** Loosely speaking, MacMahon's formula (1) for the (a, b, c)–hexagon is a product of quotients of factorials. Hence it is clear that quotients of instances of this formula will involve a lot of cancellations. For instance, by straightforward computation we obtain:

$$\frac{\mathbf{mm}(a,b,c)}{\mathbf{mm}(a-1,b,c)} = \frac{(a-1)!\,(a+b+c-1)!}{(a+b-1)!\,(a+c-1)!}, \tag{9}$$

$$\frac{\mathbf{mm}(a,b-1,c+1)}{\mathbf{mm}(a,b,c)} = \frac{c!\,(a+b-1)!}{(a+c)!\,(b-1)!}, \tag{10}$$

$$\frac{\mathbf{mm}(a,b-1,c+1)}{\mathbf{mm}(a-1,b,c)} = \frac{(a-1)!c!\,(a+b+c-1)!}{(a+c)!\,(a+c-1)!\,(b-1)!}. \tag{11}$$

Note that (10) and (11) give symmetric equations due to the obvious symmetry

$$\mathbf{mm}(a,b,c) = \mathbf{mm}(a,c,b).$$

4.2. **A very general *ansatz*.** We make the *ansatz*

$$\mathbf{e}(a,b,c,d,p) = \mathbf{mm}(a,b,c) \cdot \mathbf{F}(a,b,c,d,p). \tag{12}$$

Substituting ansatz (12) in the recursion (6) (derived from Dodgson's condensation formula (5)), we obtain the following functional equation by straightforward cancellations (see the examples of such cancellations in section 4.1):

$$\begin{aligned}(a-1)\cdot(a+b+c-1)\cdot\mathbf{F}(a-2,b,c,d,p-1)\cdot\mathbf{F}(a,b,c,d,p) = \\ (a+b-1)\cdot(a+c-1)\cdot\mathbf{F}(a-1,b,c,d,p-1)\cdot\mathbf{F}(a-1,b,c,d,p) - \\ b\cdot c\cdot\mathbf{F}(a-1,b-1,c+1,d,p)\cdot\mathbf{F}(a-1,b+1,c-1,d,p-1).\end{aligned} \tag{13}$$

For fixed d and p, this amounts to the following recursive description of $\mathbf{F}(a,b,c,d,p)$

$$\begin{aligned}\mathbf{F}(a,b,c,d,p) = \frac{1}{(a-1)\cdot(a+b+c-1)\cdot\mathbf{F}(a-2,b,c,d,p-1)} \\ \times\Big((a+b-1)\cdot(a+c-1)\cdot\mathbf{F}(a-1,b,c,d,p-1)\cdot\mathbf{F}(a-1,b,c,d,p) - \\ b\cdot c\cdot\mathbf{F}(a-1,b-1,c+1,d,p)\cdot\mathbf{F}(a-1,b+1,c-1,d,p-1)\Big).\end{aligned} \tag{14}$$

Together with the boundary values

- $\mathbf{F}(a,b,c,d,p-1)$ (for all a, b, c)
- $\mathbf{F}(0,b,c,d,p)$ and $\mathbf{F}(1,b,c,d,p)$ (for all b, c),

the recursion (14) *uniquely* determines $\mathbf{F}(a,b,c,d,p)$ for all a, b, c.

4.2.1. *First application: MacMahon's formula.* These simple observations provide a short proof of MacMahon's formula (1):

MacMahon's formula. For $d = 0$, we clearly have $\mathbf{F}(a,b,c,0,p) = \mathbf{F}(a,b,c,0,0)$ (i.e., the position p of an intrusion of length 0 is irrelevant), and

- $\mathbf{F}(0,b,c,0,0) = 1$ since $\mathbf{e}(0,b,c,0,0) = 1 = \mathbf{mm}(0,b,c)$
- and $\mathbf{F}(1,b,c,0,0) = 1$ since $\mathbf{e}(1,b,c,0,0) = \binom{b+c}{c} = \mathbf{mm}(1,b,c)$.

MacMahon's formula is equivalent to $\mathbf{F}(a,b,c,0,0) \equiv 1$, and if we want to prove this equation, we simply have to show that constant 1 is a solution of the functional equation (12). But this amounts to the simple identity

$$(a-1)\cdot(a+b+c-1) = (a+b-1)\cdot(a+c-1) - b\cdot c,$$

which is immediately verified. □

4.3. **A very general plan of action.** So our quest for a formula giving $\mathbf{e}(a,b,c,d,p)$ leads us to the following plan of action:

(1) Let $d > 0$ be fixed.
(2) Identify some p_0 for which $\mathbf{F}(a,b,c,d,p_0)$ can be derived easily.
(3) Observe that $\mathbf{F}(0,b,c,d,p) \equiv 1$, and find a formula giving $\mathbf{F}(1,b,c,d,p)$ for all $p \geq p_0$.
(4) Guess the formula giving $\mathbf{F}(a,b,c,d,p)$ and prove it by verifying that it satisfies the functional equation (14).

From the geometric situation one might suspect that formulae

- for $p \leq 0$
- and for $0 \leq p \leq a$

are of different quality (by the symmetry $\mathbf{e}(a,b,c,d,p) = \mathbf{e}(a,c,b,d,a-p)$, we may omit the case $p \geq a$): Indeed, we shall use a *specialized* ansatz for the first case, and a *modified* ansatz for the second case.

4.4. **A specialized ansatz for $p \leq 0$.** For fixed $d > 0$ and $p \leq 0$, we rewrite the function $\mathbf{F}(a,b,c,d,p)$ from (12) as follows:

$$\mathbf{F}(a,b,c,d,p) = \left(\prod_{k=0}^{d+p-1} \frac{(c-k)_{a-d-p+1+2k}}{(b+c-2d+2k+2)_{a+2d-2-3k}}\right)\cdot \mathbf{F}'(a,b,c,d,p) \quad (15)$$

Numerical experiments indicate that this specialized *ansatz* yields the factors $\mathbf{F}'(a,b,c,d,p)$ as *polynomials* for fixed d and p ($p \leq 0$).

Substituting (15) in (14), we obtain by straightforward cancellation the following functional equation for $\mathbf{F}'(a,b,c,d,p)$:

$$\begin{aligned}(a-1)\cdot \mathbf{F}'(a,b,c,d,p)\cdot \mathbf{F}'(a-2,b,c,d,-1+p) = \\ (a+b-1)\cdot \mathbf{F}'(a-1,b,c,d,-1+p)\cdot \mathbf{F}'(a-1,b,c,d,p) - \\ b\cdot \mathbf{F}'(a-1,-1+b,1+c,d,p)\cdot \mathbf{F}'(a-1,1+b,-1+c,d,-1+p).\end{aligned} \quad (16)$$

(Note that the coefficients of this functional equation do not contain the variable c.)

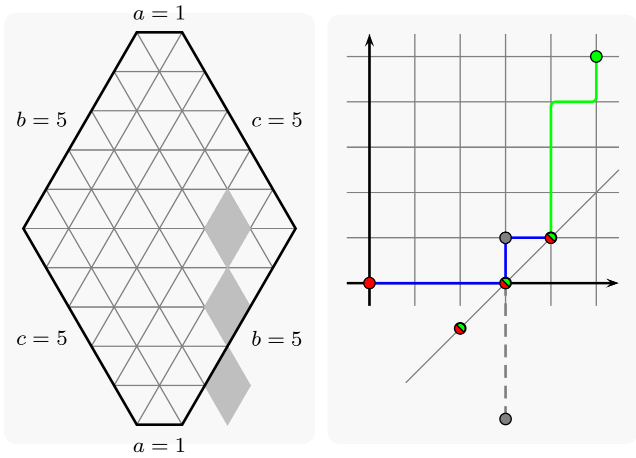
Now let $d > 0$ be arbitrary, but fixed. Observe that

$$\mathbf{F}(a,b,c,d,p) = \mathbf{F}'(a,b,c,d,p) \equiv 1 \text{ for all } p \leq -d$$

(since $\mathbf{e}(a,b,c,d,p) = \mathbf{mm}(a,b,c)$ for $p \leq -d$; see Figure 3 for an illustration): So we found our $p_0 = -d$ for which $\mathbf{F}(a,b,c,d,p_0)$ can be derived easily.

Morevover, we (trivially) have

$$\mathbf{e}(0,b,c,d,p) \equiv 1, \quad (17)$$



The left picture shows the hexagon with side lengths $(a, b, c) = (1, 5, 5)$ with an *even intrusion* of length $d = 3$ (marked as gray triangles) at position $p = -1$, and the right picture shows the starting and ending points of the nonintersecting lattice paths which correspond to lozenge tilings of this damaged hexagon. Note that the lattice paths *with* intersections are precisely those which

- run from starting point $(0, 0)$ to $(3, 0)$, and continue from there to ending point $(5, 5)$; the number of such lattice paths is
$$\binom{3}{3}\binom{7}{2},$$
- or run from starting point $(0, 0)$ to $(3, 1)$, *without* touching $(3, 0)$, then make a horizontal step to $(4, 1)$, and continue from there to ending point $(5, 5)$; by the *reflection principle* (the reflected path is shown with dashed lines), the number of such lattice paths is
$$\left(\binom{4}{3} - \binom{4}{0}\right)\binom{5}{1}.$$

So altogether, the number of nonintersecting lattice paths is
$$\binom{10}{5} - \binom{3}{3}\binom{7}{2} - \left(\binom{4}{3} - \binom{4}{0}\right)\binom{5}{1}.$$

FIGURE 4. The hexagon with side lengths $(a, b, c) = (1, 5, 5)$ and even intrusion of length $d = 3$ at position $p = -1$.

hence we have

$$\mathbf{F}'(0,b,c,d,p) = \left(\prod_{k=0}^{d+p-1} \frac{(b + c - 2d + 2k + 2)_{2d-2-3k}}{(c - k)_{-d-p+1+2k}}\right).$$

So all that is left to find is a formula which gives $\mathbf{F}'(1,b,c,d,p)$: By a straightforward application of the *reflection principle* [1] (see Figure 4 for an illustration) we obtain the following expression for $\mathbf{e}(1,b,c,d,p)$ for

$p \leq 0$ and $d \leq \frac{b+c+1}{2}$:

$$\binom{b+c}{b} - \sum_{i=0}^{d+p-1} \left(\binom{b+c-2(-p+i)-1}{b+2p-i-1} \left(\binom{2(-p+i)}{-2p+i} - \binom{2(-p+i)}{-1+i} \right) \right).$$

We may rewrite this as follows:

$$\binom{b+c}{b} - (1-2p) \sum_{i=0}^{d+p-1} \frac{(2i-2p)_{(i-1)} \binom{b+c-2i+2p-1}{b-i+2p-1}}{i!}. \tag{18}$$

This implies that $\mathbf{F'}(1,b,c,d,p)$ equals the right–hand side of (18), divided by $\binom{b+c}{b} = \mathbf{mm}(1,b,c)$ and by the product in (15).

In order to simplify notation, set

$$\mathbf{r}(a,i,x) = \mathbf{F'}(a,b+i,c-i,d,-d+x).$$

Clearly, the desired formula $\mathbf{F'}(a,b,c,d,p)$ is some function in a, b, c, $\mathbf{r}(0,i,x)$ and $\mathbf{r}(1,i,x)$: As already mentioned, numerical experiments indicate that it is, in fact, a *polynomial* for d and $p \leq 0$ fixed.

4.4.1. *Special case $p = 1 - d$ (or $x = 1$).* For $x = 1$ (equivalent to $p = 1 - d$) we claim

$$\mathbf{F'}(a,b,c,d,1-d) = \frac{\sum_{k=0}^{a-1} (-1)^{a+k-1} (-a+b+k+2)_{a-1} \binom{a-1}{k} \mathbf{r}(1,-a+k+1,1)}{(a-1)!} \tag{19}$$

for all $a, d > 0$.

First, note that formula (19) gives the correct result (namely $\mathbf{r}(1,0,1)$) for $a = 1$. In order to show its validity for $a > 1$, we must verify that it satisfies the functional equation (16), which simplifies to

$$\mathbf{F'}(a,b,c,d,-d+1) = \frac{(a+b-1) \cdot \mathbf{F'}(a-1,b,c,d,-d+1) - b \cdot \mathbf{F'}(a-1,b-1,c+1,d,-d+1)}{a-1} \tag{20}$$

since $\mathbf{F'}(a,b,c,d,-d) \equiv 1$. Now substitute (19) in (20), multiply by $(a-1)!$ and compare the coefficients of $\mathbf{r}(1,-a+k,1)$: On the right–hand side, this coefficient is

$$-b\,(-1)^{a+k-2} (-a+b+k+2)_{a-2} \binom{a-2}{k} + (a+b-1)\,(-1)^{a+k-3} (-a+b+k+2)_{a-2} \binom{a-2}{k-1}.$$

Collecting the terms with factor b and applying the recursion of binomial coefficients, we obtain

$$(-1)^{a+k-1} (-a+b+k+2)_{a-2} \left(b\binom{a-1}{k} + (a-1)\binom{a-2}{k-1} \right).$$

Now rewrite $(a-1)\binom{a-2}{k-1} = k\binom{a-1}{k}$ to arrive at

$$(-1)^{a+k-1}(b+k)(-a+b+k+2)_{a-2}\binom{a-1}{k} = \\ (-1)^{a+k-1}(-a+b+k+2)_{a-1}\binom{a-1}{k},$$

which is precisely the coefficient on the left–hand side: This proves that (19) does indeed satisfy the functional equation (16).

By combining (15), (1) and (18) we get

$$\mathbf{r}(1,i,1) = \frac{\left(\binom{b+c}{c} - \binom{b+c-2d+1}{c-i}\right)(b+c-2d+2)_{a+2d-2}}{\binom{b+c}{c}(c)_a}.$$

Inserting this in (19) leads to the following result:

Proposition 1. *Consider the damaged (a,b,c)–hexagon with an even intrusion of length $d > 0$ in position $p = 1-d$.*

For $d \geq \frac{b}{2}+1$, we have

$$\mathbf{e}(a,b,c,d,1-d) = \mathbf{mm}(a,b,c).$$

For $d < \frac{b}{2}+1$, we have

$$\mathbf{e}(a,b,c,d,1-d) = \mathbf{mm}(a,b,c)\cdot\Bigg(1 - \frac{(c)_a}{(a-1)!\,(b+c-2d+2)_{a+2d-2}}\times \\ \sum_{k=0}^{a-1}(-1)^{a+k-1}\binom{a-1}{k}\frac{(-a+b-2d+k+3)_{a+2d-2}}{a+c-k-1}\Bigg). \tag{21}$$

Proof. The first assertion is an immediate consequence of the fact that an intrusion in position $1-d$ does not inflict any actual "damage" to the hexagon if $d > \frac{b}{2}+1$, see Figure 3.

The second assertion follows from the above considerations, which immediately give

$$\mathbf{e}(a,b,c,d,1-d) = \frac{\mathbf{mm}(a,b,c)\,(c)_a}{(a-1)!\,(b+c-1)_{a-1}}\times \\ \sum_{k=0}^{a-1}(-1)^{a+k-1}\binom{a-1}{k}(-a+b+k+2)_{a-1}\left(\frac{\left(\binom{b+c}{a+c-k-1} - \binom{b+c-2d+1}{a+c-k-1}\right)}{\binom{b+c}{a+c-k-1}(a+c-k-1)}\right). \tag{22}$$

The sum in this expression is the difference of two sums, the simpler of which is

$$S_a \stackrel{\text{def}}{=} \sum_{k=0}^{a-1}(-1)^{a+k-1}\binom{a-1}{k}\frac{(-a+b+k+2)_{a-1}}{a+c-k-1}.$$

Zeilberger's algorithm [16, 14] readily gives the recursion

$$S_{a+1} = \frac{a\,(a+b+c)}{a+c} S_a,$$

from which we immediately obtain the following summation formula:

$$S_a = \frac{(a-1)!\,(c-1)!\,(a+b+c-1)!}{(a+c-1)!\,(b+c)!}$$

Use of this formula together with straightforward simplifications yields (21). ☐

4.4.2. *Very special case $x = 1$ and $d = 1$ (so $p = 0$).* The case $d = 1$, $p = 0$ is particularly simple: By the recursion for binomial coefficients and the identity $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$ (for $0 < k \leq n$), we have

$$\frac{\left(\binom{b+c}{a+c-k-1} - \binom{b+c-1}{a+c-k-1}\right)}{\binom{b+c}{a+c-k-1}\,(a+c-k-1)} = 1$$

in (22), whence the sum simplifies to

$$\sum_{k=0}^{a-1} (-1)^{a+k-1} (-a+b+k+2)_{a-1} \binom{a-1}{k} = (a-1)! \tag{23}$$

(Again, this summation formula is readily found by Zeilberger's algorithm [16, 14].)

Corollary 1. *Consider the damaged (a, b, c)–hexagon with an even intrusion of length $d = 1$ in position $p = 0$. Then we have*

$$\mathbf{e}(a,b,c,1,0) = \mathbf{mm}(a,b,c) \cdot \frac{(c)_a}{(b+c)_a} = \mathbf{mm}(a,b,c-1).$$

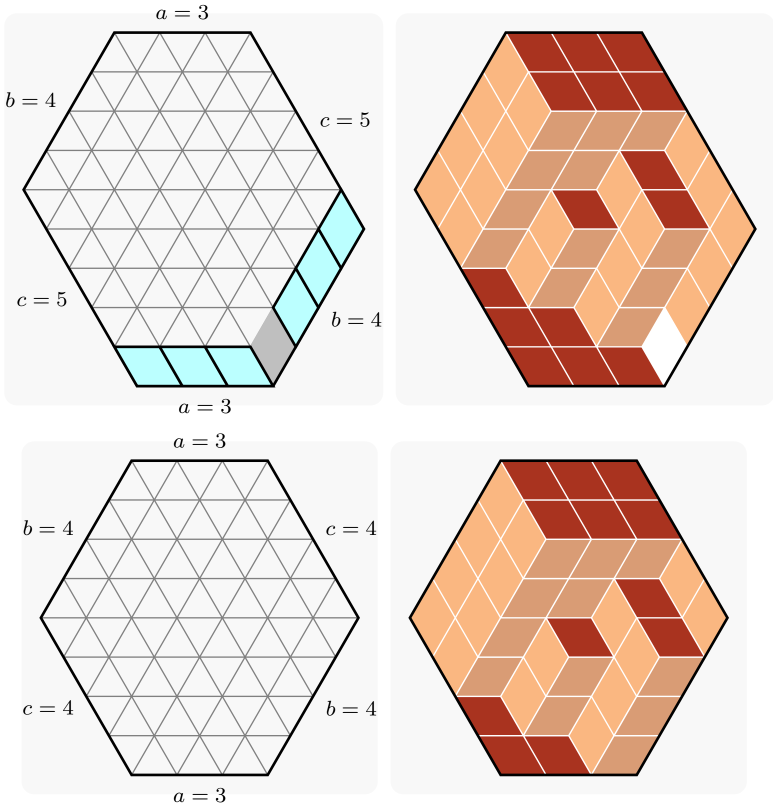
Proof. The assertion is a direct consequence of the above considerations. But there is a much simpler argument: A single intruding lozenge in position 0 implies that *all* lozenges at the baseline of the hexagon are *forced* (or, equivalently, that *all* lattice paths have to start with an *upwards* step), see Figure 5. Removing the forced lozenges gives an $(a, b, c-1)$–hexagon with *no* intrusion. ☐

4.4.3. *Special case $p = 1 - d$ (or $x = 1$), revisited.* We may choose another *ansatz*, which leads to a different formula:

Proposition 2. *Consider the damaged (a, b, c)–hexagon with an even intrusion of length $d > 0$ in position $p = 1 - d$.*

For $d \leq \left\lceil \frac{b}{2} \right\rceil$, we have the following formula:

$$\mathbf{e}(a,b,c,d,-d+1) = \mathbf{mm}(a,b,c) \times \left(1 - \frac{(b-2d+2)_c}{(2d-2)!\,(b+1)_{a+c-1}} \sum_{k=1}^{a} (b+c+k)_{a-k}\,(k)_{2d-2}\,(c)_{k-1}\right). \tag{24}$$

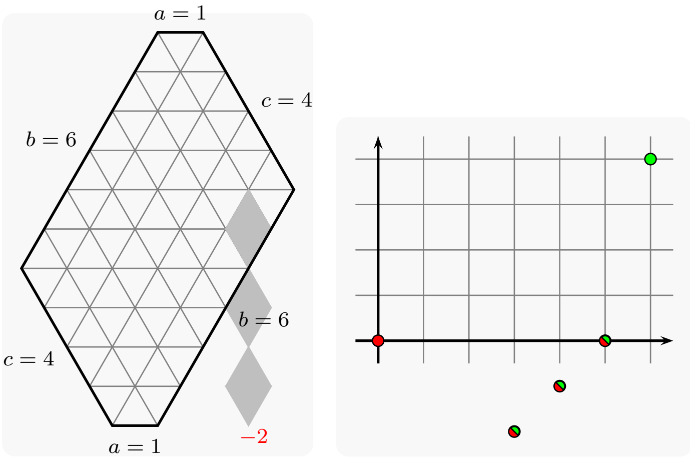


The upper left picture shows the hexagon with side lengths $(a, b, c) = (3, 4, 5)$ with an *even intrusion* of length $d = 1$ (marked as gray triangle) at position 0, and the upper right picture shows a *lozenge tiling* of this damaged hexagon. Note that the intrusion implies that certain lozenges *must* belong to *all* tilings of the damaged hexagon: These *forced* lozenges are drawn with blue colour in the upper left picture. But this means that tilings of the damaged of the upper left picture are in bijection with tilings of the (intact) hexagon with side lengths $(a, b, c) = (3, 4, 4)$ shown in the lower left picture (the lower right picture shows the tiling which is in bijection with the tiling from the upper right picture; the bijection simply "removes" the "forced lozenges").

FIGURE 5. The hexagon with side lengths $(a, b, c) = (3, 4, 5)$ and even intrusion of length $d = 1$ at position 0.

Alternatively, we have the following formula, valid for $b > d$

$$\mathbf{e}(a,b,c,d,1-d) = \frac{(c)_a \, \mathbf{mm}(a,b,c)}{(b+c-2d-2)_{a+2d-2}} \times \Bigg((b+c-2d+2)_{2d-2} - \sum_{k=2}^{d} \Big((b+c-2d+2)_{2d-2k} \, (b-2k+4)_{2k-3} \, (a)_{2k-3} \times \frac{-a(b-2k+3)+b(5-4k)-2ck+2c+8k^2-20k+13}{(2k-2)!} \Big) \Bigg), \qquad (25)$$



The left picture illustrates the situation of Proposition 2 for $a = 1$, with $d = 3$ and $p = -d + 1 = -2$. The right picture is the "translation" of this situation to the language of nonintersecting lattice paths: Observe that the lattice paths *with intersections* are *precisely* the ones

- which reach point $(2d - 1, 0) = (5, 0)$ by five horizontal steps from the origin $(0, 0)$ (and this is the *only* way to achieve this!),
- and then continue from $(5, 0)$ in an arbitrary way to the endpoint $(b, c) = (6, 4)$,

so the number of nonintersecting lattice paths in this situation is

$$\binom{b+c}{c} - \binom{b-2d+1+c}{c} = \binom{10}{4} - \binom{5}{4}.$$

FIGURE 6. Hexagon with side lengths $(a, b, c) = (1, 6, 4)$ and intrusion of length $d = 3$ at position $p = -2$.

which has the advantage that the expression after *the first line of* (25) *actually is a* polynomial *in a, b, c for fixed d.*

Proof. We make the *ansatz*

$$\mathbf{e}(a,b,c,d,1-d) = \mathbf{mm}(a,b,c) \cdot \left(1 - \frac{(b-2d+2)_c}{(2d-2)!\,(b+1)_{a+c-1}} \cdot f\,(a,b,c,d)\right). \tag{26}$$

Clearly, for proving (24) we have to show

$$f\,(a,b,c,d) = \sum_{k=1}^{a} (b+c+k)_{a-k}\,(k)_{2d-2}\,(c)_{k-1}\,. \tag{27}$$

We shall achieve this by induction on a.

For $a = 0$, we have $\mathbf{e}(0,b,c,d,1-d) = \mathbf{mm}(0,b,c) = 1$, and the sum in (27) is indeed zero.

For $a = 1$, it is easy to see that $\mathbf{e}(1,b,c,d,-d+1)$ is equal to the number of lattice paths starting in $(0, 0)$ and ending in (b, c) which *do not* pass through the lattice point $(2d-1,0)$ (see Figure 6). This number is

$$\binom{b+c}{c}-\binom{b+c-2d+1}{c},$$

which equals

$$\mathbf{mm}(1,b,c)\cdot\left(1-\frac{(b-2d+2)_c}{(b+1)_c}\right)=\frac{(b+1)_c}{c!}\cdot\left(1-\frac{(b-2d+2)_c}{(b+1)_c}\right).$$

From this we immediately obtain that (27) is true also for $a=1$:

$$\sum_{k=1}^{1}(b+c+k)_{1-k}\,(k)_{2d-2}\,(c)_{k-1}=(2d-2)!.$$

Since $\mathbf{e}(a,b,c,d,-d)=\mathbf{mm}(a,b,c)$, Dodgson's condensation (6) amounts to

$$\begin{aligned}\mathbf{e}(a,b,c,d,1-d)\cdot\mathbf{mm}(a-2,b,c)&=\mathbf{mm}(a-1,b,c)\cdot\mathbf{e}(a-1,b,c,d,1-d)\\&\quad-\mathbf{mm}(a-1,b+1,c-1)\cdot\mathbf{e}(a-1,b-1,c+1,d,1-d)\end{aligned}$$

for $a>1$, and substituting our *ansatz* (26) for $\mathbf{e}(a,b,c,d,1-d)$ gives (after straightforward cancellations) the following recursion (in a) for $f(a,b,c,d)$:

$$\begin{aligned}(a-1)\,f(a,b,c,d)=&\\&(a+b-1)\,(a+c-1)\,f(a-1,b,c,d)\\&-c\,(b-2d+1)\,f(a-1,b-1,c+1,d).\end{aligned}\tag{28}$$

So what is left to prove is that

$$\sum_{k=1}^{a}(b+c+k)_{a-k}\,(k)_{2d-2}\,(c)_{k-1}$$

actually obeys the recursion (28). Using the elementary identity

$$(a+b-1)\,(a+c-1)-b\,c=(a-1)\,(a+b+c-1)$$

we may rewrite (28) equivalently as

$$\begin{aligned}(a-1)\,f(a,b,c,d)=&\,(a-1)\,(a+b+c-1)\,f(a-1,b,c,d)\\&+b\,c\,(f(a-1,b,c,d)-f(a-1,b-1,c+1,d))\\&+c\,(2d-1)\,f(a-1,b-1,c+1,d).\end{aligned}\tag{29}$$

Now observe that

$$(a-1)(a+b+c-1)\,(b+c+k)_{a-k-1}\,(c)_{k-1}\,(k)_{-2+2d}$$

equals

$$(a-1)\,(b+c+k)_{a-k}\,(c)_{k-1}\,(k)_{2d-2}\,,$$

which is $(a-1)$ times the k–th summand of $f(a,b,c,d)$. Hence we need to show that the summand for $k=a$ in $(a-1)\,f(a,b,c,d)$,

$$(a-1)\,(c)_{a-1}\,(a)_{-2+2d}\,,\tag{30}$$

is equal to the last two summands of the right–hand side in (29), which can be simplified to

$$\sum_{k=1}^{a-1}\Bigg((b+c+k)_{a-k-1}\,(c)_{k-1}\,(k)_{2d-2}\\ \times\left(bc\left(1-\frac{c+k-1}{c}\right)+(2d-1)(c+k-1)\right)\Bigg). \tag{31}$$

Now Zeilberger's algorithm [16, 14] shows that (31) evaluates to (30) and thus concludes the proof of (24).

It is Zeilberger's algorithm [16, 14], again, which also gives a recursion for $f\,(a,b,c,d)$ in d, namely

$$\begin{gathered}(b-2d+2)(b-2d+3)f(a,b,c,d)=\\ 2(d-1)(2d-3)(b+c-2d+2)(b+c-2d+3)f(a,b,c,d-1)+\\ (a+c-1)(a+2d-4)(c)_{a-1}(a)_{2d-4}\times\\ \left(-a(b-2d+3)+b(5-4d)-2cd+2c+8d^2-20d+13\right),\end{gathered}$$

which by iteration leads to an alternative expression for f, valid for $b>d$:

$$f\,(a,b,c,d)=\frac{(2d-2)!}{(b-2d+2)_{2d-1}}\Bigg((b+c-2d+2)_{a+2d-2}\\ +(c)_a\sum_{k=1}^{d}\Bigg((b+c-2d+2)_{2d-2k}\,(b-2k+4)_{2k-3}\,(a)_{2k-3}\\ \frac{-a(b-2k+3)+b(5-4k)-2ck+2c+8k^2-20k+13}{(2k-2)!}\Bigg)\Bigg).$$

Inserting this alternative expression in our ansatz (26) gives (25) (after some straightforward cancellations and simplifications). ☐

4.5. **A modified ansatz for $0\le p\le a$.** For $0\le p\le a$, $a\ge 0$, $b>d>0$ and $c>d+p$, we define three products:

$$\mathbf{P}_p(a,b,c,d,p)\overset{\text{def}}{=}\left(\prod_{i=0}^{p-1}\frac{i!\,(b+c-d+i)!}{(b-d+i)!\,(a+c-p+i)!}\right), \tag{32}$$

$$\mathbf{P}_a(a,b,c,d,p)\overset{\text{def}}{=}\left(\prod_{i=p}^{a-1}\frac{i!\,(b+c-d+i)!}{(b+i)!\,(c-d-p+i)!}\right), \tag{33}$$

$$\mathbf{P}_d(a,b,c,d,p)\overset{\text{def}}{=}\left(\prod_{i=0}^{d-1}\frac{(a-p+1+i)_p}{(p+i)!\,(b+c-2d+1+i)_i}\right). \tag{34}$$

(Note that by the inequalities constraining the integers a,b,c,d and p, these products are well–defined: There is no factor $z!$ for $z<0$.)

We define

$$\mathbf{P}(a,b,c,d,p) \stackrel{\text{def}}{=} \mathbf{P}_p(a,b,c,d,p) \cdot \mathbf{P}_a(a,b,c,d,p) \cdot \mathbf{P}_d(a,b,c,d,p)$$

and make the modified *ansatz*

$$\mathbf{e}(a,b,c,d,p) = \mathbf{P}(a,b,c,d,p) \cdot \mathbf{Q}(a,b,c,d,p). \tag{35}$$

Inserting this modified ansatz in the condensation recursion (6) gives (after a lot of straightforward cancellations) the following functional equation for $\mathbf{Q}(a,b,c,d,p)$, valid for $1 \leq p \leq a$, $a \geq 0$, $b > d > 0$ and $c > d + p$:

$$\begin{aligned}\mathbf{Q}(a,b,c,d,p) \cdot \mathbf{Q}(a-2,b,c,d,p-1) \cdot (a+b+c-d-1) \cdot (a+d-1) = \\ \mathbf{Q}(a-1,b,c,d,p) \cdot \mathbf{Q}(a-1,b,c,d,p-1) \cdot (a+c-1) \cdot (a+b-1) \\ - \mathbf{Q}(a-1,b-1,c+1,d,p) \cdot \mathbf{Q}(a-1,b+1,c-1,d,p-1) \cdot (c-d) \cdot (b-d).\end{aligned} \tag{36}$$

The following assertion shows that this modified ansatz (35) makes sense:

Proposition 3. *For $d = 1$, the function $\mathbf{Q}(a,b,c,1,p)$ is a simple constant:*

$$\mathbf{Q}(a,b,c,1,p) \equiv 1 \text{ for all } 0 \leq p \leq a. \tag{37}$$

Proof. We shall prove (37) by induction on p: For $p = 0$, we simply have

$$\mathbf{e}(a,b,c,1,0) = \mathbf{e}(a,b,c-1,0,0) = \mathbf{mm}(a,b,c-1),$$

see Figure 5. Moreover, it is obvious that

$$\mathbf{P}(a,b,c,1,0) = \mathbf{P}_a(a,b,c,1,0) = \mathbf{mm}(a,b,c-1),$$

whence $\mathbf{Q}(a,b,c,1,0) = 1$ and (by symmetry (4)) $\mathbf{Q}(a,b,c,1,a) = 1$.

So assume (37) holds for $p-1$. By the induction hypothesis (on p), (36) simplifies to

$$\mathbf{Q}(a,b,c,1,p) = \frac{\mathbf{Q}(a-1,b,c,1,p) \cdot (a+c-1) \cdot (a+b-1) - \mathbf{Q}(a-1,b-1,c+1,1,p)\,(c-1) \cdot (b-1)}{(a+b+c-2) \cdot (a)}.$$

From this, the assertion follows by induction on $a \geq p$: Simply observe that

$$\frac{(a+c-1) \cdot (a+b-1) - (c-1) \cdot (b-1)}{(a+b+c-2) \cdot (a)} = 1$$

and $\mathbf{Q}(p,b,c,1,p) = \mathbf{Q}(p,c,b,1,0) = 1$. □

Note that the proof of Proposition 3 relied on one crucial ingredient, namely the (very simple) formula for $\mathbf{Q}(a,b,c,1,0)$ (easily obtained by the simple formula for $\mathbf{e}(a,b,c,1,0)$): It served

- as the base case for the induction on p
- *and* as the base case for the induction on $a \geq p$ (via the symmetry $\mathbf{e}(p,b,c,d,p) = \mathbf{e}(p,c,b,d,0)$).

Of course, we cannot expect that $\mathbf{Q}(a,b,c,d,p)$ is given by a simple formula for $d > 1$. But numerical experiments indicate that for d fixed, $\mathbf{Q}(a,b,c,d,p)$ is a *polynomial* in a, b, c, p. So if we can somehow *guess* this polynomial *and* are able to show that

- $\mathbf{e}(a,b,c,d,0) = \mathbf{P}(a,b,c,d,0) \cdot \mathbf{Q}(a,b,c,d,0)$
- and (36) is, in fact, a *polynomial identity*,

then we would have proved the corresponding formula.

Assuming $b \geq 2d$, the values for $p \leq a$ are partitioned in three intervals of "different quality":

- $p \in [0, a]$ (this is the interval considered in Proposition 3, for which we presented our modified *ansatz*),
- $p \in [-d + 1, -1]$,
- and $p \leq -d$: It is obvious that the intrusion does no damage to the hexagon at all in this case, whence $\mathbf{e}(a,b,c,d,p) = \mathbf{mm}(a,b,c)$ for $p \leq -d$.

So the case $p = 1 - d$ (which we already considered in sections 4.4.1 and 4.4.3) would serve as base case for the interval $[1 - d, -1]$ in the same sense as $p = 0$ served as base case for the interval $[0, a]$ in the proof of Proposition 3. So in principle, we could work with our "specialized" ansatz from $p = 1 - d$ till $p = 0$, and then continue with our "modified" ansatz: However, the formulas quickly become rather unwieldy for $d > 1$. So for now, we conclude this line of investigations with the following conjecture:

Conjecture 1. *The number* $\mathbf{e}(a,b,c,d,p)$ *of lozenge tilings of a damaged hexagon with side lengths* a, b, c *and vertical intrusion of depth* d *at* even *position* p *with* $0 \leq p \leq a$ *equals*

$$\prod_{i=0}^{d-1} \frac{(a-p+i+1)_p}{(b+c-d-i)_{d-i-1}\,(p+i)!} \times \prod_{i=0}^{p-1} \frac{i!\,(b+c-d+i)!}{(b-d+i)!\,(a+c-i-1)!} \times \prod_{i=p}^{a-1} \frac{i!\,(b+c-d+i)!}{(b+i)!\,(a+c-d-i-1)!} \times \mathbf{Q}(p,c,b,d,p), \quad (38)$$

where for fixed d *the factor* $\mathbf{Q}(p,c,b,d,p)$ *is a* polynomial *in the variables* a, b, c *and* p. *The coefficient of monomial* $b^i c^j$ *in* $\mathbf{Q}(p,c,b,d,p)$ *is a polynomial in* a *and* p *whose degree with respect to* a *is* $\leq g - j$, *and whose degree with respect to* p *is* $\leq 2g - i - j$. *For instance, in Proposition 3 we showed* $\mathbf{Q}(a,b,c,1,p) = 1$. *Numerical experiments indicate that*

$$\mathbf{Q}(a,b,c,2,p) = b \cdot (a - p + 1) + c \cdot (p + 1) + 2\left(ap - p^2 - 1\right)$$

and

$$\mathbf{Q}(a,b,c,3,p) = 98a^3 + 621a^2 + 1243a + (a+3)(2a+5)(125a+250) - (a+3)(4a+13)(75a+150) + (a+3)(375a+750) + 786.$$

A brute–force computer search yields the polynomials $\mathbf{Q}(a,b,c,d,p)$ for d up to 5: *Mathematica* shows that all these formulae factor nicely for $a = 2p$, in accordance with Byun's formula (2).

In order to specify and prove Conjecture 1, we need to find the general formula giving $\mathbf{Q}(a,b,c,d,p)$: We hope to find this formula in future work.

4.6. **Another "brute force" approach.** Consider the matrix whose determinant gives MacMahon's formula:

Lemma 1. *For $a, b, c \in \mathbb{N}$, define the following $(a \times a)$–matrices M, L, T, D and U with (i, j)–entries*

$$M_{i,j} \overset{def}{=} \binom{b+c}{b+i-j},$$
$$L_{i,j} \overset{def}{=} (-1)^{i+j} \binom{i-1}{j-1} \frac{(c)_{i-j}}{(b+j)_{i-j}},$$
$$T_{i,j} \overset{def}{=} (-1)^{i+j} \binom{j-1}{i-1} \frac{(b)_{j-i}}{(c+i)_{j-i}},$$
$$D_{i,j} \overset{def}{=} [i = j] \cdot \frac{(b+i-1)!\,(c+i-1)!}{(b+c+i-1)!\,(i-1)!},$$
$$U_{i,j} \overset{def}{=} (-i+j+1)_{i-1} \frac{b!\,(b+c+i-1)!}{(b+i-1)!\,(c+j-1)!\,(b+i-j)!}.$$

Note that M is the matrix corresponding to MacMahon's formula (i.e., $\det M = \mathbf{mm}(a,b,c)$), L is a lower triangular matrix with entries 1 on the main diagonal, T is the transpose of L with variables b and c swapped, D is a diagonal matrix (Iverson's bracket $[A]$ is 1 if assertion A is true, else 0), and U is an upper triangular matrix.

Then we have

$$U = L \cdot M \tag{39}$$

and

$$M^{-1} = T \cdot D \cdot L. \tag{40}$$

Moreover, the (i, j)–entry of the inverse M^{-1} is

$$M^{-1}_{i,j} = (-1)^{i+j} (b+j-1)!\,(c+i-1)! \times \sum_{k=1}^{a} \binom{k-1}{i-1}\binom{k-1}{j-1} \frac{(b)_{k-i}\,(c)_{k-j}}{(k-1)!\,(b+c+k-1)!} \tag{41}$$

(Note that the sum in (41) actually starts at $k = \max(i, j)$: All other summands are zero due to the binomial coefficients.)

Remark 1. *As an easy consequence of (39), we have*

$$\det M = \det U = \prod_{i=0}^{a-1} i! \frac{(b+c+i)!}{(b+i)!\,(c+i)!},$$

which is MacMahon's formula (1).

Let us call the matrix M in Lemma 1 *MacMahon's matrix*. Consider the natural decomposition of the matrix Q underlying the determinant giving $\mathtt{e}(a,b,c,d,p)$ (i.e., $\det Q = \mathtt{e}(a,b,c,d,p)$, see Example 1) into 4 submatrices Q_1, Q_2 Q_3 and Q_4,

$$Q = \begin{pmatrix} Q_2 & Q_1 \\ Q_3 & Q_4 \end{pmatrix},$$

where

- Q_1 is the submatrix of the first a rows and d last columns of Q,
- Q_2 is the submatrix of the first a rows and a first columns of Q,
- Q_3 is the submatrix of the last d rows and a first columns of Q,
- Q_4 is the submatrix of the last d rows and d last columns of Q.

Note that Q_2 is MacMahon's matrix (i.e., matrix M in Lemma 1). All the (i,j)–entries of these submatrices are binomial coefficients:

$$(Q_1)_{i,j} = \binom{2j-1}{-i+j+p} \tag{42}$$

$$(Q_2)_{i,j} = \binom{b+c}{c-i+j} \tag{43}$$

$$(Q_3)_{i,j} = \binom{b+c-2i+1}{c-i+j-p} \tag{44}$$

$$(Q_4)_{i,j} = \binom{2\,(j-i)}{j-i} \tag{45}$$

Denote by $\mathbf{1}$ and $\mathbf{0}$ the identity matrix and the zero matrix, respectively, with the "appropriate" dimensions, and observe

$$\begin{pmatrix} Q_2^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} Q_2 & Q_1 \\ Q_3 & Q_4 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & Q_2^{-1} \cdot Q_1 \\ Q_3 & Q4 \end{pmatrix}$$

Combining (41) and (42), we see that the (i,j)–entry of $Q_2^{-1} \cdot Q_1$ is

$$\left(Q_2^{-1} \cdot Q_1\right)_{i,j} = \sum_{l=1}^{a} (-1)^{i+l}\,(b+l-1)!\,(c+i-1)! \binom{2j-1}{l+j-p-1} \sum_{k=1}^{a} \binom{k-1}{i-1} \binom{k-1}{l-1} \frac{(b)_{k-i}\,(c)_{k-l}}{(k-1)!\,(b+c+k-1)!} \tag{46}$$

Clearly, by straightforward column operations we can achieve that submatrix $Q_2^{-1} \cdot Q_1$ is replaced by $\mathbf{0}$. Expressed as matrix multiplication:

$$\begin{pmatrix} \mathbf{1} & Q_2^{-1} \cdot Q_1 \\ Q_3 & Q4 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & -Q_2^{-1} \cdot Q_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ Q_3 & F \end{pmatrix},$$

where F is the product of matrices

$$F = \begin{pmatrix} Q_3 & Q_4 \end{pmatrix} \cdot \begin{pmatrix} -Q_2^{-1} \cdot Q_1 \\ \mathbf{1} \end{pmatrix} = Q_4 - Q_3 \cdot Q_2^{-1} \cdot Q_1.$$

Combining (46) and (44), we see that the (i,j)–entry of $Q_3 \cdot Q_2^{-1} \cdot Q_1$ is the triple sum

$$\begin{aligned}\left(Q_3 \cdot Q_2^{-1} \cdot Q_1\right)_{i,j} = \sum_{t=1}^{a} &\binom{b+c-2i+1}{c-i+t-p} \\ &\sum_{l=1}^{a} (-1)^{t+l} (b+l-1)!\,(c+t-1)! \\ \binom{2j-1}{l+j-p-1} &\sum_{k=1}^{a} \binom{k-1}{t-1}\binom{k-1}{l-1} \frac{(b)_{k-t}\,(c)_{k-l}}{(k-1)!\,(b+c+k-1)!}. \end{aligned} \tag{47}$$

(Note that the (i,j)–entry (49) *does not depend on d*.)

So by combining this with (44), we deduce:

Corollary 2. *Let Q_1, Q_2, Q_3 and Q_4 be the submatrices of the matrix Q underlying the determinant giving $\mathbf{e}(a,b,c,d,p)$. Then $\mathbf{e}(a,b,c,d,p)$ (and thus the number of tilings of the (a,b,c)–hexagon with an (even) intrusion of length d at position p) is given as*

$$\mathbf{e}(a,b,c,d,p) = \det F \cdot \mathbf{mm}(a,b,c) = \det F \cdot \prod_{i=0}^{a-1} i! \frac{(b+c+i)!}{(b+i)!\,(c+i)!}, \tag{48}$$

where $F = Q_4 - \left(Q_3 \cdot Q_2^{-1} \cdot Q_1\right)$.

4.6.1. *Special case $d = 1$, once again.* Note that F is the $(d \times d)$–matrix with (i,j)–entry

$$F_{i,j} = \binom{2\,(j-i)}{j-i} - \left(Q_3 \cdot Q_2^{-1} \cdot Q_1\right)_{i,j}, \tag{49}$$

where $\left(Q_3 \cdot Q_2^{-1} \cdot Q_1\right)_{i,j}$ is given by (47), so for the special case $d = 1$, the determinant of matrix F is simply $F_{1,1}$. Combining this with our result for $\mathbf{e}(a,b,c,1,p)$ (i.e., for the special case $d = 1$; see equation (37) in Proposition 3) gives (after straightforward cancellations and simplifications; observe that the sum over l only contributes two non–zero summands) the following summation formula:

$$\begin{aligned}(-1)^p\,(b+p-1)! \sum_{t=1}^{a} (-1)^t \binom{b+c-1}{c-p+t-1} (c+t-1)! \\ \sum_{k=1}^{a} \frac{\left(-\,(b+p)\binom{k-1}{p} + (c+k-p-1)\binom{k-1}{p-1}\right)(b)_{k-t}\,(c)_{k-p-1}\binom{k-1}{t-1}}{(k-1)!\,(b+c+k-1)!} \\ = 1 - \binom{a}{a-p} \frac{(b)_p\,(c)_{a-p}}{(b+c)_a}. \end{aligned} \tag{50}$$

For the special case $p = 0$, (50) reads (after some simplification)

$$b! \sum_{t=0}^{a-1} (-1)^t (b-t)_{c+t} \sum_{k=0}^{a-1} \frac{(b)_{k-t} \binom{k}{t} \binom{c+k-1}{k}}{(b+c+k)!} = 1 - \frac{(c)_a}{(b+c)_a}. \qquad (51)$$

For $p > 0$, we may rewrite (50) as

$$(-1)^p (b+p-1)! \sum_{t=1}^{a} (-1)^t \binom{b+c-1}{c-p+t-1} (c+t-1)!$$
$$\sum_{k=1}^{a} \frac{\left(-\frac{bk}{p} + b + c - 1\right) (b)_{k-t} (c)_{k-p-1} \binom{k-1}{p-1} \binom{k-1}{t-1}}{(k-1)! \, (b+c+k-1)!}$$
$$= 1 - \binom{a}{a-p} \frac{(b)_p (c)_{a-p}}{(b+c)_a}. \qquad (52)$$

As a direct consequence of Byun's formula (2), we obtain:

Proposition 4. *If we set $a = 2p$ in Corollary 2, then the determinant factors nicely:*

$$\det F|_{a=2p} = 4^{dp} \prod_{k=1}^{d} \frac{\left(k - \frac{1}{2}\right)_p (b-k+1)_p (c-k+1)_p}{(k)_p (b+c-2k+2)_{2p}}.$$

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