Cheeger inequalities on simplicial complexes

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Abstract

Cheeger-type inequalities in which the decomposability of a graph and the spectral gap of its Laplacian mutually control each other play an important role in graph theory and network analysis, in particular in the context of expander theory. The natural problem, however, to extend such inequalities to simplicial complexes and their higher order Eckmann Laplacians has been open for a long time. The question is not only to prove an inequality, but also to identify the right Cheeger-type constant in the first place. Here, we solve this problem. Our solution involves and combines constructions from simplicial topology, signed graphs, Gromov filling radii and an interpolation between the standard 2-Laplacians and the analytically more difficult 1-Laplacians, for which, however, the inequalities become equalities. It is then natural to develop a general theory for p-Laplacians on simplicial complexes and investigate the related Cheeger-type inequalities.

Keywords: Cheeger inequality; simplicial complex; Hodge Laplacian; Eckmann Laplacian; p-Laplacian

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1 Introduction and Background

Generalizing the classical Laplace operator, Laplace-type operators have been defined for functions on various geometric structures, including domains in Euclidean space or on Riemannian manifolds, and graphs. From their spectra, one can usually extract important information about the underlying structure. In particular, in a seminal paper [9], Cheeger showed that the first non-vanishing eigenvalue of the Laplace-Beltrami operator of a compact Riemannian manifold estimates how difficult it is to decompose the manifold into two pieces. This result has found many generalizations and extensions, and the discrete analogue, that is, the Cheeger-type inequality for graphs, leads to expander theory and is of fundamental importance in theoretical computer science and in the analysis of empirical networks when represented as graphs. And discrete Cheeger-type inequalities can be generalized, for instance, to weighted or signed graphs, again with diverse applications.

But there are also higher-order Laplacians, like the Hodge Laplacian operating on exterior differential forms on a Riemannian manifold, or its discrete analogue, the Eckmann Laplacian on a simplicial complex. In this paper, we look at the simplicial case. It is natural to try to generalize the classical spectral results that are known for graphs to simplicial complexes. In particular, one can ask for a version of the Cheeger inequality for higher dimensional simplicial complexes. But it turns out that estimating the first non-trivial eigenvalue of the Eckmann Laplacian on a simplicial complex is a major long-standing open problem in the field of high dimensional expander theory. (Also the analogue in Riemannian geometry, to find Cheeger inequalities for differential k-forms, is still far from being understood and solved.) And such a Cheeger-type estimate for the Eckmann Laplacian on a simplicial complex is what we shall develop in this paper. A major difficulty that we had to overcome consists already in the appropriate formulation of the inequality, and for that, we need to figure out the relevant aspects of the combinatorial structure of a simplicial complex that could support such an inequality. This then needs to be combined with insights coming from a non-linear analogue of the Laplacian, the 1-Laplacian, which involves the L^1- instead of the L^2-norm behind the ordinary Laplacian. That operator is analytically much more difficult than the ordinary Laplacian, but has the advantage that the Cheeger-type inequality here becomes an equality.

In view of the preceding, we need to recall and assemble some background material before we can develop our main results. This material will concern the general setting of Cheeger-type inequalities, simplicial complexes and the Eckmann Laplacian, signed graphs, as well as p-Laplacians, the usual Laplacians corresponding to $p = 2$, and the technically useful case being $p = 1$.

1.1 Simplicial complexes

Here, we only consider a finite set V of vertices, leaving the infinite case open. We recall some standard terminology. A *simplicial complex* Σ on V is a subset of its power set $\mathcal{P}(V)$ that is closed under taking subsets, i.e. $\forall \sigma \in \Sigma$, $\forall \sigma' \subset \sigma$, $\sigma' \in \Sigma$. The elements of Σ are called *simplices*. It follows from this setting that all the vertices constituting a simplex are different from each other. A simplex σ with $d + 1$ vertices is called a *d-simplex*, and we call d its dimension. Its subsimplices are called its *faces*, and its $(d - 1)$-dimensional faces are called its *facets*. The dimension of a simplicial complex is the largest dimension among its simplices. A 1-dimensional simplicial complex is a *graph*.

We usually assume that Σ is *connected*. This means that for any two of its non-empty simplices σ, σ', there exists a chain of simplices $\sigma_0 = \sigma, \sigma_1, \dots, \sigma_m = \sigma'$ with the property that any two adjacent simplices in this chain have at least one vertex in common. And we usually and naturally assume that all elements of the vertex set V participate in the simplicial complex Σ, that is, every vertex is contained in at least one simplex.

In order to work with orientations, we need a slight modification or amplification of our notation. Here, an *orientation* of a d-simplex is an ordering of its vertices up to even permutation. An odd permutation of the vertices changes an oriented d-simplex σ_d into the oppositely oriented simplex $-\sigma_d$. Thus, from now on, σ_d denotes an ordered simplex.

Let Σ_d be the collection of the d-simplices of Σ. In particular, Σ_0 is the vertex set V. We let $C_d = C_d(\Sigma)$ be the abelian group with coefficients in $\mathbb{R}$ generated by the elements of Σ_d. We also write $C^d = C^d(\Sigma)$ for the linear functions from C_d to $\mathbb{R}$ that satisfy

$$f(-\sigma_d) = -f(\sigma_d), \tag{1}$$

for every oriented d-simplex.

For $f \in C^{d-1}$, we define its *coboundary* $\delta f : C^d \to \mathbb{R}$ as

$$\delta f(v_0, v_1, \dots, v_d) = \sum_{i=0}^{d} (-1)^i f(v_0, \dots, \hat{v_i}, \dots, v_d), \tag{2}$$

where, as usual, a ˆ over a vertex means that it is omitted. Sometimes, we write

$$\delta_d : C^d \to C^{d+1}, \tag{3}$$

in order to specify the dimension.

The d-th *cohomology group* of the simplicial complex Σ is

$$H^d(\Sigma) := \ker \delta_d / \operatorname{image} \delta_{d-1}. \tag{4}$$

Remark 1. *More generaly, we can consider the linear space $C_d(\Sigma, \mathbb{F})$ with coefficients in an abelian group $\mathbb{F}$, generated by the elements of Σ_d, and let $C^d(\Sigma, \mathbb{F})$ be the linear functions from $C_d(\Sigma, \mathbb{F})$ to $\mathbb{F}$, satisfying* (1)*, and then we can define the cohomology group $H^d(\Sigma, \mathbb{F})$ in the same way. It is usual to take $\mathbb{F}$ to be a commutative ring (e.g. the integer ring $\mathbb{Z}$) or even a field (e.g. the field $\mathbb{C}$ of the complex numbers, or the finite field $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$). As an interesting example, we refer to [37] for the Cheeger constants defined on a simplicial complex which use the cohomology over the finite field $\mathbb{Z}_2$. In this paper, we work with real coefficients or integer coefficients.*

If we pass to the reduced cochain complex, we get the reduced cohomology $\tilde{H}^d$, which can be defined simply by the relation $\tilde{H}^0(\Sigma, \mathbb{F}) \oplus \mathbb{F} = H^0(\Sigma, \mathbb{F})$ and $\tilde{H}^d(\Sigma, \mathbb{F}) = H^d(\Sigma, \mathbb{F})$ for $d \geq 1$, where $\mathbb{F}$ can be $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{R}$.

To proceed, we choose positive definite inner products $(\cdot, \cdot)_d$ on the C^d. We can then define the adjoint $(\delta_d)^* : C^{d+1} \to C^d$ of the coboundary operator δ_d by

$$(\delta_d f_1, f_2)_{d+1} = (f_1, (\delta_d)^* f_2)_d,$$

for $f_1 \in C^d$ and $f_2 \in C^{d+1}$. We can then go back and forth between the C^d, as we have the arrows

$$C^{d-1} \underset{\delta_{d-1}{}^*}{\overset{\delta_{d-1}}{\rightleftarrows}} C^d \underset{\delta_d{}^*}{\overset{\delta_d}{\rightleftarrows}} C^{d+1}. \tag{5}$$

This allows us to define the following three operators on C^d (omitting the argument Σ, i.e., writing for instance L_d instead of $L_d(\Sigma)$, as Σ will be mostly kept fixed):

(i) The *d-dimensional up Laplace operator* or simply *d-up Laplacian* of the simplicial complex Σ is

$$L_d^{up} := (\delta_d)^* \delta_d,$$

(ii) The *d-dimensional down Laplace operator* or *d-down Laplacian* is

$$L_d^{down} := \delta_{d-1}(\delta_{d-1})^*,$$

(iii) The *d-dimensional Laplace operator* or *d-Laplacian* is the sum

$$L_d := L_d^{up} + L_d^{down} = (\delta_d)^* \delta_d + \delta_{d-1}(\delta_{d-1})^*.$$

The operators L_d^{up}, L_d^{down} and L_d are self-adjoint and non-negative. Therefore, their eigenvalues are non-negative real numbers.

The multiplicities of the eigenvalue 0 of the Laplacians $L_d(\Sigma)$ contain topological information about Σ. This is the content of Eckmann's Theorem [17], which is a discrete version of the Hodge theorem. It says that

$$\ker L_d(\Sigma) \cong H^d(\Sigma).$$

Thus, the multiplicity of the eigenvalue 0 of the operator $L_d(\Sigma)$ is equal to the Betti number b_d, the dimension of $H^d(\Sigma)$. As a corollary,

$$C^d = \text{image}\, \delta_{d-1} \oplus \text{image}\, (\delta_d)^* \oplus \ker L_d. \tag{6}$$

We point out that Eckmann's Theorem does not depend on the choice of scalar products on the spaces C^d (although the harmonic cocycles do).
While cohomology groups are defined as quotients, that is, as equivalence classes of elements of C^d, Eckmann's Theorem provides us with concrete representatives in C^d of those equivalence classes, the so-called *harmonic cocycles.* These are the eigenvectors for the eigenvalue 0 of the Laplacian. We shall now look at the non-zero part of the spectrum which will depend on the choice of the scalar products.

Since $\delta_d \delta_{d-1} = 0$ and $\delta_{d-1}{}^* \delta_d{}^* = 0$,

$$\text{image}\, L_d^{down}(\Sigma) \subset \ker L_d^{up}(\Sigma), \tag{7}$$

$$\text{image}\, L_d^{up}(\Sigma) \subset \ker L_d^{down}(\Sigma). \tag{8}$$

This implies that $\lambda \neq 0$ is an eigenvalue of $L_d(\Sigma)$ if and only if it is a eigenvalue of either $L_d^{up}(\Sigma)$ or $L_d^{down}(\Sigma)$. Therefore, the non-zero parts of the spectra satisfy

$$\text{spec}_{\neq 0}(L_d(\Sigma)) = \text{spec}_{\neq 0}(L_d^{up}(\Sigma)) \cup \text{spec}_{\neq 0}(L_d^{down}(\Sigma)). \tag{9}$$

The multiplicity of the eigenvalue 0 may be different, however.

Since $\mathrm{spec}_{\neq 0}(AB) = \mathrm{spec}_{\neq 0}(BA)$, for linear operators A and B on Hilbert spaces, we conclude

$$\mathrm{spec}_{\neq 0}(L_d^{up}(\Sigma)) = \mathrm{spec}_{\neq 0}(L_{d+1}^{down}(\Sigma)). \tag{10}$$

From (9) and (10) we conclude that each of the three families of multisets

$$\{\mathrm{spec}_{\neq 0}(L_d(\Sigma)) \mid 0 \leq d \leq m\}, \{\mathrm{spec}_{\neq 0}(L_d^{up}(\Sigma)) \mid 0 \leq d \leq m-1\}, \{\mathrm{spec}_{\neq 0}(L_d^{down}(\Sigma)) \mid 1 \leq d \leq m\}$$

determines the other two. Therefore, it suffices to consider only one of them.

We shall also make use of the following general result, the Courant-Fischer-Weyl min-imax principle.

Lemma 1.1. *Let the linear operator $A : H \to H$ on a finite dimensional vector space be self-adjoint w.r.t. the scalar product $(.,.)$. Then its eigenvalues and eigenvectors are the critical values and the critical points of the Rayleigh quotient, defined for $f \neq 0$,*

$$\frac{(Af, f)}{(f, f)}. \tag{11}$$

1.2 Cheeger inequalities

Cheeger [9] showed that the first non-vanishing eigenvalue of the Laplace-Beltrami operator of a compact connected Riemannian manifold can be bounded from below in terms of a constant introduced by him and thence called the Cheeger constant which quantifies how difficult it is to cut the manifold into two large pieces by a small hypersurface. Buser [8] then also showed an upper estimate. Thus, this eigenvalue is controlling and controlled by the Cheeger constant. It was then realized in [1, 10, 14] that an analogous estimate holds on graphs, for the first non-vanishing eigenvalue of the graph Laplacian. The analogue of Cheeger's constant had in fact already been introduced by Polya [36], without connecting it to eigenvalues. To formulate the latter inequalities, we consider an undirected and unweighted graph $\Gamma = (V, E)$ with vertex set V and edge set E. The degree $\deg v$ of a vertex v is the number of its neighbors, that is, the number of vertices directly connected it by edges. We define the volume of $S \subset V$ is $\mathrm{vol}(S) = \sum_{v \in S} \deg v$, for $V_1, V_2 \subset V$, $|E(V_1, V_2)|$ is the number of edges with one endpoint in V_1 and the other in V_2. We then put

$$\eta(S) := \frac{|E(S, V \backslash S)|}{\min(\mathrm{vol}(S), \mathrm{vol}(V \backslash S))}, \tag{12}$$

and introduce the *(Polya)-Cheeger constant*

$$h = \min_S \eta(S). \tag{13}$$

The estimate for the first non-vanishing eigenvalue λ of the normalized graph Laplacian then says

$$\frac{1}{2} h^2 \leq \lambda \leq 2h. \tag{14}$$

This estimate is important, for instance, in the theory of expander graphs, because a good expander family should have a uniformly large such λ.

In fact, one can not only bound the smallest non-vanishing eigenvalue of a graph from below, but also the largest one from above. The largest eigenvalue of the normalized Laplacian of a graph is always ≤ 2. Equality is realized for bipartite graphs, and for non-bipartite graphs, the difference $2 - \lambda$ can be controlled [6,40].

Already in the original paper by Cheeger [9], the problem was proposed to derive an estimate for the smallest non-vanishing eigenvalue of the Hodge Laplacian on differential k-forms on a closed Riemann manifold. So far, this problem is not solved. Its discrete version, a Cheeger-type inequality on simplicial complexes, is also a long-standing open problem in the area of high dimensional expanders. While some partial answers have been proposed and developed in the literature, it seems that none of them gives a complete solution to the problem. In fact, there are many different definitions of Cheeger constants on simplicial complexes. For example, it is known that the easier upper bound of (14) holds for the Cheeger constant suggested by Parzanchevski, Rosenthal and Tessler [20,35]. But none of the constants proposed so far in the literature can satisfy a full Cheeger inequality as in the graph setting. In particular, in the field of higher dimensional expanders, people use the so-called $\mathbb{Z}_2$-expander for constructing the Cheeger constants on a simplicial complex.

Thus, the problem is, and the essential purpose of this paper is to establish some good estimate for the first nontrivial eigenvalue of the discrete Eckmann Laplacian by introducing some suitable Cheeger-type constants on a simplicial complex and proving that this controls, and in turn is controlled by that eigenvalue, analogously to (14). Controlling this eigenvalue from below in terms of the Cheeger-type constant is called the Cheeger side, while controlling it from above is called the Buser side. Usually, the latter is easier than the former.

Important contributions in this direction come from Dotterrer and Kahle [16] and Steenbergen, Klivans and Mukherjee [37]. In [37], a Cheeger constant via cochain complexes is analyzed,

$$h^d(\Sigma) := \min_{\phi \in C^d(\Sigma,\mathbb{Z}_2)\setminus \operatorname{Im}\delta} \frac{\|\delta\phi\|}{\min\limits_{\psi\in\operatorname{Im}\delta} \|\phi+\psi\|} \tag{15}$$

which satisfies

$$h^d(\Sigma) = 0 \Longleftrightarrow \tilde{H}^d(\Sigma,\mathbb{Z}_2) \neq 0, \quad \forall d \geq 0,$$

where $\|\cdot\|$ is the Hamming norm on $C^d(\Sigma,\mathbb{Z}_2)$ (i.e. the l^1-norm on $\mathbb{Z}_2^n$ with $n = \#\Sigma_d$). This is a natural generalization of the classical graph Cheeger constant (13) to higher dimensions on simplicial complexes. Unfortunately, based on the results in [21] and [37], the most straightforward attempt at a higher-dimensional Cheeger inequality fails, even for the Buser side – in higher dimensions, spectral expansion (an eigenvalue gap for the Laplacian) does not imply combinatorial expansion [34]. In fact, according to the examples and theorems in [16,20,22,35,37], all the Cheeger constants defined using cohomology (or homology) with $\mathbb{Z}_2$-coefficients cannot satisfy a general two-sided Cheeger inequality as in the graph setting. This is a consequence of the relation

$$\lambda(\Delta_d^{up}) = 0 \Leftrightarrow \lambda(L_d^{up}) = 0 \Longleftrightarrow \tilde{H}^d(\Sigma,\mathbb{R}) \neq 0, \quad d \geq 0,$$

but for $d \geq 1$, the non-vanishing of $\tilde{H}^d(\Sigma,\mathbb{R})$ is not equivalent to that of $\tilde{H}^d(\Sigma,\mathbb{Z}_2)$.

We should also mention that in [35], another Cheeger-type constant is proposed, and their Theorem 1.2 generalizes the upper Cheeger inequality to higher dimensions. That modified Cheeger number is nonzero only if the simplicial complex has a complete skeleton, and the Cheeger side of the inequality includes an additive constant. We shall adopt a different definition, and therefore do not go into further detail here.

In this paper, we shall first derive Theorem 2.1 which contains an estimate for the spectral gap from $d+2$, recalling that for the vertex Laplacian of a graph, i.e., in the case $d=0$, the spectral gap at 2 can be controlled. We then turn to the more difficult estimate for the spectral gap from 0, namely, the Cheeger-type estimate for the first non-trivial eigenvalue of the Eckmann Laplacian. Since such an estimate cannot be derived for the Cheeger-type constants introduced earlier, our first contribution here is the introduction of a new Cheeger constant. The key point is that in contrast to the graph case, on higher dimensional simplices, orientations and multiplicities enter into the coboundary relations and therefore implicitly into the eigenvalues. We therefore consider generalized (i.e., with both positive and negative multiplicities) multisets of d-simplices.

1.3 Signed graphs

We consider unweighted and undirected graphs Γ. When $v, v' \in V$, the vertex set of Γ, are connected by an edge, denoted as (vv'), we write $v \sim v'$ and call v and v' neighbors. We shall need an additional structure, a sign function on the edges. A *signed graph* thus is a graph Γ equipped with a map s from its edge set to ± 1. We may *switch signs* by taking a vertex and changing the signs of all edges that it is contained in. A signed graph is called *balanced* if by switching some vertices, we can make all signs $=1$, and it is *antibalanced*, if we can make them all $=-1$.

Signed graphs have many applications in modeling biological networks, social relations, ferromagnetism, and general signed networks [4, 5, 26, 43]. The spectral theory for signed graphs has led to a number of breakthroughs in theoretical computer science and combinatorial geometry, including the solutions to the sensitive conjecture [25] and the open problem on equiangular lines [28, 29].

The Laplacian of the signed graph (Γ, s) is

$$\Delta_s f(v) = f(v) - \frac{1}{\deg v} \sum_{v' \sim v} s(vv') f(v') = \frac{1}{\deg v} \sum_{v' \sim v} (f(v) - s(vv') f(v')) \tag{16}$$

We record some basic results about the spectrum of this operator [3] that can be easily checked.

Lemma 1.2. *The eigenvalues of Δ_s are real and lie in the interval $[0,2]$. In fact, the smallest eigenvalue is $=0$ if and only if (Γ, s) is balanced, and positive otherwise. Likewise, the largest eigenvalue is $=2$ if and only if the graph is antibalanced.*

To proceed, we recall the multi-way Cheeger constant h_k^s on a signed graph (Γ, s) introduced in [3]. For disjoint $V_1, V_2 \subset V$, let $E^+(V_1, V_2) = \{\{u, v\} \in E : u \in V_1, v \in V_2, s(uv) = 1\}$ and $E^-(V_1) = \{\{u, v\} \in E : u, v \in V_1, s(uv) = -1\}$. The signed bipartiteness ratio is defined as

$$\beta^s(V_1, V_2) = \frac{2\left(|E^-(V_1)| + |E^-(V_2)| + |E^+(V_1, V_2)|\right) + |\partial(V_1 \sqcup V_2)|}{\mathrm{vol}(V_1 \sqcup V_2)}.$$

The signed Cheeger constant of the signed graph (Γ, s) is then defined as

$$h^s = \min_{(V_1,V_2)\neq(\emptyset,\emptyset)} \beta^s(V_1, V_2)$$

where the minimum is taken over all possible sub-bipartitions of V. β^s and hence also h^s is switching invariant.

The Cheeger inequality for signed graphs established in [3] says that for a signed graph (Γ, s), we have

$$\frac{\lambda_1(\Delta_s)}{2} \leq h^s \leq \sqrt{2\lambda_1(\Delta_s)}. \tag{17}$$

The k-way signed Cheeger constant is defined as

$$h^s_k = \min_{\{(V_{2i-1},V_{2i})\}_{i=1}^k} \max_{1\leq i\leq k} \beta^s(V_{2i-1}, V_{2i})$$

where the minimum is taken over the set of all possible k pairs of disjoint sub-bipartitions (V_1, V_2), (V_3, V_4), ..., (V_{2k-1}, V_{2k}). h^s_k is again switching invariant.

This definition allowed Atay and Liu to generalize and put into perspective the higher-order Cheeger inequality for ordinary graphs by Lee, Oveis Gharan, and Trevisan [33]. Their estimate is ([3])

Theorem 1.1. *There exists an absolute constant C such that for any signed graph (Γ, s), and any $k \in \{1, \dots, n\}$,*

$$\frac{\lambda_k(\Delta_s)}{2} \leq h^s_k \leq Ck^3\sqrt{\lambda_k(\Delta_s)}.$$

1.4 A relation between simplicial complexes and signed graphs

The aim of the present paper is to provide new Cheeger-type inequalities for the first nontrivial eigenvalues of $L_d(\Sigma)$, $L^{up}_d(\Sigma)$ and $L^{down}_d(\Sigma)$. As explained in Section 1.1, it suffices to consider $L^{up}_d(\Sigma)$ for every d. And as stated in that section, this operator depends on the choice of scalar products. With an appropriate choice, we obtain the *normalized* Laplacian, which we denote by Δ^{up}_d, in order to distinguish it from the general case. It is given by

$$(\Delta^{up}_d f)([\sigma]) = f([\sigma]) + \frac{1}{\deg\sigma} \sum_{\substack{\sigma'\in\Sigma_d:\sigma\neq\sigma',\\ \sigma,\sigma'\in\partial\rho}} \operatorname{sgn}([\sigma], \partial[\rho]) \operatorname{sgn}([\sigma'], \partial[\rho]) f([\sigma']),$$

Our results will be obtained for this operator, and they only partially generalize to a general L^{up}_d.

A key step is to express the up-Laplacian of a simplicial complex in terms of the Laplacian of an associated signed graph. The normalized up-Laplacian of a simplicial complex Σ can be written as

$$(\Delta^{up}_d f)([\sigma]) = f([\sigma]) - \frac{1}{\deg\sigma} \sum_{\substack{\sigma'\in\Sigma_d:\sigma\neq\sigma',\\ \sigma,\sigma'\in\partial\rho}} s([\sigma], [\sigma']) f([\sigma']), \tag{18}$$

where we have put

$$s([\sigma],[\sigma']) := -\operatorname{sgn}([\sigma],\partial[\rho])\operatorname{sgn}([\sigma'],\partial[\rho]). \tag{19}$$

Thus, we may express Δ_d^{up} in terms of the Laplacian $\Delta_{(\Gamma_d,s)}$ for the signed graph (Γ_d, s) with vertex set consisting of the d-simplices of our simplicial complex, and where two different such vertices σ, σ' are connected by an edge, $\sigma \sim \sigma'$, if there exists a $(d+1)$-simplex ρ in Σ with $\sigma, \sigma' \in \partial\rho$.

Remark 2. *This construction is very natural and essentially follows from the definition of the (up/down) combinatorial Laplacian matrices of a simplicial complex. A similar idea was already used to define the signed adjacency matrix of a triangulation on a surface [18].*

The relation between the up-Laplacian and the signed graph Laplacian (16) is

$$\Delta_d^{up} = (d+1)\Delta_{(\Gamma_d,s)} - d\ \mathrm{Id}. \tag{20}$$

By (20), the eigenvalues μ_j of Δ_d^{up} and the eigenvalues λ_j of $\Delta_{(\Gamma_d,s)}$ are related by

$$\mu_j = (d+1)\lambda_j - d. \tag{21}$$

Since the eigenvalues of $\Delta_{(\Gamma_d,s)}$ lie in the interval $[0, 2]$, those of Δ_d^{up} lie in the interval $[0, d+2]$. In fact, since $\mu_j \geq 0$ in (21), the eigenvalues of $\Delta_{(\Gamma_d,s)}$ are $\geq \frac{d}{d+1}$. Equality holds if and only if there is some non-trivial f with $\partial^d f = 0$. More precisely, the multiplicity of the eigenvalue $\frac{d}{d+1}$ of $\Delta_{(\Gamma_d,s)}$ equals the dimension of the kernel of the coboundary operator δ_d. In particular, for $d > 0$, the graph (Γ_d, s) is never balanced.

The next result then is an easy consequence of Lemma 1.2.

Proposition 1.1. *The spectrum of Δ_d^{up} contains the eigenvalue $d+2$ if and only if the signed graph (Γ_d, s) has an antibalanced component. Moreover, the multiplicity of $d+2$ equals the number of antibalanced components of (Γ_d, s).*

We consider the opposite $(\Gamma_d, -s)$ of the signed graph (Γ_d, s), with its Laplacian $\Delta_{(\Gamma_d,-s)}$. Then, the eigenvalues of the three Laplacians Δ_d^{up}, $\Delta_{(\Gamma_d,s)}$ and $\Delta_{(\Gamma_d,-s)}$ satisfy the relation:

$$\begin{array}{ccccc}
\text{Spectrum of } \Delta_d^{up} & & \text{Spectrum of } \Delta_{(\Gamma_d,s)} & & \text{Spectrum of } \Delta_{(\Gamma_d,-s)} \\
 & & & & \\
0 & & \frac{d}{d+1} & & \frac{d+2}{d+1} \\
\vdots & & \vdots & & \vdots \\
\lambda & \Longleftrightarrow & \frac{\lambda+d}{d+1} & \Longleftrightarrow & \frac{d+2-\lambda}{d+1} \\
\vdots & & \vdots & & \vdots \\
d+2 & & 2 & & 0
\end{array}$$

that is,

Proposition 1.2. *λ is an eigenvalue of Δ_d^{up} if and only if $\frac{\lambda+d}{d+1}$ is an eigenvalue of $\Delta_{(\Gamma_d,s)}$ if and only if $\frac{d+2-\lambda}{d+1}$ is an eigenvalue of $\Delta_{(\Gamma_d,-s)}$.*

In addition, analogously to Proposition 1.1

Proposition 1.3. *The multiplicity of the eigenvalue 0 of Δ_d^{up} is $\geq d+1$ (when the simplicial complex is pure, the multiplicity of the eigenvalue 0 of Δ_d^{up} is $d+1$ if and only if the simplicial complex is a simplex of dimension $d+1$). And the multiplicity of the eigenvalue $d+2$ of Δ_d^{up} agrees with the number of balanced components of $\Delta_{(\Gamma_d,-s)}$.*

1.5 p-Laplacians

An essential feature of Cheeger-type inequalities is that they connect an L^2-quantity, the smallest nontrivial eigenvalue of the Laplacian, with an L^1-quantity, the Cheeger constant. Therefore, it seems natural to interpolate between the exponents 2 and 1. This can be done, as we shall briefly explain now, but the case $p = 1$, which is the case of most interest, creates additional difficulties. But in fact, for $p = 1$, the inequalities that we are after become equalities, and this conversely is useful for deriving the inequality for $p = 2$.

Thus, similar to the up and down Laplacians on simplicial complexes (see Section 1.1), we shall now introduce the p-Laplace operators on $C^d(\Sigma)$. For $p > 1$, we put

$$\alpha_p : (t_1, t_2, \cdots) \mapsto (|t_1|^{p-2}t_1, |t_2|^{p-2}t_2, \cdots).$$

Since this becomes undetermined for $p = 1$ when $t = 0$, we need to modify the definition and let it be set valued, that is,

$$\alpha_1 : (t_1, t_2, \cdots) \mapsto \{(\xi_1, \xi_2, \cdots) : \xi_i \in \mathrm{Sgn}(t_i)\},$$

with

$$\mathrm{Sgn}(t) := \begin{cases} \{1\} & \text{if } t > 0, \\ [-1, 1] & \text{if } t = 0, \\ \{-1\} & \text{if } t < 0, \end{cases}$$

We can then define the d-th up p-Laplace operator

$$L_{d,p}^{up} := \delta_d^* \alpha_p \delta_d,$$

having for $f \in C^d(\Sigma)$,

$$L_{d,p}^{up} f = B_{d+1} \alpha_p (B_{d+1}^\top f)$$

where we identify δ_d with its standard matrix representation $B_{d+1}^\top$. Analogously, we can also define the d-th down p-Laplace operator $L_{d,p}^{down} := \delta_{d-1} \alpha_p \delta_{d-1}^*$, having for $f \in C^d(\Sigma)$, $L_{d,p}^{down} f = B_d^\top \alpha_p (B_d f)$, and the d-th p-Laplace operator as $L_{d,p} := L_{d,p}^{up} + L_{d,p}^{down}$.

The eigenvalue problem of $L_{d,p}^{up}$ is to find real numbers λ and nonzero functions $f : \Sigma_d \to \mathbb{R}$ satisfying

$$L_{d,p}^{up} f = \lambda \alpha_p(f), \text{ for the case of } p > 1,$$

or

$$0 \in L_{d,1}^{up} f - \lambda \alpha_1(f), \text{ for the case of } p = 1.$$

In the case of $d = 0$, the above nonlinear eigenproblem is actually the spectral problem for the graph p-Laplacian [2,7,12,24]. Of most interest for us will be the min-max eigenvalues, that is, those that can be obtained from Rayleigh quotients as in Lemma 1.1. Thus, we look for

$$\lambda_i(L_{d,p}^{up}) := \inf_{\gamma(S)\geq i} \sup_{f\in S} \frac{\|B_{d+1}^\top f\|_p^p}{\|f\|_p^p}, \quad i = 1, 2, \cdots, n, \tag{22}$$

where $n = \#\Sigma_d$, and

$$\gamma(S) := \begin{cases} \min\{k \in \mathbb{Z}^+ : \exists \text{ odd continuous map } \varphi : S \to \mathbb{S}^{k-1}\} & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

denotes the Krasnoselskii genus of a centrally symmetric set $S \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$. As already indicated, the important case of (22) will be $p = 1$.

Obviously, analogous constructions work for $L_{d,p}^{down}$.

2 Cheeger-type inequalities on d-faces of simplicial complexes

As explained in Section 1.1, we shall work on an abstract simplicial complex Σ with vertex set $V = \{1, \cdots, n\}$. For $\sigma = \{i_0, \cdots, i_d\} \in \Sigma$, we use $[\sigma] := [i_0, \cdots, i_d]$ to indicate the oriented d-dimensional simplex which is formed by σ when arranging its vertices in the specified order. We then let $[\Sigma_d] = \{[\sigma] : \sigma \in \Sigma_d\}$ be the set of all oriented d-simplexes.

Analogously to the cochain group $C^d(\Sigma)$, the d-th chain group $C_d(\Sigma)$ of Σ is a vector space with the basis $[\Sigma_d]$. The boundary map $\partial_d : C_d(\Sigma) \to C_{d-1}(\Sigma)$ is a linear operator defined by

$$\partial_d[i_0, \cdots, i_d] = \sum_{j=0}^{d} (-1)^j [i_0, \cdots, i_{j-1}, i_{j+1}, \cdots, i_d],$$

which can also be represented by the incidence matrix B_d of dimension $|\Sigma_{d-1}| \times |\Sigma_d|$ whose elements belong to $\{-1, 0, 1\}$.

With this notation, the d-th cochain group $C^d(\Sigma)$ is the dual of the chain group $C_d(\Sigma)$. The simplicial coboundary map $\delta_d : C^d(\Sigma) \to C^{d+1}(\Sigma)$ is a linear operator generated by $(\delta_d f)([i_0, \cdots, i_{d+1}]) = \sum_{j=0}^{d+1} (-1)^j f([i_0, \cdots, i_{j-1}, i_{j+1}, \cdots, i_{d+1}])$ for any $f \in C^d(\Sigma)$. It is obvious that $\delta_d = B_{d+1}^\top$, and we can then define the adjoint via $\delta_d^* = B_{d+1}$. We can therefore also use the incidence matrices to express the Laplace operators (see [27]):

- the d-th up Laplace operator $L_d^{up} = \delta_d^* \delta_d = B_{d+1} B_{d+1}^\top$
- the d-th down Laplace operator $L_d^{down} = \delta_{d-1} \delta_{d-1}^* = B_d^\top B_d$
- the d-th Laplace operator $L_d = L_d^{up} + L_d^{down} = \delta_d^* \delta_d + \delta_{d-1} \delta_{d-1}^* = B_d^\top B_d + B_{d+1} B_{d+1}^\top$

2.1 Spectral gap from $d + 2$

Here, we shall build upon Sections 1.1 and 1.4. Again, the key is to convert a Cheeger problem for higher dimensional simplices into one for signed graphs. We thus suggest the following Cheeger-type constants.

As always, we consider a simplicial complex Σ, and we denote the collection of its d-dimensional simplices by Σ_d. We shall need a slight modification of the construction in Section 1.4. Hereafter, we will consider the signed graph (Γ_d, s) on the vertex set Σ_d, under the up adjacency relation, and with the sign function

$$s([\tau],[\tau']) = \operatorname{sgn}([\tau],\partial[\sigma])\operatorname{sgn}([\tau'],\partial[\sigma]) \tag{23}$$

which is the opposite of the sign function defined in (19).

For disjoint $A, A' \subset \Sigma_d$, let $|E^+(A,A')| = \#\{\{\tau,\tau'\} : \tau \in A, \tau' \in A', s([\tau],[\tau']) = 1\}$ and $|E^-(A)| = \#\{\{\tau,\tau'\} : \tau,\tau' \in A, s([\tau],[\tau']) = -1\}$. Let

$$\beta(A,A') = \frac{2\left(|E^-(A)| + |E^-(A')| + |E^+(A,A')|\right) + |\partial(A \sqcup A')|}{\operatorname{vol}(A \sqcup A')}$$

where $|\partial A|$ is the number of the edges of (Γ_d, s) that cross A and $\Sigma_d \setminus A$, $\operatorname{vol}(A) = \sum_{\tau \in A} \deg \tau$ and $\deg \tau = \#\{\sigma \in \Sigma_{d+1} : \tau \subset \sigma\}$.

Then we introduce the k-th Cheeger constant on Σ_d:

$$h_k(\Sigma_d) = \min_{\text{disjoint } A_1, A_2, \ldots, A_{2k-1}, A_{2k} \text{ in } \Sigma_d} \max_{1 \le i \le k} \beta(A_{2i-1}, A_{2i}).$$

$h_k(\Sigma_d) = 0$ if and only if (Γ_d, s) has exactly k balanced components.

Remark 3. *For $d = 0$, the constant $h_k(\Sigma_0)$ reduces to the k-way Cheeger constant of a graph [33].*

Theorem 2.1. *For any simplicial complex and every $d \ge 0$,*

$$\frac{h_1(\Sigma_d)^2}{2(d+1)} \le d + 2 - \lambda_n(\Delta_d^{up}) \le 2h_1(\Sigma_d), \tag{24}$$

where $n = \#\Sigma_d$. Moreover, there exists an absolute constant C such that for any simplicial complex, and for any $k \ge 1$,

$$\frac{h_k(\Sigma_d)^2}{Ck^6(d+1)} \le d + 2 - \lambda_{n+1-k}(\Delta_d^{up}) \le 2h_k(\Sigma_d). \tag{25}$$

Proof. We first show

$$d + 2 - \lambda_{n-i+1}(\Delta_d^{up}) = (d+1)\lambda_i(\Delta_{(\Gamma_d,s)}), \quad i = 1, \ldots, n. \tag{26}$$

We have

$$\begin{aligned}&(d+2)\sum_{\tau \in \Sigma_d} \deg_\tau f(\tau)^2 - \sum_{\sigma \in \Sigma_{d+1}} \left(\sum_{\tau \in \Sigma_d, \tau \subset \sigma} \operatorname{sgn}([\tau],\partial[\sigma]) f(\tau) \right)^2 \\ = &\sum_{[\tau]\sim[\tau']} \left(f(\tau) - \operatorname{sgn}([\tau],\partial[\sigma])\operatorname{sgn}([\tau'],\partial[\sigma]) f(\tau') \right)^2.\end{aligned}$$

Recalling (23), this yields the identity

$$d + 2 - \frac{\sum\limits_{\sigma \in \Sigma_{d+1}} \left(\sum_{\tau \in \Sigma_d, \tau \subset \sigma} \operatorname{sgn}([\tau],\partial[\sigma]) f(\tau) \right)^2}{\sum_{\tau \in \Sigma_d} \deg_\tau f(\tau)^2} = (d+1) \frac{\sum_{[\tau]\sim[\tau']} \left(f(\tau) - s(\tau,\tau') f(\tau') \right)^2}{\sum_{\tau \in \Sigma_d} \widetilde{\deg_\tau} f(\tau)^2}$$

for the Rayleigh quotients, where $[\tau] \sim [\tau']$ represents an edge in the underlying signed graph (Γ_d, s), and $\widetilde{\deg}_\tau = (d+1)\deg_\tau$ is the degree of τ in (Γ_d, s). (Whenever $\tau \subset \sigma \in \Sigma_{d+1}$, this connects τ with $d+1$ other d-simplices.) Recalling Lemma 1.1, this shows (26).

$\frac{1}{d+1}h_k(\Sigma_d)$ is the k-th Cheeger constant of the signed graph (Γ_d, s). By the Cheeger inequality (17) for signed graphs, we have

$$\frac{\lambda_1(\Delta_{(\Gamma_d,s)})}{2} \le \frac{h_1(\Sigma_d)}{d+1} \le \sqrt{2\lambda_1(\Delta_{(\Gamma_d,s)})}.$$

And by Theorem 1.1, there exists an absolute constant C such that for any signed graph and any $k \ge 1$,

$$\frac{\lambda_k(\Delta_{(\Gamma_d,s)})}{2} \le \frac{h_k(\Sigma_d)}{d+1} \le Ck^3\sqrt{\lambda_k(\Delta_{(\Gamma_d,s)})}.$$

In consequence, we obtain

$$\frac{d+2-\lambda_n(\Delta_d^{up})}{2} \le h_1(\Sigma_d) \le \sqrt{2(d+1)(d+2-\lambda_n(\Delta_d^{up}))}$$

and

$$\frac{d+2-\lambda_{n+1-k}(\Delta_d^{up})}{2} \le h_k(\Sigma_d) \le Ck^3\sqrt{(d+1)(d+2-\lambda_{n+1-k}(\Delta_d^{up}))}.$$

Then, we have verified (24) and (25). □

By Theorem 2.1, $\lambda_n(\Delta_d^{up}) = d+2$ if and only if $h_1(\Sigma_d) = 0$, if and only if the associated signed graph (Γ_d, s) has a balanced component. The latter fact follows from Proposition 1.1, remembering that the sign we are currently using is the opposite of the one in that proposition.

2.2 Spectral gap from 0

Theorem 2.1 of the previous section contains the estimates for the spectral gap from $d+2$. However, the more important estimate is the one for the spectral gap from 0, namely, the Cheeger-type estimate for the first non-trivial eigenvalue of the Eckmann Laplacian. For that purpose, we shall now introduce a new Cheeger constant. The key point is that we consider generalized (i.e., with both positive and negative multiplicities) multisets of d-simplices, in order to be able to take account of (positive or negative) multiplicities, as these also enter into the coboundary relations and therefore implicitly into the eigenvalues.

(D1) A *(generalized) multiset* is a pair (S, m), where S is the underlying set of the multiset, formed from its distinct elements, and $m : S \to \mathbb{Z}$ is an integer-valued function, giving the *multiplicity*. We point out that this multiplicty is allowed to also take negative values, in order to account for orientations. For convenience, we usually write S instead of (S, m), and simply speak of a multiset, and we use $|S| := \sum_{s\in S}|m(s)|$ to indicate the *size* of the multiset S.

As the underlying set, we take Σ_d. We write $S \subset_M \Sigma_d$ when S is a multiset on the underlying set Σ_d with multiplicities in $\{-M, \ldots, 0, \ldots, M\}$. The coboundary $\partial^*_{d+1}S$ of such a multiset S is defined as the multiset of all $(d+1)$-simplices that have a member of S in its boundary, together with the appropriate multiplicities. Thus, each $\sigma \in \Sigma_{d+1}$ has the multiplicity $\sum_{\tau\in\Sigma_d} m(\tau)\mathrm{sgn}([\tau], \partial[\sigma])$, where $m(\tau)$ is the multiplicity of τ in S. And the support of $\partial^*_{d+1}S$ then consists of all such simplices with non-zero multiplicity. We define $\mathrm{vol}(S) := \sum_{\tau\in\Sigma_d} \deg_\tau |m(\tau)|$ as the volume of the multiset S.

Definition 2.1. *For $d \geq 0$,*

$$h(\Sigma_d) = \min_{\substack{S\subset_M\Sigma_d \\ S\neq\partial^*_d(T),\forall T\subset_M\Sigma_{d-1}}} \frac{|\partial^*_{d+1}S|}{\min\limits_{S'\neq\emptyset:\partial^*_{d+1}S'=\partial^*_{d+1}S} \mathrm{vol}(S')} \tag{27}$$

is constant when M is sufficiently large. And for such a large number M, we call $h(\Sigma_d)$ the Cheeger constant *on Σ_d.*

(D2) We shall now give several different definitions of $h(\Sigma_d)$, and then show that these definitions all agree. First, we describe the Cheeger constant as the $\mathbb{Z}$-expander:

Definition 2.2. *Let*

$$h(\Sigma_d) = \min_{\phi\in C^d(\Sigma,\mathbb{Z})\setminus\mathrm{Im}\,\delta} \frac{\|\delta\phi\|_1}{\min\limits_{\psi\in\mathrm{Im}\,\delta} \|\phi+\psi\|_{1,\deg}}.$$

We point out that in contrast to the definition (15) of $h^d(\Sigma)$, here we use $\mathbb{Z}$- instead of $\mathbb{Z}_2$-coefficients, and we use the (weighted) l^1-norm, where $\|\phi\|_{1,\deg} := \sum_{\tau\in\Sigma_d} \deg_\tau |\phi(\tau)|$, instead of the Hamming norm.

(D3) Anticipating Section 3, and similar to the graph 1-Laplacian, we define the up 1-Laplacian eigenvalue problem on Σ_d as the nonlinear eigenvalue problem

$$0 \in \nabla\|B^\top_{d+1}\mathbf{x}\|_1 - \lambda\nabla\|\mathbf{x}\|_{1,\deg} \tag{28}$$

where ∇ represents the usual subgradient [11]. We let $\lambda_{I_d}(\Delta^{up}_{d,1})$ be the smallest non-trivial eigenvalue of the up 1-Laplacian, where $I_d := \dim\mathrm{Image}(B^\top_d) + 1 = \mathrm{rank}(B_d) + 1$. To describe $\lambda_{I_d}(\Delta^{up}_{d,1})$, we first introduce orthogonality w.r.t. a given norm. For a norm $\|\cdot\|$ on a real linear space with an inner product $\langle\cdot,\cdot\rangle$, we say that $\mathbf{x}$ is $\|\cdot\|$-*orthogonal* to $\mathbf{y}$ if there exists $\mathbf{u} \in \nabla\|\mathbf{x}\|$ satisfying $\langle\mathbf{u},\mathbf{y}\rangle = 0$. We say $\mathbf{x}$ is $\|\cdot\|$-orthogonal to a non-empty set Y if $\mathbf{x}$ is $\|\cdot\|$-orthogonal to all $\mathbf{y} \in Y$. Clearly, if $\|\cdot\| = \|\cdot\|_2$ is the standard l^2-norm, then the $\|\cdot\|_2$-orthogonality reduces to the usual orthogonality w.r.t. the standard inner product.

Definition 2.3. *Let*

$$h(\Sigma_d) = \lambda_{I_d}(\Delta^{up}_{d,1}) = \min_{\mathbf{x}\perp^1\mathrm{Image}(B^\top_d)} \frac{\|B^\top_{d+1}\mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}}$$

where $\mathbf{x}\perp^1\mathrm{Image}(B^\top_d)$ indicates that $\mathbf{x}$ is $\|\cdot\|_{1,\deg}$-orthogonal to $\mathrm{Image}(B^\top_d)$, i.e. $\mathbf{u} \in \mathrm{Image}(B^\top_d)^\perp$ for some $\mathbf{u} \in \nabla\|\mathbf{x}\|_{1,\deg}$.

(D4) The norm $\|\cdot\|_{1,\deg}$ on $C^d(\Sigma)$ induces a quotient norm on $C^d(\Sigma)/\mathrm{image}(\delta_{d-1})$, which will be denoted by $\|\cdot\|$ for simplicity. More precisely, for any equivalence class $[\mathbf{x}]\in C^d(\Sigma)/\mathrm{image}(\delta_{d-1})$, let $\|[\mathbf{x}]\|=\inf\limits_{x'\in[x]}\|\mathbf{x}'\|_{1,\deg}$. Then

$$h(\Sigma_d)=\min_{0\neq[\mathbf{x}]\in C^d(\Sigma)/\mathrm{image}(\delta_{d-1})}\frac{\|\delta_d\mathbf{x}\|_1}{\|[\mathbf{x}]\|}=\min_{0\neq[\mathbf{x}]\in C^d(\Sigma,\mathbb{Z})/\mathrm{image}(\delta_{d-1})}\frac{\|\delta_d\mathbf{x}\|_1}{\|[\mathbf{x}]\|}.$$

Definition 2.4. *In the case of $\tilde{H}^d(\Sigma,\mathbb{R})=0$, let*

$$h(\Sigma_d)=\min_{\mathbf{y}\in\mathrm{image}(\delta_d)}\frac{\|\mathbf{y}\|_1}{\|\mathbf{y}\|_{\mathrm{fil}}}=\frac{1}{\max\limits_{\mathbf{y}\in\mathrm{image}(\delta_d)}\|\mathbf{y}\|_{\mathrm{fil}}/\|\mathbf{y}\|_1}=\frac{1}{\|\delta_d^{-1}\|_{\mathrm{fil}}}$$

where $\|\mathbf{y}\|_{\mathrm{fil}}:=\inf\limits_{x\in\delta_d^{-1}(\mathbf{y})}\|\mathbf{x}\|_{1,\deg}$ is the filling norm of $\mathbf{y}$, and $\|\delta_d^{-1}\|_{\mathrm{fil}}:=\max\limits_{\mathbf{y}\in\mathrm{image}(\delta_d)}\|\mathbf{y}\|_{\mathrm{fil}}/\|\mathbf{y}\|_1$ is called the filling profile by Gromov (see Section 2.3 in [19]).

Theorem 2.2. *The four definitions in (D1)–(D4) are equivalent.*

Proof. We start with (D3). Since $\mathrm{Image}(B_d^\top)\subset\mathrm{Ker}(B_{d+1}^\top)$, by Theorem 2.1 in [32],

$$\lambda_{I_d}(\Delta_{d,1}^{up})=\inf_{\mathbf{x}\in\mathbb{R}^n\setminus\mathrm{Image}(B_d^\top)}\frac{\|B_{d+1}^\top\mathbf{x}\|_1}{\inf\limits_{\mathbf{z}\in\mathrm{Image}(B_d^\top)}\|\mathbf{x}+\mathbf{z}\|_{1,\deg}}\tag{29}$$

$$=\inf_{[\mathbf{x}]\in\mathbb{R}^n/\mathrm{Image}(B_d^\top)}\frac{\|B_{d+1}^\top\mathbf{x}\|_1}{\|[\mathbf{x}]\|}\tag{30}$$

$$=\inf_{\mathbf{x}\in\mathbb{R}^n:\nabla\|\mathbf{x}\|_{1,\deg}\bigcap\mathrm{Image}(B_d^\top)^\perp\neq\emptyset}\frac{\|B_{d+1}^\top\mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}}\tag{31}$$

where $n=\#\Sigma_d$, $\|[\mathbf{x}]\|=\inf\limits_{\mathbf{x}'\in[\mathbf{x}]}\|\mathbf{x}+\mathbf{z}\|_{1,\deg}$ and $[\mathbf{x}]=\{\mathbf{y}\in\mathbb{R}^n:\mathbf{y}-\mathbf{x}\in\mathrm{Image}(B_d^\top)\}$. In fact, the definition of the norm $\|\cdot\|$ on the quotient space $\mathbb{R}^n/\mathrm{Image}(B_d^\top)$ implies $\|[\mathbf{x}]\|=\inf\limits_{\mathbf{z}\in\mathrm{Image}(B_d^\top)}\|\mathbf{x}+\mathbf{z}\|_{1,\deg}$. Moreover, Proposition 2.3 in [32] yields that $\|[\mathbf{x}]\|=\|\mathbf{x}\|_{1,\deg}$ if and only if $\mathbf{x}$ satisfies $\nabla\|\mathbf{x}\|_{1,\deg}\bigcap\mathrm{Image}(B_d^\top)^\perp\neq\emptyset$, that is, the minimization problem

$$\inf_{\mathbf{x}'\in\mathbb{R}^n:\mathbf{x}'-\mathbf{x}\in\mathrm{Image}(B_d^\top)}\|\mathbf{x}'\|_{1,\deg}$$

reaches its minimum in the set $\{\mathbf{x}\in\mathbb{R}^n:\nabla\|\mathbf{x}\|_{1,\deg}\bigcap\mathrm{Image}(B_d^\top)^\perp\neq\emptyset\}$. So, the above three quantities (29), (30) and (31) coincide. Using the l^1-type orthogonal notation $\perp^1$, since $\mathbf{x}\perp^1\mathrm{Image}(B_d^\top)$ means that $\mathbf{u}\perp\mathrm{Image}(B_d^\top)$ for some $\mathbf{u}\in\nabla\|\mathbf{x}\|_{1,\deg}$, the constraint $\{\mathbf{x}\in\mathbb{R}^n:\nabla\|\mathbf{x}\|_{1,\deg}\bigcap\mathrm{Image}(B_d^\top)^\perp\neq\emptyset\}$ in (31) can be reduced to $\{\mathbf{x}\in\mathbb{R}^n:\mathbf{x}\perp^1\mathrm{Image}(B_d^\top)\}$ as shown in (D3).

Similar to the proof of Proposition 3.7 in [31], we can apply Theorem 2.4 in [32] to derive that every eigenvalue of the up 1-Laplacian eigenproblem (28) has an eigenvector in the set of the extreme points associated with the function pair $(\|B_{d+1}^\top\cdot\|_1,\|\cdot\|_{1,\deg})$ since both $\|B_{d+1}^\top \cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are piecewise linear. We shall now describe these extreme points in more detail.

The unit l^1-sphere $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_{1,\deg} = 1\}$ can be represented as a union of finitely many convex polytopes of dimension $(n-1)$ such that both $\|B_{d+1}^\top \cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are linear on each convex polytope. Let k be the smallest possible number of such convex polytopes, and let $\{P_1, \cdots, P_k\}$ be the family of convex polytopes of dimension $(n-1)$, i.e. $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_{1,\deg} = 1\} = P_1 \cup \cdots \cup P_k$ and $\|B_{d+1}^\top \cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are linear when restricted on P_i for any i. Denote by $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$ the union of the vertices of P_i for all i. Clearly, $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$ is a finite set, and its elements are called the extreme points determined by the function pair $(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$.

Since all the entries of the matrix $B_{d+1}^\top$ and the degrees are rational numbers, by the theory of systems of linear equations, $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg}) \subset \mathbb{Q}^n$. Let M be a sufficiently large natural number that is greater than the least common multiple of all the denominators of the components of all points in $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$. Then, $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg}) \subset \{t\mathbf{x} : t \geq 0 \text{ and } \mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n\}$, and thus every eigenvalue has an eigenvector in $\{t\mathbf{x} : t \geq 0 \text{ and } \mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n\}$. Since both $\|B_{d+1}^\top \cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are positively one-homogeneous, we further derive that every eigenvalue has an eigenvector in the set $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n$. Moreover, the minimizations (29) and (31) can reach their minima at some points in $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n$, and the minimization problem (30) achieves its minima at some equivalence class $[\mathbf{x}]$ for some $\mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n$. That means, we can use $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n$ instead of $\mathbb{R}^n$ in the constraints of these three minimization problems (29), (30) and (31). It follows from $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n \subset \mathbb{Z}^n \subset \mathbb{R}^n$ that one can also replace $\mathbb{R}^n$ by $\mathbb{Z}^n$ in the constraints of these three minimization problems (29), (30) and (31).

We now proceed to prove the equivalence of (D1)–(D4).

Using $\mathbb{Z}^n$ instead of $\mathbb{R}^n$ in (29), and equivalently converting the notions $\mathbb{Z}^n$ to $C^d(K, \mathbb{Z})$, and $\mathrm{Image}(B_d^\top)$ to $\mathrm{Im}\,\delta$, we obtain that (D2) is a reformulation of (29). Similarly, (D4) is a reformulation of (30). And, if $\tilde{H}^d(K, \mathbb{R}) = 0$, then $\mathrm{image}(\delta_{d-1}) = \ker(\delta_d)$, which implies $C^d(K)/\mathrm{image}(\delta_{d-1}) = C^d(K)/\ker(\delta_d) \cong \mathrm{image}(\delta_d)$ and $\delta_d^{-1}(\mathbf{y}) = [\mathbf{x}]$ for any $\mathbf{y} \in \mathrm{image}(\delta_d)$. Note that the filling norm $\|\mathbf{y}\|_{\text{fil}} := \inf_{x \in \delta_d^{-1}(\mathbf{y})} \|\mathbf{x}\|_{1,\deg}$ coincides with $\|[\mathbf{x}]\|$, and $\|\mathbf{y}\|_1 = \|\delta_d \mathbf{x}\|_1 = \|B_{d+1}^\top \mathbf{x}\|_1$. So, (D4) and (30) indicate the same quantity.

Using $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n$ instead of $\mathbb{R}^n$ in (29), we can similarly identify every generalized multiset $S \subset_M \Sigma_d$ with a unique $\mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n$ by identifying x_τ with $m(\tau)$ for any $\tau \in \Sigma_d$, where $m(\tau)$ is the generalized multiplicity of τ in S. Then, for such a couple of S and $\mathbf{x}$, $\mathrm{vol}(S) = \|\mathbf{x}\|_{1,\deg}$, and $|\partial_{d+1}^* S| = \|B_{d+1}^\top \mathbf{x}\|_1$. If $\tilde{H}^d(\Sigma, \mathbb{R}) \neq 0$, then $\mathrm{image}(B_d^\top)$ is a proper subset of $\ker(B_{d+1}^\top)$, and thus (29) is zero, and in this case, there exists $S' \neq \emptyset$ such that $\partial_{d+1}^* S' = \partial_{d+1}^* S = \emptyset$, which means (27) also equals zero. If $\tilde{H}^d(\Sigma, \mathbb{R}) = 0$, then $\mathrm{image}(B_d^\top) = \ker(B_{d+1}^\top)$, and thus for such a couple of S and $\mathbf{x}$ with $\mathbf{x} \notin \ker(B_{d+1}^\top)$,

$$\inf_{\mathbf{z} \in \mathrm{image}(B_d^\top)} \|\mathbf{x} + \mathbf{z}\|_{1,\deg} = \inf_{\mathbf{x}' \in \mathbb{R}^n : \mathbf{x}' - \mathbf{x} \in \ker(B_{d+1}^\top)} \|\mathbf{x}'\|_{1,\deg} = \min_{S' \neq \emptyset : \partial_{d+1}^* S' = \partial_{d+1}^* S} \mathrm{vol}(S').$$

Therefore, (27) and (29) actually represent the same quantity which has been denoted by $h(\Sigma_d)$. The whole proof is then completed. $\square$

It is very useful that the four definitions in (D1)–(D4) represent the same Cheeger constant $h(\Sigma_d)$ from different viewpoints.

(D1) provides a combinatorial explanation of the Cheeger constant $h(\Sigma_d)$ using the language of multi-sets in combinatorics, which means that our Cheeger constant is actually a combinatorial quantity.

(D2) says that our Cheeger constant is indeed a $\mathbb{Z}$-expander, and it is clear that

$$h(\Sigma_d) = 0 \Longleftrightarrow \tilde{H}^d(\Sigma, \mathbb{R}) \neq 0, \ \ \forall d \geq 0.$$

As we have discussed, the Cheeger constant defined as an $\mathbb{Z}_2$-expander violates the Cheeger inequality on simplicial complexes. However, with a $\mathbb{Z}$-expander it is possible to get a Cheeger inequality.

(D3) says that our Cheeger constant coincides with the smallest non-trivial 1-Laplacian eigenvalue, which generalizes the equality in both graph and domain settings.

(D4) reveals the non-obvious fact that our Cheeger constant has a deep relation with Gromov's filling profile. This is an equivalent reformulation of (27) using the language of norms on cochain groups, which helps us to further understand the formula (27).

In addition, for sufficiently large numbers $M \in \mathbb{Z}_+$,

$$h(\Sigma_d) \xlongequal{\text{if } \tilde{H}^d(\Sigma,\mathbb{R})=0} \min_{\substack{S \subset_M \Sigma_d \\ \partial^*_{d+1} S \neq \emptyset}} \frac{|\partial^*_{d+1} S|}{\min\limits_{S' : \partial^*_{d+1} S' = \partial^*_{d+1} S} \mathrm{vol}(S')} > 0.$$

For the case of $d = 0$, we can take $M = 1$, and then $h(\Sigma_0)$ reduces to the usual Cheeger constant on graphs. The following result shows that the constant $h(\Sigma_d)$ satisfies Cheeger-type inequalities, and therefore provides our solution to the problem formulated in the introduction.

Proposition 2.1. *Suppose that* $\deg_\tau > 0$*,* $\forall \tau \in \Sigma_d$*. Then,*

$$\frac{h^2(\Sigma_d)}{|\Sigma_{d+1}|} \leq \lambda_{I_d}(\Delta_d^{up}) \leq \mathrm{vol}(\Sigma_d) h(\Sigma_d).$$

Proof. For simplicity, we write $h = h(\Sigma_d)$ and take $\lambda = \lambda_{I_d}(\Delta_d^{up})$. We shall prove

$$\frac{\min\limits_{\tau \in \Sigma_d} \deg_\tau}{\#\Sigma_{d+1}} h^2 \leq \lambda \leq \mathrm{vol}(\Sigma_d) h^2.$$

Let $k = \mathrm{rank}(B_d)$. Then λ and h are the $(k+1)$-th min-max eigenvalues of the d-th up Laplacian and the d-th up 1-Laplacian, respectively. We only need to prove that, for any $k \geq 1$,

$$\sqrt{\frac{1}{\sum\limits_{\tau \in \Sigma_d} \deg_\tau} \lambda_k} \leq h_k \leq \sqrt{\frac{\#\Sigma_{d+1}}{\min\limits_{\tau \in \Sigma_d} \deg_\tau} \lambda_k}.$$

In fact, it is easy to see that

$$\min_\tau \deg_\tau \leq \frac{\|\mathbf{x}\|^2_{1,\deg}}{\|\mathbf{x}\|^2_{2,\deg}} \leq \sum_{\tau \in \Sigma_d} \deg_\tau \quad \text{and} \quad 1 \leq \frac{\|B^\top_{d+1}\mathbf{x}\|_1^2}{\|B^\top_{d+1}\mathbf{x}\|_2^2} \leq \#\Sigma_{d+1}.$$

Hence

$$\frac{1}{\sum_{\tau\in\Sigma_d}\deg_\tau}\frac{\|B_{d+1}^\top\mathbf{x}\|_2^2}{\|\mathbf{x}\|_{2,\deg}^2}\le\frac{\|B_{d+1}^\top\mathbf{x}\|_1^2}{\|\mathbf{x}\|_{1,\deg}^2}\le\frac{\#\Sigma_{d+1}}{\min\limits_\tau\deg_\tau}\frac{\|B_{d+1}^\top\mathbf{x}\|_2^2}{\|\mathbf{x}\|_{2,\deg}^2}.$$

The proof of $\frac{h^2(\Sigma_d)}{\#\Sigma_{d+1}}\le\lambda_{I_d}(\Delta_d^{up})\le\mathrm{vol}(\Sigma_d)h(\Sigma_d)$ is then completed by noting that $h\le 1\le\deg_\tau$, $\forall\tau\in\Sigma_d$. □

Remark 4. *We can also define the down Cheeger constant (for $d\ge 1$)*

$$h_{down}(\Sigma_d):=\min_{x\perp^1\mathrm{Image}(B_{d+1})}\frac{\|B_d\mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}}=\lambda_{I_{d+1}}(\Delta_{d,1}^{down})$$

which possesses a combinatorial reformulation that is similar to (27)*, where $I_{d+1}:=\dim\mathrm{Image}(B_{d+1})+1=\mathrm{rank}(B_{d+1})+1$.*

Consider a d-dimensional combinatorial manifold Σ, that is, a d-dimensional topological manifold possessing a simplicial complex structure. As a manifold, we assume that Σ is connected and has no boundary. Then, B_{d+1} is a $|\Sigma_d|\times 1$ matrix of rank 1, and $I_{d+1}=\dim\mathrm{Image}(B_{d+1})+1=\mathrm{rank}(B_{d+1})+1=1+1=2$. Therefore, in particular, $\lambda_{I_{d+1}}=\lambda_2$. Moreover, the down adjacency relation induces a graph on Σ_d, and we have the Cheeger inequality:

$$\frac{h_{down}^2(\Sigma_d)}{2}\le\lambda_2(\Delta_d^{down})\le 2h_{down}(\Sigma_d).$$

In fact, Theorem 2.7 in [37] closely resembles the above inequality, and the assumption made there for the lower bound that every $(d-1)$-dimensional simplex is incident to at most 2 d-simplices is satisfied for a combinatorial manifold.

In the sequel, M will be used to denote a manifold.

Definition 2.5. *Let M be a d-dimensional orientable compact closed Riemannian manifold. A triangulation T of M is* c-uniform *if there exists $c>1$ such that for any two d-simplexes $\triangle$ and $\triangle'$ in the triangulation T,*

$$\frac{1}{c}<\frac{\mathrm{diam}(\triangle)}{\mathrm{diam}(\triangle')}<c\quad\text{and}\quad\frac{1}{c}<\frac{\mathrm{diam}(\triangle)}{\mathrm{vol}(\triangle)^{\frac{1}{d}}}<c.$$

A triangulation T of M is uniform *if there exist $N>1$ and $c>1$ such that either the number of vertices of T is smaller than N, or T is c-uniform. The constants N and c are called the* uniform parameters *of the triangulation.*

Theorem 2.3. *Let M be an orientable, compact, closed Riemannian manifold of dimension $(d+1)$. Let Σ be a simplicial complex which is combinatorially equivalent to a uniform triangulation of M. Then, there is a Cheeger inequality*

$$\frac{h^2(\Sigma_d)}{C}\le\lambda_{I_d}(\Delta_d^{up})\le C\cdot h(\Sigma_d),$$

where C is a uniform constant which is independent of the choice of Σ. In addition, $h(\Sigma_d)>0$ if and only if $H_1(\Sigma)=0$ (or equivalently, $H_1(M)=0$).

Proof. By Proposition 2.1, $\lambda_{I_d}(\Delta_d^{up}) = 0$ if and only if $h(\Sigma_d) = 0$. So, it suffices to assume that $h(\Sigma_d) > 0$, i.e., $\tilde{H}^d(M) = \tilde{H}^d(\Sigma) = 0$. Since M and Σ are of dimension $(d+1)$, Poincaré duality implies that $\tilde{H}_1(M) = \tilde{H}^d(M) = 0$.

We may assume without loss of generality that M is simply connected, and the triangulation is c-uniform for some $c > 1$, and Σ_d has n elements, where n is a sufficiently large integer.

For any $\epsilon > 0$, there exists $N > 0$ such that any c-uniform triangulation with at least N facets satisfies $\frac{1}{3c^2}\epsilon < \mathrm{diam}(\triangle) < \epsilon$, $\forall \triangle$. Here, we also regard the uniform triangulation as a uniform ϵ-net.

Claim 1 For the down Cheeger constant, we have

$$\frac{d+2}{4}h^2_{down}(\Sigma_{d+1}) \le \lambda_{I_d}(\Delta_d^{up}) \le (d+2)h_{down}(\Sigma_{d+1}).$$

Proof: This is derived by the Cheeger inequality

$$\frac{h^2_{down}(\Sigma_{d+1})}{2} \le \lambda_2(\Delta_{d+1}^{down}) \le 2h_{down}(\Sigma_{d+1})$$

proposed in Remark 4, and the duality property $\lambda_{I_d}(\Delta_d^{up}) = \frac{d+2}{2}\lambda_2(\Delta_{d+1}^{down})$.

Claim 2 The Cheeger constant $h(\Sigma_d)$ and the down Cheeger constant $h_{down}(\Sigma_{d+1})$ satisfy $h(\Sigma_d) \sim h_{down}(\Sigma_{d+1})$, i.e., there exists a uniform constant $C > 1$ such that

$$\frac{1}{C}h_{down}(\Sigma_{d+1}) \le h(\Sigma_d) \le C\, h_{down}(\Sigma_{d+1}).$$

The proof is divided into the following two claims.

Claim 2.1 $\frac{1}{\epsilon}h_{down}(\Sigma_{d+1}) \sim h(M)$

Proof: Let G be the graph with $n := \#\Sigma_{d+1}$ vertices located in the barycenters of all $(d+1)$-simplexes, such that two vertices form an edge in G if and only if these two d-simplexes are down adjacent. We may call G the underlying graph of the triangulation.

Note that $h_{down}(\Sigma_{d+1})$ also indicates the Cheeger constant of the unweighted underlying graph G. An approximation approach developed in [38, 41] implies that the Cheeger constant of a uniform triangulation should approximate the Cheeger constant of the manifold when we equip the edges of the underlying graph of the triangulation with appropriate weights (related to ϵ). In fact, since G is the underlying graph of the triangulation, we may assume that G is embedded in the manifold M, and the distribution of the vertices of G is uniform[1]. Then, according to the approximation theorems in [38,41], by adding appropriate weights (related to ϵ)[2] on G, the Cheeger constant of G (with appropriate edge weights) would approximate $h(M)$ (i.e., the difference of $h(M)$ and the Cheeger constant of the weighted graph G is bounded by $h(M)/2$ whenever ϵ

[1] The vertices of G are well-distributed on M.

[2] The weight of an edge $\{u, v\}$ is determined by the distance of u and v in M, which is about $O(\epsilon)$.

is sufficiently small). We can then adopt the same approximation approach as in [38, 41] (more precisely, a slight modification of the approximation theorem in [38, 39, 41]) to derive that $\frac{1}{\epsilon}h_{down}(\Sigma_{d+1}) \sim h(M)$.

Claim 2.2 $\frac{1}{\epsilon}h(\Sigma_d) \sim h(M)$ whenever $H_1(M) = 0$.

Proof: It is well-known that $H_1(M) = 0$ if and only if $H^d(M) = 0$ if and only if $\mathrm{Ker}(\delta_d) = \mathrm{Im}(\delta_{d-1})$, since M is a compact closed manifold of dimension $(d+1)$. Thus,

$$h(\Sigma_d) = \min_{x \notin \mathrm{Ker}(\delta_d)} \frac{\sum\limits_{\sigma \in \Sigma_{d+1}} \left| \sum\limits_{\tau \in \Sigma_d} \mathrm{sgn}([\tau], \partial[\sigma]) x_\tau \right|}{\min\limits_{z \in \mathrm{Ker}(\delta_d)} \sum\limits_{\tau \in \Sigma_d} 2|x_\tau + z_\tau|}.$$

By the duality theorem (see Lemma 2.5 and Theorem 2.1 in [32], or the main theorem in [42]), we can further obtain

$$h(\Sigma_d) = \min_{y \text{ non-constant}} \frac{\max\limits_{\sigma \overset{\text{down}}{\sim} \sigma'} \frac{1}{2}|y_\sigma - y_{\sigma'}|}{\min\limits_{t \in \mathbb{R}} \max\limits_{\sigma \in \Sigma_{d+1}} |y_\sigma + t|}$$

where $\sigma \overset{\text{down}}{\sim} \sigma'$ means σ and σ' are down adjacent, i.e., they share a common facet. And then by elementary techniques, there is no difficulty to check that the optimization in the right hand side coincides with

$$\min_{\min\limits_{\sigma} y_\sigma + \max\limits_{\sigma} y_\sigma = 0} \frac{\max\limits_{\sigma \overset{\text{down}}{\sim} \sigma'} |y_\sigma - y_{\sigma'}|}{2 \max\limits_{\sigma} |y_\sigma|} = \frac{1}{\mathrm{diam}(G)}$$

where $\mathrm{diam}(G)$ indicates the combinatorial diameter of G. We remark here that we indeed rewrite $h(\Sigma_d)$ as the smallest non-trivial eigenvalue of the ∞-Laplacian, which agrees with $1/\mathrm{diam}(G)$. This argument is similar to a theorem in [30].

Finally, since the triangulation is C-uniform, it is easy to see that

$$\frac{1}{\epsilon}h(\Sigma_d) = \frac{1}{\epsilon \cdot \mathrm{diam}(G)} \sim \frac{1}{\mathrm{diam}(M)}.$$

Hence, $\frac{1}{\epsilon}h(\Sigma_d) \sim h(M)$.

The proof is then completed by combining all the statements above. $\square$

Remark 5.

- *The constant C in Theorem 2.3 depends on the uniform parameters of the triangulation, and the ambient manifold. We hope that it is possible to find a new approach to get a uniform constant that only depends on the dimension d.*
- *Under the same condition as in Theorem 2.3, we further have $\frac{\lambda_{k_d}(\Delta_{d,1}^{up})^2}{C} \le \lambda_{k_d}(\Delta_d^{up}) \le C\lambda_{k_d}(\Delta_{d,1}^{up})$, where $k_d := \dim \mathrm{Ker}(B_{d+1}^\top) + 1$. This inequality coincides with the Cheeger inequality in Theorem 2.3 if and only if $H_1(M) = 0$.*

- *A modification of the proof can deduce that* $\frac{1}{\mathrm{diam}(G)} \sim \lambda_2(G)$ *whenever* G *can be uniformly embedded into such a typical manifold, where* $\lambda_2(G)$ *is the second smallest eigenvalue of the normalized Laplacian on* G.

- *Inspired by the approximation theory for Laplacians on triangulations of manifolds proposed by Dodziuk [13] and Dodziuk-Patodi [15], we hope that it is possible to develop an approximation theory for our Cheeger constants on triangulations of manifolds.*

3 Cheeger-type inequalities for p-Laplacians on simplicial complexes

In this section, we want to study the nonlinear eigenvalue problems for the p-Laplacians introduced in Section 1.5 on simplicial complexes. Importantly, this will provide a perspective to unify some Cheeger-type inequalities.

According to the main theorem in [42], the spectral duality that we had used for the 2-Laplacian now becomes

Proposition 3.1. *The nonzero eigenvalues of the up p-Laplacians are in one-to-one correspondence with those of the down* p^**-Laplacians:*

$$\{\lambda^{\frac{1}{p}} : \lambda \text{ is a nonzero eigenvalue of } L^{up}_{d,p}\} = \{\lambda^{\frac{1}{p^*}} : \lambda \text{ is a nonzero eigenvalue of } L^{down}_{d+1,p^*}\}.$$

Moreover, $\lambda^{\frac{1}{p}}_{n-i}(L^{up}_{d,p}) = \lambda^{\frac{1}{p^*}}_{m-i}(L^{down}_{d+1,p^*})$ *for any* $i = 0, 1, \cdots, \min\{n, m\} - 1$*, where* $n = |\Sigma_d|$ *and* $m = |\Sigma_{d+1}|$.

The case $p = 2$ of Proposition 3.1 is of course the well-known relation between up and down Laplacians that we had already noted in Section 1.1, that is, the nonzero eigenvalues of L^{up}_d and L^{down}_{d+1} coincide.

So, we can concentrate on the up p-Laplacian for investigating the spectra of simplicial complexes. To get more concise results, we will work with the *normalized up p-Laplace operator* $\Delta^{up}_{d,p}$, whose eigenvalues are determined by the critical values of the p-Rayleigh quotient

$$f \mapsto \frac{\|B^{\top}_{d+1} f\|^p_p}{\|f\|^p_{p,\deg}}$$

where $\|f\|^p_{p,\deg} = \sum_{\tau\in\Sigma_d} \deg\tau \cdot |f(\tau)|^p$. Similar to (22), we shall focus on the min-max eigenvalues

$$\lambda_i(\Delta^{up}_{d,p}) := \inf_{\gamma(S)\ge i} \sup_{f\in S} \frac{\|B^{\top}_{d+1} f\|^p_p}{\|f\|^p_{p,\deg}}, \quad i = 1, 2, \cdots, n.$$

Theorem 3.1. *For any simplicial complex and every* $d \ge 0$*, for any* $p \in (1, 2]$*, there exist uniform constants* $C_{p,d} \ge c_{p,d} > 0$ *such that*

$$c_{p,d} h_1(\Sigma_d)^p \le (d+2)^{p-1} - \lambda_n(\Delta^{up}_{d,p}) \le C_{p,d} h_1(\Sigma_d), \tag{32}$$

where $n = |\Sigma_d|$.

Proof. We need the following key claim.

Claim. For any $1 < p \le 2$, and for any integer $k \ge 2$, there exist $M_{p,k} \ge m_{p,k} > 0$ such that for any $\mathbf{x} \in \mathbb{R}^n$,

$$m_{p,k} \sum_{1\le i<j\le k} |x_i - x_j|^p \le k^{p-1} \sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p \le M_{p,k} \sum_{1\le i<j\le k} |x_i - x_j|^p$$

Proof. We only need to prove that

$$m'_{p,k} := \inf_{x \text{ non-constant}} \frac{k^{p-1} \sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p}{\sum_{i,j=1}^k |x_i - x_j|^p} > 0$$

for $p > 1$, and

$$M'_{p,k} := \sup_{x \text{ non-constant}} \frac{k^{p-1} \sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p}{\sum_{i,j=1}^k |x_i - x_j|^p} < +\infty$$

for $1 < p \le 2$. It is clear that $\sum_{i,j=1}^k |x_i - x_j|^p > 0$ if and only if $\mathbf{x}$ is non-constant. By Hölder's inequality, $k^{p-1} \sum_{i=1}^k |x_i|^p \ge |\sum_{i=1}^k x_i|^p$ and the equality holds if and only if $\mathbf{x}$ is constant. Therefore, $\sum_{i,j=1}^k |x_i - x_j|^p > 0$ if and only if $k^{p-1} \sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p > 0$. For any vector $\mathbf{x}$ satisfying $\max_i x_i - \min_i x_i = \delta > 0$, $\delta^p \le \sum_{i,j=1}^k |x_i - x_j|^p \le k^2\delta^p$.

Let $g(\mathbf{x}, t, p) = k^{p-1} \sum_{i=1}^k |x_i + t|^p - |\sum_{i=1}^k x_i + kt|^p$, for $\mathbf{x} \perp \mathbf{1}$ with $\mathbf{x} \ne \mathbf{0}$, $t \in \mathbb{R}$ and $p \ge 1$.

Since $\mathbf{x}$ is non-constant and $p > 1$, by Hölder's inequality, we have $g(\mathbf{x}, t, p) > 0$. Note that $\partial_t g(\mathbf{x}, t, p) = pk^{p-1} \sum_{i=1}^k |x_i+t|^{p-1}\mathrm{sign}(x_i+t) - pk|\sum_{i=1}^k x_i+kt|^{p-1}\mathrm{sign}(\sum_{i=1}^k x_i+kt)$. If $t > k\delta$, by Hölder's inequality, $\partial_t g(\mathbf{x}, t, p) > 0$. Similarly, if $t < -k\delta$, $\partial_t g(\mathbf{x}, t, p) < 0$. Therefore, $t \mapsto g(\mathbf{x}, t, p)$ reaches its minimum on some $t_p \in [-k\delta, k\delta]$. Therefore, $\min_{t\in\mathbb{R}} g(\mathbf{x}, t, p) = \min_{-k\delta\le t\le k\delta} g(\mathbf{x}, t, p)$ is a continuous function of $\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^k x_i = 0, \max_i x_i - \min_i x_i = \delta\}$. Hence, $\min_{\mathbf{x}\perp\mathbf{1}, \max_i x_i - \min_i x_i = \delta} \min_{t\in\mathbb{R}} g(\mathbf{x}, t, p) > 0$. Thus,

$$\begin{aligned}
\inf_{x \text{ non-constant}} \frac{k^{p-1} \sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p}{\sum_{i,j=1}^k |x_i - x_j|^p} &= \inf_{\mathbf{x}\perp\mathbf{1}, \max_i x_i - \min_i x_i = \delta} \min_{t\in\mathbb{R}} \frac{g(\mathbf{x}, t, p)}{\sum_{i,j=1}^k |x_i - x_j|^p} \\
&\ge \frac{1}{k^2\delta^p} \min_{\mathbf{x}\perp\mathbf{1}, \max_i x_i - \min_i x_i = \delta} \min_{t\in\mathbb{R}} g(\mathbf{x}, t, p) > 0.
\end{aligned}$$

Clearly, $g(\mathbf{x}, t, 2) = \sum_{\{i,j\}\subset\{1,\ldots,k\}} (x_i - x_j)^2$ and $g(\mathbf{x}, t, 1) \ge 0$.

Note that $\partial_p g(\mathbf{x}, t, p) = \frac{1}{k} \sum_{i=1}^k |kx_i+kt|^p \ln|kx_i+kt| - |\sum_{i=1}^k x_i+kt|^p \ln|\sum_{i=1}^k x_i+kt|$. Since $s \mapsto s^p \ln s$ is convex and increasing on $s \in (1, +\infty)$, by Jensen's inequality for convex functions, $\partial_p g(\mathbf{x}, t, p) > 0$ whenever $|t| > \delta + 1/k$ and $p > 1$. Therefore,

$$g(\mathbf{x}, t, 1) < g(\mathbf{x}, t, p) < \sum_{\{i,j\}\subset\{1,\ldots,k\}} (x_i - x_j)^2$$

whenever $|t| > \delta + 1/k$ and $1 < p < 2$. Consequently,

$$\sup_{x \text{ non-constant}} \frac{k^{p-1}\sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p}{\sum_{i,j=1}^k |x_i - x_j|^p} = \sup_{\mathbf{x}\perp\mathbf{1}, \max\limits_i x_i - \min\limits_i x_i = \delta} \max_{t\in\mathbb{R}} \frac{g(\mathbf{x},t,p)}{\sum_{i,j=1}^k |x_i - x_j|^p}$$
$$\le \frac{1}{\delta^p} \max_{\mathbf{x}\perp\mathbf{1}, \max\limits_i x_i - \min\limits_i x_i = \delta} \max_{t\in\mathbb{R}} g(\mathbf{x},t,p) < +\infty.$$

The claim is proved.

Now we apply the above claim to estimate the spectral gap of $\lambda_n(\Delta_{d,p}^{up})$ from $(d+2)^{p-1}$. Note that

$$\begin{aligned}
& (d+2)^{p-1} - \lambda_n(\Delta_{d,p}^{up}) \\
= \ & (d+2)^{p-1} - \sup_{f\neq 0} \frac{\sum\limits_{\sigma\in\Sigma_{d+1}} \left|\sum_{\tau\in\Sigma_d,\tau\subset\sigma} \mathrm{sgn}([\tau],\partial[\sigma]) f(\tau)\right|^p}{\sum_{\tau\in\Sigma_d} \deg\tau\cdot|f(\tau)|^p} \\
= \ & \inf_{f\neq 0} \frac{(d+2)^{p-1}\sum_{\tau\in\Sigma_d}\deg\tau\cdot|f(\tau)|^p - \sum\limits_{\sigma\in\Sigma_{d+1}} \left|\sum_{\tau\in\Sigma_d,\tau\subset\sigma} \mathrm{sgn}([\tau],\partial[\sigma]) f(\tau)\right|^p}{\sum_{\tau\in\Sigma_d} \deg\tau\cdot|f(\tau)|^p} \\
= \ & \inf_{f\neq 0} \frac{(d+2)^{p-1}\sum\limits_{\sigma\in\Sigma_{d+1}}\sum\limits_{\tau\in\Sigma_d,\tau\subset\sigma}|f(\tau)|^p - \sum\limits_{\sigma\in\Sigma_{d+1}} \left|\sum_{\tau\in\Sigma_d,\tau\subset\sigma} \mathrm{sgn}([\tau],\partial[\sigma]) f(\tau)\right|^p}{\sum_{\tau\in\Sigma_d} \deg\tau\cdot|f(\tau)|^p} \\
= \ & \inf_{f\neq 0} \frac{\sum\limits_{\sigma\in\Sigma_{d+1}}\left((d+2)^{p-1}\sum\limits_{\tau\in\Sigma_d,\tau\subset\sigma}|f(\tau)|^p - \left|\sum_{\tau\in\Sigma_d,\tau\subset\sigma} \mathrm{sgn}([\tau],\partial[\sigma]) f(\tau)\right|^p\right)}{\sum_{\tau\in\Sigma_d} \deg\tau\cdot|f(\tau)|^p} \\
\le \ & \inf_{f\neq 0} \frac{\sum\limits_{\sigma\in\Sigma_{d+1}} M_{p,d+2}\sum\limits_{\tau,\tau'\in\Sigma_d,\tau,\tau'\subset\sigma}|\mathrm{sgn}([\tau],\partial[\sigma])f(\tau) - \mathrm{sgn}([\tau'],\partial[\sigma])f(\tau')|^p}{\sum_{\tau\in\Sigma_d} \deg\tau\cdot|f(\tau)|^p} \\
= \ & M_{p,d+2}(d+1)\inf_{f\neq 0} \frac{\sum\limits_{\tau,\tau'\in\Sigma_d,\tau\sim^d\tau'}|f(\tau) - s([\tau'],[\tau])f(\tau')|^p}{\sum_{\tau\in\Sigma_d} \widetilde{\deg}\tau\cdot|f(\tau)|^p} \\
= \ & M_{p,d+2}(d+1)\lambda_1(\Delta_p(\Gamma_d,s)) \le M_{p,d+2}(d+1)2^{p-1}h(\Gamma_d,s) = M_{p,d+2}2^{p-1}h_1(\Sigma_d)
\end{aligned}$$

where $\widetilde{\deg}\,\tau = (d+1)\deg\tau$ is the degree of τ in (Γ_d,s). Similarly,

$$\begin{aligned}
(d+2)^{p-1} - \lambda_n(\Delta_{d,p}^{up}) &\ge m_{p,d+2}(d+1)\lambda_1(\Delta_p(\Gamma_d,s)) \\
&\ge m_{p,d+2}(d+1)2^{p-1}\frac{h^p(\Gamma_d,s)}{p^p} = m_{p,d+2}\frac{h^p}{p^p}\left(\frac{2}{d+1}\right)^{p-1}
\end{aligned}$$

where we used a Cheeger inequality for the p-Laplacian on signed graphs from [2, 23].

Therefore, we can always take

$$c_{p,d} = \frac{m_{p,d+2}2^{p-1}}{p^p(d+1)^{p-1}} \quad \text{and} \quad C_{p,d} = 2^{p-1}M_{p,d+2}.$$

While, for the case of $p = 2$ (already treated in Theorem 2.1), it follows from

$$k\sum_{i=1}^{k} x_i^2 - (\sum_{i=1}^{k} x_i)^2 = \sum_{1\le i<j\le k} (x_i - x_j)^2$$

that $m_{2,k} = M_{2,k} = 1$, for any $k \ge 2$, and $c_{2,d} = \frac{1}{2(d+1)}$ and $C_{2,d} = 2$ for any $d \ge 0$. $\square$

Remark 6. *In fact, we can further prove that there exist absolute constants $C'_{p,d} \ge c'_{p,d} > 0$ such that for any simplicial complex, and for any $k \ge 1$,*

$$c'_{p,d}\frac{h_k(\Sigma_d)^2}{k^6} \le (d+2)^{p-1} - \lambda'_{n+1-k}(\Delta^{up}_{d,p}) \le C'_{p,d} h_k(\Sigma_d), \tag{33}$$

where

$$\lambda'_{n+1-k}(\Delta^{up}_{d,p}) := \sup_{\gamma(S)\ge k} \inf_{f\in S} \frac{\sum\limits_{\sigma\in\Sigma_{d+1}} \left|\sum_{\tau\in\Sigma_d,\tau\subset\sigma} \mathrm{sgn}([\tau],\partial[\sigma]) f(\tau)\right|^p}{\sum_{\tau\in\Sigma_d} \deg\tau\cdot|f(\tau)|^p}$$

indicates the $(n+1-k)$-th max-min eigenvalue. Clearly, $\lambda'_n(\Delta^{up}_{d,p}) = \lambda_n(\Delta^{up}_{d,p})$ for any p.

For simplicity, and to avoid tedious processes, we just sketch the proof below. First, using the claim in the proof of Theorem 3.1, we have

$$(d+1)m_{p,d}\lambda_k(\Delta_p(\Gamma_d, s)) \le (d+2)^{p-1} - \lambda'_{n+1-k}(\Delta^{up}_{d,p}) \le (d+1)M_{p,d}\lambda_k(\Delta_p(\Gamma_d, s))$$

where $\Delta_p(\Gamma_d, s)$ represents the p-Laplacian on the signed graph (Γ_d, s). By a slightly modified variant of Theorem 1.4 in [44], and by Theorem 1.1, we can get

$$2^{p-2}c\frac{1}{k^6}(\frac{h_k(\Sigma_d)}{d+1})^2 \le 2^{p-2}\lambda_k(\Delta_{(\Gamma_d,s)}) \le \lambda_k(\Delta_p(\Gamma_d, s)) \le 2^{p-1}\frac{h_k(\Sigma_d)}{d+1}$$

in a similar manner. The proof is then finished by combining the above two inequalities.

Remark 7. *Theorem 2.1 can be recovered by taking $p = 2$ in Theorem 3.1 and Remark 6. Amazingly, for $p > 2$, we have*

$$\sup_{x \text{ non-constant}} \frac{k^{p-1}\sum_{i=1}^k |x_i|^p - |\sum_{i=1}^k x_i|^p}{\sum_{i,j=1}^k |x_i - x_j|^p} = +\infty,$$

and hence, we can only obtain a one-sided estimate $c_{p,d}h_1(\Sigma_d)^p \le (d+2)^{p-1} - \lambda_n(\Delta^{up}_{d,p})$ (or $c'_{p,d}\frac{h_k(\Sigma_d)^2}{k^6} \le (d+2)^{p-1} - \lambda'_{n+1-k}(\Delta^{up}_{d,p})$ for all max-min eigenvalues) when $p > 2$.

The last result gives a nonlinear version of the main theorem in Section 2.2. We put

$$\lambda_{I_d}(\Delta^{up}_{d,p}) = \min_{x\perp \mathrm{Image}(B_d^\top)} \frac{\|B_{d+1}^\top \boldsymbol{x}\|_p^p}{\min\limits_{y\in\mathrm{Image}(B_d^\top)} \|\boldsymbol{x}+\boldsymbol{y}\|^p_{p,\deg}}$$

which indicates the first nontrivial eigenvalue of $\Delta^{up}_{d,p}$.

Proposition 3.2. *Suppose that* $\deg_\tau > 0$, $\forall \tau \in \Sigma_d$*. Then, for any* $p \geq 1$,

$$\frac{h^p(\Sigma_d)}{|\Sigma_{d+1}|^{p-1}} \leq \lambda_{I_d}(\Delta_{d,p}^{up}) \leq \mathrm{vol}(\Sigma_d)^{p-1} h(\Sigma_d).$$

Proof. The proof is easy and very similar to Proposition 2.1. □

Theorem 3.2. *Let M be an orientable, compact, closed Riemannian manifold of dimension $(d+1)$. Let Σ be a simplicial complex which is combinatorially equivalent to a uniform triangulation of M. Then, there is a Cheeger inequality*

$$\frac{h^p(\Sigma_d)}{C} \leq \lambda_{I_d}(\Delta_{d,p}^{up}) \leq C \cdot h^{p-1}(\Sigma_d),$$

where C is a uniform constant which is independent of the choice of Σ.

Proof. The proof is essentially the same to that of Theorem 2.3, with only a small difference at Claim 1 in the proof of Theorem 2.3. In fact, we only need to use the following claim instead of Claim 1.

Claim: For the down Cheeger constant, for any $p > 1$, we have

$$(d+2)^{p-1}(\frac{h_{down}}{p^*})^p \leq \lambda_{I_d}\left(\Delta_{d,p}^{up}\right) \leq (d+2)^{p-1} h_{down}^{p-1}$$

and in particular, when p tends to $+\infty$, we have $\lim\limits_{p \to +\infty} \lambda_{I_d}(\Delta_{d,p}^{up})^{\frac{1}{p}} = (d+2)h_{down}$.

Proof: Since the down adjacency relation induces a graph on Σ_d, we can directly use the Cheeger inequality for p-Laplacian on graphs to derive

$$\frac{2^{p-1}h_{down}^p(\Sigma_{d+1})}{p^p} \leq \lambda_2(\Delta_{d+1,p}^{down}) \leq 2^{p-1}h_{down}(\Sigma_{d+1}). \tag{34}$$

Since $|\Sigma|$ is a compact piecewise flat manifold without boundary, and since the dimension of $|\Sigma|$ is $d+1$, the normalized and unnormalized versions of p-Laplacian on Σ satisfy $\lambda_i(L_{d+1,p}^{down}) = (d+2)\lambda_i(\Delta_{d+1,p}^{down})$ and $\lambda_i(L_{d,p}^{up}) = 2\lambda_i(\Delta_{d,p}^{up})$ for any i.

Together with the spectral duality $(\lambda_{I_d}(L_{d,p^*}^{up}))^{\frac{1}{p^*}} = (\lambda_2(L_{d+1,p}^{down}))^{\frac{1}{p}}$ derived by Proposition 3.1, we immediately obtain the duality equality

$$(2\lambda_{I_d}(\Delta_{d,p^*}^{up}))^{\frac{1}{p^*}} = ((d+2)\lambda_2(\Delta_{d+1,p}^{down}))^{\frac{1}{p}}. \tag{35}$$

Then substituting the duality equality (35) into the above down Cheeger inequality (34), we finally deduce that

$$(d+2)^{p^*-1}(\frac{h_{down}}{p})^{p^*} \leq \lambda_{I_d}(\Delta_{d,p^*}^{up}) \leq (d+2)^{p^*-1}h_{down}^{p^*-1}$$

The proof of the claim is completed by exchanging the positions of p and p^*.

Finally, combining the above claim with Claim 2 in the proof of Theorem 2.3, we derive the desired Cheeger-type inequality stated in Theorem 3.2. □

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