A CLUSTER EXPANSION PROOF THAT THE STOCHASTIC EXPONENTIAL OF A BROWNIAN MOTION IS A MARTINGALE

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ABSTRACT. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth and continuous real function and $\psi \in \mathrm{L}^2(\mathbb{R}^+)$. Let $B(t)$ be a standard Brownian motion defined with respect to a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and where $dB(t) = \xi(t)dt$ and $t \in \mathbb{R}^+$. The process $\xi(t)$ is a Gaussian white noise with expectation $\mathbf{E}\ \xi(t) = 0$ and with covariance $\mathbf{E}\ \xi(t)\xi(s) = \delta(t-s)$. The Dolean-Dades stochastic exponential $Z(t)$ is the solution to the linear stochastic differential equation describing a geometric Brownian motion such that $dZ(t) = \psi(t)Z(t)dB(t) = \psi(t)Z(t)\xi(t)dt$. Using a cluster expansion method, and the moment and cumulant generating functions for $\xi(t)$, it is shown that $Z(t)$ is a martingale. The original Novikov criteria for $Z(t)$ being a true martingale are reproduced and exactly satisfied, namely that

$$\mathbf{E}Z(t) = \mathbf{E}\exp\left(\int_o^t \psi(u)dB(u) - \frac{1}{2}\int_0^t |\psi(u)|^2 du\right) = 1$$

provided that $\exp\left(\int_0^t |\psi(u)|^2 du\right) < \infty$ for all $t > 0$. However, $\mathbf{E}\big[|Z(t)|^p\big] = \exp(\frac{1}{2}p(p-1)\phi(t))$, if $\phi(t) = \int_0^t |\psi(u)|^2 du$ is monotone increasing and is a submartingale for all $p > 1$.

1. Introduction

Let $B(t)$ be a standard Brownian motion with respect to a probability space $(\Omega, \mathscr{F}, \mathbf{P})$, and let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be smooth continuous real function [**1**]. A classical problem in stochastic analysis is to prove that the Dolean-Dades stochastic exponential

$$Z(t) = \exp\left(\int_o^t \psi(u)dB(u) - \frac{1}{2}\int_0^t |\psi(u)|^2 du\right) \tag{1.1}$$

$$Z(0) = 0 \tag{1.2}$$

is a true martingale [**2**]. The DDSE is the exact solution of the geometric Brownian motion [**1**]

$$dZ(t) = \psi(t)Z(t)\ \xi(t)dt \equiv \psi(t)Z(t)\ dB(t) \tag{1.3}$$

where $dB(t) = \xi(t)dt$ and $\xi(t)$ is a Gaussian white noise. The solution has the Ito stochastic integral representation

$$Z(t) = \int_0^t \psi(u)Z(u)dB(u) \tag{1.4}$$

The explicit solution (1.1) is obtained [**1**] via the Ito expansion of $\log Z(t)$. Proving that $Z(t)$ is a true martingale is actually a somewhat difficult and subtle problem and is relevant to other theorems and results [**2-7**]. Ito integrals of the form (1.4) are not always martingales. The well-known necessary and sufficient conditions for $Z(t)$ to be a true martingale are due to Novikov [**2**] and also Kazamaki [**5**]. The Novikov criteria are

$$\mathbf{E}Z(t) = \mathbf{E}\exp\left(\int_o^t \psi(u)dB(u) - \frac{1}{2}\int_0^t |\psi(u)|^2 du\right) = 1 \tag{1.5}$$

and the bound

$$\exp\left(\frac{1}{2}\int_0^t |\psi(u)|^2 du\right) < \infty,\ \forall t > 0 \tag{1.6}$$

or equivalently $\frac{1}{2}\|\psi\|^2_{L_2(\mathbb{R})} = \frac{1}{2}\int_0^t |\psi(u)|^2 du < \infty$. The function ψ can also be a random process $\psi(t,\omega)$ with respect to $\omega \in \Omega$ with Novikov criteria [**6**]

$$\mathbf{E}Z(t) = \mathbf{E}\exp\left(\int_o^t \psi(u,\omega)dB(u,\omega) - \frac{1}{2}\int_0^t |\psi(u,\omega)|^2 du\right) = 1 \tag{1.7}$$

and

$$\mathbf{E}\left\{\exp\left(\int_0^t |\psi(u,\omega)|^2 du\right)\right\} < \infty,\ \forall t > 0 \tag{1.8}$$

In this note we only consider (1.5), and a short new proof is given to establish that $Z(t)$ is a martingale. The exact criteria (1.5) and (1.6) are established, via a cluster-expansion method and utilising moment and cumulant-generating functions.

Geometric Brownian motion arises naturally in problems of stochastic exponential growth, which occurs when a quantity of interest grows exponentially but with a random multiplier and/or random waiting time between spurts of growth. Such processes are ubiquitous in both Nature and in human-created systems: bacterial growth in a sustaining medium; in nuclear and cellular fission; tissue growth in embryonic and cancer biology; in viral epidemics; in financial markets and bubbles, and the Black-Scholes option pricing model; internet growth and propagation of viral social media posts; population and extinction dynamics; Moore's law for computer processing power; and inflationary expansion of the very early Universe with random fluctuations [**8-17**]. Noise can either boost stochastic exponential growth or drive a population to extinction [**8,11,12**].

Most such models on random growth have focussed on the SDE for geometric Brownian motion since it can be solved exactly with a strong solution. Suppose a system evolves exponentially via the simple linear ODE $dX(t) = \alpha X(t)dt$ then $X(t) = X(0)\exp(\alpha t)$. If $\alpha > 0$ then the system undergoes exponential growth and $X(t) \to \infty$ as $t \to \infty$ and if $\alpha < 0$ then the system exponentially decays or collapses so that $X(t) \to 0$ as $t \to \infty$. The system is stable or static for $\alpha = 0$. If the system is randomly perturbed by white noise (additively) then

$$dX(t) = \alpha X(t)dt + \psi(t)X(t)\xi(t)dt \equiv \alpha X(t)dt + \psi(t)X(t)dB(t) \tag{1.9}$$

A strong solution exists for this SDE [**1**]. Given any C^2-functional $f(X(t))$, the Ito Lemma is

$$\begin{aligned} df(X(t)) &= \nabla_X f(X(t))dX(t) + \frac{1}{2}\nabla_X^2 f(X(t))d[X,X](t) \\ &= \nabla_X f(X(t))dX(t) + \frac{1}{2}\nabla_X^2 f(X(t))|\psi(t)|^2 dt \\ &= \nabla_X f(X(t))\big\{\alpha X(t)dt + \psi(t)X(t)dB(t)\big\} + \frac{1}{2}\nabla_X^2 f(X(t))|\psi(t)|^2 dt \end{aligned} \tag{1.10}$$

where $\nabla_X = d/dX(t)$ and $\nabla_X^2 = d^2/dX(t)^2$. Using $f(X(t)) = \log X(t)$

$$\begin{aligned} dU(t) \equiv d\log X(t) &= \frac{1}{X(t)}dX(t) + \frac{1}{2}\left(-\frac{1}{|X(t)|^2}\right)|\psi(t)|^2|X(t)|^2 dt \\ &= \frac{1}{X(t)}\big\{\alpha X(t)dt + \psi(t)X(t)dB(t)\big\} - \frac{1}{2}\psi(t)|^2 dt \\ &= \left(\alpha - \frac{1}{2}|\psi(t)|^2\right)dt + \psi(t)dB(t) \end{aligned} \tag{1.11}$$

Then the solution is

$$X(t) = X(0)\exp(\alpha t)\exp\left(\int_0^t \psi(u)dB(u) - \frac{1}{2}\int_0^t |\psi(u)|^2 du\right) = \exp(\alpha t)Z(t) \tag{1.12}$$

The expectation is

$$\mathbf{E}X(t) = X(0)\exp(\alpha t)\mathbf{E}\exp\left(\int_0^t \psi(u)dB(u) - \frac{1}{2}\int_0^t |\psi(u)|^2 du\right) = \exp(\alpha t)\mathbf{E}Z(t) \tag{1.13}$$

which is $\mathbf{E}X(t) = \exp(\alpha t)$ if $\mathbf{E}Z(t) = 1$. The martingale property of $Z(t)$ is therefore desirable and ensures there is no blow up at any finite time $t > 0$.

2. Main theorem and proof utilising a cluster expansion method

2.1. **Preliminary definitions and lemmas.** Prior to the proof, we first establish the following preliminary definitions and lemmas. The proof utilises a cluster expansion method **[18-21]**.

Definition 2.1. *White noise is (informally) defined as a zero-centred Gaussian process ξ with covariance $\mathbf{E}\ \xi(t)\xi(s) = \delta(t-s)$ and expectation $\mathbf{E}\xi(t) = 0$. A scalar or inner product $\langle \bullet, \bullet \rangle$ in $L^2(\mathbb{R})$ can be defined as*

$$\langle \psi, \phi \rangle = \mathbf{E}\big\langle \psi, \xi \big\rangle \big\langle \phi, \xi \big\rangle = \iint \psi(s)\phi(t)\mathbf{E}\ \xi(s)\xi(t)dsdt = \iint \psi(s)\phi(t)\delta(t-s)dsdt \tag{2.1}$$

Then

$$\langle \psi, \psi \rangle = \mathbf{E}\big\langle \psi, \xi \big\rangle \big\langle \psi, \xi \big\rangle = \iint \psi(s)\psi(t)\mathbf{E}\ \xi(s)\xi(t)dsdt = \iint \psi(s)\psi(t)\delta(t-s)dsdt \tag{2.2}$$

which is

$$\langle \psi, \psi \rangle = \int |\psi(s)|^2 ds = \|\psi\|^2_{L^2(\mathbb{R})} \tag{2.3}$$

The noise ξ can be formulated as a Gaussian random process on any space of distributions containing $\mathrm{L}^2(\mathbb{R})$. Stochastic integrals of the form $\int \psi(s)\xi(s)ds$ are well defined if $\psi \in \mathrm{L}^2(\mathbb{R})$. If ψ is an indicator function on $[0,t]$ then the process $B(t) = \int_0^t \xi(s)ds$ is a Brownian motion and $dB(t) = \xi(t)dt$.

Definition 2.2. *Given a time-ordered set $(t_1, ...t_m) \in (0,T)$, with $t_1 < t_2 < t_3 < ... < t_{m-1} < t_m$, the mth-order moments and cumulants for the white noise $\xi(t)$ are given by*

$$\begin{aligned} \overrightarrow{\mathbb{T}}\mathbf{E}\ \xi(t_1) \otimes ... \otimes \xi(t_m) &= \overrightarrow{\mathbb{T}}\mathbf{E}\prod_{q=1}^m \xi(t_q) \\ \overrightarrow{\mathbb{T}}\mathbf{C}\ \xi(t_1) \otimes ... \otimes \xi(t_m) &= \overrightarrow{\mathbb{T}}\mathbf{C}\prod_{q=1}^m \xi(t_q) \end{aligned} \tag{2.4}$$

where $\mathbb{T}$ is a 'time-ordering operator'. For example if $t_1 < t_2 < t_3$ and $f(t)$ is a function of t then $\overrightarrow{\mathbb{T}} f(t_2)f(t_1)f(t_3 = f(t_1)f(t_2)f(t_3)$. Since $\xi(t)$ is a Gaussian, it is defined entirely by its first two moments and all cumulants or order $m \geq 3$ vanish so that

$$\left\{\overrightarrow{\mathbb{T}}\mathbf{C}\prod_{q=1}^m \xi(t_q)\right\}_{m\geq 3} = 0 \tag{2.5}$$

For example, at second order for a Gaussian process the binary cumulant is equivalent to the binary moment

$$\mathbf{C}\xi(t_1)\xi(t_2) = \mathbf{E}\ \xi(t_1)\xi(t_2) - \mathbf{E}\ \xi(t_1)\mathbf{E}\ \xi(t_2) = \mathbf{E}\xi(t_1)\xi(t_2) = \delta(t_2 - t_1) \tag{2.6}$$

Definition 2.3. *The moment generating function (MGF) $\mathscr{M}[\xi(t)]$ and the cumulant-generating function [CGF] $\mathscr{C}[\xi(t)]$ of the white noise $\xi(t)$ are given by*

$$\mathscr{M}[\xi(t)] = \sum_{m=0}^{\infty}\frac{\beta^m}{m!}\int_0^t ...\int_0^{t_{m-1}} dt_1...dt_m \overrightarrow{\mathbb{T}}\mathsf{E}\prod_{q=1}^{m}\ \xi(t_q) \tag{2.7}$$

$$\mathscr{C}[\xi(t)] = \sum_{m=1}^{\infty}\frac{\beta^m}{m!}\int_0^t ...\int_0^{t_{m-1}} dt_1...dt_m \overrightarrow{\mathbb{T}}\mathsf{C}\prod_{q=1}^{m}\xi(t_q) \tag{2.8}$$

where β is an arbitrary real constant. It is important to note that the summation in (2.8) begins from $m = 1$ and not $m = 0$ and that

$$\overrightarrow{\mathbb{T}}\mathsf{E}\prod_{q=1}^{m}\xi(t_q) \neq \overrightarrow{\mathbb{T}}\prod_{q=1}^{m}\mathsf{E}\xi(t_q) \tag{2.9}$$

$$\overrightarrow{\mathbb{T}}\mathsf{C}\prod_{q=1}^{m}\xi(t_q) \neq \overrightarrow{\mathbb{T}}\prod_{q=1}^{m}\mathsf{C}\xi(t_q) \tag{2.10}$$

Equations (2.7) and (2.8) can also be written as

$$\mathscr{M}[\xi(t)] = \sum_{m=0}^{\infty}\frac{\beta^m}{m!}\int \mathbf{D}_m[t_1...t_m]\overrightarrow{\mathbb{T}}\mathsf{E}\prod_{q=1}^{m}\xi(t_q) \tag{2.11}$$

$$\mathscr{C}[\xi(t)] = \sum_{m=1}^{\infty}\frac{\beta^m}{m!}\int \mathbf{D}_m[t_1...t_m]\overrightarrow{\mathbb{T}}\mathsf{C}\prod_{q=1}^{m}\xi(t_q) \tag{2.12}$$

where $\int \mathbf{D}_m[t] = \int ... \int dt_1...dt_m$ is a 'path integral'. Choosing $\beta = +1$

$$\mathscr{M}[\xi(t)] = \sum_{m=0}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\overrightarrow{\mathbb{T}}\mathsf{E}\prod_{q=1}^{m}\xi(t_q) \tag{2.13}$$

$$\mathscr{C}[\xi(t)] = \sum_{m=1}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\overrightarrow{\mathbb{T}}\mathsf{C}\prod_{q=1}^{m}\xi(t_q) \tag{2.14}$$

Lemma 2.4. *The relation between the MGF and the CGF is*

$$\log\mathscr{M}[\xi(t)] = \mathscr{C}[\xi(t)] \tag{2.15}$$

so that

$$\mathscr{M}[\xi(t)] = \exp\big(\mathscr{C}[\xi(t)]\big) \tag{2.16}$$

Hence

$$\sum_{m=0}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\overrightarrow{\mathbb{T}}\mathsf{E}\prod_{q=1}^{m}\xi(t_q) \tag{2.17}$$

$$\exp\left(\sum_{m=1}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\overrightarrow{\mathbb{T}}\mathsf{C}\prod_{q=1}^{m}\xi(t_q)\right) \tag{2.18}$$

Proposition 2.5. *Given $\psi \in \mathrm{L}^2(\mathbb{R})$ and the white noise $\xi(t)$, then for all $t > 0$ define the Gaussian noise or random function*

$$\Xi(t) = \psi(t)\xi(t) \tag{2.19}$$

Then $\mathbf{E}\Xi(t) = 0$ *and*

$$\mathbf{E}\ \Xi(t)\Xi(s) = \psi(t)\psi(s)\mathbf{E}\ \xi(t)\xi(s) = \psi(t)\psi(s)\delta(t-s) \tag{2.20}$$

Lemma 2.6. *Given* $\Xi(t) = \psi(t)\xi(t)$ *then for any* $t_2 > t_1$ *and* $(t_1, t_2) \in (0, t)$

$$\langle \psi, \psi \rangle \equiv \int_0^t \int_0^{t_1} \mathbf{E}\ \Xi(t_1)\Xi(t_2) dt_1 dt_2 = \int_0^t |\psi(t_1)|^2 dt_1 = \|\psi\|_{L_2(\mathbb{R})} \tag{2.21}$$

Proof. Using the sifting property of the delta function

$$\begin{aligned}
&\int_0^t \int_0^{t_1} \mathbf{E}\ \Xi(t_1)\Xi(t_2) dt_1 dt_2 \\
&= \int_0^t \int_0^{t_1} \psi(t_1)\psi(t_2)\mathbf{E}\ \Xi(t_1)\Xi(t_2) dt_1 dt_2 \\
&= \int_0^t \psi(t_1) \left| \int_0^{t_1} \psi(t_2)\delta(t_2 - t_1) dt_2 \right| dt_1 \\
&= \int_0^t \psi(t_1)\psi(t_1) dt_1 = \int_0^t |\psi(t_1)|^2 dt_1 \equiv \|\psi\|_{L_2(\mathbb{R}^+)}
\end{aligned} \tag{2.22}$$

□

Proposition 2.7. *Given* $\Xi(t) = \beta(t)\xi(t)$ *then the Dolean-Dades stochastic exponential* $Z(t)$ *can be expressed as*

$$\begin{aligned}
Z(t) &= \exp\left(\int_0^t \psi(u) dB(u) - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right) \\
&= \exp\left(\int_0^t \psi(u)\xi(u) du - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right) \\
&= \exp\left(\int_0^t \Xi(u) du - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right)
\end{aligned} \tag{2.23}$$

2.2. **Main theorem.** The main theorem and proof are now as follows:

Theorem 2.8. *The stochastic exponential* $Z(t)$ *is a true martingale iff*

$$\mathbf{E}Z(t)\} = \mathbf{E} \exp\left(\int_0^t \psi(u) dB(u) - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right) = 1 \tag{2.24}$$

and requiring

$$\exp(\|\psi\|^2_{L^2(\mathbb{R})}) = \exp\left(\frac{1}{2} \int_0^t |\psi(u)|^2 du \right) < \infty,\ \forall t > 0 \tag{2.25}$$

Proof.

$$\begin{aligned}
\mathbf{E}Z(t) &= \mathbf{E} \exp\left(\int_0^t \psi(u) dB(u) - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right) \\
&= \mathbf{E} \exp\left(\int_0^t \psi(u)\xi(u) du - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right) \\
&= \mathbf{E} \exp\left(\int_0^t \Xi(u) du - \frac{1}{2} \int_0^t |\psi(u)|^2 du \right) \\
&= \exp\left(-\frac{1}{2} \int_0^t |\psi(u)|^2 du \right) \mathbf{E} \exp\left(\int_0^t \Xi(u) du \right)
\end{aligned}$$

$$= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right) \mathbf{E}\sum_{m=0}^{\infty}\frac{1}{m!}\int_0^t dt_1 ... \int_0^{t_{m-1}} dt_m \prod_{q=1}^m \Xi(t_q)$$

$$\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right) \mathbf{E}\sum_{m=0}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\prod_{q=1}^m \Xi(t_q)$$

$$= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right) \sum_{m=0}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\mathbf{E}\prod_{q=1}^m \Xi(t_q) \tag{2.26}$$

However, from (2.11), the MGF is

$$\mathscr{M}[\Xi(t)] = \sum_{m=0}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\mathbf{E}\prod_{q=1}^m \Xi(t_q) \tag{2.27}$$

so (2.26) becomes

$$\mathbf{E}Z(t) = \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\mathscr{M}[\Xi(t)] \tag{2.28}$$

Now using Lemma (2.4)

$$\mathscr{M}[\Xi(t)] = \exp(\mathscr{C}[\Xi(t)]) \tag{2.29}$$

giving

$$\mathbf{E}Z(t) = \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\mathscr{M}[\Xi(t)]$$

$$= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\mathscr{C}[\Xi(t)]\right)$$

$$= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\sum_{m=1}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\mathbf{C}\overrightarrow{\mathbb{T}}\prod_{q=1}^m \Xi(t_q)\right)$$

$$= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\sum_{m=1}^{\infty}\frac{1}{m!}\int \mathbf{D}_m[t_1...t_m]\mathbf{C}\overrightarrow{\mathbb{T}}\prod_{q=1}^m \psi(t_q)\xi(t_q)\right) \tag{2.30}$$

Now since $\xi(t)$ is a Gaussian process, all cumulants of order $m \geq 3$ vanish so that

$$\mathbf{C}\overrightarrow{\mathbb{T}}\left\{\prod_{q=1}^m \Xi(t_q)\right\}_{m\geq 3} = \mathbf{C}\overrightarrow{\mathbb{T}}\left\{\prod_{q=1}^m \psi(t_q)\xi(t_q)\right\}_{m\geq 3} = 0 \tag{2.31}$$

This leaves

$$\mathbf{E}Z(t) = \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)$$

$$\times \exp\left(\int \mathbf{D}_m[t_1]\psi(t_1)\mathbf{C}\xi(t_1)\} + \frac{1}{2}\int \mathbf{D}[t_1,t_2]\psi(t_1)\psi(t_2)\mathbf{C}\{\xi(t_1)\xi(t_2)\right)$$

$$\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)$$

$$\times \exp\left(\int_0^t dt_1\psi(t_1)\mathbf{C}\xi(t_1) + \frac{1}{2}\int_0^t\int_0^{t_1} dt_1 dt_2 \psi(t_1)\psi(t_2)\mathbf{C}\xi(t_1)\xi(t_2)\right)$$

$$
\begin{aligned}
&\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\\
&\times \exp\left(\int_0^t dt_1\psi(t_1)\underbrace{\mathbf{E}\,\{\xi(t_1)\}}_{=0} + \frac{1}{2}\int_0^t\int_0^{t_1} dt_1dt_2\psi(t_1)\psi(t_2)\mathbf{E}\,\xi(t_1)\xi(t_2)\right)\\
&\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}\int_0^t\int_0^{t_1} dt_1dt_2\psi(t_1)\psi(t_2)\mathbf{E}\,\xi(t_1)\xi(t_2)\right)\\
&\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}\int_0^t\int_0^{t_1} dt_1dt_2\psi(t_1)\psi(t_2)\delta(t_2-t_1)\right)\\
&= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}\int_0^t dt_1\psi(t_1)\left|\int_0^{t_1} dt_2\psi(t_2)\delta(t_2-t_1)\right|\right)\\
&= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}\int_0^t dt_1\psi(t_1)\psi(t_1)\right)\\
&= \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}\int_0^t |\psi(t_1)|^2 dt_1\right)\\
&\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\\
&\equiv \exp\left(-\frac{1}{2}\int_0^t |\psi(u)|^2 du + \frac{1}{2}\int_0^t |\psi(u)|^2 du\right) = 1 \qquad (2.32)
\end{aligned}
$$

iff

$$
\exp\left(\frac{1}{2}\int_0^t |\psi(u)|^2 du\right) < \infty \qquad (2.33)
$$

which are the Novikov criteria. If for some function $\psi(t)$, $\exists\ T > 0$ such that $\frac{1}{2}\int_0^T |\psi(u)|^2 du = \infty$ then $\exp\left(\frac{1}{2}\int_0^T |\psi(u)|^2 du\right) = \infty$ and $\exp\left(-\frac{1}{2}\int_0^T |\psi(u)|^2 du\right) = 0$ so that $\mathbf{E}Z(t) = 0$. Hence $Z(t)$ is a martingale for these criteria and the proof is complete. $\square$

As a corollary, it follows easily that $\mathbf{M}(t,p) = \mathbf{E}[|Z(t)|^p]$ is a submartingale for all $p > 1$ if $\phi(t) = \int_0^t |\psi(u)|^2 du$ is monotone increasing with t.

Corollary 2.9. *Given $\mathbf{E}Z(t) = 1$ it follows that*

$$
\mathbf{M}(t,p) = \mathbf{E}[|Z(t)|^p] = \exp\left(\tfrac{1}{2}[p(p-1)\phi(t)\right) \qquad (2.34)
$$

is a submartingale for all $p > 1$, if $\phi(t)) = \int_0^t |\psi(u)|^2 du$ is bounded but monotone increasing with t.

Proof.

$$
\begin{aligned}
|Z(t)|^p &= \left(\exp\left|\int_0^t \psi(u)dB(u) - \frac{1}{2}\int_0^t |\psi(u)|^2 du\right)\right|^p\\
&= \exp\left(p\int_0^t \psi(u)dB(u) - \frac{p}{2}\int_0^t |\psi(u)|^2 du\right)\\
&= \exp\left(-\frac{p}{2}\int_0^t |\psi(u)|^2 du\right)\exp\left(p\int_0^t \psi(u)dB(u)\right) \qquad (2.35)
\end{aligned}
$$

Then

$$\begin{aligned}\mathbf{M}(t,p) =& \mathbf{E}[|Z(t)|^p] = \exp\left(-\frac{p}{2}\int_0^t |\psi(u)|^2 du\right)\mathbf{E}\exp\left(p\int_0^t \psi(u)dB(u)\right)\\ &= \exp\left(-\frac{1}{2}p\int_0^t |\psi(u)|^2 du\right)\exp\left(\frac{1}{2}p^2\int_0^t |\psi(u)|^2 du\right)\\ &= \exp\left(\frac{1}{2}p(p-1)\int_0^t |\psi(u)|^2 du\right) = \exp\left(\tfrac{1}{2}p(p-1)\phi(t)\right) \end{aligned} \tag{2.36}$$

If $\phi(t) > \phi(t')$ for all $t > t'$ then $\mathbf{M}(t,p) > \mathbf{M}(t',p)$ so that $\mathbf{M}(t,p)$ is monotone increasing. Hence, $\mathbf{M}(t,p)$ is a submartingale on $\mathbb{R}^+$. If $p = 1$ then $\mathbf{M}(t,1) = \mathbf{E}Z(t) = 1$. $\square$

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