

Strong domination number of Hajós sum and vertex-sum of two graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a strong dominating set of G , if for every vertex $x \in V \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $\deg(x) \leq \deg(y)$. The strong domination number $\gamma_{\text{st}}(G)$ is defined as the minimum cardinality of a strong dominating set. In this paper, we study the strong domination number of Hajós sum and vertex-sum of two graphs.

Keywords: Strong domination number, strong dominating set, Hajós sum, vertex-sum.

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1 Introduction

A dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex in $V \setminus D$ is adjacent to at least one member of D . The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. This parameter has been extensively studied in the literature and there are hundreds of papers concerned with domination. We recommend a fundamental book [8] about domination in general. The various different domination concepts are well-studied now, however new concepts are introduced frequently and the interest is growing rapidly.

A set $D \subseteq V$ is a *strong dominating set* of G , if for every vertex $x \in \overline{D} = V \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $\deg(x) \leq \deg(y)$. The *strong domination number* $\gamma_{\text{st}}(G)$ is defined as the minimum cardinality of a strong dominating set. A γ_{st} -set of G is a strong dominating set of G of minimum cardinality $\gamma_{\text{st}}(G)$. If D is a strong dominating set in a graph

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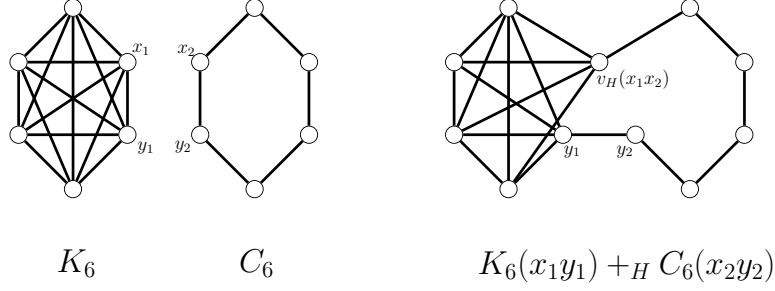


Figure 1: Hajós construction of K_6 and C_6 .

G , then we say that a vertex $u \in \overline{D}$ is *strong dominated* by a vertex $v \in D$ if $uv \in E(G)$, and $\deg(u) \leq \deg(v)$.

The strong domination number was introduced in [10] and some upper bounds on this parameter presented in [9, 10]. Similar to strong domination number, a set $D \subset V$ is a weak dominating set of G , if every vertex $v \in V \setminus D$ is adjacent to a vertex $u \in D$ such that $\deg(v) \geq \deg(u)$ (see [6]). The minimum cardinality of a weak dominating set of G is denoted by $\gamma_w(G)$. Boutrig and Chellali proved that the relation $\gamma_w(G) + \frac{3}{\Delta+1}\gamma_{st}(G) \leq n$ holds for any connected graph of order $n \geq 3$. Alikhani, Ghanbari and Zaherifard [3] examined the effects on $\gamma_{st}(G)$ when G is modified by the edge deletion, the edge subdivision and the edge contraction. Also they studied the strong domination number of k -subdivision of G .

Motivated by enumerating of the number of dominating sets of a graph and domination polynomial (see e.g. [1, 4]), the enumeration of the strong dominating sets for certain graphs has studied in [14]. Study of the strong domination number of graph operation is a natural and interesting subject and for join and corona product has studied ([14]). In this paper, we consider other kinds of graph operations which are called Hajós sum and vertex sum of two graphs. The Hajós sum is useful when either of the network is disrupted and certain node(s) is(are) not functioning. Then that node(s) is(are) to be identified(fused) with the node of a network which is functioning properly and thus new network is constructed.

2 Hajós sum

In this section, we study the strong domination number of Hajós sum of two graphs. First we recall its definition. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets, an edge $x_1y_1 \in E_1$, and an edge $x_2y_2 \in E_2$, the *Hajós sum* $G_3 = G_1(x_1y_1) +_H G_2(x_2y_2)$ is the graph obtained as follows: begin with $G_3 = (V_1 \cup V_2, E_1 \cup E_2)$; then in G_3 delete the edges x_1y_1 and x_2y_2 , identify the vertices x_1 and x_2 as $v_H(x_1x_2)$, and add the edge y_1y_2 [7]. Figure 1 shows the Hajós sum of K_6 and C_6 with respect to x_1y_1 and x_2y_2 .

The following theorem gives the lower bound and the upper bound for the strong domination number of Hajós sum of two graphs.

Theorem 2.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets, $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Also, suppose that x_1 and x_2 are not pendant vertices. Then for the Hajós sum*

$$G_3 = G_1(x_1y_1) +_H G_2(x_2y_2),$$

we have:

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) - \deg(x_1) - \deg(x_2) + 2 \leq \gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1.$$

Proof. First we find the upper bound. Since x_1 and x_2 are not pendant vertices, then by the definition of the Hajós sum we know that $\deg(v_H(x_1x_2)) = \deg(x_1) + \deg(x_2) - 2$. Also, $\deg_{G_3}(y_1) = \deg_{G_1}(y_1)$, and $\deg_{G_3}(y_2) = \deg_{G_2}(y_2)$. Suppose that D_i is a γ_{st} -set of G_i , for $i = 1, 2$. We have the following cases:

- (i) y_1 is strong dominated by x_1 , and y_2 is strong dominated by x_2 . Without loss of generality, suppose that $\deg(y_1) \geq \deg(y_2)$. Let

$$D_3 = (D_1 \setminus \{x_1\}) \cup (D_2 \setminus \{x_2\}) \cup \{v_H(x_1x_2), y_1\}.$$

D_3 is a strong dominating set of G_3 , because y_2 is strong dominated by y_1 , and every other vertices in $\overline{D_3}$ is strong dominated by the same vertices as before or $v_H(x_1x_2)$. So we have

$$\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2).$$

- (ii) y_1 is strong dominated by x_1 , and y_2 is not strong dominated by x_2 . In this case, we may have $y_2 \in D_2$ or $y_2 \in \overline{D_2}$, and we may have $x_2 \in D_2$ or $x_2 \in \overline{D_2}$. Let

$$D_3 = (D_1 \setminus \{x_1\}) \cup (D_2 \setminus \{x_2\}) \cup \{v_H(x_1x_2), y_1\}.$$

D_3 is a strong dominating set of G_3 , because if $y_2 \in \overline{D_2}$, then it is strong dominated by the same vertex as before, and every other vertices in $\overline{D_3}$ is strong dominated by the same vertices as before or $v_H(x_1x_2)$. So, in the worst case, which is $x_2 \in \overline{D_2}$, we have

$$\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1.$$

- (iii) y_1 is not strong dominated by x_1 , and y_2 is not strong dominated by x_2 . By a similar discussion as part (ii),

$$D_3 = (D_1 \setminus \{x_1\}) \cup (D_2 \setminus \{x_2\}) \cup \{v_H(x_1x_2)\},$$

is a strong dominating set of G_3 , and in the worst case, we have

$$\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1.$$

(iv) x_1 is strong dominated by y_1 , and x_2 is strong dominated by y_2 . Then clearly

$$D_3 = D_1 \cup D_2 \cup \{v_H(x_1x_2)\},$$

is a strong dominating set of G_3 , and we have

$$\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1.$$

(v) x_1 is strong dominated by y_1 , and x_2 is not strong dominated by y_2 . Then we may have y_2 is strong dominated by x_2 , which we have the result by similar argument as case (ii). Otherwise, by a similar argument as part (ii),

$$D_3 = (D_1 \cup D_2 \setminus \{x_2\}) \cup \{v_H(x_1x_2)\}.$$

is a strong dominating set of G_3 , and in the worst case, we have

$$\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1.$$

(vi) x_1 is not strong dominated by y_1 , and x_2 is not strong dominated by y_2 . Then we may have y_1 is strong dominated by x_1 , and y_2 is strong dominated by x_2 , which gives us the result by case (i), or we may have y_1 is strong dominated by x_1 , and y_2 is not strong dominated by x_2 , which gives us the result by case (ii). Otherwise, by similar argument as before,

$$D_3 = (D_1 \setminus \{x_1\}) \cup (D_2 \setminus \{x_2\}) \cup \{v_H(x_1x_2)\},$$

is a strong dominating set of G_3 , and in the worst case, we have

$$\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1.$$

So, in general, we have $\gamma_{\text{st}}(G_3) \leq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) + 1$. Now, we find the lower bound. Suppose that S_3 is a γ_{st} -set of G_3 . We find strong dominating sets of G_1 and G_2 based on S_3 . We consider the following cases:

(i) $v_H(x_1x_2) \in S_3$. Here we consider the following sub-cases:

(a) $y_1 \in S_3$ and $y_2 \in S_3$. If $v_H(x_1x_2)$ is not strong dominating any vertices in $\overline{S_3}$, then

$$S_1 = \left(S_3 \setminus \left(V(G_2) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{x_1\}$$

is a strong dominating set of G_1 , and

$$S_2 = \left(S_3 \setminus \left(V(G_1) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{x_2\}$$

is a strong dominating set of G_2 . But, if $v_H(x_1x_2)$ is strong dominating some vertices in $\overline{S_3}$, then after forming G_1 and G_2 from G_3 , then if $\deg(x_1) \geq \max\{\deg(u) \mid u \in$

$N(x_1)\}$, and $\deg(x_2) \geq \max\{\deg(v) \mid v \in N(x_2)\}$, we consider S_1 and S_2 as mentioned. If $\deg(x_1) \geq \max\{\deg(u) \mid u \in N(x_1)\}$, But $\deg(x_2) \not\geq \max\{\deg(v) \mid v \in N(x_2)\}$, we consider S_1 as mentioned, and let

$$S_2 = \left(S_3 \setminus \left(V(G_1) \cup \{v_H(x_1x_2)\} \right) \right) \cup N(x_2),$$

then one can easily check that S_2 is a strong dominating set of G_2 . If $\deg(x_1) \not\geq \max\{\deg(u) \mid u \in N(x_1)\}$, and $\deg(x_2) \not\geq \max\{\deg(v) \mid v \in N(x_2)\}$, we consider

$$S_1 = \left(S_3 \setminus \left(V(G_2) \cup \{v_H(x_1x_2)\} \right) \right) \cup N(x_1),$$

and

$$S_2 = \left(S_3 \setminus \left(V(G_1) \cup \{v_H(x_1x_2)\} \right) \right) \cup N(x_2).$$

Here S_1 and S_2 are strong dominating sets of G_1 and G_2 , respectively. So in the worst case we have

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) \leq \gamma_{\text{st}}(G_3) - 1 + \deg(x_1) - 1 + \deg(x_2) - 1.$$

- (b) $y_1 \in S_3$ and $y_2 \notin S_3$. If $v_H(x_1x_2)$ is not strong dominating any vertices in $\overline{S_3}$, then one can easily check that

$$S_1 = \left(S_3 \setminus \left(V(G_2) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{x_1\}$$

is a strong dominating set of G_1 , and one of the

$$S_2 = \left(S_3 \setminus \left(V(G_1) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{x_2\},$$

or

$$S'_2 = \left(S_3 \setminus \left(V(G_1) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{y_2\}$$

is a strong dominating set of G_2 (or possibly both are strong dominating sets of G_2). Otherwise, by similar argument as part (a), we conclude that

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) \leq \gamma_{\text{st}}(G_3) - 1 + \deg(x_1) - 1 + \deg(x_2).$$

- (c) $y_1 \notin S_3$ and $y_2 \notin S_3$. Then there exists $y'_1 \in V(G_1)$ such that y_1 is strong dominated by that, and there exists $y'_2 \in V(G_2)$ and is strong dominating y_2 . Then one can easily check that

$$S_1 = \left(S_3 \setminus \left(V(G_2) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{x_1\}$$

is a strong dominating set of G_1 , and

$$S_2 = \left(S_3 \setminus \left(V(G_1) \cup \{v_H(x_1x_2)\} \right) \right) \cup \{x_2\}$$

is a strong dominating set of G_2 , and we have

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) \leq \gamma_{\text{st}}(G_3) + 1.$$

(ii) $v_H(x_1x_2) \notin S_3$. Without loss of generality, suppose that there exists $x'_1 \in V(G_1)$ such that $v_H(x_1x_2)$ is strong dominated by x'_1 . We consider the following cases:

(a) $y_1 \in S_3$ and $y_2 \in S_3$. Then one can easily check that

$$S_1 = S_3 \setminus V(G_2)$$

is a strong dominating set of G_1 , and

$$S_2 = (S_3 \setminus V(G_1)) \cup \{x_2\}$$

is a strong dominating set of G_2 . So

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) \leq \gamma_{\text{st}}(G_3) + 1.$$

(b) $y_1 \in S_3$ and $y_2 \notin S_3$. Then

$$S_1 = S_3 \setminus V(G_2)$$

is a strong dominating set of G_1 , and

$$S_2 = (S_3 \setminus V(G_1)) \cup \{x_2\}$$

or

$$S'_2 = (S_3 \setminus V(G_1)) \cup \{y_2\}$$

is a strong dominating set of G_2 (or possibly both are strong dominating set of G_2). So

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) \leq \gamma_{\text{st}}(G_3) + 1.$$

(c) $y_1 \notin S_3$ and $y_2 \notin S_3$. Then by considering similar sets as part (a), we have

$$\gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) \leq \gamma_{\text{st}}(G_3) + 1.$$

Therefore we have $\gamma_{\text{st}}(G_3) \geq \gamma_{\text{st}}(G_1) + \gamma_{\text{st}}(G_2) - \deg(x_1) - \deg(x_2) + 2$, and we are done. $\square$

Remark 2.2. The lower bounds in Theorem 2.1 is tight. Consider Figure 2. One can easily check that the set of black vertices in each graph is a strong dominating set of that and the equality holds. This idea can be generalized and therefore there is an infinite family of graphs such that the equality of the lower bound holds. Also, the upper bounds in Theorem 2.1 is tight. Consider Figure 3. By an easy argument, the set of black vertices in each graph is a strong dominating set of that and the equality of the upper bound holds. Since this idea can be generalized, then there is an infinite family of graphs such that the equality of the upper bound holds.

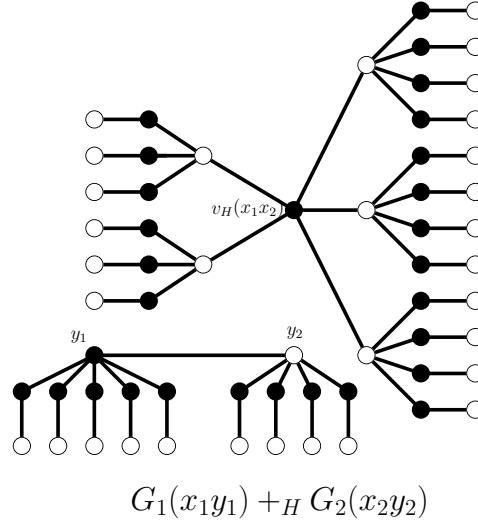
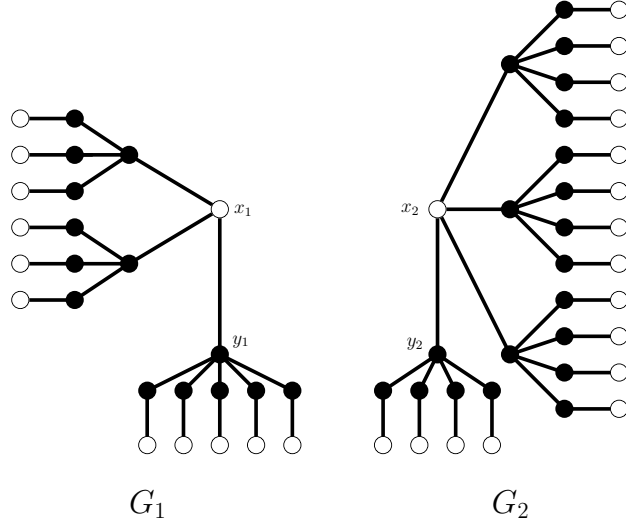


Figure 2: Hajós construction of G_1 and G_2 .

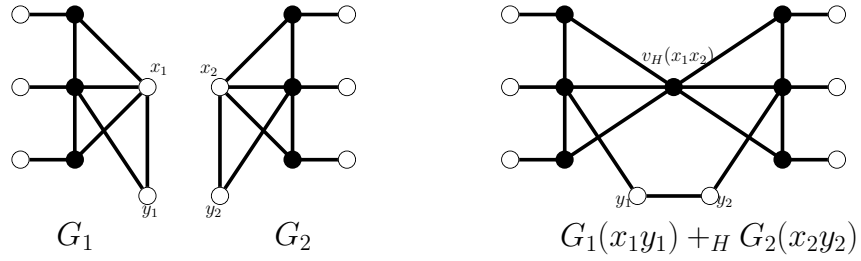


Figure 3: Hajós construction of G_1 and G_2 .

3 Vertex-Sum

In this section, we focus on the strong domination number of vertex-sum graphs. Given disjoint graphs $G_1, \dots, G_k$ with $u_i \in V(G_i)$, $i = 1, \dots, k$, the vertex-sum of $G_1, \dots, G_k$, at the vertices $u_1, \dots, u_k$, is the graph $G_1 +_u G_2 +_u \dots +_u G_k$ obtained from $G_1, \dots, G_k$ by identifying the vertices u_i , $i = 1, \dots, k$, as the same vertex u . This definition is from [5] by Barioli, Fallat and Hogben. We call u the *central vertex* of the vertex-sum. The vertex-sum of t copies of a graph G at a vertex u is denoted by G_u^t , $t \geq 2$. For the sake of simplicity, we may assume that the vertex u belongs to all the G_i . Recently the distinguishing number and the distinguishing threshold of some vertex-sum graphs studied in [11]. The following theorem gives the lower bound and the upper bound for the strong domination number of vertex-sum of two graphs.

Theorem 3.1. *For the vertex-sum of disjoint graphs $G_1, G_2, \dots, G_k$ with $u_i \in V(G_i)$, $i = 1, 2, \dots, k$, we have*

$$\left(\sum_{i=1}^k \gamma_{\text{st}}(G_i) - \deg(u_i) \right) + 1 \leq \gamma_{\text{st}}(G_1 +_u G_2 +_u \dots +_u G_k) \leq \left(\sum_{i=1}^k \gamma_{\text{st}}(G_i) \right) + 1.$$

Proof. First we find the upper bound. Suppose that D_i is a γ_{st} -set of G_i , for $i = 1, 2, \dots, k$. Then clearly

$$D = \bigcup_{i=1}^k D_i \cup \{u\},$$

is a strong dominating set of $G_1 +_u G_2 +_u \dots +_u G_k$, and we are done. Now, we consider the lower bound and prove it. Suppose that S is a γ_{st} -set of $G_1 +_u G_2 +_u \dots +_u G_k$. We find strong dominating sets of G_i , for $i = 1, 2, \dots, k$, based on S . We have two cases:

- (i) $u \notin S$. Then there exists $u' \in S$ and is strong dominating u . Without loss of generality, suppose that $u' \in V(G_1)$. Then one can easily check that

$$S_1 = S \setminus \left(\bigcup_{i=2}^k V(G_i) \right)$$

is a strong dominating set of G_1 , and for $i = 2, 3, \dots, k$,

$$S_i = (S \cup \{u_i\}) \setminus \left(\bigcup_{\substack{j=1 \\ j \neq i}}^k V(G_j) \right)$$

is a strong dominating set of G_i . So we have

$$\sum_{i=1}^k \gamma_{\text{st}}(G_i) \leq \gamma_{\text{st}}(G_1 +_u G_2 +_u \dots +_u G_k) + k - 1,$$

which is not in contradiction of the lower bound.

- (ii) $u \in S$. If after forming each G_i , for all $i = 1, 2, \dots, k$, $\deg(u_i) \geq \max\{\deg(v) \mid v \in N(u_i)\}$, then

$$S_i = \left(S \cup \{u_i\} \right) \setminus \left(\bigcup_{\substack{j=1 \\ j \neq i}}^k V(G_j) \cup \{u\} \right)$$

is a strong dominating set of G_i , for $i = 1, 2, \dots, k$. So we have

$$\sum_{i=1}^k \gamma_{\text{st}}(G_i) \leq \gamma_{\text{st}}(G_1 + G_2 + \dots + G_k) + k - 1,$$

which is not in contradiction of the lower bound. The worst case happens when after forming each G_i , for all $i = 1, 2, \dots, k$, $\deg(u_i) < \max\{\deg(v) \mid v \in N(u_i)\}$. Then by considering

$$S_i = \left(S \cup N(u_i) \right) \setminus \left(\bigcup_{\substack{j=1 \\ j \neq i}}^k V(G_j) \cup \{u\} \right),$$

one can easily check that S_i is a strong dominating set of G_i , for $i = 1, 2, \dots, k$. So we have

$$\sum_{i=1}^k \gamma_{\text{st}}(G_i) \leq \gamma_{\text{st}}(G_1 + G_2 + \dots + G_k) + \left(\sum_{i=1}^k \deg(u_i) \right) - 1.$$

Therefore we have the result. □

As an immediate result of Theorem 3.1, we have:

Corollary 3.2. *For the The vertex-sum of t copies of a graph G at a vertex u , we have*

$$t(\gamma_{\text{st}}(G) - \deg(u)) + 1 \leq \gamma_{\text{st}}(G_u^t) \leq t\gamma_{\text{st}}(G) + 1.$$

Remark 3.3. Bounds in Theorem 3.1 are tight. For the upper bound, consider G_i as shown in Figure 4. The set of black vertices is a γ_{st} -set of G_i . Now, if we consider $G_1 + G_2 + \dots + G_k$, then we need all black vertices and u in our strong dominating set. Therefore the equality holds. By generalizing this idea, we have an infinite family of graphs such that the equality of the upper bound holds. For the lower bound, consider G_i as shown in Figure 5. The set of black vertices, say S_i , is a γ_{st} -set of G_i . Now, if we consider $G_1 + G_2 + \dots + G_k$, then clearly $\left(\bigcup_{i=1}^k S_i \cup \{u\} \right) \setminus \left(\bigcup_{i=1}^k N(u_i) \right)$ is a γ_{st} -set, and we are done. By generalizing this idea, we have an infinite family of graphs such that the equality of the lower bound holds.

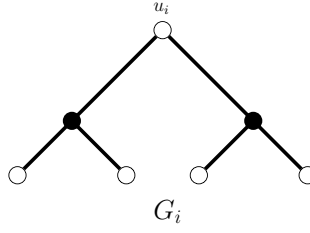


Figure 4: Graph G_i , for $i = 1, 2, \dots, k$.

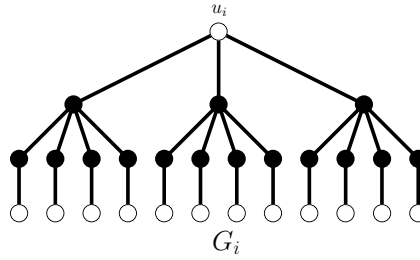


Figure 5: Graph G_i , for $i = 1, 2, \dots, k$.

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