FRIENDLY PATHS FOR FINITE SUBSETS OF PLANE INTEGER LATTICE. I

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Abstract. For a given finite subset $\mathcal{P}$ of points of the lattice $\mathbb{Z}^2$, *a friendly path* is a monotone (uphill or downhill) lattice path which splits points in half; points lying on the path itself are discarded. The purpose of this paper (and its sequel) is to fully describe all configurations of n points in $\mathbb{Z}^2$ which do not admit a friendly path. We say that such an n-set is *inseparable*. There are, up to the lattice symmetry, exactly $c(n)$ such sets. If only lattice shifts are counted, there are $\hat{c}(n)$ of them. Both sequences are new entries into OEIS (A369382 and, respectively, A367783). In particular, $n = 27$ is the first odd numbers with $c(n) = 1$. No example was known so far. This solves problem 11484(b)* posed in American Mathematical Monthly (February 2010). In this paper we also show that inseparable n-set exist for all even numbers $n \geq 12$ and almost all odd numbers.

1. Friendly paths

1.1. **Formulation.** The following problem appeared in American Mathematical Monthly ([1], Problem 11484).

Problem. *An* uphill *(NE, North-East) lattice path is the union of a (doubly infinite) sequence of directed line segments in $\mathbb{R}^2$, each connecting an integer pair (a, b) to an adjacent pair, either $(a, b+1)$ or $(a+1, b)$. A* downhill *(SE, South-East) lattice path is defined similarly, but with $b-1$ in place of $b+1$, and a* monotone *lattice path is an uphill or downhill lattice path.*

Given a finite set $\mathcal{P}$ of points in $\mathbb{Z}^2$, a friendly *path is a monotone lattice path for which there are as many points in $\mathcal{P}$ on one side of the path as on the other (points that lie on the path do not count).*

- (**a**) *Prove that if $n = b^2 + a^2 + b + a$ for some positive integers a, b such that $b \leq a \leq b + \sqrt{2b}$, then there exists a configuration of n such points that there does not exist a friendly line.*
- (**b**)* *Is it true that for every odd-sized set of points there is a friendly path?*

Asterisk means that no solution was known to the presenter of the problem (myself). None was received by the editors in two years time [2].

We remark that, as is clear from [2], the precise bound for the problem (**a**) is slightly higher:

$$n = a^2 + a + b^2 + b,\ a, b \in \mathbb{N},\ b \leq a \leq b + \sqrt{2b} + \frac{1}{2}.$$

As is easy to prove, such representation of n, if it exists, is unique. The sequence of these numbers starts from

$$4, 8, 12, 18, 24, 26, 32, 40, 42, 50, 60, 62, 72, 76, 84, 86, 98, 102, 112, 114, 128, 132, 144, 146, 162,$$
$$166, 180, 182, 188, 200, 204, 220, 222, 228, 242, 246, 264, 266, 272, 288, 292, 312, 314, 320, 338, 342.$$

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1.2. **Results.** In this paper we solve the problem and show that it further leads to more profound topics in geometric combinatorics and number theory; in particular, integer partitions.

Recall that the symmetry group of a plane square lattice is $\mathbb{G} = \mathbb{D}_4 \ltimes \mathbb{Z}^2$, denoted in crystallography as *p4m*. Since only monotone paths will be considered, the adjective "monotone" will be skipped altogether. Paths will be denoted by curly letters $\mathscr{A}, \mathscr{B}$, and so on. A finite lattice subset of size n without a friendly path will be called an *inseparable* n-set.

Theorem. *The following statements hold.*

i) *Suppose $n \in \mathbb{N}$ is odd, $n \leq 41$, and a set of n lattice points is inseparable. Then $n = 27$. Up to the $\mathbb{G}$-symmetry, the unique configuration is presented in Figure 1 (left).*
ii) *An inseparable n-set exists for every even number $n \geq 12$.*
iii) *Odd numbers n for which an inseparable n-set exists have a natural density* 1 *among all odd numbers.*

Definition. *Let $c(n)$ and $\hat{c}(n)$ be, up to a $\mathbb{G}$-symmetry, and, respectively, $\mathbb{Z}^2$-symmetry, a number of different inseparable n-sets.*

We will soon ascertain that this number is indeed finite. These two sequences are related via an identity

$$\hat{c}(n) = \sum_{\substack{\text{Over all } \mathcal{P}, |\mathcal{P}|=n, \\ \text{Up to the } \mathbb{G}\text{-symmetry}}} \frac{8}{g(\mathcal{P})},$$

where $g(\mathcal{P})$ is the number of symmetries of a set $\mathcal{P}$. Obviously, $g(\mathcal{P}) = 1, 2, 4$, or 8. *A posteriori* it appears that even-sized and odd-sized cases have no conceptual difference, save when dealing with symmetry. The construction carried out in the next subsection (namely, a quartering) will immediately show that

$$\hat{c}(n) = 8c(n) \text{ for odd } n.$$

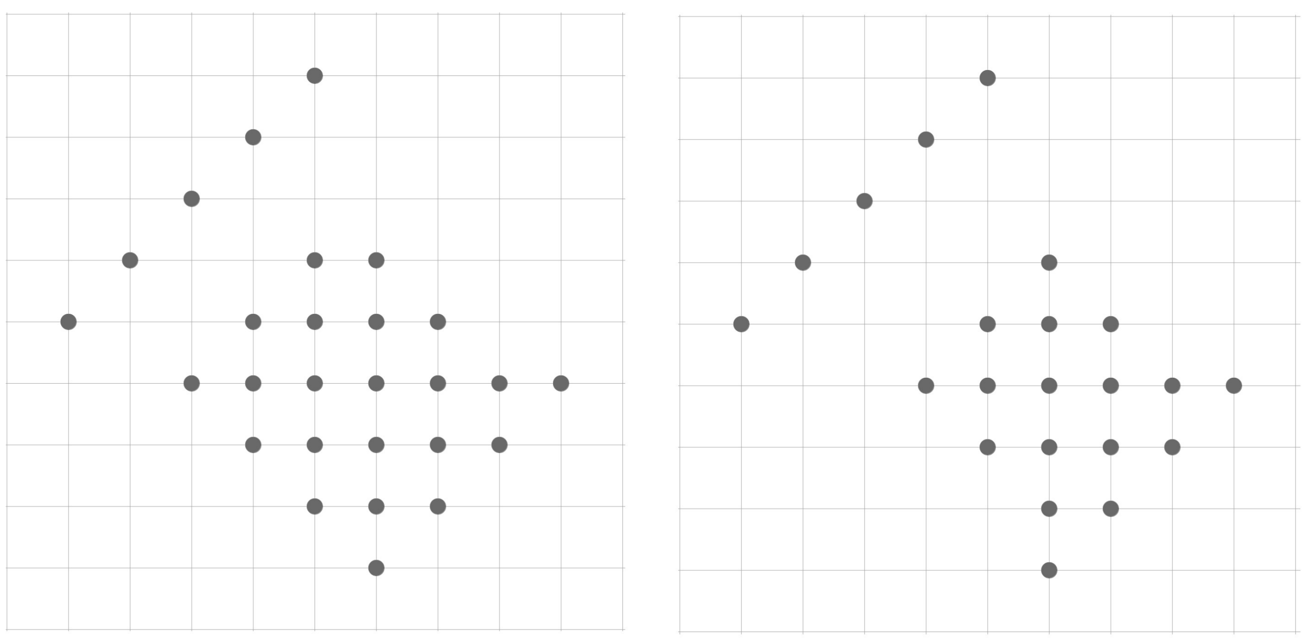
For small n, symmetric inseparable sets dominate and thus only even-sized sets exist. As n grows larger, the portion of symmetric sets diminish and both cases tend to exhibit more and more complex configurational variety. In particular, $n = 44$ is the first even number possessing an n-set with $g(\mathcal{P}) = 1$ (see Figure 7, right).

Prior to passing to proofs, we note that the given setup has a natural N-dimensional analogue for all positive integers $N \geq 2$. Also, it has a 2-dimensional analogue for a hexagonal lattice. Though a topic of lattice paths is a basic one in combinatorics and is essentially covered in [4], the interaction between finite subsets of $\mathbb{Z}^2$ and lattice paths, provided by the notion of a "friendly path", is a new and intriguing topic with several ramifications (possibly, even leading to modular forms; see Subsection 4.3).

1.3. **Preliminary quartering.** For a path $\mathscr{A}$, define *the balance* of $\mathscr{A}$ by $d(\mathscr{A}) = \ell(\mathscr{A}) - r(\mathscr{A})$, where ℓ and r are number of points from $\mathcal{P}$ on the left and, respectively, on the right shores of $\mathscr{A}$. Let $T(\mathscr{A})$ be the number of points on the path itself (this will be needed in Section 3). A *turn* on the path can be either left or right. These will be called *stop-points*. If $\mathscr{C}$ and $\mathscr{D}$ are two paths such that all stop-points of $\mathscr{C}$ lie on the left shore of $\mathscr{D}$ or $\mathscr{D}$ itself, we write $\mathscr{C} \precsim \mathscr{D}$. This implies $d(\mathscr{C}) \leq d(\mathscr{D})$. The symbol "$\precsim$" thus gives a partial ordering of paths. Let $\mathbf{S}$ be a closed rectangle in the plane bounded by lines $X = x_0$, $X = x_1$, $Y = y_0$, $Y = y_1$, $x_0 < x_1$, $y_0 < y_1$, such that $\mathcal{P} \subset \mathbf{S}$. We write $\mathscr{A} \doteq \mathscr{B}$, if $\mathscr{A} \cap \mathbf{S} = \mathscr{B} \cap \mathbf{S}$. Naturally, the structure of the path outside $\mathbf{S}$

n	$c(n)$	$\hat{c}(n)$	n	$c(n)$	$\hat{c}(n)$	n	$c(n)$	$\hat{c}(n)$	n	$c(n)$	$\hat{c}(n)$
1	0	**0**	16	1	**2**	31	0	**0**	46	9	**30**
2	0	**0**	17	0	**0**	32	3	**8**	47	0	**0**
3	0	**0**	18	2	**4**	33	0	**0**	48	5	**13**
4	1	**0**	19	0	**0**	34	6	**18**	49	0	**0**
5	0	**0**	20	2	**6**	35	0	**0**	50	7	**22**
6	0	**0**	21	0	**0**	36	10	**32**	51	0	**0**
7	0	**0**	22	2	**6**	37	0	**0**	52	8	**32**
8	1	**2**	23	0	**0**	38	6	**20**	53	0	**0**
9	0	**0**	24	4	**11**	39	0	**0**	54	12	**42**
10	0	**0**	25	0	**0**	40	9	**29**	55	0	**0**
11	0	**0**	26	4	**12**	41	0	**0**	56	18	**64**
12	1	**1**	27	1	**8**	42	12	**42**	57	0	**0**
13	0	**0**	28	3	**7**	43	1	**8**	58	14	**50**
14	1	**2**	29	0	**0**	44	18	**67**	59	0	**0**
15	0	**0**	30	2	**6**	45	2	**16**	60	17	**64**

TABLE 1. Sequences $c(n)$ and $\hat{c}(n)$ (in bold), $1 \le n \le 60$.



n=27, type [2,3,3,4] n=22, type [2,2,2,4]

FIGURE 1. The unique inseparable odd-sized set with $n \le 41$ points (left) and the smallest even-sized set not of type $[N, M, N, M]$ (right; see Proposition 1).

does not affect its relation towards $\mathcal{P}$.

Let $n \in \mathbb{N}$. Suppose that a set $\mathcal{P}$, $|\mathcal{P}| = n$, is inseparable. First, we will show that its structure can be fully described with a help of an identity $n = \sum_{j=1}^{4} Y_j$, $Y_j \in \mathbb{N}$, and four sets L_j, $1 \le j \le 4$. Each L_j is a certain partition of Y_j into distinct positive integral parts (see A000009 in [5]). More

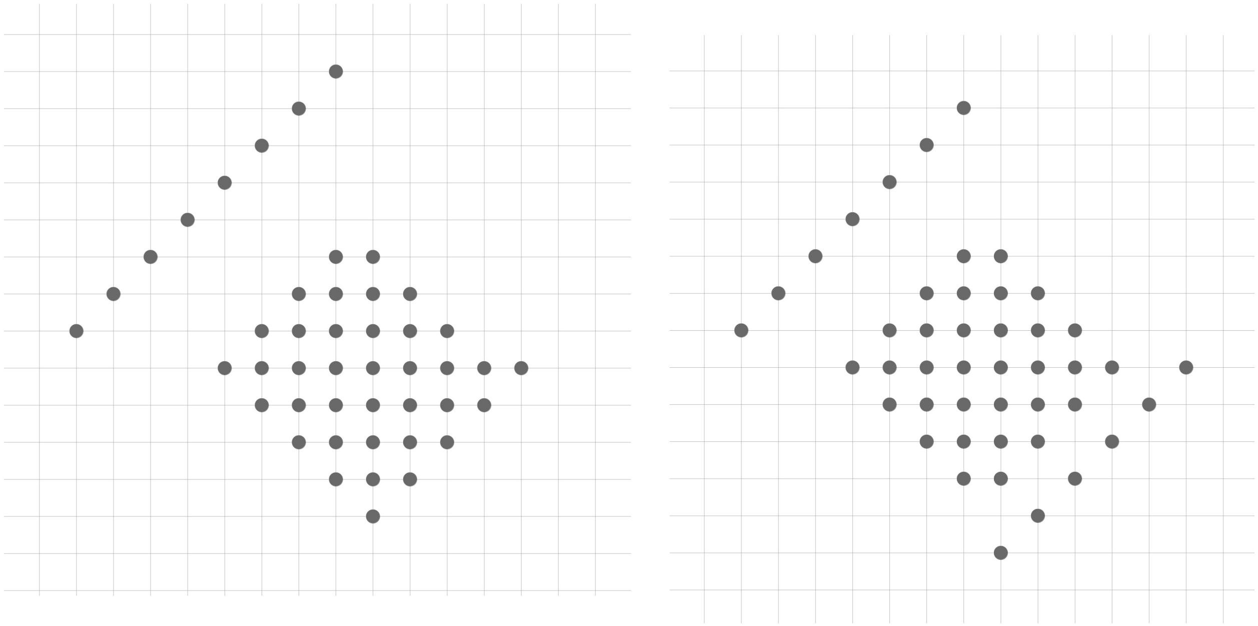


FIGURE 2. Two inseparable examples for $n = 45$

details will be given by Proposition 1.

Consider a horizontal path $y = K$, $K \in \mathbb{Z}$. Horizontal paths (going left to right) are both uphill and downhill. If K is large enough, a path has a balance $-n$. Similarly, if L is large enough, a horizontal line $y = -L$, $L \in \mathbb{Z}$, has a balance n. Since the sequence $d\big([y = Y]\big)$ for Y ranging from K to $-L$ is non-decreasing, there exists a place where this sequence changes sign. By our assumption no path has a balance 0. This implies the existence of $M \in \mathbb{Z}$ such that

$$d\Big([y = M + 1]\Big) < 0, \quad d\Big([y = M]\Big) > 0. \tag{1}$$

Let us draw a red line $y = M + \frac{1}{2}$. In the same manner, consider vertical lines going upwards. They are uphill. We thus find $T \in \mathbb{Z}$ such that

$$d\Big([x = T]\Big) < 0, \quad d\Big([x = T + 1]\Big) > 0. \tag{2}$$

Let us draw a vertical red line $x = T + \frac{1}{2}$. Let O be the intersection of these two. This unambiguous construction quarters points in $\mathcal{P}$.[1] The group of isometries of the plane which leaves the union of two red lines intact is $\mathbb{D}_4$. If $\sigma \in \mathbb{D}_4$, then trivially $\sigma(\mathcal{P})$ is inseparable, too.

2. Representations of inseparable sets

2.1. **The structure.** We now define a procedure on the path called *a shift*, which can be *negative* or *positive*. Take any left-turn of the uphill path. Two unit segments at this stop-point are $(a, b) \to (a + 1, b)$ and $(a + 1, b) \to (a + 1, b + 1)$. Let us replace them by $(a, b) \to (a, b + 1)$ and $(a, b+1) \to (a+1, b+1)$ (see Figure 3, left, left-turn at E). After this replacement the balance of

[1]This method also provides an algorithm for finding a friendly path. If vertical or horizontal line can be such, we are done. In not, there exists the unique quartering of points in $\mathcal{P}$. The second step of the algorithm becomes clear minding the analysis in Section 2.

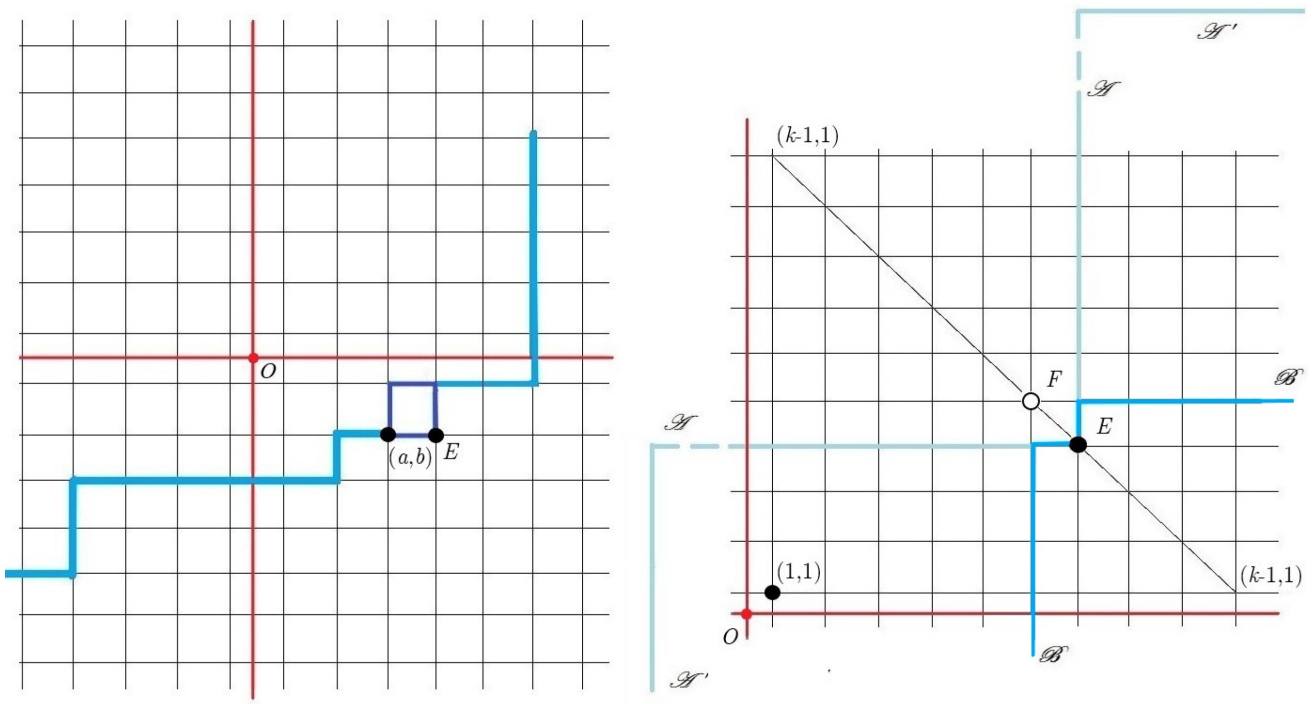


FIGURE 3. Quartering of the set $\mathcal{P}$

the path changes by 0, -1, or -2. This is a *negative shift*. Let us also define a negative shift for a downhill path. In this case, a left turn need also be taken, and two segments $(a,b)\to(a,b-1)$ and $(a,b-1)\to(a+1,b-1)$ are replaced by $(a,b)\to(a+1,b)$ and $(a+1,b)\to(a+1,b-1)$. This changes the balance also by 0, -1, or -2. On the other hand, if we consider right-turns, we can analogously introduce the notion of a *positive shift*. The latter changes the balance by 0, 1, or 2.

Let us return to the quartering of the set $\mathcal{P}$. Consider the first (top-right) quadrant $\mathcal{Q}_1$. Let the lattice points there have coordinates (x,y), $x,y\in\mathbb{N}$ (see Figure 3, right). For fixed $k\in\mathbb{N}$, $k\geq 1$, take an integer diagonal $\{(x,k+1-x):1\leq x\leq k\}$. Call this set $\Upsilon_{1,k}$. We will now prove the following statement.

Lemma. *All or none points of this diagonal belong to $\mathcal{P}$.*

For $k=1$ this is a tautology. Suppose there exists $k\geq 2$ and two neighbouring points E,F on this diagonal satisfying $E\in\mathcal{P}$, $F\notin\mathcal{P}$. Without loss of generality, assume E is to the right of F. Otherwise we just use the reflection with respect to the line $x=y$. So, $E=(x,y)$, $F=(x-1,y+1)$, $x\geq 2$.

Consider now two uphill paths $\mathscr{A}$ and $\mathscr{B}$, as marked in Figure 3, right. Both dotted half-lines extend to infinity. Outside $\mathbf{S}$, the path $\mathscr{A}$ can be altered to attain a shape of a path $\mathscr{A}'$. This does not change the balance. Now, inside $\mathbf{S}$ the path $\mathscr{A}'$ can be gradually transformed into the the path $\mathscr{B}$ by a series of positive shifts, leaving the left-turn at E intact:

$$\mathscr{A}'=\mathscr{E}_0\precsim\mathscr{E}_1\precsim\cdots\mathscr{E}_r\doteq\mathscr{B}.$$

Moreover, since $\mathscr{A}\precsim[y=M+1]$, we have $d(\mathscr{A}')=d(\mathscr{A})<0$. In a similar vein, since $[x=T+1]\precsim\mathscr{B}$, $d(\mathscr{B})>0$ (here we implicitly use a condition $x\geq 2$). By our assumption,

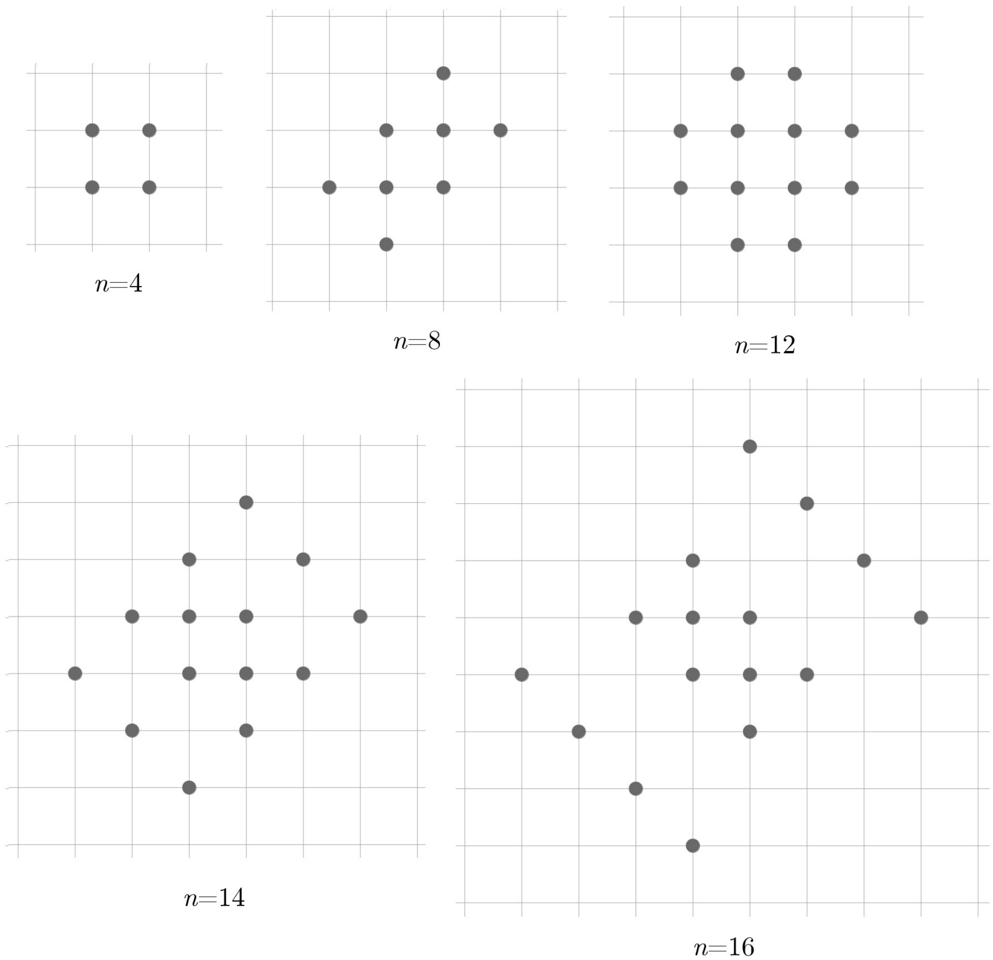


FIGURE 4. Examples of all even n with $c(n) = 1$

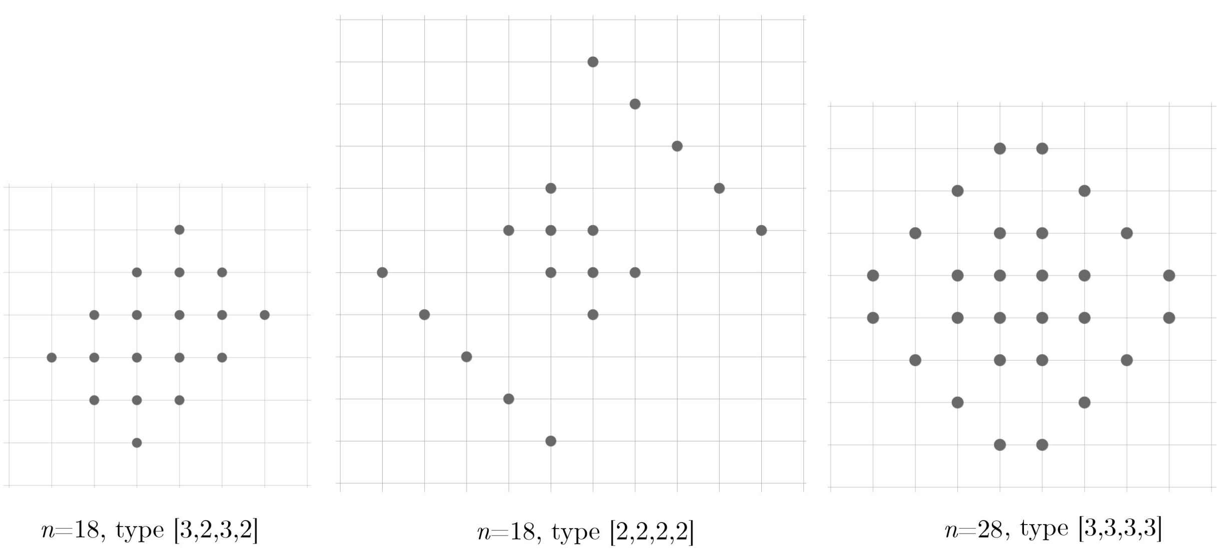


FIGURE 5. Two inseparable examples for $n = 18$ and the the smallest lacunary inseparable configuration with 8-fold symmetry ($n = 28$)

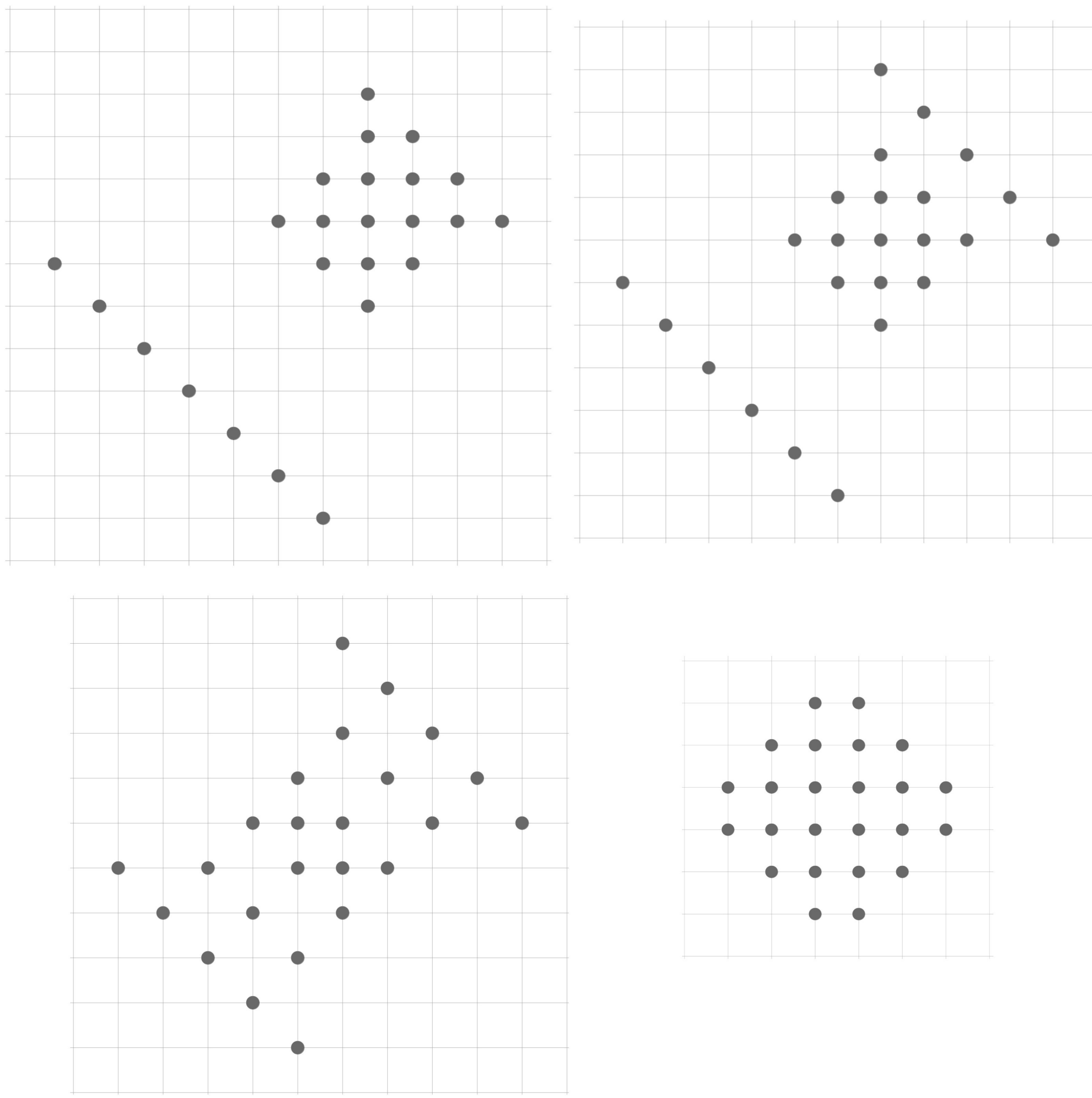


FIGURE 6. Four inseparable examples for $n = 24$

a sequence $\{d(\mathscr{E}_i) : 0 \leq i \leq r\}$ is monotone, does not contain 0 and makes integer jumps by at most 2. Since it goes from negative to a positive value, there exists s, $1 \leq s \leq r$, such that $d(\mathscr{E}_{s-1}) = -1$ and $d(\mathscr{E}_s) = 1$. Consider the path $\mathscr{E}_s$. Making a negative shift for $\mathscr{E}_s$ at E produces a path with a balance 0. A contradiction. ∎

The same conclusion must hold for every integer diagonal of any other quadrant. Thus, we have proved the main structural result.

Proposition 1. *Every inseparable set $\mathcal{P}$ can be represented in the form*

$$\begin{aligned} \theta = [L_1 L_2 L_3 L_4], \qquad & L_r = (v_{r,1}, v_{r,2}, \ldots, v_{r,N_r}), \\ 1 \leq v_{r,1} < v_{r,2} < \cdots < v_{r,N_r}, \qquad & r \in \{1,2,3,4\},\ N_r \in \mathbb{N}. \end{aligned} \tag{3}$$

Here $1,2,3$ *and* 4 *correspond, respectively, to the namesake quadrant. The notation implies that integer diagonals with numbers* $v_{r,i}$ *fully belong to* $\mathcal{P}$*, and these exhaust all of its points. We have*

$$n = n(\theta) = \sum_{r=1}^{4} Y_r, \text{ where } Y_r = \sum_{t=1}^{N_r} v_{r,t}.$$

Let us call the quadruple $[N_1, N_2, N_3, N_4]$, the *type* of the representation θ, and $\max N_i - \min N_i$ its *variation.*

Calculations will confirm that in fact $v_{r,1} = 1$ for all inseparable sets. Surely, only specific choices of such representation of an integer n produces an inseparable configuration. Computational part of this proposition will be treated in Section 3, while a theoretical part will be dealt with in [3]. We however remark that all results of this paper can be checked by hand and do not depend on computer codes.

2.2. **Examples.** The smallest integer with $c(n) \geq 2$ is $n = 18$ (see Figure 5), where these collections are, respectively,

$$\theta_1 = [(1,2,3)(1,2)(1,2,3)(1,2)] \text{ and } \theta_2 = [(1,5)(1,2)(1,5)(1,2)].$$

The smallest integer with $c(n) \geq 3$ is $n = 24$ (Figure 6), where in fact $c(24) = 4$:

$$\begin{aligned} \theta_1 &= [(1,3,5)(1,2)(1,3,5)(1,2)], & g(\theta_1) &= 4; \\ \theta_2 &= [(1,2,3,5)(1,2)(1,6)(1,2)], & g(\theta_2) &= 2; \\ \theta_3 &= [(1,2,3,4)(1,2)(1,7)(1,2)], & g(\theta_3) &= 2; \\ \theta_4 &= [(1,2,3)(1,2,3)(1,2,3)(1,2,3)], & g(\theta_4) &= 8. \end{aligned}$$

Consequently,

$$\hat{c}(24) = \frac{8}{4} + \frac{8}{2} + \frac{8}{2} + \frac{8}{8} = 11.$$

The smallest odd-sized set with a variation 3 is presented in Figure 7 (left). The initial solution in MONTHLY shows that variation for even-sized inseparable sets can be arbitrarily large. With all tools at hand, we easily see that the same holds for odd-sized sets.

3. FULL CONDITIONS FOR INSEPARABILITY

Assume, we run a computer program which enumerates all possible quadruples $[L_1 L_2 L_3 L_4]$ of finite increasing integer sequences. Which of them truly represent an inseparable set? Three tests and a preliminary requirement should be passed. This is the complete list of conditions.

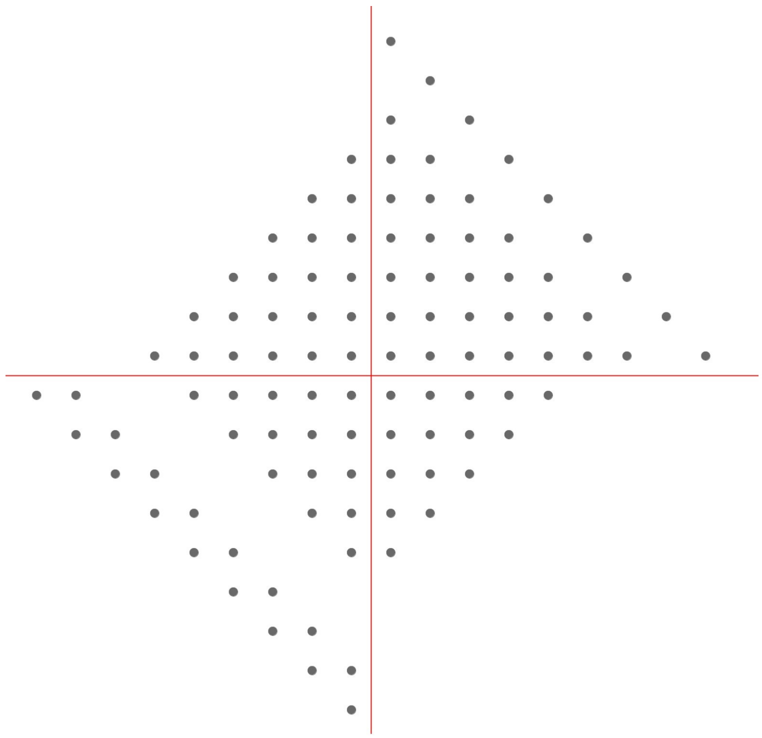
3.1. **Evenness and proper quartering.**

`Test 0`

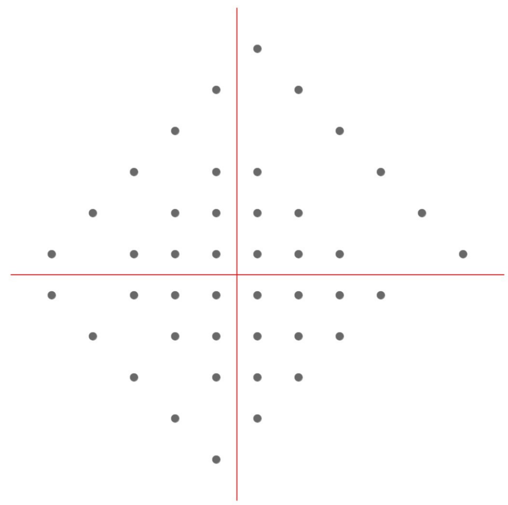
We start from type $[N_1, N_2, N_3, N_4]$ and $n(\theta)$ which (in principal) can lead to an inseparable set. By this it is meant that the following condition holds:

$$N_1 + N_3 \equiv N_2 + N_4 \equiv n \pmod 2.$$

If satisfied, this implies that the generic path starting in quarter $\mathcal{Q}_3$ and ending in quarter $\mathcal{Q}_1$ (the same applies to the pair $\mathcal{Q}_2$ and $\mathcal{Q}_4$) pass through $N_1 + N_3 + 1$ points. Therefore, the remaining $n - N_1 - N_3 - 1$ points (an odd number) cannot be divided exactly in half. A friendly path, if it exists, must not be a generic one (see **Case 3** below for a precise meaning of this).



n=105, type [8,6,7,5]



n=44
[(1,2,3,6)(1,2,3,5)(1,2,3,5)(1,2,3,4)]

FIGURE 7. The smallest odd-sized inseparable set with variation 3 (left), and the smallest even-sized inseparable set without symmetries (right).

`Test A`

Next, we must check that the quartering of the set is proper. This means

1) $-N_3 - N_4 < Y_1 + Y_2 - Y_3 - Y_4 < N_1 + N_2$;

2) $-N_2 - N_3 < Y_1 + Y_4 - Y_2 - Y_3 < N_1 + N_4$.

Using reflections with respect to both red lines in case of necessity, one can always assume that $Y_1 + Y_2 \geq Y_3 + Y_4$ and $Y_1 + Y_4 \geq Y_2 + Y_3$ holds. However, for computational purposes it is more convenient to work with conditions **1)** and **2)** directly. Depending on symmetries of a representation type $[N_1, N_2, N_3, N_4]$, additional conditions are imposed by a computer program which checks all configurations of a particular type for inseparability.

To better understand where the next two tests come from, consider an example.

3.2. **Example with** $n = 108$**.** Let $\theta = [L_1 L_2 L_3 L_4]$, where

$$L_1 = (1,2,3,5,6,9), \quad L_2 = (1,2,3,6,7,9), \quad L_3 := (1,2,3,4,6,9), \quad L_4 := (1,2,4,6,7,9).$$

Here $Y = (26, 28, 25, 29)$, $(N_1, N_2, N_3, N_4) = (6,6,6,6)$; see Figure 8.

'**Case 3**'. For this particular set, assume there exists a friendly path which enters the configuration at $\mathcal{Q}_2$, passes through $\mathcal{Q}_1$ and exists it at $\mathcal{Q}_4$. So, this friendly path have no common

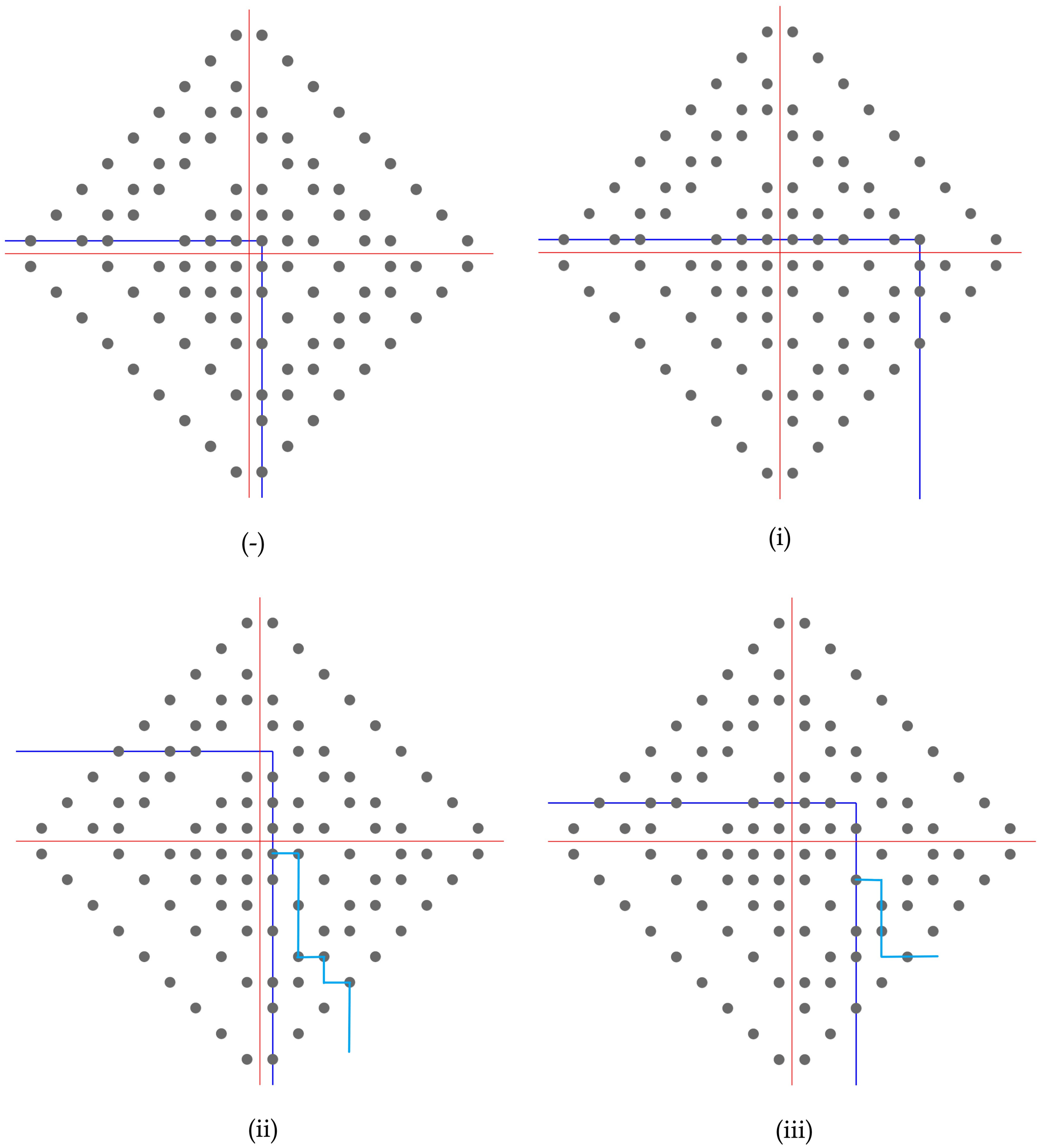


FIGURE 8. An example with $n=108$

points with $\mathcal{Q}_3$. This is exactly what the name '**Case 3**' is supposed to mean ('Case 1', 'Case 2', 'Case 4' are defined analogously). Now come several subcases.

'Subcase 3-'. Consider a downhill path which takes a single right-turn at the point $(1;1,1)$ (the first coordinate of this notation indicates the index of the quarter; see Figure 8, top-left).

There are $N_2 + N_4 + 1 = 13$ points on this path. Consequently, it cannot be friendly.

'Subcase 3i'. We now ask the following: find the smallest $x_0 \in \mathbb{N}$ such that a downhill path, which makes a single right-turn at $(1; x_0, 1)$, contains an even number of points (this is what "non generic" means). Only such paths can potentially be friendly. The answer is $x_0 = 6$ (Figure 8, top-right). For this particular path, $T(\mathscr{A}) = 14$, $r(\mathscr{A}) = 47$, $\ell(\mathscr{A}) = 47$. By a lucky chance, it is friendly.

'Subcase 3ii'. In the same vein, if a downhill path makes a single right-turn at $(1; 1, y_0)$, it contains an even amount of points, and y_0 is the smallest possible, then $y_0 = 4$ (same picture, bottom-left). For this path (shown in blue), $T(\mathscr{A}) = 12$, $r(\mathscr{A}) = 40$, $\ell(\mathscr{A}) = 56$.

'Subcase 3iii'. As a final sub-case, note that the first positive integer r not among members of L_1 is $r = 4$. Consequently, any down-hill path making a single right-turn at points $(1; 3, 2)$ or $(1; 2, 3)$ contains and even amount points (12 each). In both cases, $r(\mathscr{A}) = 44$, $\ell(\mathscr{A}) = 52$ (same picture, bottom-right). The second path is shown in blue.

Now, a negative shift made strictly inside $\mathcal{Q}_2$ or $\mathcal{Q}_4$ alters a balance by 0 or -2 (1 is excluded due to construction). Therefore, in the last two subcases we can construct a friendly path starting with a blue path by a series of negative shifts. Results in both cases are shown in sky-blue. For small n it might happen that, while performing a series of negative shifts, we are forced to leave $\mathcal{Q}_2$ or $\mathcal{Q}_4$. Yet a distinguished right turn chosen controls a positive shift by $+1$. Thus, a friendly path is always to be found.

This gives a clear picture how one checks for inseparability in a general case: in **Case 3** and all sub-cases (the path is downhill), inequalities $r(\mathscr{A}) > \ell(\mathscr{A})$ must hold.

'**Subcase 3iv**'. A special attention needs to be given if $v_{r,1} > 1$ for some r. In this case `test B` must be adjusted. Suppose, $v_{1,1} > 1$ holds (Figure 9, right). Since $10 = r(\mathscr{A}) = Y_3 > Y_1 + Y_2 + Y_4 - N_2 - N_4 = \ell(\mathscr{A}) = 8$, this does not lead to a friendly path. Assume now that instead we had an inequality $r(\mathscr{A}) < \ell(\mathscr{A})$. The tricky part is that, differently from '**Subcase 3ii**', we cannot guarantee that with a help of a series of negative shifts this path can be transformed into a friendly one. This negative outcome occurs precisely if the balance of a sky-blue path is also positive: $Y_1 > Y_2 + Y_3 + Y_4 - N_2 - N_4$. Thus, in case $v_{1,1} > 1$ a friendly path passing through $(1; 1, 1)$ does not exist if and only if

$$|Y_3 - Y_1| > Y_2 + Y_4 - N_2 - N_4. \tag{4}$$

In case of (Figure 9, right), a friendly path does exist (the one which avoids $\mathcal{Q}_2$), but the condition $v_{1,1} > 1$ cannot be ruled out for inseparable sets with small n. Nevertheless, computations confirm that indeed no such set exists.

3.3. **General check.** Consider general n and a subcase **3ii**. How we find such x_0? Checking many particular examples, we arrive at the following conclusion. For each ordered pair $(\mathcal{Q}_i, \mathcal{Q}_j)$ of neighbouring quadrants (that is, $i - j \equiv \pm 1 \pmod 4$), let $w_{i,j}$ be the smallest $p \in \mathbb{N} \setminus \{1\}$ such that

$$\text{either } p \in L_i,\ p - 1 \notin L_j,\ \text{ or } p \notin L_i,\ p - 1 \in L_j.$$

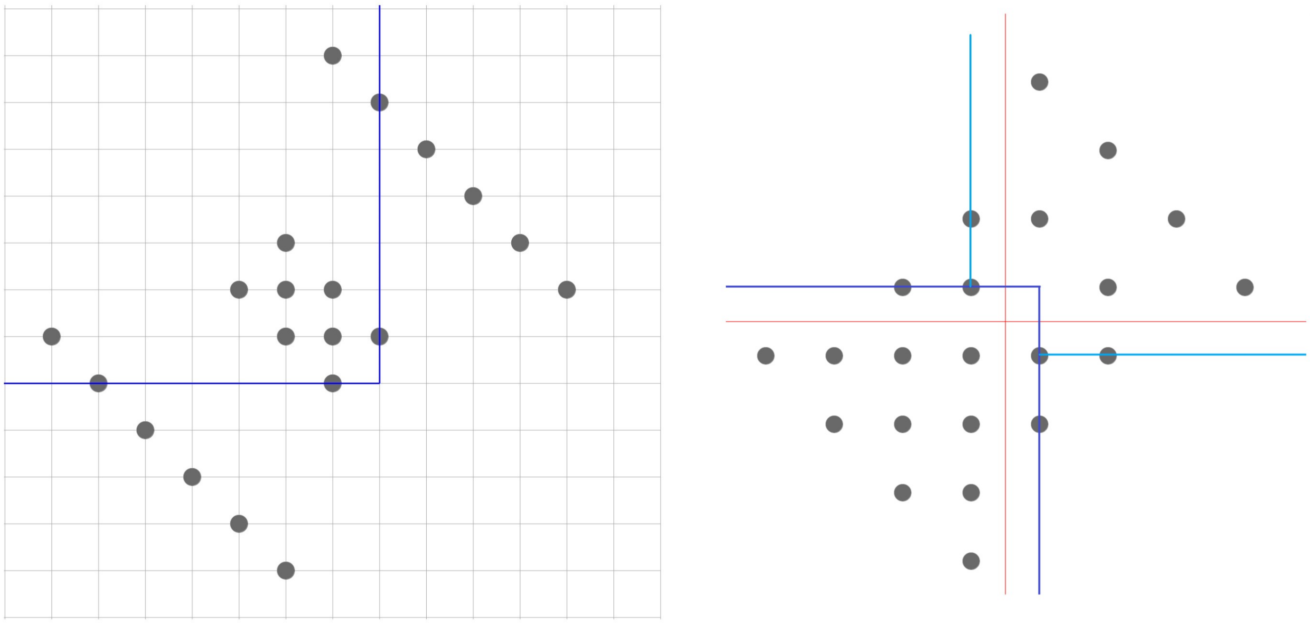


FIGURE 9. $[(1,6)(1,2)(1,6)(1,2)]$ fails `Test C` (left) and supplementary illustration for `Test B` in case $v_{1,1} > 1$ (right)

Note that such p might not exist. In this case for all paths ℓ not passing through a quadrant $j+1$, $T(\ell)$ is odd. Otherwise, if such p exists, this is exactly the quantity we are looking for (Looking back at the subcase **3ii** in the previous subsection, $w_{1,4} = 6$ corresponds exactly to the point $x_0 = 6$). As we have just witnessed, if the set is inseparable, the inequality $\ell(\mathscr{A}) < r(\mathscr{A})$ must hold for all pairs $(i,j) = (1,4), (4,3), (3,2), (2,1)$, and $r(\mathscr{A}) < \ell(\mathscr{A})$ must hold for all pairs $(i,j) = (4,1), (1,2), (2,3), (3,4)$. MAPLE program which checks these criteria is given in the appendix. This is `Test B`. Its input is representation θ and a pair of integers (j,i). The outcome is $[\mathtt{Ls}, \mathtt{Rs}, \mathtt{yra}, \mathtt{Tt}]$. Here $\mathtt{Ls} = \ell(\mathscr{A})$, $\mathtt{Rs} = r(\mathscr{A})$, $\mathtt{Tt} = T(\mathscr{A})$. Boolean variable `yra` gives value `false` if no such p exists. In case $v_{r,1} > 1$ this tests only checks the condition (4). Naturally, the boolean output of this test is sufficient in all cases. We track values `Ls`, `Rs` and `Tt` only for debugging purpose.

Equally, analogue of r in the subcase **3iii** is the following number. For quadrant $\mathcal{Q}_i$, let r_i be the smallest $p \in \mathbb{N}$ not contained in L_i. MAPLE codes which checks that no such path can be deformed to become friendly is given in the Appendix as `Test C`. Figure 9 shows a configuration which passes `Test 0`, `Test A` and `Test B`, but fails `Test C`.

4. Basic properties of both sequences

4.1. **Even numbers with $c(n) = 0$.** Let $a \in \mathbb{N}$, $a \geq 4$. Consider four types of configurations given in Figures 10 and 11 (letters on the graph refer to x-coordinate; in all specific examples, $a = 7$). All of them pass tests `0`, `A`, `B`, `C`. In particular, Figure 10 covers all even numbers in the range $[2a^2+6a+4, 2a^2+10a]$. Figure 11 covers all even numbers in the range $[2a^2+8a+10, 2a^2+12a+12]$. Now, recall that Figure 1 (right) gives a particular example of inseparable set for $n = 22$. This give the next result.

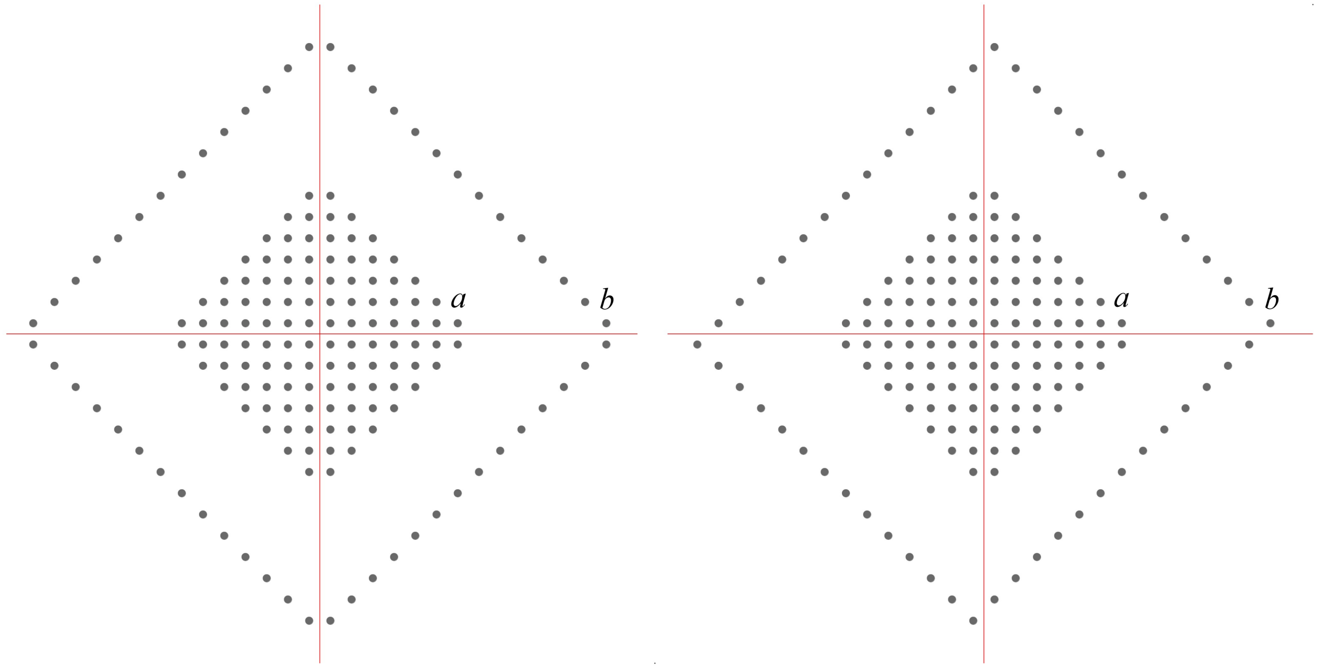


FIGURE 10. $a+1 \leq b \leq 2a$

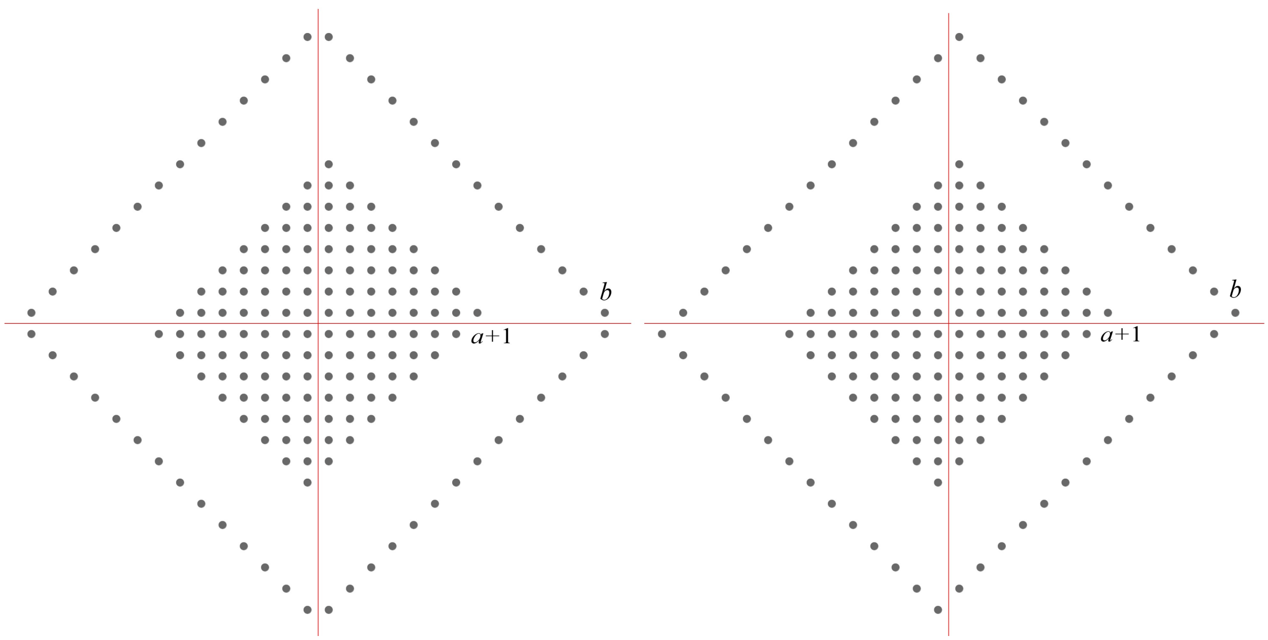


FIGURE 11. $a+2 \leq b \leq 2a$

Proposition 2. *All even numbers with* $c(n) = 0$ *are given by* $n = 2, 6, 10$.

4.2. **Odd numbers with** $c(n) > 0$**.**

Proposition 3. *Let* $K(n) = \{i \in 2\mathbb{N} - 1, i \leq n : c(i) = 0\}$. *Then*

$$\lim_{n\to\infty} \frac{K(n)}{n/2} = 0.$$

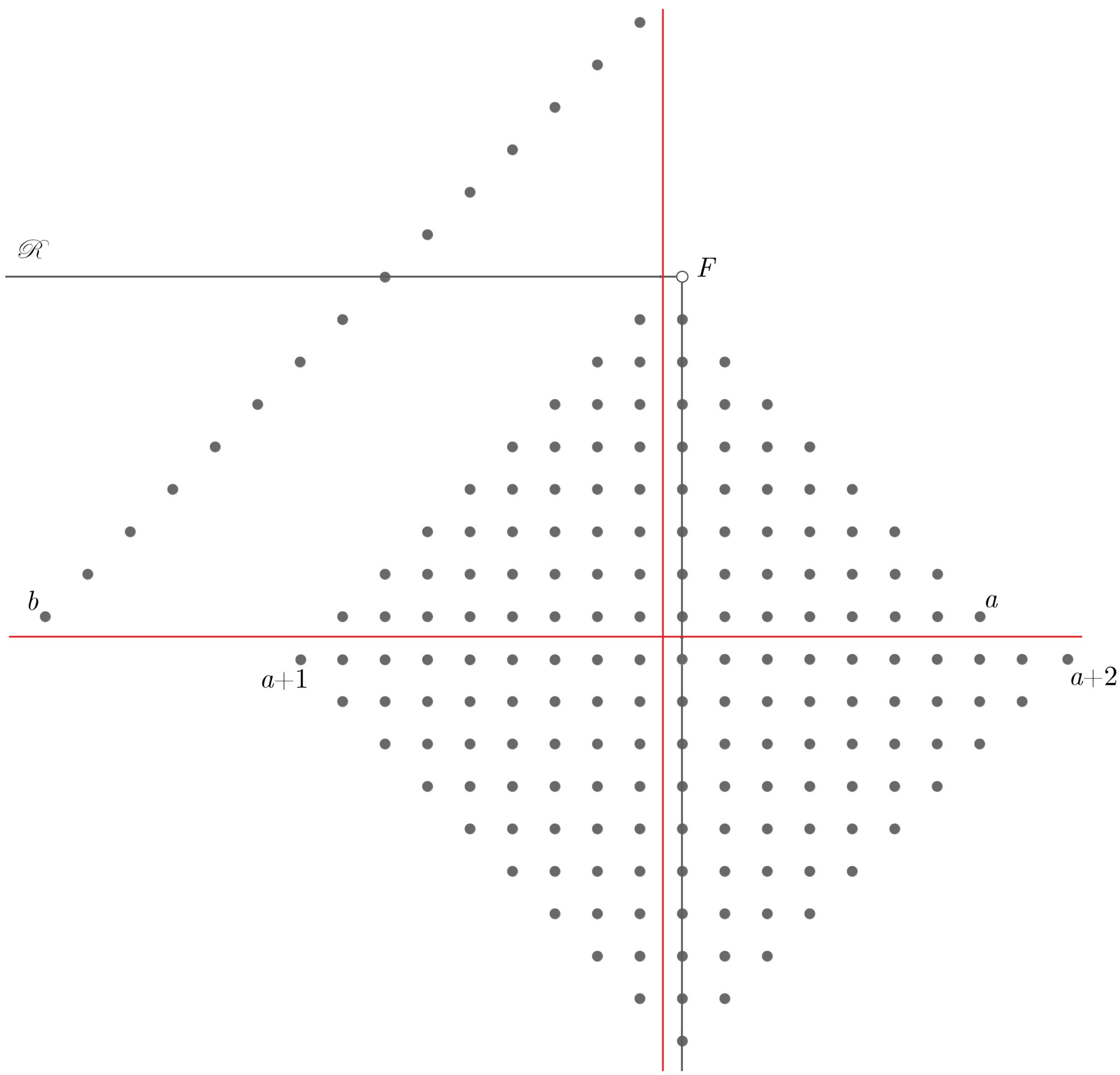


FIGURE 12. The first family (the simplest one) of inseparable sets. Here $a \geq 2$; $b-a$ is odd; $a+3 \leq b \leq 3a-1$ (type $[a, a+1, a+1, a+2]$, variation 2).

In order to show this, we will present two carefully chosen families of inseparable n-sets. Let $a, b \in \mathbb{N}$, $b > a \geq 2$, $b-a$ being odd. Consider the construction in Figure 12. Here $n = 2a^2+5a+b+4$. For a quartering to be proper, we must require $b \geq a+3$. Next, consider the path $\mathscr{R}$. Here $r(\mathscr{R}) = a^2+3a+1$, and $\ell(\mathscr{R}) = a^2+b$ points. If $b \leq 3a-1$, test `A` (subcase **3ii**) is passed. We easily calculate that test `A` then is passed in general. The construction accounts for all odd numbers in the interval $[2a^2+6a+7, 2a^2+8a+3]$. For $a = 2$ this reduces a singleton $n = 27$, and for $a = 3$ this give two numbers $n = 43$ and $n = 45$.

In the same manner, family in Figure 13 covers all odd integers in the range $[2a^2+8a+15, 2a^2+10a+5]$. Both these families cover all odd positive integers, save the ones contained in the intervals $[2a^2+8a+5, 2a^2+8a+13]$ and $[2a^2+10a+7, 2a^2+10a+13]$, $a \geq 5$. These (among odd ones) have a natural density 0.

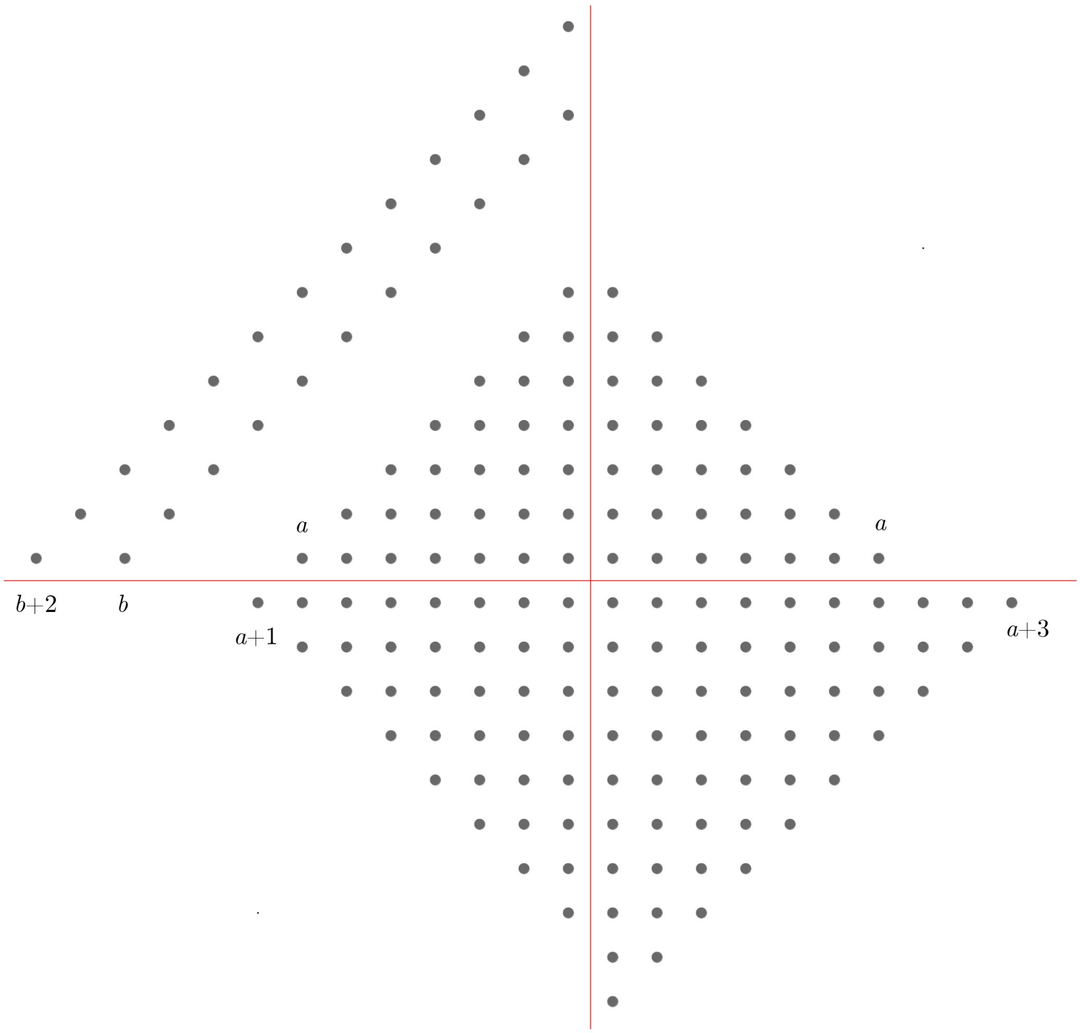


FIGURE 13. The second family of inseparable sets. Here $a \geq 5$; $a+3 \leq b \leq 2a-2$ (type $[a, a+2, a+1, a+3]$, variation 3).

4.3. **Expectations.** What kind of sequences $c(n)$ and $\hat{c}(n)$ are expected to be?

Let us get back to Proposition 1. If we were to count only quadruples (Y_1, Y_2, Y_3, Y_4) of positive integers which add up to n, the total number would be $\binom{n-1}{3}$ (sequence A000292 in [5]). This is a number of *compositions* (ordered partitions) of n into 4 positive parts. Let us now count the same compositions, only up to $\mathbb{D}_4$ symmetry (i.e. reversing order and cyclic shifts are allowed). Denote this number by $t(n)$. For example, $t(8) = 8$, and these compositions are

$$(1,1,1,5),\ (1,1,2,4),\ (1,1,3,3),\ (1,2,1,4),\ (1,3,1,3),\ (1,2,2,3),\ (1,2,3,2),\ (2,2,2,2).$$

In general, if we count separately compositions having exactly s summands equal to 1 ($s = 0, 1, 2$ or 3), we arrive to the recurrence

$$t(n) = t(n-4) + \frac{1}{2}\binom{n-5}{2} + \frac{1}{2}\left\lfloor\frac{n-5}{2}\right\rfloor + 2\left\lfloor\frac{n-4}{2}\right\rfloor + 1, \quad n \geq 5.$$

Thus, we get the sequence

$$0, 0, 0, 1, 1, 3, 4, 8, 10, 16, 20, 29, 35, 47, 56, 72, 84, 104, 120, 145, \ldots.$$

This is A005232 in [5] shifted by 4: $t(n) = A005232(n-4)$. The generating function

$$\sum_{n=1}^{\infty} t(n) z^n = \frac{z^4(1-z+z^2)}{(1-z)^2(1-z^2)(1-z^4)}.$$

But now, let us take a look at the Table 2. These are the two examples of inseparable sets for $n = 44$:

$$[(1,2,\mathbf{3},\mathbf{6})(1,2,3,4)(1,2,3,6)(1,2,3,4)],\ [(1,2,\mathbf{4},\mathbf{5})(1,2,3,4)(1,2,3,6)(1,2,3,4)].$$

They are of the forms $[L_1L_2L_3L_4]$ and $[L_1'L_2L_3L_4]$. The only difference are numbers marked in bold. In particular, this is possible due to equality $3+6=4+5$. Though being a trivial observation, it shows that the function $q(n)$ of *partitions into distinct parts* (A000009 in [5]) manifests in the structure of both sequences $c(n)$ and $\hat{c}(n)$. All these question will be treated in detail in [3].

5. Supplements

`Test B`

```
B:=proc(j,i::integer)
local Tt, t, wij, g, h, lik, v, Ls, Rs, yra;
global L,N,n,m;
yra:=true;
t:=2;
if L[i][1]>1 then
if abs(Y[i]-Y[modp(i+1,4)+1])>
Y[modp(i,4)+1]+Y[modp(i-2,4)+1]-N[modp(i,4)+1]-N[modp(i-2,4)+1]
then Ls:=1: Rs:=0: Tt:=0: else Ls:=0: Rs:=1: Tt:=0: end if: end if:
if L[i][1]=1 then
while member(t,L[i])=member(t-1,L[j]) and t<=max(m[i],m[j]) do t:=t+1: end do:
if member(t,L[i]) then Tt:=N[modp(j+1,4)+1]+N[j]+2: end if:
if member(t-1,L[j])then Tt:=N[modp(j+1,4)+1]+N[j]: end if:
if t=max(m[i],m[j])+1 then yra:=false: end if:
if yra then
wij:=t:
if L[i][N[i]]<=wij then h:=N[i] else
h:=1:
while (L[i][h]<=wij) do h:=h+1: end do:
h:=h-1:
end if:
g:=Tt-h-N[modp(j+1,4)+1]:
lik:=0:
for v from 1 to N[j] do if L[j][v]>wij then lik:=lik+(L[j][v]-wij): end if: end do:
Ls:=Y[modp(i+1,4)+1]+Y[j]-g-lik:
Rs:=n-Tt-Ls:
else Ls:=1: Rs:=0: end if: end if:
[Ls,Rs,yra,Tt]:
end proc:
```

`Test C`

```
C:=proc(i::integer)
local s, t,lik, Tt, v, Rs, Ls, Ger;
global L,N,n,m;
s:=1;
while member(s,L[i]) do s:=s+1: end do:
Ger:=true:
for t from 1 to s-2 do lik:=0: Tt:=0:
if member(s-t,L[modp(i,4)+1]) then Tt:=Tt+1: end if:
for v from 1 to N[modp(i,4)+1] do if L[modp(i,4)+1][v]>s-t then
lik:=lik+(L[modp(i,4)+1][v]-s+t): Tt:=Tt+1: end if: end do:
if member(t+1,L[modp(i-2,4)+1]) then Tt:=Tt+1: end if:
for v from 1 to N[modp(i-2,4)+1] do if L[modp(i-2,4)+1][v]>t+1 then
lik:=lik+(L[modp(i-2,4)+1][v]-t-1): Tt:=Tt+1: end if: end do:
Rs:=lik+Y[i]-(t+1)*(s-t)+1: Tt:=Tt+s-1: Ls:=n-Tt-Rs:
Ger:=Ger and (Ls>Rs):
end do:
Ger:
end proc:
```

References

[1] Problems and solutions, *Amer. Math. Monthly* **117** (2) February (2010), 182.
[2] Giedrius Alkauskas, Friendly paths, *Amer. Math. Monthly* **119** (2) February (2012), 167–168.
[3] Giedrius Alkauskas, Friendly paths for finite subsets of plane integer lattice. II (in preparation).
[4] Richard P. Stanley, *Enumerative Combinatorics*, 2nd ed., Cambridge Studies in Advanced Mathematics, 2011.
[5] The On-line Encyclopedia of Integer Sequences, Sequences A000009, A000292, A005232, A367783, A369382.

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TABLE 2. Representations of inseparable sets. Since $v_{r,1}$=1 in all cases, it is ommited in the notation.
Complete lists of inseparable sets of a given type (except when a bound $n \leqslant 60$ is imposed; then the total amount is in italic).

Type	#	L_1 L_2 L_3 L_4	n	$\frac{8}{g}$
(1,1,1,1)	1	() () () ()	4	1
(2,1,2,1)	1	(2) () (2) ()	8	2
(2,2,2,2)	4	(2) (2) (2) (2)	12	1
		(3) (2) (3) (2)	14	2
		(4) (2) (4) (2)	16	2
		(5) (2) (5) (2)	18	2
(3,2,3,2)	5	(2, 3) (2) (2, 3) (2)	18	2
		(2, 5) (2) (2, 3) (2)	20	4
		(2, 4) (2) (2, 4) (2)	20	2
		(3, 4) (2) (3, 4) (2)	22	2
		(3, 5) (2) (3, 5) (2)	24	2
(4,2,2,2)	3	(2, 3, 4) (2) (5) (2)	22	4
		(2, 3, 5) (2) (6) (2)	24	4
		(2, 3, 4) (2) (7) (2)	24	4
(4,2,4,2)	1	(2, 3, 4) (2) (2, 3, 4) (2)	26	2
(3,3,3,3)	15	(2, 3) (2, 3) (2, 3) (2, 3)	24	1
		(2, 5) (2, 3) (2, 3) (2, 3)	26	4
		(2, 4) (2, 3) (2, 4) (2, 3)	26	2
		(2, 4) (2, 4) (2, 3) (2, 3)	26	4
		(2, 6) (2, 3) (2, 4) (2, 3)	28	4
		(2, 5) (2, 3) (2, 5) (2, 3)	28	2
		(2, 4) (2, 4) (2, 4) (2, 4)	28	1
		(2, 7) (2, 3) (2, 5) (2, 3)	30	4
		(2, 6) (2, 3) (2, 6) (2, 3)	30	2
		(2, 8) (2, 3) (2, 6) (2, 3)	32	4
		(2, 7) (2, 3) (2, 7) (2, 3)	32	2
		(2, 9) (2, 3) (2, 7) (2, 3)	34	4
		(2, 8) (2, 3) (2, 8) (2, 3)	34	2
		(2, 10) (2, 3) (2, 8) (2, 3)	36	4
		(2, 9) (2, 3) (2, 9) (2, 3)	36	2
(5,3,3,3)	11	(2, 3, 4, 5) (2, 3) (2, 6) (2, 3)	36	4
		(2, 3, 4, 6) (2, 3) (2, 7) (2, 3)	38	4
		(2, 3, 4, 5) (2, 3) (2, 8) (2, 3)	38	4
		(2, 3, 4, 7) (2, 3) (2, 8) (2, 3)	40	4
		(2, 3, 4, 6) (2, 3) (2, 9) (2, 3)	40	4
		(2, 3, 4, 5) (2, 3) (2, 10) (2, 3)	40	4
		(2, 3, 4, 8) (2, 3) (2, 9) (2, 3)	42	4
		(2, 3, 4, 7) (2, 3) (2, 10) (2, 3)	42	4
		(2, 3, 5, 6) (2, 3) (2, 10) (2, 3)	42	4
		(2, 3, 4, 6) (2, 3) (2, 11) (2, 3)	42	4
		(2, 3, 4, 5) (2, 3) (2, 12) (2, 3)	42	4
(6,4,4,4) ($n \leqslant 60$)	*11*	(2, 3, 4, 5, 6)(2, 3, 4)(2, 3, 7) (2, 3, 4)	54	4
		(2, 3, 4, 5, 7)(2, 3, 4)(2, 3, 8) (2, 3, 4)	56	4
		(2, 3, 4, 5, 6)(2, 3, 4)(2, 3, 9) (2, 3, 4)	56	4
		(2, 3, 4, 5, 8)(2, 3, 4)(2, 3, 9) (2, 3, 4)	58	4
		(2, 3, 4, 5, 7)(2, 3, 4)(2, 3, 10)(2, 3, 4)	58	4
		(2, 3, 4, 5, 6)(2, 3, 4)(2, 3, 11)(2, 3, 4)	58	4
		(2, 3, 4, 5, 9)(2, 3, 4)(2, 3, 10)(2, 3, 4)	60	4
		(2, 3, 4, 5, 8)(2, 3, 4)(2, 3, 11)(2, 3, 4)	60	4
		(2, 3, 4, 6, 7)(2, 3, 4)(2, 3, 11)(2, 3, 4)	60	4
		(2, 3, 4, 5, 7)(2, 3, 4)(2, 3, 12)(2, 3, 4)	60	4
		(2, 3, 4, 5, 6)(2, 3, 4)(2, 3, 13)(2, 3, 4)	60	4
(4,3,4,3)	29	(2, 3, 4) (2, 3) (2, 3 4) (2, 3)	32	2
		(2, 3, 6) (2, 3) (2, 3, 4) (2, 3)	34	4
		(2, 4, 5) (2, 3) (2, 3, 4) (2, 3)	34	4
		(2, 3, 5) (2, 3) (2, 3, 5) (2, 3)	34	2
		(2, 3, 4) (2, 4) (2, 3, 4) (2, 4)	34	2
		(2, 3, 7) (2, 3) (2, 3, 5) (2, 3)	36	4
		(2, 4, 6) (2, 3) (2, 3, 5) (2, 3)	36	4
		(2, 3, 6) (2, 3) (2, 3, 6) (2, 3)	36	2
		(2, 4, 5) (2, 3) (2, 3, 6) (2, 3)	36	4
		(2, 4, 5) (2, 3) (2, 4, 5) (2, 3)	36	2
		(2, 3, 8) (2, 3) (2, 3, 4) (2, 3)	36	4
		(2, 3, 4) (2, 5) (2, 3, 4) (2, 5)	36	2
		(2, 4, 7) (2, 3) (2, 4, 5) (2, 3)	38	4
		(2, 5, 6) (2, 3) (2, 4, 5) (2, 3)	38	4
		(2, 4, 6) (2, 3) (2, 4, 6) (2, 3)	38	2
		(2, 3, 4) (2, 6) (2, 3, 4) (2, 6)	38	2
		(2, 4, 8) (2, 3) (2, 4, 6) (2, 3)	40	4
		(2, 5, 7) (2, 3) (2, 4, 6) (2, 3)	40	4
		(2, 4, 7) (2, 3) (2, 4, 7) (2, 3)	40	2
		(2, 5, 6) (2, 3) (2, 4, 7) (2, 3)	40	4
		(2, 5, 6) (2, 3) (2, 5, 6) (2, 3)	40	2
		(2, 5, 8) (2, 3) (2, 5, 6) (2, 3)	42	4
		(2, 6, 7) (2, 3) (2, 5, 6) (2, 3)	42	4
		(2, 5, 7) (2, 3) (2, 5, 7) (2, 3)	42	2
		(2, 5, 9) (2, 3) (2, 5, 7) (2, 3)	44	4
		(2, 6, 8) (2, 3) (2, 5, 7) (2, 3)	44	4
		(2, 5, 8) (2, 3) (2, 5, 8) (2, 3)	44	2
		(2, 6, 7) (2, 3) (2, 5, 8) (2, 3)	44	4
		(2, 6, 7) (2, 3) (2, 6, 7) (2, 3)	44	2

No configurations of types (5,5,4,4), (6,3,6,3), (5,4,3,4), (7,4,3,4) exist.

TABLE 2. Representations of inseparable sets (contd.)

Type	#	L_1 L_2 L_3 L_4	n	$\frac{8}{g}$	Type	#	L_1 L_2 L_3 L_4	n	$\frac{8}{g}$
(5,3,5,3)	8	(2, 3, 4, 5) (2, 3) (2, 3, 4, 5) (2, 3)	42	2	(4,3,3,2)	1	(2, 3, 4) (2, 3) (2, 5) (2)	27	8
		(2, 3, 4, 7) (2, 3) (2, 3, 4, 5) (2, 3)	44	4					
		(2, 3, 5, 6) (2, 3) (2, 3, 4, 5) (2, 3)	44	4	(5,4,4,3)	3	(2, 3, 4, 5) (2, 3, 4) (2, 3, 6) (2, 3)	43	8
		(2, 3, 4, 6) (2, 3) (2, 3, 4, 6) (2, 3)	44	2			(2, 3, 4, 6) (2, 3, 4) (2, 3, 7) (2, 3)	45	8
		(2, 4, 5, 6) (2, 3) (2, 4, 5, 6) (2, 3)	48	2			(2, 3, 4, 5) (2, 3, 4) (2, 3, 8) (2, 3)	45	8
		(2, 4, 5, 8) (2, 3) (2, 4, 5, 6) (2, 3)	50	4					
		(2, 4, 6, 7) (2, 3) (2, 4, 5, 6) (2, 3)	50	4					
		(2, 4, 5, 7) (2, 3) (2, 4, 5, 7) (2, 3)	50	2					
(4,4,4,4)	42	(2, 3, 4) (2, 3, 4) (2, 3, 4) (2, 3, 4)	40	1			(2, 3, 8) (2, 3, 4) (2, 3, 6) (2, 3, 4)	46	4
		(2, 3, 6) (2, 3, 4) (2, 3, 4) (2, 3, 4)	42	4			(2, 3, 7) (2, 3, 4) (2, 3, 7) (2, 3, 4)	46	2
		(2, 3, 5) (2, 3, 4) (2, 3, 5) (2, 3, 4)	42	2			(2, 3, 10) (2, 3, 4) (2, 3, 6) (2, 3, 4)	48	4
		(2, 3, 5) (2, 3, 5) (2, 3, 4) (2, 3, 4)	42	4			(2, 3, 6) (2, 3, 6) (2, 3, 6) (2, 3, 6)	48	1
		(2, 3, 8) (2, 3, 4) (2, 3, 4) (2, 3, 4)	44	4			(2, 3, 9) (2, 3, 4) (2, 3, 7) (2, 3, 4)	48	4
		(2, 3, 7) (2, 3, 4) (2, 3, 5) (2, 3, 4)	44	4			(2, 3, 8) (2, 3, 4) (2, 3, 8) (2, 3, 4)	48	2
		(2, 3, 6) (2, 3, 4) (2, 3, 6) (2, 3, 4)	44	2			(2, 3, 11) (2, 3, 4) (2, 3, 7) (2, 3, 4)	50	4
		(2, 4, 5) (2, 3, 4) (2, 3, 6) (2, 3, 4)	44	4			(2, 3, 9) (2, 3, 4) (2, 3, 9) (2, 3, 4)	50	2
		(2, 3, 7) (2, 3, 5) (2, 3, 4) (2, 3, 4)	44	8			(2, 3, 10) (2, 3, 4) (2, 3, 8) (2, 3, 4)	50	4
		(2, 3, 6) (2, 3, 5) (2, 3, 5) (2, 3, 4)	44	8			(2, 3, 12) (2, 3, 4) (2, 3, 8) (2, 3, 4)	52	4
		(2, 3, 6) (2, 3, 6) (2, 3, 4) (2, 3, 4)	44	4			(2, 3, 11) (2, 3, 4) (2, 3, 9) (2, 3, 4)	52	4
		(2, 3, 6) (2, 3, 5) (2, 3, 4) (2, 3, 5)	44	4			(2, 3, 10) (2, 3, 4) (2, 3, 10) (2, 3, 4)	52	2
		(2, 3, 5) (2, 3, 5) (2, 3, 5) (2, 3, 5)	44	1			(2, 3, 13) (2, 3, 4) (2, 3, 9) (2, 3, 4)	54	4
		(2, 4, 5) (2, 3, 4) (2, 4, 5) (2, 3, 4)	44	2			(2, 3, 12) (2, 3, 4) (2, 3, 10) (2, 3, 4)	54	4
		(2, 3, 8) (2, 3, 4) (2, 4, 5) (2, 3, 4)	46	4			(2, 3, 11) (2, 3, 4) (2, 3, 11) (2, 3, 4)	54	2
		(2, 3, 7) (2, 3, 4) (2, 4, 6) (2, 3, 4)	46	4			(2, 3, 14) (2, 3, 4) (2, 3, 10) (2, 3, 4)	56	4
		(2, 4, 6) (2, 3, 4) (2, 4, 6) (2, 3, 4)	46	2			(2, 3, 13) (2, 3, 4) (2, 3, 11) (2, 3, 4)	56	4
		(2, 3, 7) (2, 3, 5) (2, 3, 5) (2, 3, 5)	46	4			(2, 3, 12) (2, 3, 4) (2, 3, 12) (2, 3, 4)	56	2
		(2, 3, 6) (2, 3, 5) (2, 3, 6) (2, 3, 5)	46	2			(2, 3, 15) (2, 3, 4) (2, 3, 11) (2, 3, 4)	58	4
		(2, 3, 6) (2, 3, 6) (2, 3, 5) (2, 3, 5)	46	4			(2, 3, 14) (2, 3, 4) (2, 3, 12) (2, 3, 4)	58	4
		(2, 3, 9) (2, 3, 4) (2, 3, 5) (2, 3, 4)	46	4			(2, 3, 13) (2, 3, 4) (2, 3, 13) (2, 3, 4)	58	2
(5,4,5,4)	*47*	(2, 3, 4, 5) (2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	50	2			(2, 3, 6, 7) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	56	4
($n \leqslant 60$)		(2, 3, 4, 7) (2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	52	4			(2, 3, 4, 8) (2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	56	4
		(2, 3, 4, 6) (2, 3, 5)(2, 3, 4, 5)(2, 3, 4)	52	8			(2, 3, 5, 7) (2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	56	2
		(2, 3, 4, 5) (2, 3, 6)(2, 3, 4, 5)(2, 3, 4)	52	4			(2, 3, 6, 9) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	58	4
		(2, 3, 5, 6) (2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	52	4			(2, 3, 7, 8) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	58	4
		(2, 3, 4, 6) (2, 3, 4)(2, 3, 4, 6)(2, 3, 4)	52	2			(2, 3, 5, 9) (2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	58	4
		(2, 3, 4, 9) (2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	54	4			(2, 3, 6, 8) (2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	58	4
		(2, 3, 4, 7) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	54	4			(2, 3, 5, 8) (2, 3, 4)(2, 3, 5, 8)(2, 3, 4)	58	2
		(2, 3, 5, 8) (2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	54	4			(2, 3, 5, 10)(2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	58	4
		(2, 3, 6, 7) (2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	54	4			(2, 3, 5, 8) (2, 3, 4)(2, 3, 6, 7)(2, 3, 4)	58	4
		(2, 3, 4, 8) (2, 3, 4)(2, 3, 4, 6)(2, 3, 4)	54	4			(2, 3, 6, 7) (2, 3, 4)(2, 3, 6, 7)(2, 3, 4)	58	2
		(2, 3, 5, 7) (2, 3, 4)(2, 3, 4, 6)(2, 3, 4)	54	4			(2, 3, 5, 11)(2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	60	4
		(2, 3, 4, 7) (2, 3, 4)(2, 3, 4, 7)(2, 3, 4)	54	2			(2, 3, 6, 10)(2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	60	4
		(2, 3, 5, 6) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	54	2			(2, 3, 7, 9) (2, 3, 4)(2, 3, 5, 7)(2, 3, 4)	60	4
		(2, 3, 4, 10)(2, 3, 4)(2, 3, 4, 6)(2, 3, 4)	56	4			(2, 3, 5, 10)(2, 3, 4)(2, 3, 5, 8)(2, 3, 4)	60	4
		(2, 3, 4, 11)(2, 3, 4)(2, 3, 4, 5)(2, 3, 4)	56	4			(2, 3, 6, 9) (2, 3, 4)(2, 3, 5, 8)(2, 3, 4)	60	4
		(2, 3, 5, 9) (2, 3, 4)(2, 3, 4, 6)(2, 3, 4)	56	4			(2, 3, 7, 8) (2, 3, 4)(2, 3, 5, 8)(2, 3, 4)	60	4
		(2, 3, 6, 8) (2, 3, 4)(2, 3, 4, 6)(2, 3, 4)	56	4			(2, 3, 5, 9) (2, 3, 4)(2, 3, 5, 9)(2, 3, 4)	60	2
		(2, 3, 4, 9) (2, 3, 4)(2, 3, 4, 7)(2, 3, 4)	56	4			(2, 3, 5, 10)(2, 3, 4)(2, 3, 6, 7)(2, 3, 4)	60	4
		(2, 3, 5, 8) (2, 3, 4)(2, 3, 4, 7)(2, 3, 4)	56	4			(2, 3, 6, 9) (2, 3, 4)(2, 3, 6, 7)(2, 3, 4)	60	4
		(2, 3, 6, 7) (2, 3, 4)(2, 3, 4, 7)(2, 3, 4)	56	4			(2, 3, 7, 8) (2, 3, 4)(2, 3, 6, 7)(2, 3, 4)	60	4
		(2, 3, 4, 8) (2, 3, 4)(2, 3, 4, 8)(2, 3, 4)	56	2			(2, 3, 5, 9) (2, 3, 4)(2, 3, 6, 8)(2, 3, 4)	60	4
		(2, 3, 4, 9) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	56	4			(2, 3, 6, 8) (2, 3, 4)(2, 3, 6, 8)(2, 3, 4)	60	2
		(2, 3, 5, 8) (2, 3, 4)(2, 3, 5, 6)(2, 3, 4)	56	4					