

The H^∞ -control problem for parabolic systems with singular Hardy potentials

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Abstract. We solve the H^∞ -control problem with state feedback for infinite dimensional boundary control systems of parabolic type with distributed disturbances and apply the results to equations with Hardy potentials with the singularity inside or on the boundary, in the cases of a distributed control and of a boundary control.

Keywords: H^∞ -control, feedback control, robust control, abstract parabolic problems, Hardy potentials

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1 Introduction

The H^∞ -control is a technique used in control theory to design robust stabilizing feedback controllers that force a system to achieve stability with a prescribed performance even if the system output may be corrupted by perturbations. This method involves a transfer function which incorporates the effects of the input perturbations towards the output observation. The aim is to determine the optimal feedback controller which minimizes the effect of these perturbations on the output, by ensuring that the L^2 -norm of the transfer function is smaller than the L^2 -norm of the perturbation with a certain prescribed bound. This turns out in finding a suboptimal control solution constructed by means of a mathematical optimization problem. The formal H^∞ -control theory was initiated by Zames in [34], as an optimization problem with an operator norm, in particular, the H^∞ -norm. State space formulations were initially developed in [18] and [23] and continued later by the formulation of the necessary and sufficient conditions for the existence of an admissible controller in terms of solutions of algebraic Riccati equations. The state-space approach for linear infinite-dimensional H^∞ -control problems was developed in further works and we cite here e.g., [2], [3], [4], [9], [21], [22], [25], [30], [31], [32], [33], [5], the last for Navier-Stokes equations.

In this paper we discuss the H^∞ -control problem for linear infinite dimensional systems of parabolic type and give applications for equations with singular Hardy potentials, of the type $\frac{\lambda}{|x|^2}$, which as far as we know is a novel approach. Following the papers [2], [3], [4], where the H^∞ -control abstract problem was solved with assumptions proper for the hyperbolic case, we prove here a main result stating the formulation of the H^∞ -control problem in the parabolic case, relying on appropriate assumptions for parabolic operators. This is further applied to three parabolic control systems with Hardy potentials and with distributed or boundary controls. There is an extensive literature on Hardy-type inequalities with the singularity located inside the domain or on the boundary, focusing also on controllability studies (see e.g., [15], [16], [29]). Besides the high mathematical interest in such singular equations revealed in the past decades, a parabolic operator with a Hardy potential term describes a non-standard growth condition which may affect the behavior of the solutions to diffusive physical models, as for example of heat transfer or diffusion of contaminants in fluids. Also, it may represent an equivalent formulation of a system of two equations in which a state in one equation is represented as a fundamental solution by the other one. Operators with other similar potentials can arise for example in quantum mechanics, [1] or in combustion theory, [10], [16]. Linear parabolic equations with Hardy potentials have been studied in connection with stationary Schrödinger equations $-\Delta y + V(x)y + E(x)y = f$

with the singular potential $V \in L^\infty(\Omega \setminus x_0)$ arising from the uncertainty principle. The robust stabilization of the corresponding dynamic control system $y_t - \Delta y + V(x)y + E(x)y = B_1 w + B_2 u$, via the H^∞ -control method, with the control u and the exogenous perturbation w has direct implication for the equilibrium solution to the above Schrödinger equation. The content of the paper is briefly described below.

In Section 2 we present the mathematical formulation of the H^∞ -control problem. In Section 3, after specifying the work hypotheses we provide the main result stating the existence of the feedback controller determined via a Riccati equation. In Sections 4 and 5 there are given applications for parabolic equations in the N -dimensional case with a distributed control and a boundary control, respectively, and with Hardy potentials with interior singularity, while in Section 6 it is treated the 1- D case with a boundary singular Hardy potential.

2 Problem presentation and preliminaries

In this section we briefly explain the state-space approach of the H^∞ -control problem for the linear system

$$y'(t) = Ay(t) + B_1 w(t) + B_2 u(t), \quad t \in \mathbb{R}_+ := (0, +\infty) \quad (2.1)$$

$$z(t) = C_1 y(t) + D_1 u(t), \quad t \in \mathbb{R}_+, \quad (2.2)$$

$$y(0) = y_0, \quad (2.3)$$

where A, B_1, B_2, C_1, D_1 are linear operators satisfying hypotheses that will be immediately specified. Here, y is the system state, u is the control input, w is an exogenous input, or an unknown perturbation and z is the performance output.

At this point we put down a few notation, definitions and results necessary for explaining the problem. Let X be a real Hilbert space with the scalar product and norm denoted by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$, respectively and X' is its dual. The symbol $\langle \cdot, \cdot \rangle_{X', X}$ is the pairing between X' and X . Let A be a linear closed operator on X with the domain $D(A) := \{y \in X; Ay \in X\}$ dense in X . By A^* we denote the the adjoint of A . If Y is another Hilbert space, $L(X, Y)$ represent the space of all linear continuous operators from X to Y .

Let H, U, W, Z be real Hilbert spaces identified with their duals. For the beginning we assume:

- (i₁) A is the infinitesimal generator of an analytic C_0 -semigroup e^{At} on the Hilbert space H , e^{At} is compact for $t > 0$, and

$$B_1 \in L(W, H), \quad B_2 \in L(U, (D(A^*))'), \quad C_1 \in L(H, Z), \quad D_1 \in L(U, Z). \quad (2.4)$$

Here, $(D(A^*))'$ is the dual of the domain of A^* , where $D(A^*)$ is organized as a Hilbert space with the scalar product $(y_1, y_2)_{D(A^*)} = (A^* y_1, A^* y_2)_H + (y_1, y_2)_H$ for $y_1, y_2 \in D(A^*)$.

We note that the space $(D(A^*))'$ is the completion of H in the norm $\|y\| = \|(A - \lambda_0 I)^{-1} y\|_H$, $\lambda_0 \in \rho(A)$. Also, we define the extension of the operator A from H to $(D(A^*))'$, denoted for convenience still by A , by

$$\langle Ay, \psi \rangle_{(D(A^*))', D(A)} = (y, A^* \psi)_H, \quad \text{for } y \in H, \quad \psi \in D(A^*). \quad (2.5)$$

We shall work with both operators and if not seen clearly from the context which operator is used, we shall specify this.

Let us consider the uncontrolled system $y'(t) = Ay(t)$, $t \in \mathbb{R}_+$, $y(0) = y_0$, with A the infinitesimal generator of a C_0 -semigroup on H .

Definition 2.1 *The operator A generates an exponentially stable semigroup e^{At} if*

$$\|e^{At}\|_{L(H, H)} \leq C e^{-\alpha t}, \quad \text{for all } t \geq 0, \quad (2.6)$$

where α and C are positive constants.

Relation (2.6) still reads

$$\|e^{At} y\|_H \leq C e^{-\alpha t} \|y\|_H, \quad \text{for all } y \in H \text{ and all } t \geq 0. \quad (2.7)$$

Moreover, a result of Datko (see [14]) asserts that relation (2.7) is equivalent to

$$\int_0^\infty \|y(t)\|_H^2 dt < \infty. \quad (2.8)$$

Definition 2.2 *The pair (A, C_1) in system (2.1)-(2.2) is exponentially detectable if there exists $K \in L(Z, H)$ such that $A + KC_1$ generates an exponentially stable semigroup.*

In order to state our H^∞ -control problem, we recall some issues about such a problem. Assume that under certain conditions system (2.1)-(2.3) has a mild solution $y \in C([0, T]; H)$ for all $T > 0$ and u can be represented as a feedback controller $u = Fy$, where generally $F : U \rightarrow H$ is a linear closed and densely defined operator. Then, the solution $(y(t), z(t))$ becomes dependent only on $w(t)$ and reads

$$y(t) = e^{(A+B_2F)t}y_0 + \int_0^t e^{(A+B_2F)(t-s)}B_1w(s)ds, \quad t \in [0, \infty), \quad (2.9)$$

$$z(t) = (C_1 + D_1F)e^{(A+B_2F)t}y_0 + (C_1 + D_1F) \int_0^t e^{(A+B_2F)(t-s)}B_1w(s)ds. \quad (2.10)$$

The latter equation can be still written

$$z(t) = f_0(t) + (G_F w)(t), \quad t \geq 0 \quad (2.11)$$

where $f_0(t) = (C_1 + D_1F)e^{(A+B_2F)t}y_0 \in Z$, $t \geq 0$, and $G_F : L^2(\mathbb{R}_+, W) \rightarrow L^2(\mathbb{R}_+, Z)$, defined by

$$(G_F w)(t) = (C_1 + D_1F) \int_0^t e^{(A+B_2F)(t-s)}B_1w(s)ds \in Z, \quad t \geq 0, \quad (2.12)$$

shows the transfer of the influence of the perturbation input w to the output. Roughly speaking, the H^∞ -control problem means to find a feedback controller which stabilizes exponentially the system (with $y_0 = 0$), with a certain specified performance for the output $G_F w$, depending on a given constant γ . Such a feedback control F is called a suboptimal solution and the H^∞ problem can be formulated as follows: given $\gamma > 0$, find the feedback control F which exponentially stabilizes system (2.1)-(2.2) such that $\|G_F\|_{L(L^2(\mathbb{R}_+, W), L^2(\mathbb{R}_+, Z))} < \gamma$.

To be more precise in what concerns the relation with the Hardy space H^∞ , we briefly recall a well-known result property of vector-valued Hardy classes (see e.g., [26], [27], [13], Theorem A6.26). The space H^∞ is defined as the vector space of bounded holomorphic functions on the right half plane, $\mathbb{C}_+ = \{z \in \mathbb{C}; \operatorname{Re} z > 0\}$, with the norm $\|f\|_{H^\infty} = \sup_{|z| < 1} |f(z)|$. Let us take the Laplace transform in system (2.1)-(2.2) and get

$$\widehat{z}(\zeta) = C_1(\zeta I - A - B_2F)^{-1}y_0 + \widehat{G_F}(\zeta)\widehat{w}(\zeta). \quad (2.13)$$

The function $\widehat{G_F} : \mathbb{C}_+ \rightarrow L(W, Z)$,

$$\widehat{G_F}(\zeta) = (C_1 + D_1F)(\zeta I + A + B_2F)^{-1}B_1 \quad (2.14)$$

is the transfer function in the frequency domain, giving a relationship between the input and output of the system. It plays an important role in control theory by providing an insight in how disturbances in the system can affect the output. The results in the papers cited before express the fact that the L^2 -operator norm of the gain in the time domain is equal to the Hardy $H^\infty(L(W, Z))$ -norm of the transfer operator in the frequency domain, i.e.,

$$\|G_F\|_{L(L^2(\mathbb{R}_+, W), L^2(\mathbb{R}_+, Z))} := \sup_{w \in L^2(\mathbb{R}_+, W)} \frac{\|G_F w\|_{L^2(\mathbb{R}_+, Z)}}{\|w\|_{L^2(\mathbb{R}_+, W)}} = \sup_{\zeta \in \mathbb{C}_+} \left\| \widehat{G_F}(\zeta) \right\|_{L(W, Z)} =: \left\| \widehat{G_F} \right\|_{H^\infty} < \gamma. \quad (2.15)$$

Notation and some necessary results. We end this section by recalling some other notation and results necessary in the paper. We denote by $H^m(\Omega)$ the Sobolev spaces $W^{2,m}(\Omega)$, for $m \geq 1$ and by $H_0^1(\Omega)$ the space $\{y \in H^1(\Omega); \operatorname{tr}(y) = 0 \text{ on } \Gamma\}$, where $\operatorname{tr}(y)$ is the trace operator of y on $\Gamma := \partial\Omega$. Moreover, $H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega)$. Given a Banach space X and $T \in (0, \infty]$ we define by $L^p(0, T; X)$ the space of L^p X -valued functions on $(0, T)$, $p \in [1, \infty]$, by $C([0, T]; X)$ the space of continuous X -valued functions on $(0, T)$ and $W^{1,p}(0, T; X) = \{u \in L^p(0, T; X); du/dt \in L^p(0, T; X)\}$.

Let $L : D(L) \subset H \rightarrow H$ be a linear operator defined on the Hilbert space H . We say that L is m -accretive if L is accretive, meaning that $(Ly, y)_H \geq 0$, $\forall y \in D(L)$, and if $R(I + L) = H$, where R is the range. The operator L is quasi m -accretive or ω - m -accretive if $\omega I + L$ is m -accretive for some $\omega > 0$.

Hardy inequalities. Let $N > 3$ and let Ω be an open bounded subset of \mathbb{R}^N , with $0 \in \Omega$. Then we have

$$\int_{\Omega} |\nabla y(x)|^2 dx \geq H_N \int_{\Omega} \frac{|y(x)|^2}{|x|^2} dx, \text{ for all } y \in H_0^1(\Omega), \quad (2.16)$$

where $H_N = \frac{(N-2)^2}{4}$ is optimal (see [11], p. 452, Theorem 4.1).

Let $\Omega = (0, 1)$. Then we have

$$\int_0^1 |y'(x)|^2 dx \geq \frac{1}{4} \int_0^1 \frac{y(x)^2}{|x|^2} dx, \quad \forall y \in H^1(0, 1), \quad y(0) = 0, \quad (2.17)$$

see [12], p. 217, or Lemma A.1, p. 234.

We recall the Young's inequality for convolutions $(f * g)(t) = \int_0^\infty f(t - \tau)g(\tau)d\tau$,

$$\|f * g\|_{L^r(0, \infty)} \leq \|f\|_{L^p(0, \infty)} \|g\|_{L^q(0, \infty)}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \quad 1 \leq p, q, r \leq \infty. \quad (2.18)$$

For simplicity, where there is no risk of confusion, the $L^p(\Omega)$ -norm will be denoted by $\|\cdot\|_p$, $p \in [1, \infty]$, instead of $\|\cdot\|_{L^p(\Omega)}$. We set $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = (0, \infty)$. Also, $|\cdot|$ will represent the Euclidian norm in \mathbb{R}^N , for any $N = 1, 2, \dots$, accordingly. In the further calculations C, C_1, \dots, C_N, C_T denote positive constants (which may change from line to line), C_N depending on N , via $\lambda < H_N$ and C_T depending on T .

3 The main result

Besides (i_1) we assume the following hypotheses:

(i_2) the next relation takes place:

$$\left\| B_2^* e^{A^* t} \right\|_{L(H, U)} \in L^1(0, T), \text{ for all } T > 0, \quad (3.1)$$

(i_3) the pair (A, C_1) is exponentially detectable (that is there exists $K \in L(Z, H)$ such that $A + KC_1$ generates an exponentially stable semigroup) and

$$\int_0^\infty \left\| B_2^* e^{(A^* + C_1^* K^*) t} y \right\|_U dt \leq C \|y\|_H, \text{ for all } y \in H, \quad (3.2)$$

(i_4) $\|D_1^* D_1 u\|_{U^*} = \|u\|_U$ and $D_1^* C_1 = 0$.

Let us comment a little these hypotheses. The L^1 -admissibility hypothesis of the observation operator B_2^* in (i_2) is made in order to ensure the existence of a mild solution to (2.1) in $L^2(0, T; H)$ for every $T > 0$, with initial condition y_0 and inputs $u \in L^2(0, T; U)$ and $w \in L^2(0, T; W)$. In an ideal situation when $B_2 \in L(U, H)$, eqs. (2.1)-(2.3) have a unique mild solution $y \in C([0, T]; H)$, for every $T > 0$, given by

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} B_1 w(s) ds + \int_0^t e^{A(t-s)} B_2 u(s) ds, \quad t \in [0, \infty). \quad (3.3)$$

But generally, B_2 may be not continuous from U to H , in some situations its range being in a larger abstract space, indicated before to be $(D(A^*))'$. The unique solution to (2.1)-(2.3) is in this case in $C([0, \infty); (D(A^*))')$. Consequently, the previous formula should be written in a weak sense, that is for all $t \geq 0$, we have

$$(y(t), \varphi)_H = (e^{At} y_0, \varphi)_H + \int_0^t \left(e^{A(t-s)} (B_1 w(s), \varphi)_H + (u(s), B_2^* e^{A^*(t-s)} \varphi)_U \right) ds, \quad \forall \varphi \in H, \quad y_0 \in H. \quad (3.4)$$

Assumption (i_2) ensures that $y \in L^2(0, T; H)$, and this follows by proving that $\int_0^T (y(t), \varphi(t))_H dt < C \|\varphi\|_{L^2(0, T; H)}$, for $\varphi \in L^2(0, T; H)$. Indeed, this is clearly seen for the first two terms in (3.4), since $B_1 w \in L^2(\mathbb{R}_+; H)$. For the last term we calculate

$$\begin{aligned}
& \int_0^T \int_0^t \left(u(s), B_2^* e^{A^*(t-s)} \varphi(t) \right)_U ds dt = \int_0^T \int_s^T \left(u(s), B_2^* e^{A^*(t-s)} \varphi(t) \right)_U dt ds \quad (3.5) \\
& \leq \left(\int_0^T \|u(s)\|_U^2 ds \right)^{1/2} \left(\int_0^T \left\| \int_0^T B_2^* e^{A^*(t-s)} \varphi(t) dt \right\|_U^2 ds \right)^{1/2} \\
& \leq \|u\|_{L^2(0, T; U)} \left\{ \left(\int_0^T \|B_2^* e^{A^* t}\|_{L(H, U)} dt \right) \left(\int_0^T \|\varphi(t)\|_H^2 dt \right)^{1/2} \right\} \\
& \leq \|u\|_{L^2(0, T; U)} \left(\int_0^T \|B_2^* e^{A^* t}\|_{L(H, U)} dt \right) \|\varphi\|_{L^2(0, T; H)} \leq C \|\varphi\|_{L^2(0, T; H)},
\end{aligned}$$

where we used (i_2) and the Young's inequality for convolution (2.18) with $p = 1$, $q = r = 2$. Then, it follows that $y \in L^2(0, T; H)$ and the last term in (3.3) is in H .

Regarding (3.2) we mention that the corresponding result related to L^2 instead of L^1 is a particular case of Theorem 5.4.2 in [28], so that we expect that (3.1) and the detectability hypothesis imply (3.2), at least in some cases. However, we keep here relation (3.2) as a hypothesis and check it in the applications, by different proofs according the case. In applications, the first relation in hypothesis (i_4) may be weakened to $D_1^* D_1 \geq \epsilon I$ (see e.g., [5]). However, for certain choices of operators D_1 and C_1 , relations (i_4) may be proved as they are.

Theorem 3.1 below is the main result concerning the H^∞ -control problem under hypotheses (i_1) - (i_4) and it gives a representation for the feedback operator F which is a suboptimal solution to our H^∞ -control problem.

This theorem was proved, under some appropriate hypotheses for the hyperbolic case in [2] and [3]. Actually, instead of (3.1) there it was used the L^2 -admissibility condition

$$\int_0^T \|B_2^* e^{A^* t} y\|_U^2 dt \leq C_T \|y\|_H^2, \text{ for every } y \in H \text{ and } T > 0. \quad (3.6)$$

For the treatment of specific parabolic problems intended to be achieved in the paper, we have in mind to adapt that approach to the case covered by assumptions $(i_1) - (i_4)$ to obtain the following main result.

Theorem 3.1 *Let hypotheses $(i_1) - (i_4)$ hold and let $\gamma > 0$. Assume that there exists $F \in L(H, U)$ such that $A + B_2 F$ generates an analytic exponentially stable C_0 -semigroup on H and*

$$\|G_F\|_{L(L^2(\mathbb{R}_+; W), L^2(\mathbb{R}_+; Z))} < \gamma. \quad (3.7)$$

Then, there exists a Hilbert space $\mathcal{X} \subset H$ with dense and continuous injection and an operator

$$P \in L(H, H) \cap L(\mathcal{X}, D(A^*)), \quad P = P^* \geq 0, \quad (3.8)$$

which satisfies the algebraic Riccati equation

$$A^* P y + P(A - B_2 B_2^* P + \gamma^{-2} B_1 B_1^* P) y + C_1^* C_1 y = 0, \quad \forall y \in \mathcal{X}, \quad (3.9)$$

where $B_2^ P \in L(\mathcal{X}, U)$ and the operators*

$$\Lambda_P := A - B_2 B_2^* P + \gamma^{-2} B_1 B_1^* P, \quad \Lambda_P^1 := A - B_2 B_2^* P \quad (3.10)$$

with the domain \mathcal{X} generate exponentially stable semigroups on H . Moreover, the feedback control

$$\tilde{F} = -B_2^* P \quad (3.11)$$

solves the H^∞ -problem, that is $\|G_{\tilde{F}}\|_{L(L^2(\mathbb{R}_+; W), L^2(\mathbb{R}_+; Z))} < \gamma$.

Conversely, assume that there exists a solution P to equation (3.9) with the properties (3.8) and such that the corresponding operators Λ_P and Λ_P^1 generate exponentially stable semigroups on H . Then, the feedback operator $\tilde{F} = -B_2^ P$ solves the H^∞ -problem (3.7).*

The space \mathcal{X} will be defined in the theorem proof before Lemma 3.5, in (3.45). Moreover, we shall show in Lemma 3.5 that if the operator Λ_P with the domain $D(\Lambda_P) = \{y \in H; \Lambda_P y = (A - B_2 B_2^* P + \gamma^{-2} B_1 B_1^* P)y \in H\}$ is closed, then $\mathcal{X} = D(\Lambda_P)$. This will happen in all examples given the next sections.

Proof of Theorem 3.1. We assume first that there exists a solution $F \in L(H, U)$ to the H^∞ -control problem such that $A_F := A + B_2 F$ generates an analytic exponentially stable C_0 -semigroup and (3.7) holds. We must prove that there exists P satisfying (3.8)-(3.11).

The state-space approach of the above H^∞ -control problem comes back to solve the differential game

$$\sup_{w \in L^2(\mathbb{R}_+, W)} \inf_{u \in L^2(\mathbb{R}_+, U)} \frac{1}{2} \int_0^\infty (\|z(t)\|_Z^2 - \gamma^2 \|w(t)\|_W^2) dt, \quad (3.12)$$

subject to (2.1)-(2.3), which ensures a prescribed bound on the Hardy norm H^∞ of the transfer operator (see e.g., [3]).

Let $J : L^2(\mathbb{R}_+; U) \times L^2(\mathbb{R}_+; W) \rightarrow [-\infty, \infty]$ be defined as

$$J(u, w) = \frac{1}{2} \int_0^\infty \{\|C_1 y(t) + D_1 u(t)\|_Z^2 - \gamma^2 \|w(t)\|_W^2\} dt \quad (3.13)$$

and consider first a minimization problem, for a fixed $w \in L^2(\mathbb{R}_+; W)$,

$$\inf_{u \in L^2(\mathbb{R}_+; U)} J(u, w), \quad (3.14)$$

subject to system (2.1)-(2.2). By hypothesis (i_4) we see that

$$J(u, w) = \frac{1}{2} \int_0^\infty \{\|C_1 y(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2 \|w(t)\|_W^2\} dt, \quad (3.15)$$

so $u \rightarrow J(u, w)$ is strictly convex, whence it easily can be shown that (3.14) has a unique solution

$$u^* = \Gamma w \quad (3.16)$$

with $\Gamma : L^2(\mathbb{R}_+; W) \rightarrow L^2(\mathbb{R}_+; U)$.

We denote by y^{u^*} the solution to (2.1) corresponding to u^* (realizing the minimum in (3.14)) and w , that is, $y^{u^*} := y^{u^*, w}$.

Lemma 3.2 *There exists $p \in C(\mathbb{R}_+; H) \cap L^2(\mathbb{R}_+; H)$ satisfying*

$$p'(t) = -A_F^* p(t) + C_1^* C_1 y^{u^*}(t) + F^* u^*(t), \quad t \in \mathbb{R}_+, \quad (3.17)$$

$$u^*(t) = B_2^* p(t), \quad a.e. \quad t > 0. \quad (3.18)$$

Proof. We note first that the solution to the equation $y'(t) = A_F y(t) + B_1 w(t)$ with $y_0 \in H$, where $A_F = A + B_2 F$ is exponentially stable on H , is in $L^2(\mathbb{R}_+; H)$. Indeed,

$$\begin{aligned} \|y(t)\|_H &\leq C e^{-\alpha t} \|y_0\|_H + \int_0^t \|e^{A_F(t-s)} w(s)\|_H ds \\ &\leq C e^{-\alpha t} \|y_0\|_H + \int_0^t e^{-\alpha(t-s)} \|w(s)\|_W ds, \quad t \geq 0, \end{aligned}$$

and by applying the Young's inequality for convolution (2.18) with $r = 2$, $p = 1$ and $q = 2$ we obtain

$$\begin{aligned} \int_0^\infty \|y(t)\|_H^2 dt &\leq C \left\{ \|y_0\|_H^2 + \int_0^\infty \left(\int_0^t e^{-\alpha(t-s)} \|w(s)\|_W ds \right)^2 dt \right\} \\ &\leq C \|y_0\|_H^2 + C \left(\int_0^\infty e^{-\alpha t} dt \right)^2 \int_0^\infty \|w(t)\|_W^2 dt \leq C (\|y_0\|_H^2 + \|w\|_{L^2(0, \infty; W)}^2) < \infty. \end{aligned} \quad (3.19)$$

We specify that the solution to (3.17) should be understood in the following mild sense

$$p(t) = - \int_t^\infty e^{-A_F^*(s-t)} (C_1^* C_1 y^{u^*}(s) + F^* u^*(s)) ds,$$

and so $C([0, \infty); H)$ because $F^* \in L(U, H)$, $C_1^* C_1 \in L(H, H)$. Since A_F^* generates an analytic C_0 -semigroup it follows by its regularizing effect that $p \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A_F^*))$, for all $T > 0$.

We introduce $v := u - Fy$ and write problem (3.14) as

$$\inf_{v \in L^2(\mathbb{R}_+; U)} \frac{1}{2} \int_0^\infty \{ \|C_1 y(t)\|_Z^2 + \|Fy(t) + v(t)\|_U^2 - \gamma^2 \|w(t)\|_W^2 \} dt \quad (3.20)$$

subject to $y'(t) = A_F y(t) + B_1 w(t) + B_2 v(t)$, $t \geq 0$, $y(0) = y_0$. Since the functional is weakly lower semicontinuous and convex, it follows that (3.20) has a unique solution $v^* = u^* - Fy^{u^*}$, with u^* the solution to (3.14) and y^{u^*} the solution to (2.1) corresponding to u^* and w .

We set the variation $v^\lambda = v^* + \lambda V$, where $\lambda > 0$, $V \in L^2(\mathbb{R}_+; U)$ and write the system in variations

$$Y'(t) = A_F Y(t) + B_2 V(t), \quad Y(0) = 0, \quad (3.21)$$

where $Y(t) = \lim_{\lambda \rightarrow 0} \frac{y^{v^\lambda} - y^{v^*}}{\lambda}$ weakly in $L^2(\mathbb{R}_+; H)$. Eq. (3.21) can be still written as $Y'(t) = AY(t) + B_2(FY(t) + V(t))$ and so it is easily seen that it has a unique solution Y belonging to $W^{1,2}(\mathbb{R}_+; (D(A_F^*))' \cap L^2(\mathbb{R}_+; H))$, the latter following in the same way as shown before for $y(t)$ in (3.5).

Writing that v^* realizes the minimum in (3.20), in particular that $J(v^\lambda, w) \geq J(v^*, w)$, we deduce

$$\int_0^\infty \{ (C_1 y^{u^*}(t), C_1 Y(t))_Z + (Fy^{u^*}(t) + v^*(t), FY(t) + V(t))_U \} dt \geq 0.$$

If $\lambda \rightarrow -\lambda$ we obtain the reverse inequality, so that in conclusion

$$\int_0^\infty \{ (C_1^* C_1 y^{u^*}(t) + F^* Fy^{u^*}(t) + F^* v^*(t), Y(t))_H + (Fy^{u^*}(t) + v^*(t), V(t))_U \} dt = 0, \quad (3.22)$$

for all $V \in L^2(\mathbb{R}_+; U)$. By testing the first equation (3.21) by $p(t) \in D(A_F^*)$, solution to (3.17) and integrating with respect to t from 0 to ∞ , we obtain

$$\int_0^\infty (p'(t) + A_F^* p(t), Y(t))_H dt + \int_0^\infty (B_2^* p(t), V(t))_U dt = 0, \quad (3.23)$$

which by (3.17) yields

$$\int_0^\infty (C_1^* C_1 y^{u^*}(t) + F^* u^*(t), Y(t))_H dt = - \int_0^\infty (B_2^* p(t), V(t))_U dt. \quad (3.24)$$

By comparison with (3.22), where we write $v^* = u^* - Fy^{u^*}$, this yields

$$\int_0^\infty (-B_2^* p(t) + u^*(t), V(t))_U dt = 0, \quad \text{for all } V \in L^2(\mathbb{R}_+; U). \quad (3.25)$$

Therefore, we obtain (3.18) as claimed. ■

Then, the dual system (3.17) can be still written by the replacement of v^* as

$$p'(t) = -A^* p(t) + C_1^* C_1 y^{u^*}(t), \quad \text{a.e. } t \in \mathbb{R}_+. \quad (3.26)$$

Now, let us consider the function $\varphi : L^2(\mathbb{R}_+; W) \rightarrow \mathbb{R}_+$, $\varphi(w) = -J(\Gamma w, w)$, that is

$$\varphi(w) = \frac{1}{2} \int_0^\infty \left(\gamma^2 \|w(t)\|_W^2 - \|C_1 y^{u^*}(t) + D_1 u^*(t)\|_Z^2 \right) dt,$$

where y^{u^*} is the solution to (2.1) corresponding to (u^*, w) . By (2.11) and (2.12) we have

$$C_1 y^{u^*}(t) + D_1 u^*(t) = G_F w(t) - f_0(t)$$

and so

$$\begin{aligned} & \left\| (C_1 y^{u^*}(t) + D_1 u^*(t)) \right\|_Z^2 = \|G_F w(t)\|_Z^2 - 2(G_F w(t), f_0(t))_Z + \|f_0(t)\|_Z^2 \\ & \leq (1 + \delta) \|G_F w(t)\|_Z^2 + C_\delta \|f_0(t)\|_Z^2, \quad \forall t \geq 0. \end{aligned}$$

Now, we integrate from 0 to ∞ , note that $f_0 \in L^2(\mathbb{R}_+; H)$, and get,

$$\int_0^\infty \left\| (C_1 y^{u^*}(t) + D_1 u^*(t)) \right\|_Z^2 dt \leq (1 + \delta)(\gamma^2 - \varepsilon) \int_0^\infty \|w(t)\|_W^2 dt + C_\delta,$$

where ε is fixed and the last inequality is implied by (3.7). We can find δ and $\tilde{\delta}$ such that $(1 + \delta)(\gamma^2 - \varepsilon) \leq \gamma^2 - \tilde{\delta}$, which is verified with the choice $\tilde{\delta} < \varepsilon - \delta(\gamma^2 - \varepsilon)$ and $\delta < \frac{\varepsilon}{\gamma^2 - \varepsilon}$. Then

$$\varphi(w) \geq \tilde{\delta} \int_0^\infty \|w(t)\|_W^2 dt + C,$$

and it turns out that φ attains its minimum on $L^2(\mathbb{R}_+; W)$ in a unique point w^* .

Lemma 3.3 *We have*

$$w^*(t) = -\gamma^{-2} B_1^* p(t), \quad a.e. \ t > 0, \quad (3.27)$$

where $p \in W^{1,2}(0, T; H)$ is the solution to (3.26).

Proof. Recall that $u^* = \Gamma w$ and that y^{u^*} satisfies the problem

$$(y^{u^*})'(t) = A y^{u^*}(t) + B_1 w(t) + B_2 \Gamma w(t), \quad t \in \mathbb{R}_+, \quad y^*(0) = y_0$$

and proceed by giving variations to w , that is $w^\lambda = w^* + \lambda \tilde{w}$, $w \in L^2(\mathbb{R}_+; H)$. Then, the system in variations is

$$Y'(t) = A Y(t) + B_1 \tilde{w}(t) + B_2 \Gamma \tilde{w}(t), \quad t \in \mathbb{R}_+, \quad Y(0) = 0 \quad (3.28)$$

and the condition of optimality reads

$$\int_0^\infty (\gamma^2 w^*(t) - \Gamma^* \Gamma w^*(t), \tilde{w}(t))_W dt - \int_0^\infty (C_1^* C_1 y^{u^*}(t), Y(t))_H dt = 0, \quad (3.29)$$

for all $\tilde{w} \in L^2(\mathbb{R}_+; W)$. Let us recall the dual system (3.26) and test (3.28) by $p(t)$ and integrate for $t \in (0, \infty)$. We get

$$\int_0^\infty (p'(t) + A^* p(t), Y(t))_H dt + \int_0^\infty (B_1^* p(t) + \Gamma^* B_2^* p(t), \tilde{w}(t))_W dt = 0. \quad (3.30)$$

The latter and (3.29) gives

$$\int_0^\infty (\gamma^2 w^*(t) - \Gamma^* \Gamma w^*(t), \tilde{w}(t))_W dt + \int_0^\infty (B_1^* p(t) + \Gamma^* B_2^* p(t), \tilde{w}(t))_W dt = 0,$$

so that, since $-\Gamma^* \Gamma w^*(t) + \Gamma^* B_2^* p(t) = -\Gamma^* u^*(t) + \Gamma^* u^*(t) = 0$, we obtain

$$\int_0^\infty (\gamma^2 w^*(t) + B_1^* p(t), \tilde{w}(t))_W dt = 0,$$

for all $\tilde{w} \in L^2(\mathbb{R}_+; W)$, that implies (3.27), as claimed. \blacksquare

Thus, we have proved that (3.12) has a unique solution (u^*, w^*) with the corresponding state denoted y^* , characterized by the Euler-Lagrange system

$$y^{*'}(t) = A y^*(t) + B_1 w^*(t) + B_2 u^*(t), \quad t \in \mathbb{R}_+, \quad y^*(0) = y_0, \quad (3.31)$$

$$p'(t) = -A^*p(t) + C_1^*C_1y^*(t), \quad t \in \mathbb{R}_+, \quad (3.32)$$

$$u^*(t) = B_2^*p(t), \quad \text{a.e. } t > 0. \quad (3.33)$$

$$w^*(t) = -\gamma^{-2}B_1^*p(t), \quad \text{a.e. } t > 0, \quad (3.34)$$

where we already know that

$$\begin{aligned} y^* &\in C([0, \infty); (D(A^*))') \cap L^2(0, T; H), \quad \forall T > 0, \\ p &\in C([0, \infty); H) \cap L^2(\mathbb{R}_+; H). \end{aligned}$$

Lemma 3.4 *Let $y_0 \in H$. Then,*

$$y^* \in C([0, \infty); H) \cap W^{1,2}(\delta, T; H), \quad \forall \delta, 0 < \delta \leq T < \infty, \quad (3.35)$$

$$p \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A^*)), \quad \forall T > 0. \quad (3.36)$$

Proof. Since A^* generates an analytic C_0 -semigroup and $C_1^*C_1y^* \in L^2(\mathbb{R}_+; H)$ we see by (3.32) that (3.36) holds. Moreover, by (3.31) and (3.33) we have

$$\begin{aligned} y^*(t) &= e^{At}y_0 + \int_0^t e^{A(t-s)}B_1w^*(s)ds + \int_0^t e^{A(t-s)}B_2B_2^*p(s)ds \\ &= e^{At}y_0 + g_1(t) + g_2(t), \quad \forall t \geq 0. \end{aligned} \quad (3.37)$$

The first two terms are in $C([0, \infty); H) \cap W^{1,2}(\delta, T; H)$. By (3.36), $B_2B_2^*p \in W^{1,2}(0, T; (D(A^*))')$ and so we may represent it as $B_2B_2^*p = (A - \omega I)f$, with $f \in W^{1,2}(0, T; H)$, for ω sufficiently large. This yields

$$\begin{aligned} g_2(t) &= \int_0^t e^{A(t-s)}(A - \omega I)f(s)ds = -\omega \int_0^t e^{A(t-s)}f(s)ds - \int_0^t \left(\frac{d}{ds} e^{A(t-s)} \right) f(s)ds \\ &= -\omega \int_0^t e^{A(t-s)}f(s)ds - f(t) + e^{At}f(0) + \int_0^t e^{A(t-s)}f'(s)ds, \quad \forall t \geq 0. \end{aligned}$$

Since e^{At} is an analytic semigroup it follows that $g(t) = \int_0^t e^{A(t-s)}f'(s)ds$, the solution to $g'(t) = Ag(t) + f(t)$, $g(0) = 0 \in D(A)$, is in $W^{1,2}(0, T; H)$, as the first two terms. Though $f(0) \notin D(A)$, the third term is in $C([0, \infty); H) \cap W^{1,2}(\delta, T; H)$, $\forall 0 < \delta \leq T < \infty$ and so is g_2 and y^* , too. Moreover, since A^* is analytic, then (3.36) holds. ■

Proof (of Theorem 3.1, continued). Now we set

$$Py_0 := -p(0), \quad \text{for } y_0 \in H \quad (3.38)$$

and note that $P \in L(H, H)$.

Moreover, by adding (3.31) multiplied by $p(t)$ with (3.32) multiplied by $y^*(t)$ and integrating on $(0, \infty)$ we get

$$\begin{aligned} -2(y_0, p(0))_H &= \int_0^\infty \left\{ \langle Ay^*(t), p(t) \rangle_{(D(A^*))', D(A^*)} + (w^*(t), B_1^*p(t))_W + (u^*(t), B_2^*p(t))_U \right\} dt \\ &\quad + \int_0^\infty \left\{ -\langle Ay^*(t), p(t) \rangle_{(D(A^*))', D(A^*)} + (C_1^*C_1y^*(t), p(t))_H \right\} dt \\ &= \int_0^\infty \left\{ -\gamma^2 \|w^*(t)\|_W^2 + \|u^*(t)\|_U^2 + \|C_1y^*(t)\|_Z^2 \right\} dt \end{aligned} \quad (3.39)$$

whence

$$\begin{aligned} (Py_0, y_0)_H &= -(p(0), y_0)_H = \frac{1}{2} \int_0^\infty \left(\|C_1y^*(t)\|_Z^2 + \|u^*(t)\|_U^2 - \gamma^2 \|w^*(t)\|_W^2 \right) dt \\ &= \sup_{w \in L^2(\mathbb{R}_+; W)} \inf_{u \in L^2(\mathbb{R}_+; U)} \frac{1}{2} \int_0^\infty \left(\|C_1y(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2 \|w(t)\|_W^2 \right) dt \\ &\geq \inf_{u \in L^2(\mathbb{R}_+; U)} \frac{1}{2} \int_0^\infty \left(\|C_1y(t)\|_Z^2 + \|u(t)\|_U^2 \right) dt \geq 0, \end{aligned}$$

hence $P \geq 0$.

Moreover, $P = P^*$. Indeed, let $y_0, z_0 \in H$ and $(y^*, p), (z^*, q)$ be the corresponding solutions to (3.31)-(3.34). Namely, (z^*, q) satisfy

$$\begin{aligned} z^{*'}(t) &= Az^*(t) + B_1 w^*(t) + B_2 u^*(t), \quad t \in \mathbb{R}_+, \quad y^*(0) = y_0, \\ q'(t) &= -A^* q(t) + C_1^* C_1 z^*(t), \quad t \in \mathbb{R}_+. \end{aligned}$$

We see that

$$\frac{d}{dt}(p(t), z^*(t))_H = \frac{d}{dt}(q(t), y^*(t))_H, \quad \forall t \geq 0$$

and this yields $(Py_0, z_0)_H = (y_0, Pz_0)_H$, as claimed.

We recall that by the dynamic programming principle (see e.g., [6], p. 104), the minimization problem (3.14) for $w = w^*$, is equivalent with the following problem

$$\inf_{u \in L^2(\mathbb{R}_+; U)} \frac{1}{2} \int_t^\infty \left(\|C_1 y(s)\|_Z^2 + \|u(s)\|_U^2 - \gamma^2 \|w^*(s)\|_W^2 \right) ds$$

subject to (2.1)-(2.2) in $S_t = \{(t, \infty); y(t) = y^*(t)\}$, for every $t \geq 0$. Since u^* is the solution to this problem it follows by (3.38) that

$$p(t) = -Py^*(t), \quad \forall t \geq 0. \quad (3.40)$$

We denote by $T_P(t) : H \rightarrow H$ the family of operators

$$T_P(t)y_0 = y^*(t), \quad \forall t \geq 0 \quad (3.41)$$

where $y^*(t)$ is the solution to (3.31) with u^* and w^* given by (3.32)-(3.34). By (3.35) it follows that $T_P(t)$ is a C_0 -semigroup on H .

Let us denote by A_P the infinitesimal generator of $T_P(t)$, that is

$$\frac{dy^*}{dt}(t) = A_P y^*(t), \quad \forall t \geq 0, \quad y^*(0) = y_0, \quad (3.42)$$

or, equivalently

$$y^*(t) = e^{A_P t} y_0, \quad t \geq 0, \quad \forall y_0 \in H. \quad (3.43)$$

If $y_0 \in D(A_P)$ we have

$$y^* \in C^1([0, T]; H) \cap C([0, T]; D(A_P)), \quad \forall T > 0. \quad (3.44)$$

Here, $D(A_P) = \{y \in H; A_P y \in H\}$ is the domain of A_P . The space \mathcal{X} in Theorem 3.1 is actually

$$\mathcal{X} := D(A_P). \quad (3.45)$$

Now, replacing in the right-hand side of (3.31) u^* and w^* by (3.33)-(3.34), (3.32) and (3.40) we get

$$y^{*'}(t) = \widetilde{\Lambda}_P y^*(t) \quad (3.46)$$

where $\widetilde{\Lambda}_P$ is the operator

$$\widetilde{\Lambda}_P : H \rightarrow (D(A^*))', \quad \widetilde{\Lambda}_P y = Ay - B_2 B_2^* P y + \gamma^{-2} B_1 B_1^* P y \in (D(A^*))'$$

and A is the extension from H to $(D(A^*))'$.

We define by $\Lambda_P : D(\Lambda_P) \subset H \rightarrow H$ the restriction of the operator $\widetilde{\Lambda}_P$ to H , namely

$$\begin{aligned} \Lambda_P y &= (A - B_2 B_2^* P + \gamma^{-2} B_1 B_1^* P)y, \quad y \in D(\Lambda_P), \\ D(\Lambda_P) &= \{y \in H; (A - B_2 B_2^* P + \gamma^{-2} B_1 B_1^* P)y \in H\}. \end{aligned} \quad (3.47)$$

Lemma 3.5 *We have*

$$P \in L(\mathcal{X}, D(A^*)), \quad (3.48)$$

$$B_2^* P \in L(\mathcal{X}; U), \quad (3.49)$$

$$A_P y = \Lambda_P y, \text{ for all } y \in \mathcal{X} \subset D(\Lambda_P) \quad (3.50)$$

and Λ_P generates a C_0 -semigroup on H .

Moreover, if Λ_P is closed in H , then

$$\mathcal{X} = D(A_P) = D(\Lambda_P). \quad (3.51)$$

Proof. Let $y_0 \in D(A_P)$. We know by (3.44) that $y^* \in C^1([0, T]; H)$ and $A_P y^* \in C([0, T]; H)$ for all $T > 0$, and so by (3.32) it follows therefore that $p' \in C([0, T]; H)$ and so $A^* p \in C([0, T]; H)$. Hence, $A^* p(0) \in H$. It follows that $p(0) \in D(A^*)$ and so $P y_0 \in D(A^*)$. This implies (3.48). Since $P \in (\mathcal{X}, D(A^*))$ and $B_2^* \in L(D(A^*), U)$ it follows (3.49).

We have by (3.42) and (3.44) that

$$\frac{d}{dt}(y^*(t), \varphi)_H = (A_P y^*(t), \varphi)_H, \quad \forall t \geq 0, \quad \varphi \in H.$$

On the other hand, by (3.46) we have (see the weak form (3.4) applied to $\widetilde{\Lambda}_P : H \rightarrow (D(A^*))'$)

$$\frac{d}{dt}(y^*(t), \varphi)_H = \left\langle \frac{dy^*}{dt}(t), \varphi \right\rangle_{(D(A^*))', D(A^*)} = \left\langle \widetilde{\Lambda}_P y^*(t), \varphi \right\rangle_{(D(A^*))', D(A^*)}, \quad \forall t \geq 0, \quad \forall \varphi \in D(A^*).$$

Hence,

$$(A_P y^*(t), \varphi)_H = \left\langle \widetilde{\Lambda}_P y^*(t), \varphi \right\rangle_{(D(A^*))', D(A^*)}, \quad \forall t \geq 0, \quad \forall \varphi \in D(A^*).$$

Recalling that $y^* \in C^1([0, \infty); H) \subset C([0, \infty); (D(A^*))')$ and letting $t \rightarrow 0$ we get

$$(A_P y_0, \varphi)_H = \left\langle \widetilde{\Lambda}_P y_0, \varphi \right\rangle_{(D(A^*))', D(A^*)}, \quad \forall \varphi \in D(A^*).$$

This implies that $\widetilde{\Lambda}_P y_0 \in H$, namely $y_0 \in D(\Lambda_P)$, and $A_P y_0 = \Lambda_P y_0$ on $D(A_P) \subset D(\Lambda_P)$, that is (3.50).

Since these two operators coincide on $D(A_P)$ then Λ_P generates a C_0 -semigroup on H .

Now, $D(A_P) \subset D(\Lambda_P) \subset H$ and since $D(A_P)$ is dense in H it follows that $D(\Lambda_P)$ is dense in H and $D(A_P)$ is dense in $D(\Lambda_P)$.

Assume that Λ_P is closed and let $y_0 \in D(\Lambda_P)$. There exists $(y_0^n)_n \subset D(A_P)$, $y_0^n \rightarrow y_0$ in H and by (3.50) we have

$$(A_P y_0^n, \varphi)_H = (\Lambda_P y_0^n, \varphi)_H, \quad \varphi \in H,$$

which implies (using the adjoint of A_P^* which is the generator of a C_0 -semigroup on H) that

$$(y_0^n, A_P^* \varphi)_H = (\Lambda_P y_0^n, \varphi)_H, \quad \varphi \in D(A_P^*) \subset H.$$

Since Λ_P is closed, by letting $n \rightarrow \infty$ we obtain

$$(y_0, A_P^* \varphi)_H = (\Lambda_P y_0, \varphi)_H, \quad \varphi \in D(A_P^*).$$

Then, $\varphi \rightarrow (y_0, A_P^* \varphi)_H$ is a linear continuous functional on H and $|(y_0, A_P^* \varphi)_H| \leq C \|\varphi\|_H$, so that $y_0 \in D(A_P)$ and (3.50) is proved. ■

Proof. (of Theorem 3.1, continued). To prove that P is a solution to the Riccati equation (3.9) we use the relation

$$\frac{d}{dt}(y^*(t), p(t))_H = \langle (y^*)'(t), p(t) \rangle_{(D(A^*))', D(A^*)} + (y^*(t), p'(t))_H$$

and calculate by (3.31)-(3.34) and (3.40) a relation as done for (3.39) but integrating from t to ∞ . We get

$$\begin{aligned} (P y^*(t), y^*(t))_H &= (-p(t), y^*(t))_H \\ &= \frac{1}{2} \int_t^\infty \left(\|C_1 y^*(t)\|_Z^2 + \|u^*(t)\|_U^2 - \gamma^2 \|w^*(t)\|_W^2 \right) dt, \quad t \geq 0. \end{aligned}$$

If $y_0 \in D(A_P)$ this implies by differentiating (by using (3.42) and (3.44)) that

$$\begin{aligned} & (Py^*(t), A_P y^*(t))_H + (PA_P y^*(t), y^*(t))_H + \|C_1 y^*(t)\|_Z^2 \\ & + \|B_2^* P y^*(t)\|_U^2 - \gamma^2 \|\gamma^{-2} B_1^* P y^*(t)\|_W^2 = 0, \quad t \geq 0 \end{aligned}$$

and since, by (3.48), $B_2^* P \in L(D(A_P), D(A^*))$ we obtain for $t \rightarrow 0$ the equation

$$2(Py_0, A_P y_0)_H + \|C_1 y_0\|_Z^2 + \|B_2^* P y_0\|_U^2 - \gamma^2 \|B_1^* P y_0\|_W^2 = 0, \quad \forall y_0 \in D(A_P). \quad (3.52)$$

By differentiating along $z \in D(A_P)$ we get

$$(P y_0, A_P z)_H + (P z, A_P y_0)_H + ((B_2 B_2^* - \gamma^{-2} B_1 B_1^*) P z, P y_0)_H + (C_1^* C_1 y_0, z)_H = 0,$$

for all $y_0, z \in D(A_P)$. But here $A_P y = \Lambda_P y$ for $y \in D(A_P)$ and we can replace A_P by Λ_P in the previous equation obtaining after all calculations

$$(A^* P y_0, z)_H + (P(A - B_2 B_2^* + \gamma^{-2} B_1 B_1^*) P y_0, z)_H + (C_1^* C_1 y_0, z)_H = 0$$

for all $y_0, z \in D(A_P)$, namely (3.9).

For proving that the semigroup $e^{A_P t}$ is exponentially stable we use the detectability assumption (i_3). Let us take $K \in L(Z, H)$ and write eq. (3.31) in the following form

$$y^{*'}(t) = (A + K C_1) y^*(t) + B_2 u^*(t) + B_1 w^*(t) - K C_1 y^*(t), \quad t \geq 0,$$

or equivalently,

$$\begin{aligned} y^*(t) &= e^{(A+K C_1)t} y_0 + \int_0^t e^{(A+K C_1)(t-s)} (B_2 u^*(s) + B_1 w^*(s)) ds \\ &\quad - \int_0^t e^{(A+K C_1)(t-s)} K C_1 y^*(s) ds, \quad \text{for all } t \geq 0. \end{aligned}$$

Since $B_1 w^*, K C_1 y^* \in L^2(\mathbb{R}_+; H)$ and $e^{(A+K C_1)t}$ is exponentially stable it remains to show that

$$t \rightarrow \int_0^t e^{(A+K C_1)(t-s)} B_2 u^*(s) ds \in L^2(\mathbb{R}_+; H). \quad (3.53)$$

To this end, for each $\psi \in L^2(\mathbb{R}_+; H)$, using the Young's inequality (2.18) (with $p = 1, q = r = 2$) and (3.2) we calculate

$$\begin{aligned} & \int_0^\infty \left(\psi(t), \int_0^t e^{(A+K C_1)(t-s)} B_2 u^*(s) ds \right)_H dt \\ &= \int_0^\infty \left(\int_s^\infty B_2^* e^{(A^* + C_1^* K^*)(t-s)} \psi(t) dt, u^*(s) \right)_U ds \\ &\leq \left(\int_0^\infty \left(\int_s^\infty \|B_2^* e^{(A^* + C_1^* K^*)(t-s)} \psi(t)\|_U dt \right)^2 ds \right)^{1/2} \left(\int_0^\infty \|u^*(s)\|_U^2 ds \right)^{1/2} \\ &\leq \|u^*\|_{L^2(0, \infty; U)} \left(\int_0^\infty \left(\int_0^\infty \|B_2^* e^{(A^* + C_1^* K^*)(t-s)}\|_{L(H, U)} \|\psi(t)\|_H dt \right)^2 ds \right)^{1/2} \\ &\leq \|u^*\|_{L^2(0, \infty; U)} \left(\int_0^\infty \|B_2^* e^{(A^* + C_1^* K^*)s}\|_{L(H, U)} ds \right) \left(\int_0^\infty \|\psi(s)\|_H^2 ds \right)^{1/2} \\ &\leq C \|u^*\|_{L^2(0, \infty; U)} \|\psi\|_{L^2(0, \infty; U)} \leq C_1 \|\psi\|_{L^2(0, \infty; U)}, \end{aligned}$$

and this implies (3.53), as claimed.

We shall prove now that the operator $\Lambda_P^1 := A - B_2 B_2^* P$ generates an exponentially stable C_0 -semigroup in H with the domain $\{y \in H; (A - B_2 B_2^* P)y \in H\} = D(A_P)$. The solution $y^*(t)$ to (3.31) is in $L^2(\mathbb{R}_+; H)$ can be written also as

$$y^*(t) = e^{\Lambda_P^1 t} y_0 + \gamma^{-2} \int_0^t e^{\Lambda_P^1 (t-s)} B_1 B_1^* P y^*(s) ds$$

and since the second term on the right-hand side is in $L^2(\mathbb{R}_+; H)$, it follows that $e^{\Lambda_P^1 t} y_0 \in L^2(\mathbb{R}_+; H)$.

Now, we shall prove (3.7). Let us consider the equation

$$y'(t) = (A - B_2 B_2^* P)y(t) + B_1 w(t), \quad t \geq 0, \quad y(0) = 0, \quad (3.54)$$

with $w \in L^2(\mathbb{R}_+; W)$. As seen earlier, this equation has a unique mild solution and by (3.4) we have

$$\frac{d}{dt}(y(t), \varphi)_H = (y(t), (A^* - B_2 B_2^* P)\varphi)_H + (B_1 w(t), \varphi)_H, \quad \forall \varphi \in D(A^*). \quad (3.55)$$

Let $p(t) = -Py(t)$, $t > 0$. Since by (3.48) $P \in L(D(A_P), D(A^*))$ it follows that $p(t) \in D(A^*)$ and p is the solution to eq. (3.32) with y^* replaced by y . Moreover, as seen earlier by (3.32) it follows that $A^* p$, $p' \in L^2(\mathbb{R}_+; H)$ and we have by (3.55)

$$\frac{d}{dt}(y(t), p(t))_H = (y(t), (A^* - B_2 B_2^* P)p(t))_H + (B_1 w(t), p(t))_H + (y'(t), p(t))_H.$$

Then we calculate using (3.40) and (3.9)

$$\begin{aligned} & \frac{d}{dt}(Py(t), y(t))_H = 2(Py(t), y'(t))_H \\ &= 2(Py(t), Ay(t))_H - 2\|B_2^* Py(t)\|_H^2 + 2(B_1 w(t), Py(t))_H \\ &= \|B_2^* Py(t)\|_H^2 - \gamma^{-2} \|B_1^* Py(t)\|_H^2 - \|C_1 y(t)\|_H^2 - 2\|B_2^* Py(t)\|_H^2 + 2(B_1 w(t), Py(t))_H \\ &= -\|B_2^* Py(t)\|_H^2 - \|C_1 y(t)\|_H^2 - \gamma^{-2} \|B_1^* Py(t)\|_H^2 + 2(w(t), B_1^* Py(t))_W, \quad \text{a.e. } t > 0. \end{aligned}$$

Integrating this from 0 to ∞ we obtain

$$0 = \int_0^\infty \left(-\|B_2^* Py(t)\|_H^2 - \|C_1 y(t)\|_H^2 - \gamma^{-2} \|B_1^* Py(t)\|_H^2 + 2(w(t), B_1^* Py(t))_W \right) dt,$$

since $y(0) = 0$ and $\lim_{t \rightarrow \infty} (Py(t), y(t))_H = 0$. Therefore,

$$\begin{aligned} & \int_0^\infty \left(\|C_1 y(t)\|_H^2 + \|B_2^* Py(t)\|_H^2 \right) dt \\ &= \int_0^\infty \left(-\gamma^{-2} \|B_1^* Py(t)\|_H^2 + 2(w(t), B_1^* Py(t))_H - \gamma^2 \|w(t)\|_W^2 \right) dt + \int_0^\infty \gamma^2 \|w(t)\|_W^2 dt \\ &= \int_0^\infty \gamma^2 \|w(t)\|_W^2 dt - \int_0^\infty \gamma^2 \|\tilde{w}(t)\|_W^2 dt, \end{aligned}$$

where

$$\tilde{w}(t) = w(t) - \gamma^{-2} B_1^* Py(t). \quad (3.56)$$

If we prove that there exists $\alpha > 0$ such that

$$\|\tilde{w}\|_{L^2(0, \infty; W)} \geq \alpha \|w\|_{L^2(0, \infty; W)}, \quad \forall w \in L^2(\mathbb{R}_+; W), \quad (3.57)$$

it follows that

$$\gamma^2 \left(\|w\|_{L^2(\mathbb{R}_+; W)}^2 - \|\tilde{w}\|_{L^2(\mathbb{R}_+; W)}^2 \right) \leq \gamma^2 (1 - \alpha) \|w\|_{L^2(\mathbb{R}_+; W)}^2$$

and therefore

$$\int_0^\infty \left(\|C_1 y(t)\|_H^2 + \|B_2^* Py(t)\|_H^2 \right) dt \leq (\gamma^2 - \delta) \|w\|_{L^2(\mathbb{R}_+; W)}^2$$

with $\delta > 0$ independent on w . Therefore,

$$\int_0^\infty \left(\|C_1 y(t)\|_H^2 + \|B_2^* P y(t)\|_H^2 \right) dt \leq (\gamma^2 - \delta) \int_0^\infty \|w(t)\|_W^2 dt,$$

which by (2.12) implies (3.7). We note that once (3.57) proved, α can be chosen smaller such that $\alpha < 1$. It remains to prove (3.57) and this will be done in Lemma 3.6 given at the end of this section.

Therefore, G corresponding to $\tilde{F} := -B_2^* P$ has the property $\|G_{\tilde{F}} w\|_{L^2(\mathbb{R}_+; Z)} < \gamma \|w\|_{L^2(\mathbb{R}_+; W)}$, that is \tilde{F} is the feedback operator which solves the H^∞ -control problem. This ends the proof of the the first part of Theorem 3.1.

Assume now that P is a solution to equation (3.9), satisfying (3.8), such that $\Lambda_P = A - (B_2 B_2^* - \gamma^{-2} B_1 B_1^*) P$ generates an exponentially stable semigroup on \mathcal{X} . We set $y^*(t) = e^{\Lambda_P t} y_0$, for $y_0 \in H$, so that $y^* \in C([0, \infty); H) \cap L^2(\mathbb{R}_+; H)$. Let us define $p(t) = -P y^*(t)$, for $t \geq 0$. Then, $p \in L^2(\mathbb{R}_+; H) \cap C([0, \infty); H)$ and by replacing $P y^*(t)$ in (3.9) we get that p satisfies equation (3.32) with the regularity obtained in Lemma 3.4. The , as before. Finally, we show that the operator Λ_P^1 generates an exponentially stable semigroup and that the controller $\tilde{F} y = -B_2^* P y$ stabilizes equation $y'(t) = (A + B_2 \tilde{F}) y(t) + B_1 w(t)$, $y(0) = 0$, arguing as before beginning from (3.54). This ends the proof of Theorem 3.1. ■

It remains to prove (3.57). We set

$$\Phi(w) = \|\tilde{w}\|_{L^2(\mathbb{R}_+; W)}^2. \quad (3.58)$$

Lemma 3.6 *We have*

$$\Phi(w) \geq \alpha \|w\|_{L^2(0, \infty; W)}^2, \text{ for all } w \in L^2(\mathbb{R}_+; W), \quad (3.59)$$

where $\alpha > 0$.

Proof. We proceed by reduction to absurdity. Assume that (3.59) does not hold and argue from this a contradiction. Thus, let $(w_n)_n \subset L^2(\mathbb{R}_+; W)$ be such that $\|w_n\|_{L^2(\mathbb{R}_+; W)} = 1$, $\forall n \in \mathbb{N}$ and $\Phi(w_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence, by (3.58) and eq. (3.54) we have

$$\Phi(w_n) = \left\| w_n - \gamma^{-2} B_1^* P \int_0^t e^{(A - B_2 B_2^* P)s} B_1 w_n(s) ds \right\|_{L^2(\mathbb{R}_+; W)}^2 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (3.60)$$

On the other hand, on a subsequence, we have $w_n \rightarrow \bar{w}$ weakly in $L^2(\mathbb{R}_+; W)$, and since Φ is weakly lower semicontinuous in $L^2(\mathbb{R}_+; W)$ (because it is continuous and convex) we have by (3.60) that $\Phi(\bar{w}) = 0$ which implies that

$$\bar{w}(t) = \gamma^{-2} B_1^* P \int_0^t e^{(A - B_2 B_2^* P)s} B_1 \bar{w}(s) ds, \quad \forall t \geq 0.$$

By Gronwall's lemma we deduce that $\bar{w}(t) = 0$. Now, if we prove that

$$w_n \rightarrow \bar{w} \text{ strongly in } L^2(\mathbb{R}_+; W) \text{ as } n \rightarrow \infty$$

(namely, that $(w_n)_n$ is compact in $L^2(\mathbb{R}_+; W)$) we arrive to a contradiction because, the choice $\|w_n\|_{L^2(\mathbb{R}_+; W)} = 1$ implies $\|\bar{w}\|_{L^2(\mathbb{R}_+; W)} = 1$, which was found before to be 0.

To prove that $(w_n)_n$ is compact in $L^2(\mathbb{R}_+; W)$, by (3.60) it suffices to show that the sequence

$$z_n(t) = \gamma^{-2} B_1^* P \int_0^t e^{(A - B_2 B_2^* P)s} B_1 w_n(s) ds, \quad t \geq 0$$

is compact in $L^2(\mathbb{R}_+; W)$, that is, it contains a convergent subsequence. Taking into account that

$$\|z_n\|_{L^2(T, \infty; W)} \rightarrow 0 \text{ as } T \rightarrow \infty, \text{ uniformly in } n, \quad (3.61)$$

since $A - B_2 B_2^* P$ generates an exponentially stable semigroup, it suffices to prove that $(z_n)_n$ is compact in $L^2(0, T; W)$, for each $T > 0$. We set

$$S(t) = e^{(A - B_2 B_2^* P)t}, \quad t \geq 0 \quad (3.62)$$

and prove that $\{S(t)\}$ is compact for each $t > 0$. This means that the set $\{S(t)y_0; y_0 \in H; \|y_0\|_H \leq M\}$ is relatively compact in H . Since $A - B_2B_2^*P = \Lambda_P - \gamma^{-2}B_1B_1^*P$ and $B_1B_1^*P \in L(H, H)$ and $\Lambda_P = A_P$ on $D(A_P)$ it suffices to show that $T_P(t) = e^{A_P t}$ is compact for each $t > 0$. This follows by density by showing first that $\{T_P(t)y_0; y_0 \in D(A_P), \|A_P y_0\|_H + \|y_0\|_H \leq M\}$ is relatively compact in H . To this end, for $\varepsilon > 0$, we write $T_P(t)y_0$ in the following form

$$\begin{aligned} T_P(t)y_0 &= e^{At}y_0 - \int_0^t e^{A(t-s)}(B_2B_2^*Py(s) - \gamma^{-2}B_1B_1^*Py(s))ds \\ &= e^{At}y_0 - e^{A\varepsilon} \int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)}(B_2B_2^*Py(s) - \gamma^{-2}B_1B_1^*Py(s))ds \\ &\quad - \int_{t-\varepsilon}^t e^{A(t-s)}(B_2B_2^*Py(s) - \gamma^{-2}B_1B_1^*Py(s))ds, \end{aligned} \quad (3.63)$$

where $y(t) = T_P(t)y_0$. If $\mathcal{M} = \{y_0 \in D(A_P); \|A_P y_0\|_H + \|y_0\|_H \leq M\}$, relation (3.63) yields

$$\begin{aligned} T_P(t)\mathcal{M} &= \{e^{At}y_0; y_0 \in \mathcal{M}\} \\ &\quad - \left\{ e^{A\varepsilon} \int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)}(B_2B_2^*Py(s) - \gamma^{-2}B_1B_1^*Py(s))ds; y_0 \in \mathcal{M} \right\} \\ &\quad - \left\{ \int_{t-\varepsilon}^t e^{A(t-s)}(B_2B_2^*Py(s) - \gamma^{-2}B_1B_1^*Py(s))ds; y_0 \in \mathcal{M} \right\} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3. \end{aligned}$$

In the sum above, \mathcal{M}_1 is relatively compact because e^{At} is compact by (i_1) . Next, we write $\mathcal{M}_2 = \mathcal{M}_{21} + \mathcal{M}_{22}$ where $\mathcal{M}_{2i} = \left\{ e^{A\varepsilon} \int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)} B_i B_i^* Py(s) ds; y_0 \in \mathcal{M} \right\}$, $i = 1, 2$. \mathcal{M}_{21} is relatively compact because $e^{A\varepsilon}$ is compact and $\int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)} B_1 B_1^* Py(s) ds$ is bounded,

$$\begin{aligned} \left\| \int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)} B_1 B_1^* Py(s) ds \right\|_H &\leq C \int_0^t \|B_1 B_1^* Py(s)\|_U ds \\ &\leq C \int_0^t \|B_1^* Py(s)\|_H ds \leq Ct \|y_0\|_H. \end{aligned}$$

Then,

$$\begin{aligned} &\left\| \int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)} B_2 B_2^* Py(s) ds \right\|_H = \sup_{\varphi \in H, \|\varphi\|_H \leq 1} \left(\int_0^{t-\varepsilon} e^{A(t-s-\varepsilon)} B_2 B_2^* Py(s) ds, \varphi \right)_H \\ &\leq \sup_{\|\varphi\|_H \leq 1} \int_0^{t-\varepsilon} \left(B_2^* Py(s), B_2^* e^{A^*(t-s-\varepsilon)} \varphi \right)_U ds \leq \sup_{\|\varphi\|_H \leq 1} \int_0^{t-\varepsilon} \|B_2^* Py(s)\|_U \|B_2^* e^{A^*(t-s-\varepsilon)} \varphi\|_U ds \\ &\leq \sup_{\|\varphi\|_H \leq 1} \int_0^{t-\varepsilon} \|Py(s)\|_{D(A^*)} \|B_2^* e^{A^*(t-s-\varepsilon)}\|_{L(H,U)} \|\varphi\|_H ds \\ &\leq \int_0^{t-\varepsilon} \|y(s)\|_{D(A_P)} \|B_2^* e^{A^*(t-s-\varepsilon)}\|_{L(H,U)} ds \\ &\leq \int_0^{t-\varepsilon} \|A_P y(s)\|_H \|B_2^* e^{A^*(t-s-\varepsilon)}\|_{L(H,U)} ds \leq C \|A_P y_0\|_H \int_0^{t-\varepsilon} \|B_2^* e^{A^*(t-s-\varepsilon)}\|_{L(H,U)} ds \leq C_T, \end{aligned}$$

hence \mathcal{M}_{22} is relatively compact, too. We also have

$$\left\| \int_{t-\varepsilon}^t e^{A(t-s)} B_1 B_1^* Py(s) ds \right\|_H \leq C \int_{t-\varepsilon}^t \|B_2 B_2^* Py(s)\|_H ds \leq C\varepsilon$$

and similarly we estimate that the term corresponding to $B_2 B_2^* P$ is bounded by $C\varepsilon$. Since ε is arbitrary it follows that $T_P(t)\mathcal{M}$ is compact and, as mentioned earlier, it follows by density that the set $\{T_P(t)y_0; \|y_0\|_H \leq M\}$ is compact for each M and $t > 0$, fixed. Now, coming back to z_n we write

$$z_n(t) = \gamma^{-2} B_1^* P S(\varepsilon) \left(\int_0^{t-\varepsilon} S(t-s-\varepsilon) B_1 w_n(s) ds \right) + \gamma^{-2} B_1^* P \int_{t-\varepsilon}^t S(t-s) B_1 w_n(s) ds$$

and get

$$\begin{aligned} & \left\| \int_0^{t-\varepsilon} S(t-s-\varepsilon)B_1w_n(s)ds \right\|_H \leq C \left\| \int_0^{t-\varepsilon} e^{-\beta(t-s-\varepsilon)}B_1w(s)ds \right\|_H \\ & \leq C \int_0^{t-\varepsilon} \|B_1w_n(s)\|_H ds \leq C \|w\|_{L^2(\mathbb{R}_+;W)} \leq C, \quad \forall t \geq 0, \end{aligned}$$

hence, $\left\{ S(\varepsilon) \left(\int_0^{t-\varepsilon} S(t-s-\varepsilon)B_1w_n(s)ds \right) \right\}$ is compact in H .

Taking into account that $\left\| \int_{t-\varepsilon}^t S(t-s)B_1w(s)ds \right\|_H \leq C\varepsilon$, it follows that $(z_n(t))_n$ is compact in H , for every $t > 0$. Also, it is equi-uniformly continuous, that is $\|z_n(t+h) - z_n(t)\|_H \leq \varepsilon$ if $|h| \leq \delta(\varepsilon)$, for any t . The latter follows because the semigroup $S(t)$ is continuous for $t > 0$ in the uniform operator topology (see [24], p. 48, Theorem 3.2), and this means that $\|(S(t+h) - S(t))\theta\|_H \leq \delta_1(h) \|\theta\|_H$, where $\delta_1(h) \rightarrow 0$, and $\theta \in H$. Then,

$$\begin{aligned} \|z_n(t+h) - z_n(t)\|_H & \leq C_1 \int_t^{t+h} \|S(t+h-s)B_1Pw_n(s)\|_H ds \\ & + C_2 \int_0^t \|(S(t+h-s) - S(t-s))B_1Pw_n(s)\|_H ds \\ & \leq C_1 \int_t^{t+h} e^{-\beta(t+h-s)} \|w_n(s)\|_H ds + C_2\delta_2(h), \end{aligned}$$

where $\delta_2(h) \rightarrow 0$ as $h \rightarrow 0$. Then, by Ascoli-Arzelà's theorem, $(z_n)_n$ is compact in $C([0, T]; H)$, for every $T > 0$ and so $z_n \rightarrow z$ strongly in $L^2(0, T; H)$, for every $T > 0$. Recalling (3.61) we note that

$$\|z_n - z\|_{L^2(\mathbb{R}_+; H)}^2 = \int_0^T \|z_n(t) - z(t)\|_H^2 dt + \int_T^\infty \|z_n(t) - z(t)\|_H^2 dt \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

because the first term tends to 0 by the compactness argument developed before and

$$\int_T^\infty \|z_n(t) - z(t)\|_H^2 dt \leq 2 \int_T^\infty \|z_n(t)\|_H^2 dt + \int_T^\infty \|z(t)\|_H^2 dt \rightarrow 0$$

by (3.61) and the fact that $A - B_2B_2^*P$ generates an exponentially stable semigroup.

Going back to (3.60), it follows that $\|(w_n - z_n)(t)\|_W \rightarrow 0$, a.e. $t > 0$, and so $(w_n)_n$ is compact, as claimed. This ends the proof of Lemma 3.5 and also of Theorem 3.1. \blacksquare

Remark 3.7 *Theorem 3.1 reduces the existence of a robust feedback controller F satisfying (3.7) to the existence of a solution P to (3.9) in the same way as for $B_1 = B_2 = D_1 = 0$, $C_1 = I$, the Lyapunov equation $A^*P + PA = I$ is related to the stability of the semigroup e^{At} . In the specific examples discussed in the next sections we shall show that the operatorial equation (3.9) reduces to a nonlinear integro-differential elliptic equation.*

4 The case of a N - D distributed control

Let Ω be an open bounded subset of \mathbb{R}^N , $N > 3$ with the boundary $\Gamma = \partial\Omega$ sufficiently smooth and assume that $0 \in \Omega$. We consider the following singular system

$$y_t - \Delta y - \frac{\lambda y}{|x|^2} - a(x)y = B_1w + B_2u, \quad \text{in } (0, \infty) \times \Omega, \quad (4.1)$$

$$y = 0, \quad \text{on } (0, \infty) \times \Gamma, \quad (4.2)$$

$$y(0) = y_0, \quad \text{in } \Omega, \quad (4.3)$$

$$z = C_1y + D_1u, \quad \text{in } (0, \infty) \times \Omega, \quad (4.4)$$

where $\lambda > 0$, $|\cdot|$ denotes the Euclidian norm in \mathbb{R}^N , for any $N = 1, 2, \dots$, according the case and a has the expression

$$a(x) = a_0\chi_{\Omega_0}(x), \quad a_0 > 0, \quad \Omega_0 \subset \Omega. \quad (4.5)$$

In this problem

$$y_0 \in L^2(\Omega) \quad (4.6)$$

and we choose

$$H = W = Z = L^2(\Omega), \quad U = \mathbb{R}, \quad (4.7)$$

$$\begin{aligned} B_1 w &= \chi_{\omega_1}(x)w, & B_2 u &= b(x)u, \\ C_1 y &= \chi_{\Omega_C}(x)y, & D_1 u &= d(x)u, \quad x \in \Omega, \quad u \in \mathbb{R}, \end{aligned} \quad (4.8)$$

where $\Omega_0, \Omega_C, \omega_1$ are open sets of Ω , χ_ω is characteristic functions of the set $\omega \subset \Omega$,

$$\omega_1 \sqsubseteq \Omega, \quad \Omega_0 \sqsubseteq \Omega_C \subset \Omega, \quad (4.9)$$

and

$$b \in L^2(\Omega), \quad d \in L^2(\Omega), \quad d(x) = \chi_{\Omega \setminus \Omega_C}. \quad (4.10)$$

We begin by checking the hypotheses $(i_1) - (i_4)$.

(i_1) By their expressions we see that

$$B_1, C_1 \in L(L^2(\Omega), L^2(\Omega)), \quad B_2, D_1 \in L(\mathbb{R}, L^2(\Omega))$$

and $B_2^* : L^2(\Omega) \rightarrow \mathbb{R}$ is defined by

$$B_2^* v = \int_{\Omega} b(x)v(x)dx, \quad \text{for } v \in L^2(\Omega). \quad (4.11)$$

We recall the Hardy inequality (2.16) and consider $\lambda < H_N$. We introduce the self-adjoint operator

$$A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad Ay = \Delta y + \frac{\lambda y}{|x|^2} + ay, \quad (4.12)$$

with

$$D(A) = \{y \in H_0^1(\Omega); Ay \in L^2(\Omega)\}. \quad (4.13)$$

It is clear that $\overline{D(A)} = L^2(\Omega)$ because $D(A)$ contains $C_0^\infty(\Omega \setminus \{0\})$. Then, equation (4.1) can be equivalently written

$$y'(t) = Ay(t) + B_1 w(t) + B_2 u(t), \quad t \geq 0. \quad (4.14)$$

In order to show that A generates a C_0 -semigroup on $L^2(\Omega)$, we have to prove that A is ω - m -dissipative on $L^2(\Omega)$, or that $-A$ is ω - m -accretive on $L^2(\Omega)$ (see [7], p. 155).

Lemma 4.1 *Let $\lambda < H_N$. The operator $-A$ is ω - m -accretive on $L^2(\Omega)$, for $\omega > a_0$.*

Proof. This means to show that $-A$ is ω -accretive, that is $((\omega I - A)y, y)_2 \geq 0$ for some $\omega > 0$ and all $y \in L^2(\Omega)$ and that $\omega I - A$ is surjective. To this end we shall use several times the Hardy inequality (2.16) which ensures that $\frac{y}{x} \in L^2(\Omega)$ if $y \in H_0^1(\Omega)$. We have

$$\begin{aligned} ((\omega I - A)y, y)_2 &= \omega \int_{\Omega} |y|^2 dx + \int_{\Omega} |\nabla y|^2 dx - \lambda \int_{\Omega} \frac{|y|^2}{|x|^2} dx - a_0 \int_{\Omega} |y|^2 dx \\ &\geq \left(1 - \frac{\lambda}{H_N}\right) \int_{\Omega} |\nabla y|^2 dx + (\omega - a_0) \int_{\Omega} |y|^2 dx \\ &\geq \frac{1}{2} \left(1 - \frac{\lambda}{H_N}\right) \|\nabla y\|_2^2 + \frac{H_N}{2} \left(1 - \frac{\lambda}{H_N}\right) \left\| \frac{y}{x} \right\|_2^2 + (\omega - a_0) \|y\|_2^2 \end{aligned}$$

which shows that $-A$ is ω -accretive on $L^2(\Omega)$ for $\lambda < H_N$ and $\omega > a_0$.

To prove the surjectivity of $\omega I - A$, we show that the range $R(\omega I - A) = L^2(\Omega)$. Thus, let $f \in L^2(\Omega)$ and prove that the equation

$$\omega y - Ay = f \quad (4.15)$$

has a solution $y \in D(A)$, by the equivalent variational formulation expressed by the minimization problem

$$\min_{y \in H_0^1(\Omega)} \left\{ J(y) = \int_{\Omega} \left(\frac{1}{2} |\nabla y|^2 - \frac{\lambda}{2} \frac{y^2}{|x|^2} - \frac{\omega - a(x)}{2} y^2 - f y \right) dx \right\}, \quad (4.16)$$

subject to (4.14) and $y(0) = y_0 \in L^2(\Omega)$. For $\omega > a_0$ we have

$$\frac{1}{2} \left(1 - \frac{\lambda}{H_N} \right) \int_{\Omega} |\nabla y|^2 dx + (\omega - a_0) \int_{\Omega} y^2 dx - \frac{1}{2(\omega - a_0)} \int_{\Omega} |f|^2 dx \leq J(\varphi) < \infty$$

so that J has an infimum d . Taking a minimizing sequence $(y_n)_n$ we have

$$d \leq J(y_n) \leq d + \frac{1}{n} \quad (4.17)$$

and so

$$\|\nabla y_n\|_2 + \|y_n\|_2 + \left\| \frac{y_n}{x} \right\|_2 \leq C_N \text{ for } \omega > a_0.$$

Further, C, C_N, C_T denote some constants (which may change from line to line), C_N depending on N , via $\lambda < H_N$ and C_T depending on T .

We deduce that on a subsequence denoted still by n it follows that

$$y_n \rightarrow y \text{ weakly in } H_0^1(\Omega), \quad \frac{y_n}{x} \rightarrow l \text{ weakly in } L^2(\Omega)$$

and by compactness $y_n \rightarrow y$ strongly in $L^2(\Omega)$. Then $\frac{y_n}{x} \rightarrow \frac{y}{x}$ a.e. on Ω and $l = \frac{y}{x}$ by the Vitali's theorem. We can now pass to the limit in (4.17), relying on the weakly lower semicontinuity of J and get that $J(y) = d$, that is y realizes the minimum in (4.16).

Next, we give a variation $y^\sigma = y + \sigma\eta$, for $\sigma > 0$ and $\eta \in H_0^1(\Omega)$, and particularize the condition of optimality, namely $J(\tilde{y}) \geq J(y)$ for any $\tilde{y} \in H_0^1(\Omega)$ for $\tilde{y} = y^\sigma$. We calculate

$$\lim_{\sigma \rightarrow 0} \frac{J(y^\sigma) - J(y)}{\sigma} = \int_{\Omega} \left((\omega - a(x))y\eta + \nabla y \cdot \nabla \eta - \frac{\lambda y \eta}{|x|^2} - f\eta \right) dx \geq 0.$$

Repeating the calculus for $\sigma \rightarrow -\sigma$ we get the reverse inequality, so that finally we can write

$$\int_{\Omega} \left\langle (\omega - a(x))y - \Delta y - \frac{\lambda y}{|x|^2} - f, \eta \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dx = 0 \text{ for all } \eta \in H_0^1(\Omega),$$

which implies that y is the weak solution to the equation (4.15). The solution is also unique because J is strictly convex and the system is linear. By (4.15) we see that $Ay \in L^2(\Omega)$, so that $y \in D(A)$. ■

In conclusion, A generates an analytic C_0 -semigroup on $L^2(\Omega)$ for $\lambda < H_N$.

Moreover, since as earlier seen, the operator $(\omega I - A)^{-1}$ is a compact operator for $\omega > a_0$, it follows that e^{At} is compact for all $t > 0$.

(i₂) Let $y_0 \in L^2(\Omega)$, $u \in L^2(\mathbb{R}_+, \mathbb{R})$, $w \in L^2(\mathbb{R}_+; L^2(\Omega))$. Since $B_1 w + B_2 u \in L^2(0, T; L^2(\Omega))$ and $y_0 \in \overline{D(A)} = L^2(\Omega)$, eq. (4.14) with $y(0) = y_0$ has a unique mild solution $y \in C([0, T], L^2(\Omega))$, given by (3.3) for any $T > 0$ (see [7], p. 131, Corollary 4.1). The solution also satisfies $y \in L^2(0, T; H_0^1(\Omega)) \cup W^{1,2}(0, T; H^{-1}(\Omega))$.

In order to prove (i₃) we provide the following lemma.

Lemma 4.2 *Let $\lambda < H_N$. Then, the pair (A, C_1) is exponentially detectable.*

Proof. Let $K \equiv -kI$, with $k \geq a_0$ and set $A_1 = A + KC_1$. This is still ω - m -accretive, so that A_1 generates a C_0 -semigroup on $L^2(\Omega)$, $S_1(t) = e^{A_1 t}$. Hence $y(t) = e^{A_1 t} y_0$ satisfies

$$\frac{dy}{dt}(t) = A_1 y(t), \quad t \geq 0, \quad y(0) = y_0. \quad (4.18)$$

Recalling the expression of C_1 , multiplying (4.18) by $y(t)$ and applying again (2.16) we get

$$\frac{1}{2} \frac{d}{dt} \|y(t)\|_2^2 + \left(1 - \frac{\lambda}{H_N}\right) \|\nabla y(t)\|_2^2 + k \int_{\Omega_C} |y(t)|^2 ds \leq a_0 \int_{\Omega_0} |y(t)|^2 dx. \quad (4.19)$$

We take into account that $\Omega_0 \sqsubseteq \Omega_C$ and $k \geq a_0$, and integrate from 0 to t . We obtain

$$\frac{1}{2} \|y(t)\|_2^2 + \left(1 - \frac{\lambda}{H_N}\right) \int_0^t \|\nabla y(s)\|_2^2 ds + (k - a_0) \int_{\Omega_0} |y(t)|^2 ds \leq \frac{1}{2} \|y_0\|_2^2, \quad \forall t > 0.$$

From here and the Poincaré inequality it follows that

$$\int_0^t \|y(s)\|_2^2 ds \leq C_N \|y_0\|_2^2, \quad \text{for all } t > 0, \quad (4.20)$$

with C_N a constant depending on H_N . Letting $t \rightarrow \infty$ in (4.20) we finally get that

$$\int_0^\infty \|y(s)\|_2^2 ds \leq C_N \|y_0\|_2^2. \quad (4.21)$$

This means by Datko's result, previously recalled, that e^{A+KC_1} generates an exponentially stable semigroup, that is there exists $\alpha > 0$ such that

$$\left\| e^{(A+KC_1)t} y \right\|_2 \leq C e^{-\alpha t} \|y\|_2 \quad \text{for all } y \in L^2(\Omega).$$

Then,

$$\begin{aligned} & \int_0^\infty \left\| B_2^* e^{(A^*+C_1^*K^*)t} y \right\|_U dt \leq C \int_0^\infty \left\| e^{(A^*+C_1^*K^*)t} y \right\|_2 dt \\ & \leq C \|y\|_2 \int_0^\infty e^{-\alpha t} dt = C \|y\|_2, \quad \forall y \in L^2(\Omega), \end{aligned}$$

that is (3.2) is verified. ■

(i₄) By (4.10) we have

$$\|D_1 u\|_2^2 = u^2 \|d\|_{L^2(\Omega \setminus \Omega_C)}^2 = u^2$$

and

$$D_1^* C_1 y = \int_{\Omega} d(x) \chi_{\Omega_C}(x) y(x) dx = 0.$$

The hypotheses being checked, we can formulate the H^∞ -control problem for system (4.1)-(4.4) as in Theorem 3.1.

In order to explicit Theorem 3.1 and to give a differential formulation for it, as announced in Remark 3.7, we recall that the linear continuous operator $P \in L(L^2(\Omega), L^2(\Omega))$ can be represented by the L. Schwartz kernel theorem (see e.g., [20], p. 166) as an integral operator with a kernel $P_0 \in L^2(\Omega \times \Omega)$, namely

$$P\varphi(x) = \int_{\Omega} P_0(x, \xi) \varphi(\xi) d\xi, \quad \text{for all } \varphi \in C_0^\infty(\Omega). \quad (4.22)$$

By (4.8) and (4.11) we have

$$\begin{aligned} B_1 B_1^* \varphi(x) &= \chi_{\omega_1}(x) \varphi(x), \quad C_1 C_1^* \varphi(x) = \chi_{\Omega_C}(x) \varphi(x), \\ B_1 B_1^* P \varphi(x) &= \chi_{\omega_1}(x) \int_{\Omega} P_0(x, \xi) \varphi(\xi) d\xi, \\ P B_1 B_1^* P \varphi(x) &= \int_{\Omega} \int_{\Omega} \chi_{\omega_1}(\bar{\xi}) P_0(x, \bar{\xi}) P_0(\bar{\xi}, \xi) \varphi(\xi) d\bar{\xi} d\xi \end{aligned} \quad (4.23)$$

$$\begin{aligned}
B_2 B_2^* \varphi(x) &= b(x) \int_{\Omega} b(\bar{x}) \varphi(\bar{x}) d\bar{x}, \quad x \in \Omega, \\
B_2 B_2^* P \varphi(x) &= b(x) \int_{\Omega} \int_{\Omega} b(\bar{x}) P_0(\bar{x}, \xi) \varphi(\xi) d\bar{x} d\xi, \quad x \in \Omega, \\
P B_2 B_2^* P \varphi(x) &= \int_{\Omega} \int_{\Omega} \int_{\Omega} P_0(x, \bar{\xi}) P_0(\bar{x}, \xi) b(\bar{\xi}) b(\bar{x}) \varphi(\xi) d\bar{x} d\bar{\xi} d\xi.
\end{aligned} \tag{4.24}$$

Moreover, by a straightforward calculation we obtain

$$A^* P \varphi(x) = \int_{\Omega} \left(\Delta_x P_0(x, \xi) + \frac{\lambda P_0(x, \xi)}{|x|^2} + a(x) P_0(x, \xi) \right) \varphi(\xi) d\xi, \tag{4.25}$$

$$P A \varphi(x) = \int_{\Omega} \varphi(\xi) \left(\Delta_{\xi} P_0(x, \xi) + \frac{\lambda P_0(x, \xi)}{|\xi|^2} + a(\xi) P_0(x, \xi) \right) d\xi, \tag{4.26}$$

and by denoting $E := B_2 B_2^* - \gamma^{-2} B_1 B_1^*$, we have

$$\begin{aligned}
P E P \varphi(x) &= \int_{\Omega} \varphi(\xi) d\xi \int_{\Omega} \int_{\Omega} P_0(x, \bar{\xi}) P_0(\bar{x}, \xi) b(\bar{\xi}) b(\bar{x}) d\bar{x} d\bar{\xi} \\
&\quad - \gamma^{-2} \int_{\Omega} \varphi(\xi) d\xi \int_{\Omega} \chi_{\omega_1}(\bar{\xi}) P_0(x, \bar{\xi}) P_0(\bar{\xi}, \xi) d\bar{\xi}.
\end{aligned}$$

For $x \in \Omega$ we define the distribution $\mu_x \in \mathcal{D}'(\Omega)$ by

$$\mu_x(\varphi) = \chi_{\Omega_c}(x) \varphi(x) = \int_{\Omega} \delta(x - \xi) \chi_{\Omega_c}(\xi) \varphi(\xi) d\xi, \quad \forall \varphi \in C_0^\infty(\Omega),$$

where δ is the Dirac distribution. Then, by replacing all these in (3.9), we deduce the equation

$$\begin{aligned}
&\Delta_x P_0(x, \xi) + \Delta_{\xi} P_0(x, \xi) + \lambda P_0(x, \xi) \left(\frac{1}{|x|^2} + \frac{1}{|\xi|^2} \right) + (a(x) + a(\xi)) P_0(x, \xi) \\
&- \int_{\Omega} \int_{\Omega} P_0(x, \bar{\xi}) P_0(\bar{x}, \xi) b(\bar{\xi}) b(\bar{x}) d\bar{x} d\bar{\xi} + \gamma^{-2} \int_{\Omega} \chi_{\omega_1}(\bar{\xi}) P_0(x, \bar{\xi}) P_0(\bar{\xi}, \xi) d\bar{\xi} \\
&= -\delta(x - \xi) \chi_{\Omega_c}(\xi), \quad \text{in } \mathcal{D}'(\Omega \times \Omega).
\end{aligned} \tag{4.27}$$

This equation is accompanied by the conditions

$$P_0(x, \xi) = 0, \quad \forall (x, \xi) \in \Gamma \times \Gamma, \tag{4.28}$$

$$P_0(x, \xi) = P(\xi, x), \quad \forall (x, \xi) \in \Omega \times \Omega, \tag{4.29}$$

$$P_0(x, \xi) \geq 0, \quad \forall (x, \xi) \in \Omega \times \Omega \tag{4.30}$$

and so we can enounce the following

Theorem 4.3 *Let $\gamma > 0$ and let A , B_1 , B_2 , C_1 and D_1 be given by (4.12) and (4.8), respectively. Then there exists $\tilde{F} \in L(L^2(\Omega), \mathbb{R})$ which solves the H^∞ -control problem for system (4.1)-(4.4) if and only if there exists a solution $P_0 \in D(A) \times D(A)$ to (4.27)-(4.28), satisfying (4.29)-(4.30). Moreover, in this case*

$$\tilde{F}y = - \int_{\Omega} \int_{\Omega} b(x) P_0(x, \xi) y(\xi) d\xi dx, \quad \forall y \in L^2(\Omega), \tag{4.31}$$

is a feedback controller which solves the H^∞ -problem for system (4.1)-(4.4).

In this case it is easily seen that $\Lambda_P = A - B_2 B_2^* P + \gamma^{-2} B_1 B_1^* P$ has the domain $D(\Lambda_P) = D(A)$, and since Λ_P is closed it follows that $\mathcal{X} = D(A)$. Moreover, by (4.31) we see that $\tilde{F} \in L(L^2(\Omega), \mathbb{R})$.

A direct approach of problem (4.27)-(4.30) is an interesting problem by itself but is beyond the objective of this work.

5 Dirichlet boundary control

As in the previous section let Ω be an open bounded subset of \mathbb{R}^N , $N > 3$ with the boundary $\Gamma = \partial\Omega$ sufficiently smooth and such that $0 \in \Omega$. Consider the following system

$$y_t - \Delta y - \frac{\lambda y}{|x|^2} - a(x)y = B_1 w, \quad \text{in } (0, \infty) \times \Omega, \quad (5.32)$$

$$y = \tilde{u}, \quad \text{on } (0, \infty) \times \Gamma, \quad (5.33)$$

$$y(0) = y_0, \quad \text{in } \Omega, \quad (5.34)$$

$$z = C_1 y + D_1 u, \quad \text{in } (0, \infty) \times \Omega, \quad (5.35)$$

where $y_0 \in L^2(\Omega)$, a is again given by (4.5) and

$$\tilde{u}(t, x) = \sum_{j=1}^m \alpha_j(x) u_j(t), \quad u_j(t) \in \mathbb{R} \text{ a.e. } t \in (0, \infty), \quad j = 1, \dots, m, \quad (5.36)$$

$$\alpha = (\alpha_1, \dots, \alpha_m) \in (L^2(\Gamma))^m, \quad \alpha_j \geq 0 \text{ a.e. } x \in \Gamma.$$

We assume in addition that

$$\frac{D_0 \alpha_j}{x} \in L^2(\Omega), \quad j = 1, \dots, m. \quad (5.37)$$

The expression (5.36) allows the possibility to consider combinations of conditions on subsets of the boundary for the controls $u_j(t) \in \mathbb{R}$. The hypothesis (5.37) will be justified later.

(i₁) For this problem we choose

$$H = W = Z = L^2(\Omega), \quad U = \mathbb{R}^m, \quad (5.38)$$

$$B_1 w = \chi_{\omega_1}(x)w, \quad C_1 y = \chi_{\Omega_C}(x)y, \quad D_1 u = \sum_{j=1}^m d_j(x)u_j, \quad x \in \Omega, \quad (5.39)$$

$u = (u_1, \dots, u_m)$, with the conditions $\omega_1 \sqsubseteq \Omega$, $\Omega_0 \sqsubseteq \Omega_C$, and

$$d_j \in L^2(\Omega), \quad d_j(x) = 0 \text{ on } \Omega_C, \quad \int_{\Omega \setminus \Omega_C} d_j d_k dx = \delta_{jk}. \quad (5.40)$$

Thus, $B_1 \in L(L^2(\Omega), L^2(\Omega))$, $C_1 \in L(L^2(\Omega), L^2(\Omega))$ and $D_1 : U \rightarrow L^2(\Omega)$. The operator B_2 will be further defined. The operator A is the same as before, that is

$$A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad Ay = \Delta y + \frac{\lambda y}{|x|^2} + a(x)y, \quad (5.41)$$

$$D(A) = \{y \in H_0^1(\Omega); Ay \in L^2(\Omega)\}. \quad (5.42)$$

By Lemma 4.1, for $\lambda < H_N$ and $\omega > a_0$, it follows that $-A$ is ω - m -accretive on $L^2(\Omega)$ and self-adjoint, so that A generates a C_0 compact semigroup e^{At} on $L^2(\Omega)$. Moreover, as we shall see later, if $y \in D(A)$ then $y \in H^2(\Omega \setminus \{0\})$.

In order to write equation (5.32) in the operatorial form, we need some preliminaries. Let us consider the problem

$$\Delta \theta = 0 \text{ in } \Omega, \quad \theta = v \text{ on } \Gamma, \quad \text{for } t > 0. \quad (5.43)$$

The boundary condition is meant in the sense of the trace of θ on Γ , generally denoted by $tr(\theta)$. But, if any confusion is avoided we shall no longer indicate the trace by the symbol tr . The unique solution to this problem is the well-known Dirichlet map, $v \rightarrow \theta$, here denoted by $D_0 v$. If $v \in L^2(\Gamma)$, then $D_0 : L^2(\Gamma) \rightarrow H^{1/2}(\Omega)$ and it satisfies $\|D_0 v\|_{H^{1/2}(\Omega)} \leq C \|v\|_{L^2(\Gamma)}$ (see e.g. [19]).

In our case, $v = \tilde{u} \in L^2(\mathbb{R}_+; L^2(\Gamma))$ and so $D_0 \tilde{u}(t) \in H^{1/2}(\Omega)$ and

$$\|D_0 \tilde{u}(t)\|_{H^{1/2}(\Omega)} \leq C \|\tilde{u}(t)\|_{L^2(\Gamma)}, \quad \text{a.e. } t > 0.$$

Moreover, since \tilde{u} is given by (5.36) and D_0 is linear we have

$$D_0\tilde{u}(t) = \sum_{j=1}^m u_j(t)D_0\alpha_j, \quad t > 0. \quad (5.44)$$

Let us introduce the operator

$$A_0 : D(A_0) = D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad A_0 y = \Delta y + \frac{\lambda y}{|x|^2}. \quad (5.45)$$

This operator is m -dissipative on $L^2(\Omega)$ by a similar proof as in Lemma 4.1. Let us determine the Dirichlet mapping $v \rightarrow Dv$ corresponding to A_0 , that is

$$\Delta Dv + \frac{\lambda Dv}{|x|^2} = 0 \text{ in } \Omega, \quad Dv = v \text{ on } \Gamma. \quad (5.46)$$

Lemma 5.1 *For $\lambda < H_N$, Dv associated to A_0 exists and it is unique for $v \in L^2(\Gamma)$ satisfying $\frac{D_0 v}{x} \in L^2(\Omega)$. Moreover, one has*

$$Dv \in H^{1/2}(\Omega) \text{ and } \|Dv\|_{H^{1/2}(\Omega)} \leq C \left(\|v\|_{L^2(\Gamma)} + \left\| \frac{D_0 v}{x} \right\|_{L^2(\Omega)} \right). \quad (5.47)$$

Proof. Let t be fixed and denote $\varphi = Dv - D_0 v$ and consider the equation

$$\Delta \varphi + \frac{\lambda \varphi}{|x|^2} = -\frac{\lambda D_0 v}{|x|^2} \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Gamma. \quad (5.48)$$

We assert that problem (5.48) has a unique solution in $D(A)$ and prove it via a variational technique, by showing that the solution to (5.48) is given by the minimization of the functional $\Psi(\varphi)$,

$$\min_{\varphi \in H_0^1(\Omega)} \left\{ \Psi(\varphi) = \int_{\Omega} \left(\frac{1}{2} |\nabla \varphi|^2 - \frac{1}{2} \frac{\lambda \varphi^2}{|x|^2} - \frac{\lambda \varphi D_0 v}{|x|^2} \right) dx \right\}. \quad (5.49)$$

It is easily seen that

$$\left(\frac{1}{2} - \frac{\lambda}{H_N} \right) \int_{\Omega} |\nabla \varphi|^2 dx - \lambda \int_{\Omega} \left| \frac{D_0 v}{x} \right|^2 dx \leq \Psi(\varphi) < \infty,$$

so that Ψ has an infimum d . We note here the necessity of the assumption $\frac{D_0 v}{x} \in L^2(\Omega)$. Next, we proceed as in Lemma 4.1 and show that $\varphi \in H_0^1(\Omega)$ is the unique weak solution to the equation (5.48). By (5.48) we note that by multiplying by φ we get

$$\|\nabla \varphi\|_2^2 + \frac{\lambda}{2} \left\| \frac{\varphi}{x} \right\|_2^2 \leq \frac{\lambda}{2} \left\| \frac{D_0 v}{x} \right\|_2^2.$$

Then, it follows that $Dv = \varphi + D_0 v$ which is the Dirichlet map for (5.46), has the properties $Dv \in H^{1/2}(\Omega)$, $\frac{Dv}{x} = \frac{\varphi}{x} + \frac{D_0 v}{x} \in L^2(\Omega)$ and $\|Dv\|_{H^{1/2}(\Omega)} \leq \|\varphi + D_0 v\|_{H^{1/2}(\Omega)} \leq C \left(\|\varphi\|_{H_0^1(\Omega)} + \|D_0 v\|_{H^{1/2}(\Omega)} \right)$, implying (5.47). ■

Lemma 5.1 implies that the operator $D : L^2(\Gamma) \rightarrow L^2(\Omega)$ with the domain $\{v \in L^2(\Gamma); \frac{D_0 v}{x} \in L^2(\Omega)\}$ is closed and densely defined. We denote by $D^* : L^2(\Omega) \rightarrow L^2(\Gamma)$ its adjoint.

Now, we can write the operatorial form of the system. Let $u = (u_1, \dots, u_m)$ and assume for the beginning that

$$u \in W^{1,2}(0, T; \mathbb{R}^m), \quad w \in W^{1,2}(0, T; L^2(\Omega)), \quad T \geq 0$$

and note that $D\tilde{u}(t)$ is well defined due to (5.37), $D\tilde{u}(t) \in H^{1/2}(\Omega)$ and

$$\|D\tilde{u}(t)\|_{H^{1/2}(\Omega)} \leq C \sum_{j=1}^m |u_j(t)| \left(\|\alpha_j\|_{L^2(\Gamma)} + \left\| \frac{D_0 \alpha_j}{x} \right\|_{L^2(\Omega)} \right), \quad \text{a.e. } t > 0.$$

We and write the difference system (5.32) and (5.46),

$$\begin{aligned} & (y - D\tilde{u})_t - \Delta(y - D\tilde{u}) - \frac{\lambda(y - D\tilde{u})}{|x|^2} - a(x)(y - D\tilde{u}) \\ &= B_1 w - (D\tilde{u})_t + a(x)D\tilde{u}, \text{ in } (0, \infty) \times \Omega, \\ y - D\tilde{u} &= 0, \text{ on } (0, \infty) \times \Gamma, \quad (y - D\tilde{u})(0) = y_0 - \tilde{\theta}_0 \text{ in } \Omega, \end{aligned}$$

where $\tilde{\theta}_0 = D\tilde{u}(0)$. The solution to the previous system reads

$$(y - D\tilde{u})(t) = e^{At}(y_0 - \tilde{\theta}_0) + \int_0^t e^{A(t-s)}(B_1 w + aD\tilde{u})(s)ds - \int_0^t e^{A(t-s)}(D\tilde{u})_t(s)ds.$$

Integrating by parts the last right-hand side term we obtain

$$\begin{aligned} y(t) - D\tilde{u}(t) &= e^{At}y_0 - e^{At}\tilde{\theta}_0 + \int_0^t e^{A(t-s)}(B_1 w + a(x)D\tilde{u})(s)ds \\ &\quad - D\tilde{u}(t) + e^{At}\tilde{\theta}_0 - \int_0^t e^{A(t-s)}AD\tilde{u}(s)dx \end{aligned}$$

which yields

$$y(t) = e^{At}y_0 - \int_0^t e^{A(t-s)}AD\tilde{u}(s)dx + \int_0^t e^{A(t-s)}(B_1 w + a(x)D\tilde{u})(s)ds.$$

The formula is preserved by density if $u \in L^2(0, T; \mathbb{R}^m)$ and $w \in L^2(0, T; L^2(\Omega))$ and this represents the solution to the equation

$$y'(t) = Ay(t) + B_1 w(t) - AD\tilde{u}(t) + a(x)D\tilde{u}(t), \quad y(0) = y_0. \quad (5.50)$$

Since $D\tilde{u}(t)$ is not in $D(A)$ one must interpret $AD\tilde{u}(t)$ by using the extension \tilde{A} of A to the whole space $L^2(\Omega)$ by

$$\tilde{A} : L^2(\Omega) \rightarrow (D(A))', \quad \left\langle \tilde{A}y, \psi \right\rangle_{(D(A))', D(A)} = (y, A\psi), \quad \forall \psi \in D(A), \quad (5.51)$$

see (2.5). Now, we can define $B_2 : U \rightarrow (D(A))'$,

$$B_2 u = -\tilde{A} \left(\sum_{j=1}^m u_j D\alpha_j \right) + a(x) \sum_{j=1}^m u_j D\alpha_j = - \sum_{j=1}^m u_j A_0 D\alpha_j, \quad (5.52)$$

where $u = (u_1, \dots, u_m) \in U = \mathbb{R}^m$. Expression (5.52) is well defined since $D\alpha_j \in H^{1/2}(\Omega) \subset L^2(\Omega)$ and $a \in L^\infty(\Omega)$. Eventually, we can express equations (5.32)-(5.33) as

$$\begin{aligned} y'(t) &= Ay(t) + B_1 w(t) + B_2 \tilde{u}(t), \quad t \geq 0, \\ y(0) &= y_0 \end{aligned} \quad (5.53)$$

with \tilde{A} defined in (5.51), B_2 defined in (5.52) and \tilde{u} defined in (5.36).

(i₂) For verifying (3.1) we need to calculate B_2^* and D^* . We denote by $\frac{\partial v}{\partial \nu}$ the normal derivative of v on the boundary Γ . We give the following lemma.

Lemma 5.2 *The operator $B_2^* : D(A) \rightarrow \mathbb{R}^m$ is given by*

$$(B_2^* v)_j = - \left(\alpha_j, \frac{\partial v}{\partial \nu} \right)_{L^2(\Gamma)}, \quad \text{for } v \in D(A), \quad j = 1, \dots, m, \quad (5.54)$$

where $\frac{\partial v}{\partial \nu} \in L^2(\Gamma)$.

The operator $D^* : L^2(\Omega) \rightarrow L^2(\Gamma)$, is defined by

$$D^* p = \frac{\partial}{\partial \nu}(A_0^{-1} p) \text{ on } \Gamma, \quad \text{for } p \in L^2(\Omega). \quad (5.55)$$

Proof. We use the definition of B_2 and for $v \in D(A)$ we calculate

$$\begin{aligned}
\langle B_2 u, v \rangle_{(D(A))', D(A)} &= \left\langle -\tilde{A} \left(\sum_{j=1}^m u_j D\alpha_j \right) + a \sum_{j=1}^m u_j D\alpha_j, v \right\rangle_{(D(A))', D(A)} \\
&= - \sum_{j=1}^m (u_j D\alpha_j, Av)_{L^2(\Omega)} + \sum_{j=1}^m (u_j D\alpha_j, av)_{L^2(\Omega)} = \sum_{j=1}^m u_j (D\alpha_j, -Av + av)_{L^2(\Omega)} \\
&= u \cdot (D\alpha, -Av + av)_{L^2(\Omega)} = u \cdot (D\alpha, -A_0 v)_{L^2(\Omega)},
\end{aligned} \tag{5.56}$$

where $(D\alpha, A_0 v)_{L^2(\Omega)}$ denotes the vector with the components $(D\alpha_j, -A_0 v)_{L^2(\Omega)}$ for $v \in D(A)$. Here we took into account that $-Av + av = -A_0 v$ with A_0 defined in (5.45). Hence, we can define the components of $B_2^* : D(A) \rightarrow U^* = U = \mathbb{R}^m$ by

$$(B_2^* v)_j = (D\alpha_j, -A_0 v)_{L^2(\Omega)}, \quad v \in D(A), \quad j = 1, \dots, m. \tag{5.57}$$

For the computation of $(D\alpha_j, -A_0 v)_{L^2(\Omega)}$ let us consider the generic systems

$$\Delta D\beta + \frac{\lambda D\beta}{|x|^2} = 0, \quad D\beta = \beta \text{ on } \Gamma, \quad \beta \in L^2(\Gamma), \tag{5.58}$$

$$-\Delta v - \frac{\lambda v}{|x|^2} = p, \quad v = 0 \text{ on } \Gamma, \quad p \in L^2(\Omega). \tag{5.59}$$

The second system has a unique solution $v \in H_0^1(\Omega)$. In order to make a rigorous calculus we assume first that $\beta \in H^1(\Gamma)$ and $-A_0$ is replaced by for $\varepsilon > 0$ by

$$-A_{0,\varepsilon} = -\Delta - \frac{\lambda}{|x|^2 + \varepsilon}, \quad D(A_{0,\varepsilon}) = H^2(\Omega) \cap H_0^1(\Omega). \tag{5.60}$$

Thus, the equation $-A_{0,\varepsilon} v = p$ has a unique solution $v_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ and all operations below make sense. We multiply the approximating equation for $D\beta$ by the solution v_ε . By applying the Green's formula we obtain

$$\int_{\Omega} \left(D\beta \Delta v_\varepsilon + \frac{\lambda v_\varepsilon D\alpha_j}{|x|^2 + \varepsilon} \right) dx + \int_{\Gamma} \left(v_\varepsilon \frac{\partial D\beta}{\partial \nu} - D\beta \frac{\partial v_\varepsilon}{\partial \nu} \right) dx = 0$$

which implies, by using (5.59) and the boundary condition for $D\alpha_j$, that

$$- \int_{\Omega} p D\beta dx = \int_{\Gamma} \beta \frac{\partial v_\varepsilon}{\partial \nu} d\sigma, \quad \forall \beta \in L^2(\Gamma).$$

Therefore, we have for each $p \in L^2(\Omega)$

$$(D\beta, p)_{L^2(\Omega)} = \left(\beta, \frac{\partial}{\partial \nu} (A_{0,\varepsilon}^{-1} p) \right)_{L^2(\Gamma)} \quad \forall \beta \in H^1(\Gamma),$$

which can be written also as

$$(D\beta, -A_{0,\varepsilon} v_\varepsilon)_{L^2(\Omega)} = - \left(\beta, \frac{\partial v_\varepsilon}{\partial \nu} \right)_{L^2(\Gamma)} \quad \text{for } \beta \in H^1(\Gamma), \quad v_\varepsilon \in D(A_{0,\varepsilon}).$$

These remain true at limit as $\varepsilon \rightarrow 0$, hence

$$(D\beta, p)_{L^2(\Omega)} = \left(\beta, \frac{\partial}{\partial \nu} (A_0^{-1} p) \right)_{L^2(\Gamma)} \quad \text{for } \beta \in H^1(\Gamma), \quad p \in L^2(\Omega), \tag{5.61}$$

$$(D\beta, -A_0 v)_{L^2(\Omega)} = - \left(\beta, \frac{\partial v}{\partial \nu} \right)_{L^2(\Gamma)} \quad \text{for } \beta \in H^1(\Gamma), \quad v \in D(A_0) = D(A) \tag{5.62}$$

and the latter makes sense since $\frac{\partial v}{\partial \nu} \in H^{-1/2}(\Gamma)$. We note that both $A_{0,\varepsilon}$ and A_0 are surjective, because they are m -accretive and coercive. Then, by (5.61) we can define $D^* : L^2(\Omega) \rightarrow L^2(\Gamma)$, by (5.55).

Going back to (5.57) and using (5.62) in which we set $\beta := \alpha_j$ it turns out that we can define $B_2^* : D(A) \rightarrow U$ by

$$(B_2^*v)_j = (D\alpha_j, -A_0v)_{L^2(\Gamma)} = - \left(\alpha_j, \frac{\partial v}{\partial \nu} \right)_{L^2(\Gamma)}, \text{ for } v \in D(A). \quad (5.63)$$

It remains to show that $\frac{\partial v}{\partial \nu}$ belongs to $L^2(\Gamma)$ if $v \in D(A)$. Indeed, there exists $(v_\varepsilon)_\varepsilon \subset H^2(\Omega) \cap D(A)$ such that $v_\varepsilon \rightarrow v$ strongly in $D(A)$, $\frac{\partial v_\varepsilon}{\partial \nu} \rightarrow \frac{\partial v}{\partial \nu}$ strongly in $H^{-1/2}(\Gamma)$ as $\varepsilon \rightarrow 0$ and

$$(B_2^*v_\varepsilon)_j = - \left(\alpha_j, \frac{\partial v_\varepsilon}{\partial \nu} \right)_{L^2(\Gamma)}. \quad (5.64)$$

We recall that $0 \in \Omega$. We consider $\varphi \in C^4(\overline{\Omega})$ defined by

$$\varphi(x) = \begin{cases} 0, & \text{if } x \in \Omega_\delta \\ 1, & \text{if } x \in \Omega \setminus \Omega_{2\delta} \end{cases}$$

where $\delta > 0$ is such that $\Omega_\delta = \{x \in \Omega; \|x\| < \delta\}$ and $0 \in \Omega_\delta$. The function $\varphi v_\varepsilon \in H^2(\Omega \setminus \Omega_{2\delta})$. Indeed, since $v_\varepsilon \in H^2(\Omega)$ it follows that there exists $f \in L^2(\Omega)$ such that $f = Av_\varepsilon$ and so $\Delta v_\varepsilon = f - \frac{\lambda v_\varepsilon}{|x|^2} \in L^2(\Omega \setminus \Omega_{2\delta})$. We have

$$\Delta(\varphi v_\varepsilon) = \varphi \Delta v_\varepsilon + 2\nabla \varphi \cdot \nabla v_\varepsilon + v_\varepsilon \Delta \varphi \in L^2(\Omega).$$

This together with the boundary condition $\varphi v_\varepsilon = 0$ on Γ implies that $\varphi v_\varepsilon \in H^2(\Omega \setminus \Omega_{2\delta})$ and so $v_\varepsilon \in H^2(\Omega \setminus \Omega_{2\delta})$, too, because $\varphi = 1$ on $\Omega \setminus \Omega_{2\delta}$. Consequently, $\frac{\partial v_\varepsilon}{\partial \nu} \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$. This is preserved by density nearby the boundary. Finally, (5.63) remains true by density for $\alpha_j \in L^2(\Gamma)$ and so this implies (5.54). ■

Now, we pass to the proof of (i_2) . Such a result is proved for the Laplace operator in [8], p. 320, Proposition 4.39, but here we give a complete different proof under our hypotheses.

To this end, we recall that $Ay = A_0y + ay$ with A_0 defined in (5.45) and consider the problem

$$\frac{dy}{dt}(t) + B_0y(t) - ay = 0, \text{ in } (0, T) \times \Omega, \quad y(0) = y_0 \in L^2(\Omega) \quad (5.65)$$

where

$$B_0 = -A_0, \quad B_0 = -\Delta - \frac{\lambda}{|x|^2}, \quad B_0 : D(B_0) = D(A_0) \rightarrow L^2(\Omega). \quad (5.66)$$

The operator B_0 is m -accretive, $B_0 = B_0^*$ and $B_0 - aI$ is ω - m -accretive. The unique solution to problem (5.65) has also the property $y(t) \in D(A) = D(A_0)$ a.e. $t \in (0, T)$ by the regularizing effect (see [7], p. 158 Theorem 4.11).

First, we determine two estimates. We multiply equation (5.65) first by $y(t)$ and integrate over $(0, t)$. We obtain, using Gronwall's lemma

$$\|y(t)\|_2^2 + \int_0^t (B_0y(s), y(s))_2 ds = C_T \|y_0\|_2^2, \quad \forall t \in [0, T]. \quad (5.67)$$

Then, we multiply (5.65) by $tB_0y(t)$ which yields

$$\frac{1}{2} \frac{d}{dt} (tB_0y(t), y(t))_2 + t \|B_0y(t)\|_2^2 = \frac{1}{2} (B_0y(t), y(t))_2 + (ay(t), B_0y(t))_2. \quad (5.68)$$

We integrate this and by (5.67) we get

$$t(B_0y(t), y(t))_2 + \int_0^t s \|B_0y(s)\|_2^2 ds \leq C \int_0^t (B_0y(s), y(s))_2 ds \leq C_T \|y_0\|_2^2. \quad (5.69)$$

To prove (i₂) we have to estimate

$$\begin{aligned} \|B_{2,\varepsilon}^* e^{At} y_0\|_{\mathbb{R}^m} &= \|B_{2,\varepsilon}^* y(t)\|_{\mathbb{R}^m} = \left\| \left(- \left(\alpha_j, \frac{\partial y(t)}{\partial \nu} \right)_{L^2(\Gamma)} \right)_{j=1}^m \right\|_{\mathbb{R}^m} \\ &\leq \sum_{j=1}^m \|\alpha_j\|_{L^2(\Gamma)} \left\| \frac{\partial y(t)}{\partial \nu} \right\|_{L^2(\Gamma)}, \end{aligned} \quad (5.70)$$

thus, actually we have to estimate $\left\| \frac{\partial y(t)}{\partial \nu} \right\|_{L^2(\Gamma)}$ for $t > 0$. Since we shall relate this to the fractional powers of the operator B_0 , for a rigorous computation involving its fractional powers we shall rely again on the approximation, $B_{0,\varepsilon} = -A_{0,\varepsilon}$, see (5.60). We proceed with all calculations for the approximating equation (5.65) with $B_{0,\varepsilon}$ instead of B_0 and pass to the limit at the end. Thus, $D(B_{0,\varepsilon}) = H^2(\Omega) \cap H_0^1(\Omega)$, $B_{0,\varepsilon} : D(B_{0,\varepsilon}) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ and it is m -accretive and self-adjoint.

Therefore, we recall that the fractional powers are defined by $B_{0,\varepsilon}^s : D(B_{0,\varepsilon}^s) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, $s \geq 0$, see [24]. Then, $D(B_{0,\varepsilon}^s) \subset H^{2s}(\Omega)$ with equality iff $2s < 3/2$, see e.g., [17]. We have the interpolation inequality

$$\|B_{0,\varepsilon}^s w\|_2 \leq C \|B_{0,\varepsilon}^{s_1} w\|_2^\lambda \|B_{0,\varepsilon}^{s_2} w\|_2^{1-\lambda}, \quad \text{for } s = \lambda s_1 + (1-\lambda)s_2, \quad (5.71)$$

and the relations

$$\|B_{0,\varepsilon}^s w\|_2 \leq C \|B_{0,\varepsilon}^{s_1} w\|_2 \quad \text{if } s < s_1, \quad (5.72)$$

$$\|B_{0,\varepsilon}^s w\|_{H^m(\Omega)} \leq C \|B_{0,\varepsilon}^{s+m/2} w\|_2. \quad (5.73)$$

Now, we come back to $\frac{\partial y}{\partial \nu}(t)$ and using the trace theorem and (5.73) applied to $B_{0,\varepsilon}$ we write for the approximating solution

$$\left\| \frac{\partial y_\varepsilon}{\partial \nu}(t) \right\|_{L^2(\Gamma)} \leq C \|y_\varepsilon(t)\|_{H^{3/2}(\Omega)} \leq C \|B_{0,\varepsilon}^{3/4} y(t)\|_{L^2(\Omega)}, \quad (5.74)$$

so that we must estimate $\|B_{0,\varepsilon}^{3/4} y(t)\|_H$.

Next, we use (5.71) and write

$$\|B_{0,\varepsilon}^{3/4} y_\varepsilon(t)\|_2 \leq C \|B_{0,\varepsilon} y_\varepsilon(t)\|_2^{3/4} \|y_\varepsilon(t)\|_2^{1/4}. \quad (5.75)$$

Further, we calculate via Hölder's inequality

$$\begin{aligned} \int_0^t \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^{3/4} ds &= \int_0^t s^p \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^{3/4} s^{-p} ds \\ &\leq \left(\int_0^t s^{8p/3} \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^2 ds \right)^{3/8} \left(\int_0^t s^{-8p/5} ds \right)^{5/8} \\ &= \left(\int_0^t s \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^2 ds \right)^{3/8} \left(\int_0^t s^{-3/5} ds \right)^{5/8} \\ &\leq C \left(\int_0^t s \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^2 ds \right)^{3/8} (t^{2/5})^{5/8}, \end{aligned} \quad (5.76)$$

where we chose $p = \frac{3}{8}$. This together with (5.70), (5.74), (5.75) and (5.67) implies

$$\begin{aligned} \int_0^t \|B_{2,\varepsilon}^* e^{As} y_0\|_{\mathbb{R}^m} ds &\leq C \int_0^t \left\| \frac{\partial y_\varepsilon}{\partial \nu}(s) \right\|_{L^2(\Gamma)} ds \leq C \int_0^t \|B_{0,\varepsilon} y_\varepsilon(s)\|_{L^2(\Omega)}^{3/4} ds \\ &\leq C \int_0^t \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^{3/4} \|y_\varepsilon(s)\|_2^{1/4} ds \leq C_T \|y_0\|_2^{1/4} \int_0^t \|B_{0,\varepsilon} y_\varepsilon(s)\|_2^{3/4} ds \\ &\leq C_T \|y_0\|_2^{1/4} \|y_0\|_2^{3/4} (t^{2/5})^{5/8} \leq C_T \|y_0\|_2, \quad \forall t \in [0, T]. \end{aligned} \quad (5.77)$$

Passing to the limit by recalling (5.62) we get (i_2) as claimed.

This hypothesis has also an important consequence. We note that (5.53) with the initial condition $y(0) = y_0 \in L^2(\Omega)$ has a unique solution $y \in C([0, T]; (D(A))')$,

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}(B_1w(s) + B_2u(s))ds, \quad t \in [0, \infty). \quad (5.78)$$

We are going to show first that (i_2) ensures in addition that $y \in L^2(0, T; L^2(\Omega))$.

Actually, we shall prove the following assertion: if (3.1) takes place then the solution y to (5.53) belongs to $L^2(0, T; L^2(\Omega))$ if $u \in L^2(0, T; U)$. Since in (5.78) the sum between the first and the last term corresponding to the contribution of w is already in $C([0, T]; L^2(\Omega))$ we focus only on the term $Y(t) := \int_0^t e^{A(t-s)}B_2u(s)ds$ and show as in (3.5) that $\|Y\|_{L^2(0, T; L^2(\Omega))} \leq C \|u\|_{L^2(0, T; U)}$. In conclusion, equation (5.53) with the initial condition $y_0 \in L^2(\Omega)$ has a mild solution $y \in L^2(0, T; L^2(\Omega))$.

(i_3) The first part of hypothesis (i_3) , that is the detectability of the pair (A, C_1) follows as in Lemma 4.2. Now we prove (3.2). We recall that

$$A_1y = A_0y + a_0\chi_{\Omega_0}(x)y - k\chi_{\Omega_C}(x)y$$

with A_0 defined in (5.45) and consider the problem

$$\frac{dy}{dt}(t) + B_0y(t) = a_0\chi_{\Omega_0}(x)y - k\chi_{\Omega_C}(x)y, \quad \text{in } (0, T) \times \Omega, \quad y(0) = y_0 \in L^2(\Omega) \quad (5.79)$$

where $B_0 = -A_0$ is m -accretive, $B_0 = B_0^*$ and A_1 is m -accretive. Then, problem (5.79) has a unique solution $y(t) = S_1(t)y_0$, where $S_1(t)$ is the C_0 -semigroup generated by A_1 . The solution $y \in L^2(0, T; H_0^1(\Omega))$ and $y(t) \in D(A)$ a.e. $t \in (0, T)$.

Since $A_1 = A + KC_1$ generates an exponentially stable semigroup we have

$$\|y(t)\|_2 \leq e^{-\alpha t} \|y_0\|_2, \quad \alpha = k - a_0. \quad (5.80)$$

Moreover, $S_1(t)$ is analytic and so

$$\|A_1y(t)\|_2 \leq \frac{C_T}{t} \|y(t)\|_2, \quad \forall t \in (0, T). \quad (5.81)$$

Since $\|B_0y\|_H \leq \|A_1y\|_H + C \|y\|_H$ it follows that

$$\|B_0y(t)\|_2 \leq \frac{C_T}{t} \|y(t)\|_2, \quad \forall t \in (0, T). \quad (5.82)$$

The previous calculations for proving point (i_2) hold here too, and by (5.77) we have

$$\begin{aligned} \left\| B_2^* e^{(A+KC_1)t} y_0 \right\|_{\mathbb{R}^m} &= \left\| B_2^* y(t) \right\|_{\mathbb{R}^m} = \left\| \left(- \left(\alpha_j, \frac{\partial y(t)}{\partial \nu} \right)_{L^2(\Gamma)} \right)_{j=1}^m \right\|_{\mathbb{R}^m} \\ &\leq \sum_{j=1}^m \|\alpha_j\|_{L^2(\Gamma)} \left\| \frac{\partial y(t)}{\partial \nu} \right\|_{L^2(\Gamma)} \leq C \|B_0y(t)\|_2^{3/4} \|y_0\|_2^{1/4} e^{-\alpha t/4}, \end{aligned} \quad (5.83)$$

where $y(t) = S_1(t)y_0$ is the solution to (5.79). Thus,

$$\int_0^T \left\| B_2^* e^{(A+KC_1)t} y_0 \right\|_{\mathbb{R}^m} dt \leq C_T \|y_0\|_2, \quad \text{for } T \geq 0. \quad (5.84)$$

On the other hand, for $t > T$ we have

$$\|A_1y(t)\|_2 = \|A_1S_1(T)S_1(t-T)y_0\|_2 \leq \frac{C_T}{T} \|S_1(t-T)y_0\|_2 \leq \frac{C_T}{T} e^{-\alpha(t-T)} \|y_0\|_2.$$

Then we calculate

$$\begin{aligned} & \|B_0 y(t)\|_H^{3/4} \leq (\|A_1 y(t)\|_H + C \|y(t)\|_H)^{3/4} \leq C \|A_1 y(t)\|_H^{3/4} + C \|y(t)\|_H^{3/4} \\ & \leq \frac{C_T}{T^{3/4}} e^{-3\alpha(t-T)/4} \|y_0\|_2^{3/4} + C \|y_0\|_2^{3/4}, \end{aligned}$$

hence, by (5.83)

$$\begin{aligned} \left\| B_2^* e^{(A^* + KC_1)t} y_0 \right\|_U & \leq \left(\frac{C_T}{T^{3/4}} e^{-3\alpha(t-T)/4} + 1 \right) \|y_0\|_2^{3/4} \|y_0\|_2^{1/4} e^{-\alpha t/4} \\ & = \left(\frac{C_T}{T^{3/4}} e^{-3\alpha(t-T)/4} + e^{-\alpha t/4} \right) \|y_0\|_2, \text{ for } t > T. \end{aligned} \quad (5.85)$$

In particular, let $T = 1$ and by (5.84) and (5.85) we finally get

$$\begin{aligned} & \int_0^\infty \left\| B_2^* e^{(A^* + KC_1)t} y_0 \right\|_U dt \\ & = \int_0^1 \left\| B_2^* e^{(A^* + KC_1)t} y_0 \right\|_U dt + \int_1^\infty \left\| B_2^* e^{(A^* + KC_1)t} y_0 \right\|_U dt \\ & \leq C_1 \|y_0\|_2 + \|y_0\|_2 \int_1^\infty \left(C_1 e^{-3\alpha(t-1)/4} + e^{-\alpha t/4} \right) dt \leq C \|y_0\|_2, \end{aligned} \quad (5.86)$$

for all $y_0 \in L^2(\Omega)$. In conclusion, we have obtained (3.2) as claimed.

(i₄) The adjoint of D_1 is $D_1^* : L^2(\Omega) \rightarrow \mathbb{R}^m$

$$D_1^* v = \left(\int_\Omega d_1(x) v(x) dx, \dots, \int_\Omega d_m(x) v(x) dx \right).$$

Then, by (5.40), $\|D_1 u\|_{L^2(\Omega)}^2 = \int_\Omega \left(\sum_{j=1}^m d_j(x) \right)^2 dx = 1$, and $\int_\Omega d_j(x) \chi_{\Omega_C}(x) y dx = 0$, hence $D_1^* C_1 y(\xi) = 0$.

Then, calculating the operators in (3.9) we see that formulae (4.23), (4.25)-(4.26) are the same and using (5.63) we get

$$PB_2 B_2^* P \varphi(x) = \int_\Omega \varphi(\xi) \left(\sum_{j=1}^m A_j(\xi) A_j(x) \right) d\xi$$

where

$$A_j(\xi) = \int_\Gamma \alpha_j(\sigma) \frac{\partial P_0}{\partial \nu_\sigma}(\sigma, \xi) d\sigma.$$

Proceedings with all calculations as in Section 4 we have

Theorem 5.3 *Let $\gamma > 0$ and let A , B_1 , C_1 and D_1 be given by (5.41) and (5.39), respectively and B_2 , B_2^* be given by (5.52) and (5.54). Assume that $P_0 \in D(A) \times D(A)$ is a solution to equation*

$$\begin{aligned} & \Delta_x P_0(x, \xi) + \Delta_\xi P_0(x, \xi) + \lambda P_0(x, \xi) \left(\frac{1}{|x|^2} + \frac{1}{|\xi|^2} \right) + (a(x) + a(\xi)) P_0(x, \xi) \\ & - \sum_{j=1}^m A_j(x) A_j(\xi) + \gamma^{-2} \int_\Omega \chi_{\omega_1}(\bar{\xi}) P_0(x, \bar{\xi}) P_0(\bar{\xi}, \xi) d\bar{\xi} \\ & = -\delta(x - \xi) \chi_{\Omega_C}(\xi), \text{ in } \mathcal{D}'(\Omega \times \Omega), \end{aligned} \quad (5.87)$$

with conditions (4.28)-(4.30). Then, the feedback control $\tilde{F} \in L(L^2(\Omega), \mathbb{R}^m)$,

$$(\tilde{F}y)_j = \int_\Omega y(\xi) \left(\alpha_j, \frac{\partial P_0}{\partial \nu}(\cdot, \xi) \right)_{L^2(\Gamma)} d\xi, \quad j = 1, \dots, m, \quad \forall y \in L^2(\Omega) \quad (5.88)$$

solves the H^∞ -problem.

In this case, by (5.51), (5.52) and (5.57) we have

$$\Lambda_P y = A_0 \left(y + \int_{\Omega} y(\xi) \sum_{j=1}^m \left(\int_{\Gamma} \alpha_j(\sigma) \frac{\partial P_0}{\partial \nu_{\sigma}}(\sigma, \xi) d\sigma \right) D\alpha_j d\xi \right) + ay + \chi_{\omega_1} \int_{\Omega} P_0(x, \xi) y(\xi) d\xi$$

and we see that

$$D(\Lambda_P) = \left\{ y \in H; y + \int_{\Omega} y(\xi) \sum_{j=1}^m \left(\int_{\Gamma} \alpha_j(\sigma) \frac{\partial P_0}{\partial \nu_{\sigma}}(\sigma, \xi) d\sigma \right) D\alpha_j d\xi \in D(A) \right\}.$$

Moreover, Λ_P is closed because if $y_n \rightarrow y$ in H , since A_0 is closed we see that $\Lambda_P y_n \rightarrow \Lambda_P y$ in H . Then, by Lemma 3.5 we deduce that $\mathcal{X} = D(\Lambda_P)$.

6 Dirichlet boundary control in an 1D domain with a boundary singularity

We briefly discuss here the H^{∞} -boundary control problem for an one-dimensional parabolic equation with the singularity on the boundary. Namely, let $\Omega = (0, 1)$ and consider the system

$$y_t - \Delta y - \frac{\lambda y}{|x|^2} - a(x)y = B_1 w, \quad \text{in } (0, \infty) \times \Omega, \quad (6.1)$$

$$y(t, 0) = 0, \quad y(t, 1) = u \quad \text{for } t \geq 0, \quad (6.2)$$

$$y(0) = y_0, \quad \text{in } \Omega, \quad (6.3)$$

$$z = C_1 y + D_1 u, \quad \text{in } (0, \infty) \times \Omega, \quad (6.4)$$

where $y_0 \in L^2(\Omega)$, $u \in \mathbb{R}$.

(i_1) For this problem we choose $H = W = Z = L^2(\Omega)$, $U = \mathbb{R}$,

$$B_1 w = \chi_{\omega_1}(x)w, \quad C_1 y = \chi_{\Omega_C}(x)y, \quad D_1 u = d(x)u, \quad x \in \Omega, \quad (6.5)$$

with the conditions $\omega_1 \sqsubseteq \Omega$, $\Omega_0 \subset \Omega_C$, and

$$d \in L^2(\Omega), \quad d(x) = 0 \text{ on } \Omega_C, \quad \int_{\Omega \setminus \Omega_C} d^2(x) dx = 1. \quad (6.6)$$

Thus, $B_1 \in L(L^2(\Omega), L^2(\Omega))$, $C_1 \in L(L^2(\Omega), L^2(\Omega))$ and $D_1 : U \rightarrow L^2(\Omega)$.

We deal again with the operator $A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, $Ay = \Delta y + \frac{\lambda y}{|x|^2}$, which is ω - m -accretive on $L^2(\Omega)$ and generates a compact C_0 -semigroup on $L^2(\Omega)$. The difference here is that in the calculus of the accretivity of $-A$ we use the Hardy inequality (2.17) instead of (2.16). Next, we define

$$B : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, \quad Bu = (0, u) \quad (6.7)$$

and consider problem $\Delta \theta = 0$, $\theta = Bu$ on $\Gamma = \{0, 1\}$ which provides the Dirichlet map $D_0 u$, associated to Δ and Bu , expressed in this case by

$$D_0 u = ux. \quad (6.8)$$

Next, the problem

$$\Delta Du + \frac{\lambda Du}{|x|^2} = 0, \quad Du = Bu \text{ on } \Gamma, \quad (6.9)$$

provides the Dirichlet map associated to A_0 defined in (5.45). Making the difference $\varphi = Du - D_0 u$ we write the equation

$$\Delta \varphi + \frac{\lambda \varphi}{|x|^2} = -\frac{\lambda u}{x}, \quad \varphi = 0 \text{ on } \Gamma.$$

By a similar calculus as in Lemma 5.1, where we note that in this case while solving (5.49) we have

$$\left(\frac{1}{2} - \frac{\lambda}{H_N}\right) \int_{\Omega} |\nabla\varphi|^2 dx - |u|^2 \leq \Psi(\varphi) < \infty,$$

we deduce that Ψ has a minimum. Thus, we find that $\varphi \in H_0^1(\Omega)$, $\frac{\varphi}{x} \in L^2(\Omega)$ and

$$Du = \varphi + ux \in H^1(\Omega), \quad \frac{Du}{x} \in L^2(\Omega). \quad (6.10)$$

We define

$$B_2 : U = \mathbb{R} \rightarrow L^2(\Omega), \quad B_2 u = -\tilde{A}Du + a(x)Du = -uA_0D(0,1) \quad (6.11)$$

where \tilde{A} is defined as in (5.51) and $D(0,1)$ is the Dirichlet map corresponding to the boundary data $y(t,0) = 1$, $y(t,1) = 1$. Then, $B_2^* : D(A) \rightarrow \mathbb{R}$ and Lemma 5.2 implies that

$$B_2^* v = -v'(1), \quad v \in D(A), \quad D^* p = p'(1), \quad (6.12)$$

where $D^* : L^2(\Omega) \rightarrow \mathbb{R}$. We recall that p is in H^2 in the neighborhood of the boundary $x = 1$.

Hypotheses (i_2) , (i_3) and (i_4) are proved as in Section 5.

Finally, we calculate the term $PB_2B_2^*P\varphi(x)$, the other terms being the same as in the previous sections,

$$PB_2B_2^*P\varphi(x) = \int_{\Omega} \int_{\Omega} \frac{\partial P_0}{\partial x}(1, \xi) \frac{\partial P_0}{\partial \xi}(x, 1) \varphi(\xi) d\xi$$

and replacing in (3.9) we get

Theorem 6.1 *Let $\gamma > 0$ and let A , B_1 , C_1 and D_1 be given by (5.41) and (6.5), respectively and B_2 , B_2^* be given by (6.7) and (6.12). Assume that $P_0 \in D(A) \times D(A)$ is a solution to equation*

$$\begin{aligned} & \Delta_x P_0(x, \xi) + \Delta_{\xi} P_0(x, \xi) + \lambda P_0(x, \xi) \left(\frac{1}{|x|^2} + \frac{1}{|\xi|^2} \right) + (a(x) + a(\xi)) P_0(x, \xi) \\ & + \int_{\Omega} \frac{\partial P_0}{\partial x}(1, \xi) \frac{\partial P_0}{\partial \xi}(x, 1) d\xi + \gamma^{-2} \int_{\Omega} \chi_{\omega_1}(\bar{\xi}) P_0(x, \bar{\xi}) P_0(\bar{\xi}, \xi) d\bar{\xi} \\ & = -\delta(x - \xi) \chi_{\Omega_C}(\xi), \quad (x, \xi) \in \Omega \times \Omega, \end{aligned} \quad (6.13)$$

with the boundary conditions $P_0(x, 0) = P_0(x, 1) = 0$ for $x \in (0, 1)$ and by symmetry $P_0(0, \xi) = P_0(1, \xi) = 0$. Then, the feedback control $\tilde{F} \in L(D(A), \mathbb{R})$,

$$\tilde{F}y = \int_{\Omega} y(\xi) \frac{\partial P_0}{\partial x}(1, \xi) d\xi, \quad y \in L^2(\Omega) \quad (6.14)$$

solves the H^∞ -problem.

In this case

$$\Lambda_P y = A_0 \left(y + \int_{\Omega} \frac{\partial P_0}{\partial x}(1, \xi) D(0, 1) y(\xi) d\xi \right) + ay + \chi_{\omega_1} \int_{\Omega} P_0(x, \xi) y(\xi) d\xi$$

which is closed, so that

$$\mathcal{X} = D(\Lambda_P) = \left\{ y \in L^2(\Omega); y + \int_{\Omega} \frac{\partial P_0}{\partial x}(1, \xi) D(0, 1) y(\xi) d\xi \in D(A) \right\}.$$

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