One-quasihomomorphisms from the integers into symmetric matrices

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ABSTRACT

A function f from $\mathbb{Z}$ to the symmetric matrices over an arbitrary field K of characteristic 0 is a 1-quasihomomorphism if the matrix $f(x+y)-f(x)-f(y)$ has rank at most 1 for all $x, y \in \mathbb{Z}$. We show that any such 1-quasihomomorphism has distance at most 2 from an actual group homomorphism. This gives a positive answer to a special case of a problem posed by Kazhdan and Ziegler.

KEYWORDS

Quasihomomorphisms, rank metric, linear approximation

1 INTRODUCTION

We continue the program initiated in [1] of studying particular instances of a problem posed by Kazhdan and Ziegler in their work on approximate cohomology [2]. We are given a function f that behaves roughly like a homomorphism, in the following manner.

Definition 1.1. Let $(H, +)$ be an abelian group. A *norm* on H is a map $\|\cdot\| : H \to \mathbb{R}$ such that

- $\|x\| \geq 0$ for all $x \in H$, with equality if and only if $x = 0$,
- $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in H$,
- $\|-x\| = \|x\|$ for all $x \in H$.

Note that equipping H with a norm is equivalent to equipping it with an equivariant metric d, that is, a metric such that $d(x, y) = d(x+z, y+z)$ for all $x, y, z \in H$; the connection is given by $d(x, y) = \|x - y\|$.

Definition 1.2. Let $(G, +)$ and $(H, +)$ be abelian groups, where H is equipped with a norm $\|\cdot\|$. A map $f : G \to H$ is a *c-quasihomomorphism* (where $c \in \mathbb{R}_{\geq 0}$) if for all $x, y \in \mathbb{Z}$ we have that

$$\|f(x+y) - f(x) - f(y)\| \leq c. \tag{1}$$

The natural question is whether every c-quasihomomorphism can be approximated by an actual group homomorphism.

Question 1.3. *Fix G, H and c. Does there exist a constant $C \in \mathbb{R}_{\geq 0}$ such that for every c-quasihomomorphism $f : G \to H$, there exists a group homomorphism $\varphi : G \to H$ such that*

$$\forall x \in G : \quad \|f(x) - \varphi(x)\| \leq C.$$

A variant of this question, where $G = H$ and G can be nonabelian, was asked already by Ulam [3, Chapter VI.1] in 1960. Our case of interest is when $G = \mathbb{Z}$ is the additive group of integers, and H is the additive group of matrices over some field $\mathbb{K}$, where the norm is given by the rank. The argument from [1, Remark 1.11] shows that in this case the answer is affirmative for fields of positive characteristic. For the rest of the paper, we will fix a field $\mathbb{K}$ of characteristic 0. Note that every group morphism $\varphi : \mathbb{Z} \to H$ is of the form $\varphi(x) = x \cdot A$, where $A \in H$ is a fixed element.

Question 1.4. *Fix $c \in \mathbb{N}$. Does there exist a constant $C \in \mathbb{R}_{\geq 0}$ such that for every natural number n and every c-quasihomomorphism $f : \mathbb{Z} \to \mathrm{Mat}(n \times n, \mathbb{K})$, there exists a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{K})$ such that*

$$\forall x \in \mathbb{Z} : \quad \mathrm{rk}\,(f(x) - x \cdot A) \leq C.$$

This is the instance of Question 1.3 asked by Kazhdan and Ziegler. It was answered affirmatively in [1] under the assumption that f lands in the space of diagonal matrices and by choosing $C = 28c$. In this paper, we study the case $c = 1$. We are able to prove a much better bound than the predicted $C = 28$: indeed, the constant C can be chosen equal to 2. Moreover, we can weaken the assumption that f lands in the space of diagonal matrices.

Theorem 1.5. *Assume $\mathrm{char}(\mathbb{K}) = 0$ and let $\mathrm{Sym}(n \times n, \mathbb{K})$ be the space of symmetric matrices. If $f : \mathbb{Z} \to \mathrm{Sym}(n \times n, \mathbb{K})$ is assumed to be a 1-quasihomomorphism, there is an $A \in \mathrm{Sym}(n \times n, \mathbb{K})$ such that*

$$\mathrm{rk}\,(f(x) - x \cdot A) \leq 2 \quad \forall x \in \mathbb{Z}. \tag{2}$$

The rest of this paper is devoted to proving Theorem 1.5. The strategy is to prove that the sequence of consecutive differences $\Delta_f(i) = f(i+1) - f(i)$ satisfies two kinds of symmetries. One is a reflection symmetry in a local sense, which we call palindromicity. The other is a periodicity. By expressing f as a sum of deltas, and applying the symmetries, we arrive to the result.

2 LEMMAS ABOUT SYMMETRIC MATRICES

In this section we prove some elementary lemmas about symmetric matrices that we will use later during the proof. Let $(\cdot, \cdot)$ be the bilinear form on $\mathbb{K}^n$ given by

$$(x, y) := x_1 y_1 + \cdots + x_n y_n$$

for all $x, y \in \mathbb{K}^n$. Then an $n \times n$ matrix over $\mathbb{K}$ is symmetric if that $(Ax, y) = (x, Ay)$ for all $x, y \in \mathbb{K}^n$.

Lemma 2.1. *Let $A \in \mathrm{Sym}(n \times n, \mathbb{K})$. Then $\mathrm{im}(A) = \ker(A)^{\perp}$.*

Proof. Since our bilinear form is nondegenerate we see that

$$Ax = 0 \iff (Ax, y) = 0 \ \forall y \iff (x, Ay) = 0 \ \forall y.$$

Therefore,

$$x \in \ker(A) \iff x \perp \mathrm{im}(A),$$

which means that $\mathrm{im}(A) = \ker(A)^{\perp}$. □

Lemma 2.2. *Let A, B be symmetric matrices. Moreover, suppose that $\mathrm{im}(A) \cap \mathrm{im}(B) = 0$. Then $\mathrm{rk}\,(A+B) = \mathrm{rk}\,(A) + \mathrm{rk}\,(B)$.*

Proof. We always have inequalities

$$\begin{aligned}\mathrm{rk}\,(A+B) &= \dim\mathrm{im}(A+B)\\ &\le \dim(\mathrm{im}(A)+\mathrm{im}(B))\\ &\le \dim(\mathrm{im}(A))+\dim(\mathrm{im}(B))\\ &= \mathrm{rk}\,(A)+\mathrm{rk}\,(B)\,.\end{aligned}$$

Our assumption $\mathrm{im}(A)\cap\mathrm{im}(B)=0$ implies that the second "$\le$" is an equality. We show that the first "$\le$" is an equality as well. For this we need to show that $\mathrm{im}(A+B)=\mathrm{im}(A)+\mathrm{im}(B)$. Taking $\perp$ of both sides and applying Lemma 2.1, this is equivalent to showing $\ker(A+B)=\ker(A)\cap\ker(B)$. But this again follows from our assumption $\mathrm{im}(A)\cap\mathrm{im}(B)=0$:

$$\begin{aligned}v\in\ker(A+B) &\implies Av=-Bv \implies\\ Av=Bv=0 &\implies v\in\ker(A)\cap\ker(B).\end{aligned}$$

□

In fact, we will only need the following corollaries:

Corollary 2.3. *Let A,B be symmetric matrices. If $\mathrm{rk}\,(B)=1$ and $\mathrm{im}(B)\not\subset\mathrm{im}(A)$, then $\mathrm{rk}\,(A+B)=\mathrm{rk}\,(A)+1$.*

Proof. This is just the main claim for B of rank one. □

Corollary 2.4. *Let $A\in\mathrm{Sym}(n\times n,\mathbb{K})$ with $\mathrm{rk}\,(A)\le 2$. Assume there are three rank-1 symmetric matrices B_i ($i=1,2,3$) such that $\dim(\mathrm{im}(B_1)+\mathrm{im}(B_2)+\mathrm{im}(B_3))=3$ and $\mathrm{rk}\,(A-B_i)\le 1$ for $i=1,2,3$. Then $A=0$.*

Proof. Suppose by contradiction that $\mathrm{rk}\,(A)\ge 1$. Then

$$\mathrm{rk}\,(A-B_i)\le 1<2\le \mathrm{rk}\,(A)+\mathrm{rk}\,(B_i)\,,$$

thus by the contraposition of Lemma 2.2, it follows that $\mathrm{im}(B_i)\subseteq\mathrm{im}(A)$. However, this would imply that

$$3=\dim(\mathrm{im}(B_1)+\mathrm{im}(B_2)+\mathrm{im}(B_3))\le\dim(\mathrm{im}(A))=\mathrm{rk}\,(A)\le 2,$$

which is a contradiction. □

3 DELTA SEQUENCE

We begin by arguing that without loss of generality, we can assume that $f(1)=0$. This follows from the following observation.

Observation 3.1. Let H be a normed abelian group and $f:\mathbb{Z}\to H$ any function. If g is defined by

$$g(x)=f(x)+x\cdot C,$$

where $C\in H$, then:

- f is a 1-quasihomomorphism if and only if g is.
- We have that
$$\|f(x)-x\cdot A\|\le 2 \iff \|g(x)-x\cdot A'\|\le 2,$$
where $A'=A-C$

Hence, by choosing $C=-f(1)$, we see that proving Theorem 1.5 under the additional assumption $f(1)=0$ is enough to prove it in general.

From now on we always assume $f(1)=0$. This allows us to reformulate the condition of f being a 1-quasihomomorphism in terms of a difference operator on f.

Definition 3.2. Given a function $f:\mathbb{Z}\to H$, we define its *delta map* $\Delta_f(x):\mathbb{Z}\to H$ as

$$\Delta_f(x)=f(x+1)-f(x).$$

Remark 3.3. If $f(1)=0$, we can write f in terms of Δ_f:

$$f(x)=\sum_{i=1}^{x-1}\Delta_f(i)\text{ for }x\ge 1,\tag{3}$$

and

$$f(x)=-\sum_{i=0}^{x}\Delta_f(i)\text{ for }x\le 0.\tag{4}$$

△

Lemma 3.4. *Let $f:\mathbb{Z}\to H$ be a map with $f(1)=0$. The map f is a c-quasihomomorphism if and only if for all $k\in\mathbb{Z}_{\ge 0}$ and $z\in\mathbb{Z}$ we have*

$$\left\|\sum_{i=1}^{k}\Delta_f(i)-\sum_{i=0}^{k}\Delta_f(z-i)\right\|\le c,\tag{5}$$

$$\left\|\sum_{i=0}^{k}\Delta_f(-i)-\sum_{i=0}^{k-1}\Delta_f(z-i)\right\|\le c.\tag{6}$$

Proof. In essence, this is just plugging in Equations (3) and (4) into Equation (1). We present the proof in a slightly different way, to avoid doing case distinctions on the signs of x, y, and $x+y$. Calculate:

$$\begin{aligned}&\sum_{i=1}^{k}\Delta(i)-\sum_{i=0}^{k}\Delta(z-i)=\\ &\sum_{i=1}^{k}[f(i+1)-f(i)]-\sum_{i=0}^{k}[f(z-i+1)-f(z-i)]=\\ &f(k+1)+f(z-k)-f(z+1).\end{aligned}$$

By setting $x=k+1, y=z-k$, we see that Equation (5) holds if and only if the c-quasihomomorphism condition (1) is fullfilled for $x\in\mathbb{Z}_{\ge 1}$ and $y\in\mathbb{Z}$. Similarly, calculate:

$$\begin{aligned}&\sum_{i=0}^{k}\Delta(-i)-\sum_{i=0}^{k-1}\Delta(z-i)=\\ &\sum_{i=0}^{k}[f(-i+1)-f(-i)]-\sum_{i=0}^{k-1}[f(z-i+1)-f(z-i)]=\\ &-f(-k)-f(z+1)+f(z+1-k).\end{aligned}$$

By setting $x=-k, y=z+1$, we see that Equation (6) is equivalent to the quasihomomorphism condition for $x\in\mathbb{Z}_{\le 0}$ and $y\in\mathbb{Z}$, and we are done. □

In particular, Condition (5) for $k=0$ states that $\|\Delta(y)\|\le c$ for all $y\in\mathbb{Z}$.

Notation 3.5. For the rest of this paper, f will denote a 1-quasihomomorphism $\mathbb{Z}\to\mathrm{Sym}(n\times n,\mathbb{K})$ with $f(1)=0$; its delta map Δ_f will be denoted by Δ. We will denote $\mathrm{im}(\Delta(i))$ by L_i. Since $\mathrm{rk}\,(\Delta(i))\le 1$, we have that $\dim(L_i)\le 1$.

Note that if $\dim(\sum_{i\in\mathbb{Z}}L_i)\le 2$, then by (3) and (4) we also have $\mathrm{rk}\,(f(x))\le 2$ for all $x\in\mathbb{Z}$, and Theorem 1.5 is true with $A=0$. So from now on we will assume:

Assumption 3.6. $\dim(\sum_{i\in\mathbb{Z}}L_i)\ge 3$.

Then we can make the following observation.

Lemma 3.7. *If Assumption 3.6 holds, then* $\Delta(0) + \Delta(-1) = 0$.

Proof. Note that Equation (6) for $k = 1$ tells us that for all $z \in \mathbb{Z}$ we have

$$\operatorname{rk}(\Delta(0) + \Delta(-1) - \Delta(z)) \leq 1.$$

By Assumption 3.6, we can apply Corollary 2.4 to conclude that $\Delta(0) + \Delta(-1) = 0$. □

Observation 3.8. Still working under Assumption 3.6, now Equation (6) for $k \geq 0$ can be rewritten as

$$\operatorname{rk}\left(\sum_{i=2}^{k+1} \Delta(-i) - \sum_{i=0}^{k} \Delta(y - i)\right) \leq 1. \tag{7}$$

Note the symmetry: if we define $\tilde{\Delta}(x) := \Delta(-1-x)$, then Δ satisfies the assumptions (5) and (7) if and only if $\tilde{\Delta}$ does.

$$\dots \underbrace{\boxed{\Delta(-k-1) \ \cdots \ \Delta(-2)}}_{k} \ \Delta(-1) \ \Big| \ \Delta(0) \ \underbrace{\boxed{\Delta(1) \ \cdots \ \Delta(k)}}_{k} \ \dots \ \underbrace{\boxed{\Delta(z-k) \ \cdots \ \Delta(z)}}_{k+1} \ \dots$$

Figure 1: Equation (5) says that the sum of the right red block and the sum of the blue block differ by a rank one matrix. Similarly, Equation (6) says that the sum of the left red block and the sum of the blue block differ by a rank one matrix.

Next, note that if $\dim(\sum_{i\in\mathbb{Z}} L_i) \geq 3$ but $\dim(\sum_{i\in\mathbb{Z}\setminus\{0,-1\}} L_i) \leq 2$, it still holds that $\operatorname{rk}(f(x)) \leq 2$ for all $x \in \mathbb{Z}$. So we will replace Assumption 3.6 with something slightly stronger:

Assumption 3.9. $\dim(\sum_{i\in\mathbb{Z}\setminus\{0,-1\}} L_i) \geq 3$.

Under this assumption, we will show that Δ needs to have a very specific structure.

4 PALINDROMICITY

Now we show that Δ satisfies a property reminiscent of palindromes.

Notation 4.1. For $m \in \mathbb{N}$, write

$$V_m = L_{-m-1} + \cdots + L_{-2} + L_1 + \ldots + L_m. \tag{8}$$

Note that L_{-1}, L_0 are not part of the sum. Assumption 3.9 precisely says that there exists an m with $\dim V_m \geq 3$.

Lemma 4.2. *Let m be such that $V_m \supsetneq V_{m-1}$.*

(1) For all $i \in \{1, \ldots, m-1\}$ we have that

$$\Delta(i) = \Delta(m-i) = \Delta(-i-1) = \Delta(i-m-1). \tag{9}$$

(2) Moreover, if $\dim V_m \geq 3$, it holds that

$$\Delta(m+1) = -\Delta(m) \quad \text{and} \quad \Delta(-m-2) = -\Delta(-m-1).$$

In particular, $L_m = L_{m+1}$ and $L_{-m-2} = L_{-m-1}$.

Remark 4.3. To state Lemma 4.2 more visually: if $V_m \supsetneq V_{m-1}$ and $\dim V_m \geq 3$, then Δ has the following structure:

i	$-m-2$	$-m-1$			$\cdots$			-1	0			$\cdots$			m	$m+1$
$\Delta(i)$	α	$-\alpha$	a	b	$\cdots$	b	a	β	$-\beta$	a	b	$\cdots$	b	a	γ	$-\gamma$

△

Proof. For Item (1) we show 3 equalities for $i \in \{1, \ldots, m-1\}$:

- $\Delta(i) = \Delta(m-i)$, which encodes palindromicity of the right blue block;
- $\Delta(m-i) = \Delta(-i-1)$, which encodes equality of the blocks;
- $\Delta(-i-1) = \Delta(i-m-1)$, which encodes palindromicity of the left blue block.

Note that the third equality follows from the first two by substituting i for $m-i$. By symmetry (cfr. Observation 3.8) we may assume that $L_m \not\subset V_{m-1}$.

We first prove the identity $\Delta(i) = \Delta(m-i)$ by induction on i. For the base case $i = 1$, observe that setting $k = 1$ and $z = m$ in Equation (5) gives

$$\operatorname{rk}(\Delta(1) - \Delta(m-1) - \Delta(m)) \leq 1.$$

By Corollary 2.3 we get that $\Delta(1) = \Delta(m-1)$. For the case $i = 2$, we put $k = 2$ and $z = m$ in Equation (5):

$$\operatorname{rk}(\Delta(1) + \Delta(2) - \Delta(m-2) - \Delta(m-1) - \Delta(m)) \leq 1.$$

Using $\Delta(1) = \Delta(m-1)$ and Corollary 2.3 we find $\Delta(2) = \Delta(m-2)$.

One proceeds in a similar fashion for higher i. Namely, if the equality is true for i, one gets the equality for $i+1$ from Equation (5) by setting $k = i+1$ and $z = m$. The equality $\Delta(m-i) = \Delta(-i-1)$ is proven analogously, using Equation (7).

For Item (2), we want to show that $\Delta(m+1) + \Delta(m) = 0$. If i is in $\{1, \ldots m-1, m\}$, Equation (5) for $z = m+1$ and $k = i$, combined with (9), imply that

$$\operatorname{rk}(\Delta(m+1) + \Delta(m) - \Delta(i)) \leq 1.$$

When i is in $\{-m-1, -m, \ldots, -2\}$ the same equation can be derived from Equation (7) for $z = m+1$ and $k = -i-1$. Since $\dim V_m \geq 3$, by Corollary 2.4 this implies that $\Delta(m+1) + \Delta(m) = 0$. The proof that $\Delta(-m-2) + \Delta(-m-1) = 0$ is analogous. Finally we have that $L_m = \operatorname{im}(\Delta(m)) = \operatorname{im}(\Delta(m+1)) = L_{m+1}$, and analogously for the other one. □

5 APAP SEQUENCES

Now, our aim is to show that the finite pattern observed in Section 4 can be extended to infinity. We call a sequence satisfying this pattern APAP, meaning *almost periodic almost palindromic*. In this section, we define APAP sequences and prove some general lemmas; in the next section we will show that our delta sequence is APAP. For the purposes of this section, H can be any abelian group.

Definition 5.1. A sequence $(\Delta(i))_{i=-N}^{N-1}$, with $\Delta(i) \in H$ is APAP with period $p \in [2, N]$, if

$$\Delta(i+p) = \Delta(i) \qquad \text{if } i \not\equiv -1 \text{ or } 0 \mod p, \tag{10}$$

$$\Delta(j-1) + \Delta(j) = 0 \qquad \forall j \in \{-N+1, \ldots, N-1\} \text{ with } p|j, \tag{11}$$

$$\Delta(p-1-i) = \Delta(i) \qquad \forall i = 1, \ldots, p-2. \tag{12}$$

From now on we will refer to the respective Conditions (10), (11), (12). We call $\Delta(1), \ldots, \Delta(p-3), \Delta(p-2)$ the *palindromic block*, and will write B_Δ for the "block sum" $\Delta(1) + \cdots + \Delta(p-2)$.

Remark 5.2. The next two pictures illustrate how an APAP sequence looks like. First we see a global picture:

The blue box represents the palindromic block, whereas the red circles represent the p-cancellation. Each box has length $p-2$. Note that while the blue box is always meant to be the same, the red circles are not.

Next, we see the same picture but now zoomed in:

$$\begin{array}{c|cccccccc} i & (a-1)p-1 & (a-1)p & & & \cdots & & & ap-1 \\ \hline \Delta(i) & \alpha & -\alpha & a & b & \cdots & b & a & \beta \end{array}$$

$$\begin{array}{c|cccccccc} i & ap & & & \cdots & & & (a+1)p-1 & (a+1)p \\ \hline \Delta(i) & -\beta & a & b & \cdots & b & a & \gamma & -\gamma \end{array}.$$

In this picture we see the cancellation in red and the palindromic block in blue. △

The following result is a quick calculation that uses the three properties of being an APAP sequence.

Lemma 5.3. *Let $(\Delta(i))_{i=-N}^{N-1}$ be an APAP sequence with period p. For any $k \in \mathbb{Z}_{\geq 0}$, the sum of any kp consecutive elements in $(\Delta(i))_{i=-N}^{N-1}$, where the index of the first element is not a multiple of p is constant. Moreover, this constant equals $k \cdot B_\Delta$.* □

Our first source of APAP sequences is Lemma 4.2:

Lemma 5.4. *Let m be such that $V_m \supsetneq V_{m-1}$ and $\dim V_m \geq 3$. The sequence $(\Delta(i))_{-m-2}^{m+1}$ is APAP with period $m+1$. Moreover, for any other period p that makes this sequence APAP we have that $p|m+1$.*

Proof. Since $\dim V_m \geq 3$, the sequence $(\Delta(i))_{-m-2}^{m+1}$ is APAP with period $m+1$ by the two items of Lemma 4.2. Now suppose that $(\Delta(i))_{-m-2}^{m+1}$ is APAP with period p. Since $V_m \supsetneq V_{m-1}$ at least one of $L_m \not\subset V_{m-1}$ or $L_{-m-1} \not\subset V_{m-1}$ is true. By the symmetry from Observation 3.8 we assume the former.

Suppose that p does not divide $m+1$, so $m \not\equiv -1 \mod p$. If additionally we have that $m \not\equiv 0 \mod p$, then $\Delta(m) = \Delta(j)$ with j the residue of m divided by p. Since $j < p < m+1$, we get $\Delta(m) = \Delta(j)$ is in V_{m-1}, a contradiction. To finish, assume that $m \equiv 0 \mod p$, so then $\Delta(m-1) + \Delta(m) = 0$, which again implies that $\Delta(m)$ is in V_{m-1}, a contradiction. □

Next, we use the last claim from Lemma 5.4 to study how two distinct APAP structures on the same sequence interact. We apply this result in Claim 6.3.

Lemma 5.5. *Let $(\Delta(i))_{i=-N}^{N-1}$ be APAP with period p. Suppose there is a $q \in [2, p-1]$ such that*

(1) $\Delta(i) = \Delta(i+q)$ for $i = 1, \ldots, p-q-2$ (q-periodicity),
(2) $\Delta(i) = \Delta(q-1-i)$ for $i = 1, \ldots, q-2$ (palindromicity of the first $q-2$ elements),
(3) $\Delta(q-1) + \Delta(q) = 0$,

Write $g = \gcd(p, q)$. If $g > 1$ then Δ is APAP with period g. If $g = 1$ then all $\Delta(i)$ are the same up to a sign.

We will deduce this using the following easy number-theoretic lemma:

Lemma 5.6. *Let $q < p$ be integers and write $g = \gcd(p, q)$. Consider the equivalence relation $\sim$ on $\mathbb{Z}$ generated by:*

- *$x \sim y$ if $x \equiv y \mod q$ (q-periodic),*
- *$x \sim q-1-x$ for x in $\{0, \ldots, q-1\}$ (q-palindromic),*
- *$x \sim p-1-x$ for x in $\{0, \ldots, p-1\}$ (p-palindromic).*

Then we have that $x \sim y$ if and only if $x \equiv y \mod g$ or $x + y \equiv -1 \mod g$.

Proof of Lemma 5.6. We first show $\sim$ is also p-periodic. For this, take any $x \in \mathbb{Z}$, and let $m \in \mathbb{Z}$ be the unique integer for which $p - q \leq x - mq \leq p-1$. Indeed,

$$\begin{aligned} x &\sim x - mq \sim p-1-(x-mq) \sim q-1-(p-1-x+mq) \\ &= x - p - (m-1)q \sim x - p. \end{aligned}$$

In the previous calculation, $x - mq$ is contained in $\{0, \ldots, p-1\}$ and $p-1-(x-mq)$ is in $\{0, \ldots, q-1\}$, so the operations are valid. The combination of q-periodicity and p-periodicity is equivalent to g-periodicity, namely $x \sim y$ when $x \equiv y \mod g$. Additionally, palindromicity gives $x \sim y$ when $x + y \equiv -1 \mod g$. Indeed, by g-periodicity we may assume that x is in $\{1, \ldots, g-1\}$, then by q-palindromicity and periodicity we have that $x \sim q-1-x \sim -1-x \sim y$. □

Proof of Lemma 5.5. We first consider the case $g > 1$. Let us write $[a]_q$ for the unique integer in $\{1, \ldots, q\}$ that is congruent to a modulo q.

Claim 5.7. It suffices to check the APAP property on the interval $[1, q]$. In other words: if we verify the identities

(a) $\Delta(i) = \Delta(i+g)$ for $i \in \{1, \ldots, q-g-2\}$ with $i \not\equiv -1$ or $0 \mod g$,
(b) $\Delta(kg-1) + \Delta(kg) = 0$ for $k = 1, \ldots, q/g$,
(c) $\Delta(i) = \Delta(g-1-i)$ for $i = 1, \ldots, g-2$,

then Δ is APAP with period g.

Proof. For palindromicity there is nothing to prove. For periodicity: given any $i \neq -1$ or $0 \mod g$, we have

$$\Delta(i) = \Delta([i]_p) = \Delta([[i]_p]_q) = \Delta([i]_g),$$

where we used p-periodicity, q-periodicity, and (a).

Cancellation is similar: if $p|j$ then $\Delta(j) = -\Delta(j-1)$ by p-cancellation; if $g \mid j$ but $p \nmid j$ then we can use p-periodicity, q-periodicity, and (b) to find

$$\begin{aligned} \Delta(j-1) + \Delta(j) &= \Delta([j-1]_p) + \Delta([j]_p) \\ &= \Delta([[j-1]_p]_q) + \Delta([[j]_p]_q) = 0. \end{aligned}$$ □

We now verify the conditions (a), (b), (c) above. For this, we formally define the q-periodic map $\tilde{\Delta} : \mathbb{Z} \to H$ by $\tilde{\Delta}(i) = \tilde{\Delta}([i]_q)$. Since Δ and $\tilde{\Delta}$ agree on the interval $[1, q]$, by Claim 5.7 we may work now with $\tilde{\Delta}$ instead. We consider the equivalence relation $\sim$ from the previous lemma. Then showing (a) and (c) amounts to showing that $\tilde{\Delta}$ is constant on every equivalence class except for the one generated by 0. Indeed, two numbers x and y in the same equivalence class can be connected by a chain as in Lemma 5.6, and the only case this doesn't imply an equality of $\tilde{\Delta}$ is when $x = 0$, $q-1$, or $p-1$, but then we are in the bad equivalence class.

We are left with showing (b). For this, we in fact will prove the stronger claim that $\tilde{\Delta}(kg-1) = \tilde{\Delta}(kg) = 0$ for $k = 1, \ldots, q/g$. Viewing $\tilde{\Delta}$ as a map $\mathbb{Z}/q\mathbb{Z} \to H$, we claim that

$$\tilde{\Delta}(i) = \tilde{\Delta}(p-1-i) \tag{13}$$

for every $i \in \mathbb{Z}/q\mathbb{Z}$. The only nontrivial case is $i = 0$: if $q = p-1$ then $\tilde{\Delta}(0) = \tilde{\Delta}(q) = \tilde{\Delta}(p-1)$, and if $q < p-1$ then by q-periodicity and p-palindromicity we get $\tilde{\Delta}(0) = \tilde{\Delta}(q) = \tilde{\Delta}(p-1-q) = \tilde{\Delta}(p-1)$.

This naturally leads us to the sequence

$$\tilde{\Delta}(0), \tilde{\Delta}(p-1), \tilde{\Delta}(-p), \ldots, \tilde{\Delta}(-(k-1)p), \tilde{\Delta}(kp-1), \tilde{\Delta}(-kp), \ldots$$

Besides having $\tilde{\Delta}(-(k-1)p) = \tilde{\Delta}(kp-1)$ by Equation (13), we also have $\tilde{\Delta}(kp-1) = \tilde{\Delta}(-kp)$ by $\Delta(i)$ being APAP with period p, except when $[kp]_q - 1 = 0$ or $[kp]_q = 0$. Since $g > 1$, we never have $[kp]_q = 1$. Thus, we let $b = \frac{q}{g}$ be the smallest natural number such that $[bp]_q = 0$, so we get

$$\tilde{\Delta}(0) = \tilde{\Delta}(p-1) = \tilde{\Delta}(-p) = \cdots = \tilde{\Delta}(-(b-1)p) = \tilde{\Delta}(bp-1).$$

Note that the set of arguments in the above chain of equalities contains every $x \in \mathbb{Z}/q\mathbb{Z}$ that is congruent to 0 or -1 modulo g. Moreover, since $\Delta(q-1) + \Delta(q) = 0$, we get $\tilde{\Delta}(bp-1) = -\tilde{\Delta}(0)$, rendering the whole sequence equal to 0, as desired.

Now assume that g equals 1. By q-periodicity it suffices to show that $\Delta(1), \ldots, \Delta(q)$ are equal up to a sign. We define $\tilde{\Delta}$ as above, and let a be the smallest natural number such that $[ap]_q = 1$. Then, by similar arguments, we find that

$$\begin{aligned} \tilde{\Delta}(0) = \tilde{\Delta}(p-1) = \tilde{\Delta}(-p) = \cdots = \tilde{\Delta}(-(a-1)p) = \\ = \tilde{\Delta}(ap-1) = -\tilde{\Delta}(ap) = \cdots \\ = -\tilde{\Delta}(qp-1) = \tilde{\Delta}(0). \end{aligned}$$

Note that above, since $g = 1$ we have that $[ap]_q = 0$ for the first time when $a = q$. So we find that all the $\tilde{\Delta}(x)$, for $x \in \mathbb{Z}/q\mathbb{Z}$, are equal up to a sign, as desired. □

6 THE DELTA SEQUENCE IS APAP

In the following theorem we use the same notation as before, i.e. given a 1-quasihomomorphism f we denote by L_i the space $\mathrm{im}(\Delta_f(i))$.

Theorem 6.1. *Let $f : \mathbb{Z} \to \mathrm{Sym}(n \times n, \mathbb{K})$ be a 1-quasihomomorphism. Assume that $\dim(\sum_{i \in \mathbb{Z} \setminus \{0,-1\}} L_i) \geq 3$. We can find a natural number p such that Δ_f is APAP with period p. Moreover, p can be chosen such that $\dim(L_1 + \cdots + L_{p-2}) \leq 2$; hence in particular* $\mathrm{rk}\,(B_\Delta) \leq 2$.

Proof of Theorem 6.1. Let m be minimal such that $\dim V_m > 2$. By Lemma 4.2 we have that the sequence $(\Delta(i))_{i=-m-2}^{m+1}$ is APAP with period $m+1$. Let p be the minimal positive integer such that $(\Delta(i))_{i=-m-2}^{m+1}$ is APAP with period p. By Lemma 5.4 we have that p is a divisor of $m+1$. We will show that the entire sequence $(\Delta(i))_{i=-\infty}^{\infty}$ is APAP with period p. Then, using minimality of m, we get that

$$\dim(L_1 + \cdots + L_{p-2}) = \dim V_{p-2} \leq 2,$$

which implies that $\mathrm{rk}\,(B_\Delta) \leq 2$.

Now, assume that for some N the sequence $(\Delta(i))_{i=-N}^{N-1}$ is APAP with period p. We will simultaneously extend the sequence by one on both sides, and show $(\Delta(i))_{i=-N-1}^{N}$ is still APAP with period p; thus proving the theorem by induction.

We have three cases. If $N \equiv -1 \mod p$ there is nothing to prove, as illustrated in the following picture.

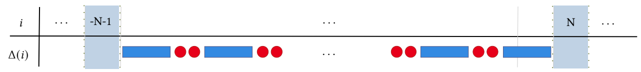

Figure 2: Here we see that for $i = -N$ and $i = N-1$ we start and end with the palindromic block. Our Definition of APAP is not dependent of what entry we put next for $i = N$ and $i = -N-1$.

Next, assume that $N \equiv 0 \mod p$. This case is illustrated as follows:

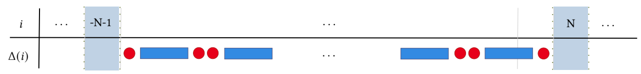

Figure 3: Here our Δ starts for $i = N$ with an "end-cancellation", e.g. $-\alpha$ and it ends with another "start-cancellation", e.g. γ. Keeping our global picture in mind, we can see that the sum of the entries $\Delta(-N-1)$ and $\Delta(-N)$, resp. $\Delta(N-1)$ and $\Delta(N)$, should be zero.

So we need to show $\Delta(N-1) + \Delta(N) = 0$ and $\Delta(-N-1) + \Delta(-N) = 0$. We reason analogously to the proof of Item (2) from Lemma 4.2. Equations (5) and (7) yield

$$\mathrm{rk}\,(\Delta(i) - \Delta(N-1) - \Delta(N)) \leq 1$$

for i in $\{-N, \ldots, N-1\} \setminus \{-1, 0\}$. Since $\dim V_m > 2$, there are three indices i with linearly independent L_i, so by Corollary 2.4 we get that $\Delta(N-1) + \Delta(N) = 0$. The other equality follows analogously.

For the last case, assume that $N \not\equiv -1, 0 \mod p$.

Let i be the residue of N when dividing by p. Now $\Delta(N)$ and $\Delta(-N-1)$ are both in a palindromic block, and we want to show that $\Delta(N) = \Delta(i) = \Delta(-N-1)$. We only prove the first equality, the second one being analogous.

We will prove that $\Delta(N) = \Delta(i)$ by contradiction in two steps:

(1) Suppose that $\Delta(N) \neq \Delta(i)$, then $L_N \not\subseteq V_2$.
(2) $L_N \not\subseteq V_2$ leads to a contradiction with minimality of p.

Claim 6.2. Suppose that $\Delta(N) \neq \Delta(i)$, then $L_N \not\subseteq V_2$.

Proof. Apply Equation (5) with $k = m$ and $z = N$ to get:

$$\mathrm{rk}\left(\sum_{j=1}^{m} \Delta(j) - \sum_{j=0}^{m} \Delta(N-j)\right) \leq 1.$$

Rewrite the sum inside the previous expression as

$$\left(-\Delta(m+1) + \sum_{j=1}^{m+1} \Delta(j)\right) - \left(\Delta(N) - \Delta(N-m-1) + \sum_{j=1}^{m+1} \Delta(N-j)\right).$$

Note that by induction hypothesis both $\sum_{j=1}^{m+1} \Delta(j)$ and $\sum_{j=1}^{m+1} \Delta(N-j)$ are sums of $m+1$ consecutive elements in an APAP sequence, and recall that $m+1$ is a multiple of p, so Lemma 5.3 implies that both sums cancel each other. Since $N - m - 1 \equiv i \mod p$, we have

$$\mathrm{rk}\,(\Delta(i) + \Delta(m) - \Delta(N)) \leq 1.$$

Note that $L_m \not\subseteq V_2$ but $L_i \subseteq V_2$. So if also $L_N \subseteq V_2$ this would imply $L_m \not\subseteq \mathrm{im}(\Delta(i) - \Delta(N))$, but then Corollary 2.3 yields

$$\mathrm{rk}\,(\Delta(i) + \Delta(m) - \Delta(N)) = \mathrm{rk}\,(\Delta(i) - \Delta(N)) + 1 \geq 2,$$

which is a contradiction. □

Claim 6.3. If $L_N \not\subset V_2$, we get a contradiction with the minimality of p.

Proof. We write $q = i + 1$, where i is still the residue of N modulo p. We will apply Lemma 5.5 to show that $(\Delta(i))_{i=-N}^{N-1}$ is APAP with period equal to $\gcd(p, q)$. For this, we need to verify the three conditions.

Write $N = ap + q - 1$. We apply Equation (5) with $k = 1$ and $z = N$:

$$\operatorname{rk}(\Delta(1) - (\Delta(N) + \Delta(N-1))) \le 1. \tag{14}$$

Since our sequence is APAP with period p, we find that $\Delta(N-1) = \Delta(ap + q - 2) = \Delta(q-2)$, hence

$$\operatorname{rk}(\Delta(1) - (\Delta(N) + \Delta(q-2))) \le 1. \tag{15}$$

Since $L_N \not\subset V_2$ but $L_1, L_{q-2} \subset V_2$, we can apply Corollary 2.3 to $A = \Delta(1) - \Delta(q-2)$ and $B = \Delta(N)$ to conclude that $\Delta(1) - \Delta(q-2) = 0$. Repeating the argument for $k = 2, \ldots, q-2$, we find that

$$\Delta(k) = \Delta(q - 1 - k) \text{ for } k = 1, \ldots, q-2,$$

showing Condition (2) of Lemma 5.5.

For $k = q - 1$, we find

$$\operatorname{rk}(\Delta(q-1) - (\Delta(N) + \Delta(ap))) \le 1,$$

but now L_{ap} need not be in V_2 and we don't get any new information. However, for $k = q$, we get

$$\operatorname{rk}(\Delta(q-1) + \Delta(q) - (\Delta(N) + \Delta(ap) + \Delta(ap-1))) \le 1.$$

Now we know that $\Delta(ap) + \Delta(ap-1) = 0$ and conclude that

$$\Delta(q-1) + \Delta(q) = 0,$$

which shows Condition (3) of Lemma 5.5.

Now we continue with $k = q + 1$:

$$\operatorname{rk}(\Delta(q+1) - (\Delta(N) + \Delta(ap-2))) \le 1.$$

But we know that $\Delta(ap-2) = \Delta(p-2) = \Delta(1)$, and hence we conclude

$$\Delta(q+1) = \Delta(1).$$

We can continue this up to $k = p - 2$, and find that

$$\Delta(k) = \Delta(k-q) \text{ for } k = q+1, \ldots, p-2, \tag{16}$$

which is Condition (1) of Lemma 5.5. We have verified all conditions, hence it holds that $(\Delta(i))_{i=-N}^{N-1}$ is APAP with period $g := \gcd(p, q)$. Hence, the shorter sequence $(\Delta(i))_{i=-m-2}^{m+1}$ is APAP with period g strictly less than p; contradicting our choice of p. □

This finishes our induction, and thus the proof. □

7 PROOF OF THE MAIN RESULT

Putting everything together we get:

Proof of Theorem 1.5. By Theorem 6.1 we find that Δ is APAP with period p. Define

$$A = \frac{B_\Delta}{p} = \frac{\Delta(1) + \cdots + \Delta(p-2)}{p} = \frac{f(p-1)}{p}.$$

We will show that Equation (2) holds with this A. We restrict to the case $x \ge 1$; the other case being analogous. Write $x = ap + r$ with $1 \le r \le p$. If $x \ge 1$,we have that $f(x) = \Delta(1) + \cdots + \Delta(x-1)$. Applying Lemma 5.3 we get that:

$$\begin{aligned} f(x) =& \Delta(1) + \cdots + \Delta(x-1) \\ =& aB_\Delta + \sum_{j=1}^{r-1} \Delta(ap+j) \\ =& apA + \sum_{j=1}^{r-1} \Delta(ap+j). \end{aligned} \tag{17}$$

We have two cases. First, we assume that $r = p$. Equation (17) becomes

$$\begin{aligned} f(x) &= apA + \sum_{j=1}^{p-1} \Delta(ap+j) \\ &= apA + \sum_{j=1}^{p-2} \Delta(ap+j) + \Delta(x-1) \\ &= apA + pA + \Delta(x-1) = xA + \Delta(x-1). \end{aligned}$$

It follows that

$$\operatorname{rk}(f(x) - xA) = \operatorname{rk}(\Delta(x-1)) \le 1.$$

If $r < p$, Equation (17) becomes

$$f(x) = apA + \sum_{j=1}^{r-1} \Delta(j).$$

In particular $\operatorname{im}(f(x) - x \cdot A) \subseteq \sum_{i=1}^{p-2} L_i$. But by Theorem 6.1, $\dim \sum_i L_i \le 2$, and hence $\operatorname{rk}(f(x) - xA) \le 2$. □

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