

SMOOTH STRUCTURES ON PL-MANIFOLDS OF DIMENSIONS BETWEEN 8 AND 10

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ABSTRACT. In this paper, we identify the concordance classes of smooth structures on PL -manifolds of dimension between 8 and 10 in terms of the cohomology and Steenrod operations. This leads to the computation of the homotopy inertia groups. Finally we discuss the special cases of Lens spaces and real projective spaces.

1 Introduction

The study of smooth manifolds and their piece-wise linear (PL)-triangulations is one of the essential and active topics in differential topology. Shortly after Milnor discovered exotic smooth 7-spheres[10], Kervaire constructed a PL 10-manifold without any smooth manifold in its homotopy type, and a new exotic 9-sphere[9]. This motivates the problem of classifying all smooth structures on a PL -manifold if exists, compatible with the underlying PL -structure, up to some suitable equivalence relation. In higher dimensions, the classification of compatible smooth structures up to several equivalence relations has been studied (see [11, 14, 15, 20, 23]). In this paper, we consider one of such equivalence relations, called concordance.

Convention: We work in the category of oriented smooth manifolds such that all morphisms are PL -smooth, implicitly assuming all manifolds are closed connected smooth oriented of dimension ≥ 5 , and that all maps are orientation preserving.

Definition 1.1. Let M be a closed smooth manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a PL -homeomorphism $f : N \rightarrow M$. Two such pairs (N_1, f_1) and (N_2, f_2) are PL -concordant provided there exists a diffeomorphism $g : N_1 \rightarrow N_2$ and a PL homeomorphism $F : N_1 \times [0, 1] \rightarrow M \times [0, 1]$ such that $F|_{N_1 \times 0} = f_1$ and $F|_{N_1 \times 1} = f_2 \circ g$.

The set of all such PL -concordance classes is denoted by $\mathcal{C}(M)$. The PL -concordance class of (N, f) is denoted by $[N, f]$, and the class $[M, \text{Id}]$ of the identity $\text{Id} : M \rightarrow M$ can be considered as the base point of $\mathcal{C}(M)$. The study of $\mathcal{C}(M)$ typically proceeds by reducing to bundle theory and then to homotopy theory. In fact, Cairns-Hirsch-Mazur [15] proved that, if M admits a smooth structure then there is a set bijection

$$\mathcal{C}(M) \cong [M, PL/O], \quad (1.1)$$

where PL/O is an H-space, (actually an infinite loop space) that is a homotopy fiber of the forgetful map $BO \rightarrow BPL$. Note that, the spaces BO and BPL have compatible commutative H-space structures arising from the Whitney sum of bundles [11, p.92]. Hence $[M, PL/O]$ has a group structure. The bijection in (1.1) has some immediate consequences. One consequence is that $\mathcal{C}(M)$ admits an abelian group structure, with $[M, \text{Id}]$ acting as the identity element. Another consequence is the isomorphism between the groups $\mathcal{C}(\mathbb{S}^n)$ and Θ_n , representing h -cobordism classes of smooth homotopy spheres. For $n \geq 5$, the group Θ_n can also be identified with the set of all (oriented) diffeomorphism classes of smooth homotopy spheres. Explicit calculations of concordance groups $\mathcal{C}(M)$ have been performed for certain manifolds M , including the product of standard spheres $\mathbb{S}^i \times \mathbb{S}^j$, an \mathbb{S}^j -bundle over \mathbb{S}^i [20], as well as complex and quaternionic projective spaces [8, 2]. Moreover, through obstruction theory and the fact that PL/O is 6-connected, one can establish

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that the group $\mathcal{C}(M) \cong H^7(M; \Theta_7)$ for a closed smooth 7-manifold M , where $\Theta_7 \cong \mathbb{Z}/28$ is the group of homotopy 7-spheres. In this paper, we extend this result to manifolds of dimensions $n = 8, 9, 10$, utilizing the structure of the 10th-Postnikov section of PL/O (see 3.2).

Theorem A. *The smooth concordance structure set $\mathcal{C}(M)$ for manifolds M with $8 \leq \dim(M) \leq 10$ is explicitly determined in terms of the action of Steenrod operations on the cohomology of M .*

A detailed discussion of these results is presented in Theorem 2.1, Theorem 2.2, and Theorem 3.1. The underlying idea is to utilize cohomology operations to gain sufficient knowledge of the cell attachments of M in degrees 7 through 10, enabling the computation of $[M, PL/O]$ through the initial stages of the Postnikov tower of PL/O .

Recall that the group $\Theta_n \cong \mathcal{C}(S^n)$ acts on the smooth structure set $S^{\text{Diff}}(M)$ [3, 25], given by

$$\begin{aligned} \Theta_n \times S^{\text{Diff}}(M) &\longrightarrow S^{\text{Diff}}(M) \\ ([\Sigma, f], [N, g]) &\mapsto [\Sigma \# N, f \# g] \end{aligned} \quad (1.2)$$

The stabilizer of this action at the base point $[M, \text{Id}]$ is known as the homotopy inertia group, denoted by $I_h(M)$. It follows from [10] that, for dimensions $8 \leq n \leq 10$, the group Θ_n fits into the following split short exact sequence

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \longrightarrow \pi_n^s / \text{Im}(J) \longrightarrow 0,$$

where $\pi_n^s / \text{Im}(J) \subseteq \pi_n(G/O)$. Note that, $\pi_8^s / \text{Im}(J) = \mathbb{Z}/2\{\epsilon\}$, $\pi_9^s / \text{Im}(J) = \mathbb{Z}/2\{\mu\} \oplus \mathbb{Z}/2\{\eta \circ \epsilon\}$, and $\pi_{10}^s / \text{Im}(J) = \mathbb{Z}/2\{\eta \circ \mu\} \oplus \mathbb{Z}/3\{\beta_1\}$. In this paper, we prove the following by using the structure of $\mathcal{C}(M)$ given in Theorem A together with the Postnikov section of PL/O :

Theorem B. *Let M be a closed oriented smooth n -manifold for $8 \leq n \leq 10$. Then the stabilizer of the action of Θ_n / bP_{n+1} given in (1.2) on the base point $[M, \text{Id}]$ is explicitly determined in terms of Steenrod operations on the cohomology of M (see Theorem 4.5, Theorem 4.9, and Theorem 4.14).*

Recall the lens space $L^{2n+1}(m) = \mathbb{S}^{2n+1} / \mathbb{Z}_m$, where the group action is given by $(z_0, z_1, \dots, z_n) \mapsto (\alpha z_0, \alpha z_1, \dots, \alpha z_n)$ with $\alpha = \exp \frac{2\pi i}{m}$. The following theorem yields the computation of the inertia group of $L^9(m)$ and $\mathbb{R}P^n$ for $n = 8$ and 10.

Theorem C. *Let m be a positive integer and n be a non-negative integer.*

- (i) *Let $m = 2n + 1$. Then for any exotic sphere $\Sigma \in \Theta_9$, the connected sum $L^9(m) \# \Sigma$ is not diffeomorphic to $L^9(m)$.*
- (ii) *Let $m = 4n + 2$. Then there is a unique exotic sphere $\Sigma \in \Theta_9$ such that $L^9(m) \# \Sigma$ is diffeomorphic to $L^9(m)$.*
- (iii) *Let $m = 4n$. Then, there are exactly four exotic spheres $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \in \Theta_9$ such that no two of the manifolds $L^9(m)$, $L^9(m) \# \Sigma_1$, $L^9(m) \# \Sigma_2$, $L^9(m) \# \Sigma_3$, and $L^9(m) \# \Sigma_4$ are diffeomorphic.*
- (iv) *Let $\Sigma \in \Theta_8$ be the exotic sphere. Then $\mathbb{R}P^8 \# \Sigma$ is not diffeomorphic to $\mathbb{R}P^8$.*
- (v) *For any homotopy 10-sphere $\Sigma \in \Theta_{10}$, the connected sum $\mathbb{R}P^{10} \# \Sigma$ is diffeomorphic to $\mathbb{R}P^{10}$.*

Theorem C (i), (ii), and (iii) are immediate consequences of Theorem 4.9 and [12, Theorem 4.2]. The last two assertions of Theorem C will be proved in Theorem 4.5 and Theorem 4.16.

1.1 Notation

- Let O_n be the orthogonal group, $PL_n \subset O_n$ is the group of piece-wise linear homeomorphisms, and G_n be the set of homotopy equivalences. Denote by $O = \text{colim}_{n \rightarrow \infty} O_n$, $PL = \text{colim}_{n \rightarrow \infty} PL_n$, and $G = \text{colim}_{n \rightarrow \infty} G_n$ [13, 16].
- Let G/O be the homotopy fiber of the canonical map $BO \rightarrow BG$ between the classifying spaces for stable vector bundles and stable spherical fibrations [18, §2 and §3], and G/PL be the homotopy fiber of the canonical map $BPL \rightarrow BG$ between the classifying spaces for PL \mathbb{R}^n -bundles and stable spherical fibrations [22].
- For an infinite loop space X we use the small letter x to denote the connective spectrum such that $\Omega^\infty(x) \simeq X$. We use this notation to define the spectra g , o , pl , pl/o , g/o , g/pl .

- The Eilenberg MacLane spectrum for an Abelian group A is denoted by HA .
- The notation $\{-, -\}$ is used to denote the stable homotopy classes of maps between spectra.
- The notation $\tau_{\leq m}$ is reserved for the m^{th} Postnikov section. It satisfies $\pi_i \tau_{\leq m}(E) = \pi_i(E)$ for $i \leq m$ and 0 if $i > m$. The notation ${}^{>m}\tau$ refers to the m -connected cover, which is also the fiber of $X \rightarrow \tau_{\leq m}X$.
- The Moore space $\bar{M}(A, n)$ is the space whose reduced homology is concentrated in degree n , whence it is isomorphic to A .
- The notations used for the generators of the groups Θ_n/bP_{n+1} and $\pi_n(G/O)$ are the same and are as given in [24] and [21].

1.2 Organization

In Section 2 and 3, we give the homotopy splitting of the 10^{th} Postnikov section of pl/o and compute the set $[M^n, PL/O]$ for $8 \leq n \leq 10$. In Section 4 we discuss the concordance and homotopy inertia group of smooth manifold M^n , in particular, compute $I_h(\mathbb{R}P^n)$, for $8 \leq n \leq 10$.

2 Smooth structures on 8, 9-manifolds

In this section, we use the structure of the Postnikov section $\tau_{\leq 9}PL/O$ given in [7], and compute $[M, PL/O]$ for $\dim(M) = 8, 9$. For an 8-dimensional manifold, we deduce the following theorem. This is also implied by the computations in [7]; however, here we independently confirm this result through a direct calculation.

Theorem 2.1. *Let M^8 be a closed smooth manifold. Then*

$$[M^8, PL/O] \cong H^7(M^8; \mathbb{Z}/28) \oplus H^8(M^8; \mathbb{Z}/2)$$

Proof. We prove that the 8^{th} -Postnikov section of PL/O splits as a product $K(\mathbb{Z}/28, 7) \times K(\mathbb{Z}/2, 8)$, implying the required isomorphism. It suffices to prove splitting in a p -local category for every prime p . As the k -invariants lie in the stable range, we work stably using Eilenberg-MacLane spectra instead of their underlying spaces. Note that, the homotopy groups of pl/o in degrees at most 8 have p -torsion only for $p = 2$ and 7. For the case $p = 7$, the homotopy group is non-zero only in degree 7. Therefore, it suffices to work 2-locally.

The stable 8^{th} -Postnikov section of pl/o is the fiber of a map $\Sigma^7 H\mathbb{Z}/4 \rightarrow \Sigma^9 H\mathbb{Z}/2$. Up to homotopy, this map is either 0 or $(\Sigma^7 Sq^2) \circ q$, that is

$$\Sigma^7 H\mathbb{Z}/4 \xrightarrow{q} \Sigma^7 H\mathbb{Z}/2 \xrightarrow{\Sigma^7 Sq^2} \Sigma^9 H\mathbb{Z}/2.$$

This can be readily seen from the following diagram, wherein for any $\phi \in \{\Sigma^7 H\mathbb{Z}/4, \Sigma^9 H\mathbb{Z}/2\}$, the observation $\{H\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}/2\} \cong \{0, Sq^2\}$ gives us

$$\begin{array}{ccccc} \Sigma^6 H\mathbb{Z}/2 & \xrightarrow{\beta} & \Sigma^7 H\mathbb{Z}/2 & \longrightarrow & \Sigma^7 H\mathbb{Z}/4 & \xrightarrow{q} & \Sigma^7 H\mathbb{Z}/2 \\ & & \searrow \psi & & \downarrow \phi & \swarrow \zeta & \\ & & & & \Sigma^9 H\mathbb{Z}/2 & & \end{array}$$

where $\psi = \Sigma^7 Sq^2$ or 0, and $\beta = Sq^1$. Since $Sq^2 \circ \beta \neq 0$, the only possibility for the map ψ is 0, which implies the existence of the ζ map. We intend to show that $(\Sigma^7 Sq^2) \circ q$ does not occur as the k -invariant. The idea is to exhibit a map $\Sigma^7 H\mathbb{Z}/4 \rightarrow \tau_{\leq 8} pl/o$ such that its composition with the fibration map $b : \tau_{\leq 8} pl/o \rightarrow \Sigma^7 H\mathbb{Z}/4$ is an equivalence. This implies that the k -invariant is 0.

Consider the natural fibration map $\Sigma^{-1}g/pl \rightarrow pl/o$ and restrict it to the 6-connected cover ${}^{>6}\tau \Sigma^{-1}g/pl$ of $\Sigma^{-1}g/pl$. Since $\pi_7({}^{>6}\tau \Sigma^{-1}g/pl) \cong \mathbb{Z}$ and $\pi_8({}^{>6}\tau \Sigma^{-1}g/pl) = 0$, implies

$${}^{>6}\tau_{\leq 8}(\tau \Sigma^{-1}g/pl) \cong \Sigma^7 H\mathbb{Z} \xrightarrow{q} \tau_{\leq 8} pl/o.$$

Consequently, we have the following diagram

$$\begin{array}{ccccc} \Sigma^7 H\mathbb{Z} & \xrightarrow{\times 4} & \Sigma^7 H\mathbb{Z} & \xrightarrow{p} & \Sigma^7 H\mathbb{Z}/4 \\ & & \downarrow g & & \downarrow \\ \Sigma^8 H\mathbb{Z}/2 & \longrightarrow & \tau_{\leq 8} pl/o & \xrightarrow{b} & \Sigma^7 H\mathbb{Z}/4 \end{array}$$

where p is the natural projection from \mathbb{Z} onto $\mathbb{Z}/4$. Observe that the composition $b \circ g \circ (\times 4)$ is 0, since the composition $p \circ (\times 4)$ is 0. Therefore we get a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^7 H\mathbb{Z} & \xrightarrow{\times 4} & \Sigma^7 H\mathbb{Z} & \xrightarrow{p} & \Sigma^7 H\mathbb{Z}/4 \\ \downarrow & & \downarrow g & & \downarrow \cong \\ \Sigma^8 H\mathbb{Z}/2 & \longrightarrow & \tau_{\leq 8} pl/o & \xrightarrow{b} & \Sigma^7 H\mathbb{Z}/4 \end{array}$$

Since every map from $\Sigma^7 H\mathbb{Z} \rightarrow \Sigma^8 H\mathbb{Z}/2$ is null homotopic, we have the following diagram

$$\begin{array}{ccccc} \Sigma^7 H\mathbb{Z} & \xrightarrow{\times 4} & \Sigma^7 H\mathbb{Z} & \xrightarrow{p} & \Sigma^7 H\mathbb{Z}/4 \\ & \searrow 0 & \downarrow g & \swarrow \tilde{b} & \\ & & \tau_{\leq 8} pl/o & & \end{array}$$

where $g \circ (\times 4) = 0$ implies the existence of a map \tilde{b} having the desired property.

Thus, the map \tilde{b} is a homotopy section for the map b , implying the following decomposition

$$\tau_{\leq 8} pl/o \simeq \Sigma^7 H\mathbb{Z}/4 \vee \Sigma^8 H\mathbb{Z}/2. \quad (2.1)$$

This complete the proof. \square

Theorem 2.1 shows how a decomposition result for the 8th Postnikov section facilitates the computation of $\mathcal{C}(M) = [M, PL/O]$ for 8-manifolds M . We now recall from [7] the formula for ${}^{>6}\tau_{\leq 9} pl/o$. Note that, $\text{Ext}(\mathbb{Z}/4, \mathbb{Z}/2) \cong \mathbb{Z}/2$ along with the universal coefficient theorem gives $H^{n+1}(\tilde{K}(\mathbb{Z}/4, n); \mathbb{Z}/2) \cong \mathbb{Z}/2$. Let us fix the notation d_2 for the corresponding stable map $H\mathbb{Z}/4 \rightarrow \Sigma H\mathbb{Z}/2$, and define

$$\begin{aligned} \mathcal{F} &:= \text{Fibre}(H\mathbb{Z}/2 \xrightarrow{Sq^2} \Sigma^2 H\mathbb{Z}/2) \\ \mathcal{F}_2 &:= \text{Fibre}(H\mathbb{Z}/4 \xrightarrow{Sq^2 \circ d_2} \Sigma^3 H\mathbb{Z}/2). \end{aligned} \quad (2.2)$$

With this, the 9th Postnikov section of pl/o is given by

$$\tau_{\leq 9} pl/o \simeq \Sigma^8 \mathcal{F} \vee \Sigma^7 \mathcal{F}_2 \vee \Sigma^7 H\mathbb{Z}/\tau \vee \Sigma^9 H\mathbb{Z}/2. \quad (2.3)$$

Let M^9 be a closed oriented smooth manifold. Then the minimal cell structure [6, §4.C] on $M^9/M^{(6)}$ is of the following form

$$M^9/M^{(6)} \simeq \left(\bigvee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee (\mathbb{S}^7)^{\vee_l} \vee (\mathbb{S}^8)^{\vee_k} \right) \bigcup e^9,$$

where $M(\mathbb{Z}/p^r, n)$ stands for the Moore space for the group \mathbb{Z}/p^r in degree n , and J is some finite indexing set. In this case, the attaching map of the 9-cell onto 8-cells is null homotopic. Therefore,

$$M^9/M^{(6)} \simeq \left(\left(\bigvee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee (\mathbb{S}^7)^{\vee_l} \right) \bigcup_f e^9 \right) \vee (\mathbb{S}^8)^{\vee_k}, \quad (2.4)$$

and hence the attaching map f lies in $\pi_8 \left(\bigvee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee (\mathbb{S}^7)^{\vee_l} \right)$. By the connectivity argument,

$$\pi_8 \left(\bigvee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee (\mathbb{S}^7)^{\vee_l} \right) \cong \bigoplus_{(p,r) \in J} \pi_8(M(\mathbb{Z}/p^r, 7)) \oplus_l \pi_8(\mathbb{S}^7)$$

with $\pi_8(\mathbb{S}^7) \cong \mathbb{Z}/2\{\eta\}$ and

$$\pi_8(M(\mathbb{Z}/p^r, 7)) \cong \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \mathbb{Z}/2\{\iota \circ \eta\} & \text{if } p=2, \end{cases}$$

where $\iota \circ \eta$ is the composite $S^8 \xrightarrow{\eta} S^7 \xrightarrow{\iota} M(\mathbb{Z}/2^7, 7)$.

Consider the Steenrod square operation $Sq^2 : H^7(M^9; \mathbb{Z}/2) \rightarrow H^9(M^9; \mathbb{Z}/2)$, and for each $r \geq 1$, there is a higher order Bockstein operation $\beta_r : H^*(M^9; \mathbb{Z}/2) \rightarrow H^{*+1}(M^9; \mathbb{Z}/2)$. Now, depending on the attaching map we have following possibilities (for more details see [17]):

- (1) If M^9 is a spin manifold then the attaching map $S^8 \rightarrow \bigvee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee (\mathbb{S}^7)^{\vee \iota}$ is null homotopic, since $Sq^2 : H^7(M^9; \mathbb{Z}/2) \rightarrow H^9(M^9; \mathbb{Z}/2)$ is zero. Thus

$$M^9/M^{(6)} \simeq (\mathbb{S}^7)^{\vee \iota} \vee (\mathbb{S}^8)^{\vee k} \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee \mathbb{S}^9, \quad (2.5)$$

- (2) If M^9 is a non-spin manifold then $Sq^2 : H^7(M^9; \mathbb{Z}/2) \rightarrow H^9(M^9; \mathbb{Z}/2)$ is non-zero. In addition,

- (a) Suppose that for any $u \in H^7(M^9; \mathbb{Z}/2)$ with $Sq^2(u) \neq 0$ and any $v \in \text{Ker}(Sq^2)$, we have $\beta_r(u+v) = 0$ and $u+v \notin \text{Im}(\beta_s)$, $\forall s \geq 1$. Then, the non-trivial factor of the attaching map f in (2.4) is η , which implies

$$M^9/M^{(6)} \simeq C(\eta) \vee M', \quad (2.6)$$

where $M' \simeq (\mathbb{S}^7)^{\vee \iota-1} \vee (\mathbb{S}^8)^{\vee k} \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7)$.

- (b) Suppose that for any $u \in H^7(M^9; \mathbb{Z}/2)$ with $Sq^2(u) \neq 0$ and any $v \in \text{Ker}(Sq^2)$, we have $u+v \notin \text{Im}(\beta_s)$, $\forall s \geq 1$, while there exist $u' \in H^7(M^9; \mathbb{Z}/2)$ with $Sq^2(u') \neq 0$ and $v' \in \text{Ker}(Sq^2)$ such that $\beta_r(u'+v') \neq 0$ for some $r \geq 1$. Then the only non-trivial factor of the attaching map f in (2.4) is $\iota \circ \eta$, and

$$M^9/M^{(6)} \simeq C(\iota \circ \eta) \vee M'' \quad (2.7)$$

where $M'' \simeq (\mathbb{S}^7)^{\vee \iota} \vee (\mathbb{S}^8)^{\vee k} \vee_{(p,r) \in J'} M(\mathbb{Z}/p^r, 7)$.

The following theorem applies the splitting of the Postnikov section (2.3) to the case of 9-manifolds.

Theorem 2.2. *Let M^9 be a closed oriented smooth manifold and let $Sq^2 \circ d_2 : H^6(M^9; \mathbb{Z}/4) \rightarrow H^9(M^9; \mathbb{Z}/2)$.*

- (1) *If M^9 is a spin manifold then*

$$[M^9, PL/O] \cong H^7(M^9; \mathbb{Z}/28) \oplus H^8(M^9; \mathbb{Z}/2) \oplus H^9(M^9; \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2).$$

- (2) *Let M^9 be a non-spin manifold.*

- (a) *If $Sq^2 \circ d_2$ is non-trivial then*

$$[M^9, PL/O] \cong H^7(M^9; \mathbb{Z}/28) \oplus H^8(M^9; \mathbb{Z}/2) \oplus H^9(M^9; \mathbb{Z}/2).$$

- (b) *If $Sq^2 \circ d_2$ is trivial then*

$$[M^9, PL/O] \cong H^7(M^9; \mathbb{Z}/7) \oplus H^8(M^9; \mathbb{Z}/2) \oplus H^9(M^9; \mathbb{Z}/2) \oplus [M^9, \Sigma^7 \mathcal{F}_2],$$

where $[M, \Sigma^7 \mathcal{F}_2] \cong \tilde{K} \oplus \tilde{A}$, with \tilde{K} is a part of the short exact sequence

$$0 \longrightarrow \text{Ker}(Sq^2) \longrightarrow \tilde{K} \longrightarrow \text{Ker}(Sq^2 \oplus Sq^1) \longrightarrow 0,$$

$Sq^1 : H^7(M; \mathbb{Z}/2) \rightarrow H^8(M; \mathbb{Z}/2)$ and $Sq^2 : H^7(M; \mathbb{Z}/2) \rightarrow H^9(M; \mathbb{Z}/2)$, and \tilde{A} is the non-trivial extension satisfying the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \tilde{A} \longrightarrow A \longrightarrow 0,$$

where $A = \mathbb{Z}/4$ or $\mathbb{Z}/2$.

Proof. Using (2.3), we have $[M^9, PL/O] \cong [M^9, \tau_{\leq 9} pl/o] \cong H^7(M^9; \mathbb{Z}/7) \oplus H^9(M^9; \mathbb{Z}/2) \oplus [M^9, \Sigma^8 \mathcal{F}] \oplus [M^9, \Sigma^7 \mathcal{F}_2]$. Thus, it is enough to compute $[M^9, \Sigma^7 \mathcal{F}_2]$ and $[M^9, \Sigma^8 \mathcal{F}]$.

Note that, $\Sigma^8 \mathcal{F}$ is 7-connected, thus

$$[M^9, \Sigma^8 \mathcal{F}] \cong [M^9/M^{(6)}, \Sigma^8 \mathcal{F}]. \quad (2.8)$$

The space $\Sigma^7 \mathcal{F}_2$ is 6-connected and the group $[M^9, \Sigma^7 \mathcal{F}_2]$ can be computed using the following commutative diagram:

$$\begin{array}{ccccccc} [M^9, \Sigma^6 H\mathbb{Z}/4] & \xrightarrow{Sq^2 \circ d_2} & [M^9, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M^9, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M^9, \Sigma^7 H\mathbb{Z}/4] \\ \uparrow & & \uparrow \cong & & \uparrow & & \uparrow \cong \\ 0 & \longrightarrow & [M^9/M^{(5)}, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M^9/M^{(5)}, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M^9/M^{(5)}, \Sigma^7 H\mathbb{Z}/4] \end{array} \quad (2.9)$$

(1) Let M^9 be a spin manifold. Then (2.5) and (2.8) together gives

$$[M^9, \Sigma^8 \mathcal{F}] \cong [M^9/M^{(6)}, \Sigma^8 \mathcal{F}] \cong H^8(M^9/M^{(6)}; \mathbb{Z}/2) \oplus H^9(M^9/M^{(6)}; \mathbb{Z}/2).$$

For $[M^9, \Sigma^7 \mathcal{F}_2]$, since $Sq^2 = 0$, using (2.9) we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & [M^9, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M^9, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M^9, \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0 \\ & & \uparrow \cong & & \uparrow & & \uparrow \cong \\ 0 & \longrightarrow & [M^9/M^{(5)}, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M^9/M^{(5)}, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M^9/M^{(5)}, \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0. \end{array} \quad (2.10)$$

This implies

$$[M^9, \Sigma^7 \mathcal{F}_2] \cong [M^9/M^{(5)}, \Sigma^7 \mathcal{F}_2].$$

In (2.10), we demonstrate the splitting of the short exact sequence in the second row, thereby implying the splitting of the sequence in the first row. For that purpose, consider the cofiber sequence $M^9/M^{(5)} \rightarrow M^9/M^{(6)} \xrightarrow{\Psi} \Sigma(M^{(6)}/M^{(5)})$, and the following commutative diagram induced from it

$$\begin{array}{ccccccc} 0 & \longrightarrow & [M^9/M^{(5)}, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M^9/M^{(5)}, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M^9/M^{(5)}, \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0 \\ & & \uparrow \cong & & \uparrow & & \uparrow \\ 0 & \longrightarrow & [M^9/M^{(6)}, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M^9/M^{(6)}, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M^9/M^{(6)}, \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0 \\ & & \uparrow \phi^* & & \uparrow \Psi^* & & \uparrow \gamma^* \\ 0 & \longrightarrow & [\Sigma(M^{(6)}/M^{(5)}), \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [\Sigma(M^{(6)}/M^{(5)}), \Sigma^7 \mathcal{F}_2] & \longrightarrow & [\Sigma(M^{(6)}/M^{(5)}), \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0. \end{array} \quad (2.11)$$

Observe that due to (2.5), the map $\Psi : M^9/M^{(6)} \rightarrow \Sigma(M^{(6)}/M^{(5)}) \simeq \vee_i \mathbb{S}^7$ decomposes in $\gamma : \vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \rightarrow \vee_i \mathbb{S}^7$ and $\phi : \mathbb{S}^9 \rightarrow \vee_i \mathbb{S}^7$. Up to homotopy, the map ϕ is either 0 or η^2 ($\pi_2^9 \cong \mathbb{Z}/2\{\eta^2\}$). If it is $\eta^2 = \eta \circ \eta$, then the map ϕ^* is 0 due to the following.

$$\begin{array}{ccc} [\mathbb{S}^7, \Sigma^7 \mathcal{F}_2] & \xrightarrow{(\eta^2)^*} & [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \\ & \eta^* \searrow & \nearrow \eta^* \\ & & [\mathbb{S}^8, \Sigma^7 \mathcal{F}_2] = 0 \end{array}$$

Thus, we obtain $\text{Im}(\Psi^*) = \text{Im}(\gamma^*)$. This shows that in (2.11), the exact sequence in the second row induces a split exact sequence at $[M^9/M^{(6)}, \Sigma^7 \mathcal{F}_2]_{\text{Im}(\Psi^*)}$. Therefore, we get

$$[M^9, \Sigma^7 \mathcal{F}_2] \cong H^7(M; \mathbb{Z}/4) \oplus H^9(M; \mathbb{Z}/2). \quad (2.12)$$

As a result, we obtain the following in the spin case,

$$[M^9, PL/O] \cong H^7(M^9; \mathbb{Z}/28) \oplus H^8(M^9; \mathbb{Z}/2) \oplus H^9(M^9; \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2).$$

(2) Let M^9 be a non-spin manifold. Using (2.8), we have $[M^9, \Sigma^8 \mathcal{F}] \cong [M^9/M^{(6)}, \Sigma^8 \mathcal{F}]$, and by (2.6) and (2.7),

$$[M^9/M^{(6)}, \Sigma^8 \mathcal{F}] \cong [C(\eta), \Sigma^8 \mathcal{F}] \oplus [M', \Sigma^8 \mathcal{F}] \text{ or } [C(\iota \circ \eta), \Sigma^8 \mathcal{F}] \oplus [M'', \Sigma^8 \mathcal{F}]. \quad (2.13)$$

Further, $[\mathbb{S}^8, \Sigma^8 \mathcal{F}] \xrightarrow{\cong} [M(\mathbb{Z}/2^r, 7), \Sigma^8 \mathcal{F}] \cong \mathbb{Z}/2$, and $[M(\mathbb{Z}/p^r, 7), \Sigma^8 \mathcal{F}] = 0$ for all odd prime p . Hence

$$[M', \Sigma^8 \mathcal{F}] \cong H^8(M'; \mathbb{Z}/2) \text{ and } [M'', \Sigma^8 \mathcal{F}] \cong H^8(M''; \mathbb{Z}/2). \quad (2.14)$$

It remains to compute $[C(\eta), \Sigma^8 \mathcal{F}]$ and $[C(\iota \circ \eta), \Sigma^8 \mathcal{F}]$.

The computation of $[C(\eta), \Sigma^8 \mathcal{F}]$ follows easily from the following exact sequence

$$\cdots [\mathbb{S}^8, \Sigma^8 \mathcal{F}] \xrightarrow{\cong} [\mathbb{S}^9, \Sigma^8 \mathcal{F}] \longrightarrow [C(\eta), \Sigma^8 \mathcal{F}] \longrightarrow [\mathbb{S}^7, \Sigma^8 \mathcal{F}] = 0.$$

Therefore, we get

$$[C(\eta), \Sigma^8 \mathcal{F}] = 0 \quad (2.15)$$

For the computation of $[C(\iota \circ \eta), \Sigma^8 \mathcal{F}]$, consider the following commutative diagram obtained from the cofiber sequence $\mathbb{S}^8 \xrightarrow{\iota \circ \eta} M(\mathbb{Z}/2^r, 7) \longrightarrow C(\iota \circ \eta)$,

$$\begin{array}{ccccccc} & & [\mathbb{S}^8, \Sigma^8 \mathcal{F}] & & & & \\ & \nearrow \Sigma i^* & \downarrow \cong & & & & \\ [\Sigma M(\mathbb{Z}/2^r, 7), \Sigma^8 \mathcal{F}] & \longrightarrow & [\mathbb{S}^9, \Sigma^8 \mathcal{F}] & \longrightarrow & [C(\iota \circ \eta), \Sigma^8 \mathcal{F}] & \longrightarrow & [M(\mathbb{Z}/2^r, 7), \Sigma^8 \mathcal{F}] \xrightarrow{(\iota \circ \eta)^*} [\mathbb{S}^8, \Sigma^8 \mathcal{F}] \\ & & & & & \nearrow d^* \uparrow \cong & \\ & & & & & [\mathbb{S}^8, \Sigma^8 \mathcal{F}] & \xrightarrow{\times 2} \end{array} \quad (2.16)$$

where the map Σi^* is a part of the exact sequence obtained from the cofiber sequence $\mathbb{S}^7 \xrightarrow{2^r} \mathbb{S}^7 \xrightarrow{i} M(\mathbb{Z}/2^r, 7)$. Also, the map $(\iota \circ \eta)^*$ is trivial which implies that $[C(\iota \circ \eta), \Sigma^8 \mathcal{F}] \cong [M(\mathbb{Z}/2^r, 7), \Sigma^8 \mathcal{F}] \cong \mathbb{Z}/2$. In conclusion, we get

$$[C(\eta), \Sigma^8 \mathcal{F}] \cong H^8(C(\eta); \mathbb{Z}/2) \text{ and } [C(\iota \circ \eta), \Sigma^8 \mathcal{F}] \cong H^8(C(\iota \circ \eta); \mathbb{Z}/2). \quad (2.17)$$

Therefore, combining (2.14) and (2.17) we obtain

$$[M, \Sigma^8 \mathcal{F}] \cong H^8(M; \mathbb{Z}/2). \quad (2.18)$$

Finally, let us compute $[M^9, \Sigma^7 \mathcal{F}_2]$, by taking into account two cases depending on the nature of the attaching map: whether the map $Sq^2 \circ d_2 : H^6(M; \mathbb{Z}/4) \longrightarrow H^9(M; \mathbb{Z}/2)$ is non-trivial or trivial.

In the case when $Sq^2 \circ d_2$ is non-trivial, it is clear from the diagram (2.9) that

$$[M^9, \Sigma^7 \mathcal{F}_2] \cong [M^9, \Sigma^7 H\mathbb{Z}/4]. \quad (2.19)$$

Now, for the case $Sq^2 \circ d_2 = 0$, we need $[C(\eta), \Sigma^7 \mathcal{F}_2]$ and $[C(\iota \circ \eta), \Sigma^7 \mathcal{F}_2]$. So let us first calculate these groups.

We have $\eta : \mathbb{S}^8 \longrightarrow \mathbb{S}^7$, $\mathcal{F}_2 = \text{Fiber}(H\mathbb{Z}/4 \xrightarrow{Sq^2 \circ d_2} \Sigma^3 H\mathbb{Z}/2)$, and the following commutative square

$$\begin{array}{ccc} H\mathbb{Z}/2 & \xrightarrow{Sq^2 \circ Sq^1} & \Sigma^3 H\mathbb{Z}/2 \\ \downarrow & & \downarrow = \\ H\mathbb{Z}/4 & \xrightarrow{Sq^2 \circ d_2} & \Sigma^3 H\mathbb{Z}/2 \end{array} \quad (2.20)$$

Let $\mathcal{D} = \text{Fiber}(H\mathbb{Z}/2 \xrightarrow{Sq^2 \circ Sq^1} \Sigma^3 H\mathbb{Z}/2)$. We have the following commutative diagram

$$\begin{array}{ccccc} \Sigma^7 H\mathbb{Z}/4 & \longrightarrow & \Sigma^7 \mathcal{D} & \longrightarrow & \Sigma^8 \mathcal{F} \\ \downarrow = & & \downarrow & & \downarrow \\ \Sigma^7 H\mathbb{Z}/4 & \longrightarrow & \Sigma^7 H\mathbb{Z}/2 & \xrightarrow{\Sigma^7 Sq^1} & \Sigma^8 H\mathbb{Z}/2 \\ \downarrow & & \downarrow \Sigma^7(Sq^2 \circ Sq^1) & & \downarrow \Sigma^7 Sq^2 \\ 0 & \longrightarrow & \Sigma^{10} H\mathbb{Z}/2 & \xrightarrow{=} & \Sigma^{10} H\mathbb{Z}/2 \end{array}$$

in which the rows and columns are cofiber sequences.

Note that $[C(\eta), \Sigma^7 \mathcal{F}] = 0 = [C(\eta), \Sigma^8 \mathcal{F}]$ implies

$$[C(\eta), \Sigma^7 \mathcal{D}] \cong [C(\eta), \Sigma^7 H\mathbb{Z}/4] \cong \mathbb{Z}/4.$$

Now, using (2.20) we have the following commutative diagram of short exact sequences, wherein observe that if the bottom row splits, then so does the top row.

$$\begin{array}{ccccccc}
0 & \longrightarrow & [C(\eta), \Sigma^9 H^{\mathbb{Z}/2}] & \longrightarrow & [C(\eta), \Sigma^7 \mathcal{D}] & \longrightarrow & [C(\eta), \Sigma^7 H^{\mathbb{Z}/2}] \longrightarrow 0 \\
& & \downarrow = & & \downarrow & & \downarrow \\
0 & \longrightarrow & [C(\eta), \Sigma^9 H^{\mathbb{Z}/2}] & \longrightarrow & [C(\eta), \Sigma^7 \mathcal{F}_2] & \longrightarrow & [C(\eta), \Sigma^7 H^{\mathbb{Z}/4}] \longrightarrow 0
\end{array}$$

Therefore,

$$[C(\eta), \Sigma^7 \mathcal{F}_2] \cong \mathbb{Z}/8. \quad (2.21)$$

To compute $[C(\iota \circ \eta), \Sigma^7 \mathcal{F}_2]$, consider the following commutative diagram

$$\begin{array}{ccccccc}
& & \mathbb{Z}/2 & & & & \\
& & \parallel & & & & \\
0 & \longrightarrow & [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [C(\iota \circ \eta), \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M(\mathbb{Z}/2^r, 7), \Sigma^7 \mathcal{F}_2] \longrightarrow 0 \\
& & \downarrow = & & \downarrow & & \downarrow \\
0 & \longrightarrow & [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [C(\eta), \Sigma^7 \mathcal{F}_2] & \longrightarrow & [\mathbb{S}^7, \Sigma^7 \mathcal{F}_2] \longrightarrow 0
\end{array} \quad (2.22)$$

where the rows are exact sequences induced from the cofiber sequences of $C(\iota \circ \eta)$ and $C(\eta)$. To compute $[M(\mathbb{Z}/2^r, 7), \Sigma^7 \mathcal{F}_2]$, we use the fiber sequence (2.2) of \mathcal{F}_2 that gives

$$[M(\mathbb{Z}/2^r, 7), \Sigma^7 \mathcal{F}_2] \cong H^7(M(\mathbb{Z}/2^r, 7); \mathbb{Z}/4),$$

and further, compute the cohomology using cofiber sequence $\mathbb{S}^7 \xrightarrow{\times 2^r} \mathbb{S}^7 \rightarrow M(\mathbb{Z}/2^r, 7)$, that gives

$$[M(\mathbb{Z}/2^r, 7), \Sigma^7 \mathcal{F}_2] \cong \begin{cases} \mathbb{Z}/4 & \text{if } r = 1, \\ \mathbb{Z}/2 & \text{if } r > 1. \end{cases} \quad (2.23)$$

A straightforward diagram chasing in (2.22) along with (2.21) shows the non-splitting of the short exact sequence at $[C(\iota \circ \eta), \Sigma^7 \mathcal{F}_2]$. Furthermore,

$$[C(\iota \circ \eta^2), \Sigma^7 \mathcal{F}_2] \cong \begin{cases} \mathbb{Z}/8 & \text{if } r = 1, \\ \mathbb{Z}/4 & \text{if } r > 1. \end{cases} \quad (2.24)$$

Now, consider the case when $Sq^2 \circ d_2$ is trivial. This implies $Sq^2 Sq^1 : H^6(M; \mathbb{Z}/2) \rightarrow H^9(M; \mathbb{Z}/2)$ is 0. Therefore

$$Sq^2 : \text{Coker}(Sq^1 : H^6(M; \mathbb{Z}/2) \rightarrow H^7(M; \mathbb{Z}/2)) \rightarrow H^9(M; \mathbb{Z}/2)$$

is well-defined and non-zero. Therefore, we have

$$\text{Coker}(Sq^1) = \text{Ker}(Sq^2) \oplus \mathbb{Z}/2.$$

Let $q : H^*(M; \mathbb{Z}/4) \rightarrow H^*(M; \mathbb{Z}/2)$ be the map induced by the non-trivial morphism $\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$.

Note that $\text{Im}(H^7(\widehat{M}; \mathbb{Z}/4) \rightarrow H^7(M; \mathbb{Z}/4)) \subseteq \text{Ker}(Sq^2 \circ q)$, where $\widehat{M} = M'$ or M'' as mentioned in (2.6) and (2.7). Then, using the exact sequence

$$0 \rightarrow \text{Coker}(Sq^1) \rightarrow H^7(M; \mathbb{Z}/4) \xrightarrow{q} \text{Ker}(Sq^1 : H^7(M; \mathbb{Z}/2) \rightarrow H^8(M; \mathbb{Z}/2)) \rightarrow 0,$$

observe that $H^7(M; \mathbb{Z}/4) = \widetilde{K} \oplus A$ where $A = \mathbb{Z}/2$ or $\mathbb{Z}/4$, and \widetilde{K} fits into the following possible non-trivial extension which is determined from the structure of $H^7(M; \mathbb{Z}/4)$

$$0 \longrightarrow \text{Ker}(Sq^2) \longrightarrow \widetilde{K} \longrightarrow \text{Ker}(Sq^2 \oplus Sq^1) \longrightarrow 0.$$

Additionally, note that $\widetilde{K} \subseteq H^7(M; \mathbb{Z}/4)$, is in fact the image of $H^7(\widehat{M}; \mathbb{Z}/4)$. Thus, we have

$$[M, \Sigma^7 \mathcal{F}_2] \cong \widetilde{K} \oplus \widetilde{A} \quad (2.25)$$

where \widetilde{A} is the non-trivial extension in

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0.$$

This completes the proof. \square

3 Smooth structures on 10-manifolds

We now consider at the analogue of Theorem 2.2 for simply-connected 10-manifolds. To address this, we need information about the 10th-Postnikov section $\tau_{\leq 10} pl/o$ of pl/o computed in [7]. In this context, observe that $Sq^2 \circ Sq^2 \circ d_2 = 0$ leads to the construction of a class $\Phi : \mathcal{F}_2 \rightarrow \Sigma^4 H\mathbb{Z}/2$ using the following diagram

$$\begin{array}{ccc} \Sigma^{-1} H\mathbb{Z}/4 & \xrightarrow{Sq^2 \circ d_2} & \Sigma^2 H\mathbb{Z}/2 & \longrightarrow & \mathcal{F}_2 \\ & & \downarrow Sq^2 & \nearrow \Phi & \\ & & \Sigma^4 H\mathbb{Z}/2 & & \end{array}$$

The operation Φ defines a secondary cohomology operation from $\text{Ker}(Sq^2 \circ d_2) (\subseteq H^i(M; \mathbb{Z}/4)) \rightarrow H^{i+4}(M; \mathbb{Z}/2)$. Now, let

$$\mathcal{E} := \text{Fibre}(\mathcal{F}_2 \xrightarrow{\Phi} \Sigma^4 H\mathbb{Z}/2). \quad (3.1)$$

With this, the 10th Postnikov section of pl/o is given by

$$\tau_{\leq 10} pl/o \simeq \Sigma^8 \mathcal{F} \vee \Sigma^7 \mathcal{E} \vee \Sigma^7 H\mathbb{Z}/7 \vee \Sigma^9 H\mathbb{Z}/2 \vee \Sigma^{10} H\mathbb{Z}/3. \quad (3.2)$$

Let M^{10} be a closed smooth manifold with $H_1(M^{10}) = 0$. Then there is a minimal cell structure [6, §4.C] on $M^{10}/M^{(6)}$ of the form

$$M^{10}/M^{(6)} \simeq \left(\vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \right) \bigcup_f e^{10},$$

where the attaching map f lies in $\pi_9 \left(\vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \right)$. By the connectivity argument,

$$\pi_9 \left(\vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \right) \cong \oplus_l \pi_9(\mathbb{S}^7) \oplus_k \pi_9(\mathbb{S}^8) \oplus_{(p,r) \in J} \pi_9(M(\mathbb{Z}/p^r, 7)), \quad (3.3)$$

with $\pi_9(\mathbb{S}^8) \cong \mathbb{Z}/2\{\eta\}$, $\pi_9(\mathbb{S}^7) \cong \mathbb{Z}/2\{\eta^2\}$ and $\pi_9(M(\mathbb{Z}/p^r, 7)) \cong \mathbb{Z}/2\{\iota \circ \eta^2\}$.

If M^{10} is a spin manifold, then there exists a higher order cohomology operation $\psi : H^6(M; \mathbb{Z}/4) \rightarrow H^{10}(M; \mathbb{Z}/2)$ corresponding to $(Sq^2 \circ Sq^2) + (Sq^3 \circ Sq^1) = 0$ in order to detect the map η^2 [19, Corollary 2, pg177]. Depending on either ψ is trivial or not, we have following possibilities:

(1) If ψ is trivial, then

$$M^{10}/M^{(6)} \simeq M^{10}/M^{(6)} \simeq \vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \vee \mathbb{S}^{10}. \quad (3.4)$$

(2) If ψ is non-trivial, then

$$M^{10}/M^{(6)} \simeq C(\eta^2) \vee M', \quad (3.5)$$

or

$$M^{10}/M^{(6)} \simeq C(\iota \circ \eta^2) \vee M'', \quad (3.6)$$

with $M' \simeq \vee_{l-1} \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7)$ and $M'' \simeq \vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J'} M(\mathbb{Z}/p^r, 7)$.

The following theorem applies the splitting of the Postnikov section (3.2) to the case of 10-manifolds.

Theorem 3.1. *Let M^{10} be a closed smooth 10-manifold with $H_1(M) = 0$, and $\Phi, \psi : H^6(M; \mathbb{Z}/4) \rightarrow H^{10}(M; \mathbb{Z}/2)$ be the secondary operations described in (3.1) and (3.4).*

(1) *Let M^{10} be a spin manifold.*

(a) *If $\Phi = 0$ then*

$$\begin{aligned} [M^{10}, PL/O] &\cong H^7(M^{10}; \mathbb{Z}/7) \oplus H^8(M^{10}; \mathbb{Z}/2) \oplus H^9(M^{10}; \mathbb{Z}/2) \oplus H^{10}(M^{10}; \mathbb{Z}/3) \\ &\oplus [M^{10}, \Sigma^7 \mathcal{E}]. \end{aligned}$$

Furthermore, if the higher order cohomology operation $\psi = 0$ then

$$[M^{10}, \Sigma^7 \mathcal{E}] \cong H^{10}(M; \mathbb{Z}/2) \oplus H^7(M; \mathbb{Z}/4).$$

On the other hand, if $\psi \neq 0$, then $[M, \Sigma^7 \mathcal{E}] = \tilde{K} \oplus \tilde{A}$, where $\tilde{K} \subseteq H^7(M; \mathbb{Z}/4)$, and \tilde{A} is the non-trivial extension satisfying the following sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \tilde{A} \longrightarrow A \longrightarrow 0.$$

with $A = \mathbb{Z}/4$ or $\mathbb{Z}/2$.

(b) If $\Phi \neq 0$ then

$$[M^{10}, PL/O] \cong H^7(M^{10}; \mathbb{Z}/28) \oplus H^8(M^{10}; \mathbb{Z}/2) \oplus H^9(M^{10}; \mathbb{Z}/2) \oplus H^{10}(M^{10}; \mathbb{Z}/3).$$

(2) If M^{10} is a non-spin manifold then

$$[M^{10}, PL/O] \cong H^7(M^{10}; \mathbb{Z}/7) \oplus H^{10}(M^{10}; \mathbb{Z}/3) \oplus \text{Ker}(Sq^2) \oplus \text{Ker}(Sq^2 \circ d_2),$$

where $Sq^2 : H^8(M^{10}; \mathbb{Z}/2) \rightarrow H^{10}(M^{10}; \mathbb{Z}/2)$, and $Sq^2 \circ d_2 : H^7(M; \mathbb{Z}/4) \rightarrow H^{10}(M; \mathbb{Z}/2)$.

Proof. Using the decomposition in (3.2), it is enough to compute $[M^{10}, \Sigma^8 \mathcal{F}]$ and $[M^{10}, \Sigma^7 \mathcal{E}]$.

For $[M^{10}, \Sigma^8 \mathcal{F}]$, we have a long exact sequence from fibration (2.2)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [M, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M, \Sigma^8 \mathcal{F}] & \longrightarrow & [M, \Sigma^8 H\mathbb{Z}/2] \xrightarrow{Sq^2} [M, \Sigma^{10} H\mathbb{Z}/2] \\ & & \parallel & & & & \parallel \\ & & 0 & & & & 0 \end{array}$$

which implies

$$[M^{10}, \Sigma^8 \mathcal{F}] = \begin{cases} H^8(M; \mathbb{Z}/2) & \text{if } M \text{ is spin,} \\ \text{Ker}(Sq^2) & \text{if } M \text{ is not spin.} \end{cases} \quad (3.7)$$

For $[M, \Sigma^7 \mathcal{E}]$, we have the following exact sequence obtained from (3.1)

$$\begin{array}{ccccccc} [M, \Sigma^8 H\mathbb{Z}/2] & & & & & & (3.8) \\ \downarrow & \searrow^{Sq^2} & & & & & \\ [M, \Sigma^6 \mathcal{F}_2] & \xrightarrow{\Phi} & [M, \Sigma^{10} H\mathbb{Z}/2] & \longrightarrow & [M, \Sigma^7 \mathcal{E}] & \longrightarrow & [M, \Sigma^7 \mathcal{F}_2] \xrightarrow{\Phi} [M, \Sigma^{11} H\mathbb{Z}/2] \\ & & \parallel & & & & \parallel \\ & & \mathbb{Z}/2 & & & & 0 \end{array}$$

If either M is non-spin or the map $\Phi \neq 0$, it is evident that $[M, \Sigma^7 \mathcal{E}] \cong [M, \Sigma^7 \mathcal{F}_2]$. The latter group can be computed using the long exact sequence

$$\begin{array}{ccccccc} \cdots & [M, \Sigma^9 H\mathbb{Z}/2] & \longrightarrow & [M, \Sigma^7 \mathcal{F}_2] & \longrightarrow & [M, \Sigma^7 H\mathbb{Z}/4] \xrightarrow{Sq^2 \circ d_2} [M, \Sigma^{10} H\mathbb{Z}/2], & (3.9) \\ & \parallel & & & & & \\ & 0 & & & & & \end{array}$$

which together with (3.8) gives

$$[M, \Sigma^7 \mathcal{E}] \cong [M, \Sigma^7 \mathcal{F}_2] = \text{Ker}(Sq^2 \circ d_2). \quad (3.10)$$

Thus

$$[M, \Sigma^7 \mathcal{E}] = \begin{cases} H^7(M; \mathbb{Z}/4) & \text{if } M \text{ is spin and } \Phi \neq 0 \\ \text{Ker}(Sq^2 \circ d_2) & \text{if } M \text{ is non-spin.} \end{cases} \quad (3.11)$$

Now, assume that M is spin and $\Phi = 0$. In this case, from (3.8) and (3.9), we obtain the following short exact sequence

$$0 \longrightarrow H^{10}(M; \mathbb{Z}/2) \longrightarrow [M, \Sigma^7 \mathcal{E}] \longrightarrow H^7(M; \mathbb{Z}/4) \longrightarrow 0. \quad (3.12)$$

In order to compute $[M, \Sigma^7 \mathcal{E}]$, we first compute $[C(\eta^2), \Sigma^7 \mathcal{E}]$ and $[C(\iota \circ \eta^2), \Sigma^7 \mathcal{E}]$. In this regard, consider the following commutative diagram whose rows and columns are cofiber sequences

$$\begin{array}{ccccc} \mathbb{S}^9 & \xrightarrow{\eta^2} & \mathbb{S}^7 & \longrightarrow & C(\eta^2) \\ \eta \downarrow & & \parallel & & \downarrow \\ \mathbb{S}^8 & \xrightarrow{\eta} & \mathbb{S}^7 & \longrightarrow & C(\eta) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma C(\eta) & \longrightarrow & * & \longrightarrow & \Sigma^2 C(\eta) \end{array}$$

This, together with (3.12) gives the following

$$\begin{array}{ccccccc}
& & & [C(\eta), \Sigma^7 \mathcal{E}] & & & \\
& & & \downarrow & & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & [C(\eta^2), \Sigma^7 \mathcal{E}] & \longrightarrow & \mathbb{Z}/4 \longrightarrow 0 \\
& & & & \downarrow & & \\
& & & & [\Sigma C(\eta), \Sigma^7 \mathcal{E}] & &
\end{array}$$

For the non-spin case, we have $[C(\eta), \Sigma^7 \mathcal{E}] \cong [C(\eta), \Sigma^7 \mathcal{F}_2] \cong \mathbb{Z}/8$, and $[\Sigma C(\eta), \Sigma^7 \mathcal{E}] = 0$ using (2.21). This implies that the short exact sequence for $C(\eta^2)$ does not split, and we get

$$[C(\eta^2), \Sigma^7 \mathcal{E}] \cong \mathbb{Z}/8. \quad (3.13)$$

To compute $[C(\iota \circ \eta^2), \Sigma^7 \mathcal{E}]$, consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] & \longrightarrow & [C(\iota \circ \eta^2), \Sigma^7 \mathcal{E}] & \longrightarrow & [M(\mathbb{Z}/2^r, 7), \Sigma^7 \mathcal{E}] \longrightarrow 0 \\
& & \uparrow \cong & & \downarrow & & \downarrow \\
0 & \longrightarrow & [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] & \longrightarrow & [C(\eta^2), \Sigma^7 \mathcal{E}] & \longrightarrow & [\mathbb{S}^7, \Sigma^7 \mathcal{E}] \longrightarrow 0 \\
& & & & & & \uparrow \cong \\
& & & & & & \mathbb{Z}/4
\end{array} \quad (3.14)$$

which is obtained from the cofiber sequences of $C(\eta^2)$ and $C(\iota \circ \eta^2)$. Since the short exact sequence at $[C(\eta^2), \Sigma^7 \mathcal{E}]$ does not split, we conclude that the short exact sequence at $[C(\iota \circ \eta^2), \Sigma^7 \mathcal{E}]$ does not split as well.

Thus, depending on r in $M(\mathbb{Z}/2^r, 7)$, (3.14) implies

$$[C(\iota \circ \eta^2), \Sigma^7 \mathcal{E}] \cong \begin{cases} \mathbb{Z}/8 & \text{if } r = 1, \\ \mathbb{Z}/4 & \text{if } r > 1. \end{cases} \quad (3.15)$$

Note that the 10-manifold spin case bifurcates into two sub-cases: whether the map $\psi : H^6(M; \mathbb{Z}/4) \rightarrow H^{10}(M; \mathbb{Z}/2)$ (as described in (3.4)) is non-trivial or trivial.

Assume that $\psi \neq 0$ for spin 10-manifold. This implies that the attaching map of the top cell $\mathbb{S}^9 \rightarrow M/M^{(6)} \simeq (\mathbb{S}^7)^{\vee \iota} \vee (\mathbb{S}^8)^{\vee s} \vee_{p,r} M(\mathbb{Z}/p^r, 7)$ attaches by η^2 onto some $(\mathbb{S}^7)^{\vee \iota}$ or some $M(\mathbb{Z}/p^r, 7)$.

Let $\tilde{K} \subseteq H^7(M; \mathbb{Z}/4)$ be $\text{Ker}(H^7(M; \mathbb{Z}/4) \rightarrow H^7(M; \mathbb{Z}/2) \xrightarrow{\psi} H^{10}(M; \mathbb{Z}/2))$. Further, let $H^7(M; \mathbb{Z}/4) = \tilde{K} \oplus A$ where $A = \mathbb{Z}/2$ or $\mathbb{Z}/4$ (similar to the 9-manifold scenario). Then, $[M, \Sigma^7 \mathcal{E}] = \tilde{K} \oplus \tilde{A}$ where \tilde{A} is the non-trivial extension in $0 \rightarrow \mathbb{Z}/2 \rightarrow \tilde{A} \rightarrow A \rightarrow 0$.

On the other hand, assume $\psi = 0$. Analogous to the 9-manifold spin case, we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & [M^{10}, \Sigma^{10} H^{\mathbb{Z}/2}] & \longrightarrow & [M^{10}, \Sigma^7 \mathcal{E}] & \longrightarrow & [M^{10}, \Sigma^7 H^{\mathbb{Z}/4}] \longrightarrow 0 \\
& & \uparrow \cong & & \uparrow & & \uparrow \cong \\
0 & \longrightarrow & [M^{10}/M^{(5)}, \Sigma^{10} H^{\mathbb{Z}/2}] & \longrightarrow & [M^{10}/M^{(5)}, \Sigma^7 \mathcal{E}] & \longrightarrow & [M^{10}/M^{(5)}, \Sigma^7 H^{\mathbb{Z}/4}] \longrightarrow 0.
\end{array} \quad (3.16)$$

This implies that

$$[M^{10}, \Sigma^7 \mathcal{E}] \cong [M^{10}/M^{(5)}, \Sigma^7 \mathcal{E}].$$

For the computation of $[M^{10}/M^{(5)}, \Sigma^7 \mathcal{E}]$, consider the following

$$\begin{array}{ccccccc}
0 & \longrightarrow & [M^{10}/M^{(5)}, \Sigma^{10} H\mathbb{Z}/2] & \longrightarrow & [M^{10}/M^{(5)}, \Sigma^7 \mathcal{E}] & \longrightarrow & [M^{10}/M^{(5)}, \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0 \\
& & \uparrow \cong & & \uparrow & & \uparrow \\
0 & \longrightarrow & [M^{10}/M^{(6)}, \Sigma^{10} H\mathbb{Z}/2] & \longrightarrow & [M^{10}/M^{(6)}, \Sigma^7 \mathcal{E}] & \longrightarrow & [M^{10}/M^{(6)}, \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0 \\
& & \uparrow \phi^* & & \uparrow \Psi^* & & \uparrow \gamma^* \\
0 & \longrightarrow & [\Sigma(M^{(6)}/M^{(5)}), \Sigma^{10} H\mathbb{Z}/2] & \longrightarrow & [\Sigma(M^{(6)}/M^{(5)}), \Sigma^7 \mathcal{E}] & \longrightarrow & [\Sigma(M^{(6)}/M^{(5)}), \Sigma^7 H\mathbb{Z}/4] \longrightarrow 0.
\end{array} \tag{3.17}$$

Using (3.4), the map $\Psi : M^{10}/M^{(6)} \rightarrow \Sigma(M^{(6)}/M^{(5)}) \simeq \vee_i \mathbb{S}^7$ decomposes in $\gamma : \vee_l \mathbb{S}^7 \vee_k \mathbb{S}^8 \vee_{(p,r) \in J} M(\mathbb{Z}/p^r, 7) \rightarrow \vee_i \mathbb{S}^7$ and $\phi : \mathbb{S}^{10} \rightarrow \vee_i \mathbb{S}^7$. Moreover, note that the attaching map of the 10-cell onto the 6-cell is a multiple of $\nu \in \pi_3^s$. Thus, we need to compute $\nu^* : [\mathbb{S}^7, \Sigma^7 \mathcal{E}] \rightarrow [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}]$.

Using (3.2), we get the following commutative diagram (observe that it suffices to work 2-locally)

$$\begin{array}{ccc}
[\mathbb{S}^7, \Sigma^7 \mathcal{E}] & \xrightarrow{\nu^*} & [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] \\
\parallel & & \parallel \\
[\mathbb{S}^7, \tau_{\leq 10} pl/o] & \xrightarrow{\nu^*} & [\mathbb{S}^{10}, \tau_{\leq 10} pl/o] \\
\downarrow & & \downarrow \\
[\mathbb{S}^7, >^6 \tau_{\leq 10} g/o] & \xrightarrow{\nu^*} & [\mathbb{S}^{10}, >^6 \tau_{\leq 10} g/o]
\end{array}$$

Since the map $[\mathbb{S}^{10}, \tau_{\leq 10} pl/o] \rightarrow [\mathbb{S}^{10}, >^6 \tau_{\leq 10} g/o]$ is an isomorphism, and $[\mathbb{S}^7, >^6 \tau_{\leq 10} g/o] = 0$, the map $\nu^* = 0$, and hence in (3.17) the induced map $\phi^* = 0$. Thus, $\text{Im}(\Psi^*) = \text{Im}(\gamma^*)$, which implies in the diagram (3.17) the sequence splits at $[M^{10}/M^{(6)}, \Sigma^7 \mathcal{E}]/\text{Im}(\Psi^*)$ and consequently at $[M^{10}/M^{(5)}, \Sigma^7 \mathcal{E}]$. Therefore the exact sequence in (3.16) splits for M .

This completes the proof for all 10 dimensional manifold M^{10} with $H_1(M^{10}) = 0$. \square

4 Inertia group of M^n for $8 \leq n \leq 10$

In the classification of smooth structures of manifolds, the group Θ_n plays an important role. Specifically, if we take a connected sum of a smooth manifold with an exotic sphere, the resulting manifold remains homeomorphic to the underlying topological manifold; however, it might change the diffeomorphism class. Therefore, the determination of subgroups of Θ_n namely, the inertia group, the homotopy inertia group, and the concordance inertia group correlates with the classification problem. We begin with the definitions of these groups.

Definition 4.1. *Let M^n be a closed oriented smooth manifold.*

- (i) *The inertia group of M^n is the subgroup $I(M^n) \subseteq \Theta_n$ of homotopy spheres Σ^n such that $M^n \# \Sigma^n$ is diffeomorphic to M^n .*
- (ii) *The homotopy inertia group $I_h(M^n)$ consists of $\Sigma^n \in I(M^n)$ such that there exists a diffeomorphism from $M^n \# \Sigma^n \rightarrow M^n$ that is homotopic to the canonical homeomorphism $h_{\Sigma^n} : M^n \# \Sigma^n \rightarrow M^n$.*
- (iii) *The concordance inertia group $I_c(M^n)$ consists of $\Sigma^n \in I_h(M^n)$ such that there exists a diffeomorphism from $M^n \# \Sigma^n \rightarrow M^n$ that is concordant to the canonical homeomorphism $h_{\Sigma^n} : M^n \# \Sigma^n \rightarrow M^n$.*

This section discusses the concordance and homotopy inertia groups of smooth manifold M^n for $8 \leq n \leq 10$.

4.1 Concordance inertia group

Recall that, if $d : M^n \rightarrow \mathbb{S}^n$ is a degree one map then the kernel of the induced map $d^* : [\mathbb{S}^n, PL/O] \rightarrow [M^n, PL/O]$ can be identified with $I_c(M^n)$.

Theorem 4.1. *Let M^8 be a closed manifold. Then*

$$I_c(M^8) = \{0\}.$$

Proof. The statement follows directly from the 8th Postnikov decomposition (2.1) of pl/o , wherein the map $d^* : [\mathbb{S}^8, \tau_{\leq 8} pl/o] \rightarrow [M^8, \tau_{\leq 8} pl/o]$ induced by the collapse map $d : M^8 \rightarrow M^8 \setminus \text{int}(\mathbb{D}^8)$ is injective. \square

Theorem 4.2. *Let M^9 be a closed oriented smooth manifold with $Sq^2 \circ d_2 : H^6(M; \mathbb{Z}/4) \rightarrow H^9(M; \mathbb{Z}/2)$.*

(1) *If M^9 is spin then*

$$I_c(M^9) = \{0\}.$$

(2) *If M^9 is non-spin and $Sq^2 \circ d_2 = 0$ then*

$$I_c(M^9) = \mathbb{Z}/2\{\eta \circ \epsilon\}.$$

(3) *If M^9 is non-spin and $Sq^2 \circ d_2 \neq 0$ then*

$$I_c(M^9) = \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\}.$$

Proof. Using the splitting (2.3), we have the following decomposition

$$[M^9, PL/O] \cong H^7(M^9; \mathbb{Z}/7) \oplus H^9(M^9; \mathbb{Z}/2) \oplus [M^9, \Sigma^8 \mathcal{F}] \oplus [M^9, \Sigma^7 \mathcal{F}_2]. \quad (4.1)$$

Since, the concordance inertia group of a manifold is isomorphic to the kernel of $d^* : [\mathbb{S}^9, PL/O] \rightarrow [M^9, PL/O]$, it is enough to check the $\text{Ker}(d^*)$ in (4.1) componentwise. As M^9 is oriented, the map $d^* : H^9(\mathbb{S}^9; \mathbb{Z}/2) \rightarrow H^9(M^9; \mathbb{Z}/2)$ is injective on the top cohomology. Thus, we focus on the maps $d^* : [\mathbb{S}^9, \Sigma^8 \mathcal{F}] \rightarrow [M^9, \Sigma^8 \mathcal{F}]$ and $d^* : [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$.

- (1) If M^9 is a spin manifold, then by the proof of Theorem 2.2(1), we have $[M^9, \Sigma^8 \mathcal{F}] \cong H^8(M^9; \mathbb{Z}/2) \oplus H^9(M^9; \mathbb{Z}/2)$ and $[M^9, \Sigma^7 \mathcal{F}_2] \cong H^7(M; \mathbb{Z}/4) \oplus H^9(M; \mathbb{Z}/2)$. Thus, the induced maps $d^* : [\mathbb{S}^9, \Sigma^8 \mathcal{F}] \rightarrow [M^9, \Sigma^8 \mathcal{F}]$ and $d^* : [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$ are injective, with their image isomorphic to $H^9(M^9; \mathbb{Z}/2)$. Therefore, the kernel of $d^* : [\mathbb{S}^9, PL/O] \rightarrow [M^9, PL/O]$ is trivial for the spin manifold M^9 .
- (2) Let M^9 be a non-spin manifold. From (2.6) and (2.7) we have $M^9/M^{(6)} \simeq C(\eta) \vee M'$ or $C(\iota \circ \eta) \vee M''$. Let $X = C(\eta)$ or $C(\iota \circ \eta)$. Since M' or M'' is in the 8-skeleton of $M^9/M^{(6)}$, the degree one map factors through X , and we have the following commutative diagram

$$\begin{array}{ccc} [\mathbb{S}^9, \Sigma^8 \mathcal{F}] & \xrightarrow{d^*} & [M^9, \Sigma^8 \mathcal{F}] \\ & \searrow d^* & \nearrow \\ & [X, \Sigma^8 \mathcal{F}] & \end{array}$$

It follows from (2.15) and (2.16) that the map $[\mathbb{S}^9, \Sigma^8 \mathcal{F}] \cong \mathbb{Z}/2\{\eta \circ \epsilon\} \rightarrow [X, \Sigma^8 \mathcal{F}]$ is trivial. Therefore, for the component $\Sigma^8 \mathcal{F}$, the induced map $d^* : [\mathbb{S}^9, \Sigma^8 \mathcal{F}] \rightarrow [M^9, \Sigma^8 \mathcal{F}]$ is trivial as well.

For the map $d^* : [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$, consider the following commutative diagram

$$\begin{array}{ccc} [\mathbb{S}^9, \Sigma^9 H\mathbb{Z}/2] & \xrightarrow{\cong} & [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \cong \mathbb{Z}/2\{\mu\} \\ \cong \downarrow d^* & & \downarrow d^* \\ [M^9, \Sigma^6 H\mathbb{Z}/4] & \xrightarrow{Sq^2 d_2} & [M^9, \Sigma^9 H\mathbb{Z}/2] \longrightarrow [M^9, \Sigma^7 \mathcal{F}_2] \end{array} \quad (4.2)$$

whose rows are part of exact sequences obtained from the fibration sequence for \mathcal{F}_2 as in (2.2). In the case $Sq^2 \circ d_2 = 0$, the map $[M^9, \Sigma^9 H\mathbb{Z}/2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$ becomes injective, and so is the $d^* : [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$.

Further, the map $Sq^2 \circ d_2$ in (4.2) being non-trivial implies the map $[M^9, \Sigma^9 H\mathbb{Z}/2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$ is trivial. As a consequence, the map $d^* : [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \rightarrow [M^9, \Sigma^7 \mathcal{F}_2]$ is trivial as well.

This completes the proof in all cases. \square

Theorem 4.3. *Let M^{10} be a closed simply-connected smooth manifold, and let $\Phi : H^6(M^{10}; \mathbb{Z}/4) \rightarrow H^{10}(M^{10}; \mathbb{Z}/2)$ be the secondary cohomology operation mentioned in (3.1).*

(1) If M^{10} is a spin manifold and $\Phi = 0$, then

$$I_c(M^{10}) = \{0\}.$$

(2) If M^{10} is either a spin manifold with $\Phi \neq 0$, or a non-spin manifold, then

$$I_c(M^{10}) = \mathbb{Z}/2\{\eta \circ \mu\}.$$

Proof. The splitting (3.2) gives the following

$$\begin{aligned} [M^{10}, PL/O] &\cong [M^{10}, \Sigma^8 \mathcal{F}] \oplus [M^{10}, \Sigma^7 \mathcal{E}] \oplus H^7(M^{10}; \mathbb{Z}/7) \oplus H^9(M^{10}; \mathbb{Z}/2) \\ &\oplus H^{10}(M^{10}; \mathbb{Z}/3). \end{aligned}$$

By using Theorem 2.2, we have $[\mathbb{S}^{10}, PL/O] \cong [\mathbb{S}^{10}, \Sigma^{10} H\mathbb{Z}/3] \oplus [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}]$. Analogous to Theorem 4.2, it is enough to check the componentwise kernel of $d^* : [\mathbb{S}^{10}, \Sigma^{10} H\mathbb{Z}/3] \oplus [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] \rightarrow [M^{10}, \Sigma^{10} H\mathbb{Z}/3] \oplus [M^{10}, \Sigma^7 \mathcal{E}]$.

Since M^{10} is simply-connected, the map $d^* : [\mathbb{S}^{10}, \Sigma^{10} H\mathbb{Z}/3] \rightarrow [M^{10}, \Sigma^{10} H\mathbb{Z}/3]$ is injective. This implies that the generator β_1 of $[\mathbb{S}^{10}, \Sigma^{10} H\mathbb{Z}/3] = \mathbb{Z}/3\{\beta_1\}$ does not belong to $I_c(M^{10})$.

So it remains to compute the kernel of $d^* : [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] = \mathbb{Z}/2\{\eta \circ \mu\} \rightarrow [M^{10}, \Sigma^7 \mathcal{E}]$. For that, let us consider the following commutative diagram

$$\begin{array}{ccccc} [M^{10}, \Sigma^8 H\mathbb{Z}/2] & & [\mathbb{S}^{10}, \Sigma^{10} H\mathbb{Z}/2] & \xrightarrow{\cong} & [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] \\ & \searrow Sq^2 & \cong \downarrow d^* & & \downarrow d^* \\ [M^{10}, \Sigma^{(6)} \mathcal{F}_2] & \xrightarrow{\Phi} & [M^{10}, \Sigma^{10} H\mathbb{Z}/2] & \longrightarrow & [M^{10}, \Sigma^7 \mathcal{E}] \end{array} \quad (4.3)$$

where the bottom row is a part of the exact sequence (3.8). From the diagram, it is clear that the map $d^* : [\mathbb{S}^{10}, \Sigma^7 \mathcal{E}] \rightarrow [M^{10}, \Sigma^7 \mathcal{E}]$ is injective or trivial if and only if the map from $[M^{10}, \Sigma^{10} H\mathbb{Z}/2] \rightarrow [M^{10}, \Sigma^7 \mathcal{E}]$ is injective or trivial, respectively. For the spin case, whenever $\Phi = 0$, the map $[M^{10}, \Sigma^{10} H\mathbb{Z}/2] \rightarrow [M^{10}, \Sigma^7 \mathcal{E}]$ is injective, else it is trivial. On the other hand, Φ is always non-trivial for the non-spin case. Hence the map $[M^{10}, \Sigma^{10} H\mathbb{Z}/2] \rightarrow [M^{10}, \Sigma^7 \mathcal{E}]$ is trivial. This completes the proof in all cases. \square

4.2 Homotopy inertia group

In surgery theory, there is a natural map $f_{M^n} : \Theta_n \rightarrow S^{\text{Diff}}(M^n)$, where $S^{\text{Diff}}(M^n)$ is the homotopy smooth structure set of M^n [3, 25]. Then, the homotopy inertia group $I_h(M^n)$ can be identified with $\text{Ker}(f_{M^n})$. If M^n is a closed oriented smooth manifold, then we have the following commutative square

$$\begin{array}{ccc} [\mathbb{S}^n, PL/O] & \xrightarrow{\psi_*} & [\mathbb{S}^n, G/O] \\ f_{M^n} \downarrow & & \downarrow d^* \\ S^{\text{Diff}}(M^n) & \xrightarrow{g'} & [M^n, G/O], \end{array} \quad (4.4)$$

where $\psi : PL/O \rightarrow G/O$ is a natural fibration, d^* is induced from the degree one map, and g' is the part of surgery exact sequence. Note that, if n is even and M^n is simply-connected then the maps ψ_* and g' are injective.

Recall that, $I_c(M^n) \subseteq I_h(M^n)$. Hence it is enough to discuss those elements in $\Theta_n \equiv [\mathbb{S}^n, PL/O]$ which are not in $I_c(M^n)$.

We observe the fact that follows from straightforward diagram chasing. Consider the following commutative diagram whose rows and columns are part of exact sequences of abelian groups, and ξ and β are surjective homomorphisms.

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \downarrow \xi & & \downarrow \phi & & \downarrow \psi \\ E & \xrightarrow{\gamma} & F & \xrightarrow{\eta} & G \\ & & \downarrow \theta & & \\ & & H & & \end{array} \quad (4.5)$$

Claim: If $y \notin \text{Im}(\phi)$ and $\eta(y) = z \neq 0$ then $z \notin \text{Im}(\psi)$.

We prove this by contradiction. Suppose $z \in \text{Im}(\psi)$. Then there exist elements b, c such that $\beta(b) = c$ and $\psi(c) = z$. For $\phi(b) = x$, we have $\eta(x) = z$ and $\theta(x) = 0$. Hence $\eta(x - y) = 0$, which also implies that there exists an element e such that $\gamma(e) = x - y$. As ξ is surjective, commutativity of diagram allows us to affirm the existence of b' such that $\phi(b') = x - y$. Consequently, $\theta(x - y) = \theta(x) - \theta(y) = 0$, implying that $\theta(y) = 0$. However, this contradicts the fact that $y \notin \text{im}(\phi)$. Therefore, the claim is established.

Let $M^{(6)}$ be the 6-skeleton of an n -manifold M for $n \leq 10$. Consider the following commutative diagram,

$$\begin{array}{ccc} [\Sigma M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \xlongequal{\quad} & [\Sigma M^{(6)}, K(\mathbb{Z}_2, 7)] \\ \downarrow j_* & & \downarrow j_* \\ [\Sigma M^{(6)}, \tau_{\leq 10} pl/o] & \xlongequal{\quad} & [\Sigma M^{(6)}, K(\mathbb{Z}_4, 7)] \end{array}$$

where the map j_* is induced from the fibration

$$>^6\tau_{\leq 10}(\Sigma^{-1}g/pl) \xrightarrow{j} \tau_{\leq 10} pl/o \xrightarrow{\psi} >^6\tau_{\leq 10} g/o. \quad (4.6)$$

Observe that, the map $j_* : [\Sigma M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [\Sigma M^{(6)}, \tau_{\leq 10} pl/o]$ is surjective.

Further, consider the following commutative diagram

$$\begin{array}{ccccc} [\Sigma M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \rightarrow & [M/M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \xrightarrow{q^*} & [M, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \\ \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\ [\Sigma M^{(6)}, \tau_{\leq 10} pl/o] & \longrightarrow & [M/M^{(6)}, \tau_{\leq 10} pl/o] & \xrightarrow{q^*} & [M, \tau_{\leq 10} pl/o] \\ & & \downarrow \psi_* & & \downarrow \psi_* \\ & & [M/M^{(6)}, >^6\tau_{\leq 10} g/o] & \xrightarrow{q^*} & [M, >^6\tau_{\leq 10} g/o] \end{array}$$

whose columns and rows are part of exact sequences of abelian groups obtained from the fibration in (4.6), and the cofibration $M^{(6)} \hookrightarrow M \xrightarrow{q} M/M^{(6)}$, respectively. Furthermore, the maps $j_* : [\Sigma M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [\Sigma M^{(6)}, \tau_{\leq 10} pl/o]$ and $q^* : [M/M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [M, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)]$ are surjective. Thus, the Claim 4.2 holds for this commutative diagram, and hence we get the next proposition:

Proposition 4.4. *If $y \notin \text{Im}(j_* : [M/M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [M/M^{(6)}, \tau_{\leq 10} pl/o])$ then $q^*(y) \notin \text{Im}(j_* : [M, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [M, \tau_{\leq 10} pl/o])$ provided $q^*(y) \neq 0$.*

The map of our interest from (4.4), especially is $d^* : [\mathbb{S}^n, >^6\tau_{\leq 10} g/o] \rightarrow [M, >^6\tau_{\leq 10} g/o]$. For that, the way we are going to use Proposition 4.4 is as follows. Consider the following diagram

$$\begin{array}{ccccc} & & [M/M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \xrightarrow{q^*} & [M, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \\ & & \downarrow j_* & & \downarrow j_* \\ [\mathbb{S}^n, \tau_{\leq 10} pl/o] & \xrightarrow{d^*} & [M/M^{(6)}, \tau_{\leq 10} pl/o] & \xrightarrow{q^*} & [M, \tau_{\leq 10} pl/o] \\ \downarrow \psi_* & & \downarrow \psi_* & & \downarrow \psi_* \\ [\mathbb{S}^n, >^6\tau_{\leq 10} g/o] & \xrightarrow{d^*} & [M/M^{(6)}, >^6\tau_{\leq 10} g/o] & \xrightarrow{q^*} & [M, >^6\tau_{\leq 10} g/o] \\ & & \searrow & \nearrow & \\ & & & d^* & \end{array} \quad (4.7)$$

whose columns are part of exact sequence using (4.6), the map d^* is induced degree one maps, and q^* is induced quotient map. For $n = 8, 9$ and 10 the map $\psi_* : [\mathbb{S}^n, \tau_{\leq 10} pl/o] \rightarrow [\mathbb{S}^n, >^6\tau_{\leq 10} g/o]$ is injective, surjective and isomorphism respectively. In most of the cases discussed below, we applied the Proposition 4.4. For that, we use diagrams (4.2) and (4.7) together, and show that if $0 \neq y \in d^*([\mathbb{S}^n, \tau_{\leq 10} pl/o]) \subseteq [M/M^{(6)}, \tau_{\leq 10} pl/o]$, then $y \notin \text{Im}(j_* : [M/M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [M/M^{(6)}, \tau_{\leq 10} pl/o])$.

Theorem 4.5. *Let M^8 be a closed manifold. Then*

$$I_h(M^8) = \{0\}.$$

Proof. In (4.4), as $[\mathbb{S}^8, PL/O] = \mathbb{Z}/2\{\epsilon\}$ and the map $\psi_* : [\mathbb{S}^8, PL/O] \rightarrow [\mathbb{S}^8, G/O]$ is injective, it is enough to prove that the image of $\psi_*(\epsilon)$ under $d^* : [\mathbb{S}^8, G/O] \rightarrow [M^8, G/O]$ is non-zero. By Theorem 4.1, the map $d^* : [\mathbb{S}^8, \tau_{\leq 8} pl/o] \rightarrow [M^8, \tau_{\leq 8} pl/o]$ is injective. Recall that, ${}^{>6}\tau_{\leq 8}(\Sigma^{-1}g/pl)_{(2)} \simeq \Sigma^7 H\mathbb{Z}_{(2)}$ and $\tau_{\leq 8}(pl/o) \simeq \Sigma^7 H\mathbb{Z}/4 \vee \Sigma^8 H\mathbb{Z}/2$, such that the map $\Sigma^7 H\mathbb{Z}_{(2)} \rightarrow \Sigma^8 H\mathbb{Z}/2$ is null homotopic. So we get that $d^*([\mathbb{S}^8, \tau_{\leq 8} pl/o]) \cap \text{Im}(j_* : [M^8, {}^{>6}\tau_{\leq 8}(\Sigma^{-1}g/pl)] \rightarrow [M^8, \tau_{\leq 8} pl/o]) = \{0\}$. Therefore, in (4.7), $\psi_* : [M^8, \tau_{\leq 8} pl/o] \rightarrow [M^8, {}^{>6}\tau_{\leq 8} g/o]$ maps $d^*([\mathbb{S}^8, \tau_{\leq 8} pl/o])$ injectively. Using this in (4.4), completes the proof. \square

The following corollary uses the fact that every orientation preserving self-homotopy equivalence of $\mathbb{R}P^8$ is homotopic to a diffeomorphism.

Corollary 4.6. $I(\mathbb{R}P^8) = \mathbb{Z}/2$.

In the 9-dimensional case, using (4.7) we first discuss the map $d^* : [\mathbb{S}^9, {}^{>6}\tau_{\leq 9} g/o] \rightarrow [M^9/M^{(6)}, {}^{>6}\tau_{\leq 9} g/o]$ in the next lemma.

Lemma 4.7. *Let M^9 be a closed oriented smooth manifold with $Sq^2 \circ d_2 : H^6(M^9; \mathbb{Z}/4) \rightarrow H^9(M^9; \mathbb{Z}/2)$.*

- (1) *If M^9 is spin then $d^* : [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \rightarrow [M^9/M^{(6)}, {}^{>6}\tau_{\leq 10} g/o]$ is injective.*
- (2) *Suppose M^9 be non-spin and $Sq^2 \circ d_2$ be trivial.*
 - (a) *If M^9 satisfies (2.6) then $d^* : [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \rightarrow [M^9/M^{(6)}, {}^{>6}\tau_{\leq 10} g/o]$ is trivial.*
 - (b) *If M^9 satisfies (2.7) then $d^* : [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \rightarrow [M^9/M^{(6)}, {}^{>6}\tau_{\leq 10} g/o]$ has $\text{Ker}(d^*) = \mathbb{Z}/2\{\eta \circ \epsilon\}$ and $\text{Im}(d^*) = \mathbb{Z}/2\{\mu\}$.*

Proof. Recall that $[\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \cong [\mathbb{S}^9, \Sigma^8 \mathcal{F}] \oplus [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2]$. Additionally, $[\mathbb{S}^9, \Sigma^8 \mathcal{F}] = \mathbb{Z}/2\{\eta \circ \epsilon\}$ and $[\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] = \mathbb{Z}/2\{\mu\}$.

- (1) If M^9 is spin manifold then the $M^9/M^{(6)}$ decomposition (2.5) directly gives the injectivity of the map $d^* : [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \rightarrow [M^9/M^{(6)}, {}^{>6}\tau_{\leq 10} g/o]$.
- (2) Let M^9 be a non-spin manifold and $Sq^2 \circ d_2 = 0$. Note that, from (2.6), (2.7) the degree map induces the following

$$\begin{array}{ccc} [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] & \xrightarrow{d^*} & [M^9/M^{(6)}, {}^{>6}\tau_{\leq 10} g/o] \\ & \searrow d^* & \swarrow p^* \\ & [X, {}^{>6}\tau_{\leq 10} g/o] & \end{array}$$

where $X = C(\eta)$ or $C(\iota \circ \eta)$, and $p : M^9/M^{(6)} \rightarrow X$ is the projection map. Now, we will discuss the image of $\{\mu\} \in [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2]$ under $d^* : [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \rightarrow [X, {}^{>6}\tau_{\leq 10} g/o]$.

- (a) If M^9 satisfies (2.6) then $X = C(\eta)$. Consider the induced long exact sequence due to the cofiber sequence for $\eta : \mathbb{S}^8 \rightarrow \mathbb{S}^7$

$$\dots \longrightarrow [\mathbb{S}^8, {}^{>6}\tau_{\leq 10} g/o] \xrightarrow{\eta^*} [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o] \longrightarrow [C(\eta), {}^{>6}\tau_{\leq 10} g/o]$$

$$\begin{array}{ccc} & \swarrow & \\ & [\mathbb{S}^7, {}^{>6}\tau_{\leq 10} g/o] & \xrightarrow{\eta^*} [\mathbb{S}^8, {}^{>6}\tau_{\leq 10} g/o] \end{array}$$

Since $[\mathbb{S}^7, {}^{>6}\tau_{\leq 10} g/o] = 0$ and $\eta^* : [\mathbb{S}^8, {}^{>6}\tau_{\leq 10} g/o] \rightarrow [\mathbb{S}^9, {}^{>6}\tau_{\leq 10} g/o]$ is surjective, gives

$$[C(\eta), {}^{>6}\tau_{\leq 10} g/o] = 0$$

Therefore, in (2) the map $d^* : [\mathbb{S}^9, \Sigma^7 \mathcal{F}_2] \rightarrow [M^9/M^{(6)}, {}^{>6}\tau_{\leq 10} g/o]$ is trivial.

- (b) If M^9 satisfies (2.7) then $X = C(\iota \circ \eta)$. Now consider the induced long exact sequence from the cofiber sequence for $\iota \circ \eta : \mathbb{S}^8 \rightarrow \mathbb{S}^7 \rightarrow M(\mathbb{Z}/2^r, 7)$

$$\begin{array}{ccccc}
& & [\mathbb{S}^8, >^6\tau_{\leq 10} g/o] & & \\
& \swarrow \text{dashed} & \downarrow \eta^* & & \\
[\Sigma M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10} g/o] & \longrightarrow & [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] & \xrightarrow{d^*} & [C(\iota \circ \eta), >^6\tau_{\leq 10} g/o] \\
& \nwarrow \text{solid} & \downarrow \text{solid} & & \\
[M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10} g/o] & \longrightarrow & [\mathbb{S}^8, >^6\tau_{\leq 10} g/o] & & \\
& \swarrow \text{dashed} & \uparrow \eta^* & & \\
& & [\mathbb{S}^7, >^6\tau_{\leq 10} g/o] & &
\end{array}$$

where $\text{Im}([\Sigma M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10} g/o] \longrightarrow [\mathbb{S}^8, >^6\tau_{\leq 10} g/o]) \cong \mathbb{Z}/2\{\epsilon\} \subseteq \mathbb{Z}/2\{\bar{\nu}\} \oplus \mathbb{Z}/2\{\epsilon\}$. This implies

$$\text{Ker}(d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \longrightarrow [C(\iota \circ \eta), >^6\tau_{\leq 10} g/o]) = \mathbb{Z}/2\{\eta \circ \epsilon\} \quad (4.8)$$

and

$$\text{Im}(d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \longrightarrow [C(\iota \circ \eta), >^6\tau_{\leq 10} g/o]) = \mathbb{Z}/2\{\mu\}. \quad (4.9)$$

This completes the proof in all cases. \square

Theorem 4.8. *Let M^9 be a closed oriented smooth manifold with $Sq^2 \circ d_2 : H^6(M^9; \mathbb{Z}/4) \rightarrow H^9(M^9; \mathbb{Z}/2)$.*

- (1) *If M^9 is spin then the map $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ is injective.*
- (2) *Let M^9 be non-spin and the map $Sq^2 \circ d_2$ be trivial.*
 - (a) *If M^9 satisfies (2.6) then $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ maps $\{\mu\}$ to 0.*
 - (b) *If M^9 satisfies (2.7) then $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ maps $\{\mu\}$ injectively.*

Proof. (1) Let M^9 be a spin manifold. Applying Proposition 4.4, using Lemma 4.7(1) and Theorem 4.2(1) together, we obtain the map $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ is injective.

- (2) Let M^9 be non-spin and the map $Sq^2 \circ d_2$ be trivial.
 - (a) If M^9 satisfies (2.6), then from Lemma 4.7(2)(a) we know the map $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9/M^{(6)}, >^6\tau_{\leq 10} g/o]$ is trivial. Since, the degree one map $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ factors through $[M^9/M^{(6)}, >^6\tau_{\leq 10} g/o]$, makes the statement clear in this case.
 - (b) Now consider the case when M^9 satisfies (2.7). From Lemma 4.7(2)(b) it follows that $\mathbb{Z}/2\{\mu\}$ is in the $\text{Im}(d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9/M^{(6)}, >^6\tau_{\leq 10} g/o])$. As, under $\psi_* : [\mathbb{S}^9, \tau_{\leq 10} pl/o] \rightarrow [\mathbb{S}^9, >^6\tau_{\leq 10} g/o]$ the group $\mathbb{Z}/2\{\mu\}$ is mapped injectively, implies in (4.7), $d^*(\mu) \notin \text{Im}(j_* : [M^9/M^{(6)}, >^6\tau_{\leq 10} \Sigma^{-1}(g/pl)] \rightarrow [M^9/M^{(6)}, \tau_{\leq 10} pl/o])$. Also, it follows from Theorem 4.2(2)(a) that $d^*(\mu)$ is non-zero in $[M^9, \tau_{\leq 10} pl/o]$. Finally using Proposition 4.4, we get that $d^*(\mu)$ gets mapped non-trivially under $\psi_* : [M^9, \tau_{\leq 10} pl/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$. Therefore, μ gets mapped non-trivially under $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ as well. Hence the statement. \square

Theorem 4.9. *Let M^9 be a closed oriented smooth manifold with the map $Sq^2 \circ d_2 : H^6(M^9; \mathbb{Z}/4) \rightarrow H^9(M^9; \mathbb{Z}/2)$.*

- (1) *If M^9 is a spin manifold then*

$$I_h(M^9) \bigcap \bigcap \Theta_9/bP_{10} = \{0\}.$$

- (2) *If M^9 is a spin simply-connected manifold then*

$$I_h(M^9) = \{0\}.$$

- (3) *Suppose M^9 is non-spin and the map $Sq^2 \circ d_2 = 0$.*

(a) If M^9 satisfies 2.6 then

$$I_h(M^9) \supseteq \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\}.$$

(b) If M^9 satisfies 2.7 then

$$I_h(M^9) \supseteq \mathbb{Z}/2\{\eta \circ \epsilon\} \text{ and } I_h(M^9) \not\supseteq \mathbb{Z}/2\{\mu\}.$$

(4) If M^9 is non-spin with $Sq^2 \circ d_2 \neq 0$ then

$$I_h(M^9) \supseteq \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\}.$$

Proof. Recall that, $I_h(M^n) = \text{Ker}(f_{M^n} : [\mathbb{S}^n, PL/O] \rightarrow S^{\text{Diff}}(M^n))$, where f_{M^n} is part of the commutative square (4.4).

- (1) Let M^9 be a spin manifold. Recollect that, $\Theta_9/bP_{10} = \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\} \subseteq [\mathbb{S}^9, \tau_{\leq 10} pl/o]$, and in (4.4) the map $\psi_* : [\mathbb{S}^9, \tau_{\leq 10} pl/o] \rightarrow [\mathbb{S}^9, >^6\tau_{\leq 10} g/o]$ maps $\mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\}$ injectively. Furthermore, by Theorem 4.8(1), the map $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$ is injective, thereby making the statement clear in this case.
- (2) If M^9 is simply-connected spin manifold then the result [4, Proposition II.2] says that $I_h(M^9) \subseteq \text{Coker}(J_9) = \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\}$. Hence, from statement (1) we get $I_h(M^9) = 0$.
- (3) Let M^9 be a non-spin manifold and the map $Sq^2 \circ d_2 = 0$.
 - (a) Suppose M^9 satisfies 2.6. Consider the following commutative diagram

$$\begin{array}{ccccc} & & & & [\mathbb{S}^9, PL/O] \\ & & & \nearrow \mathbf{d} & \downarrow \psi_* \\ [\Sigma M_0^9, G/O] & \xrightarrow{(\Sigma h)^*} & [\mathbb{S}^9, G/O] & \xrightarrow{d^*} & [M^9, G/O] \end{array}$$

where $M_0^9 = M^9 \setminus \text{int}(\mathbb{D}^9)$, and $\mathbf{d} : [\Sigma M_0^9, G/O] \rightarrow [\mathbb{S}^9, PL/O]$ is the map given in [4, Proposition 3.1], such that $I_h(M^9) = \text{Im}(\mathbf{d})$. The bottom row is obtained from cofiber sequence $M^9 \xrightarrow{d} \mathbb{S}^9 \xrightarrow{\Sigma h} \Sigma M_0^9$ such that, h is the top cell attaching map of M^9 and d is the degree one map.

Using Theorem 4.2(2)(a) and Theorem 4.8(1)(a), together gives $d^* : [\mathbb{S}^9, G/O] \rightarrow [M^9, G/O]$ is trivial. Hence, in the above diagram, $(\Sigma h)^* : [\Sigma M_0^9, G/O] \rightarrow [\mathbb{S}^9, G/O]$ is surjective. As $\mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\} \subseteq [\mathbb{S}^9, PL/O]$ makes the statement clear in this case.

- (b) If M^9 satisfies 2.7, then by Theorem 4.2(2)(a) we know that $\mathbb{Z}/2\{\eta \circ \mu\} \subseteq I_h(M^9)$. Now, in (4.4) recall that, $\psi_*(\mu)$ is non-zero in $[\mathbb{S}^9, >^6\tau_{\leq 10} g/o]$, and the Theorem 4.8(2)(b) shows that, $\mathbb{Z}/2\{\mu\}$ is mapped injectively under the map $d^* : [\mathbb{S}^9, >^6\tau_{\leq 10} g/o] \rightarrow [M^9, >^6\tau_{\leq 10} g/o]$. Therefore, $f_{M^9}(\mu)$ is also non-zero, which complete the proof for this case.
- (4) The statement for the case when M^9 is non-spin and the map $Sq^2 \circ d_2 \neq 0$ is clear from Theorem 4.2(2)(b) since $\mathbb{Z}/2\{\eta \circ \epsilon\} \oplus \mathbb{Z}/2\{\mu\} = I_c(M^9) \subseteq I_h(M^9)$.

□

Recall that, $L^9(m)$ is a closed oriented smooth 9-manifold, and its inertia group, which depends on m , is discussed below.

Theorem 4.10. *Let m be a positive integer and n be a non-negative integer.*

(1) If $m = 2n + 1$ then

$$I(L^9(m)) = \{0\}.$$

(2) If $m = 4n + 2$ then

$$I(L^9(m)) = \mathbb{Z}/2\{\eta \circ \epsilon\}.$$

(3) If $m = 4n$ then

$$I(L^9(m)) = \mathbb{Z}/2\{\eta \circ \epsilon\} \oplus bP_{10}.$$

Proof. Note that, if m is odd then $L^9(m)$ is a spin manifold; otherwise, it is non-spin. Observe that, if $L^9(m)$ is non-spin, then $Sq^2 \circ d_2 = 0$, and $L^9(m)$ satisfies (2.7). Additionally, $bP_{10} \subseteq I_h(L^9(m))$ if and only if $4|m$ by [12, Theorem 4.2]. Combining this together with the fact that; an

orientation-preserving self-homotopy equivalence of a lens space is homotopic to the identity, and using Theorem 4.9, completes the proof. \square

Note that when $m = 2$, $L^{2n+1}(m)$ is nothing but the usual real projective space $\mathbb{R}P^{2n+1}$. The following remark is part of Theorem 4.9 and serves as a special case of $m = 2$ in Theorem 4.10.

Remark 4.11. *There exists an unique exotic sphere $\Sigma \in \Theta_9$ such that $\mathbb{R}P^9 \# \Sigma$ is diffeomorphic to $\mathbb{R}P^9$.*

Now, in the 10-dimensional case for the homotopy inertia group we just need to check the case when M^{10} is spin and $\Phi = 0$. For the same, recall the homotopy decompositions of $M^{10}/M^{(6)}$ (3.4), (3.5) and (3.6). Since $\pi_{10}(pl/o) \cong \pi_{10}(g/o) \cong \mathbb{Z}_2\{\eta \circ \mu\} \oplus \mathbb{Z}_3\{\beta_1\}$, it suffices to work locally at prime 2 and 3. Let us first discuss the 3-localized case.

Theorem 4.12. *Let M^{10} be a closed orientated smooth manifold. Then the induced degree one map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(3)} \rightarrow [M^{10}, >^6\tau_{\leq 10} g/o]_{(3)}$ is injective.*

Proof. Recall that $g/o_{(3)} \simeq \text{cok}(J)_{(3)} \times bso_{(3)}$. Since $[\mathbb{S}^{10}, bso]_{(3)} = 0$; we get $[\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(3)} = [\mathbb{S}^{10}, \text{cok}(J)]_{(3)}$. Note that

$$\pi_i(\text{cok}(J))_{(3)} \begin{cases} 0 & \text{if } i \leq 9 \\ \mathbb{Z}/3 & \text{if } i = 10 \end{cases}$$

This implies $\tau_{\leq 10} \text{cok}(J)_{(3)} = K(\mathbb{Z}/3, 10)$. Therefore the statement is true because $d^* : H^{10}(\mathbb{S}^{10}; \mathbb{Z}/3) \rightarrow H^{10}(M^{10}; \mathbb{Z}/3)$ is injective. \square

Now let us work 2-locally.

Theorem 4.13. *Let M^{10} be a closed smooth 10-manifold with $H_1(M) = 0$. Then the induced degree one map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [M^{10}, >^6\tau_{\leq 10} g/o]_{(2)}$ is injective if M^{10} satisfy (3.4) or (3.6), and is trivial if M^{10} satisfy (3.5).*

Proof. Consider the case when M^{10} satisfies (3.4). Since $[\mathbb{S}^{10}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)]_{(2)} = 0$, $d^*([\mathbb{S}^{10}, \tau_{\leq 10} pl/o]_{(2)}) \cap j_*([M^{10}/M^{(6)}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)]_{(2)}) = \{0\}$. From Theorem 4.3, we know that $d^*(\eta \circ \mu)$ is non-zero in $[M^{10}, \tau_{\leq 10} pl/o]$. Therefore, by Proposition 4.4 the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [M^{10}, >^6\tau_{\leq 10} g/o]_{(2)}$ is injective.

For the remaining cases, the homotopy decompositions of $M^{10}/M^{(6)}$ in (3.5) or (3.6), gives the following commutative diagram

$$\begin{array}{ccc} [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] & \xrightarrow{d^*} & [M^{10}/M^{(6)}, >^6\tau_{\leq 10} g/o] \\ & \searrow d^* & \swarrow p^* \\ & [X, >^6\tau_{\leq 10} g/o] & \end{array}$$

where $X = C(\eta^2)$ or $C(\iota \circ \eta^2)$, and $p : M^{10}/M^{(6)} \rightarrow X$ is the projection map. According to our interest for homotopy inertia group, as p^* is injective, so first we will check the image of $\{\eta \circ \mu\} \in [\mathbb{S}^{10}, \Sigma^7\mathcal{E}]$ under $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \rightarrow [X, >^6\tau_{\leq 10} g/o]$.

Now, suppose M^{10} satisfies (3.5). Consider the cofiber sequence for $\eta^2 : \mathbb{S}^9 \rightarrow \mathbb{S}^7$

$$\begin{array}{ccc} [\mathbb{S}^8, >^6\tau_{\leq 10} g/o]_{(2)} & \xrightarrow{(\eta^2)^*} & [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \longrightarrow [C(\eta^2), >^6\tau_{\leq 10} g/o]_{(2)} \\ & & \swarrow \\ [\mathbb{S}^7, >^6\tau_{\leq 10} g/o]_{(2)} & \xrightarrow{(\eta^2)^*} & [\mathbb{S}^9, >^6\tau_{\leq 10} g/o]_{(2)} \end{array}$$

Here, the group $[\mathbb{S}^7, >^6\tau_{\leq 10} g/o]_{(2)} = 0$. Further, we have

$$\begin{array}{ccc} [\mathbb{S}^8, >^6\tau_{\leq 10} g/o]_{(2)} & \cong & [\mathbb{S}^8, \text{cok}(J)]_{(2)} \oplus [\mathbb{S}^8, bso]_{(2)} \\ \downarrow (\eta^2)^* & & \downarrow (\eta^2)^* \quad \downarrow (\eta^2)^* \\ [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} & \cong & [\mathbb{S}^{10}, \text{cok}(J)]_{(2)} \oplus [\mathbb{S}^{10}, bso]_{(2)} \end{array}$$

where $[\mathbb{S}^{10}, \text{cok}(J)]_{(2)} = 0$ and $(\eta^2)^* : [\mathbb{S}^8, \text{bso}]_{(2)} \rightarrow [\mathbb{S}^{10}, \text{bso}]_{(2)}$ is non-zero [1, Proposition 7.1]. Therefore, the map $(\eta^2)^* : [\mathbb{S}^8, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)}$ is surjective, proving that in the exact sequence 4.2

$$[C(\eta^2), >^6\tau_{\leq 10} g/o]_{(2)} = 0. \quad (4.10)$$

This gives in (4.2) the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [M^{10}/M^{(6)}, >^6\tau_{\leq 10} g/o]_{(2)}$ is trivial. As a result, in (4.7) the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [M^{10}, >^6\tau_{\leq 10} g/o]_{(2)}$ is trivial.

For the case when M^{10} satisfies (3.6), consider the exact sequence

$$\begin{array}{ccc} [\Sigma M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \rightarrow & [\mathbb{S}^{10}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \xrightarrow{d^*} [C(\iota \circ \eta^2), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \\ & \searrow & \swarrow \\ [M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \xrightarrow{\quad} & [\mathbb{S}^9, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \\ & \searrow \text{dashed} & \swarrow \text{dashed} \\ & & [\mathbb{S}^7, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \xrightarrow{(\eta^2)^*} \end{array}$$

where $[\mathbb{S}^{10}, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)]_{(2)} = 0$, and the map $(\eta^2)^* : [\mathbb{S}^7, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \rightarrow [\mathbb{S}^9, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)]$ is trivial. Thus,

$$[C(\iota \circ \eta^2), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \cong [M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)]$$

The cofiber sequence for $M(\mathbb{Z}/2^r, 7)$ induces the following

$$\begin{array}{ccc} [\mathbb{S}^8, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \xrightarrow{\times 2^r} [\mathbb{S}^8, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & \rightarrow & [M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \\ & \searrow & \swarrow \\ [\mathbb{S}^7, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \xrightarrow{\times 2^r} [\mathbb{S}^7, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] & & \end{array}$$

Note that $[\mathbb{S}^8, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] = 0$ and $[\mathbb{S}^7, >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] \cong \mathbb{Z}$ together implies $[M(\mathbb{Z}/2^r, 7), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] = 0$, which gives,

$$[C(\iota \circ \eta^2), >^6\tau_{\leq 10}(\Sigma^{-1}g/pl)] = 0 \quad (4.11)$$

Therefore, it follows from (4.6) that the map

$$\psi_* : [C(\iota \circ \eta^2), \tau_{\leq 10} pl/o] \rightarrow [C(\iota \circ \eta^2), >^6\tau_{\leq 10} g/o] \quad (4.12)$$

is injective. Consequently, in (4.7), $d^*(\eta \circ \mu) \notin j_*([M^{10}/M^{(6)}, >^6\tau_{\leq 10}\Sigma^{-1}(g/pl)])$. Since, according to Theorem 4.3(1)(a), the element $d^*(\eta \circ \mu)$ is non-zero in $[M^{10}, \tau_{\leq 10} pl/o]$, and by Proposition 4.4, the element $d^*(\eta \circ \mu)$ gets mapped non-trivially under $\psi_* : [M^{10}, \tau_{\leq 10} pl/o] \rightarrow [M^{10}, >^6\tau_{\leq 10} g/o]$. Therefore, the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \rightarrow [M^{10}, >^6\tau_{\leq 10} g/o]$ maps $\eta \circ \mu$ non-trivially, completing the proof. \square

Theorem 4.14. *Let M^{10} be a closed simply-connected smooth 10-manifold.*

(1) *If M^{10} is a spin manifold and $\Phi = 0$ then*

$$I_h(M^{10}) = \begin{cases} 0 & \text{if } M^{10} \text{ satisfies (3.4) or (3.6),} \\ \mathbb{Z}/2\{\eta \circ \mu\} & \text{if } M^{10} \text{ satisfies (3.5).} \end{cases}$$

(2) *If M^{10} is a spin manifold and $\Phi \neq 0$ or is a non-spin manifold then*

$$I_h(M^{10}) = \mathbb{Z}/2\{\eta \circ \mu\}.$$

Proof. The proof follows from (4.4) using for the simply-connected 10-manifolds together with Theorem 4.12 and Theorem 4.13. \square

Remark 4.15. (i) *The Theorem 4.12 gives more general result: Let M^{10} be a closed orientated smooth manifold. Then $I_h(M^{10}) \cap (\Theta_{10})_{(3)} = \{0\}$.*

(ii) *Note that, if M^{10} is non-oriented then $H^{10}(M^{10}; \mathbb{Z}/3) = 0$; therefore the similar proof of Theorem 4.12 gives $I_h(M^{10}) \supseteq \mathbb{Z}/3\{\beta_1\}$.*

4.3 Inertia group of $\mathbb{R}P^{10}$

Now, we compute the homotopy inertia group of a non oriented manifold $\mathbb{R}P^{10}$ by considering the proof techniques used for Theorem 4.14 with minimal adjustments.

Theorem 4.16. $I_h(\mathbb{R}P^{10}) = \mathbb{Z}/3\{\beta_1\} \oplus \mathbb{Z}/2\{\eta \circ \mu\}$.

Proof. For $I_h(\mathbb{R}P^{10})$, as mentioned in remark 4.15(ii), we have $\mathbb{Z}/3\{\beta_1\} \subseteq I_h(\mathbb{R}P^{10})$. Therefore, it is enough to work 2-locally. Consider the diagram (4.4), in which note that $\psi_* : [\mathbb{S}^{10}, PL/O] \rightarrow [\mathbb{S}^{10}, G/O]$ and $g' : S^{\text{Diff}}(M^n) \rightarrow [M^n, G/O]$ are injective. So it is enough to show that the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [\mathbb{R}P^{10}, >^6\tau_{\leq 10} g/o]_{(2)}$ is trivial. Note that, this map d^* factors through $[\mathbb{R}P^{10}/\mathbb{R}P^6, >^6\tau_{\leq 10} g/o]$, as shown in the diagram below,

$$\begin{array}{ccc} [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] & \xrightarrow{d^*} & [\mathbb{R}P^{10}, >^6\tau_{\leq 10} g/o] \\ & \searrow d^* & \swarrow q^* \\ & & [\mathbb{R}P^{10}/\mathbb{R}P^6, >^6\tau_{\leq 10} g/o] \end{array} \quad (4.13)$$

where $q : \mathbb{R}P^{10} \rightarrow \mathbb{R}P^{10}/\mathbb{R}P^6$ is the quotient map. To complete the proof, we claim that the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \rightarrow [\mathbb{R}P^{10}/\mathbb{R}P^6, >^6\tau_{\leq 10} g/o]$ is trivial.

As $\mathbb{R}P^{10}/\mathbb{R}P^7 \simeq \mathbb{S}^8 \vee M(\mathbb{Z}/2, 9)$ and $\Sigma\mathbb{R}P^7/\mathbb{R}P^6 \simeq \mathbb{S}^8$, consider the following commutative diagram

$$\begin{array}{ccccc} [\Sigma\mathbb{R}P^7/\mathbb{R}P^6, >^6\tau_{\leq 10} g/o] & \xrightarrow{f^*} & [\mathbb{R}P^{10}/\mathbb{R}P^7, >^6\tau_{\leq 10} g/o] & \xrightarrow{q^*} & [\mathbb{R}P^{10}/\mathbb{R}P^6, >^6\tau_{\leq 10} g/o] \\ \parallel & & \parallel & & \uparrow d^* \\ [\mathbb{S}^8, >^6\tau_{\leq 10} g/o] & \xrightarrow{f^*} & [\mathbb{S}^8 \vee M(\mathbb{Z}/2, 9), >^6\tau_{\leq 10} g/o] & \xleftarrow{d^*} & [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \end{array}$$

whose top row is a part of exact sequence induced from the cofiber sequence $\mathbb{R}P^{10}/\mathbb{R}P^6 \xrightarrow{q} \mathbb{R}P^{10}/\mathbb{R}P^7 \xrightarrow{f} \Sigma\mathbb{R}P^7/\mathbb{R}P^6$. Furthermore, the injectivity of the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o]_{(2)} \rightarrow [\mathbb{R}P^{10}/\mathbb{R}P^6, >^6\tau_{\leq 10} g/o]_{(2)}$ can be verified using the exact sequence (??) by replacing $\Sigma^7\mathcal{E}$ with $>^6\tau_{\leq 10} g/o$.

Observe that, the connecting map $f : \mathbb{R}P^{10}/\mathbb{R}P^7 \rightarrow \Sigma\mathbb{R}P^7/\mathbb{R}P^6$ is homotopic to the map $(2, \phi) : \mathbb{S}^8 \vee M(\mathbb{Z}/2, 9) \rightarrow \mathbb{S}^8$. Here, the map ϕ fits in the following commutative triangle

$$\begin{array}{ccc} M(\mathbb{Z}/2, 9) & \xrightarrow{\phi} & \mathbb{S}^8 \\ \uparrow \wr & \nearrow \eta & \\ \mathbb{S}^9 & & \end{array}$$

According to [5, Lemma 2.3], there is a non-split short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & [\mathbb{S}^{10}, \mathbb{S}^8] & \xrightarrow{d^*} & [M(\mathbb{Z}/2, 9), \mathbb{S}^8] & \longrightarrow & [\mathbb{S}^9, \mathbb{S}^8] \longrightarrow 0 \\ & & \wr & & \wr & & \\ & & \mathbb{Z}/2 = \{\eta^2\} & & \mathbb{Z}/2 = \{\eta\} & & \end{array}$$

which makes ϕ the generator of $[M(\mathbb{Z}/2, 9), \mathbb{S}^8] = \mathbb{Z}/4$. In (4.3), observe that $d^*(\eta^2) = 2\phi$. In the following commutative triangle, this forces the map $[\mathbb{S}^8, >^6\tau_{\leq 10} g/o] \rightarrow [M(\mathbb{Z}/2, 9), >^6\tau_{\leq 10} g/o]$ to be $(2\phi)^*$

$$\begin{array}{ccc} [\mathbb{S}^8, >^6\tau_{\leq 10} g/o] & & \\ \downarrow (\eta^2)^* & \searrow & \\ [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] & \xrightarrow{d^*} & [M(\mathbb{Z}/2, 9), >^6\tau_{\leq 10} g/o]. \end{array}$$

The surjectivity of $(\eta^2)^*$ is surjective implies

$$\begin{aligned} \text{Im}(d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \rightarrow [M(\mathbb{Z}/2, 9), >^6\tau_{\leq 10} g/o]) &\subseteq \text{Im}((2\phi)^*) \\ &\subseteq \text{Im}(\phi^*) \subseteq \text{Im}((2, \phi)^*). \end{aligned}$$

Now, it follows from the top row exact sequence in (4.3) that $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \rightarrow [\mathbb{R}\mathbb{P}^{10}/\mathbb{R}\mathbb{P}^6, >^6\tau_{\leq 10} g/o]$ is trivial. Therefore, in (4.13), the map $d^* : [\mathbb{S}^{10}, >^6\tau_{\leq 10} g/o] \rightarrow [\mathbb{R}\mathbb{P}^{10}, >^6\tau_{\leq 10} g/o]$ is trivial which completes the proof. \square

We conclude this section with the following result using Theorem 4.16

Corollary 4.17. $I(\mathbb{R}\mathbb{P}^{10}) = \Theta_{10}$.

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