arXiv:2302.02576v1 [cond-mat.stat-mech] 6 Feb 2023

Critical behaviors of cascading dynamics on multiplex two-dimensional lattices

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(Dated: February 7, 2023)

We study the critical phenomena of viable clusters in multiplex two-dimensional lattices using numerical simulations. We identify viable sites on multiplex lattices using two cascading algorithms: the cascade of activations (CA) and deactivations (CD). We found that the giant viable clusters identified by CA and CD processes exhibit different critical behaviors. Specifically, the critical phenomena of CA processes are consistent with the ordinary bond percolation on a single layer but CD processes exhibit the critical behaviors consistent with mutual percolation on multiplex lattices. In addition, we computed the susceptibility of cascading dynamics by using the concept of ghost field. Our results suggest that the CA and CD processes generate viable clusters in different ways.

I. INTRODUCTION

The elements of a complex system often function properly only if multiple resources are provided through multiple layers of networks [1–4]. For example, computercontrolled systems function properly only when the Internet and power grids are supplied interdependently [2, 3] In addition, for a city to function properly, resources such as water, gas, and electricity must be supplied by separate channels [1, 4, 5]. For this reason, the mutual connectivity of multiple layers of networks has received much attention for several years [3, 4, 6–10]. A mutually-connected component, which is a central concept in mutual percolation, is defined as a set of nodes that every node pair in the component has at least one path, composed of nodes within the same cluster, in each and every layer of networks [3, 7]. Pioneering works on the mutual percolation have studied the robustness of interdependent networks by using the notion of a mutually connected giant component and showed that an abrupt transition can appear between percolating and non-percolating phases [3, 6, 7, 11]. These studies imply that a small perturbation can cause an abrupt collapse of the entire system, posing a potentially catastrophe to complex systems [3, 4, 6, 12-15]

Since many real-world systems are embedded in low dimensions, percolation problems on multi-layered twodimensional lattices have been also of interest [16–18]. Unlike multiplex networks, the mutually connected giant cluster on two-dimensional lattices emerges continuously as the probability of bond occupation increases [16, 17]. However, the studies on the critical behaviors of the mutual percolation on two-dimensional lattices yielded conflicting results [16, 19–21]. Son *et al.* demonstrated that the percolation transition for interdependent diluted lattices belongs to a different universality class with a larger order parameter exponent than that of ordinary percolation [16, 20, 21]. However, another study reported that the mutual percolation on two-dimensional lattices belongs to the same universality class of ordinary percolation [19]. In addition to mutual percolation, several percolation problems in multi-layer lattices have been also studied such as history dependent percolation [22–24], and percolation in porous media [25, 26].

From a perspective different from topological aspect, there have been studies on the viability of nodes as a consequence of cascading dynamics following the iterative activations or deactivations of nodes [4, 5, 14]. A model for the viability deals with systems that demand more than one type of vital resource to be produced and distributed by source nodes in multiplex networks [4]. Viable nodes in this model are identified by using two different dynamical processes: cascade of activations (CA) and deactivations (CD). The CA process starts with all the sites in the deactivated state, and finds mutuallyconnected clusters through the diffusion of activations from the source nodes; in contrast, the CD process finds mutually-connected clusters by iteratively removing any sites that are not mutually-connected. On multiplex networks, these processes can produce two different final configurations of viable nodes, corresponding to different stable solutions of a single mean-field equation [4, 14]. Viability is strongly related to mutual percolation on multiplex networks because the dynamical consequence of CD becomes identical to mutual percolation in the limit when the fraction of initial source nodes goes to be zero. While cascading dynamics on multiplex networks has been studied [4, 5], it has not been explored on lowdimensional systems, especially the critical behaviors of cascading dynamics.

In this work, we study the critical phenomena of cascading dynamics on multiplex two-dimensional lattices using extensive Monte-Carlo simulations. We found that two cascading processes, CA and CD, can lead to different critical behaviors in viable clusters statistics. While CD exhibits the same set of critical exponents reported for mutual percolation on interdependent networks [16],

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FIG. 1. (a) A multiplex lattice consists of double-layered two-dimensional diluted lattices. (b,c) Examples of the final configurations of (b) CA and (c) CD processes are depicted. Viable, non-viable, and source sites are respectively denoted by black filled circles, black open circles, and grey filled circles.

CA exhibits the critical behaviors akin to the ordinary percolation on two-dimensional lattices. In addition, we computed the susceptibility of cascading dynamics by using the ghost field concept [27, 28] and confirmed that the hyperscaling relation is satisfied.

II. MODEL

Let us imagine an example for cascading processes on a multiplex lattice. A site needs two resources, say \mathcal{A} and \mathcal{B} , to be viable, e.g., a city requires water and electricity for functioning. Each resource is delivered along the corresponding layer of the lattice. Initially, the resources are generated from a few source sites in each layer. The important point is that only a viable site can convey resources to the neighboring sites along connected bonds. In this setting, we consider the following two processes for finding viable sites from the given lattice structures and initial source configurations. i) Assume that all sites except the source sites are initially in the non-viable state. Sites that satisfy the viable condition change into the viable state. These viable sites are able to convey resources to their neighbors. There will be sites that change to the viable state in turn and we call this process the cascade of activation (CA). ii) On the other hand, one may start at the beginning when all sites are putatively in a viable state. In this case, sites that do not satisfy the viable condition must change from the viable to non-viable state iteratively until no further deactivation is necessary. This process is called the cascade of deactivation (CD).

The specific simulation rules are as follows. There are two layers of two-dimensional lattices, say \mathcal{L}_A and \mathcal{L}_B

with $N = L \times L$, where L denotes the size of the lattice. Each bond in the lattice is occupied with probability p, and empty with probability 1-p. On top of multiplex diluted lattices, resources \mathcal{A} and \mathcal{B} are distributed at their source sites. In this study, we assume that the sources of resources \mathcal{A} and \mathcal{B} are located at every site on the boundary of lattices \mathcal{L}_A and \mathcal{L}_B , respectively [Fig. 1(a)]. We use cylindrical boundary conditions. Then, each site on a multiplex lattice is viable only if both resources \mathcal{A} and \mathcal{B} are supplied. Moreover, only viable sites can deliver resources to their neighbors connected by occupied bonds. Otherwise a site becomes non-viable and cannot convey resources further. Note that resources \mathcal{A} and \mathcal{B} are respectively supplied only through the chain of viable sites on \mathcal{L}_A and \mathcal{L}_B . Using the two processes, CA and CD, we find viable sites in the steady state. We then identify the viable clusters as a set of viable sites for every pair of sites in the cluster has at least one path composed of viable sites within each and every layer of the lattice. We also define the fraction of the giant viable cluster as $V_{\rm CA}$ for CA as shown in Fig. 1(b), and $V_{\rm CD}$ for CD as shown in Fig. 1(c).

III. RESULTS

We conducted Monte Carlo simulations for CA and CD processes on multiplex lattices with various sizes L. We then identify the configuration of the viable clusters and analyze their critical behaviors at the steady state. The fractions V of the giant viable cluster for the two processes are in general different from each other as shown in Fig. 2(a), as in the case of multiplex networks [4]. The giant viable cluster for CD process appears at a smaller p compared with that for CA. In addition, the fraction $V_{\rm CD}$ of the giant viable cluster for CD is generally larger than $V_{\rm CA}$.

The critical behaviors of percolation problems are typically characterized by the divergence of the average cluster size χ and correlation length ξ at the critical point p_c in the thermodynamic limit $L \to \infty$. Near the critical point, the order parameter V, susceptibility χ , and correlation length ξ follow power laws with the critical exponents β , γ , and ν as follows:

$$V(p) \sim (p - p_c)^{\beta},\tag{1}$$

$$\chi(p) \sim |p - p_c|^{-\gamma},\tag{2}$$

$$\xi(p) \sim |p - p_c|^{-\nu}.$$
 (3)

In order to obtain the critical exponents, we use the conventional scaling ansatz for finite-size systems [29],

$$V(p,L) = L^{-\beta/\nu} f_V[(p-p_c)L^{1/\nu}], \qquad (4)$$

$$\chi(p,L) = L^{\gamma/\nu} f_{\chi}[(p - p_c(L))L^{1/\nu}].$$
 (5)

where $p_c(L)$ is the location of the peak of susceptibility. While computing the fraction V of the giant viable cluster is straightforward, how to measure susceptibility



FIG. 2. (a) Numerical results of the fraction V for the CA (red) and CD (blue) processes as a function of the bond occupation probability p on multiplex lattices with L = 800, averaged over 100 different realizations are shown. (b) The relation $|p_c - p_c(L)| \sim L^{-1/\nu}$ for CA (red) and CD (blue) is shown. The points represent numerical simulation results averaged over 400 samples.

 χ is far from trivial. For the ordinary percolation, the susceptibility χ is directly measured by the average size of the finite clusters. In our model, however, the quantity corresponding to the susceptibility is not readily provided by the size of the average finite viable clusters. To obtain the susceptibility in CA and CD, we use the concept of "ghost field" analogous to magnetic susceptibility [27, 28, 30]. We add one additional ghost site that is connected to each site in the system with a probability H. By applying a small amount of H corresponding to the external magnetic field in spin systems, we measure the change in order parameter V such that $\chi = \partial V / \partial H$. In practice, we choose one site that does not belong to the giant viable cluster. Next, we assume that this site has become part of the giant viable cluster through the connection to the ghost site. We then identify a set of sites that newly entered the giant viable cluster because of the presence of the ghost site. We call the set of sites as "susceptible cluster" and measure the size of the susceptible cluster as susceptibility. In the numerical simulations, we attempted to select every site that does not belong to the giant viable cluster and measured the distribution $\phi(s)$ that a site belongs to a susceptible cluster with size s. The susceptibility χ is obtained as

$$\chi(p,L) = \frac{\sum_{s} s\phi(s,p)}{\sum_{s} \phi(s,p)},\tag{6}$$

where the sum is over all cluster sizes excluding the giant viable cluster.

We estimated the viability threshold and critical exponent ν for both CA and CD using a finite size scaling ansatz. We first obtained the threshold as $p_c^{\text{CA}} \simeq 0.6743$ and $p_c^{\text{CD}} \simeq 0.5761$. As in the case of CA and CD in multiplex networks [4], the giant viable clusters in CA processes emerges at a larger p compared to CD. Next, we obtained the exponent of the correlation length ν from the relation $|p_c(L) - p_c| \sim L^{-1/\nu}$ as shown in Fig. 2(b). Here, $p_c(L)$ is a threshold with size L and is estimated by the value of p at the maximum value of susceptibility χ . The fitted values of ν for CA and CD are respectively



FIG. 3. (a) Log-log plot of $V_{\rm CA}L^{\beta/\nu}$ at $p_c^{\rm CA}$ with respect to L for various p. (b) Data collapse of the scaled order parameter $V_{\rm CA}L^{\beta/\nu}$ vs. $(p-p_c)L^{1/\nu}$ with $\beta = 0.137(1)$ and $\nu = 1.342(3)$. (c) Log-log plot of $\chi_{\rm CA}$ at $p_c(L)$ with respect to L, showing the relation $\chi \sim L^{\gamma/\nu}$. (d) Data collapse of the scaled susceptibility $\chi_{\rm CA}L^{-\gamma/\nu}$ vs. $(p-p_c)L^{1/\nu}$ with $\gamma = 2.41(1)$ and $\nu = 1.342(3)$. Numerical results of CA were obtained for various lattice sizes L from 50 to 800, averaged over 400 realizations.

1.342(3) and 1.197(7).

We then obtained β/ν and γ/ν using the scalings respectively $V \sim L^{-\beta/\nu}$ and $\chi \sim L^{\gamma/\nu}$. The results of critical behaviors for CA processes are shown in Fig. 3. Figure 3(a) shows the scaling of $V \sim L^{-\beta/\nu}$ with various values of p. The scaling shows the estimates of the exponents $\beta = 0.137(1)$ and $\nu = 1.342(3)$. Data collapse curves of $VL^{\beta/\nu}$ with respect to $(p - p_c)L^{1/\nu}$ in Fig. 3(b) confirm the estimated values of the critical exponents. Figure 3(c) shows the relationship $\chi \sim L^{\gamma/\nu}$ with $\gamma/\nu = 1.794(6)$ which is confirmed by the finite-size-scaled data collapse of $\chi L^{-\gamma/\nu}$ with respect to $(p - p_c)L^{1/\nu}$, as shown in Fig. 3(d). We found that the critical exponents for CA processes are consistent within error bars for ordinary bond percolation on two-dimensional lattices.

For CD processes, we also obtain β and γ using scaling $V \sim L^{-\beta/\nu}$ and $\chi \sim L^{\gamma/\nu}$ as shown in Fig. 4. The estimated values of the critical exponents for the CD processes are $\beta = 0.163(2)$ and $\nu = 1.197(7)$ [Fig. 4(a)]. In addition, figure 4(c) exhibits the relationship $\chi \sim L^{\gamma/\nu}$ with $\gamma/\nu = 1.73(1)$. We also confirm the critical exponents by using the data collapse of $VL^{\beta/\nu}$ with respect to $(p - p_c)L^{1/\nu}$ [Fig. 4(b)] and $\chi L^{-\gamma/\nu}$ with respect to $(p - p_c)L^{1/\nu}$ [Fig. 4(d)]. The values of the exponents β , γ , and ν are distinct from those for CA. In addition, the critical behaviors of CD are consistent within error bars of the mutual percolation on two-dimensional lattices reported in [16, 21].

We also obtain the size distribution $\phi(s)$ that a site



FIG. 4. (a) Log-log plot of $V_{\rm CD}L^{\beta/\nu}$ at $p_c^{\rm CD}$ with respect to L for various p. (b) Data collapse of the scaled order parameter $V_{\rm CD}L^{\beta/\nu}$ vs. $(p-p_c)L^{1/\nu}$ with $\beta = 0.163(2)$ and $\nu = 1.197(7)$. (c) Log-log plot of $\chi_{\rm CD}$ at $p_c(L)$ with respect to L, showing the relation $\chi \sim L^{\gamma/\nu}$. (d) Data collapse of the scaled susceptibility $\chi_{\rm CD}L^{-\gamma/\nu}$ vs. $(p-p_c)L^{1/\nu}$ with $\gamma = 2.07(1)$ and $\nu = 1.197(7)$. Numerical results of CD were obtained for various lattice sizes L from 50 to 800, averaged over 400 realizations.

belongs to susceptible clusters with size s. At p_c , the size distribution follows the power-law form $\phi(s) \sim s^{1-\tau}$ in the thermodynamic limit. We apply a scaling ansatz to the distribution

$$\phi(s) = s^{1-\tau} g(s/L^{d_f}) \tag{7}$$

where g is a scaling function and d_f is the fractal dimension given by $d_f = d - \beta/\nu$ where d is the spatial dimension of lattices. Figures 5(a,b) show the power-law decay of $\phi(s)$ at p_c for (a) CA and (b) CD, respectively. We found the estimate of critical exponents as $\tau = 2.06(3)$ for CA and $\tau = 2.07(1)$ for CD. We confirm the scaling ansatz using data collapses, as shown in Figs. 5(c,d).

The obtained values of the threshold and critical exponents are presented in Table 1. We observed that the critical exponents for CA and CD are distinct. Moreover, all critical exponents of CA are consistent within error bars with ordinary bond percolation on two-dimensional lattices. However, the critical exponents of CD are consistent with those of the mutual percolation on two-dimensional lattices, which shows a discrepancy with CA. In addition, the sets of exponents for both CA and CD satisfy the hyperscaling relations $d\nu = 2\beta + \gamma$ and $\tau = d/d_f + 1$.

To understand the differences between the underlying mechanisms forming susceptible clusters, we examine the joint probability distribution between the sizes of susceptible clusters s and that of double-bond clusters $s_{\rm dbc}$, and that of mutually connected clusters $s_{\rm mcc}$. A double bond



FIG. 5. Probability distributions $\phi(s)$ that a node belongs to a susceptible cluster with size s for (a) CA and (b) CD. Data collapses of $\phi(s)s^{\tau-1}$ vs. s/L^{d_f} for (c) CA with $\tau =$ 2.06(3) and $d_f = 1.8976$ and for (d) CD with $\tau = 2.07(1)$ and $d_f = 1.864$. The fractal dimension d_f is estimated by $d_f = d - \beta/\nu$. Monte-carlo simulations were performed at the critical point p = 0.6743 for CA and p = 0.5761 for CD. Numerical results were obtained for various lattice sizes Lfrom 50 to 800, averaged over 400 realizations.

refers to a connection in which a bond between two adjacent sites exists in both layers, and the double-bond cluster stands for a set of sites that are connected solely via double bonds. The mutually connected component is defined as a set of sites that every pair of sites in the cluster has at least one path within each and every layer of the lattice.

Figures 6(a,b) show the joint probability distribution $J(s, s_{dbc})$ for (a) CA and (b) CD to examine the effect of double bonds in forming susceptible clusters. The joint probability distributions are computed at the critical point, p = 0.6743 for CA and p = 0.5761 for CD. We found that the size of the double-bond clusters are strongly correlated to that of the susceptible clusters for

TABLE I. Critical points and exponents

	CA	Ordinary Percolation	CD	Mutual Percolation [21]
p_c	0.6743(1)	1/2	0.5761(1)	0.576132(5)
ν	1.342(3)	4/3	1.197(7)	1.200(7)
β	0.137(1)	5/36	0.163(2)	0.163(2)
γ	2.41(1)	43/18	2.07(2)	
au	2.06(3)	187/91	2.07(1)	
$1/\nu$	0.745(2)	3/4	0.836(5)	0.833
β/ν	0.1024(5)	5/48	0.136(1)	0.1358
γ/ν	1.794(6)	43/24	1.73(1)	1.728



FIG. 6. Joint probability distributions $J(s, s_{\rm dbc})$ where s is the size of susceptible clusters and $s_{\rm dbc}$ is that of double-bond clusters are shown for (a) CA and (b) CD. Joint probability distributions $J(s, s_{\rm mcc})$ where $s_{\rm mcc}$ is the size of mutually connected clusters are shown for (c) CA and (d) CD. Numerical results are calculated at the critical points p = 0.6743 for CA and p = 0.5761 for CD with L = 100, averaged over 5000 realizations.

CA. This means that the double bonds play a dominant role in the CA process. For CD process, however, small susceptible clusters are formed by double bonds, while large susceptible clusters are not. We found that largesized susceptible clusters in CD are constructed by mutually connected clusters as shown in Fig. 6(d). On the other hand, for the CA process there is already a mutually connected giant component near the critical point as shown in Fig. 6(c). Therefore, mutually connected clusters cannot contribute crucially the emergence of the large-sized susceptible clusters in CA process. In conclusion, we found that large-sized susceptible clusters are mainly constructed by double bonds in the CA process whereas they are formed by mutual connections in the CD process.

IV. DISCUSSION

In this work, we study the critical phenomena of cascading dynamics on multiplex lattices considering two different processes to identify viable clusters. We found that viable clusters identified by the CA and CD processes exhibits different critical behaviors. The CA process exhibits critical phenomena consistent to ordinary bond percolation but the CD process exhibits those to mutual percolation on multiplex lattices. In addition, we explicitly calculate susceptibility by using the ghost field. Our study shows that viability that requires multiple connectivity can be maintained by diverse mechanisms in a multiplex low-dimensional system. The cascading dynamics that we consider here is an example of cooperative couplings between multiple layers. Similar cooperative interactions can be realized in real-world systems in a variety of ways such as cooperative infections [31, 32], social contagions [33, 34], and interdependencies in multilayered systems [3, 35]. In this regard, our research shows a glimpse of the complexity generated by cooperative interactions in multiplex low-dimensional systems. Further studies may be required to examine the cascading dynamics on an arbitrary number of layers and partially coupled layers.

ACKNOWLEDGMENTS

This work was supported by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MSIT) (No. 2020R1I1A3068803 (BM) and No. 2020R1A2C2003669 (K-IG)).

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