

SUM FORMULAS FOR SCHUR MULTIPLE ZETA VALUES

HENRIK BACHMANN, SHIN-YA KADOTA, YUTA SUZUKI, SHUJI YAMAMOTO,
AND YOSHINORI YAMASAKI

ABSTRACT. In this paper, we study sum formulas for Schur multiple zeta values and give a generalization of the sum formulas for multiple zeta(-star) values. We show that for ribbons of certain types, the sum of Schur multiple zeta values over all admissible Young tableaux of this shape evaluates to a rational multiple of the Riemann zeta value. For arbitrary ribbons with n corners, we show that such a sum can be always expressed in terms of multiple zeta values of depth $\leq n$. In particular, when $n = 2$, we give explicit, what we call, bounded type sum formulas for these ribbons. Finally, we show how to evaluate this sum when the corresponding Young diagram has exactly one corner and also prove bounded type sum formulas for them. This will also lead to relations among sums of Schur multiple zeta values over all admissible Young tableaux of different shapes.

1. INTRODUCTION

The purpose of this note is to present several different types of sum formulas for Schur multiple zeta values, which can be seen as generalizations of classical sum formulas for multiple zeta(-star) values. Schur multiple zeta values are real numbers introduced in [7], and they can be seen as a simultaneous generalization of the multiple zeta values (MZVs) and multiple zeta-star values (MZSVs), which are defined for an index $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_{\geq 1}^d$ with $k_d \geq 2$ by

$$\zeta(\mathbf{k}) := \sum_{0 < m_1 < \dots < m_d} \frac{1}{m_1^{k_1} \dots m_d^{k_d}}, \quad \zeta^\star(\mathbf{k}) := \sum_{0 < m_1 \leq \dots \leq m_d} \frac{1}{m_1^{k_1} \dots m_d^{k_d}}.$$

Here the condition $k_d \geq 2$ ensures the convergence of the above sums, and the index $\mathbf{k}$ is called admissible in this case. For an index $\mathbf{k} = (k_1, \dots, k_d)$ we write $\text{wt}(\mathbf{k}) = k_1 + \dots + k_d$ to denote its weight and $\text{dep}(\mathbf{k}) = d$ for its depth. A classical result is then ([2],[3]), that the sum of MZ(S)V over all admissible indices of fixed weight w and depth d evaluates to (an integer multiple of) $\zeta(w)$, i.e., for $d \geq 1$ and $w \geq d + 1$

$$(1.1) \quad \sum_{\substack{\mathbf{k} \text{ admissible} \\ \text{wt}(\mathbf{k})=w \\ \text{dep}(\mathbf{k})=d}} \zeta(\mathbf{k}) = \zeta(w), \quad \sum_{\substack{\mathbf{k} \text{ admissible} \\ \text{wt}(\mathbf{k})=w \\ \text{dep}(\mathbf{k})=d}} \zeta^\star(\mathbf{k}) = \binom{w-1}{d-1} \zeta(w).$$

Schur MZVs generalize MZ(S)V by replacing an index $\mathbf{k}$ by a skew Young tableau (see Definition 1.1 for the exact definition). For example, if we have a skew Young diagram $\lambda/\mu = (2, 2)/(1) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$ and $k_1, k_3 \geq 1, k_2 \geq 2$ the Schur MZV of *shape* λ/μ for the Young

Date: June 5, 2023.

2020 Mathematics Subject Classification. Primary 11M32; Secondary 05E05.

Key words and phrases. Schur multiple values, Sum formulas, Integrals associated with 2-posets, Jacobi-Trudi formula.

tableau $\mathbf{k} = \begin{array}{|c|} \hline k_1 \\ \hline k_3 \quad k_2 \\ \hline \end{array} \in \text{YT}(\lambda/\mu)$ is defined by

$$\zeta(\mathbf{k}) = \zeta \left(\begin{array}{|c|} \hline k_1 \\ \hline k_3 \quad k_2 \\ \hline \end{array} \right) = \sum_{\substack{m_1 \\ \wedge \\ m_3 \leq m_2}} \frac{1}{m_1^{k_1} m_2^{k_2} m_3^{k_3}},$$

where in the sum $m_1, m_2, m_3 \geq 1$. Schur MZVs generalize MZVs and MZSVs in the sense that we recover them as the special cases where the Young diagrams are respectively given by $\underbrace{(1, \dots, 1)}_d$ and (d) , i.e.,

$$\zeta(k_1, \dots, k_d) = \zeta \left(\begin{array}{|c|} \hline k_1 \\ \vdots \\ k_d \\ \hline \end{array} \right) \quad \text{and} \quad \zeta^*(k_1, \dots, k_d) = \zeta \left(\begin{array}{|c|c|c|} \hline k_1 & \cdots & k_d \\ \hline \end{array} \right).$$

The classical sum formulas (1.1) state that for the shapes $\lambda = (1, \dots, 1)$ or $\lambda = (d)$ the sum of Schur MZVs over all admissible Young tableaux of shape λ and weight $w \geq d + 1$ evaluates to (a multiple of) $\zeta(w)$. Therefore, it is natural to ask, how this situation is for a general shape λ/μ , i.e., if there exist explicit evaluations of

$$(1.2) \quad S_w(\lambda/\mu) := \sum_{\substack{\mathbf{k} \in \text{YT}(\lambda/\mu) \\ \mathbf{k} \text{ admissible} \\ \text{wt}(\mathbf{k}) = w}} \zeta(\mathbf{k}).$$

The optimistic guess that $S_w(\lambda/\mu)$ is always a rational multiple of $\zeta(w)$ seems to be wrong since we will see that, for example, for $\lambda = (2, 2) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$ and $\mu = \emptyset$ we have for $w \geq 5$

$$(1.3) \quad \begin{aligned} S_w \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) &= -(w-2)\zeta(1, w-1) + (w-4)\zeta(2, w-2) + 2\zeta(3, w-3) \\ &\quad - 2\zeta(3)\zeta(w-3) + (w-2)\zeta(2)\zeta(w-2), \end{aligned}$$

which, by computer experiments, seems not to evaluate to a rational multiple of $\zeta(w)$ for arbitrary w . Nevertheless, we see that (1.3) gives a representation as a sum of products of MZVs, where the number of terms does not depend on w . That such an expression exists is not obvious since a priori, the number of terms in (1.2) increases with the weight w . It is, therefore, also reasonable to call (1.3) a sum formula. Since it is of a different type to the sum formula of MZVs, we introduce the following types of sum formulas for shape λ/μ :

- single type*: $S_w(\lambda/\mu)$ evaluates to a rational multiple of $\zeta(w)$.
- polynomial type*: $S_w(\lambda/\mu)$ evaluates to a polynomial in $\zeta(m)$ with $m \leq w$.
- bounded type*: $S_w(\lambda/\mu)$ is expressed as a $\mathbb{Q}$ -linear combination of MZVs, where the number of terms does not depend¹ on w , but just on λ/μ .

With this terminology, the classical sum formulas are of single type, and the formula (1.3) is of bounded type. This list is not intended to be exhaustive, i.e., we do not claim that any shape λ/μ has a sum formula of one of these three types. It is also not exclusive, for instance, a single type formula is also both of polynomial and bounded type.

In this note, we will focus on the case when λ/μ has either just one corner (see Section 1.1 for the definition of corners) or λ/μ is a ribbon, by which we mean a (skew) Young

¹By this, we mean that for a fixed λ/μ we can find $a(\mathbf{k}, w) \in \mathbb{Q}$ for admissible indices $\mathbf{k}$ and $w \geq 1$ with $S_w(\lambda/\mu) = \sum_{\mathbf{k}} a(\mathbf{k}, w) \zeta(\mathbf{k})$ such that there exists a positive real number C with $|\{\mathbf{k} \mid a(\mathbf{k}, w) \neq 0\}| < C$ for all $w \geq 1$.

diagram which is connected and contains no 2×2 block of boxes. For example, if λ is just a column or a row, we obtain the single type sum formulas (1.1). As a generalization we will show (Theorem 3.8) that for $\lambda/\mu = \underbrace{(d, \dots, d)}_r / \underbrace{(d-1, \dots, d-1)}_{r-1}$, i.e., when λ/μ is an anti-hook, we have

$$S_w(\lambda/\mu) = S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \vdots \\ \square \\ \hline \end{array} \right) = \binom{w-1}{d-1} \zeta(w).$$

This can be seen as a unification of the two classical sum formulas (1.1). In Theorem 3.9, we will show that also the “stairs of tread one”, where the height of each stair is the same, will give rise to single type sum formulas. For example, as special cases, we get the following formulas, valid for all $w \geq 8$

$$(1.4) \quad S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \right) = \frac{(w-8)(w-1)}{2} \zeta(w), \quad S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \right) = \frac{(w-9)(w-8)(w-1)}{6} \zeta(w).$$

For arbitrary ribbons with n corners, we will see in Corollary 3.5 that the corresponding S_w can be always expressed in terms of MZVs of depth $\leq n$. Especially, in Theorem 3.13 we give an explicit bounded type sum formula for any ribbon with two corners in terms of double zeta values.

In the last section, we consider shapes that have exactly one corner but which are not necessarily ribbons. We will show that arbitrary shapes with one corner always give rise to sum formulas of bounded type (Theorem 4.2) similar to (1.3). This will also lead to relations among $S_w(\lambda/\mu)$ for different shapes λ/μ . For example, as a special case of Theorem 4.8, we will see that for any $w \geq 1$

$$2S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \right) - S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \right) - 4S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \right) = (w-7)S_w \left(\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \right).$$

Notice that the diagrams appearing in the left-hand side are obtained by adding one box (the gray ones) to the diagram in the right-hand side.

The structure of this paper is as follows: In Section 2 we give a recursive formula for a certain weighted sums of MZVs, which will be used in later sections. For this, we will recall the notion of 2-posets and their associated integrals. In Section 3, we consider $S_w(\lambda/\mu)$ where λ/μ are ribbons. Section 4 will be devoted to the case of shapes with exactly one corner.

1.1. Notation and definition of Schur MZVs. We will use the following notation in this work. A tuple $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_{\geq 1}^d$ will be called an *index* and we write $\mathbf{k} = \emptyset$ when $d = 0$. We call $\mathbf{k}$ *admissible* if $k_d \geq 2$ or if $\mathbf{k} = \emptyset$. For $n \geq 1$ and $k \in \mathbb{Z}_{\geq 1}$ we use for the n -time repetition of k the usual notation $\{k\}^n = \underbrace{k, \dots, k}_n$.

A *partition* of a natural number n is a tuple $\lambda = (\lambda_1, \dots, \lambda_h)$ of positive integers $\lambda_1 \geq \dots \geq \lambda_h \geq 1$ with $n = |\lambda| = \lambda_1 + \dots + \lambda_h$. We will also use the notation $\lambda = (n^{m_n}, \dots, 2^{m_2}, 1^{m_1})$, where $m_i = m_i(\lambda)$ is the multiplicity of i in λ . For another partition $\mu = (\mu_1, \dots, \mu_r)$ we write $\mu \subset \lambda$ if $r \leq h$ and $\mu_i \leq \lambda_i$ for $i = 1, \dots, r$, and we

define the *skew Young diagram* $D(\lambda/\mu)$ of λ/μ by

$$D(\lambda/\mu) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq h, \mu_i < j \leq \lambda_i\},$$

where $\mu_i = 0$ for $i > r$. In the case where $\mu = \emptyset$ is the empty partition (i.e., the unique partition of zero) we just write $\lambda/\mu = \lambda$.

A *Young tableau* $\mathbf{k} = (k_{i,j})_{(i,j) \in D(\lambda/\mu)}$ of shape λ/μ is a filling of $D(\lambda/\mu)$ obtained by putting $k_{i,j} \in \mathbb{Z}_{\geq 1}$ into the (i, j) -entry of $D(\lambda/\mu)$. For shorter notation, we will also just write $(k_{i,j})$ in the following if the shape λ/μ is clear from the context. A Young tableau $(m_{i,j})$ is called *semi-standard* if $m_{i,j} < m_{i+1,j}$ and $m_{i,j} \leq m_{i,j+1}$ for all possible i and j . The set of all Young tableaux and all semi-standard Young tableaux of shape λ/μ are denoted by $\text{YT}(\lambda/\mu)$ and $\text{SSYT}(\lambda/\mu)$, respectively.

An entry $(i, j) \in D(\lambda/\mu)$ is called an (outer) *corner* of λ/μ if $(i, j+1) \notin D(\lambda/\mu)$ and $(i+1, j) \notin D(\lambda/\mu)$. We denote the set of all corners of λ/μ by $C(\lambda/\mu)$. For a Young tableau $\mathbf{k} = (k_{i,j}) \in \text{YT}(\lambda/\mu)$ we define its *weight* by $\text{wt}(\mathbf{k}) = \sum_{(i,j) \in D(\lambda/\mu)} k_{i,j}$ and we call it *admissible* if $k_{i,j} \geq 2$ for all $(i, j) \in C(\lambda/\mu)$.

Definition 1.1. For an admissible $\mathbf{k} = (k_{i,j}) \in \text{YT}(\lambda/\mu)$ the *Schur multiple zeta value* (Schur MZV) is defined by

$$(1.5) \quad \zeta(\mathbf{k}) := \sum_{(m_{i,j}) \in \text{SSYT}(\lambda/\mu)} \prod_{(i,j) \in D(\lambda/\mu)} \frac{1}{m_{i,j}^{k_{i,j}}}.$$

Note that the admissibility of $\mathbf{k}$ ensures the convergence of (1.5) ([7, Lemma 2.1]). For the empty tableau $\mathbf{k} = \emptyset$, we have $\zeta(\emptyset) = 1$.

Finally, we mention that the convention for the binomial coefficients we use in this work is for $n, k \in \mathbb{Z}$ given by

$$\binom{n}{k} := \begin{cases} \frac{n(n-1)\cdots(n-(k-1))}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Acknowledgement. The authors thank the referee for his/her careful reading and valuable comments which improved the quality of the manuscript. The first author was partially supported by JSPS KAKENHI Grant Numbers JP19K14499, JP21K13771. The third author was partially supported by JSPS KAKENHI Grant Numbers JP19K23402, JP21K13772. The fourth author was partially supported by JSPS KAKENHI Grant Numbers JP18H05233, JP18K03221, JP21K03185. The fifth author was partially supported by JSPS KAKENHI Grant Numbers JP21K03206.

2. WEIGHTED SUM FORMULAS

When evaluating sums of Schur MZVs we will often encounter weighted sums of MZVs, which we will discuss in this section. For indices $\mathbf{n} = (n_1, \dots, n_d)$, $\mathbf{k} = (k_1, \dots, k_d)$ and an integer $l \geq 0$, define

$$P_l(\mathbf{n}; \mathbf{k}) := \sum_{\substack{\mathbf{w}=(w_1, \dots, w_d): \text{adm.} \\ w_i \geq n_i \ (i=1, \dots, d) \\ \text{wt}(\mathbf{w}) = \text{wt}(\mathbf{k}) + l}} \prod_{i=1}^d \binom{w_i - n_i}{k_i - 1} \cdot \zeta(w_1, \dots, w_d).$$

Notice that by definition $P_l(\emptyset; \emptyset) = 1$ if $l = 0$ and 0 otherwise. In particular, we put $P_l(\mathbf{k}) := P_l((1, \dots, 1); \mathbf{k})$ and $Q_l(\mathbf{k}) := P_l((1, \dots, 1, 2); \mathbf{k})$. The aim of this section is to obtain explicit bounded expressions of $P_l(\mathbf{k})$ and $Q_l(\mathbf{k})$, which play important roles

throughout the present paper. Here, we say that an expression of $P_l(\mathbf{k})$ or $Q_l(\mathbf{k})$ is *bounded* if the number of terms appearing in the expression does not depend on l but only on $\mathbf{k}$. Notice that, since $\binom{w-2}{k-1} = \sum_{j=0}^{k-1} (-1)^j \binom{w-1}{k-j-1}$, we have

$$(2.1) \quad Q_l(\mathbf{k}) = \sum_{j=0}^{k_d-1} (-1)^j P_{l+j}(k_1, \dots, k_{d-1}, k_d - j)$$

and in particular $Q_l(\mathbf{k}) = P_l(\mathbf{k})$ if $\mathbf{k}$ is non-admissible. By (2.1), it is sufficient to study only $P_l(\mathbf{k})$. Moreover, we may assume that $l > 0$ because the case $l = 0$ is trivial: $P_0(\mathbf{k}) = \zeta(\mathbf{k})$ if $\mathbf{k}$ is admissible and 0 otherwise, and $Q_0(\mathbf{k}) = 0$ for any index $\mathbf{k}$. Furthermore, when $d = 1$, we have for $k \geq 1$ and $l > 0$

$$P_l(k) = \binom{l+k-1}{k-1} \zeta(l+k), \quad Q_l(k) = \binom{l+k-2}{k-1} \zeta(l+k)$$

and therefore also assume that $d \geq 2$.

The next proposition asserts that $P_l(\mathbf{k})$ satisfies recursive formulas with respect to l , which can be described by using the Schur MZVs of anti-hook shape

$$(2.2) \quad \begin{aligned} \zeta \left(\begin{smallmatrix} \mathbf{l} \\ \mathbf{k} \end{smallmatrix} \right) &= \zeta \left(\begin{smallmatrix} l_1, \dots, l_s \\ k_1, \dots, k_r \end{smallmatrix} \right) \\ &:= \zeta \left(\begin{array}{|c|c|c|c|} \hline & & & k_1 \\ \hline & & & \vdots \\ \hline & & & k_r \\ \hline l_1 & \cdots & l_s & k_r \\ \hline \end{array} \right) = \sum_{0 < a_1 < \dots < a_r \geq b_s \geq \dots \geq b_1 > 0} \frac{1}{a_1^{k_1} \cdots a_r^{k_r} b_1^{l_1} \cdots b_s^{l_s}} \end{aligned}$$

with $\mathbf{l} = (l_1, \dots, l_s)$ being an index and $\mathbf{k} = (k_1, \dots, k_r)$ a non-empty admissible index.

Proposition 2.1. *Let $d \geq 2$ and $l > 0$.*

(i) *If $\mathbf{k} = (k_1, \dots, k_d)$ is admissible, then it holds that*

$$(2.3) \quad P_l(\mathbf{k}) = \sum_{i=1}^d \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\dots+k_{i-1}+a_i} P_{k_i-1-a_i}(k_{i-1}, \dots, k_1, l+1) P_{a_i}(k_{i+1}, \dots, k_d).$$

(ii) *If $\mathbf{k} = (k_1, \dots, k_d)$ is non-admissible (i.e., $k_d = 1$), then it holds that*

$$(2.4) \quad \begin{aligned} P_l(\mathbf{k}) &= \sum_{i=1}^d \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\dots+k_{i-1}+a_i} P_{k_i-1-a_i}(k_{i-1}, \dots, k_1, l+1) P_{a_i}(k_{i+1}, \dots, k_{d-1}, 1) \\ &+ \sum_{i=1}^{d-1} (-1)^{l+d+k_i} \sum_{\substack{(b_0, \dots, b_{d-1}) \in \mathbb{Z}_{\geq 1}^d \\ b_i=2 \\ b_0+\dots+b_{d-1}=\text{wt}(\mathbf{k})+l+1}} (-1)^{b_0+b_1+\dots+b_{i-1}} \binom{b_0-1}{l} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^{d-1} \binom{b_j-1}{k_j-1} \right\} \\ &\times \sum_{j=i}^{d-1} \sum_{c_j=1}^{b_j-1} (-1)^{c_j+j+b_{j+1}+\dots+b_{d-1}} \zeta \left(\begin{smallmatrix} c_j, b_{j+1}, \dots, b_{d-1} \\ b_{i-1}, \dots, b_1, b_0 \end{smallmatrix} \right) \zeta(b_{i+1}, \dots, b_{j-1}, b_j - c_j + 1). \end{aligned}$$

Remark 2.2. The number of terms in the expression (2.4) is actually bounded by a constant depending on $\mathbf{k}$ but not on l by the following reason: By the non-vanishing of the binomial coefficient $\binom{b_0-1}{l}$, the summation variable b_0 can be restricted by $b_0 \geq l+1$ and then the other summation variables are bounded independently of l . Also, by applying the formula (4.1) to an anti-hook, we can expand each term

$$\zeta \left(\begin{smallmatrix} c_j, b_{j+1}, \dots, b_{d-1} \\ b_{i-1}, \dots, b_1, b_0 \end{smallmatrix} \right)$$

into a sum of MZVs. The number of appearing MZVs is independent of $b_0, \dots, b_{d-1}$ and c_j . Finally, by using the harmonic product formula, we can rewrite the products of MZVs into a sum of MZVs. The resulting number of terms, after the application of the harmonic product formula, depends only on the number of entries of MZVs appearing as above and so it is independent of l .

To prove Proposition 2.1, we first recall the notion of 2-posets and the associated integrals introduced by the fourth-named author in [9].

Definition 2.3 ([9, Definition 2.1]).

- (i) A 2-poset is a pair (X, δ_X) , where $X = (X, \leq)$ is a finite partially ordered set (poset for short) and δ_X is a map (called the label map of X) from X to $\{0, 1\}$. We often omit δ_X and simply say “a 2-poset X ”. Moreover, a 2-poset X is called admissible if $\delta_X(x) = 0$ for all maximal elements x and $\delta_X(x) = 1$ for all minimal elements x .
- (ii) For an admissible 2-poset X , the associated integral $I(X)$ is defined by

$$(2.5) \quad I(X) = \int_{\Delta_X} \prod_{x \in X} \omega_{\delta_X(x)}(t_x),$$

where $\Delta_X = \{(t_x)_x \in [0, 1]^X \mid t_x < t_y \text{ if } x < y\}$ and $\omega_0(t) = \frac{dt}{t}$ and $\omega_1(t) = \frac{dt}{1-t}$.

We depict a 2-poset as a Hasse diagram in which an element x with $\delta_X(x) = 0$ (resp. $\delta_X(x) = 1$) is represented by $\circ$ (resp. $\bullet$). For example, the diagram



represents the 2-poset $X = \{x_1, \dots, x_{10}\}$ with order $x_1 < x_2 < x_3 < x_4 < x_5 > x_6 < x_7 < x_8 > x_9 < x_{10}$ and label $(\delta_X(x_1), \dots, \delta_X(x_{10})) = (1, 0, 1, 1, 0, 1, 0, 0, 1, 0)$.

In [5], it is shown that the Schur MZVs of anti-hook shape has the following expression by the associated integral of a 2-poset. This can be regarded as simultaneous generalization of the integral expressions of MZVs and MZSVs.

Theorem 2.4 ([5, Theorem 4.1]). *For an index $\mathbf{l} = (l_1, \dots, l_s)$ and a non-empty admissible index $\mathbf{k} = (k_1, \dots, k_r)$, we have*

$$\zeta \begin{pmatrix} \mathbf{l} \\ \mathbf{k} \end{pmatrix} = I \left(\begin{array}{c} \text{Diagram with } r \text{ vertical chains of } k_r-1, k_{r-1}-1, \dots, k_1-1 \text{ nodes, connected by } s \text{ zig-zag paths of } l_s-1, \dots, l_1-1 \text{ nodes.} \end{array} \right).$$

For example, for the 2-poset X given by (2.6), we have $\zeta \begin{pmatrix} 2, 3 \\ 2, 1, 2 \end{pmatrix} = I(X)$.

In our proof of Proposition 2.1, we consider a kind of extension of the integral $I(X)$ to non-admissible 2-posets X . This extension is given by using the notion of “admissible part” which we define below.

Let $\mathcal{X}$ be the set of isomorphism classes of 2-posets, and $\mathbb{Q}\mathcal{X}$ denote the $\mathbb{Q}$ -vector space freely generated by this set. We equip $\mathbb{Q}\mathcal{X}$ with a $\mathbb{Q}$ -algebra structure by setting

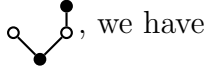
$[X] \cdot [Y] := [X \sqcup Y]$. If we let $\mathcal{X}^0 \subset \mathcal{X}$ be the subset consisting of admissible 2-posets, its $\mathbb{Q}$ -span $\mathbb{Q}\mathcal{X}^0$ becomes a $\mathbb{Q}$ -subalgebra of $\mathbb{Q}\mathcal{X}$ and the integral (2.5) defines a $\mathbb{Q}$ -algebra homomorphism $I: \mathbb{Q}\mathcal{X}^0 \rightarrow \mathbb{R}$.

Let $\mathcal{T} \subset \mathcal{X}$ be the subset of totally ordered 2-posets. Then a $\mathbb{Q}$ -linear map $\mathbb{Q}\mathcal{X} \rightarrow \mathbb{Q}\mathcal{T}$, which we call the *totally ordered expansion*, is defined by

$$[X] = [X, \leq, \delta] \mapsto [X]^{\text{tot}} := \sum_{\leq'} [X, \leq', \delta],$$

where $[X] = [X, \leq, \delta]$ is the isomorphism class of any 2-poset X and $\leq'$ runs over the total orders on the set X which are refinements of the original partial order $\leq$. We have $[X]^{\text{tot}} = [X_a^b]^{\text{tot}} + [X_b^a]^{\text{tot}}$ for any 2-poset X and non-comparable elements $a, b \in X$, where X_a^b denotes the 2-poset obtained from X by adjoining the relation $a < b$. Note also that the integration map $I: \mathbb{Q}\mathcal{X}^0 \rightarrow \mathbb{R}$ factors through the totally ordered expansion, i.e., we have $I([X]) = I([X]^{\text{tot}})$ for any $[X] \in \mathcal{X}^0$.

For any 2-poset X , we define its *admissible part* $[X]^{\text{adm}}$ to be the partial sum of the totally ordered expansion $[X]^{\text{tot}}$ consisting of the admissible terms. For example, if $X =$



$$[X]^{\text{tot}} = \begin{bmatrix} \circ \\ \bullet \\ \circ \\ \bullet \end{bmatrix} + 2 \begin{bmatrix} \bullet \\ \circ \\ \bullet \\ \bullet \end{bmatrix} \quad \text{and} \quad [X]^{\text{adm}} = \begin{bmatrix} \circ \\ \bullet \\ \circ \\ \bullet \end{bmatrix}.$$

Then the $\mathbb{Q}$ -linear map

$$\mathbb{Q}\mathcal{X} \longrightarrow \mathbb{R}; [X] \mapsto I([X]^{\text{adm}})$$

is an extension of $I: \mathbb{Q}\mathcal{X}^0 \rightarrow \mathbb{R}$. Notice that this map is *not* an algebra homomorphism (for example, $[\circ] \cdot [\bullet] \mapsto I([\circ \bullet])$ but $[\circ], [\bullet] \mapsto 0$), unlike the map defined by using the shuffle regularization (see Remark at the end of §2 of [9]).

In the following computations, we omit the symbol ‘tot’; for example, we write $[X] = [X_a^b] + [X_b^a]$ instead of $[X]^{\text{tot}} = [X_a^b]^{\text{tot}} + [X_b^a]^{\text{tot}}$. In other words, we compute in the quotient space $\mathbb{Q}\mathcal{X} / \text{Ker}([X] \mapsto [X]^{\text{tot}})$.

Proof of Proposition 2.1. Set $m_i = k_i - 1$ for $1 \leq i \leq d$. Define

$$X = X_l(\mathbf{k}) = \begin{array}{c} \circ \dots \circ m_d \\ \vdots \\ \bullet \dots \bullet m_2 \\ \vdots \\ \circ \dots \circ m_1 \\ \vdots \\ \bullet \end{array}.$$

Then we have

$$(2.7) \quad P_l(\mathbf{k}) = I([X]^{\text{adm}}),$$

which is the key ingredient of the proof. On the other hand, we see that

$$[X] = \sum_{a=0}^{m_1} (-1)^a \left[\begin{array}{c} \circ \dots \circ m_d \\ \vdots \\ \bullet \dots \bullet m_2 \\ \vdots \\ \circ \dots \circ m_1 - a \\ \vdots \\ \bullet \end{array} \right] + (-1)^{k_1} \left[\begin{array}{c} \circ \dots \circ m_d \\ \vdots \\ \bullet \dots \bullet m_2 \\ \vdots \\ \circ \dots \circ m_1 \\ \vdots \\ \bullet \end{array} \right].$$

By repeating similar computations, we have

$$[X] = \sum_{i=1}^{d-1} \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\dots+k_{i-1}+a_i} [X_{i,a_i}] + (-1)^{k_1+\dots+k_{d-1}} [X_{d,0}],$$

where

$$X_{i,a} = \text{diagram}, \quad X_{d,0} = \text{diagram},$$

that is,

$$\begin{aligned} X_{i,a} &= X_{k_{i-1}-a}(k_{i-1}, \dots, k_1, l+1) \sqcup X_a(k_{i+1}, \dots, k_d), \\ X_{d,0} &= X_{k_{d-1}}(k_{d-1}, \dots, k_1, l+1). \end{aligned}$$

Notice that $X_{d,0}$ is always admissible because $l > 0$. By taking the admissible parts and making the integrals associated with these 2-posets, we have

$$\begin{aligned} (2.8) \quad P_l(\mathbf{k}) &= \sum_{i=1}^{d-1} \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\dots+k_{i-1}+a_i} I([X_{i,a_i}]^{\text{adm}}) \\ &\quad + (-1)^{k_1+\dots+k_{d-1}} P_{k_{d-1}}(k_{d-1}, \dots, k_1, l+1). \end{aligned}$$

If $m_d > 0$ (i.e., $\mathbf{k}$ is admissible), since $X_{i,a}$ is also admissible, the formula (2.3) is immediately obtained from (2.7) and (2.8).

If $m_d = 0$ (i.e., $\mathbf{k}$ is non-admissible), noticing that $X_{i,a}$ is admissible if and only if $i = d-1$ and $a > 0$, we have² from (2.8)

$$\begin{aligned} P_l(\mathbf{k}) &= \sum_{i=1}^{d-1} \sum_{a_i=0}^{k_i-1} \overline{\mathbb{1}}_{i=d-1} (-1)^{k_1+\dots+k_{i-1}+a_i} I([X_{i,a_i}]^{\text{adm}}) \\ &\quad + \sum_{i=d-1}^d \sum_{a_i=0}^{k_i-1} \overline{\mathbb{1}}_{i=d-1} (-1)^{k_1+\dots+k_{i-1}+a_i} P_{k_{i-1}-a_i}(k_{i-1}, \dots, k_1, l+1) P_{a_i}(k_{i+1}, \dots, k_{d-1}, 1). \end{aligned}$$

Now, we compute $I([X_{i,a}]^{\text{adm}})$. Observe that

$$[X_{i,a}]^{\text{adm}} = [X_{i,a}^{(1)}] + [X_{i,a}^{(2)}] + [X_{i,a}^{(3)}],$$

²For a condition P , we let $\mathbb{1}_P$ denote the indicator function on P , that is, $\mathbb{1}_P = 1$ if P is satisfied and 0 otherwise. We also put $\overline{\mathbb{1}}_P = 1 - \mathbb{1}_P$. Condition with multiple lines stands for the conjunction of all lines.

where

$$X_{i,a}^{(1)} = \begin{array}{c} \begin{array}{c} l \\ \bullet \\ \vdots \\ \bullet \\ m_1 \\ \bullet \\ \vdots \\ \bullet \\ m_{i-1} \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_i - a \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_{d-1} \end{array} \\ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_{i+1} \end{array} \end{array},$$

$$X_{i,a}^{(2)} = \begin{array}{c} \begin{array}{c} l \\ \bullet \\ \vdots \\ \bullet \\ m_1 \\ \bullet \\ \vdots \\ \bullet \\ m_{i-1} \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_i - a \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_{d-1} \end{array} \\ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_{i+1} \end{array} \end{array},$$

and

$$X_{i,a}^{(3)} = \begin{array}{c} \begin{array}{c} l \\ \bullet \\ \vdots \\ \bullet \\ m_1 \\ \bullet \\ \vdots \\ \bullet \\ m_{i-1} \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_i - a \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_{d-1} \end{array} \\ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ m_{i+1} \end{array} \end{array}.$$

Here we understand that $X_{i,m_i}^{(2)} = X_{i,0}^{(3)} = 0$.

It is easy from (2.7) again to see that

$$I([X_{i,a}^{(3)}]) = P_{k_i-1-a}(k_{i-1}, \dots, k_1, l+1) P_a(k_{i+1}, \dots, k_{d-1}, 1)$$

and hence

$$(2.9) \quad \begin{aligned} P_l(\mathbf{k}) &= \sum_{i=1}^{d-1} \sum_{a_i=0}^{k_i-1} \mathbb{I}_{i=d-1} (-1)^{k_1+\dots+k_{i-1}+a_i} I([X_{i,a_i}^{(1)}] + [X_{i,a_i}^{(2)}]) \\ &+ \sum_{i=1}^d \sum_{a_i=0}^{k_i-1} (-1)^{k_1+\dots+k_{i-1}+a_i} P_{k_i-1-a_i}(k_{i-1}, \dots, k_1, l+1) P_{a_i}(k_{i+1}, \dots, k_{d-1}, 1). \end{aligned}$$

Moreover, we see that $[X_{i,a}^{(1)}] + [X_{i,a}^{(2)}]$ can be written in terms of 2-posets

$$Y_i(p_0, \dots, \check{p}_i, \dots, p_{d-1}) = \begin{array}{c} \begin{array}{c} p_0 \\ \bullet \\ \vdots \\ \bullet \\ p_1 \\ \bullet \\ \vdots \\ \bullet \\ p_{i-1} \end{array} \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ p_{d-1} \end{array} \\ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ p_{i+1} \end{array} \end{array}$$

for various values $p_0, \dots, \check{p}_i, \dots, p_{d-1}$ ($\check{p}_i$ means that p_i is skipped). Actually, setting $m_0 = l$, we have

$$[X_{i,a}^{(1)}] = \sum_{\substack{\mathbf{b}_i = (b_0, \dots, \check{b}_i, \dots, b_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1} \\ \text{wt}(b_0, \dots, b_{i-1}) = m_i - a_i \\ \text{wt}(b_{i+1}, \dots, b_{d-1}) = a_i}} \binom{m_0 + b_0 - 1}{m_0 - 1} \prod_{\substack{j=1 \\ j \neq i}}^{d-1} \binom{m_j + b_j}{m_j} [Y_i(\mathbf{b}_i + \mathbf{m}_i)],$$

$$[X_{i,a}^{(2)}] = \sum_{\substack{\mathbf{b}_i = (b_0, \dots, \check{b}_i, \dots, b_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1} \\ \text{wt}(b_0, \dots, b_{i-1}) = m_i - a_i \\ \text{wt}(b_{i+1}, \dots, b_{d-1}) = a_i}} \binom{m_0 + b_0 - 1}{m_0} \prod_{\substack{j=1 \\ j \neq i}}^{d-1} \binom{m_j + b_j}{m_j} [Y_i(\mathbf{b}_i + \mathbf{m}_i)],$$

where $\mathbf{m}_i = (m_0, \dots, \check{m}_i, \dots, m_{d-1})$. Hence, using the identity $\binom{s-1}{t-1} + \binom{s-1}{t} = \binom{s}{t}$ for $s, t \geq 0$, we have

$$[X_{i,a}^{(1)}] + [X_{i,a}^{(2)}] = \sum_{\substack{\mathbf{b}_i = (b_0, \dots, \check{b}_i, \dots, b_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1} \\ \text{wt}(b_0, \dots, b_{i-1}) = m_i - a_i \\ \text{wt}(b_{i+1}, \dots, b_{d-1}) = a_i}} \prod_{\substack{j=0 \\ j \neq i}}^{d-1} \binom{m_j + b_j}{m_j} [Y_i(\mathbf{b}_i + \mathbf{m}_i)].$$

Substituting this into (2.9) and changing the order of summations, we see that

$$P_l(\mathbf{k}) = \sum_{i=1}^{d-1} (-1)^{l+k_i} \sum_{\substack{\mathbf{b}_i = (b_0, \dots, \check{b}_i, \dots, b_{d-1}) \in \mathbb{Z}_{\geq 1}^{d-1} \\ \text{wt}(\mathbf{b}_i) = \text{wt}(\mathbf{k}) + l - 1}} (-1)^{b_0 + \dots + b_{i-1}} \prod_{\substack{j=0 \\ j \neq i}}^{d-1} \binom{b_j - 1}{m_j} I[Y_i(\mathbf{b}_i - \{1\}^{d-1})]$$

$$+ \sum_{i=1}^d \sum_{a_i=0}^{k_i-1} (-1)^{k_1 + \dots + k_{i-1} + a_i} P_{k_i-1-a_i}(k_{i-1}, \dots, k_1, l+1) P_{a_i}(k_{i+1}, \dots, k_{d-1}, 1).$$

Therefore, one obtains (2.4) by employing the expression of $I([Y_i(\mathbf{b}_i - \{1\}^{d-1})])$ given in Lemma 2.5 below. This completes the proof. $\square$

Lemma 2.5. For $1 \leq i \leq d-1$ and $\mathbf{p} = (p_0, \dots, \check{p}_i, \dots, p_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$ with $p_0 > 0$, we have

$$I([Y_i(\mathbf{p})]) = \sum_{j=i}^{d-1} \sum_{c_j=0}^{p_j-1} (-1)^{c_j + p_{j+1} + \dots + p_{d-1}} \zeta \left(\begin{matrix} c_j + 1, p_{j+1} + 1, \dots, p_{d-1} + 1 \\ p_{i-1} + 1, \dots, p_1 + 1, p_0 + 1 \end{matrix} \right)$$

$$\times \zeta(p_{i+1} + 1, \dots, p_{j-1} + 1, p_j - c_j + 1),$$

where we set $p_i = 1$.

Proof. When $i = d-1$, the formula reads as

$$I \left(\begin{array}{c} p_0 \\ \vdots \\ p_1 \\ \vdots \\ p_{d-2} \end{array} \right) = \zeta \left(\begin{matrix} 1 \\ p_{d-2} + 1, \dots, p_1 + 1, p_0 + 1 \end{matrix} \right),$$

which is a special case of Theorem 2.4. For $i < d - 1$, we compute as

$$\begin{aligned}
[Y_i(\mathbf{p})] &= \sum_{c_{d-1}=0}^{p_{d-1}-1} (-1)^{c_{d-1}} \left[\begin{array}{c} \text{Diagram 1: } p_0, \dots, p_{i-1}, p_{d-1}-c_{d-1}, p_{i+1} \end{array} \right] + (-1)^{p_{d-1}} \left[\begin{array}{c} \text{Diagram 2: } p_0, \dots, p_{i-1}, p_{d-2}, p_{i+1} \end{array} \right] \\
&= \dots \\
&= \sum_{j=i+1}^{d-1} \sum_{c_j=0}^{p_j-1} (-1)^{c_j+p_{j+1}+\dots+p_{d-1}} \left[\begin{array}{c} \text{Diagram 3: } p_0, \dots, p_{i-1}, p_j-c_j, p_{i+1} \end{array} \right] \\
&\quad + (-1)^{p_{i+1}+\dots+p_{d-1}} \left[\begin{array}{c} \text{Diagram 4: } p_0, \dots, p_{i-1}, p_{i+1} \end{array} \right].
\end{aligned}$$

This together with Theorem 2.4 shows the desired result. $\square$

In the following we give explicit expressions of $P_l(\mathbf{k})$ for the cases $d = 2$ and $d = 3$ which are valid whether $\mathbf{k}$ is admissible or not.

Corollary 2.6. *For $k_1, k_2 \geq 1$ and $l > 0$, we have*

$$\begin{aligned}
P_l(k_1, k_2) &= (-1)^{k_2} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1 + w_2 = k_1 + k_2 + l}} (-1)^{w_1} \binom{w_1 - 1}{k_2 - 1} \binom{w_2 - 1}{l} \zeta(w_1) \zeta(w_2) \\
&\quad + (-1)^{k_1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = k_1 + k_2 + l}} \binom{w_1 - 1}{k_1 - 1} \binom{w_2 - 1}{l} \zeta(w_1, w_2) + \mathbb{1}_{k_2=1} \binom{l + k_1 - 1}{k_1 - 1} \zeta \left(\frac{1}{l + k_1} \right).
\end{aligned}$$

Corollary 2.7. *For $k_1, k_2, k_3 \geq 1$ and $l > 0$, we have*

$$\begin{aligned}
P_l(k_1, k_2, k_3) &= (-1)^{k_1 + k_2} \sum_{\substack{w_1, w_2 \geq 1, w_3 \geq 2 \\ w_1 + w_2 + w_3 = k_1 + k_2 + k_3 + l}} \binom{w_1 - 1}{k_2 - 1} \binom{w_2 - 1}{k_1 - 1} \binom{w_3 - 1}{l} \zeta(w_1, w_2, w_3) \\
&\quad + (-1)^{k_2 + k_3} \sum_{\substack{w_1 \geq 1, w_2 \geq 2, w_3 \geq 2 \\ w_1 + w_2 + w_3 = k_1 + k_2 + k_3 + l}} (-1)^{w_1 + w_2} \binom{w_1 - 1}{k_2 - 1} \binom{w_2 - 1}{k_3 - 1} \binom{w_3 - 1}{l} \zeta(w_1, w_2) \zeta(w_3) \\
&\quad + (-1)^{k_1 + k_3} \sum_{\substack{w_1 \geq 2, w_2 \geq 1, w_3 \geq 2 \\ w_1 + w_2 + w_3 = k_1 + k_2 + k_3 + l}} (-1)^{w_1} \binom{w_1 - 1}{k_3 - 1} \binom{w_2 - 1}{k_1 - 1} \binom{w_3 - 1}{l} \zeta(w_1) \zeta(w_2, w_3)
\end{aligned}$$

$$\begin{aligned}
& + \mathbb{1}_{k_3=1} (-1)^{k_2+1} \sum_{\substack{b_0 \geq 2, b_2 \geq 1 \\ b_0+b_2=k_1+k_2+l}} (-1)^{b_2} \binom{b_0-1}{l} \binom{b_2-1}{k_2-1} \\
& \quad \times \left\{ (-1)^{b_2} \zeta \left(\begin{smallmatrix} 1, b_2 \\ b_0 \end{smallmatrix} \right) + \sum_{c_2=1}^{b_2-1} (-1)^{c_2} \zeta \left(\begin{smallmatrix} c_2 \\ b_0 \end{smallmatrix} \right) \zeta(b_2 - c_2 + 1) \right\} \\
& + \mathbb{1}_{k_3=1} (-1)^{k_1} \sum_{\substack{b_0 \geq 2, b_1 \geq 1 \\ b_0+b_1=k_1+k_2+l}} \binom{b_0-1}{l} \binom{b_1-1}{k_1-1} \zeta \left(\begin{smallmatrix} 1 \\ b_1, b_0 \end{smallmatrix} \right).
\end{aligned}$$

3. RIBBONS

3.1. Preparation. In this section, we study the sums of Schur MZVs for ribbon diagrams. Recall that a skew Young diagram is called a *ribbon* if it is connected and contains no 2×2 block of boxes. Explicitly, such a can be drawn as

$$(3.1) \quad \begin{array}{c} \text{Diagram showing a ribbon structure with three steps. The first step has a horizontal segment labeled } s_1 \text{ and a vertical segment labeled } r_1. \text{ The second step has a horizontal segment labeled } s_2 \text{ and a vertical segment labeled } r_2. \text{ The third step has a horizontal segment labeled } s_n \text{ and a vertical segment labeled } r_n. \text{ A dashed line connects the top of the first vertical segment to the bottom of the second horizontal segment.} \end{array}$$

where the integers $s_1 \geq 0, s_2, \dots, s_n, r_1, \dots, r_n > 0$ indicate the numbers of boxes.

Definition 3.1. For integers $w, s_1, \dots, s_n \geq 0$ and $r_1, \dots, r_n > 0$, we write

$$S_w \left(\begin{smallmatrix} s_1, \dots, s_n \\ r_1, \dots, r_n \end{smallmatrix} \right) := \sum_{\substack{\mathbf{l}_1, \dots, \mathbf{l}_n \\ \text{dep}(\mathbf{l}_i) = s_i \\ \mathbf{k}_1, \dots, \mathbf{k}_n: \text{adm.} \\ \text{dep}(\mathbf{k}_i) = r_i \\ \sum_i \text{wt}(\mathbf{k}_i) + \sum_i \text{wt}(\mathbf{l}_i) = w}} \zeta \left(\begin{smallmatrix} \mathbf{l}_1, \dots, \mathbf{l}_n \\ \mathbf{k}_1, \dots, \mathbf{k}_n \end{smallmatrix} \right),$$

where we define as a generalization of (2.2)

$$(3.2) \quad \zeta \left(\begin{smallmatrix} \mathbf{l}_1, \dots, \mathbf{l}_n \\ \mathbf{k}_1, \dots, \mathbf{k}_n \end{smallmatrix} \right) := \sum_{\substack{0 < b_{i,1} \leq \dots \leq b_{i,s_i+1} \\ 0 < a_{i,1} < \dots < a_{i,r_i} \\ b_{i,s_i+1} = a_{i,r_i} \ (i=1, \dots, n) \\ b_{i+1,1} < a_{i,1} \ (i=1, \dots, n-1)}} \prod_{i=1}^n \frac{1}{a_{i,1}^{k_{i,1}} \dots a_{i,r_i}^{k_{i,r_i}} b_{i,1}^{l_{i,1}} \dots b_{i,s_i}^{l_{i,s_i}}}$$

for indices $\mathbf{l}_i = (l_{i,1}, \dots, l_{i,s_i})$ of depth s_i and $\mathbf{k}_i = (k_{i,1}, \dots, k_{i,r_i})$ of depth r_i . Note that the series (3.2) is meaningful even if some s_i is zero, i.e., $\mathbf{l}_i = \emptyset$.

Remark 3.2. Notice that only for $s_2, \dots, s_n > 0$ the $S_w \binom{s_1, \dots, s_n}{r_1, \dots, r_n}$ gives the sum over all admissible tableaux of shape (3.1) as in (1.2). For example, we have

$$S_w \binom{2}{2} = \sum_{\substack{a, b, d \geq 1, c \geq 2 \\ a+b+c+d=w}} \zeta \left(\begin{array}{|c|c|c|} \hline & & d \\ \hline a & b & c \\ \hline \end{array} \right) = S_w \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \right)$$

but

$$S_w \binom{2, 0}{1, 1} = \sum_{\substack{a, b \geq 1, c, d \geq 2 \\ a+b+c+d=w}} \zeta \left(\begin{array}{|c|c|c|} \hline & & d \\ \hline a & b & c \\ \hline \end{array} \right) \neq S_w \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \right).$$

In the latter, the index (d) is required to be admissible, i.e., $d \geq 2$.

Note that $S_w \binom{s_1, \dots, s_n}{r_1, \dots, r_n}$ is nonzero only when

$$w \geq s_1 + \dots + s_n + r_1 + \dots + r_n + n.$$

Our basic strategy of computing these sums on ribbons is to reduce the number of corners n by using the following formula:

Proposition 3.3. *Let $s_1, \dots, s_n \geq 0$ and $r_1, \dots, r_n > 0$ be integers. For $1 \leq i \leq n-1$ with $r_i \geq 2$, we have*

$$(3.3) \quad S_w \binom{s_1, \dots, s_i, s_{i+1}, \dots, s_n}{r_1, \dots, r_i, r_{i+1}, \dots, r_n} + S_w \binom{s_1, \dots, s_i, s_{i+1}+1, \dots, s_n}{r_1, \dots, r_i-1, r_{i+1}, \dots, r_n} \\ = \sum_{\substack{w_1+w_2=w \\ w_1 \geq s_1+\dots+s_i+r_1+\dots+r_i+i \\ w_2 \geq s_{i+1}+\dots+s_n+r_{i+1}+\dots+r_n+n-i}} S_{w_1} \binom{s_1, \dots, s_i}{r_1, \dots, r_i} \cdot S_{w_2} \binom{s_{i+1}, \dots, s_n}{r_{i+1}, \dots, r_n}.$$

Proof. By switching the inequality $b_{i+1,1} < a_{i,1}$ in (3.2) (for the given i) to the opposite $a_{i,1} \leq b_{i+1,1}$ and adding the corresponding Schur MZVs, we deduce that

$$(3.4) \quad \zeta \left(\begin{array}{|c|c|c|} \hline \mathbf{l}_1, \dots, \mathbf{l}_i, \mathbf{l}_{i+1}, \dots, \mathbf{l}_n \\ \hline \mathbf{k}_1, \dots, \mathbf{k}_i, \mathbf{k}_{i+1}, \dots, \mathbf{k}_n \\ \hline \end{array} \right) + \zeta \left(\begin{array}{|c|c|c|} \hline \mathbf{l}_1, \dots, \mathbf{l}_i, \mathbf{l}'_{i+1}, \dots, \mathbf{l}_n \\ \hline \mathbf{k}_1, \dots, \mathbf{k}'_i, \mathbf{k}_{i+1}, \dots, \mathbf{k}_n \\ \hline \end{array} \right) \\ = \zeta \left(\begin{array}{|c|c|c|} \hline \mathbf{l}_1, \dots, \mathbf{l}_i \\ \hline \mathbf{k}_1, \dots, \mathbf{k}_i \\ \hline \end{array} \right) \cdot \zeta \left(\begin{array}{|c|c|c|} \hline \mathbf{l}_{i+1}, \dots, \mathbf{l}_n \\ \hline \mathbf{k}_{i+1}, \dots, \mathbf{k}_n \\ \hline \end{array} \right),$$

where $\mathbf{k}'_i = (k_{i,2}, \dots, k_{i,r_i})$ and $\mathbf{l}'_{i+1} = (k_{i,1}, l_{i+1,1}, \dots, l_{i+1,s_{i+1}})$. Then (3.3) follows. $\square$

By using Proposition 3.3 repeatedly, the sums on general ribbons are expressed in terms of the values of the type $S_w \binom{s, 0, \dots, 0}{r_1, r_2, \dots, r_n}$. For example, we have

$$S_w \binom{s, 1}{r_1, r_2} = \sum_{\substack{w_1+w_2=w \\ w_1 \geq s+r_1+2 \\ w_2 \geq r_2+1}} S_{w_1} \binom{s}{r_1+1} S_{w_2} \binom{0}{r_2} - S_w \binom{s, 0}{r_1+1, r_2}, \\ S_w \binom{s, 2}{r_1, r_2} = \sum_{\substack{w_1+w_2=w \\ w_1 \geq s+r_1+2 \\ w_2 \geq r_2+2}} S_{w_1} \binom{s}{r_1+1} S_{w_2} \binom{1}{r_2} - S_w \binom{s, 1}{r_1+1, r_2} \\ = \sum_{\substack{w_1+w_2=w \\ w_1 \geq s+r_1+2 \\ w_2 \geq r_2+2}} S_{w_1} \binom{s}{r_1+1} S_{w_2} \binom{1}{r_2}$$

$$- \sum_{\substack{w_1+w_2=w \\ w_1 \geq s+r_1+e \\ w_2 \geq r_2+1}} S_{w_1} \binom{s}{r_1+2} S_{w_2} \binom{0}{r_2} + S_w \binom{s, 0}{r_1+2, r_2}$$

and so on (cf. Lemma 3.11). For the latter type sums, the following formula holds:

Theorem 3.4. *For $w \geq 0$, $s \geq 0$ and $r_1, \dots, r_n > 0$, we have*

$$(3.5) \quad S_w \binom{s, 0, \dots, 0}{r_1, r_2, \dots, r_n} = \sum_{\substack{t_1, \dots, t_n \geq 0 \\ t_1 + \dots + t_n = s}} \sum_{\substack{w_i \geq r_i + t_i + 1 \\ w_1 + \dots + w_n = w}} \prod_{i=1}^n \binom{w_i - 1}{t_i} \zeta(w_1, \dots, w_n).$$

Proof. Put $r := r_1 + \dots + r_n$. Then the left hand side is the sum of the series

$$(3.6) \quad \zeta \left(\begin{array}{|c|c|c|c|} \hline & & k_1 & \\ \hline & & \vdots & \\ \hline l_1 & \dots & l_s & k_r \\ \hline \end{array} \right) = \sum_{0 < a_1 < \dots < a_r \leq b_s \leq \dots \leq b_1 > 0} \frac{1}{a_1^{k_1} \dots a_r^{k_r} b_1^{l_1} \dots b_s^{l_s}},$$

where $(l_1, \dots, l_s)$ runs through indices of depth s and $(k_1, \dots, k_r)$ runs through indices of depth r such that $k_u \geq 2$ for $u \in \{r_n, r_n + r_{n-1}, \dots, r\}$, satisfying $l_1 + \dots + l_s + k_1 + \dots + k_r = w$. By “stuffing”, i.e., classifying all possible orders of a_p ’s and b_q ’s, this series is expanded into a certain sum of MZVs $\zeta(x_1, \dots, x_{r+u})$ of weight w with $0 \leq u \leq s$. Here each of entries of $(x_1, \dots, x_{r+u})$ is of the form

$$(3.7) \quad k_p + l_q + \dots + l_{q'} \text{ or } l_q + \dots + l_{q'},$$

where $l_q, \dots, l_{q'}$ are some consecutive members of $l_1, \dots, l_s$, possibly zero members in the former case and at least one member in the latter.

Let us fix an index $(x_1, \dots, x_{r+u})$ of weight w with $0 \leq u \leq s$ and count how many times $\zeta(x_1, \dots, x_{r+u})$ appears in $S_w \binom{s, 0, \dots, 0}{r_1, r_2, \dots, r_n}$ when all series (3.6) are expanded as above. First, given $u_1, \dots, u_n \geq 0$ satisfying $u_1 + \dots + u_n = u$, consider the cases that $x_{r_n+u_n+\dots+r_i+u_i}$ contains $k_{r_n+\dots+r_i}$ as a summand for $i = 1, \dots, n$. Then the number of possibilities of the places where other k_p ’s appear is $\prod_{i=1}^n \binom{r_i-1+u_i}{u_i}$. Moreover, each entry of $(x_1, \dots, x_{r+u})$ is decomposed into a sum of type (3.7) and the total number of the plus symbol ‘+’ is $s - u$. The number of possible such decompositions is

$$\begin{aligned} & \left(\sum_{i=1}^n \{ (x_{h_{i+1}+1} - 1) + \dots + (x_{h_i-1} - 1) + (x_{h_i} - 2) \} \right)_{s-u} \\ &= \binom{w - n - r - u}{s - u}, \end{aligned}$$

where we set $h_i = r_n + u_n + \dots + r_i + u_i$ and $h_{n+1} = 0$. Therefore we obtain that

$$\begin{aligned} S_w \binom{s, 0, \dots, 0}{r_1, r_2, \dots, r_n} &= \sum_{\substack{u_1, \dots, u_n \geq 0 \\ u := u_1 + \dots + u_n \leq s}} \binom{w - n - r - u}{s - u} \prod_{i=1}^n \binom{r_i + u_i - 1}{u_i} \\ &\quad \times \sum_{\substack{w_1, \dots, w_{r+u} \geq 1 \\ w_p \geq 2 \ (p=r_n+u_n+\dots+r_i+u_i) \\ w_1 + \dots + w_{r+u} = w}} \zeta(w_1, \dots, w_{r+u}). \end{aligned}$$

By applying Ohno's relation to the last sum, we see that this is equal to

$$(3.8) \quad S_w \begin{pmatrix} s, 0, \dots, 0 \\ r_1, r_2, \dots, r_n \end{pmatrix} = \sum_{\substack{u_1, \dots, u_n \geq 0 \\ u := u_1 + \dots + u_n \leq s}} \binom{w - n - r - u}{s - u} \prod_{i=1}^n \binom{r_i + u_i - 1}{u_i} \\ \times \sum_{\substack{w_i \geq r_i + u_i + 1 \\ w_1 + \dots + w_n = w}} \zeta(w_1, \dots, w_n).$$

By means of the identity

$$\binom{w - n - r - u}{s - u} = \sum_{\substack{v_1, \dots, v_n \geq 0 \\ v_1 + \dots + v_n = s - u}} \prod_{i=1}^n \binom{w_i - 1 - r_i - u_i}{v_i},$$

one can rewrite the expression (3.8) as

$$\sum_{\substack{u_1, \dots, u_n \geq 0 \\ v_1, \dots, v_n \geq 0 \\ u_1 + v_1 + \dots + u_n + v_n = s}} \sum_{\substack{w_i \geq r_i + u_i + 1 \\ w_1 + \dots + w_n = w}} \prod_{i=1}^n \binom{w_i - 1 - r_i - u_i}{v_i} \binom{r_i + u_i - 1}{u_i} \zeta(w_1, \dots, w_n) \\ = \sum_{\substack{t_1, \dots, t_n \geq 0 \\ t_1 + \dots + t_n = s}} \sum_{\substack{w_i \geq r_i + t_i + 1 \\ w_1 + \dots + w_n = w}} \prod_{i=1}^n \left(\sum_{u_i=0}^{t_i} \binom{w_i - 1 - r_i - u_i}{t_i - u_i} \binom{r_i + u_i - 1}{u_i} \right) \zeta(w_1, \dots, w_n).$$

Here we put $t_i = u_i + v_i$ and use that

$$\binom{w_i - 1 - r_i - u_i}{t_i - u_i} = 0 \quad \text{when } t_i > u_i \text{ and } r_i + u_i + 1 \leq w_i \leq r_i + t_i.$$

Thus the theorem follows from the identity

$$\sum_{u_i=0}^{t_i} \binom{w_i - 1 - r_i - u_i}{t_i - u_i} \binom{r_i + u_i - 1}{u_i} = \binom{w_i - 1}{t_i}. \quad \square$$

Corollary 3.5. *For $n \geq 1$ and integers $s_1, \dots, s_n \geq 0$ and $r_1, \dots, r_n > 0$ the sum $S_w(s_1, \dots, s_n)$ can be written as a $\mathbb{Q}$ -linear combination of MZVs of weight w and depth $\leq n$.*

Proof. This is a direct consequence of Proposition 3.3 and Theorem 3.4. $\square$

Corollary 3.6. *For any $s \geq 0$ and $r_1, \dots, r_n > 0$, the symmetric sum*

$$\sum_{\sigma \in \mathfrak{S}_n} S_w \begin{pmatrix} s, 0, \dots, 0 \\ r_{\sigma(1)}, r_{\sigma(2)}, \dots, r_{\sigma(n)} \end{pmatrix}$$

is a polynomial in single zeta values. In particular, $S_w(s, 0, \dots, 0)$ is a polynomial in single zeta values.

Proof. By (3.5), this symmetric sum is equal to

$$\sum_{\substack{t_1, \dots, t_n \geq 0 \\ t_1 + \dots + t_n = s}} \sum_{\substack{w_i \geq r_i + t_i + 1 \\ w_1 + \dots + w_n = w}} \prod_{i=1}^n \binom{w_i - 1}{t_i} \sum_{\sigma \in \mathfrak{S}_n} \zeta(w_{\sigma(1)}, \dots, w_{\sigma(n)}).$$

Thus the claim follows from Hoffman's symmetric sum formula [3, Theorem 2.2]. $\square$

Example 3.7. The case of $n = 1$ will be treated in Theorem 3.8. Here let us consider the case of $n = 2$ and $r_1 = r_2 = r$. We have

$$\begin{aligned}
S_w \begin{pmatrix} s, 0 \\ r, r \end{pmatrix} &= \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s}} \sum_{\substack{w_i \geq r + t_i + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \zeta(w_1, w_2) \\
&= \frac{1}{2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s}} \sum_{\substack{w_i \geq r + t_i + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} (\zeta(w_1, w_2) + \zeta(w_2, w_1)) \\
&= \frac{1}{2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s}} \sum_{\substack{w_i \geq r + t_i + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} (\zeta(w_1)\zeta(w_2) - \zeta(w)).
\end{aligned}$$

This is a kind of sum formula of polynomial type. For a sum formula of bounded type, see §3.3 (for instance see Example 3.16 for a hook case).

3.2. Sum formulas of single type. In this subsection, we present sum formulas of single type for two special types of ribbons. The first is a simultaneous generalization of the classical sum formulas for the MZVs and MZSVs stated in (1.1).

Theorem 3.8. *For any integers $r \geq 1$, $s \geq 0$ and $w \geq s + r + 1$, we have*

$$(3.9) \quad S_w \begin{pmatrix} s \\ r \end{pmatrix} = \binom{w-1}{s} \zeta(w).$$

Proof. This immediately follows from Theorem 3.4. $\square$

The second formula is for the “stair of tread one” shape. The proof is an application of Proposition 3.3 and Theorem 3.4.

Theorem 3.9. *For any integers $r \geq 1$, $n \geq 1$ and $w \geq (r+2)n + 1$, we have*

$$(3.10) \quad S_w \begin{pmatrix} \{1\}^{n-1}, 1 \\ \{r\}^{n-1}, r+1 \end{pmatrix} = c_{w,r}(n) \zeta(w),$$

where

$$c_{w,r}(n) := \frac{w-1}{n} \binom{w - (r+1)n - 2}{n-1}.$$

See (1.4) in the introduction for examples in the cases $r = n = 2$ and $r = 1$, $n = 3$.

Remark 3.10. The coefficient $c_{w,r}(n)$ is a positive integer. In fact,

$$c_{w,r}(n) = (r+1) \binom{w - (r+1)n - 2}{n-1} + \binom{w - (r+1)n - 1}{n}.$$

Proof of Theorem 3.9. We prove (3.10) by induction on n . For $n = 1$, this is a special case of (3.9). Let us assume $n > 1$ and for $1 \leq i \leq n$, put

$$S_{w,r}(n, i) := S_w \begin{pmatrix} \{1\}^{i-1}, 1, \{0\}^{n-i} \\ \{r\}^{i-1}, r+1, \{r+1\}^{n-i} \end{pmatrix}, \quad T_{w,r}(n) := S_w \begin{pmatrix} \{0\}^n \\ \{r+1\}^n \end{pmatrix}.$$

Then, for $1 \leq i \leq n-1$, Proposition 3.3 shows that

$$S_{w,r}(n, i) + S_{w,r}(n, i+1) = \sum_{\substack{w_1 \geq (r+2)i+1 \\ w_2 \geq (r+2)(n-i)}} S_{w_1,r}(i, i) \cdot T_{w_2,r}(n-i)$$

(here and in what follows, we omit the “total weight = w ” condition like $w_1 + w_2 = w$). The induction hypothesis gives $S_{w_1,r}(i, i) = c_{w_1,r}(i)\zeta(w_1)$ for $1 \leq i \leq n-1$, while Theorem 3.4 shows that

$$T_{w_2,r}(n-i) = \sum_{\substack{w'_1, \dots, w'_{n-i} \geq r+2 \\ w'_1 + \dots + w'_{n-i} = w_2}} \zeta(w'_1, \dots, w'_{n-i}).$$

Hence we obtain

$$(3.11) \quad \begin{aligned} S_{w,r}(n, i) + S_{w,r}(n, i+1) &= \sum_{\substack{w_1 \geq (r+2)i+1 \\ w'_1, \dots, w'_{n-i} \geq r+2}} c_{w_1,r}(i) \zeta(w_1) \zeta(w'_1, \dots, w'_{n-i}) \\ &= A_i + B_i, \end{aligned}$$

where A_i and B_i are defined by

$$(3.12) \quad \begin{aligned} A_i &:= \sum_{\substack{w_1 \geq (r+2)i+1 \\ w'_1, \dots, w'_{n-i} \geq r+2}} c_{w_1,r}(i) \{ \zeta(w_1, w'_1, \dots, w'_{n-i}) + \dots + \zeta(w'_1, \dots, w'_{n-i}, w_1) \} \\ &= \sum_{j=1}^{n-i+1} \sum_{\substack{w_1, \dots, w_{n-i+1} \geq r+2 \\ w_j \geq (r+2)i+1}} c_{w_j,r}(i) \zeta(w_1, \dots, w_{n-i+1}) \end{aligned}$$

and

$$\begin{aligned} B_i &:= \sum_{\substack{w_1 \geq (r+2)i+1 \\ w'_1, \dots, w'_{n-i} \geq r+2}} c_{w_1,r}(i) \{ \zeta(w_1 + w'_1, \dots, w'_{n-i}) + \dots + \zeta(w'_1, \dots, w_1 + w'_{n-i}) \} \\ &= \sum_{j=1}^{n-i} \sum_{\substack{w_1, \dots, w_{n-i} \geq r+2 \\ w_j \geq (r+2)(i+1)+1}} \sum_{a=(r+2)i+1}^{w_j-(r+2)} c_{a,r}(i) \zeta(w_1, \dots, w_{n-i}). \end{aligned}$$

Note that the definition of A_i works for $i = n$ and gives $A_n = c_{w,r}(n)\zeta(w)$.

For $1 \leq i \leq n-1$, we have $A_{i+1} = B_i$ since

$$\begin{aligned} \sum_{a=(r+2)i+1}^{w_j-(r+2)} c_{a,r}(i) &= \sum_{b=0}^k \frac{(r+2)i+b}{i} \binom{b+i-1}{i-1} \quad (k := w_j - (r+2)(i+1) - 1) \\ &= (r+1) \sum_{b=0}^k \binom{b+i-1}{i-1} + \sum_{b=0}^k \binom{b+i}{i} \\ &= (r+1) \binom{k+i}{i} + \binom{k+i+1}{i+1} = c_{w_j,r}(i+1). \end{aligned}$$

Thus (3.11) says that

$$S_{w,r}(n, i) + S_{w,r}(n, i+1) = A_i + A_{i+1}.$$

This implies inductively that $S_{w,r}(n, i) = A_i$, starting from

$$S_{w,r}(n, 1) = \sum_{j=1}^n \sum_{\substack{w_1, \dots, w_n \geq r+2 \\ w_j \geq r+3}} (w_j - 1) \zeta(w_1, \dots, w_n) = A_1,$$

which is a consequence of (3.5) and (3.12). The final identity $S_{w,r}(n, n) = A_n = c_{w,r}(n)\zeta(w)$ is exactly the desired formula. $\square$

3.3. Two corners. We next study ribbon shapes with two corners which give a bounded type sum formula. The key ingredient is the sum formula weighted by the binomial coefficients (Proposition 2.1). For ribbons with two corners, we only use the case $d = 2$. For ease of calculation, we rewrite Corollary 2.6 in the following form with $m_1 = k_1 - 1$ and $m_2 = k_2 - 1$: For integers $m_1, m_2 \geq 0$ and $w \geq m_1 + m_2 + 3$, we have

$$\begin{aligned}
& \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{m_1} \binom{w_2 - 1}{m_2} \zeta(w_1, w_2) \\
&= (-1)^{m_2+1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1 + w_2 = w}} (-1)^{w_1} \binom{w_1 - 1}{m_2} \binom{w_2 - 1}{m_1 + m_2 + 1 - w_1} \zeta(w_1) \zeta(w_2) \\
&\quad + (-1)^{m_1+1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{m_1} \binom{w_2 - 1}{m_1 + m_2 + 1 - w_1} \zeta(w_1, w_2) \\
&\quad + \mathbb{1}_{m_2=0} \binom{w-2}{m_1} (\zeta(w) + \zeta(1, w-1)).
\end{aligned} \tag{3.13}$$

We then start with a preliminary calculation:

Lemma 3.11. *For $s_1, s_2 \geq 0$, $r_1, r_2 > 0$ and $w \geq s_1 + s_2 + r_1 + r_2 + 2$, we have*

$$\begin{aligned}
S_w \left(\begin{smallmatrix} s_1, s_2 \\ r_1, r_2 \end{smallmatrix} \right) &= \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1 \geq s_1 + s_2 + r_1 - i + 1 \\ w_2 \geq r_2 + i + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \binom{w_2 - 1}{i} \zeta(w_1) \zeta(w_2) \\
&\quad + (-1)^{s_2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s_1}} \sum_{\substack{w_1 \geq s_2 + r_1 + t_1 + 1 \\ w_2 \geq r_2 + t_2 + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \zeta(w_1, w_2).
\end{aligned}$$

Proof. By a repeated application of Proposition 3.3, we obtain

$$\begin{aligned}
S_w \left(\begin{smallmatrix} s_1, s_2 \\ r_1, r_2 \end{smallmatrix} \right) &= \sum_{w_1 + w_2 = w} S_{w_1} \left(\begin{smallmatrix} s_1 \\ r_1 + 1 \end{smallmatrix} \right) S_{w_2} \left(\begin{smallmatrix} s_2 - 1 \\ r_2 \end{smallmatrix} \right) - S_w \left(\begin{smallmatrix} s_1, s_2 - 1 \\ r_1 + 1, r_2 \end{smallmatrix} \right) \\
&= \dots \\
&= \sum_{i=1}^{s_2} (-1)^{i-1} \sum_{w_1 + w_2 = w} S_{w_1} \left(\begin{smallmatrix} s_1 \\ r_1 + i \end{smallmatrix} \right) S_{w_2} \left(\begin{smallmatrix} s_2 - i \\ r_2 \end{smallmatrix} \right) + (-1)^{s_2} S_w \left(\begin{smallmatrix} s_1, 0 \\ s_2 + r_1, r_2 \end{smallmatrix} \right) \\
&= \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{w_1 + w_2 = w} S_{w_1} \left(\begin{smallmatrix} s_1 \\ s_2 + r_1 - i \end{smallmatrix} \right) S_{w_2} \left(\begin{smallmatrix} i \\ r_2 \end{smallmatrix} \right) + (-1)^{s_2} S_w \left(\begin{smallmatrix} s_1, 0 \\ s_2 + r_1, r_2 \end{smallmatrix} \right).
\end{aligned}$$

By applying Theorem 3.8 to the first sum with taking care of admissible range and by applying Theorem 3.4 to the second sum, we obtain the lemma. $\square$

As an immediate consequence of the expression above, one finds that there is indeed a polynomial type sum formula for a specific class of ribbons with two corners, that is, the case $r_2 = s_2 + r_1$.

Theorem 3.12. *For $s_1, s_2 \geq 0$, $r_1 > 0$ and $w \geq s_1 + 2s_2 + 2r_1 + 2$, we have*

$$S_w \left(\begin{smallmatrix} s_1, s_2 \\ r_1, s_2 + r_1 \end{smallmatrix} \right)$$

is a polynomial in single zeta values.

Proof. By Lemma 3.11 with $r_2 = s_2 + r_1$, it suffices to show

$$\sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s_1}} \sum_{\substack{w_1 \geq s_2 + r_1 + t_1 + 1 \\ w_2 \geq s_2 + r_1 + t_2 + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \zeta(w_1, w_2)$$

can be written as a polynomial of single zeta values. By symmetry, this sum is

$$= \frac{1}{2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s_1}} \sum_{\substack{w_1 \geq s_2 + r_1 + t_1 + 1 \\ w_2 \geq s_2 + r_1 + t_2 + 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} (\zeta(w_1, w_2) + \zeta(w_2, w_1)).$$

Then, the result follows by the harmonic product formula. $\square$

The next theorem is a sum formula for general ribbons with two corners, which is of bounded type. Although the explicit formula itself is rather complicated, it is a direct consequence of (3.13) and Lemma 3.11.

Theorem 3.13. *For $s_1, s_2 \geq 0$, $r_1, r_2 > 0$ and $w \geq s_1 + s_2 + r_1 + r_2 + 2$, we have*

$$(3.14) \quad S_w \begin{pmatrix} s_1, s_2 \\ r_1, r_2 \end{pmatrix} = \binom{w-2}{s_1+s_2} \zeta(w) + \sum_{\substack{w_1, w_2 \geq 2 \\ w_1 + w_2 = w}} A_{w_1, w_2}^{s_1, s_2, r_1, r_2} \zeta(w_1) \zeta(w_2) + \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} B_{w_1, w_2}^{s_1, s_2, r_1, r_2} \zeta(w_1, w_2),$$

where the integers $A_{w_1, w_2}^{s_1, s_2, r_1, r_2}$ and $B_{w_1, w_2}^{s_1, s_2, r_1, r_2}$ are given by

$$\begin{aligned} A_{w_1, w_2}^{s_1, s_2, r_1, r_2} &:= (-1)^{w_1} C_{w_1, w_2}^{s_1, s_2} \\ &\quad - \mathbb{1}_{w_1 \leq s_1 + r_1 \text{ or } w_2 \leq s_2 + r_2 - 1} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_2 - 1} \\ &\quad + \mathbb{1}_{w_1 > s_1 + r_1} (-1)^{s_1 + r_1 + w_1} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_1 + s_2 + r_1 - w_1} \\ &\quad + \mathbb{1}_{r_2 < w_2 \leq s_2 + r_2 - 1} (-1)^{s_2 + r_2 + w_2} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{r_2 - 1}, \\ B_{w_1, w_2}^{s_1, s_2, r_1, r_2} &:= C_{w_1, w_2}^{s_1, s_2} - (-1)^{s_2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 \geq w_1 - (s_2 + r_1) \text{ or } t_2 \geq w_2 - r_2 \\ t_1 + t_2 = s_1}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \end{aligned}$$

with

$$C_{w_1, w_2}^{s_1, s_2} := (-1)^{s_2} \sum_{\substack{0 \leq i \leq s_1 \\ 1 \leq j \leq s_2 \\ i+j=w_1}} \binom{w_1 - 1}{i} \binom{w_2 - 1}{s_1 - i} - (-1)^{s_1} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_1 + s_2 - w_1}.$$

Remark 3.14. By our convention on binomial coefficients, $A_{w_1, w_2}^{s_1, s_2, r_1, r_2} = B_{w_1, w_2}^{s_1, s_2, r_1, r_2} = 0$ unless

$$w_1 \leq s_1 + s_2 + r_1 \quad \text{or} \quad w_2 \leq \max(s_2 + r_2 - 1, s_1 + r_2)$$

and so (3.14) is a bounded type sum formula after expanding the product $\zeta(w_1)\zeta(w_2)$ by the harmonic product formula.

Proof of Theorem 3.13. Write the equation in Lemma 3.11 as

$$S_w \begin{pmatrix} s_1, s_2 \\ r_1, r_2 \end{pmatrix} = S_1 + S_2$$

with

$$S_1 := \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1 \geq s_1+s_2+r_1-i+1 \\ w_2 \geq r_2+i+1 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{i} \zeta(w_1) \zeta(w_2),$$

$$S_2 := (-1)^{s_2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1+t_2=s_1}} \sum_{\substack{w_1 \geq s_2+r_1+t_1+1 \\ w_2 \geq r_2+t_2+1 \\ w_1+w_2=w}} \binom{w_1-1}{t_1} \binom{w_2-1}{t_2} \zeta(w_1, w_2).$$

The sum S_1 can be rewritten as $S_1 = S_{11} - S_{12} - S_{13}$ where

$$S_{11} := \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{i} \zeta(w_1) \zeta(w_2)$$

$$S_{12} := \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{2 \leq w_1 \leq s_1+s_2+r_1-i \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{i} \zeta(w_1) \zeta(w_2)$$

$$S_{13} := \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{2 \leq w_2 \leq r_2+i \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{i} \zeta(w_1) \zeta(w_2).$$

By the harmonic product formula, we have $S_{11} = S_{111} + S_{112} + S_{113}$ with

$$S_{111} := \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{i} \zeta(w_1, w_2),$$

$$S_{112} := \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{i} \binom{w_2-1}{s_1} \zeta(w_1, w_2),$$

$$S_{113} := \left\{ \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{i} \right\} \zeta(w).$$

For the sum S_{111} , after supplementing the term with $w_1 = 1$, by (3.13), we get

$$S_{111} = (-1)^{s_2} \sum_{i=0}^{s_2-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} (-1)^{w_1} \binom{w_1-1}{i} \binom{w_2-1}{s_1+i+1-w_1} \zeta(w_1) \zeta(w_2)$$

$$+ \sum_{i=0}^{s_2-1} (-1)^{s_1+s_2-i} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{s_1+i+1-w_1} \zeta(w_1, w_2)$$

$$+ \mathbb{1}_{s_2>0} (-1)^{s_2-1} \binom{w-2}{s_1} (\zeta(w) + \zeta(1, w-1))$$

$$- \mathbb{1}_{s_1=0} \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \binom{w-2}{i} \zeta(1, w-1),$$

where we should note that the sum S_{111} is empty if $s_2 = 0$ and $\mathbb{1}_{s_2>0}$ is inserted to cover such a degenerate case. By swapping the summation and changing variable via $i \rightsquigarrow w_1 - i - 1$, we have

$$\begin{aligned} & (-1)^{s_2} \sum_{i=0}^{s_2-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} (-1)^{w_1} \binom{w_1-1}{i} \binom{w_2-1}{s_1+i+1-w_1} \zeta(w_1) \zeta(w_2) \\ &= (-1)^{s_2} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} (-1)^{w_1} \zeta(w_1) \zeta(w_2) \sum_{\substack{0 \leq i \leq s_1 \\ 1 \leq j \leq s_2 \\ i+j=w_1}} \binom{w_1-1}{i} \binom{w_2-1}{s_1-i}. \end{aligned}$$

Since $\binom{w_2-1}{s_1+i+1-w_1} = 0$ if $i < w_1 - s_1 - 1$, by changing variable via $i \rightsquigarrow i + w_1 - s_1 - 1$, we have

$$\begin{aligned} & \sum_{i=0}^{s_2-1} (-1)^{s_1+s_2-i} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-1}{s_1+i+1-w_1} \zeta(w_1, w_2) \\ &= -(-1)^{s_1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-2}{s_1+s_2-w_1} \zeta(w_1, w_2). \end{aligned}$$

Combining the above two formulas with $\sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \binom{w-2}{i} = \binom{w-3}{s_2-1}$, we have

$$\begin{aligned} S_{111} &= (-1)^{s_2} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} (-1)^{w_1} \zeta(w_1) \zeta(w_2) \sum_{\substack{0 \leq j \leq s_1 \\ 1 \leq i \leq s_2 \\ i+j=w_1}} \binom{w_1-1}{j} \binom{w_2-1}{s_1-j} \\ &\quad - (-1)^{s_1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{s_1} \binom{w_2-2}{s_1+s_2-w_1} \zeta(w_1, w_2) \\ &\quad - \mathbb{1}_{s_2>0} (-1)^{s_2} \binom{w-2}{s_1} (\zeta(w) + \zeta(1, w-1)) - \mathbb{1}_{s_1=0} \binom{w-3}{s_2-1} \zeta(1, w-1). \end{aligned}$$

Similarly, for the sum S_{112} , we can apply (3.13) to get

$$\begin{aligned} S_{112} &= \sum_{i=0}^{s_2-1} (-1)^{s_1+s_2-i} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1+w_2=w}} (-1)^{w_1} \binom{w_1-1}{s_1} \binom{w_2-1}{s_1+i+1-w_1} \zeta(w_1) \zeta(w_2) \\ &\quad + (-1)^{s_2} \sum_{i=0}^{s_2-1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1+w_2=w}} \binom{w_1-1}{i} \binom{w_2-1}{s_1+i+1-w_1} \zeta(w_1, w_2) \\ &\quad + \mathbb{1}_{s_1=0} \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \binom{w-2}{i} (\zeta(w) + \zeta(1, w-1)) \\ &\quad + \mathbb{1}_{s_2>0} (-1)^{s_2} \binom{w-2}{s_1} \zeta(1, w-1). \end{aligned}$$

By a calculation similar to that of S_{111} , we have

$$\begin{aligned}
S_{112} = & -(-1)^{s_1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1 + w_2 = w}} (-1)^{w_1} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_1 + s_2 - w_1} \zeta(w_1) \zeta(w_2) \\
& + (-1)^{s_2} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \zeta(w_1, w_2) \sum_{\substack{0 \leq i \leq s_1 \\ 1 \leq j \leq s_2 \\ i+j=w_1}} \binom{w_1 - 1}{i} \binom{w_2 - 1}{s_1 - i} \\
& + \mathbb{1}_{s_1=0} \binom{w-3}{s_2-1} (\zeta(w) + \zeta(1, w-1)) + \mathbb{1}_{s_2>0} (-1)^{s_2} \binom{w-2}{s_1} \zeta(1, w-1).
\end{aligned}$$

For the sum S_{113} , by $\sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \binom{w_2-1}{i} = \binom{w_2-2}{s_2-1}$ we have

$$\begin{aligned}
S_{113} = & \left\{ \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \sum_{\substack{w_1, w_2 \geq 2 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \binom{w_2 - 1}{i} \right\} \zeta(w) \\
= & \left\{ \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_2 - 1} - \mathbb{1}_{s_1=0} \binom{w-3}{s_2-1} \right\} \zeta(w) \\
= & \left\{ \mathbb{1}_{s_2>0} \binom{w-2}{s_1 + s_2} - \mathbb{1}_{s_1=0} \binom{w-3}{s_2-1} \right\} \zeta(w).
\end{aligned}$$

We next consider the sum S_{12} and S_{13} . By swapping the summation, we have

$$\begin{aligned}
S_{12} = & \sum_{\substack{2 \leq w_1 \leq s_1 + s_2 + r_1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \zeta(w_1) \zeta(w_2) \sum_{i=0}^{\min(s_2-1, s_1+s_2+r_1-w_1)} (-1)^{s_2-i-1} \binom{w_2-1}{i} \\
= & \sum_{\substack{2 \leq w_1 \leq s_1 + r_1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \zeta(w_1) \zeta(w_2) \sum_{i=0}^{s_2-1} (-1)^{s_2-i-1} \binom{w_2-1}{i} \\
& + \sum_{\substack{s_1 + r_1 < w_1 \leq s_1 + s_2 + r_1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \zeta(w_1) \zeta(w_2) \sum_{i=0}^{s_1 + s_2 + r_1 - w_1} (-1)^{s_2-i-1} \binom{w_2-1}{i} \\
= & \sum_{\substack{2 \leq w_1 \leq s_1 + r_1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_2 - 1} \zeta(w_1) \zeta(w_2) \\
& - \sum_{\substack{s_1 + r_1 < w_1 \leq s_1 + s_2 + r_1 \\ w_1 + w_2 = w}} (-1)^{s_1 + r_1 + w_1} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_1 + s_2 + r_1 - w_1} \zeta(w_1) \zeta(w_2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
S_{13} = & \sum_{\substack{2 \leq w_2 \leq s_2 + r_2 - 1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{s_2 - 1} \zeta(w_1) \zeta(w_2) \\
& - \sum_{\substack{r_2 < w_2 \leq s_2 + r_2 - 1 \\ w_1 + w_2 = w}} (-1)^{s_2 + r_2 + w_2} \binom{w_1 - 1}{s_1} \binom{w_2 - 2}{r_2 - 1} \zeta(w_1) \zeta(w_2).
\end{aligned}$$

For the sum S_2 , we can dissect as $S_2 = S_{21} - S_{22} - S_{23}$ by writing

$$\begin{aligned} S_{21} &:= (-1)^{s_2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s_1}} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \zeta(w_1, w_2), \\ S_{22} &:= (-1)^{s_2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s_1}} \sum_{\substack{1 \leq w_1 \leq s_2 + r_1 + t_1 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \zeta(w_1, w_2), \\ S_{23} &:= (-1)^{s_2} \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 + t_2 = s_1}} \sum_{\substack{2 \leq w_2 \leq r_2 + t_2 \\ w_1 + w_2 = w}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2} \zeta(w_1, w_2). \end{aligned}$$

For S_{21} , we can calculate the sum over binomial coefficients and get

$$S_{21} = (-1)^{s_2} \binom{w-2}{s_1} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \zeta(w_1, w_2) = (-1)^{s_2} \binom{w-2}{s_1} \zeta(w)$$

by the usual sum formula. For S_{22}, S_{23} , note that

$$\begin{aligned} S_{22} &= (-1)^{s_2} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \zeta(w_1, w_2) \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 \geq w_1 - (s_2 + r_1) \\ t_1 + t_2 = s_1}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2}, \\ S_{23} &= (-1)^{s_2} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \zeta(w_1, w_2) \sum_{\substack{t_1, t_2 \geq 0 \\ t_2 \geq w_2 - r_2 \\ t_1 + t_2 = s_1}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2}. \end{aligned}$$

Since the range of (t_1, t_2) for S_{22} and S_{23} are disjoint, we have

$$S_{22} + S_{23} = (-1)^{s_2} \sum_{\substack{w_1 \geq 1, w_2 \geq 2 \\ w_1 + w_2 = w}} \zeta(w_1, w_2) \sum_{\substack{t_1, t_2 \geq 0 \\ t_1 \geq w_1 - (s_2 + r_1) \text{ or } t_2 \geq w_2 - r_2 \\ t_1 + t_2 = s_1}} \binom{w_1 - 1}{t_1} \binom{w_2 - 1}{t_2}.$$

By combining the above calculations, we obtain the theorem. $\square$

As a special case of Theorem 3.13, we obtain the following sum formula of bounded type for the hook shape. Note that the left hand side of the next corollary corresponds to the hook shape $(s, 1^r)$ when $s \geq 2$.

Corollary 3.15. *For $s, r \geq 1$ and $w \geq s + r + 2$, we have*

$$\begin{aligned} S_w \begin{pmatrix} 0, s-1 \\ r, 1 \end{pmatrix} &= \binom{w-2}{s-1} \zeta(w) - \sum_{k=1}^{s-1} \binom{w-k-2}{s-k-1} \zeta(k, w-k) + (-1)^s \sum_{k=s}^{s+r-1} \zeta(k, w-k) \\ &\quad - \sum_{k=2}^{s-1} (-1)^k \binom{w-k-2}{s-k-1} \zeta(k) \zeta(w-k) - \sum_{k=2}^r \binom{w-k-2}{s-2} \zeta(k) \zeta(w-k) \\ &\quad + (-1)^r \sum_{k=r+1}^{s+r-1} (-1)^k \binom{w-k-2}{r+s-1-k} \zeta(k) \zeta(w-k). \end{aligned}$$

Proof. Follows from Theorem 3.13 and some rearrangement of terms. $\square$

Example 3.16. For the shape $\lambda = (3, 1, 1)$ we obtain for $w \geq 7$ the sum formula

$$\begin{aligned} S_w \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \right) &= S_w \begin{pmatrix} 0, 2 \\ 2, 1 \end{pmatrix} \\ &= \binom{w-2}{2} \zeta(w) - (w-3)\zeta(2)\zeta(w-2) - (w-5)\zeta(3)\zeta(w-3) + \zeta(4)\zeta(w-4) \\ &\quad - (w-3)\zeta(1, w-1) - \zeta(2, w-2) - \zeta(3, w-3) - \zeta(4, w-4) \end{aligned}$$

from Corollary 3.15 by taking $s = 3$ and $r = 2$.

4. DIAGRAMS WITH ONE CORNER

In this section, we consider general shapes with one corner. Recall that in Section 2 we defined for an index $\mathbf{k} = (k_1, \dots, k_d)$ and $l \geq 0$

$$Q_l(\mathbf{k}) = \sum_{\substack{\mathbf{w}=(w_1, \dots, w_d): \text{adm.} \\ \text{wt}(\mathbf{w})=\text{wt}(\mathbf{k})+l}} \binom{w_1-1}{k_1-1} \cdots \binom{w_{d-1}-1}{k_{d-1}-1} \binom{w_d-2}{k_d-1} \zeta(\mathbf{w}).$$

We will show in Theorem 4.2 that our target $S_w(\lambda/\mu)$ defined in (1.2) with λ/μ having one corner can be expressed explicitly in terms of $Q_{w-|\lambda/\mu|}$. This will follow from a purely combinatorial argument and we will see that this statement is already true for truncated Schur MZVs. For an integer $M \geq 1$ and an arbitrary, not necessarily admissible, Young tableau $\mathbf{k}$, these are defined by

$$\zeta_M(\mathbf{k}) = \sum_{\substack{(m_{i,j}) \in \text{SSYT}(\lambda/\mu) \\ m_{i,j} < M}} \prod_{(i,j) \in D(\lambda/\mu)} \frac{1}{m_{i,j}^{k_{i,j}}}.$$

In particular, for an integer $M \geq 1$ and an index $\mathbf{k} = (k_1, \dots, k_d)$ with $k_1, \dots, k_d \geq 1$ the truncated MZVs are given by

$$\zeta_M(k_1, \dots, k_d) = \sum_{0 < m_1 < \dots < m_d < M} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}.$$

To give the precise statement of above mentioned theorem, we will need to introduce some algebraic setup following [4]. Denote by $\mathfrak{H}^1 = \mathbb{Q}\langle z_k \mid k \geq 1 \rangle$ the non-commutative polynomial ring in the variables z_k for $k \geq 1$. A monic monomial in $\mathfrak{H}^1$ is called a word and the empty word will be denoted by $\mathbf{1}$. For $k \geq 1$ and $n \in \mathbb{Z}$, we define

$$z_k^n = \begin{cases} \underbrace{z_k \cdots z_k}_n & \text{if } n > 0, \\ \mathbf{1} & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

There is a one-to-one correspondence between indices and words; to each index $\mathbf{k} = (k_1, \dots, k_d)$ corresponds the word $z_{\mathbf{k}} := z_{k_1} \cdots z_{k_d}$. Thus we can extend various functions on indices to $\mathbb{Q}$ -linear maps on $\mathfrak{H}^1$. For example, we define $Q_l: \mathfrak{H}^1 \rightarrow \mathbb{R}$ by setting $Q_l(z_{\mathbf{k}}) = Q_l(\mathbf{k})$ and extending it linearly.

We define the stuffle product $*$ and the index shuffle product $\sqcup$ on $\mathfrak{H}^1$ as the $\mathbb{Q}$ -bilinear products, which satisfy $\mathbf{1} * w = w * \mathbf{1} = w$ and $\mathbf{1} \sqcup w = w \sqcup \mathbf{1} = w$ for any word $w \in \mathfrak{H}^1$ and for any $i, j \geq 1$ and words $w_1, w_2 \in \mathfrak{H}^1$,

$$z_i w_1 * z_j w_2 = z_i(z_j w_2) + z_j(z_i w_1) + z_{i+j}(w_1 * w_2),$$

$$z_i w_1 \sqcup z_j w_2 = z_i(w_1 \sqcup z_j w_2) + z_j(z_i w_1 \sqcup w_2).$$

By [4, Theorem 2.1] we obtain a commutative $\mathbb{Q}$ -algebra $\mathfrak{H}_*^1$.

Lemma 4.1. *Let $D_1, \dots, D_r$ be non-empty subsets of the skew Young diagram $D(\lambda/\mu)$ which gives a disjoint decomposition of $D(\lambda/\mu)$, i.e., $D(\lambda/\mu) = D_1 \sqcup \dots \sqcup D_r$. Then the following conditions are equivalent:*

- (i) *If we set $t_{ij} = a$ for $(i, j) \in D_a$ with $a = 1, \dots, r$, then (t_{ij}) is a semi-standard Young tableau of shape λ/μ .*
- (ii) *There exists a semi-standard Young tableau (m_{ij}) of shape λ/μ such that*

$$m_{ij} < m_{kl} \iff a < b$$

holds for any $(i, j) \in D_a$ and $(k, l) \in D_b$.

Proof. Obviously, (i) implies (ii). Conversely, assume that a semi-standard tableau (m_{ij}) satisfies the condition in (ii). If (t_{ij}) is defined as in (i), one has

$$m_{ij} < m_{kl} \iff a < b \iff t_{ij} < t_{kl}$$

for any $(i, j) \in D_a$ and $(k, l) \in D_b$, which shows that (t_{ij}) is also semi-standard. Thus (ii) implies (i). $\square$

We call a tuple $(D_1, \dots, D_r)$ of non-empty subsets of $D(\lambda/\mu)$ satisfying the equivalent conditions (i) and (ii) of Lemma 4.1 a *semi-standard decomposition*. Let $\text{SSD}(\lambda/\mu)$ denote the set of all semi-standard decompositions of $D(\lambda/\mu)$. Then we define an element $\varphi(\lambda/\mu)$ of $\mathfrak{H}^1$ by

$$\varphi(\lambda/\mu) := \sum_{(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)} z_{|D_1|} \cdots z_{|D_r|},$$

where $|D_i|$ denotes the number of elements of D_i . This element is related to the sum formula as follows.

Theorem 4.2. *When λ/μ has only one corner, we have for $w > |\lambda/\mu|$*

$$S_w(\lambda/\mu) = Q_{w-|\lambda/\mu|}(\varphi(\lambda/\mu)).$$

Proof. For any admissible Young tableau $\mathbf{k} = (k_{ij})$ of shape λ/μ , we see that

$$(4.1) \quad \zeta(\mathbf{k}) = \sum_{(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)} \zeta \left(\sum_{(i,j) \in D_1} k_{ij}, \dots, \sum_{(i,j) \in D_r} k_{ij} \right)$$

by classifying the semi-standard tableaux (m_{ij}) of shape λ/μ according to the semi-standard decompositions $(D_1, \dots, D_r)$ determined as in Lemma 4.1 (ii). Then, from the definition of $S_w(\lambda/\mu)$, we have

$$\begin{aligned} S_w(\lambda/\mu) &= \sum_{\substack{\mathbf{k} \in \text{YT}(\lambda/\mu) \\ \mathbf{k}: \text{adm.} \\ \text{wt}(\mathbf{k})=w}} \zeta(\mathbf{k}) = \sum_{\substack{k_{ij} \geq 1, (i,j) \in D(\lambda/\mu) \\ k_{ij} \geq 2, (i,j) \in C(\lambda/\mu) \\ \sum_{(i,j)} k_{ij} = w}} \sum_{(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)} \zeta \left(\sum_{(i,j) \in D_1} k_{ij}, \dots, \sum_{(i,j) \in D_r} k_{ij} \right) \\ &= \sum_{(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)} \sum_{\substack{w_1, \dots, w_{r-1} \geq 1 \\ w_r \geq 2 \\ w_1 + \dots + w_r = w}} \binom{w_1 - 1}{|D_1| - 1} \cdots \binom{w_{r-1} - 1}{|D_{r-1}| - 1} \binom{w_r - 2}{|D_r| - 1} \zeta(w_1, \dots, w_r). \end{aligned}$$

Here recall that $C(\lambda/\mu)$ is the set of corners of λ/μ and note that, for any $(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)$, the unique corner of λ/μ belongs to D_r . The last expression above is equal to

$$\sum_{(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)} Q_{w-|\lambda/\mu|}(|D_1|, \dots, |D_r|) = Q_{w-|\lambda/\mu|}(\varphi(\lambda/\mu)).$$

Thus the proof is complete. $\square$

By Remark 2.2 we see that Theorem 4.2 gives a sum formula of bounded type for shapes with one corner. In order to evaluate the sum $S_w(\lambda/\mu)$ in the one corner case, one therefore needs to find an expression of $\varphi(\lambda/\mu)$. For this purpose, the following expression is useful:

Proposition 4.3. *For any skew shape λ/μ , let $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ and $\mu' = (\mu'_1, \dots, \mu'_s)$ be the conjugates of λ and μ , respectively. Then we have the identity*

$$\varphi(\lambda/\mu) = \det_* [z_1^{\lambda'_i - \mu'_j - i + j}]_{1 \leq i, j \leq s},$$

where $\det_*$ denotes the determinant performed in the stuffle algebra $\mathfrak{H}_*^1$.

Proof. For any integer $M > 0$, we have

$$\zeta_M(\varphi(\lambda/\mu)) = \sum_{(D_1, \dots, D_r) \in \text{SSD}(\lambda/\mu)} \zeta_M(|D_1|, \dots, |D_r|) = \zeta_M(\mathbf{O}_{\lambda/\mu}),$$

where $\mathbf{O}_{\lambda/\mu}$ denotes the tableau of shape λ/μ all entries of which are 1. Indeed, the first equality is obvious from the definition of $\varphi(\lambda/\mu)$, and the second follows in the same way as (4.1).

On the other hand, by the Jacobi-Trudi type formula for truncated Schur MZVs ([7, Theorem 1.1], [1, Theorem 4.7]), we have

$$\zeta_M(\mathbf{O}_{\lambda/\mu}) = \det[\zeta_M(z_1^{\lambda'_i - \mu'_j - i + j})]_{1 \leq i, j \leq s} = \zeta_M\left(\det_* [z_1^{\lambda'_i - \mu'_j - i + j}]_{1 \leq i, j \leq s}\right).$$

Since the map $\mathfrak{H}^1 \rightarrow \prod_{M \geq 1} \mathbb{Q}$ given by $w \mapsto (\zeta_M(w))_{M \geq 1}$ is injective ([8, Theorem 3.1]), we obtain the desired identity in $\mathfrak{H}^1$. $\square$

In some cases, one can compute $\varphi(\lambda/\mu)$ explicitly by using Proposition 4.3.

Theorem 4.4. *For $n \geq 1$ and $k \geq 0$, it holds that*

$$\varphi((2^{n+k})/(1^k)) = \sum_{l=0}^n \frac{k+1}{l+k+1} \binom{2l+k}{l} z_2^{n-l} \sqcup z_1^{2l+k}.$$

Proof. By Proposition 4.3, we have

$$\varphi((2^{n+k})/(1^k)) = \begin{vmatrix} z_1^n & z_1^{n+k+1} \\ z_1^{n-1} & z_1^{n+k} \end{vmatrix} = z_1^{n+k} * z_1^n - z_1^{n+k+1} * z_1^{n-1}.$$

By [6, Lemma 1] we get for $m \geq n \geq 1$

$$(4.2) \quad z_1^m * z_1^n = \sum_{l=0}^n \binom{m+n-2l}{m-l} \sum_{\mathbf{k} \in G_l^{m+n-2l}} z_{\mathbf{k}} = \sum_{l=0}^n \binom{m+n-2l}{n-l} z_2^l \sqcup z_1^{m+n-2l},$$

where G_b^a is the set of all possible indices containing a times 1 and b times 2. This gives

$$\begin{aligned} & \varphi((2^{n+k})/(1^k)) \\ &= z_1^{n+k} * z_1^n - z_1^{n+k+1} * z_1^{n-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^n \binom{2n+k-2l}{n-l} z_2^l \sqcup z_1^{2n+k-2l} - \sum_{l=0}^{n-1} \binom{2n+k-2l}{n-l-1} z_2^l \sqcup z_1^{2n+k-2l} \\
&= \sum_{l=0}^n \left(\binom{2n+k-2l}{n-l} - \binom{2n+k-2l}{n-l-1} \right) z_2^l \sqcup z_1^{2n+k-2l} \\
&= \sum_{l=0}^n \left(\binom{2l+k}{l} - \binom{2l+k}{l-1} \right) z_2^{n-l} \sqcup z_1^{2l+k} \\
&= \sum_{l=0}^n \frac{k+1}{l+k+1} \binom{2l+k}{l} z_2^{n-l} \sqcup z_1^{2l+k}. \quad \square
\end{aligned}$$

For some shapes λ/μ , the $\varphi(\lambda/\mu)$ will contain sums over all indices over a fixed weight and depth. The corresponding sums of Q applied to these can be evaluated by using the following lemma.

Lemma 4.5. *For $k \geq d \geq 1, w \geq d+1$ and an index $\mathbf{n} = (n_1, \dots, n_d)$, we have*

$$\sum_{\substack{\mathbf{k}=(k_1, \dots, k_d) \\ \text{wt}(\mathbf{k})=k}} P_{w-k}(\mathbf{n}; \mathbf{k}) = \binom{w - \text{wt}(\mathbf{n})}{k-d} \zeta(w).$$

In particular,

$$\sum_{\substack{\mathbf{k}=(k_1, \dots, k_d) \\ \text{wt}(\mathbf{k})=k}} P_{w-k}(\mathbf{k}) = \binom{w-d}{k-d} \zeta(w), \quad \sum_{\substack{\mathbf{k}=(k_1, \dots, k_d) \\ \text{wt}(\mathbf{k})=k}} Q_{w-k}(\mathbf{k}) = \binom{w-d-1}{k-d} \zeta(w).$$

Proof. This follows directly by using the Chu-Vandermonde identity and the usual sum formula for MZVs. $\square$

Remark 4.6. For $s \geq 0$ and $r \geq 1$, one can show the identity

$$\varphi(((s+1)^r)/(s^{r-1})) = \sum_{l=0}^s \binom{r+l-1}{l} \sum_{\substack{\mathbf{k}=(k_1, \dots, k_{r+l}) \\ \text{wt}(\mathbf{k})=r+s}} z_{\mathbf{k}}$$

by counting how many times $z_{\mathbf{k}}$ appears for each $\mathbf{k}$, as in the proof of Theorem 3.4. (One can also show this by induction on s via the expression

$$\varphi(((s+1)^r)/(s^{r-1})) = \det_* \begin{bmatrix} z_1 & z_1^2 & \cdots & z_1^s & z_1^{r+s} \\ 1 & z_1 & \cdots & z_1^{s-1} & z_1^{r+s-1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & z_1 & z_1^{r+1} \\ 0 & \cdots & 0 & 1 & z_1^r \end{bmatrix}$$

obtained from Proposition 4.3 together with (4.2).) Using Theorem 4.2 and Lemma 4.5 this gives another way of proving the anti-hook sum formula in Theorem 3.8

$$\begin{aligned}
S_w(((s+1)^r)/(s^{r-1})) &= Q_{w-(r+s)}(\varphi(((s+1)^r)/(s^{r-1}))) \\
&= \sum_{l=0}^s \binom{r+l-1}{l} \sum_{\substack{\mathbf{k}=(k_1, \dots, k_{r+l}) \\ \text{wt}(\mathbf{k})=r+s}} Q_{w-(r+s)}(\mathbf{k})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^s \binom{r+l-1}{l} \binom{w-r-l-1}{s-l} \zeta(w) \\
&= \binom{w-1}{s} \zeta(w).
\end{aligned}$$

We can summarize the general strategy to give a bounded expression of $S_w(\lambda/\mu)$ for the case when λ/μ has one corner as follows:

- (i) Give an expression for $\varphi(\lambda/\mu)$, by evaluating the determinant in Proposition 4.3 by the stuffle product. Then use Theorem 4.2 to get $S_w(\lambda/\mu) = Q_{w-|\lambda/\mu|}(\varphi(\lambda/\mu))$.
- (ii) If sums of Q_l over all indices of a fixed weight and depth appear, use Lemma 4.5 to write them in terms of Riemann zeta values.
- (iii) For other terms involving Q write them in terms of P , by using (2.1), i.e.,

$$Q_l(k_1, \dots, k_d) = \sum_{j=0}^{k_d-1} (-1)^j P_{l+j}(k_1, \dots, k_{d-1}, k_d - j).$$

Then use Proposition 2.1 to get recursive (bounded) expressions of P_{l+j} in terms of MZVs. For depth 2 and 3 explicit expressions are given by Corollary 2.6 and 2.7.

Example 4.7. Using above strategy we get formula (1.3) for $S_w \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right)$ in the introduction and the following examples:

- (i) For $w \geq 6$ we have

$$\begin{aligned}
S_w \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right) &= \binom{w-2}{2} \zeta(2) \zeta(w-2) - \frac{5}{4} \zeta(4) \zeta(w-4) + \zeta(2) \zeta(2, w-4) \\
&\quad - \zeta(2) \zeta(1, w-3) - \binom{w-2}{2} \zeta(1, w-1) + \binom{w-3}{2} \zeta(2, w-2) \\
&\quad + (w-3) \zeta(3, w-3) + (w-3) \zeta(1, 1, w-2) - (w-5) \zeta(1, 2, w-3) \\
&\quad - 2\zeta(1, 3, w-4) + \zeta(2, 1, w-3) - \zeta(2, 2, w-4).
\end{aligned}$$

- (ii) For $w \geq 6$ we have

$$\begin{aligned}
S_w \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right) &= (w-2) \zeta(2) \zeta(w-2) + (w-5) \zeta(3) \zeta(w-3) - \frac{5}{4} \zeta(4) \zeta(w-4) \\
&\quad - \zeta(2) \zeta(1, w-3) + \zeta(2) \zeta(2, w-4) + (2-w) \zeta(1, w-1) + (w-4) \zeta(2, w-2) \\
&\quad + 2\zeta(3, w-3) + (w-3) \zeta(1, 1, w-2) - (w-5) \zeta(1, 2, w-3) \\
&\quad - 2\zeta(1, 3, w-4) + \zeta(2, 1, w-3) - \zeta(2, 2, w-4).
\end{aligned}$$

Comparing (i) and (ii) with (1.3), we see that for all $w \geq 1$ we have

$$(4.3) \quad 2S_w \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right) - 2S_w \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right) = (w-5) S_w \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right).$$

We will now show that the relation (4.3) among S_w for different shapes is a special case of a more general family of relations. For this we first notice that any skew Young diagram $D(\lambda/\mu)$ with one corner can be written as

$$\lambda = (n^m) = \underbrace{(n, \dots, n)}_m, \quad \mu = (\mu_1, \dots, \mu_m)$$

with $\mu_1 = n$, $\mu_m > 0$ (for example, the one-box diagram is represented as $(2, 2)/(2, 1)$). Then we write I for the set of $i = 1, \dots, m$ that

$$\mu[i] := (\mu_1, \dots, \mu_i - 1, \dots, \mu_m)$$

is non-increasing. Then, for $i \in I$, $D(\lambda/\mu[i]) = D(\lambda/\mu) \cup \{(i, \mu_i)\}$ is still a skew Young diagram with one corner. As a generalization of (4.3) we obtain the following.

Theorem 4.8. *If λ/μ is a skew Young diagram with one corner then for all $w \geq 1$*

$$(4.4) \quad \sum_{i \in I} ((i - \mu_i) - (m - n)) S_w(\lambda/\mu[i]) = (w - |\lambda/\mu| - 1) S_w(\lambda/\mu).$$

To prove this, we need the following Lemma. Define a linear map $\partial: \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$ by

$$\partial(z_{k_1} \cdots z_{k_d}) := \sum_{a=1}^d k_a z_{k_1} \cdots z_{k_{a+1}} \cdots z_{k_d}.$$

In particular, $\partial(1) = 0$.

Lemma 4.9.

- (i) ∂ is a derivation with respect to the stuffle product.
- (ii) For any $N \in \mathbb{Z}$, we have

$$\partial(z_1^N) = z_1 * z_1^N - (N + 1) z_1^{N+1}.$$

- (iii) We have

$$(l - 1) Q_l(v) = Q_{l-1}(\partial(v))$$

for any $v \in \mathfrak{H}^1$.

Proof. (i) is verified, e.g., by induction on the depth. It is also easy to show (ii) from the definition. Finally, the identity in (iii) follows from the definition of Q_l and the identity

$$\begin{aligned} & (w - k - 1) \binom{w_1 - 1}{k_1 - 1} \cdots \binom{w_d - 2}{k_d - 1} \\ &= ((w_1 - k_1) + \cdots + (w_d - 1 - k_d)) \binom{w_1 - 1}{k_1 - 1} \cdots \binom{w_d - 2}{k_d - 1} \\ &= \sum_{a=1}^d k_a \binom{w_1 - 1}{k_1 - 1} \cdots \binom{w_a - 1}{k_a} \cdots \binom{w_d - 2}{k_d - 1}, \end{aligned}$$

where $w = w_1 + \cdots + w_d$ and $k = k_1 + \cdots + k_d$. □

Proof of Theorem 4.8. By Theorem 4.2 and Proposition 4.3, we have

$$S_w(\lambda/\mu) = Q_{w-|\lambda/\mu|} \left(\det_* [z_1^{m-\mu'_j-i+j}]_{1 \leq i, j \leq n} \right).$$

Here $(\mu'_1, \dots, \mu'_n)$ denotes the transpose of the partition $\mu = (\mu_1, \dots, \mu_m)$. Note that $\mu'_1 = m$ and $\mu'_n > 0$. By Lemma 4.9 (iii) and (i), we see that

$$\begin{aligned} (4.5) \quad (w - |\lambda/\mu| - 1) S_w(\lambda/\mu) &= (w - |\lambda/\mu| - 1) Q_{w-|\lambda/\mu|} \left(\det_* [z_1^{m-\mu'_j-i+j}]_{1 \leq i, j \leq n} \right) \\ &= Q_{w-|\lambda/\mu|-1} \left(\partial \det_* [z_1^{m-\mu'_j-i+j}]_{1 \leq i, j \leq n} \right) \\ &= \sum_{k=1}^n Q_{w-|\lambda/\mu|-1} \left(\det_* [\partial^{\delta_{jk}} (z_1^{m-\mu'_j-i+j})]_{1 \leq i, j \leq n} \right). \end{aligned}$$

Here $\partial^{\delta_{jk}}$ means the operator ∂ if $j = k$ and the identity operator otherwise. Moreover, since the identity

$$\begin{aligned}\partial(z_1^{m-\mu'_k-i+k}) &= z_1 * z_1^{m-\mu'_k-i+k} - (m - \mu'_k - i + k + 1)z_1^{m-\mu'_k-i+k+1} \\ &= ((\mu'_k - k) - (m - n))z_1^{m-(\mu'_k-1)-i+k} \\ &\quad + z_1 * z_1^{m-\mu'_k-i+k} - (n + 1 - i)z_1^{m-(\mu'_k-1)-i+k}\end{aligned}$$

holds by Lemma 4.9 (ii), we have

$$\begin{aligned}(4.6) \quad \det_*[\partial^{\delta_{jk}}(z_1^{m-\mu'_j-i+j})] \\ &= ((\mu'_k - k) - (m - n)) \det_*[z_1^{m-(\mu'_j-\delta_{jk})-i+j}] \\ &\quad + z_1 * \det_*[z_1^{m-\mu'_j-i+j}] - \det_*[(n + 1 - i)\delta_{jk} z_1^{m-(\mu'_j-\delta_{jk})-i+j}].\end{aligned}$$

On the other hand, the left-hand side of (4.4) is equal to

$$(4.7) \quad \sum_{k=1}^n ((\mu'_k - k) - (m - n)) Q_{w-|\lambda/\mu|-1} \left(\det_*[z_1^{m-(\mu'_j-\delta_{jk})-i+j}] \right).$$

Here, a priori, k runs only over the indices such that $\mu'_k > \mu'_{k+1}$. However, if $\mu'_k = \mu'_{k+1}$, the k -th and $(k+1)$ -st columns of the matrix $(z_1^{m-(\mu'_j-\delta_{jk})-i+j})_{i,j}$ are equal, so the determinant is zero.

Comparing (4.5), (4.6) and (4.7), it suffices to prove the equality

$$(4.8) \quad n z_1 * \det_*[z_1^{m-\mu'_j-i+j}] \stackrel{?}{=} \sum_{k=1}^n \det_*[(n + 1 - i)\delta_{jk} z_1^{m-(\mu'_j-\delta_{jk})-i+j}].$$

Let us compute the right-hand side by the cofactor expansion with respect to the k -th column.

$$\begin{aligned}&\sum_{k=1}^n \det_*[(n + 1 - i)\delta_{jk} z_1^{m-(\mu'_j-\delta_{jk})-i+j}] \\ &= \sum_{k=1}^n \sum_{l=1}^n (-1)^{k+l} (n + 1 - l) z_1^{m-(\mu'_k-1)-l+k} * \det_*[z_1^{m-\mu'_j-i+j}]_{i \neq l, j \neq k} \\ &= \sum_{l=1}^n (n + 1 - l) \sum_{k=1}^n (-1)^{k+l} z_1^{m-(\mu'_k-1)-l+k} * \det_*[z_1^{m-\mu'_j-i+j}]_{i \neq l, j \neq k}.\end{aligned}$$

Then, by the cofactor expansion with respect to the l -th row, we have

$$\sum_{k=1}^n (-1)^{k+l} z_1^{m-(\mu'_k-1)-l+k} * \det_*[z_1^{m-\mu'_j-i+j}]_{i \neq l, j \neq k} = \det_*[z_1^{m-\mu'_j-(i-\delta_{il})+j}].$$

This is zero for $l = 2, \dots, n$ since the l -th and $(l-1)$ -st rows are equal. Thus we have shown that the right-hand side of (4.8) is $n \det_*[z_1^{m+\delta_{i1}-\mu'_j-i+j}]$, and now it is enough to prove

$$z_1 * \det_*[z_1^{m-\mu'_j-i+j}] \stackrel{?}{=} \det_*[z_1^{m+\delta_{i1}-\mu'_j-i+j}].$$

But this is obvious since two matrices here are of the form

$$\begin{pmatrix} 1 & * \\ \mathbf{0} & Z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 & * \\ \mathbf{0} & Z \end{pmatrix},$$

respectively, with the common $(n-1) \times (n-1)$ matrix $Z = [z_1^{m-\mu'_i-i+j}]_{2 \leq i, j \leq n}$. Hence the proof is complete. $\square$

REFERENCES

- [1] H. Bachmann, Interpolated Schur multiple zeta values, *J. Aust. Math. Soc.* **104** (2018), 289–307.
- [2] A. Granville, A decomposition of Riemann’s zeta-function, *Analytic number theory* (Kyoto, 1996), 95–101, London Math. Soc. Lecture Note Ser., 247, Cambridge Univ. Press, Cambridge, 1997.
- [3] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (2) (1992), 275–290.
- [4] M. E. Hoffman, The algebra of multiple harmonic series, *J. Algebra* **194** (1997), 477–495.
- [5] M. Kaneko and S. Yamamoto, A new integral-series identity of multiple zeta values and regularizations. *Sel. Math. New Ser.* **24** (2018), 2499–2521.
- [6] Kwang-Wu Chen, Applications of stuffle product of multiple zeta values, *Journal of Number Theory*, **153** (2015), 107–116.
- [7] M. Nakasuji, O. Phuksuwan and Y. Yamasaki, On Schur multiple zeta functions: a combinatoric generalization of multiple zeta functions, *Adv. Math.* **333** (2018), 570–619.
- [8] S. Yamamoto, Explicit evaluation of certain sums of multiple zeta-star values, *Functiones et Approximatio* **49** (2013), 283–289.
- [9] S. Yamamoto, Multiple zeta-star values and multiple integrals, Various aspects of multiple zeta values 2016, *RIMS Kôkyûroku Bessatsu* **B68** (2017), 3–14.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, JAPAN.

Email address: henrik.bachmann@math.nagoya-u.ac.jp

FACULTY OF FUNDAMENTAL SCIENCE, NATIONAL INSTITUTE OF TECHNOLOGY (KOSEN), NIIHAMA COLLEGE, NIIHAMA, JAPAN.

Email address: s.kadota@niihama-nct.ac.jp

DEPARTMENT OF MATHEMATICS, RIKKYO UNIVERSITY, TOKYO, JAPAN

Email address: suzuyu@rikkyo.ac.jp

FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY, YOKOHAMA, JAPAN

Email address: yamashu@math.keio.ac.jp

GRADUATE SCHOOL OF SCIENCE AND ENGINEERING, EHIME UNIVERSITY, MATSUYAMA, JAPAN

Email address: yamasaki@math.sci.ehime-u.ac.jp